Exercise 2.5 Assume that the density function of a distribution $\mathcal{D}$ is $(1-p)U(-1,1)+pU(\frac{-1}{p},\frac{1}{p})$ for $p \in (0,0.5)$, where $U(.)$ denoted the density of the uniform distribution. Let $X_1, ..., X_n$ be iid samples from $\mathcal{D}$. For $\epsilon > 0$, estimate the probability

$$Pr(\frac{1}{n}\sum_{i=1}^{n} X_i \geq \epsilon) \tag{1}$$

Solution:

According to Bernstein's Inequality, we have:

$$\Pr\left[\bar{X} \geq \mu + \epsilon\right] \leq \exp\left(\frac{-n\epsilon^2}{2V + \frac{2\epsilon b}{3}}\right)$$

where μ is the mean of X, ϵ is a constant, V is the variance of X, and b is the upper bound of $|X-\mu|$.

So we need to calculate μ, V, and b:

- Mean:

$$\begin{aligned}
\mathbb{E}[X] &= \mathbb{E}[(1-p)U(-1,1) + p\,U(-1/p, 1/p)] \\
&= (1-p)\,\mathbb{E}[U(-1,1)] + p\,\mathbb{E}[U(-1/p,1/p)] \\
&= 0
\end{aligned}$$

- Variance:

$$\begin{aligned}
Var[X] &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \\
&= \mathbb{E}[X^2] \\
&= (1-p)\,\mathbb{E}[(U(-1,1))^2] + p\,\mathbb{E}[(U(-1/p,1/p))^2] \\
&= (1-p)\frac{(1-(-1))^2}{12} + p\frac{(\frac{1}{p}-(-\frac{1}{p}))^2}{12} \\
&= \frac{1-p}{3} + \frac{1}{3p} \\
&= \frac{1}{3}(1-p+\frac{1}{p})
\end{aligned}$$

- Upper bound b:

  We have that :

$$X \in [-1,1] \cup \left[-\frac{1}{p},\frac{1}{p}\right], \quad p \in (0,0.5) \implies \frac{1}{p} > 2 > 1$$

  Thus,

$$|X| \leq \frac{1}{p} \quad \Rightarrow \quad b = \frac{1}{p}$$

Subtituting to Bernstein's inequality we have that:

$$\Pr\left[\bar{X} \geq \mu + \epsilon\right] \leq \exp\left(\frac{-n\epsilon^2}{2V + \frac{2\epsilon b}{3}}\right)$$

$$\Pr\left[\bar{X} \geq \epsilon\right] \leq \exp\left(\frac{-n\epsilon^2}{\frac{2}{3}\left(1 - p + \frac{1}{p}\right) + \frac{2\epsilon}{3p}}\right)$$

$$\boxed{\Pr\left[\bar{X} \geq \epsilon\right] \leq \exp\left(\frac{-3n\epsilon^2}{2(1 - p + \frac{1+\epsilon}{p})}\right)}$$