FPT UNIVERSITY
DREAM OF INNOVATION

The Foundations: Logic and Infererence

4.1. NESTED QUANTIFIERS

4.2. THE NEGATION

4.1. NESTED QUANTIFIERS

$\forall x \exists y(x + y = 0)$

is the same thing as $\forall x Q(x)$, where $Q(x)$ is $\exists y P(x, y)$, where $P(x, y)$ is $x + y = 0$.

"The product of a positive real number and a negative real number is always a negative real number."

THINKING OF QUANTIFICATION AS LOOPS

to see whether $\exists x \exists y P(x, y)$ is true, we loop through the values for x, where for each x we loop through the values for y until we hit an x for which we hit a y for which $P(x, y)$ is true. The statement $\exists x \exists y P(x, y)$ is false only if we never hit an x for which we hit a y such that $P(x, y)$ is true.

To see whether $\exists x \forall y P(x, y)$ is true, we loop through the values for x until we find an x for which $P(x, y)$ is always true when we loop through all values for y. Once we find such an x, we know that $\exists x \forall y P(x, y)$ is true. If we never hit such an x, then we know that $\exists x \forall y P(x, y)$ is false.

THE ORDER of QUANTIFIERs

Many mathematical statements involve multiple quantifications of propositional functions involving more than one variable. It is important to note that the order of the quantifiers is important, unless all the quantifiers are universal quantifiers or all are existential quantifiers.

Let $P(x, y)$ be the statement "$x + y = y + x$."

$\forall x \forall y P(x, y)$

$\forall y \forall x P(x, y)$

Let $Q(x, y)$ denote "$x + y = 0$."

$\exists y \forall x Q(x, y)$ "There is a real number y such that for every real number x, $Q(x, y)$."

$\forall x \exists y\, x + y = 0$

EXAMPLEs

Let U be the real number and P(x,y) denote “x.y=0”. Find the truth values of the following:

$\forall x \forall y$ “x.y=0”

$\forall x \exists y$ “x.y=0”

Let U be the real number and P(x,y) denote “x/y=1”. Find the truth values of the following:

$\forall x \exists y$ “x/y=1”

$\exists x \exists y$ “x/y=1”

Let P(x,y) denote "x = -y". Find the negation of

$\forall x \exists y \, P(x, y).$

DE MORGAN'S LAWS

Use quantifiers to express the statement that "There does not exist a woman who has taken a flight on every airline in the world."

"There is a woman who has taken a flight on every airline in the world"

$$\exists w \forall a \exists f (P(w, f) \wedge Q(f, a))$$

$$\neg \exists w \forall a \exists f (P(w, f) \wedge Q(f, a))$$
$$\equiv \forall w \neg \forall a \exists f (P(w, f) \wedge Q(f, a))$$
$$\equiv \forall w \exists a \neg \exists f (P(w, f) \wedge Q(f, a))$$
$$\equiv \forall w \exists a \forall f \neg (P(w, f) \wedge Q(f, a))$$
$$\equiv \forall w \exists a \forall f (\neg P(w, f) \vee \neg Q(f, a)).$$

The Foundations: Logic and Infererence

1. Propositional logic [1]
2. Propositional equivalences [2]
3. Predicates and quantifiers [3]
4. Nested quantifiers and negations [3]
5. **Rules of Inference for Propositional Logic** [4]
6. Rules of Inference for Quantified Statements [4]

5.1. ARGUMENTs

5.2. SOME INFERENCE RULEs

5.3. USING INFERENCE RULES to BUILD ARGUMENTs

5.1. ARGUMENTs

An *argument* in propositional logic is a sequence of propositions.

$$(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \rightarrow q$$

premises   *conclusion*

If it is raining, I will need an umbrella.
It is raining.

$\therefore$ I will need an umbrella

Modus Ponens

$p \rightarrow q$
p

$\therefore q$

If it rains today, then we will not have a barbecue today. If we do not have a barbecue today, then we will have a barbecue tomorrow. Therefore, if it rains today, then we will have a barbecue tomorrow.

Hypothetical Syllogism

$p \rightarrow q$
$q \rightarrow r$

$\therefore p \rightarrow r$

FALLACIEs

Several common fallacies arise in incorrect arguments. These fallacies resemble rules of inference, but are based on contingencies rather than tautologies.

From $p \to q$ we can form new conditional statements .

- $q \to p$ is the converse of $p \to q$
- $\neg p \to \neg q$ is the inverse of $p \to q$
- $\neg q \to \neg p$ is the contrapositive of $p \to q$

If you do every problem in this book, then you will learn discrete mathematics. You learned discrete mathematics.

Therefore, you did every problem in this book.

Hắn chửi như những người say rượu hát. Giá **hắn biết hát** thì hắn có lẽ **hắn không cần chửi**. Khổ cho hắn và khổ cho người, **hắn lại không biết hát**. Thì hắn chửi, cũng như chiều nay hắn chửi..... (Nam Cao, Chí Phèo, trang 78)

5.2. SOME INFERENCE RULEs

Modus Tollens

$$\begin{array}{l} p \to q \\ \neg q \\ \hline \therefore \neg p \end{array}$$

She did not get a prize

If she is good at learning she will get a prize

∴ She is not good at learning

Disjunctive Syllogism

$$\begin{array}{l} p \lor q \\ \neg p \\ \hline \therefore q \end{array}$$

Power puts off or the lamp is malfunctional

Power doesn't put off

the lamp is malfunctional

Addition

$$\begin{array}{l} p \\ \hline \therefore p \lor q \end{array}$$

Conjunction

$$\begin{array}{l} p \\ q \\ \hline \therefore p \land q \end{array}$$

Simplification

$$\begin{array}{l} p \land q \\ \hline \therefore q \end{array}$$

SOME INFERENCE RULEs

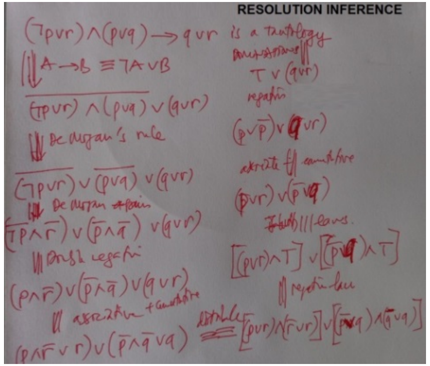
Resolution

$$\neg p \lor r$$
$$p \lor q$$
$$\therefore q \lor r$$

Jasmin is skiing OR it is not snowing

It is snowing OR Bart is playing hockey

Jasmin is skiing OR Bart is playing hockey



Show that the premises "It is not sunny this afternoon and it is colder than yesterday," "We will go swimming only if it is sunny," "If we do not go swimming, then we will take a canoe trip," and "If we take a canoe trip, then we will be home by sunset" lead to the conclusion "We will be home by sunset."

$$\neg p \wedge q$$
$$r \rightarrow p$$
$$\neg r \rightarrow s$$
$$s \rightarrow t$$
$$t.$$

Step	Reason
1. $\neg p \wedge q$	Premise
2. $\neg p$	Simplification using (1)
3. $r \rightarrow p$	Premise
4. $\neg r$	Modus tollens using (2) and (3)
5. $\neg r \rightarrow s$	Premise
6. s	Modus ponens using (4) and (5)
7. $s \rightarrow t$	Premise
8. t	Modus ponens using (6) and (7)

USING INFERENCE RULES to BUILD ARGUMENTs

Show that the premises "If you send me an e-mail message, then I will finish writing the program," "If you do not send me an e-mail message, then I will go to sleep early," and "If I go to sleep early, then I will wake up feeling refreshed" lead to the conclusion "If I do not finish writing the program, then I will wake up feeling refreshed."

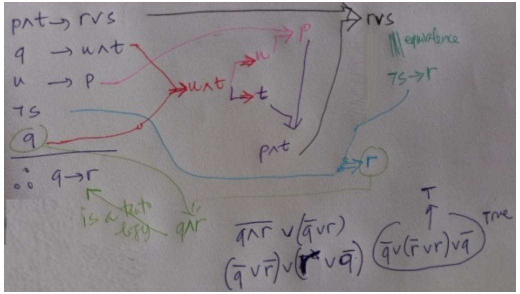
Step	Reason
1. $p \rightarrow q$	Premise
2. $\neg q \rightarrow \neg p$	Contrapositive of (1)
3. $\neg p \rightarrow r$	Premise
4. $\neg q \rightarrow r$	Hypothetical syllogism using (2) and (3)
5. $r \rightarrow s$	Premise
6. $\neg q \rightarrow s$	Hypothetical syllogism using (4) and (5)

USING INFERENCE RULES to BUILD ARGUMENTs

Show that the argument with premises

$(p \wedge t) \rightarrow (r \vee s)$, $q \rightarrow (u \wedge t)$, $(u \rightarrow p)$, $\neg s$, q

and conclusion $q \rightarrow r$ is valid.



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RULES of INFERENCE for QUANTIFIERs

Step	Reason
1. $\forall x(D(x) \rightarrow C(x))$	Premise
2. $D(\text{Marla}) \rightarrow C(\text{Marla})$	Universal instantiation from (1)
3. $D(\text{Marla})$	Premise
4. $C(\text{Marla})$	Modus ponens from (2) and (3)

Rule of Inference	*Name*
$\dfrac{\forall x P(x)}{\therefore\ P(c)}$	Universal instantiation
$\dfrac{P(c) \text{ for an arbitrary } c}{\therefore\ \forall x P(x)}$	Universal generalization
$\dfrac{\exists x P(x)}{\therefore\ P(c) \text{ for some element } c}$	Existential instantiation
$\dfrac{P(c) \text{ for some element } c}{\therefore\ \exists x P(x)}$	Existential generalization

"Everyone in this discrete mathematics class has taken a course in computer science"

"Marla is a student in this class"

"Marla has taken a course in computer science."

RULES of INFERENCE for QUANTIFIERs

Show that the premises "A student in this class has not read the book," and "Everyone in this class passed the first exam" imply the conclusion "It is not the case that Everyone who passed the first exam has read the book."

Let $C(x)$

The premises are

The conclusion

Step	Reason
1. $\exists x(C(x) \wedge \neg B(x))$	Premise
2. $C(a) \wedge \neg B(a)$	Existential instantiation from (1)
3. $C(a)$	Simplification from (2)
4. $\forall x(C(x) \rightarrow P(x))$	Premise
5. $C(a) \rightarrow P(a)$	Universal instantiation from (4)
6. $P(a)$	Modus ponens from (3) and (5)
7. $\neg B(a)$	Simplification from (2)
8. $P(a) \wedge \neg B(a)$	Conjunction from (6) and (7)
9. $\exists x(P(x) \wedge \neg B(x))$	Existential generalization from (8)

Prove that

$$\left.\begin{array}{l} \forall x,\ \overline{Q(x)} \to P(x) \\ \exists x,\ \overline{P(x)} \\ \forall x,\ \overline{q(x)} \vee r(x) \\ \forall x,\ s(x) \to \overline{r(x)} \end{array}\right. \quad \therefore\ \exists x,\ \overline{s(x)}$$

$$\overline{P(a)} \qquad \overline{Q(a)} \to P(a)$$

$$\overline{\overline{Q(a)}} \equiv q(a)$$

$$\overline{q(a)} \vee r(a)$$

$$\text{III}$$

$$q(a) \to r(a)$$

$$r(a)$$

$$s(a) \to \overline{r(a)}$$

$$\overline{s(a)} \qquad \Delta$$