Chapter 2
Basic Structures Set, Functions, Sequences and Sums

MAD101

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Definitions and notation

Definition. A set is an **unordered** collection of elements.

Examples.

- $\{1, 2, 3\}$ is the set containing "1" and "2" and "3."
- $\{1, 1, 2, 3, 3\} = \{1, 2, 3\}$ since repetition is irrelevant.
- $\{1, 2, 3\} = \{3, 2, 1\}$ since sets are unordered.
- $\{1, 2, 3, \dots\}$ is a way we denote an infinite set (in this case, the natural numbers).
- $\emptyset =$ is the empty set, or the set containing no element.

Note. $\emptyset \neq \{\emptyset\}$

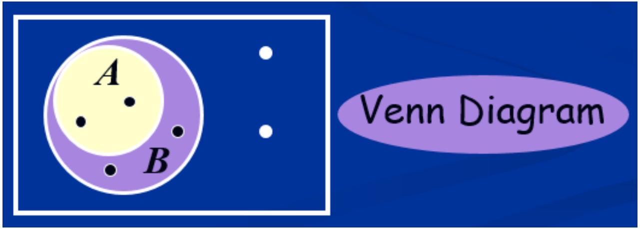
Definitions and notation

- $x \in S$ means "x is an element of set S."
- $x \notin S$ means "x is not an element of set S."
- $A \subseteq B$ means "A is a subset of B."

  or, "B contains A."

  or, "every element of A is also in B."

  or, $\forall x((x \in A) \rightarrow (x \in B))$



Definitions and notation

- $A \subseteq B$ means "A is a subset of B."
- $A \supseteq B$ means "A is a superset of B."
- A = B if and only if A and B have exactly the same elements

  Iff, $A \subseteq B$ and $B \subseteq A$

  Iff, $A \subseteq B$ and $A \supseteq B$

  Iff, $\forall x((x \in A) \leftrightarrow (x \in B))$

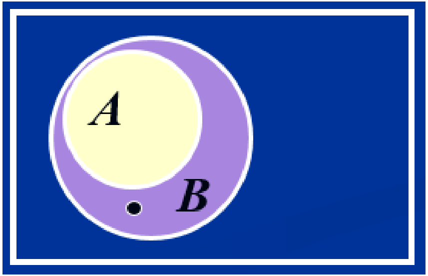
Note. To show equality of sets A and B, show: $A \subseteq B$ and $B \subseteq A$

Definitions and notation

$A \subset B$ means "A is a proper subset of B." That means

$A \subseteq B$ and $A \neq B$

$\forall x((x \in A) \rightarrow (x \in B)) \wedge \exists x((x \in B) \wedge (x \notin A))$



Example.

- $\{1, 2, 3\} \subseteq \{1, 2, 3, 4, 5\}$

Example.

- $\{1, 2, 3\} \subseteq \{1, 2, 3, 4, 5\}$ OK
- $\{1, 2, 3\} \subset \{1, 2, 3, 4, 5\}$

Example.

1 Sets

- $\{1, 2, 3\} \subseteq \{1, 2, 3, 4, 5\}$ OK
- $\{1, 2, 3\} \subset \{1, 2, 3, 4, 5\}$OK
- $\emptyset \subseteq \{1, 2, 3\}$

Example.

- $\{1,2,3\} \subseteq \{1,2,3,4,5\}$ OK
- $\{1,2,3\} \subset \{1,2,3,4,5\}$OK
- $\emptyset \subseteq \{1,2,3\}$OK $(\forall x((x \in \emptyset) \rightarrow (x \in \{1,2,3\}))$ and $x \in \emptyset$ FALSE)
- Is $\emptyset \in \{1,2,3\}$?

Example.

- $\{1, 2, 3\} \subseteq \{1, 2, 3, 4, 5\}$ OK
- $\{1, 2, 3\} \subset \{1, 2, 3, 4, 5\}$OK
- $\emptyset \subseteq \{1, 2, 3\}$OK $(\forall x((x \in \emptyset) \rightarrow (x \in \{1, 2, 3\}))$ and $x \in \emptyset$ FALSE)
- Is $\emptyset \in \{1, 2, 3\}$?NO
- Is $\emptyset \subseteq \{\emptyset, 1, 2, 3\}$?

Example.

- $\{1,2,3\} \subseteq \{1,2,3,4,5\}$ OK
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- Is $\emptyset \in \{1,2,3\}$?NO
- Is $\emptyset \subseteq \{\emptyset,1,2,3\}$?OK
- Is $\emptyset \in \{\emptyset,1,2,3\}$?

Example.

- $\{1,2,3\} \subseteq \{1,2,3,4,5\}$ OK
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Example.

- Is $\{x\} \subseteq \{x\}$?

Example.

- Is $\{x\} \subseteq \{x\}$?OK
- Is $\{x\} \in \{x, \{x\}\}$?

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- Is $\{x\} \in \{x\}$?

Example.

- Is $\{x\} \subseteq \{x\}$?OK
- Is $\{x\} \in \{x, \{x\}\}$?OK
- Is $\{x\} \subseteq \{x, \{x\}\}$OK
- Is $\{x\} \in \{x\}$?NOT OK

Ways to define sets

- Explicitly: {John, Paul, George, Ringo}
- Implicitly: $\{1, 2, 3, \ldots\}, or \{2, 3, 5, 7, 11, 13, 17, \ldots\}$
- Set builder: $\{x : x \text{ is prime}\}, \{x | x \text{ is odd}\}$
- In general $\{x : P(x)\}$, where P(x) is some predicate. We read *"the set of all x such that $P(x)$"*

Cardinality

If S is finite, then the **cardinality** of S, |S|, is the number of distinct elements in S.

Example.

$S = \{1, 2, 3\} \rightarrow$ |S|=3

$S = \{2, 4, 1, 7\} \rightarrow$ |S|=4

$S = \emptyset \rightarrow$ |S|=0

$S = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ |S|=3

$S = \{1, 2, 3, ...\} \rightarrow$ |S| is infinite.

Power sets

If S is a set, then the power set of S is $P(S) = 2^S = \{x : x \subseteq S\}$. We say, *"P(S) is the set of all subsets of S."*

Example.

- If $S = \{a\}$ then $2^S = \{\emptyset, \{a\}\}$
- If $S = \{a, b\}$ then $2^S = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$
- If $S = \emptyset$ then $2^S = \{\emptyset\}$
- If $S = \{\emptyset, \{\emptyset\}\}$ then $2^S = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$

Note. $\boxed{\text{If S finite, then } |2^S| = 2^{|S|} \text{ (If } |S| = n\text{, then } |2^S| = 2^n)}$

- If $S = \{a\}$ then $2^S = \{\emptyset, \{a\}\} \rightarrow 2^S = 2$
- If $S = \{a, b\}$ then $2^S = \{\emptyset, \{a\}, \{b\}, \{a, b\}\} \rightarrow |2^S| = 2^2 = 4$
- If $S = \emptyset$ then $2^S = \{\emptyset\} \rightarrow |2^S| = 2^0 = 1$
- If $S = \{\emptyset, \{\emptyset\}\}$ then $2^S = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\} \rightarrow |2^S| = 2^2 = 4$

The **Cartesian Product** of two sets A and B is $A \times B = \{(a, b) : a \in A \wedge b \in B\}$

Example.

If A = {Lâm, Bình, Chi}, and B = {Xung, Ca}, then

$A \times B$= {(Lâm, Xung), (Lâm, Ca), (Bình, Xung), (Bình, C), (Chi, Xung), (Chi, Ca)}

Notes.

- (a, b) = (c, d) iff a = c, and b = d
- $\boxed{\text{A,B finite} \rightarrow |A \times B| = |A||B|}$

If A = {Lâm, Bình, Chi} → |A|=3, and B = {Xung, Ca} → |B|=2

$A \times B$= {(Lâm, Xung), (Lâm, Ca), (Bình, Xung), (Bình, C), (Chi, Xung), (Chi, Ca)} $\rightarrow |A \times B| = 3.2 = 6$

Cartesian Product

The Cartesian Product of n sets $A_1, A_2, \ldots, A_n$ is:

$$A_1 \times A_2 \times ..... \times A_n = \{(a_1, a_2, ..., a_n) : a_i \in A_i, \forall i = 1, 2, ..., n\}$$

Note.

$$A^n = A \times A \times A \times ... \times A(n \text{ times}) = \{(a_1, a_2, ..., a_n) : a_i \in A, \forall i = 1, 2, ..., n\}$$

Example.

$A = \{a, b\}, B = \{1, 2, 3\}, C = \{0, 1\}$

$A \times B \times C = \{(a, 1, 0), (a, 1, 1), (a, 2, 0), (a, 2, 1), (a, 3, 0), (a, 3, 1),$
$(b, 1, 0), (b, 1, 1), (b, 2, 0), (b, 2, 1), (b, 3, 0), (b, 3, 1)\}$

$|A \times B \times C| = |A||B||C| = 2.3.2 = 12$

Given A={0,Ø}. Find the cardinality of P(AxA).

Select one:

- a. 2
- b. {(0,Ø),(0,0),(Ø,Ø), (Ø,0)}
- c. 4
- d. 16

Quizz

Given A={0,Ø}. Find the cardinality of P(AxA).

Select one:

- a. 2
- b. {(0,Ø),(0,0),(Ø,Ø), (Ø,0)}
- c. 4
- d. 16

Ans: d ($|A|=2 \rightarrow |A \times A| = 4 \rightarrow P(A \times A)=2^4 = 16$)

Quizz

(i) How many bit strings of length 8 are there?

(ii) How many bit strings of length 8 begin with 10 or 01?

Select one:

- a. (i) 2^7 (ii) 2^7
- b. (i) 2^8 (ii) 2^6
- c. (i) 2^8 (ii) 2^7
- d. (i) 2^7 (ii) 2^8

Quizz

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Select one:

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- b. (i) 2^8 (ii) 2^6
- c. (i) 2^8 (ii) 2^7
- d. (i) 2^7 (ii) 2^8

Ans: C (abcdefgh from 1 and 0)

Quizz

How many bit strings of length 8 begin with 11 or end with 00?

Select one:

- a. $2^6 - 2^4$
- b. $2.2^6 - 2^4$
- c. 2.2^4
- d. 2^4

How many bit strings of length 8 begin with 11 or end with 00?

Select one:

- a. $2^6 - 2^4$
- b. $2.2^6 - 2^4$
- c. 2.2^4
- d. 2^4

Ans: B (11abcdef+abcdef00-11abcd00)

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2 Set operations

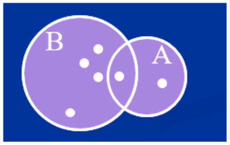
Union

The *union* of two sets A and B is:

$$A \cup B = \{ x : x \in A \lor x \in B \}$$

If A = {Charlie, Lucy, Linus}, and
B = {Lucy, Desi}, then

$A \cup B$ = {Charlie, Lucy, Linus, Desi}

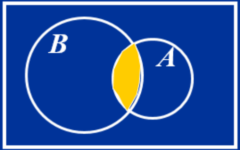


Intersection

The *intersection* of two sets A and B is:

$$A \cap \mathrm{B} = \{x : x \in A \wedge x \in B\}$$

If A = {Charlie, Lucy, Linus}, and
B = {Lucy, Desi}, then
$A \cap B$ = {Lucy}

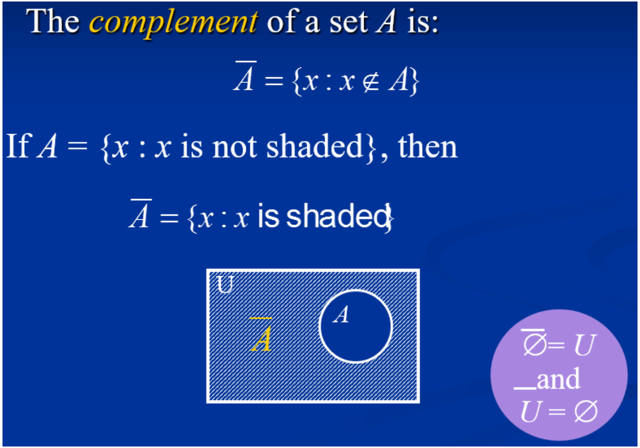


Example.

If A = {x : x is a US president}, and B = {x : x is in this room}, then $A \cap B$ = {x : x is a US president in this room} $= \emptyset$.

Complement

2 Set operations



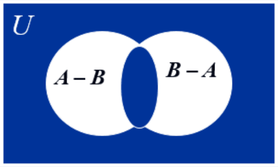
Difference and symmetric difference

The ***symmetric difference***, $A \oplus B$, is:

$$A \oplus B = \{x : (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)\}$$

$$= (A - B) \cup (B - A)$$

$$= \{x : x \in A \oplus x \in B\}$$



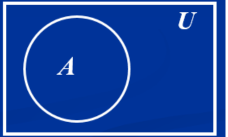
Set Identities

2 Set operations

- **Identity**

$$A \cap U = A$$

$$A \cup \varnothing = A$$



- **Domination**

$$A \cup U = U$$

$$A \cap \varnothing = \varnothing$$

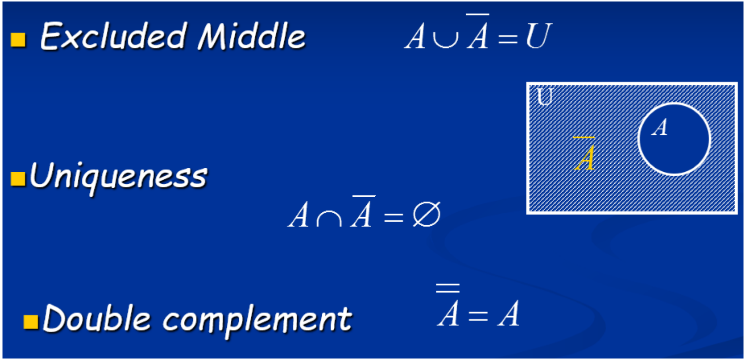
- **Idempotent**

$$A \cup A = A$$

$$A \cap A = A$$

Set Identities

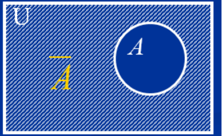
2 Set operations



Set Identities

2 Set operations

- Excluded Middle $A \cup \overline{A} = U$



- Uniqueness

$$A \cap \overline{A} = \varnothing$$

- Double complement $\overline{\overline{A}} = A$

Set Identities

- *Commutativity*

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

- *Associativity*

$$(A \cup B) \cup C = A \cup (B \cup C)$$
$$(A \cap B) \cap C = A \cap (B \cap C)$$

- *Distributivity*

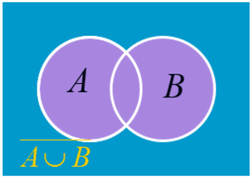
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Set Identities

2 Set operations

- DeMorgan's I $\overline{A \cup B} = \overline{A} \cap \overline{B}$

- DeMorgan's II $\overline{A \cap B} = \overline{A} \cup \overline{B}$



Computer Representation of Sets

- Let $U = \{x_1, x_2, \ldots, x_n\}$, and choose an arbitrary order of the elements of U, say

$$x_1, x_2, \ldots, x_n$$

- Let $A \subseteq U$. Then the **bit string representation** of A is the bit string of length n : $a_1 a_2 \ldots a_n$ such that $a_i = 1$ if $x_i \in A$, and 0 otherwise.

Example. Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and $A = \{x_1, x_3, x_5, x_6\}$. Then the bit string representation of A is 101011

Computer Representation of Sets

Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and $A = \{x_1, x_3, x_5, x_6\}, B = \{x_2, x_3, x_6\}$. Then we have a quick way of finding the bit string corresponding to of $A \cup B$ and $A \cap B$

	A	1	0	1	0	1	1
	B	0	1	1	0	0	1
Bit-wise OR	$A \cup B$	1	1	1	0	1	1
Bit-wise AND	$A \cap B$	0	0	1	0	0	1

Quizz

2 Set operations

Let U = {0,1,2,3,4,5,6,7,8,9}.

Given the subsets A = {1,2,3,4,8}, B = {0,5,6,7,9}. The bit string representing the subset A - B is ...

Select one:

- a. 00 1110 0010
- b. 01 1110 0110
- c. 01 1110 0010
- d. 00 1011 0010

Let U = {0,1,2,3,4,5,6,7,8,9}.

Given the subsets A = {1,2,3,4,8}, B = {0,5,6,7,9}. The bit string representing the subset A - B is ...

Select one:

- a. 00 1110 0010
- b. 01 1110 0110
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Ans: C

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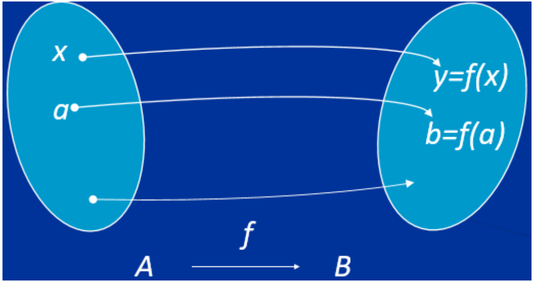
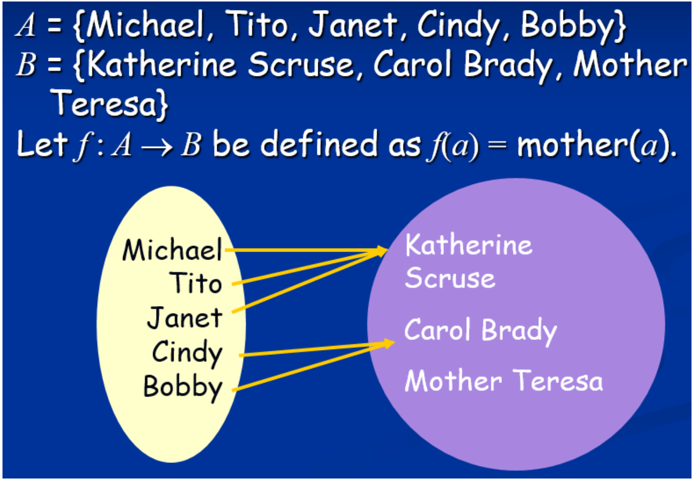
3 Functions

Definition. A **function** f is a rule that assigns to each element x in a set A exactly **one** element y=f(x) in a set B.

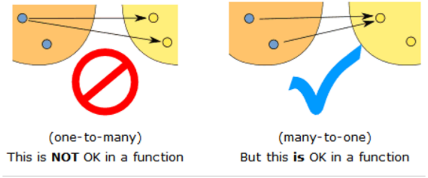


- A is the **domain**, B is the **codomain** of f.
- b = f(a) is the **image** of a and a is the **preimage** of b.
- The **range** of f is the set $\{f(a), a \in A\}$

Example.


A = {Michael, Tito, Janet, Cindy, Bobby}
B = {Katherine Scruse, Carol Brady, Mother Teresa}
Let $f : A \rightarrow B$ be defined as $f(a)$ = mother(a).
Michael
Tito
Janet
Cindy
Bobby
Katherine Scruse
Carol Brady
Mother Teresa

Example.
3 Functions

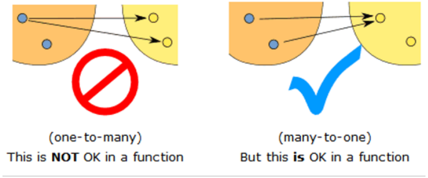


What are functions?

1. $f : \mathbb{Z} \to \mathbb{R} : f(x) = x^2 + 2 \to$

Example.

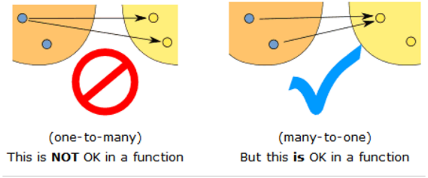
3 Functions



What are functions?

1. $f : \mathbb{Z} \to \mathbb{R} : f(x) = x^2 + 2 \to$ YES
2. $f : \mathbb{Z} \to \mathbb{R} : f(x) = \dfrac{1}{(x-1)^2} + 5x \to$

Example.
3 Functions



What are functions?

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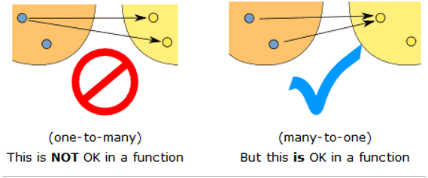
2. $f : \mathbb{Z} \to \mathbb{R} : f(x) = \dfrac{1}{(x-1)^2} + 5x \to$ NO
3. $f : \mathbb{R} \to \mathbb{R} : f(x) = \dfrac{2x+5}{7} \to$

Example.



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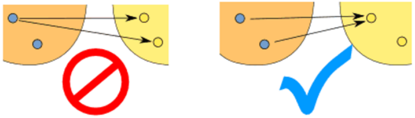
3. $f : \mathbb{R} \to \mathbb{R} : f(x) = \dfrac{2x+5}{7} \to$ YES
4. $f : \mathbb{Z} \to \mathbb{R} : f(x) = \dfrac{(2x+5)^2}{7-2x} \to$

Example.
3 Functions



(one-to-many)
This is **NOT** OK in a function

(many-to-one)
But this **is** OK in a function

What are functions?

1. $f : \mathbb{Z} \to \mathbb{R} : f(x) = x^2 + 2 \to$ YES
2. $f : \mathbb{Z} \to \mathbb{R} : f(x) = \dfrac{1}{(x-1)^2} + 5x \to$ NO
3. $f : \mathbb{R} \to \mathbb{R} : f(x) = \dfrac{2x+5}{7} \to$ YES
4. $f : \mathbb{Z} \to \mathbb{R} : f(x) = \dfrac{(2x+5)^2}{7-2x} \to$ YES

Functions as sets of ordered pairs

A function can be defined as a set of ordered pairs: $\{(a, b) | b = f(a), a \in A\}$

Example. $\{(2, 4), (3, 5), (7, 3)\}$ is a function that says "2 is related to 4", "3 is related to 5", "7 is related to 3"

Notes.

- The domain is $\{2, 3, 7\}$ (input values)
- The range is $\{4, 5, 3\}$ (output values)

1. **Ceiling.** $f(x) = \lceil x \rceil$ the **least integer** y so that $x \leq y$. **Example.**
   a. $\lceil 1.2 \rceil =$

Some Important Functions

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   a. $\lfloor 1.8 \rfloor =$

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Some Important Functions

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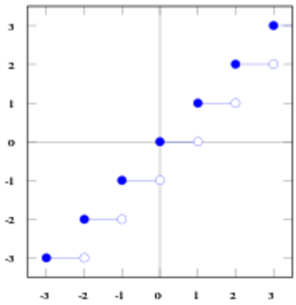
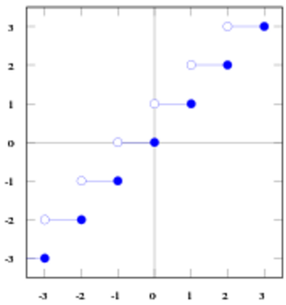
   b. $\lfloor -1.8 \rfloor = -2$
   c. $\lfloor -5 \rfloor = -5$

Note. $\boxed{\lfloor x \rfloor \leq x \leq \lceil x \rceil}$. What is $\lceil -1.1 + \lfloor 1.1 \rfloor \rceil$

Note. $\boxed{\lfloor x \rfloor \leq x \leq \lceil x \rceil}$. What is $\lceil -1.1 + \lfloor 1.1 \rfloor \rceil \ = 0$



Hàm Floor



Hàm Ceiling

Quizz

Let f be floor function and g be ceiling function.
Which of the following is true ?

Select one:

- a. f(-3.1)= - 3
- b. g(-4.5) = - 4
- c. g(7) = 8
- d. f(5.3) = 6

Let f be floor function and g be ceiling function.
Which of the following is true ?

Select one:

- a. f(-3.1)= - 3
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- c. g(7) = 8
- d. f(5.3) = 6

Ans: b

Quizz

3 Functions

Study relations in the set of real numbers R:

(i) f: R → R; f(x)= $(x+1)/(x^2 + 3)$

(ii) f:R → R; f(x) = $x/(2x^2 - 6x - 1)$

Select correct statement(s)

Select one:

- a. (i) is not a function, (ii) is not a function
- b. (i) is a function, (ii) is a function
- c. (i) is not a function, (ii) is a function
- d. (i) is a function, (ii) is not a function

Quizz

3 Functions

Study relations in the set of real numbers R:

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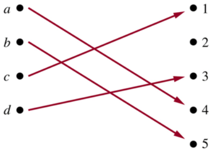
- d. (i) is a function, (ii) is not a function

Ans: d

One-to-One Functions
3 Functions



Definition.A function $f : A \to B$ is **one-to-one** (injective, an injection) if

$$\forall x, y(f(x) = f(y) \to x = y)$$

Remarks.

- A function $f : A \to B$ is **one-to-one** (injective, an injection) iff

$$\forall x, y(x \neq y) \to f(x) \neq f(y)$$

- A **strictly increasing** (tăng) or **strictly decreasing** function on an interval I is one-to-one on I

Example.

The following functions are one to one or not:

1. $f : \mathbb{Z} \to \mathbb{Z}, f(x) = x^2$

Example.

The following functions are one to one or not:

1. $f : \mathbb{Z} \to \mathbb{Z}, f(x) = x^2$ (No)
2. $f : [0, +\infty) \to \mathbb{R}, f(x) = x^2 + 1$

Example.

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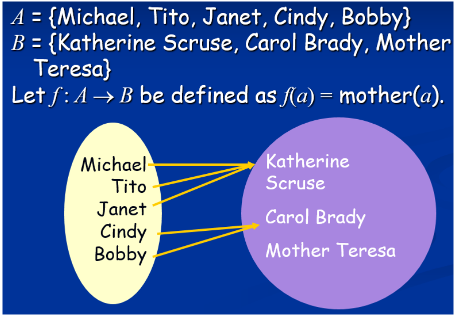
3. Function



Example.

3 Functions

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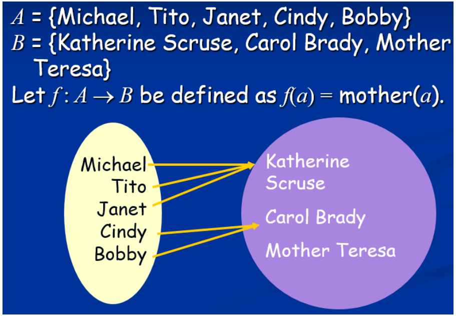
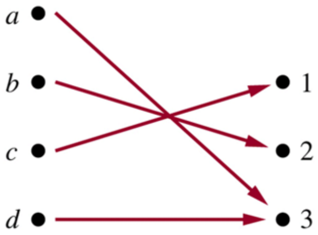


(No)

Onto Functions



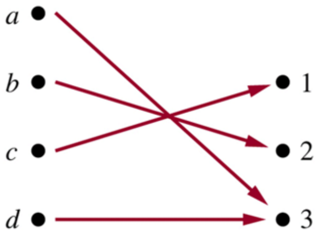
Definition. A function $f : A \to B$ is **onto** (surjective, a surjection) if

$$\forall b \in B, \exists a \in A | f(a) = b$$

Example. The following functions are onto or not

1. $f : \mathbb{Z} \to \mathbb{Z}, f(x) = x^2$

Onto Functions
3 Functions



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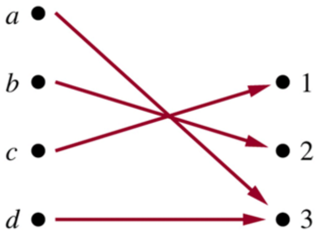
Onto Functions
3 Functions



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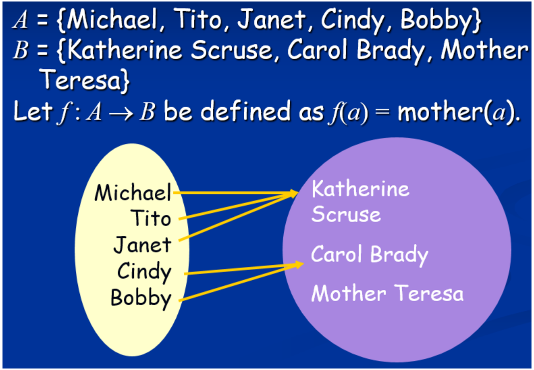
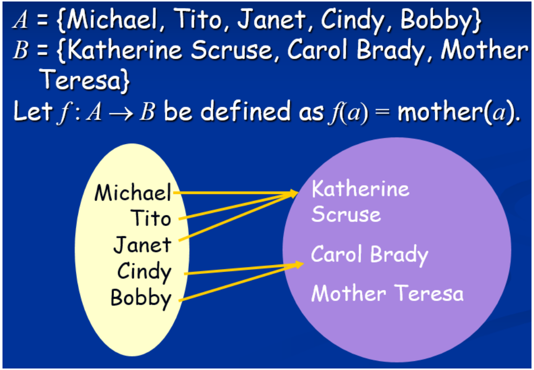
Example.

Function



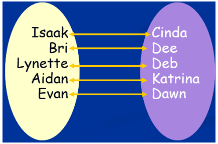
Example.
3 Functions

Function



(No)

Bijection



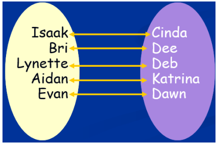
Definition. A function $f : A \to B$ is **bijective** if it is **one-to-one and onto**. We also say that f is a **bijection**.

Example. The following functions are bijection or not

1. $f : \mathbb{R} \to \mathbb{R}, f(x) = x^3 + 1$

Bijection

3 Functions



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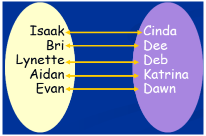
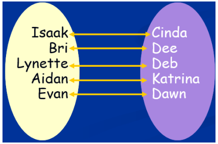
Example. The following functions are bijection or not

1. $f : \mathbb{R} \to \mathbb{R}, f(x) = x^3 + 1$ (Yes)
2. Function



Bijection

3 Functions



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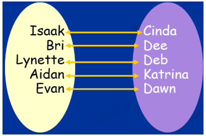
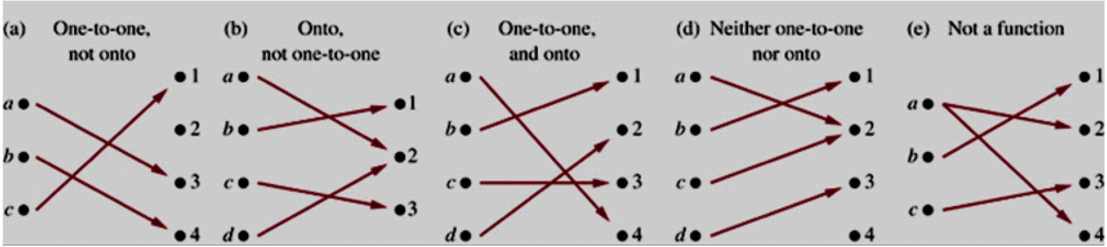
Example. The following functions are bijection or not

1. $f : \mathbb{R} \to \mathbb{R}, f(x) = x^3 + 1$ (Yes)
2. Function



(Yes)

Example.
3 Functions



If f: Z → N; f(x) = $(2 - x)^2$.

Which of the following statements is true?

(i) f is one-to-one

(ii) f is onto

Select one:

- a. (i)
- b. Both
- c. (ii)
- d. None

Ans: d $(x = \pm\sqrt{y} + 2, y = 2 \rightarrow \nexists x \in \mathbb{Z})$

Quizz

How many one-to-one functions are there from the set {1, 2, 3} to the set {1, 2, 3, 4, 5, 6}?

Select one:

- a. 6.5.4
- b. 6^3
- c. 0
- d. 18

Quizz

How many one-to-one functions are there from the set {1, 2, 3} to the set {1, 2, 3, 4, 5, 6}?

Select one:

- a. 6.5.4
- b. 6^3
- c. 0
- d. 18

Ans: 6.5.4

Let B be the set {a, b}. How many functions are there from B^2 to B?

Select one:

- a. 16
- b. 8
- c. 2
- d. 4

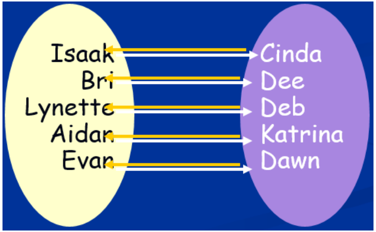
Let B be the set {a, b}. How many functions are there from B^2 to B?

Select one:

- a. 16
- b. 8
- c. 2
- d. 4

Ans: 16 ($|B \times B| = |B||B| = 2.2 = 4$,mỗi phần tử thuộc B^2 có 2 cách chọn ảnh)

Inverse Functions



Definition. Let $f : A \to B$ be a **bijection**. Then the **inverse function** of f , denoted by f^{-1} is the function that assigns each element b in B the unique element a in A such that f(a) = b. Thus $f^{-1}(b) = a$.

Example.

$$f^{-1}(\text{Cinda}) = \text{Isaak}, f^{-1}(\text{Dee}) = Bri, \ldots, f^{-1}(\text{Dawn}) = \text{Evan}$$

Example.

- Is the function $f(x) = x^2$ from Z to Z invertible? (i.e. the inverse function exists)
  The function f is not onto. Therefore it is not a bijection, and hence not invertible
- Is the function $\mathrm{f(x) = x + 1}$ from Z to Z invertible?
  The function f is a bijection so it is invertible.

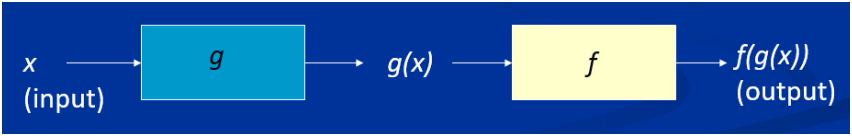
Example.

Is the function f(x) = x + 1 from Z to Z invertible? What is its inverse?

To find the inverse, let y be any element in **Z,** we find the element x in **Z** such that $y = f(x) = x + 1$. Solving this equation we obtain $x = y - 1$. Hence $f^{-1}(y) = y - 1$.

We also write $f^{-1}(x) = x - 1$.

Compositions of Functions

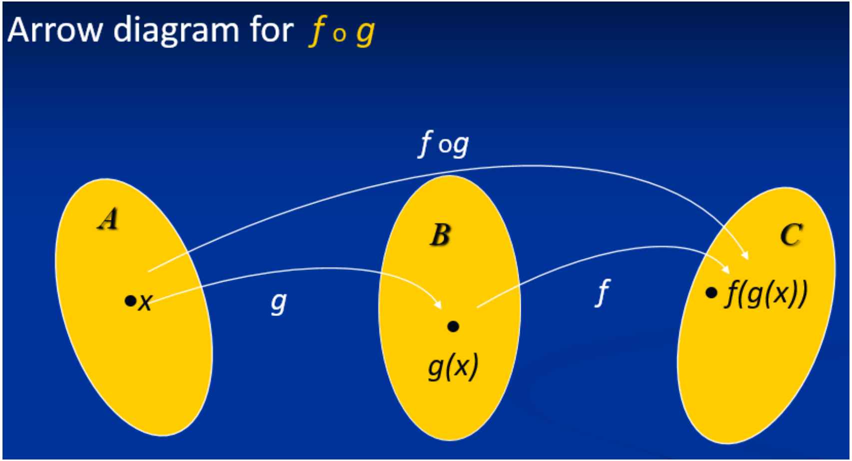


Definition. The **composition** of a function $g : A \to B$ and a function $f : B \to C$ is the function $f \circ g : A \to C$ defined by

$$f \circ g(x) = f(g(x))$$

Note. The domain of fog is also the domain of g, and the codomain of fog is also the codomain of f.

Arrow diagram for $f \circ g$



Example.

let $f(x) = x^2$ and g(x) = x – 3 are functions from R to R. Find $f \circ g$ and $g \circ f$

$$(f \circ g)(x) = f(g(x)) = f(x - 3) = (x - 3)^2$$

$$(g \circ f)(x) = g(f(x)) = g(x^2) = x^2 - 3$$

This shows that in general: $f \circ g \neq g \circ f$

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Definition.

A **sequence** $\{a_i\}$ is a function $f : \mathbb{N} \to \mathbb{R}$, where we write a_i to indicate $f(i)$.

Example.

- $1, 1/2, 1/3, ..., 1/n, ...$
- **Finite** sequence $\{a_i\}$, where $a_i = i, i = 0, 1, 2$: $a_0 = 0, a_1 = 1, a_2 = 2$
- **Infinite** sequence $\{a_i\}$, where $a_i = i^2$: $a_0 = 0, a_1 = 1, a_2 = 4, ...$
- $a_0 = 1, a_n = 2a_{n-1} - 3, n = 1, 2, ... \to a_1 = -1\ a_2 = -5....$
- **Geometric progression**: $a, ar, ar^2, ..., ar^n, ...$
- **Arithmetic progression**: $a, a + d, a + 2d, ..., a + nd, ...$

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Introduction

5 Summations

- **Notation.** $\sum_{i=1}^{k} a_i = a_1 + a_2 + \ldots + a_k$
- **Properties.**
  1. $\sum_{i=1}^{k} (ca_i + db_i) = c\sum_{i=1}^{k} a_i + d\sum_{i=1}^{k} b_i$
  2. $\sum_{i=1}^{k} a = a + a + \ldots. + a = ka$

Familiar Summation Formulae

- $\sum_{k=1}^{n} k = 1 + 2 + \ldots + n = \dfrac{n(n+1)}{2}$

Let $\quad S = 1 + 2 + 3 + \ldots + n$

Then $\quad S = n + (n-1) + (n-2) + \ldots + 1$

$$2S = (n+1) + (n+1) + \ldots + (n+1) = n(n+1)$$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

- $\sum_{k=0}^{n} ar^k = a\dfrac{1 - r^{n+1}}{1 - r} (r \neq 1)$

Some Useful Summation Formulae

TABLE 2 Some Useful Summation Formulae.

Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x\rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x\rvert < 1$	$\frac{1}{(1-x)^2}$

Cardinality

- Cardinality = number of elements in a set.
- The sets A and B have the same cardinality if and only if there is a one-to-one correspondence from A to B
- A set that is either finite or has the same cardinality as the set of positive integers is called ***countable***.

  a1, a2, a3,....., an,......
- A set that is not countable is called ***uncountable.***
- When a infinite set S is countable, we denote the cardinality of S is $|S| = \aleph_0$ (aleph null)
- For example, $|\mathbb{N}| = \aleph_0$ because $\mathbb{N}$ is countable and infinite but $\mathbb{R}$ is uncountable and infinite, and we say $|\mathbb{R}| = 2^{\aleph_0}$

Quizz

5 Summations

Let $a_n = -a_{n-2}$ for all $n > 1$. If $a_0 = 3$ and $a_1 = 5$, find a_7.

Select one:

- a. 3
- b. 7
- c. -5
- d. -3

Let $a_n = -a_{n-2}$ for all $n > 1$. If $a_0 = 3$ and $a_1 = 5$, find a_7.

Select one:

- a. 3
- b. 7
- c. -5
- d. -3

Ans: C

Suppose a_n is defined recursively by: $a_0=3$, $a_{n+1}=3.a_n$, $n>0$. What is a_n ?

Select one:

a. $a_n=3^n$

b. $a_n=3^{n+1}$

c. $a_n=3n$

d. $a_n=3n+3$

Ans: b

Find $f(2)$ and $f(3)$ if

$f(n) = f(n - 1) \times f(n - 2) + 1$, and $f(0) = 1$, $f(1) = 4$

Select one:

- a. f(2) = 36, f(3) = 60
- b. f(2) = 30, f(3) = 66
- c. f(2) = 5, f(3) = 21
- d. f(2) = 15, f(3) = 20

Ans: C

Quizz

5 Summations

Study the following sequences:

$a_n = 3n - 2, n = 1, 2, 3, \ldots$

$b_n = b_{n-1} + 3$ for $n>1$ and $b_1 = 1$

Select true statements.

Select one or more:

- a. $b_3 = 7$
- b. $b_3 = 9$
- c. $a_n = b_n$ for all $n > 0$
- d. We can't compute b_n for all $n > 0$

Ans: a & c

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1. List the members of these sets.

a) {x | x is a real number such that $x^2 = 1$}

b) {x | x is a positive integer less than 12}

c) {x | x is the square of an integer and x < 100}

d) {x | x is an integer such that $x^2 = 2$}

2. For each of the following sets, determine whether 2 is an element of that set.

a) {$x \in R$ | x is an integer greater than 1}

b) {$x \in R$ | x is the square of an integer}

c) {2,{2}}

d) {{2},{{2}}}

e) {{2},{2,{2}}}

f) {{{2}}}

3. Determine whether each of these statements is true or false.

a) $0 \in \emptyset$ b) $\emptyset \in \{0\}$ c) $\{0\} \subset \emptyset$ d) $\emptyset \subset \{0\}$

e) $\{0\} \in \{0\}$ f) $\{0\} \subset \{0\}$ g) $\{\emptyset\} \subseteq \{\emptyset\}$

4. Determine whether each of these statements is true or false.

a) $x \in \{x\}$ b) $\{x\} \subseteq \{x\}$ c) $\{x\} \in \{x\}$

d) $\{x\} \in \{\{x\}\}$ e) $\emptyset \subseteq \{x\}$ f) $\emptyset \in \{x\}$

5. What is the cardinality of each of these sets?

a) $\{a\}$ b) $\{\{a\}\}$ c) $\{a, \{a\}\}$ d) $\{a, \{a\}, \{a, \{a\}\}\}$

6. What is the cardinality of each of these sets?

a) $\emptyset$ b) $\{\emptyset\}$ c) $\{\emptyset, \{\emptyset\}\}$ d) $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$

7. Find the power set of each of these sets, where a and b are distinct elements.

a) $\{a\}$ b) $\{a, b\}$ c) $\{\emptyset, \{\emptyset\}\}$

8. How many elements does each of these sets have where a and b are distinct elements?

a) $P(\{a, b, \{a, b\}\})$ b) $P(\{\emptyset, a, \{a\}, \{\{a\}\}\})$ c) $P(P(\emptyset))$

9. Find A^2 and A^3 if

a) $A = \{1, 3\}$ b) $A = \{1, a\}$

10. Let $A=\{1,2,3\}$ and $B=\{1,a\}$. What is the cardinality of each of these sets?

a) $A\times B$ b) A^2 c) $P(B)$ d) $P(B\times A)$ e) $A\cup B$

11. Find the truth set of each of these predicates where the domain is the set of integers.

a) $P(x)$: $x^2 < 3$ b) $Q(x)$: $x^2 > x$ c) $R(x)$: $2x + 1 = 0$

Set operations

6 Problems

1. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{0, 3, 6\}$. Find

a) $A \cup B$ b) $A \cap B$ c) $A - B$ d) $B - A$.

2. Let $A = \{a, b, c, d, e\}$ and $B = \{a, b, c, d, e, f, g, h\}$. Find

a) $A \cup B$ b) $A \cap B$ c) $A - B$ d) $B - A$.

3. Find the sets A and B if $A - B = \{1, 5, 7, 8\}$, $B - A = \{2, 10\}$, and $A \cap B = \{3, 6, 9\}$.

4. Let A and B be sets. Show that

a) $(A \cap B) \subseteq A$ b) $A \subseteq (A \cup B)$ c) $A - B \subseteq A$

d) $A \cap (B - A) = \emptyset$ e) $A \cup (B - A) = A \cup B$ f) $A \oplus B = (A \cup B) - (A \cap B)$.

5. Suppose that the universal set is $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Express each of these sets with bit strings where the i^{th} bit in the string is 1 if i is in the set and 0 otherwise.

a) $\{3, 4, 5\}$ b) $\{1, 3, 6, 10\}$ c) $\{2, 3, 4, 7, 8, 9\}$

1. Why is f not a function from R to R if

a) $f(x) = 1/x$?

b) $f(x) = \sqrt{x}$?

c) $f(x) = \pm\sqrt{x^2+1}$?

2. Determine whether f is a function from Z to R if

a) $f(n) = \pm n$

b) $f(n) = \sqrt{n^2+1}$

c) $f(n) = \dfrac{1}{n^2-4}$

3. Find these values

a) $\lceil 1.1 \rceil$

b) $\lceil -0.1 \rceil$

c) $\lceil 4 \rceil$

d) $\lfloor 3.2 \rfloor$

Functions

6 Problems

4. Determine whether each of these functions from {a, b, c, d} to itself is one-to-one (onto)

a) f (a) = b, f (b) = a, f (c) = c, f (d) = d

b) f (a) = b, f (b) = b, f (c) = d, f (d) = c

c) f (a) = d, f (b) = b, f (c) = c, f (d) = d

5. Determine whether each of these functions from Z to Z is one-to-one (onto)

a) $f(n) = n - 1$ b) $f(n) = n^2 + 1$ c) $f(n) = n^3$ d) $f(n) = \left\lceil \frac{n}{2} \right\rceil$

7. Determine whether each of these functions is a bijection from R to R.

a) $f(x) = -3x + 4$ b) $f(x) = -3x^2 + 7$ c) $f(x) = (x + 1)/(x + 2)$ d) $f(x) = x^5 + 1$

8. Let $S = \{-1, 0, 2, 4, 7\}$. Find $f(S)$ if

a) $f(x) = 1$ b) $f(x) = 2x + 1$ c) $f(x) = \left\lceil \frac{x}{5} \right\rceil$

9. Let f be the function from R to R defined by $f(x) = x^2$. Find

a) $f^{-1}(\{1\})$ b) $f^{-1}(\{x \mid 0 < x < 1\})$ c) $f^{-1}(\{x \mid x > 4\})$

1. 1. Find these terms of the sequence $\{a_n\}$, where $a_n = 2\ (-3)^n + 5n$.

a) a_0 b) a_1 c) a_4 d) a_5

2. What is the term a8 of the sequence {an} if an equals

a) $2n-1$? b) 7? c) $1+(-1)^n$? d) $-(-2)^n$?

3. Find the first five terms of the sequence defined by each of these recurrence relations and initial conditions.

a) $a_n = 6a_{n-1}$, $a_0 = 2$ b) $a_n = a_{n-1}^2, a_1 = 2$ c) $a_n = a_{n-1} + 3a_{n-2}$, $a_0 = 1$, $a_1 = 2$

5. What are the values of these sums?

a) $\sum_{k=1}^{5}(k+1)$ b) $\sum_{j}^{4}(j+2)^2$ c) $\sum_{i=0}^{2}\sum_{j=1}^{3}(2i-3j)$ d) $\sum_{i=1}^{3}\sum_{j=2}^{4} ij$

6. What are the values of these sums, where S = {1, 3, 5, 7}?

a) $\sum_{j\in S}\left(j+\frac{1}{j}\right)$ b) $\sum_{j\in S} j^2$ c) $\sum_{j\in S} 2$

7. What are the values of the following products?

a) $\prod_{i=0}^{10} i$ b) $\prod_{i=1}^{100}(-1)^i$ c) $\prod_{i=0}^{4} i!$

Q&A

Thank you for listening!