CHAPTER 2

Basic Structures: Sets, Functions, Sequences, and Sums

Much of discrete mathematics is devoted to the study of discrete structures, used to represent discrete objects.

Many important discrete structures are built using sets.

The concept of a function is extremely important in discrete mathematics.

Introduce some important types of sequences, identify a sequence, develop formulae for certain types of summations

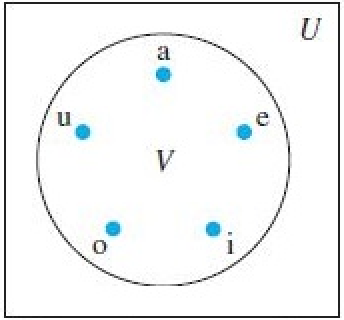
2.1 Sets

DEFINITION 1

A *set* is an unordered collection of objects, called *elements* or *members* of the set. A set is said to *contain* its elements. We write $a \in A$ to denote that a is an element of the set A. The notation $a \notin A$ denotes that a is not an element of the set A.

Several ways to describe a set.

1. Roster method
2. Set builder
3. Venn diagram



Venn Diagram for the Set of Vowels.

$\mathbf{N} = \{0, 1, 2, 3, \ldots\}$, the set of **natural numbers**
$\mathbf{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, the set of **integers**
$\mathbf{Z}^+ = \{1, 2, 3, \ldots\}$, the set of **positive integers**
$\mathbf{Q} = \{p/q \mid p \in \mathbf{Z}, q \in \mathbf{Z}, \text{ and } q \neq 0\}$, the set of **rational numbers**
$\mathbf{R}$, the set of **real numbers**

$$[a, b] = \{x \mid a \le x \le b\}$$
$$[a, b) = \{x \mid a \le x < b\}$$
$$(a, b] = \{x \mid a < x \le b\}$$
$$(a, b) = \{x \mid a < x < b\}$$

$\{\mathbf{N}, \mathbf{Z}, \mathbf{Q}, \mathbf{R}\}$

DEFINITION 2

Two sets are *equal* if and only if they have the same elements. Therefore, if A and B are sets, then A and B are equal if and only if $\forall x(x \in A \leftrightarrow x \in B)$. We write $A = B$ if A and B are equal sets.

The sets {1, 3, 5} and {3, 5, 1} are equal, because they have the same elements.

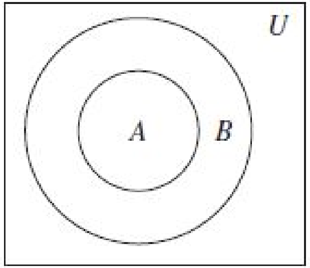
THE EMPTY SET

Subsets

DEFINITION 3

The set A is a *subset* of B if and only if every element of A is also an element of B. We use the notation $A \subseteq B$ to indicate that A is a subset of the set B.

$$\forall x (x \in A \rightarrow x \in B)$$



Venn Diagram Showing that A Is a Subset of B.

Showing that A is a Subset of B To show that $A \subseteq B$, show that if x belongs to A then x also belongs to B.

Showing that A is Not a Subset of B To show that $A \nsubseteq B$, find a single $x \in A$ such that $x \notin B$.

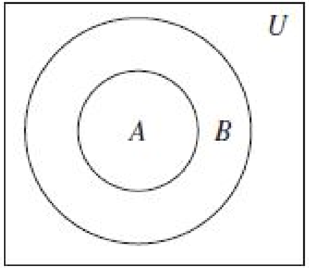
Subsets

THEOREM 1

For every set S, (*i*) $\emptyset \subseteq S$ and (*ii*) $S \subseteq S$.

$A \subset B$ if and only if

$$\forall x(x \in A \rightarrow x \in B) \wedge \exists x(x \in B \wedge x \notin A)$$



Venn Diagram Showing that A Is a Subset of B.

Showing Two Sets are Equal To show that two sets A and B are equal, show that $A \subseteq B$ and $B \subseteq A$.

$A = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $B = \{x \mid x \text{ is a subset of the set } \{a, b\}\}$

Also note that $\{a\} \in A$, but $a \notin A$.

The Size of a Set

DEFINITION 4

Let S be a set. If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is a *finite set* and that n is the *cardinality* of S. The cardinality of S is denoted by $|S|$.

Let A be the set of odd positive integers less than 10.

Let S be the set of letters in the English alphabet.

The null set

DEFINITION 5

A set is said to be *infinite* if it is not finite.

$\mathbf{N}$ $\mathbf{Z}$ $\mathbf{Z}^+$ $\mathbf{Q}$

$\mathbf{R}$

$[a, b]$

Power Sets

DEFINITION 6

Given a set S, the *power set* of S is the set of all subsets of the set S. The power set of S is denoted by $\mathcal{P}(S)$.

What is the power set of the set $\{0, 1, 2\}$?

If a set has n elements, then its power set has 2^n elements.

What is the power set of the empty set?

What is the power set of the set $\{\emptyset\}$?

Cartesian Products

DEFINITION 7

The *ordered n-tuple* $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element, ..., and a_n as its nth element.

$$(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n)$$

DEFINITION 8

Let A and B be sets. The *Cartesian product* of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. Hence,

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}.$$

$A = \{1, 2\}$ and $B = \{a, b, c\}$'

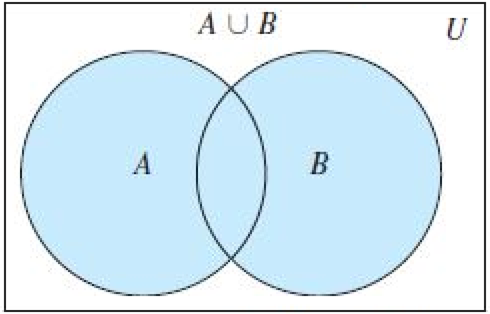
The Cartesian product $A \times B$ is

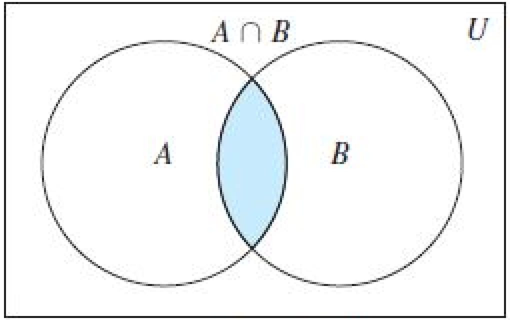
The Cartesian product $B \times A$ is

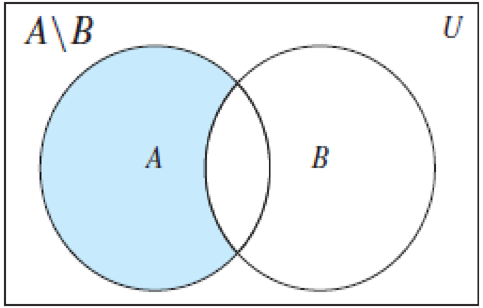
Note that the Cartesian products $A \times B$ and $B \times A$ are not equal,

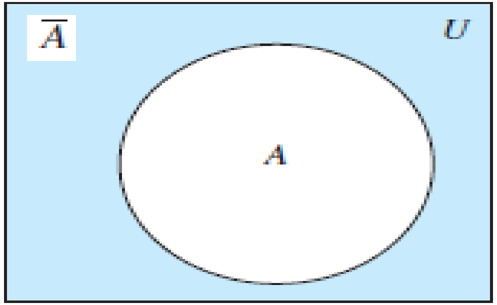
Set Operations

$A \cup B = \{x \mid x \in A \vee x \in B\}$. $A \cap B = \{x \mid x \in A \wedge x \in B\}$.









Set Identities

Identity	*Name*
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

De Morgan law

$$
\begin{aligned}
\overline{A \cap B} &= \{x \mid x \notin A \cap B\} \\
&= \{x \mid \neg(x \in (A \cap B))\} \\
&= \{x \mid \neg(x \in A \wedge x \in B)\} \\
&= \{x \mid \neg(x \in A) \vee \neg(x \in B)\} \\
&= \{x \mid x \notin A \vee x \notin B\} \\
&= \{x \mid x \in \overline{A} \vee x \in \overline{B}\} \\
&= \{x \mid x \in \overline{A} \cup \overline{B}\} \\
&= \overline{A} \cup \overline{B}
\end{aligned}
$$

Distributive laws

Logical Equivalences.

Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

Identity	*Name*
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

$$(A \cap B) \cup [B \cap ((C \cap D) \cup (C \cap \overline{D}))]$$

$$= (A \cap B) \cup [B \cap (C \cap (D \cup \overline{D}))]$$

$$= (A \cap B) \cup [B \cap (C \cap \mathcal{U})]$$

$$= (A \cap B) \cup (B \cap C)$$

$$= (B \cap A) \cup (B \cap C)$$

$$= B \cap (A \cup C)$$

$$\{\overline{C} \cup [A \cap (B \cap \overline{A})]\} \cup [(\overline{C} \cap A) \cup \overline{D}]$$

$$\overline{C \cap D}$$

Computer Representation of Sets

First, specify an arbitrary ordering of the elements of U, for instance $a_1, a_2, \ldots, a_n$.

Represent a subset A of U with the bit string of length n,
where the i^{th} bit in this string is 1 if a_i belongs to A and is 0 if a_i does not belong to A.

Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$A = \{1, 3, 5, 7, 9\}$

$B = \{1, 2, 3, 4, 5\}$

$A \cap B =$

$A \cap C =$

$\overline{A} =$