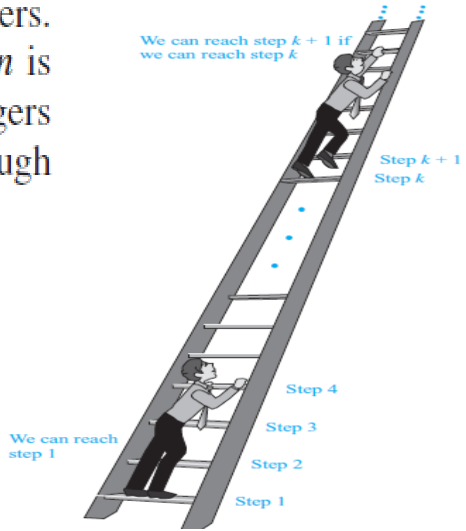
Induction

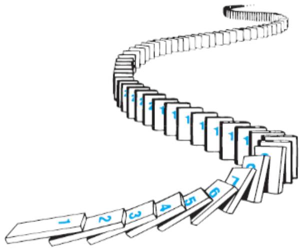
Many mathematical statements assert that a property is true for all positive integers. Examples of such statements are that for every positive integer n: $n! \leq n^n$, $n^3 - n$ is divisible by 3; a set with n elements has 2^n subsets; and the sum of the first n positive integers is $n(n+1)/2$. A major goal of this chapter, and the book, is to give the student a thorough understanding of mathematical induction, which is used to prove results of this kind.



PRINCIPLE OF MATHEMATICAL INDUCTION To prove that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, we complete two steps:

BASIS STEP: We verify that $P(1)$ is true.

INDUCTIVE STEP: We show that the conditional statement $P(k) \to P(k+1)$ is true for all positive integers k.



Show that if n is a positive integer,

$$1+2+\cdots+n=\frac{n(n+1)}{2}.$$

BASIS STEP: $P(1)$ is true, because $1=\frac{1(1+1)}{2}$. (The left-hand side of this equation is 1 because 1 is the sum of the first positive integer. The right-hand side is found by substituting 1 for n in $n(n+1)/2$.)

INDUCTIVE STEP: For the inductive hypothesis we assume that $P(k)$ holds for an arbitrary positive integer k. That is, we assume that

$$1+2+\cdots+k=\frac{k(k+1)}{2}.$$

Under this assumption, it must be shown that $P(k+1)$ is true, namely, that

$$1+2+\cdots+k+(k+1)=\frac{(k+1)[(k+1)+1]}{2}=\frac{(k+1)(k+2)}{2}$$

is also true. When we add $k+1$ to both sides of the equation in $P(k)$, we obtain

$$\begin{aligned}1+2+\cdots+k+(k+1) &\overset{\text{IH}}{=} \frac{k(k+1)}{2}+(k+1)\\ &= \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}.\end{aligned}$$

This last equation shows that $P(k+1)$ is true under the assumption that $P(k)$ is true. This completes the inductive step.

We have completed the basis step and the inductive step, so by mathematical induction we know that $P(n)$ is true for all positive integers n. That is, we have proven that $1+2+\cdots+n=n(n+1)/2$ for all positive integers n.

Conjecture a formula for the sum of the first n positive odd integers. Then prove your conjecture using mathematical induction.

$$1 = 1, \qquad 1 + 3 = 4, \qquad 1 + 3 + 5 = 9,$$
$$1 + 3 + 5 + 7 = 16, \qquad 1 + 3 + 5 + 7 + 9 = 25.$$

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

Often, we will need to show that $P(n)$ is true for $n = b,\ b+1,\ b+2, \ldots,$ where b is an integer other than 1, we show that $P(b)$ is true in the basis step.
In the inductive step, we show that the conditional statement

$$P(k) \rightarrow P(k+1) \text{ is true for } k = b, b+1, b+2, \ldots.$$

EXAMPLE $$1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$$

for all nonnegative integers n.

PROVING INEQUALITIES Mathematical induction can be used to prove a variety of inequalities that hold for all positive integers greater than a particular positive integer, as

EXAMPLE 5 $n < 2^n$

for all positive integers n.

EXAMPLE 7 The **harmonic numbers** H_j, $j = 1, 2, 3, \ldots$, are defined by

$$H_j = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{j}.$$

Use mathematical induction to show that

$$H_{2^n} \geq 1 + \frac{n}{2},$$

whenever n is a nonnegative integer.

PROVING DIVISIBILITY RESULTS Mathematical induction can be used to prove divisibility results about integers.

EXAMPLE

"$n^3 - n$ is divisible by 3."

PROVING RESULTS ABOUT SETS Mathematical induction can be used to prove many results about sets.

EXAMPLE if S is a finite set with n elements,
where n is a nonnegative integer,
then S has 2^n subsets.

$$1^2+2^2+3^2+\ldots+n^2=[n(n+1)(2n+1)] / 6, n \geq 1$$

$$1^3+2^3+3^3+\ldots+n^3=[n^2(n+1)^2] / 4$$

$$n! < n^n$$