Chapter 01: The Foundation: Logic and Proofs

VO TRAN DUY

Course: Discrete Mathematics - $MAD101$

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Outline

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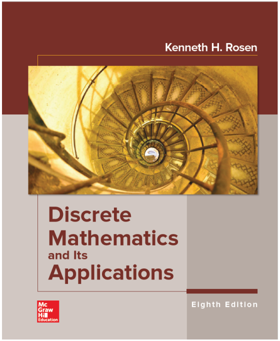
Score Evaluation

1. Must attend at least 80% of contact hours; **otherwise, you are not allowed to take the exam**.
2. Score structure:
   - 10% for Attendance
   - 50% for 5 Progress Tests (PT).
   - 40% for Final exam (FE) $\longrightarrow$ 50 questions in 60 minutes.
   - Bonus score for weekly homework will be added directly to the PT's with an appropriate rate.
3. To pass this subject, you must have
   - Total score ≥ 5
   - Final Examination ≥ 4 (of 10)
   - Every ongoing assessment score > 0.
4. Retake the FE when you do not pass or want to improve your previous result.

Outline

Course Description

1. Chapter 01 : The Foundations: Logic and Proofs.
2. Chapter 02 : Basic Structures: Sets, Function, Sequences, and Sums.
3. Chapter 03 : Algorithms.
4. Chapter 04 : Number Theory and Cryptography.
5. Chapter 05 : Induction and Recursion.
6. Chapter 06 : Counting and Advanced Counting Techniques.
7. Chapter 07 : Graphs.
8. Chapter 08 : Trees.



Outline

Propositional Logic

Rusell's Paradox

Once upon a time, there was a barber who lived in the village of Seville. In that village, all men shave themselves or ask a barber. And this barber declared:

" *I only shave the men of the village of Seville who do not shave themselves.* "

Propositional Logic

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A **proposition** is a declarative sentence (that is, a sentence that declares a fact) that is either true or false, but not both.

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Examples

- VND is the official currency of Vietnam.
- The speed of light in vacuum is a universal physical constant that is exactly equal to $299,792,458$ metres per second.
- The sun rises in the North and sets in the South.
- The largest whale (and largest mammal, as well as the largest animal known ever to have existed) is the blue whale, a baleen whale (Mysticeti).

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Counter-examples

- Please stop it, Alex!
- Do you have any idea for decorating the Christmas tree?
- We will hang out with friends at a famous restaurant in town this weekend.

Propositional Logic

$\hookrightarrow$ We usually fix the following notations:

- $p, q, r, s, \ldots$: to represent propositions (*propositional variables*);
- $\boldsymbol{T}$ or $\mathbf{1}$: to denote the truth value of a true proposition;
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Definition (Negation)

Let p be a proposition. The negation of p, denoted by $\neg p$ (also denoted by $\overline{p}$), is the statement "It is not the case that p."
The proposition $\neg p$ is read "not p". The truth value of $\neg p$, is the opposite of the truth value of p.

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Examples

Let p : "Earth is an oblate spheroid". Then the negation of p is:

$\overline{p}$: Earth is **not** an oblate spheroid.

Propositional Logic

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Let p and q be propositions. The conjunction of p and q, denoted by $p \wedge q$, is the proposition "p and q". The conjunction $p \wedge q$ is true when both p and q are true and is false otherwise.

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Definition (Disjunction)

Let p and q be propositions. The disjunction of p and q, denoted by $p \vee q$, is the proposition "p or q". The disjunction $p \vee q$ is false when both p and q are false and is true otherwise.

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p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Figure: The truth table for the conjunction and the disjunction of two propositions.

Propositional Logic

Definition (Exclusive or)

Let p and q be propositions. The *exclusive* or of p and q, denoted by $p \bigoplus q$ (or p XOR q), is the proposition that is true when exactly one of p and q is true and is false otherwise.

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Let p and q be propositions. The conditional statement $p \to q$ is the proposition "if p, then q". The conditional statement $p \to q$ is false when p is true and q is false, and true otherwise.

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p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Propositional Logic

Remarks

- See the ROSEN's book, page 7 for other expression of the conditional statement.
- In the conditional statement $p \rightarrow q$, p is called the hypothesis (or antecedent or premise) and q is called the conclusion (or consequence).
- We say that p is a *sufficient* condition for q, and q is a *necessary* condition for p.
- The proposition $q \rightarrow p$ is called the *converse* of $p \rightarrow q$.
- The proposition $\overline{q} \rightarrow \overline{p}$ is called the *contrapositive* of $p \rightarrow q$.
- The proposition $\overline{p} \rightarrow \overline{q}$ is called the *inverse* of $p \rightarrow q$.

Propositional Logic

Definition (Biconditional statement)

Let p and q be propositions. The biconditional statement $p \leftrightarrow q$ is the proposition "p if and only if q". The biconditional statement $p \leftrightarrow q$ is true when p and q have the same truth values, and is false otherwise.

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p	q	$p \leftrightarrow q$
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Propositional logic

$\hookrightarrow$ The precedence of logical operators basically obeys two rules:

- Parentheses from inner to outer.
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Operator	*Precedence*
$\neg$	1
$\land$ $\lor$	2 3
$\rightarrow$ $\leftrightarrow$	4 5

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Examples

(i). $p \vee q \rightarrow p \wedge r$ means that $(p \vee q) \rightarrow (p \wedge r)$.
(ii). $p \wedge \overline{q} \vee q \wedge \overline{p} \rightarrow \overline{r}$ means that $((p \wedge (\overline{q})) \vee (q \wedge (\overline{p}))) \rightarrow (\overline{r})$.

Propositional Logic

- We can use a **bit** to represent a truth value: **bit 1** for **true** and **bit 0** for **false.**
- A **Boolean variable** has value either true or false, and can be represented by a bit.
- By replacing true by 1 and false by 0 in the truth tables of logical operators, we obtain the corresponding tables for bit operations.
- The operators $\neg, \wedge, \vee$ and $\oplus$ are also denoted by NOT, AND, OR and XOR.

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Examples

The bitwises AND, OR and XOR of 0110010110 and 1100011101 are

$$0100010100 \text{ (AND)}; \quad 1110011111 \text{ (OR)}; \quad 1010001011 \text{ (XOR)}.$$

Outline

Propositional Equivalences

Definition

- A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, is called a *tautology*.
- A compound proposition that is always false is called a *contradiction*.
- A compound proposition that is neither a tautology nor a contradiction is called a *contingency*.

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Examples

• It is clear that $p \vee \overline{p}$ and $p \wedge \overline{p}$ are a tautology and a contradiction, respectively while the proposition p itself is a contingency.

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Definition

The compound propositions p and q are called logically equivalent if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that p and q are logically equivalent.

Propositional Equivalences

$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws

$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

Figure: Some logical equivalences.

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Exercises: Prove the following claims:

- $p \to q \equiv \overline{p} \vee q$;
- $(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$;
- $\overline{p \vee (\overline{p} \wedge q)} \equiv \overline{p} \wedge \overline{q}$.

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Predicates and Quantifiers

Examples

- Which statements are propositions:
  1. $3 + 2 = 5$
  2. $X + 2 = 5$
  3. $X + 2 = 5$ for any choice of X in $\{1, 2, 3\}$
  4. $X + 2 = 5$ for some X in $\{1, 2, 3\}$

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- Let $Q(x, y)$: "$x > y$" with $x, y \in \mathbb{R}$. Which statements are propositions:
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  3. $Q(x, 9)$

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  1. No; 2. Yes; 3. No.

$\hookrightarrow$ In general, a statement involving n variables $x_1, x_2, \ldots, x_n$ can be denoted by $P(x_1, x_2, \ldots, x_n)$.

Predicates and Quantifiers

Definition (n-place Predicate)

A statement of the form $P(x_1, x_2, \ldots, x_n)$ is the value of the propositional function P (defined on a set U) at the n-tuple $(x_1, x_2, \ldots, x_n)$ and P is also called an n-place predicate or an n-ary predicate.

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- We often denote a predicate by $P(x_1, x_2, \ldots, x_n)$.
- Note that $P(x_1, x_2, \ldots, x_n)$ is not a proposition, but $P(x_1^0, x_2^0, \ldots, x_n^0)$ with $(x_1^0, x_2^0, \ldots, x_n^0) \in U$, is a proposition with a determined truth value.

Outline

Time for practice

***Ex*1.** Use De Morgan's laws to find the negation of each of the following statements:
a) Lena will take a job in industry or go to graduate school.
b) Yoshiko knows Java and calculus.
c) James is young and strong.
d) Rita will move to Oregon or Washington.

***Ex*2.** Show that each of these conditional statements is a tautology:
a) $[\overline{p} \wedge (p \vee q)] \to q$.
b) $[(p \vee q) \wedge (p \to r) \wedge (q \to r)] \to r$.
c) $[p \wedge (p \to q)] \to q$.

***Ex*3.** Let $P(x)$ be the statement "The word x contains the letter a." What are these truth values?
a) P(orange)
b) P(lemon)
c) P(true)
d) P(false).

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Examples

Consider the following program, designed to determine the maximum and minimum values of two given variables x and y:

$$\text{If } x \geq y \text{ then } \max := x,\ \min := y$$
$$\text{Otherwise } \max := y,\ \min := x.$$

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$\hookrightarrow$ Let $P_1(x, y)$ and $P_2(x, y)$ be the statements "$x = a$ and $y = b$ and $x \geq y$ " and "$x = a$ and $y = b$ and $x < y$ ", where a, b are the input values of x and y, respectively $\Longrightarrow P_1(x, y) \oplus P_2(x, y)$ is the precondition.
$\hookrightarrow$ Let $Q_1(x, y)$ and $Q_2(x, y)$ be the statements "$\max = a$ and $\min = b$" and "$\max = b$ and $\min = a$" $\Longrightarrow Q_1(x, y) \oplus Q_2(x, y)$ is the postcondition.

Predicates and Quantifiers

Question

Study the following code segment:

$$\text{If } x \geq 2 \text{ OR } y < x^2 \text{ then } x := 2y$$

$$\text{If } x < 0 \text{ XOR } y > \min\{\sqrt{|x|}, 0.5\} \text{ then } x := y \text{ and } y := -x.$$

What are the values of x and y after the codes execute under the following inputs:

- $x = 1, \ y = -1$
- $x = 6, \ y = 0.4$
- $x = 0.08, \ y = -5$
- $x = 0, \ y = 1.$

Answer:

- $x = -2, \ y = -1$ (after first code) $\longrightarrow x = -1, \ y = 2$ (after second code)

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- $x = -10,\ y = -5$ (after first code) $\longrightarrow x = -5,\ y = 10$ (after second code)

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- $x = -10,\ y = -5$ (after first code) $\longrightarrow x = -5,\ y = 10$ (after second code)
- $x = 0,\ y = 1$ (after first code) $\longrightarrow x = 1,\ y = 0$ (after second code).

Predicates and Quantifiers

$\hookrightarrow$ Let $P(x)$ be a predicate defined on some domain (of discourse) U. Then there are two ways to form a proposition from $P(x)$:

- The first way is to consider $P(x)$ at some fixed element $x \in U$.
- The second way is to add the **quantifier** in $P(x)$.

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Definition

1. The universal quantification $\forall \boldsymbol{xP(x)}$ of $P(x)$ is the statement

"$P(x)$ for all values of $x \in U$."

Here $\forall$ is called the universal quantifier.

2. The *existential quantification* $\exists \boldsymbol{xP(x)}$ of $P(x)$ is the proposition

"There exists an element $x \in U$ s.t. $P(x)$."

Here $\exists$ is called the existential quantifier.

3. The *uniqueness quantifier* $\exists! \boldsymbol{xP(x)}$ of $P(x)$ is the proposition

"There exists a unique $x \in U$ such that $P(x)$ is true."

Here $\exists!$ is called the uniqueness quantifier.

Predicates and Quantifiers

Example

Suppose that all creatures on earth are considered. Express these statements by using suitable quantifiers:

- “All lions are fierce.”
- “Some lions do not drink coffee.”
- “Some fierce creatures do not drink coffee.”

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• Note that $\exists x(P(x) \to \overline{R(x)})$ and $\exists x(Q(x) \to \overline{R(x)})$ are wrong expressions for the second and the third statements, respectively.

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TABLE 1 **Quantifications of Two Variables.**

Statement	*When True?*	*When False?*
$\forall x \forall y P(x, y)$ $\forall y \forall x P(x, y)$	$P(x, y)$ is true for every pair x, y.	There is a pair x, y for which $P(x, y)$ is false.
$\forall x \exists y P(x, y)$	For every x there is a y for which $P(x, y)$ is true.	There is an x such that $P(x, y)$ is false for every y.
$\exists x \forall y P(x, y)$	There is an x for which $P(x, y)$ is true for every y.	For every x there is a y for which $P(x, y)$ is false.
$\exists x \exists y P(x, y)$ $\exists y \exists x P(x, y)$	There is a pair x, y for which $P(x, y)$ is true.	$P(x, y)$ is false for every pair x, y.

Figure: The order of quantifiers.

Predicates and Quantifiers

Negating nested quantifiers

- Move the negation ($\neg$) to the right.
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Example

Express the negation of these statements if the domain for all variables consists of al real numbers:

- $\forall x\ \exists y\ (x = y^3)$
- $\forall m\ \forall n\ \exists p\ (p > \sqrt{m^2 + n^2} \vee p < (m + n)/2)$
- $\forall x\ (\exists y\ \forall z\ P(x, y, z) \wedge \exists z\ \forall y\ Q(x, y, z))$.

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Solution.

- $\exists x\ \forall y\ (x \neq y^3)$

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- $\exists x\ \forall y\ (x \neq y^3)$
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- $\exists x\ (\forall y\ \exists z\ \overline{P(x,y,z)} \vee \forall z\ \exists y\ \overline{Q(x,y,z)})$.

Outline

Rule of Inference

Rule of Inference	***Tautology***	***Name***
p $p \to q$ $\therefore q$	$(p \wedge (p \to q)) \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	$(\neg q \wedge (p \to q)) \to \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore p \to r$	$((p \to q) \wedge (q \to r)) \to (p \to r)$	Hypothetical syllogism
$p \vee q$ $\neg p$ $\therefore q$	$((p \vee q) \wedge \neg p) \to q$	Disjunctive syllogism
p $\therefore p \vee q$	$p \to (p \vee q)$	Addition
$p \wedge q$ $\therefore p$	$(p \wedge q) \to p$	Simplification
p q $\therefore p \wedge q$	$((p) \wedge (q)) \to (p \wedge q)$	Conjunction
$p \vee q$ $\neg p \vee r$ $\therefore q \vee r$	$((p \vee q) \wedge (\neg p \vee r)) \to (q \vee r)$	Resolution

Rule of Inference

Rule	Name
$\dfrac{\forall x P(x)}{\therefore P(c)}$	Universal Instantiation
$\dfrac{P(c) \text{ for arbitrary } c}{\therefore \forall x P(x)}$	Universal generalization
$\dfrac{\exists x P(x)}{\therefore P(c) \text{ for some element } c}$	Existential instantiation
$\dfrac{P(c) \text{ for some element } c}{\therefore \exists x P(x)}$	Existential generalization

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$\underline{P(c) \text{ for arbitrary } c}$ $\therefore \forall x P(x)$	Universal generalization
$\underline{\exists x P(x)}$ $\therefore P(c)$ for some element c	Existential instantiation
$\underline{P(c) \text{ for some element } c}$ $\therefore \exists x P(x)$	Existential generalization

$\hookrightarrow$ **Universal Instantiation:** If $\forall x P(x)$ is true, then $P(c)$ is true for any choice of c in the universe of discourse.

$\hookrightarrow$ **Universal generalization.** If $P(c)$ is true for any choice of c in the universe of discourse, then $\forall x P(x)$ is true.

Rule of Inference

Example

Rule of Inference

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- $\forall x(MAD(x) \to MAE(x)) \to$ **Premise**
- MAD(Khang) $\to$ MAE(Khang) $\to$ **Universal Instantiation**
- MAD (Khang) $\to$ **Premise**
- MAE(Khang) $\to$ **Modus Ponens**

Therefore, Khang has passed the MAE course.

Exercises

1. Let $P(x), Q(x), R(x)$, and $S(x)$ be the statements "x is a duck," "x is one of my poultry," "x is an officer," and "x is willing to waltz," respectively. Express each of these statements using quantifiers; logical connectives; and $P(x), Q(x), R(x)$, and $S(x)$.
   a) No ducks are willing to waltz.
   b) No officers ever decline to waltz.
   c) All my poultry are ducks.
   d) My poultry are not officers.
   e) Does (d) follow from (a), (b), and (c)? If not, is there a correct conclusion?
2. Let $P(x), Q(x), R(x)$, and $S(x)$ be the statements "x is a baby" "x is logical," "x is able to manage a crocodile," and "x is despised," respectively. Suppose that the domain consists of all people. Express each of these statements using quantifiers; logical connectives; and $P(x), Q(x), R(x)$, and $S(x)$.
   a) Babies are illogical.
   b) Nobody is despised who can manage a crocodile.
   c) Illogical persons are despised.
   d) Babies cannot manage crocodiles.
   e) Does (d) follow from (a), (b), and (c)? If not, is there a correct conclusion?

Exercises

1. Express the negation of each of these statements in terms of quantifiers without using the negation symbol.
   1. $\forall x\ (-2 < x < 3)$.
   2. $\exists x\ (0 \leq x < 5)$.
   3. $\forall x\ ((x \geq -2) \vee (x < 10))$.
   4. $\exists x\ ((x < 4) \wedge (x > -4))$.
2. Determine the truth value of each of these statements if the domain consists of all real numbers.
   1. $\exists x\ (x^3 = 1)$.
   2. $\exists x\ (x^4 < x^2)$.
   3. $\forall x\ ((-x)^2 = x^2)$.
   4. $\forall x\ (2x > x)$.

Thank you for your attention !