# Chapter 8: Test of Hypotheses for a Single Sample

**Course Name: PROBABILITY & STATISTICS**

**Lecturer: Huong Pham**

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# Content

# Content

## Problem

An air crew escape system (hệ thống thoát hiểm phi hành đoàn) that consists of an ejection seat (ghế phóng) and a rocket motor that powers the seat. The rocket motor contains a propellant (nhiên liệu đẩy).

In order for the ejection seat to function properly, the propellant should have a mean burning rate of 50 cm/s.

So the practical engineering question that must be answered is: Does the mean burning rate of the propellant equal 50 cm/s, or is it some other value (either higher or lower)?

# Statistical Hypotheses

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# Null hypothesis and alternative hypothesis

## Example 1

Consider the aircrew escape system described in the introduction. Specifically, we are interested in deciding whether or not the mean burning rate is 50 centimeters per second.

# Null hypothesis and alternative hypothesis

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$\mu$ : the mean burning rate.
Null hypothesis $H_0$: $\mu = 50$ cm/s.
Alternative hypothesis $H_1$: $\mu \neq 50$ cm/s.

# Null hypothesis and alternative hypothesis

**Example 2**

A researcher claims that less than 63% of voters favor gun (súng) control. Express the null hypothesis and the alternative hypothesis.

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A researcher claims that less than 63% of voters favor gun (súng) control. Express the null hypothesis and the alternative hypothesis.

$p$: percentage of voters favor gun control.
Null hypothesis $H_0$: $p = 0.63$
Alternative hypothesis $H_1$: $p < 0.63$.

# Null hypothesis and alternative hypothesis

**Question 1**

The owner of a football team claims that the average attendance at games is over 45,000. Express the null hypothesis and the alternative hypothesis in symbolic form.
$\mu$ : the average attendance at games.

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## Question 2

A cereal company (công ty ngũ cốc) claims that the mean weight of the cereal in its packets is 14.2 oz. Express the null hypothesis and the alternative hypothesis in symbolic form.

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A cereal company (công ty ngũ cốc) claims that the mean weight of the cereal in its packets is 14.2 oz. Express the null hypothesis and the alternative hypothesis in symbolic form.
$\mu$ : the mean weight of the cereal in its packets.
**Answer**: $H_0 : \mu = 14.2 \quad H_1 : \mu \neq 14.2.$
This is a two-sided alternative hypothesis.

# Null hypothesis and alternative hypothesis

We have 2 form on the alternative hypothesis:

(1) **two-sided alternative hypothesis**

$$H_0 : \mu = \mu_0 \quad H_1 : \mu \neq \mu_0$$

(2) **one-sided alternative hypothesis**

$$H_0 : \mu = \mu_0 \qquad H_1 : \mu > \mu_0 \text{ or}$$
$$H_0 : \mu = \mu_0 \qquad H_1 : \mu < \mu_0.$$

# Tests of Statistical Hypotheses

## What is Hypothesis testing?

A procedure leading to a decision about a particular hypothesis is called a **test of a hypothesis**. Hypothesis-testing procedures rely on using the information in a random sample from the population of interest. If this information is consistent (phù hợp) with the hypothesis, we will not reject the hypothesis; however, if this information is inconsistent with the hypothesis, we will conclude that the hypothesis is false.

# Tests of Statistical Hypotheses

Suppose we wish to test
$H_0$: $\mu = 50$ cm/s.
$H_1$: $\mu \neq 50$ cm/s.
Taking a sample of size $n = 10$.
$\bar{x}$: sample mean burning rate. $\bar{x}$ can take on many different values.
Suppose that

- If $48.5 \leq \bar{x} \leq 51.5$, not reject $H_0$.
- If either $\bar{x} < 48.5$ or $\bar{x} > 51.5$, reject $H_0$.

**Definition**

- The **critical region** for the test: ($\bar{x} < 48.5$ or $\bar{x} > 51.5$) is the region that rejects $H_0$.
- The **acceptance region** for the test: ($48.5 \leq \bar{x} \leq 51.5$) is the region that fails to reject $H_0$.
- The **critical values**: (48.5 and 51.5) are the boundaries between the critical regions and the acceptance region.

## Type I error and Type II error

- Rejecting the null hypothesis $H_0$ when it is true is defined as a **type I error**.
- Failing to reject the null hypothesis when it is false is defined as a **type II error**.

| Decision | $H_0$ Is True | $H_0$ Is False |
|---|---|---|
| Fail to reject $H_0$ | no error | type II error |
| Reject $H_0$ | type I error | no error |

## Probability of Type I Error

$$\alpha = P(\text{type I error}) = P(\text{reject } H_0 \text{ when } H_0 \text{ is true}).$$

Sometimes the type I error probability is called the **significance level** (mức ý nghĩa), or the $\alpha$**-error,** or the **size** of the test.

## Probability of Type II Error

$$\beta = P(\text{type II error}) = P(\text{fail to reject } H_0 \text{ when } H_0 \text{ is false})$$

# Computing the Type I Error Probability

In the propellant burning rate example, a type I error will occur when either $\bar{x} > 51.5$ or $\bar{x} < 48.5$ when the true mean burning rate really is 50 centimeters per second.

$$\alpha = P(\bar{X} < 48.5 \ \text{ when } \ \mu = 50) + \mathrm{P}(\bar{\mathrm{X}} > 51.5 \ \text{ when } \ \mu = 50).$$

Suppose that the standard deviation of burning rate is $\sigma = 2.5$ centimeters per second. According to the central limit theorem, the distribution of the sample mean is approximately normal with mean $\mu = 50$ and standard deviation $\sigma/\sqrt{n} = 2.5/\sqrt{10} = 0.79$. Therefore,

$$\begin{aligned} \alpha &= P(Z < \frac{48.5 - 50}{0.79}) + P(Z > \frac{51.5 - 50}{0.79}) \\ &= P(Z < -1.90) + P(Z > 1.90) = 0.0287 + 0.0287 = 0.0574 \end{aligned}$$

This implies that 5.74% of all random samples would lead to rejection of the hypothesis when the true mean burning rate is really 50 centimeters per second.

# Computing the Type I Error Probability

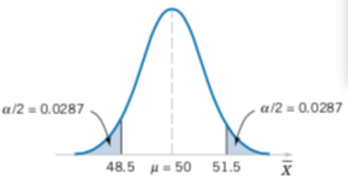


**Figure 9-2** The critical region for $H_0$: μ = 50 versus $H_1$: μ ≠ 50 and $n$ = 10.

From inspection of Fig. 9-2, notice that we can reduce $\alpha$ by widening the acceptance region (vùng chấp nhận/ miền chấp nhận). For example, if we make the critical values 48 and 52, the value of is

$$\alpha = P(Z < \frac{48-50}{0.79}) + P(Z > \frac{52-50}{0.79})$$
$$= P(Z < -2.53) + P(Z > 2.53) = 0.0057 + 0.0057 = 0.0114$$

# Computing the Type I Error Probability

We could also reduce $\alpha$ by increasing the sample size. If $n = 16, \sigma/\sqrt{n} = 2.5/\sqrt{16} = 0.625$ and using the original critical region (critical values are 48.5 and 51.5), we find

$$\begin{aligned}\alpha &= P(Z < \frac{48.5 - 50}{0.625}) + P(Z > \frac{51.5 - 50}{0.625}) \\ &= P(Z < -2.4) + P(Z > 2.4) = 0.0082 + 0.0082 = 0.0164\end{aligned}$$

# Computing the Type II Error Probability

To calculate $\beta$ (sometimes called the $\beta-$error), we must have a specific alternative hypothesis; that is, we must have a particular value of $\mu$.

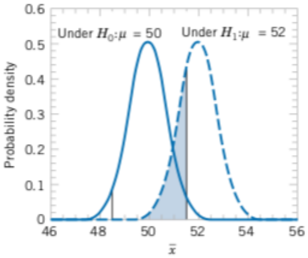


**Figure 9-3** The probability of type II error when $\mu = 52$ and $n = 10$.

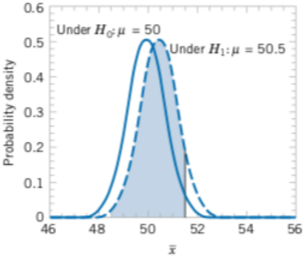


**Figure 9-4** The probability of type II error when $\mu = 50.5$ and $n = 10$.

# Computing the Type II Error Probability

Figure 9-3 will help us calculate the probability of type II error. We find that

$$\begin{aligned}\beta &= P(48.5 \leq \bar{X} \leq 51.5 \quad \text{when} \quad \mu = 52) \\ &= P(\frac{48.5 - 52}{0.79} \leq Z \leq \frac{51.5 - 52}{0.79}) \\ &= P(-4.43 \leq Z \leq -0.63) = P(Z \leq -0.63) - P(Z \leq -4.43) \\ &= 0.2643 - 0.000 = 0.2643\end{aligned}$$

Thus, if we are testing $H_0 : \mu = 50$ against $H_1 : \mu \neq 50$ with $n = 10$, and the true value of the mean is $\mu = 52$, the probability that we will fail to reject the false null hypothesis is 0.2643.

# Computing the Type II Error Probability

The probability of making a type II error increases rapidly as the true value of $\mu$ approaches the hypothesized value. For example, see Fig. 9-4, where the true value of the mean is $\mu = 50.5$ and the hypothesized value is $H_0 : \mu = 50$. Then

$$\begin{aligned}
\beta &= P(48.5 \leq \bar{X} \leq 51.5 \quad \text{when} \quad \mu = 50.5) \\
&= P(\frac{48.5 - 50.5}{0.79} \leq Z \leq \frac{51.5 - 50.5}{0.79}) \\
&= P(-2.53 \leq Z \leq 1.27) = P(Z \leq 1.27) - P(Z \leq -2.53) \\
&= 0.8980 - 0.0057 = 0.8923
\end{aligned}$$

# Computing the Type II Error Probability

The type II error probability also depends on the sample size $n$. Suppose that the null hypothesis is $H_0 : \mu = 50$ centimeters per second and that the true value of the mean is $\mu = 52$. When $n = 16$, we have

$$\begin{aligned}
\beta &= P(48.5 \leq \bar{X} \leq 51.5 \quad \text{when} \quad \mu = 52) \\
&= P(\frac{48.5 - 52}{0.625} \leq Z \leq \frac{51.5 - 52}{0.625}) \\
&= P(-5.60 \leq Z \leq -0.80) = P(Z \leq -0.80) - P(Z \leq -5.60) \\
&= 0.2119 - 0.0000 = 0.2119
\end{aligned}$$

# Computing the Type II Error Probability

The results from this section and a few other similar calculations are summarized in the following table.

| Acceptance Region | Sample Size | $\alpha$ | $\beta$ at $\mu = 52$ | $\beta$ at $\mu = 50.5$ |
|---|---|---|---|---|
| $48.5 < \bar{x} < 51.5$ | 10 | 0.0576 | 0.2643 | 0.8923 |
| $48 < \bar{x} < 52$ | 10 | 0.0114 | $\boxed{0.5000}$ | $\boxed{0.9705}$ |
| $48.81 < \bar{x} < 51.19$ | 16 | $\boxed{0.0576}$ | $\boxed{0.0966}$ | $\boxed{0.8606}$ |
| $48.42 < \bar{x} < 51.58$ | 16 | $\boxed{0.0114}$ | $\boxed{0.2515}$ | $\boxed{0.9578}$ |

# Test of Statistical Hypotheses

## Remark

- The size of the critical region, and consequently the probability of a type I error $\alpha$, can always be reduced by appropriate selection of the critical values.
- Type I and type II errors are related. A decrease in the probability of one type of error always results in an increase in the probability of the other, provided that the sample size ndoes not change.
- An increase in sample size reduces $\beta$, provided that $\alpha$ is held constant.
- When the null hypothesis is false, $\beta$ increases as the true value of the parameter approaches the value hypothesized in the null hypothesis. The value of $\beta$ decreases as the difference between the true mean and the hypothesized value increases.

# Test of Statistical Hypotheses

Generally, the analyst controls the type I error probability $\alpha$ when he or she selects the critical values. Thus, it is usually easy for the analyst to set the type I error probability at (or near) any desired value.

**Notice**

A widely used procedure in hypothesis testing is to use a type 1 error or significance level of $\alpha = 0.05$. This value has evolved through experience, and may not be appropriate for all situations.

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**Notice**

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An important concept that we will make use of is the **power** of a statistical test.

**Power**

The **power** of a statistical test is the probability of rejecting the null hypothesis $H_0$ when the alternative hypothesis is true.
The **power** is computed as $1 - \beta$, and power can be interpreted as the probability of correctly rejecting a false null hypothesis.

# P-Values in Hypothesis Tests

**P-Value**

The $P-$value is the smallest level of significance that would lead to rejection of the null hypothesis $H_0$ with the given data.

# Question 1

An entomologist writes an article in a scientific journal which claims that fewer than 21 in five thousand male fireflies are unable to produce light due to a genetic mutation. Use the parameter p, the true proportion of fireflies unable to produce light. Express the null hypothesis $H_0$ and the alternative hypothesis $H_1$ in symbolic form.

Select one:

- a. $H_0$: $p < 0.0041$
  $H_1$: $p \geq 0.0042$
- b. $H_0$: $p = 21$
  $H_1$: $p < 21$
- c. $H_0$: $p > 0.0011$
  $H_1$: $p \leq 0.0011$
- d. $H_0$: $p = 0.0042$
  $H_1$: $p < 0.0042$

# Question 2

An entomologist writes an article in a scientific journal which claims that fewer than 16 in four thousand male fireflies are unable to produce light due to a genetic mutation. Assume that a hypothesis test of the given claim will be conducted. Identify the type I error for the test.

Select one:

- a. The error of rejecting the claim that the true proportion is less than 16 in four thousand when it really is less than 16 in four thousand.
- b. The error of failing to reject the claim that the true proportion is at least 16 in four thousand when it really is at least 16 in four thousand.
- c. None of the other choices is true
- d. The error of rejecting the claim that the true proportion is at least 16 in ten thousand when it really is at least 16 in ten thousand.

# Question 3

Carter Motor Company claims that its new sedan, the Libra, will average better than 27 miles per gallon in the city. Assume that a hypothesis test of the given claim will be conducted. Identify the type I error for the test.

Select one:

- a. None of the other choices is true
- b. The error of rejecting the claim that the mean is at most 27 miles per gallon when it really is at most 27 miles per gallon.
- c. The error of failing to reject the claim that the mean is at most 27 miles per gallon when it is actually greater than 27 miles per gallon.
- d. The error of rejecting the claim that the true proportion is more than 27 miles per gallon when it really is more than 27 miles per gallon.

# Question 4

The owner of a football team claims that the average attendance at games is over 67,000, and he is therefore justified in moving the team to a city with a larger stadium. Assume that a hypothesis test of the given claim will be conducted. Identify the type I error for the test.

Select one:

- a. The error of rejecting the claim that the mean attendance is more than 67,000, when it is actually less than 67,000.
- b. The error of failing to reject the claim that the mean attendance is at most 67,000, when it is actually greater than 67,000.
- c. None of the other choices is true
- d. The error of rejecting the claim that the mean attendance is at most 67,000, when it really is at most 67,000.

# Question 5

A psychologist claims that more than 13 percent of the population suffers from professional problems due to extreme shyness. Assume that a hypothesis test of the given claim will be conducted. Identify the type I error for the test.

Select one:

- a. The error of rejecting the claim that the true proportion is more than 13 percent when it really is more than 13 percent.
- b. The error of rejecting the claim that the true proportion is at most 13 percent when it is actually at most 13 percent.
- c. None of the other choices is true
- d. The error of failing to reject the claim that the true proportion is at most 13 percent when it really is greater than 13 percent.

# Content

## 2. Tests on the Mean of a Normal Distribution, Variance Known

• We will assume that a random sample $X_1, X_2, ..., X_n$ has been taken from the population with $EX_i = \mu, DX_i = \sigma^2$. Based on our previous discussion, the sample mean $\bar{X}$ is an **unbiased point estimator** (best point estimator) of $\mu$ and $\bar{X} \simeq N(\mu, \sigma^2/n)$.

• $\mu_0$ is given value of $\mu$.

# 2.1 Hypothesis Tests on the Mean

- Suppose that we wish to test the hypotheses

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu \neq \mu_0.$$

- Since $\bar{X} \simeq N(\mu_0, \sigma^2/n)$ if the null hypothesis is true, we could calculate a P-value or construct a critical region based on the computed value of the sample mean $\bar{X}$.
- Test Statistic

$$Z_0 = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \simeq N(0, 1).$$

# 2.1 Hypothesis Tests on the Mean

• The hypothesis testing procedure is as follows:

+) Take a random sample of size $n$ and compute the value of the sample mean $\bar{x}$.

+) The standard normal z-value that corresponds to $\bar{x}$ is

$$z_0 = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$$

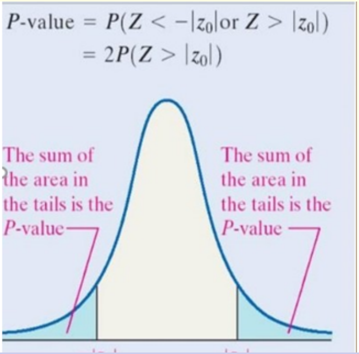
The hypothesis testing methods:

- P-value approach
- Fixed significance level approach with the z-test

# P-value approach

- If $P = P-$value $P < \alpha$ ( sufficiently small value), reject the null hypothesis $H_0$.
- If $P - value > \alpha$, then fail to reject $H_0$.
  - For the two-sided alternative hypothesis, the P-value is

$$P = 2[1 - \Phi(|z_0|)]$$



# P-value approach

- The upper-tailed test involves the hypotheses

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu > \mu_0.$$

has P-value as $P = 1 - \Phi(z_0)$.

- The lower-tailed test involves the hypotheses

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu < \mu_0.$$

has P-value as $P = \Phi(z_0)$.

# Conclusion

**Testing Hypotheses on the Mean, Variance Known (*Z*-Tests)**

Null hypothesis: $H_0\colon \mu = \mu_0$

Test statistic: $Z_0 = \dfrac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$

| **Alternative Hypotheses** | ***P*-Value** | **Rejection Criterion for Fixed-Level Tests** |
|---|---|---|
| $H_1\colon \mu \neq \mu_0$ | Probability above $\lvert z_0 \rvert$ and probability below $-\lvert z_0 \rvert$, $P = 2[1 - \Phi(\lvert z_0 \rvert)]$ | $z_0 > z_{\alpha/2}$ or $z_0 < -z_{\alpha/2}$ |
| $H_1\colon \mu > \mu_0$ | Probability above $z_0$, $P = 1 - \Phi(z_0)$ | $z_0 > z_\alpha$ |
| $H_1\colon \mu < \mu_0$ | Probability below $z_0$, $P = \Phi(z_0)$ | $z_0 < -z_\alpha$ |

The *P*-values and critical regions for these situations are shown in Figs. 9-7 and 9-8.

# Conclusion

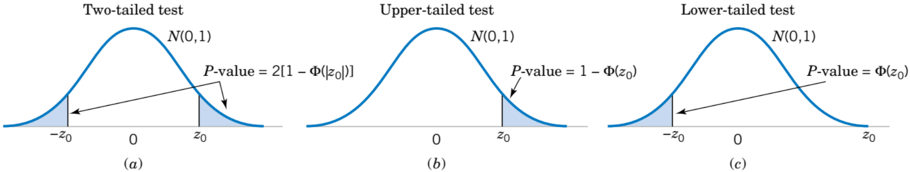



**Figure 9-7** The $P$-value for a $z$-test. (a) The two-sided alternative $H_1 : \mu \neq \mu_0$. (b) The one-sided alternative $H_1 : \mu > \mu_0$. (c) The one-sided alternative $H_1 : \mu < \mu_0$.

# Conclusion

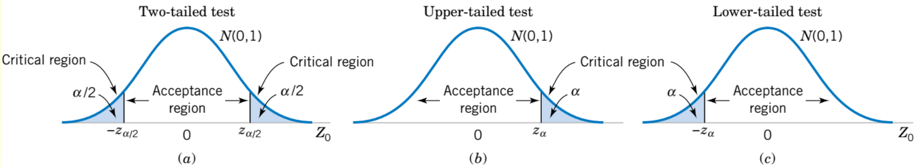



**Figure 9-8** The distribution of $Z_0$ when $H_0$: $\mu = \mu_0$ is true, with critical region for (a) the two-sided alternative $H_1 : \mu \neq \mu_0$. (b) The one-sided alternative $H_1$: $\mu > \mu_0$. (c) the one-sided alternative $H_1 : \mu < \mu_0$.

**Example 2**

You wish to test the claim that $\mu > 45$ at a level of significance of $\alpha = 0.025$ and are given sample statistics $n = 44; \sigma = 5; \bar{x} = 45.8$. Compute the value of the test statistic.

# 2.1 Hypothesis Tests on the Mean

## Example 2

You wish to test the claim that $\mu > 45$ at a level of significance of $\alpha = 0.025$ and are given sample statistics $n = 44; \sigma = 5; \bar{x} = 45.8$. Compute the value of the test statistic.

**Answer**: Test statistic value

$$z_0 = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{45.8 - 45}{5/\sqrt{44}} = 1.061$$

# Example

The heights of all adults in a community is known to have standard deviation of 0.03m. A random sample of 43 adults are collected, and their average height is 1.64m. Test the hypothesis that the true average height of all adults in the community is 1.7m, at $\alpha = 0.05$.

# Example

The heights of all adults in a community is known to have standard deviation of 0.03m. A random sample of 43 adults are collected, and the average height of this sample is 1.64m. Use both tradition and P-value methods, at the significance level of 5%, test the hypothesis that the average height of all adults in the community is greater than 1.6m.

## 2.2 Type II Error and Choice of Sample Size

- Consider the two-sided hypotheses

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu \neq \mu_0.$$

Suppose that the null hypothesis is false and that the true value of the mean is $\mu = \mu_0 + \delta,\ \delta > 0$

- A type II error will be made only if $-z_{\alpha/2} \leq Z_0 \leq z_{\alpha/2}$

**Probability of a Type II Error for a Two-Sided Test on the Mean, Variance Known**

$$\beta = \Phi\left(z_{\alpha/2} - \frac{\delta\sqrt{n}}{\sigma}\right) - \Phi\left(-z_{\alpha/2} - \frac{\delta\sqrt{n}}{\sigma}\right)$$

It is also possible to derive an equation similar for a one-sided alternative hypothesis.

# 2.2 Type II Error and Choice of Sample Size

**Sample Size for a Two-Sided Test on the Mean, Variance Known**

$$n = \frac{(z_{\alpha/2} + z_\beta)^2 \sigma^2}{\delta^2} \text{ where } \delta = \mu - \mu_0.$$

**Sample Size for a One-Sided Test on the Mean, Variance Known**

$$n = \frac{(z_\alpha + z_\beta)^2 \sigma^2}{\delta^2} \text{ where } \delta = \mu - \mu_0.$$

*Note:* If $n$ is not an integer, the convention is to round the sample size up to the next integer.

# Question 1

It is desired to estimate the average total compensation of CEOs in the Service industry. Data were randomly collected from 28 CEOs and the 99% confidence interval was calculated to be ($2,181,260, $5,836,180). Based on the interval above, do you believe the average total compensation of CEOs in the Service industry is less than $3,000,000?

Select one:

- a. I cannot conclude that the average is less than $3,000,000 at the 99% confidence level.
- b. No, and I am 99% confident that the average compensation more than $3,000,000.
- c. Yes, and I am 99% confident of it.
- d. I am 99% confident that the average compensation is $3,000,000.

# Question 2

Assume that the data has a normal distribution and the number of observations is greater than fifty. Find the critical z value used to test a null hypothesis.

$\alpha = 0.09$ for a two-tailed test.

Select one:

- a. $\pm 1.34$
- b. $\pm 1.4805$
- c. $\pm 1.695$
- d. $\pm 1.645$
- e. None of the other choices is true

# Question 3

Determine whether the given conditions justify testing a claim about a population mean μ. If so, what is formula for test statistic?

The sample size is $n = 25$, $\sigma = 5.93$, and the original population is normally distributed.

Select one:

- a. Yes, test statistic = $(\overline{x} - \mu)/(\sigma)$
- b. Yes, test statistic = $(\overline{x} - \mu)/(s/\sqrt{n})$
- c. No
- d. Yes, test statistic = $(\overline{x} - \mu)/(\sigma/\sqrt{n})$

# Question 5

Assume that the data has a normal distribution and the number of observations is greater than fifty. Find the critical z value used to test a null hypothesis.

$\alpha = 0.01$ for a two-tailed test.

Select one:

- a. $\pm 1.764$
- b. $\pm 2.575$
- c. $\pm 1.645$
- d. None of the other choices is true
- e. $\pm 1.96$

# Content

# Tests on the Mean of a Normal Distribution, Variance Unknown

Consider testing the hypotheses

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu \neq \mu_0.$$

Test Statistic

$$T_0 = \frac{\bar{X} - \mu_0}{S/\sqrt{n}}$$

If the null hypothesis is true, $T_0$ has a $t$ distribution with $n-1$ degrees of freedom. We can calculate the P-value from this distribution, or, if we use a fixed significance level approach, we can locate the critical region to control the type I error probability at the desired level.

# 3.1 Hypothesis Tests on the Mean

To test $H_0$ against the two-sided alternative $H_0 \neq 0$, the value of the test statistic $t_0 = \frac{\bar{x}-\mu_0}{s/\sqrt{n}}$ is calculated.

## P-value approach

For the two-sided alternative hypothesis, the P-value is

$$P = 2P(T_{n-1} > |t_0|)$$

Similarly, the one-sided alternatives:
The upper-tailed test involves the hypotheses

$$H_0 : \mu = \mu_0; \quad H_1 : \mu > \mu_0,$$

has P-value as $P = P(T_{n-1} > t_0)$.
The lower-tailed test involves the hypotheses

$$H_0 : \mu = \mu_0; \quad H_1 : \mu < \mu_0,$$

has P-value as $P = P(T_{n-1} < t_0)$.

# Fixed Significance Level Approach with the z-test

For the two-sided alternative hypothesis:

1) Reject $H_0$ if $t_0 > t_{\alpha/2,n-1}$ or $t_0 < -t_{\alpha/2,n-1}$;

2) Fail to reject $H_0$ if $-t_{\alpha/2,n-1} < t_0 < t_{\alpha/2,n-1}$;
where, $t_{\alpha/2,n-1}$ is the $\alpha/2$ percentage points of the $t$ distribution with $n-1$ degrees of freedom.

Similarly, for the upper-tailed test, we would reject $H_0$ if $t_0 > t_{\alpha,n-1}$.

For the lower-tailed test, we reject $H_0$ if $t_0 < -t_{\alpha,n-1}$.

# Conclusion

**Testing Hypotheses on the Mean of a Normal Distribution, Variance Unknown**

Null hypothesis: $H_0$: $\mu = \mu_0$

Test statistic: $$T_0 = \frac{\overline{X} - \mu_0}{S/\sqrt{n}}$$

| Alternative Hypotheses | *P*-Value | Rejection Criterion for Fixed-Level Tests |
|---|---|---|
| $H_1$: $\mu \neq \mu_0$ | Probability above $\lvert t_0 \rvert$ and probability below $-\lvert t_0 \rvert$ | $t_0 > t_{\alpha/2,n-1}$ or $t_0 < -t_{\alpha/2,n-1}$ |
| $H_1$: $\mu > \mu_0$ | Probability above $t_0$ | $t_0 > t_{\alpha,n-1}$ |
| $H_1$: $\mu < \mu_0$ | Probability below $t_0$ | $t_0 < -t_{\alpha,n-1}$ |

The calculations of the *P*-values and the locations of the critical regions for these situations are shown in Figs. 9-10 and 9-12, respectively.

# Conclusion

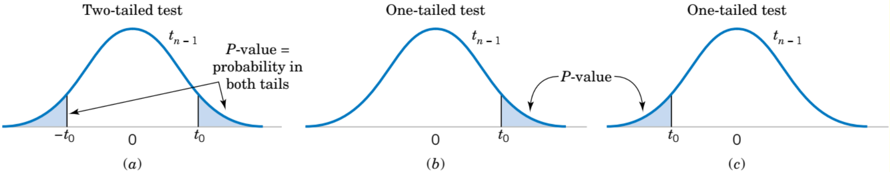



**Figure 9-10** Calculating the $P$-value for a $t$-test: (a) $H_1$: $\mu \neq \mu_0$: (b) $H_1$: $\mu > \mu_0$; (c) $H_1$: $\mu < \mu_0$.

# Conclusion

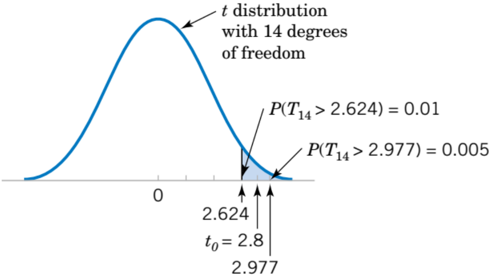



**Figure 9-11** $P$-value for $t_0 = 2.8$; an upper-tailed test is shown to be between 0.005 and 0.01.

# Conclusion

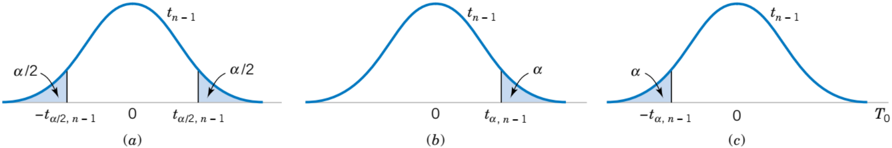



**Figure 9-12** The distribution of $T_0$ when $H_0$: $\mu = \mu_0$ is true, with critical region for (a) $H_1$: $\mu \neq \mu_0$, (b) $H_1$: $\mu > \mu_0$, and (c) $H_1$: $\mu < \mu_0$.

# Tests on the Mean of a Normal Distribution, Variance Unknown

## Example 3

Find the test statistic $t_0$ for a sample with $n = 9; \bar{x} = 5.6; s = 0.88$ and if $H_1 : \mu > 5.7$.

# Tests on the Mean of a Normal Distribution, Variance Unknown

## Example 3

Find the test statistic $t_0$ for a sample with $n = 9; \bar{x} = 5.6; s = 0.88$ and if $H_1 : \mu > 5.7$.

**Answer**: We have $n = 9; \bar{x} = 5.6; s = 0.88; \mu_0 = 5.7$. Hence

$$t_0 = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{5.6 - 5.7}{0.88/\sqrt{9}} = -0.341$$

# Tests on the Mean of a Normal Distribution, Variance Unknown

## Example 4

Find the critical values for a sample with $n = 10; \bar{x} = 7.9; s = 1.2$ if $H_1 : \mu < 8.2$ and the level of significance $\alpha = 0.05$. Let $t_{0.05,9} = 1.833; t_{0.025,9} = 2.262$

# Tests on the Mean of a Normal Distribution, Variance Unknown

## Example 4

Find the critical values for a sample with $n = 10; \bar{x} = 7.9; s = 1.2$ if $H_1 : \mu < 8.2$ and the level of significance $\alpha = 0.05$. Let $t_{0.05,9} = 1.833; t_{0.025,9} = 2.262$

**Answer**: This is a one-sided alternative hypothes is $H_1 : \mu < 8.2$ on mean with unknown variance. The critical value is

$$-t_{\alpha,n-1} = -t_{0.05,9} = -1.833$$

# Tests on the Mean of a Normal Distribution, Variance Unknown

## Example 5

Given a sample with $n = 10; \bar{x} = 7.9; s = 1.2$ if $H_1 : \mu \neq 8.2$ and the level of significance $\alpha = 0.05$. Let $t_{0.05,9} = 1.833; t_{0.025,9} = 2.262$. Should we reject $H_0$?

# Tests on the Mean of a Normal Distribution, Variance Unknown

## Example 5

Given a sample with $n = 10; \bar{x} = 7.9; s = 1.2$ if $H_1 : \mu \neq 8.2$ and the level of significance $\alpha = 0.05$. Let $t_{0.05,9} = 1.833; t_{0.025,9} = 2.262$. Should we reject $H_0$?

**Answer**: This is a two-sided alternative hypothesis on population mean with unknown variance. Test statistic value

$$t_0 = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{7.9 - 8.2}{1.2/\sqrt{10}} = -0.79$$

The critical values $2.262; -2.262$. Since $-2.262 < -0.79 < 2.262$, we should not reject $H_0$.

# Example

The heights of all adults in a community is known to have a normal distribution. A random sample are collected and the heights (in meters) are recorded as follows:

$$1.55 \quad 1.60 \quad 1.58 \quad 1.62 \quad 1.65 \quad 1.70 \quad 1.68,$$

Test the hypothesis that the average height of all adults in the community is at most 1.60(m), at the significance level of 5%.

# Question 1

Determine whether the given conditions justify testing a claim about a population mean $\mu$. If so, what is formula for test statistic?

The sample size is $n = 17$, $\sigma$ is not known, and the original population is normally distributed.

Select one:

- a. No
- b. Yes, test statistic = $(\overline{x} - \mu)/s$
- c. Yes, test statistic = $(\overline{x} - \mu)/(\sigma\sqrt{n})$
- d. Yes, test statistic = $(\overline{x} - \mu)/(s/\sqrt{n})$

# Question 2

You wish to test the claim that $\mu \neq 17$ at a level of significance of $\alpha = 0.05$ and sample statistics are given n = 36, s = 2.5, $\overline{x} = 16.1$ . Compute the value of the test statistic. Round your answer to two decimal places.

Select one:

- a. None of the other choices is true
- b. -2.86
- c. -2.16
- d. -1.97
- e. -1.83

# Question 3

You wish to test the claim that $\mu = 1200$ at a level of significance of $\alpha = 0.01$ and sample statistics are given n = 37, s =80, $\overline{x} = 1207$. Compute the value of the test statistic. Round your answer to two decimal places.

Select one:

- a. -5.18
- b. -2.16
- c. None of the other choices is true
- d. 4.67
- e. 0.53

# Content

# 4.1 Large-Sample Tests on a Proportion

Consider the two-sided hypotheses

$$H_0 : p = p_0$$
$$H_1 : p \neq p_0.$$

Then, if the null hypothes is true, we have $N \sim N(np_0, np_0(1 - p_0))$
The test statistic is

$$Z_0 = \frac{X - np_0}{\sqrt{np_0(1 - p_0)}}$$

# P-value approach

- For the two-sided alternative hypothesis, the P-value is

$$P = 2[1 - \Phi(|z_0|)]$$

- The upper-tailed test involves the hypotheses

$$H_0 : p = p_0$$
$$H_1 : p > p_0.$$

has P-value as $P = 1 - \Phi(z_0)$.

- The lower-tailed test involves the hypotheses

$$H_0 : p = p_0$$
$$H_1 : p < p_0.$$

has P-value as $P = \Phi(z_0)$.

# Fixed Significance Level Approach with the z-test

- For the two-sided alternative hypothesis:
  1) Reject $H_0$ if the observed value of the test statistic $z_0$ is either $z_0 > z_{\alpha/2}$ or $z_0 < -z_{\alpha/2}$;
  2) Fail to reject $H_0$ if $-z_{\alpha/2} < z_0 < z_{\alpha/2}$;
  where, $z_{\alpha/2}$ is the $100\alpha/2$ percentage point of the standard normal distribution.
- Similarly, for the upper-tailed test, we would reject $H_0$ if $z_0 > z_\alpha$.
- For the lower-tailed test, we reject $H_0$ if $z_0 < -z_\alpha$.

# Conclusion

**Testing Hypotheses on a Binomial Proportion**

Null hypotheses: $H_0\colon p = p_0$

Test statistic: $$Z_0 = \frac{X - np_0}{\sqrt{np_0(1 - p_0)}}$$

| **Alternative Hypotheses** | ***P*-Value** | **Rejection Criterion for Fixed-Level Tests** |
|---|---|---|
| $H_1\colon p \neq p_0$ | Probability above $\lvert z_0\rvert$ and probability below $-\lvert z_0\rvert$ $P = 2[1 - \Phi(z_0)]$ | $z_0 > z_{\alpha/2}$ or $z_0 < -z_{\alpha/2}$ |
| $H_1\colon p > p_0$ | Probability above $z_0$, $P = 1 - \Phi(z_0)$ | $z_0 > z_\alpha$ |
| $H_1\colon p < p_0$ | Probability below $z_0$, $P = \Phi(z_0)$ | $z_0 < -z_\alpha$ |

## Example 6

A random sample of 200 circuits generated 9 defectives. Use the data to test $H_0 : p = 0.05$ versus $H_1 : p \neq 0.05$. Use $\alpha = 0.05$. Let
$z_{0.05} = 1.65,\ z_{0.025} = 1.96$

a) Find the critical values for this test.

b) Should we reject $H_0$?

## Example 6

A random sample of 200 circuits generated 9 defectives. Use the data to test $H_0 : p = 0.05$ versus $H_1 : p \neq 0.05$. Use $\alpha = 0.05$. Let $z_{0.05} = 1.65,\ z_{0.025} = 1.96$

a) Find the critical values for this test.

b) Should we reject $H_0$?

**Answer**: This is a two-tail alternative hypothesis on proportion population $p_0 = 0.05$. The sample has $n = 200; X = 9$ and the level of significance $\alpha = 0.05$

a) The critical values: $z_{\alpha/2} = 1.96, -z_{\alpha/2} = -1.96$

b) Test statistic value

$$z_0 = \frac{x - np_0}{\sqrt{np_0(1-p_0)}} = \frac{9 - 200 * 0.05}{\sqrt{200 * 0.05 * (1 - 0.05)}} = -0.324$$

We see that $-z_{0.025} < z_0 < z_{0.025}$, hence we fail to reject $H_0$.

# Example

A company claims that the percentage of defective products is kept under control, that is less than 3%. In a random sample of 135 products it is found out that 6 of them are defective. Test the claim of the company at the significance level of 5%.

# Question 1

A claim is made that the proportion of children who play sports is less than 0.5, and the sample statistics include n =1200 subjects with 40% saying that they play a sport. Find the value of the test statistic z using $z = \frac{\bar{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$

Select one:

- a. -13.61
- b. 6.928
- c. 13.61
- d. -6.928
- e. None of the other choices is true

# Question 2

The claim is that the proportion of drowning deaths of children attributable to beaches is more than 0.25, and the sample statistics include n= 690 drowning deaths of children with 35% of them attributable to beaches. Find the value of the test statistic z using $z = \frac{\overline{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$.

Select one:

- a. None of the other choices is true
- b. 6.07
- c. 2.94
- d. -6.07
- e. -2.94

# Question 3

A skeptical paranormal researcher claims that the proportion of Americans that have seen a UFO, p, is less than 1 in every one thousand. Assume that a hypothesis test of the given claim will be conducted. Identify the type I error for the test.

Select one:

- a. None of the other choices is true
- b. The error of rejecting the claim that the true proportion is at least 1 in one thousand when it really is at least 1 in one thousand.
- c. The error of rejecting the claim that the true proportion is less than 1 in one thousand when it really is less than 1 in one thousand.
- d. The error of failing to reject the claim that the true proportion is at least 1 in one thousand when it is actually less than 1 in one thousand.

# Question 4

Determine the critical values to test the claim about the population proportion p ≠ 0.325 given n = 42 and $\overline{p} = 0.247$. Use $\alpha = 0.01$.

Select one:

- a. 2.33 and -2.33
- b. 1.645 and 1.645
- c. None of the other choices is true
- d. 1.96 and -1.96
- e. 2.575 and -2.575

# Question 5

Determine the test statistic to test the claim about the population proportion p > 0.51 given n = 50 and $\overline{p} = 0.61$. Use $\alpha = 0.05$.

Select one:

- a. 1.96
- b. -1.96
- c. -1.645
- d. None of the other choices is true
- e. 1.645

# Question 6

An airline claims that the no-show rate for passengers is less than 3%. In a sample of 420 randomly selected reservations, 21 were no-shows. At $\alpha = 0.01$, compute the value of the test statistic to test the airline's claim.

Select one:

- a. 3.1
- b. 1.45
- c. 2.4
- d. 1

# Question 7

A telephone company claims that 25% of its customers have at least two telephone lines. The company selects a random sample of 500 customers and finds that 108 have two or more telephone lines. At $\alpha = 0.05$, compute the value of the test statistic to test the company's claim.

Select one:

- a. -1.76
- b. 2.33
- c. -2.33
- d. 1.05
- e. 1.76

# Question 8

The FPT university claims that 20% of its graduates are women. In a graduating class of 250 students, 60 were women. At $\alpha = 0.05$, does this suggest that the school is believable? Let $z_{0.025} = 1.96$ and $z_{0.05} = 1.65$.

Select one:

- a. No, because $z_0 = 2.31 > -z_{0.05}$
- b. Yes, because $|z_0| = 1.52 < z_{0.05}$
- c. Yes, because $|z_0| = 1.58 < z_{0.025}$
- d. No, because $z_0 = 1.12 < z_{0.05}$

# Question 9

A recent study claimed that at most 15% of junior high students are overweight. In a sample of 175 students, 28 were found to be overweight. At $\alpha$ = 0.03, determine the critical values to test the claim.

Select one:

- a. 2.17
- b. 1.88
- c. -2.17
- d. -1.88

# Question 10

A recent study claimed that at least 17% of junior high students are overweight. In a sample of 175 students, 28 were found to be overweight. At $\alpha = 0.01$, determine the value of the test statistic to test the claim.

Select one:

- a. 1.96
- b. 2.33
- c. -2.17
- d. -0.35