Real Analysis - P1

Learning Theory and Applications Group

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Contents

Part I

Lebesgue Intergration

Key definitions here:

1 The Real Number: Sets, Sequences, and Functions

1.1 The Field, Positivity, and Completeness Axioms

1.1.1 Excercise

Ex 1. For $a \neq 0$ and $b \neq 0$, show that $(ab)^{-1} = a^{-1}b^{-1}$

$$(a^{-1}b^{-1})(ab) = a^{-1}b^{-1}ba = a^{-1}1a = 1$$

As a result, $a^{-1}b^{-1} = (ab^{-1})$
Ex 2. Verify the following:

- For each real number $a \neq 0$, $a^2 > 0$. In particular, $1 > 0$ since $1 \neq 0$ and $1 = 1^2$
- For each positive number a, its multiplicative inverse a^{-1} also is positive
- If $a > b$, then
$$ac > bc \text{ if } c > 0 \text{ and } ac < bc \text{ if } c < 0$$

For the first point, we first need to prove that, for any a, then $-a = (-1)a$,

$$a + (-a) = 0 = (1 + (-1))a = a + (-1)a$$

Next, for each $a \neq 0$, if a is positive, then a^2 is positive by definition of positiveness. On the other hand, if $a < 0$, then let $a = -b$ with $b > 0$,

$$a^2 = (-b)^2 = (-1)b(-1)b = (-1)(-b)b = (-1)^2b^2 > 0 \tag{1}$$

For the second point, assuming by contradiction that $a^{-1} < 0$ for any $a > 0$, then let $a^{-1} = -b$ with $b > 0$. then

$$1 = a(a^{-1}) = a(-b) = (-1)ab < 0$$

Here $ab > 0$ since both a and b are positive, and we know from previous point that $0 > -(ab) = (-1)ab$
The last point is straighforward from the definition of $>$.

$$ac - bc = \underbrace{(a-b)}_{>0}\underbrace{c}_{>0} > 0$$

$$ac - bc = \underbrace{(a-b)}_{>0}\underbrace{c}_{<0} = (-1)\underbrace{(a-b)}_{>0}\underbrace{d}_{>0} < 0 \text{ with } d = -c$$

Ex 3. For a nonempty set of real numbers E, show that $\inf E = \sup E$ if and only if E consists of a single point.
If the set E has a single element, then the least upper bound equal to that single element. This similarly applies to lowerbound. In other word, its sup and inf coincides.
On the other direction, if a set E has its sup and inf equal, and assuming by contradiction that E has at least 2 distinct elements, then the gap between these two points $\neq 0$. The difference between sup and inf is lowerbounded by this gap, so they cannot equal.
Ex 4. Let a and b be real numbers.

i Show that if $ab = 0$, then $a = 0$ or $b = 0$

ii Verify that $a^2 - b^2 = (a-b)(a+b)$ and conclude from part (i) that if $a^2 = b^2$, then $a = b$ or $a = -b$.

iii Let c be a positive real number. Define $E = \{x \in \mathbb{R} | x^2 < c\}$ verify that E is nonempty and bounded above. Define $x_0 = \sup E$. Show that $x_0^2 = c$. Use part (ii) to show that there is a unique $x > 0$ for which $x^2 = c$. It is denoted by $\sqrt{c}$

For the first point, suppose that $ab = 0$ and both a and b are not 0, then there exists a^{-1} and b^{-1}, then we have

$$abb^{-1}a^{-1} = 1$$

which means that $b^{-1}a^{-1} = (ab)^{-1}$, but since $ab = 0$, no such number exists.

The second point is a straighforward application of distributive property,

$$(a-b)(a+b) = a(a+b) + (-b)(a+b) = a^2 + ab - ba - b^2 = a^2 - b^2$$

Then from part (i), since $(a-b)(a+b) = 0$, one of the two terms must be 0.

In part (iii), we see that $0^2 = 0 < c$ for all $c > 0$, so E is nonempty. By contradiction, suppose E is not bounded from above, that is, for every $b > 0$, we can always choose some $x \in E$ such that $x > b$, letting $b > c$ lead to a contradiction with the definition of E.

Next, since E is bounded from above, then it has a supremum by completeness axiom. Denote $x_0 = \sup E$. We will show that $x_0^2 \geq c$ and $x_0^2 \leq c$ to conclude that $x_0^2 = c$.

Since $x^2 < c, \forall x \in E$, c is an upperbound of E^2, and because $\sup E$ is the smallest/least upperbound, then $\sup(E)^2 \leq c$. On the otherhand, $x_0 \geq x, \forall x \in E$ and E contains **all** real numbers whose square less than c, so $x_0^2 \geq c$.

Finally, we need to show that x_0 is a unique positive real number such that $x_0^2 = c$. By contradiction, suppose there is some $x > 0$ such that $x \neq x_0$ and $x^2 = c$, then by part (ii), since $x_0^2 = x^2$, we have either $x = x_0$ or $x = -x_0$, but x is positive and $-x_0$ is negetive, so $x = x_0$.

Ex 5. Let a, b, c be real bumbers such that $a \neq 0$ and consider the quadratic equation

$$ax^2 + bx + c = 0, x \in \mathbb{R}$$

i Suppose $b^2 - 4ac > 0$, use the Field Axiom and the preceding problem to complete the square and thereby show that this equation has exactly two solutions given by

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

ii Now suppose $b^2 - 4ac < 0$. Show that the quadratic equation fails to have any solution.

Suppose that $b^2 - 4ac > 0$, then from previous problem, there exists a unique positive number $\sqrt{b^2 - 4ac}$. we can verify that

$$\left(x - \frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(x - \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right) = x^2 - x\frac{-b - \sqrt{b^2 - 4ac}}{2a} - x\frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{b^2 - b^2 + 4ac}{4a^2}$$
$$= x^2 + x\frac{b}{a} + \frac{c}{a} = 0.$$

Then also from the previous problem, either one of the two terms equal 0. As a result, the equation has exactly two solutions.

On the other hand, if $b^2 - 4ac < 0$, then the equation can be rewritten as

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} + \frac{4ac - b^2}{4a}\right) = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4} > 0,$$

which does not have any solution.

Ex 6. Use the Completeness Axiom to show that every nonempty set of real numbers that is bounded below has an infimum and that

$$\inf E = -\sup\{-x | x \in E\}.$$

The set E is bounded below, which means that the set $E' = \{-x | x \in E\}$ is bounded from above, then its supremum exists by completeness axiom. Denote $x_0 = \sup E'$, then $x_0 \geq -x, \forall x \in E \rightleftarrows -x_0 \leq x, \forall x \in E$. As a result, $-x_0 \leq \inf E$.

Suppose that there exists some x' such that $x' > -x_0$ and $x' \leq x, \forall x \in E$; i.e. x' is a "greater" lowerbound of E than x_0. Then we can show that $-x'$ is a "smaller" upperbound of E', which contradicts with the definition of supremum. As a result, no such x' exists, and $-x_0$ is the infimum of E

Ex 7. For real bumbers a and b, verify the following:

i $|ab| = |a||b|$

ii $|a+b| \leq |a| + |b|$

iii For $\epsilon > 0$,

$$|x-a| < \epsilon \text{ if and only if } a - \epsilon < x < a + \epsilon$$

First we define the sign operator as $\text{sg}(x) \in \{1, -1\}, x \neq 0$. The absolute value can be written as the product with the sign operator

$$|a| = a\text{sg}(a)$$

Then the first claim can be verified as

$$|ab| = ab\text{sg}(ab) = a\text{sg}(a)b\text{sg}(b) = |a||b|$$

by noting $\text{sg}(ab) = \text{sg}(a)\text{sg}(b)$, and

$$|a+b| = (a+b)\text{sg}(a+b) = a\text{sg}(a+b) + b\text{sg}(a+b) \leq a\text{sg}(a) + b\text{sg}(b) = |a| + |b|$$

by noting $a\text{sg}(a) = \max(a, -a) \geq a\text{sg}(c), \forall c$

Final point: if $x - a > 0$, then $|x-a| = x - a$ and $|x - a| < \epsilon \rightleftarrows a < x < a + \epsilon$

Similar, if $x - a < 0$, then $|x - a| < \epsilon \rightleftarrows a > x > a - \epsilon$, combining the both cases and with the zero case yield the desired claim.

1.2 The Natural and Rational Numbers

1.2.1 Excercise

1.2.2 Excercise

Exercise 9:
a) We need to prove that If $n > 1$ is a natural number, then $n - 1$ is also a natural number.
Let P(n) be the assertion that $n \in \mathbb{N}$ and $n > 1 => n - 1 \in \mathbb{N}$
Base Case: Let $n = 2$. Then:

$$n - 1 = 2 - 1 = 1 \in \mathbb{N}.$$

Thus, the base case holds.
Inductive Step: Assume that $P(k)$ is true for some natural number $k \geq 2$, i.e., assume that:

$$k - 1 \in \mathbb{N}.$$

We need to show that $P(k+1)$ is also true, meaning:

$$(k+1) - 1 \in \mathbb{N}.$$

Since:

$$(k+1) - 1 = k,$$

and by our inductive hypothesis, $k \in \mathbb{N}$, it follows that $P(k+1)$ is true.
By the principle of mathematical induction, for all $n > 1$, we conclude that $n - 1$ is a natural number.
b) We prove that the given statement is true for a fixed n.
Let $P(m)$ be the assertion that for a given natural number n and $m < n$, then $n - m$ is a natural number.
Base case: $P(1)$ is true since $n - 1$ is a natural number, according to part a).
Inductive step: Assume that $P(k)$ is true for some natural number $k \geq 2$ and $k < n$, i.e $n - k \in \mathbb{N}$. We need to show that $P(k+1)$ is also true, meaning that

$$n - (k+1) \in \mathbb{N}$$

Since

$$n - (k+1) = n - k - 1 = (n-k) - 1$$

and given our assumption, $n - k \in \mathbb{N}$, it follows that $(n-k) - 1 \in \mathbb{N}$ i.e. $P(k+1)$ is true.
By the principle of mathematical induction, for a fixed $n \in \mathbb{N}$ and $m < n$, $n - m$ is a natural number. The same can be proven given a fixed m instead of n.

Ex 13. Show that each real number is the supremum of a set of rational numbers and also supremum of a set of irrational numbers.

Let x be any real number. We want to show that x is the supremum of both a set of rational numbers and a set of irrational numbers.

Define a set of rational numbers as: $S = \{q \in \mathbb{Q} : q < x\}$. According to Theorem 2, rational numbers are dense in $\mathbb{R}$, therefore there are rational numbers arbitrarily close to x, meaning S is nonempty. The upper bound of S is x, since every rational number $q \in S$ must satisfies $q < r$. To prove x is the least upper bound of S, we use The density of the rational (and irrational) numbers in R, which guarantees that between any number s that is less than a given real number x, there exists a rational number. This means there is a number $q \in S$ that satisfies $s < q < x$. Thus, no number smaller than x can be an upper bound of S, which confirms that $x = \sup(S)$ is indeed the least upper bound.

Similarly, for irrational numbers, we define a set $T = \{t \in \mathbb{R}/\mathbb{Q} : t < x\}$. We have to prove T is dense in $\mathbb{R}$, and the proof for rational numbers can be applied for irrational numbers. We can prove T is dense in $\mathbb{R}$ through irrational numbers are dense in $\mathbb{R}$. Since Q are dense in R, therefore $Q + \sqrt{2}$ are dense in $R + \sqrt{2}$. We know that $Q + \sqrt{2}$ is a subset in of the irrational numbers, therefore irrational numbers are dense in R. From this, we can prove there exists an irrational number t satisfies $s < t < x$. This mean $x = \sup(T)$ is indeed the least upper bound.

1.3 The Countable and Uncountable Sets

Hoang Anh

Exercise 16: Consider the mapping from **N** to **Z** defined by

$$f(n) = \begin{cases} 0 & \text{if } n = 1 \\ \frac{n}{2} & \text{if } n \text{ is even} \\ -\frac{n+1}{2} & \text{if } n \text{ is odd and } n > 1 \end{cases}$$

If n is a natural number, then $f(2n) = n$ and $f(2n-1) = -n$. We also have $f(1) = 0$. Therefore f is onto.

Now suppose $f(n) = f(n')$. If $f(n)$ equals 0, then $n = n' = 1$. If $f(n)$ is positive, then $\frac{n}{2} = \frac{n'}{2} \implies n = n'$. If $f(n)$ is negative, then $-\frac{n+1}{2} = -\frac{n'+1}{2} \implies n = n'$. Therefore f is one-to-one.

Exercise 18: As a preliminary result, I rst show that every nite set of numbers contains a maximal element. **S(n):** Let $S \subset \mathbb{R}$ be a non-empty set. If there exists a one-to-one correspondence between $\{1, \cdots, n\}$ and S, then S contains a maximal element.

Suppose there exists a one-to-one correspondence f between $\{1\}$ and S. Then $S = \{f(1)\}$, so $s \leq f(1)$ for all $s \in S$. Thus $S(1)$ is true.

Now assume $S(k)$ is true and suppose there exists a one-to-one correspondence between $\{1, \cdots, k+1\}$ and S. Then $S = \{f(i) | 1 \leq i \leq k\} \cup \{f(k+1)\}$. By the induction hypothesis, $\{f(i) | 1 \leq i \leq k\}$ has a maximal element $\hat{s}$. If $\hat{s} \geq f(k+1)$, then $\hat{s}$ is a maximal element of S. If $\hat{s} < f(k+1)$, then $f(k+1)$ is a maximal element of S. We conclude that $S(k+1)$ must be true.

S(n): The Cartesian product $\underbrace{\mathbb{N} \times \cdots \times \mathbb{N}}_{n \text{ times}}$ is countably infinite.

The identity function establishes a one-to-one correspondence between $\mathbb{N}$ and $\mathbb{N}$, so $\mathbb{N}$ is countable. Now suppose $\mathbb{N}$ were finite. Then by the preliminary result, there would exist a maximal element m of $\mathbb{N}$. But $m + 1$ would then be a natural number larger than m, a contradiction. We conclude that $\mathbb{N}$ is countably infinite, so $S(1)$ is true.

Suppose $S(k)$ is true. Then there exists a one-to-one mapping f of $\mathbb{N}$ onto $\underbrace{\mathbb{N} \times \cdots \times \mathbb{N}}_{k \text{ times}}$. Consider the mapping from $\underbrace{\mathbb{N} \times \cdots \times \mathbb{N}}_{k+1 \text{ times}}$ to $\mathbb{N}$ defined by

$$g(n_1, \cdots, n_k, n_{k+1}) = (f^{-1}(n_1, \cdots, n_k) + n_{k+1})^2 + n_{k+1}$$

It is straightforward to check that g is one-to-one using the argument in the text. Thus $\underbrace{\mathbb{N} \times \cdots \times \mathbb{N}}_{k+1 \text{ times}}$ is equipotent to $g(\underbrace{\mathbb{N} \times \cdots \times \mathbb{N}}_{k+1 \text{ times}})$, a subset of the countable set $\mathbb{N}$. We infer from Theorem 3 that $\underbrace{\mathbb{N} \times \cdots \times \mathbb{N}}_{k+1 \text{ times}}$ is countable.

Now suppose $\underbrace{\mathbb{N}\times\cdots\times\mathbb{N}}_{k+1\text{ times}}$ is finite. Then there exists a one-to-one mapping f from $\{1,\cdots,n\}$ onto $\underbrace{\mathbb{N}\times\cdots\times\mathbb{N}}_{k+1\text{ times}}$ for some $n\in\mathbb{N}$. Consider the mapping from $\underbrace{\mathbb{N}\times\cdots\times\mathbb{N}}_{k\text{ times}}$ to $\{1,\cdots,n\}$ defined by

$$g(n_1,\cdots,n_k)=f^{-1}(n_1,\cdots,n_k,1)$$

This establishes a one-to-one correspondence between $\underbrace{\mathbb{N}\times\cdots\times\mathbb{N}}_{k\text{ times}}$ and a subset of $\{1,\cdots,n\}$, implying that $\underbrace{\mathbb{N}\times\cdots\times\mathbb{N}}_{k\text{ times}}$ is finite. This contradicts the assumption that $\underbrace{\mathbb{N}\times\cdots\times\mathbb{N}}_{k\text{ times}}$ is countably infinite. We conclude that $\underbrace{\mathbb{N}\times\cdots\times\mathbb{N}}_{k+1\text{ times}}$ is countably infinite, so $S(k+1)$ is true. **Exercise 20:**

Suppose $g(f(a))=g(f(a'))$. Since g is one-to-one, we must have $f(a)=f(a')$. Since f is one-to-one, we must also have $a=a'$. But this means $g\circ f$ is one-to-one. Now fix $c\in C$. Since g is onto, there exists $b\in B$ such that $g(b)=c$. Since f is onto, there also exists $a\in A$ such that $f(a)=b$. But this means $g(f(a))=c$, so $g\circ f$ is onto.

Suppose $f^{-1}(b)=f^{-1}(b')$. Then $b=f(f^{-1}(b))=f(f^{-1}(b'))=b'$, so f^{-1} must be one-to-one. Now suppose $a\in A$. Then $a=f^{-1}(f(a))$, so f^{-1} is onto.

Exercise 22: Suppose $2^{\mathbb{N}}$ is countable. Let $\{X_n|n\in\mathbb{N}\}$ denote an enumeration of $2^{\mathbb{N}}$ and define

$$D=\{n\in\mathbb{N}|n\text{ is not in }X_n\}$$

Then $D\in 2^{\mathbb{N}}$, so $D=X_d$ for some $d\in\mathbb{N}$. If d is not in D, then we would have a contradiction because d would have to be in D by construction. Likewise if d is in D, then we have a contradiction because d could not be in D by construction. We can conclude that no enumeration can exist, so $2^{\mathbb{N}}$ is uncountable. **Exercise 26:** Let G denote the set of irrational numbers in $(0,1)$ and let $\{q_n|n\in\mathbb{N}\}$ denote an enumeration of the rationals in $(0,1)$. Define

$$i_n=\frac{\sqrt{2}}{2^n}$$

and construct the mapping $f:(0,1)\to G$ as

$$f(x)=\begin{cases} i_{2n} & \text{if } x=q_n \\ i_{2n-1} & \text{if } x=i_n \\ x & \text{otherwise}\end{cases}$$

f defines a one-to-one correspondence between $(0,1)$ and G, so $|(0,1)|=|G|$.

In Problem 25 we showed that $|\mathbb{R}|=|(0,1)|$, so the above result implies $|\mathbb{R}|=|G|$. This means we can find a one-to-one mapping g from $\mathbb{R}$ onto G. Now consider the mapping $h:\mathbb{R}\times\mathbb{R}\to G\times G$ defined by

$$h(x,y)=(g(x),g(y))$$

h defines a one-to-one mapping from $\mathbb{R}\times\mathbb{R}$ onto $G\times G$, so $|\mathbb{R}\times\mathbb{R}|=|G\times G|$.

Recall that if x is an irrational number in $(0,1)$, it can be uniquely written as

$$x=\frac{1}{a_1+\frac{1}{a_2+\frac{1}{a_3+\cdots}}}=[a_1,a_2,a_3,\cdots]$$

where $a_1,a_2,a_3,\cdots$ is an infinite sequence of natural numbers. (This representation is called the continued fraction expansion of x.) Let $x=[a_1,a_2,\cdots]$ and $y=[b_1,b_2,\cdots]$ denote two elements of G and consider the mapping $m:G\times G\to G$ defined by

$$m(x,y)=[a_1,b_1,a_2,b_2,\cdots]$$

Then m defines a one-to-one correspondence between $G\times G$ and G, so $|G\times G|=|G|$. Combining the above results, we have $|\mathbb{R}\times\mathbb{R}|=|G\times G|=|G|=|\mathbb{R}|$.

1.3.1 Excercise

1.4 Open Sets, Closed Sets, and Borel Sets of Real Numbers

Exercise 28: Suppose A is a non-empty, proper subset of $\mathbf{R}$ that is both open and closed. Then there exists $x \in A$ and $y \in \mathbf{R} \setminus A$. Suppose without loss of generality that $x < y$ and define

$$E = \{x \in A : x < y\}$$

Then E is non-empty ($x \in E$) and bounded above (by y). The completeness axiom implies that there exists a least upper bound of E. Let $x^* = \sup E$ and suppose $x^* \in A$. Since $y \notin A$ and y is an upper bound of E, we must have $x^* < y$. Therefore there exists $r > 0$ such that $x^* + r < y$. But since A is open, we can also find $r^* \in (0, r)$ such that that $(x^* - r^*, x^* + r^*) \subset A$. But this implies $x^* + \frac{r}{2} \in E$, so x^* is not an upper bound for E. This contradicts the definition of x^*. Now suppose $x^* \in \mathbf{R} \setminus A$. Since A is closed, $\mathbf{R} \setminus A$ is open. Therefore there exists $r > 0$ such that $(x^* - r, x^* + r) \subset \mathbf{R} \setminus A$. Thus if $x \in A$, $x \leq x^* - r$. But this means $x^* - r$ is an upper bound of E, contradicting the assumption that x^* is the least upper bound.

The above argument shows that E cannot have a least upper bound, a contradiction of the completeness axiom. We conclude that no non-empty, proper subset of $\mathbf{R}$ that is both open and closed can exist. **Exercise 31:** Suppose E is a set containing only isolated points. For each $x \in E$, define $f(x) = (p, q)$ where p and q are rational numbers such that $p < x < q$ and $(p, q) \cap E = \{x\}$. f defines a one-to-one mapping from E to $\mathbf{Q} \times \mathbf{Q}$. By Corollary 4 and Problem 23, $\mathbf{Q} \times \mathbf{Q}$ is a countable set. This means there exists a one-to-one mapping g from $\mathbf{Q} \times \mathbf{Q}$ onto $\mathbf{N}$. The composition $g \circ f$ defines a one-to-one mapping from E to $\mathbf{N}$ (see Problem 20), which implies E is countable (see Problem 17). **Exercise 32:** (i) Suppose E is open and $x \in E$. Then there exists an $r > 0$ such that the interval $(x - r, x + r)$ is contained in E. But this means $x \in \text{int } E$, so $E \subseteq \text{int } E$. Since $\text{int } E \subseteq E$ by definition, $E = \text{int } E$.

Conversely, suppose $E = \text{int } E$. If x is a point in E, then $x \in \text{int } E$. But this means there exists an $r > 0$ such that the interval $(x - r, x + r)$ is contained in E, so E is open.

(ii) Let E be dense in $\mathbf{R}$ and suppose $x \in \text{int}(\mathbf{R} \setminus E)$. Then there exists $r > 0$ such that $(x - r, x + r) \subseteq \mathbf{R} \setminus E$. But this means there does not exist an element of E between any two numbers in $(x - r, x + r)$, contradicting the assumption that E is a dense set. We conclude that no such x can be found, so $\text{int}(\mathbf{R} \setminus E) = \emptyset$.

Conversely, suppose $\text{int}(\mathbf{R} \setminus E) = \emptyset$. Let x and y be two real numbers satisfying $x < y$ and suppose $(x, y) \subset \mathbf{R} \setminus E$. Let $z \in (x, y)$ and choose $r \in (0, \min(z - x, y - z))$. Then $(z - r, z + r) \subset (x, y)$, so $(z - r, z + r) \subset \mathbf{R} \setminus E$. But this means $z \in \text{int}(\mathbf{R} \setminus E)$, contradicting the assumption that $\text{int}(\mathbf{R} \setminus E) = \emptyset$. Therefore $(x, y) \not\subset \mathbf{R} \setminus E$, which means there must be an element of E between z and y. But since x and y were arbitrary, this means E is dense in $\mathbf{R}$.

1.4.1 Excercise

1.5 Sequences of Real Numbers

1.5.1 Summary

A sequence is a function $f : \mathbb{N} \to \mathbb{R}$ with customary notation $\{a_n\}$ where n is called the index, the number a_n is the nth term.
A sequence $\{a_n\}$ is said to be

- bounded if $\exists c \geq 0$ s.t. $|a_n| \leq c \forall n$
- increasing if $a_n < a_{n+1} \forall n$
- decreasing if the sequence $\{-a_n\}$ is increasing
- monotone if it's either increasing or decreasing

For any sequence $\{a_n\}$ and a strictly increase sequence $\{n_k\} \in \mathbb{N}$, call the sequence $\{a_{n_k}\}$ a subsequence of $\{a_n\}$.

Definition 1. *A sequence $\{a_n\}$ converges to it's limit a (write $\lim_{n\to\infty} a_n = a$ or $\{a_n\} \to a$) if $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t.*

$$n \geq N \implies |a - a_n| < \epsilon.$$

Proposition 1. *If $\{a_n\} \to a$, then the limit is unique, the sequence is bounded, and, $\forall c \in \mathbb{R}$,*

$$a_n \leq c \forall n \implies a \leq c.$$

Proof ***<u>Ex Extra.</u>***

Theorem 1. *A monotone sequence of real numbers converges if and only if it is bounded.*

Theorem 2 (Bolzano-Weierstrass)**.** *Every bounded sequence of real numbers has a convergent subsequence.*

Definition 2. *A sequence of real numbers $\{a_n\}$ is Cauchy if $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t.*

$$n, m \geq N \implies |a_m - a_n| < \epsilon.$$

Theorem 3. *A sequence of real numbers converges if and only if it is Cauchy.*

Theorem 4. *Convergent real sequences are linear and monotonic.*

Definition 3. *A sequence $\{a_n\}$ converges to infinity (write $\lim_{n\to\infty} a_n = \infty$ or $\{a_n\} \to \infty$) if $\forall c \in \mathbb{R}, \exists N \in \mathbb{N}$ s.t.*

$$n \geq N \implies a_n \geq c.$$

Similar definitions are made at $-\infty$.

Definition 4. *The limit superior and limit inferior of a sequence $\{a_n\}$ is defined as,*

$$\limsup \{a_n\} = \lim_{n\to\infty} [\sup \{a_k | k \geq n\}]$$
$$\liminf \{a_n\} = \lim_{n\to\infty} [\inf \{a_k | k \geq n\}]$$

Proposition 2. *Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers*

(i) $\limsup \{a_n\} = \ell \in \mathbb{R}$ *if and only if for each $\epsilon > 0$, there are infinitely many indices n for which $a_n > \ell - \epsilon$ and only finitely many indices n for which $a_n > \ell + \epsilon$.*

(ii) $\limsup \{a_n\} = \infty$ *if and only if $\{a_n\}$ is not bounded above.*

(iii) $\limsup \{a_n\} = -\liminf \{-a_n\}$.

(iv) A sequence of real numbers $\{a_n\}$ converges to $a \in \mathbb{R}$ if and only if $\liminf \{a_n\} = \limsup \{a_n\} = a$.

(v) $a_n \leq b_n \forall n \implies \limsup \{a_n\} \leq \liminf \{b_n\}$.

Proof ***Ex 39.***

Definition 5. *For every sequence $\{a_k\}$ of real numbers, define a sequence of partial sums $\{s_n\}$ where $s_n = \sum_{k=1}^{n}$. The series $\sum_{k=1}^{\infty} a_k$ is summable to $s \in \mathbb{R}$ when $\{s_n\} \to s$.*

Proposition 3. *Let $\{a_n\}$ be a sequence of real numbers.*

(i) The series $\sum_{k=1}^{\infty} a_k$ is summable if and only if for each $\epsilon > 0, \exists N \in \mathbb{N}$ s.t.

$$\sum_{k=n}^{n+m} a_k < \epsilon \forall n \geq N, m \in \mathbb{N}.$$

(ii) If the series $\sum_{k=1}^{\infty} |a_k|$ is summable, then $\sum_{k=1}^{\infty} a_k$ also is summable.

(iii) If each term a_k is nonnegative, then the series $\sum_{k=1}^{\infty} a_k$ is summable if and only if the sequence of partial sums is bounded.

Proof ***Ex 45.***

1.5.2 Excercise

Problems done: 38, 39, 40, 41, 45. and proved the first Proposition (i.e. Ex Extra.).

Ex 38.

Lemma 1. *For any set $X \subseteq \mathbb{R}$, $\forall d > 0 \in \mathbb{R}, \exists x \in X$ s.t. $x < \inf X + d$.*

Proof We prove by contradiction. Assume there exists $d > 0 \in \mathbb{R}$ s.t. $\forall x \in X, \inf X + d \leq x$. There is now a greater lower bound $\inf X + d$, which contradicts the definition of infimum.

We use the above lemma to solve this excercise.
Let $\liminf \{a_n\} = L$.

- $\liminf \{a_n\}$ is a cluster point.

  By the above lemma, for every n, we can pick the smallest index $k_n \geq n$ satisfying $a_{k_n} \leq \inf \{a_k | k \geq n\} + \frac{1}{n}$
  Now, $\forall \epsilon > 0, \exists N \in \mathbb{N}, N \geq 1/\epsilon$ s.t. $n \geq N \implies a_{k_n} - L < 1/N \leq \epsilon$. The subsequence $\{a_{k_n}\}$ converges to L by defintion.

- There does not exist a cluster point M satisfying $M < \liminf \{a_n\}$.

  We argue by contradiction. Assume there exists such a cluster point, this means there also exists a subsequence $\{a_{m_j}\}$ that converges to M.
  Let $\epsilon = \frac{M-L}{2}$, by definition, $\exists J \in \mathbb{N}$ s.t.

$$j \geq J \implies a_{m_j} - M < \epsilon \iff a_{m_j} < M + \epsilon = \frac{L+M}{2}.$$

  Also, by definition, $L = \liminf \{a_n\} = \lim_{n\to\infty} \{\inf \{a_k | k \geq n\}\}$, as such $\exists N \in \mathbb{N}, N > J$ s.t.

$$n \geq N \implies L - \inf \{a_k | k \geq n\} < \epsilon \iff \inf \{a_k | k \geq n\} > L - \epsilon = \frac{L+M}{2}.$$

  This is a contradiction, as there exists $N \in \mathbb{N}$ satisfying

$$n \geq N \implies \begin{cases} a_{m_n} < \frac{L+M}{2} \\ \inf \{a_k | k > n\} \geq \frac{L+M}{2} \end{cases}.$$

Proof is similar for $\limsup \{a_n\}$
Ex 39. Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers

(i) $\limsup \{a_n\} = \ell \in \mathbb{R}$ if and only if for each $\epsilon > 0$, there are infinitely many indices n for which $a_n > \ell - \epsilon$ and only finitely many indices n for which $a_n > \ell + \epsilon$.

Trivial. Use definition of suprimum and the fact that the collection of sequences $\{\{a_k | k \geq n\}\}_{n=1}^{\infty}$ is decending.

(ii) $\limsup \{a_n\} = \infty$ if and only if $\{a_n\}$ is not bounded above.

We prove the above through showing that $\limsup \{a_n\} < \infty$ if and only if $\{a_n\}$ is bounded above. Note that the limit superior of a sequence always exists.

- If $\{a_n\}$ is bounded above, then $\exists M < \infty \in \mathbb{R}$ s.t. $a_n \leq M \forall n$. As a result, $\sup \{a_k | k \geq n\} \leq M$.
- If $\limsup \{a_n\} < \infty$, then $\sup \{a_k \; k \geq n\}$ is bounded.
  And because there exists $c > 0$ satisfying $a_n \leq \sup \{a_k | k \geq 1\} \leq c$ for all n, the sequence $\{a_n\}$ is also bounded above.

(iii) $\limsup \{a_n\} = -\liminf \{-a_n\}$.

$$\limsup \{a_n\} = \lim_{n\to\infty} \sup \{a_k | k \geq n\} = -\lim_{n\to\infty} \inf \{-a_k | k \geq n\} = -\liminf \{-a_n\}.$$

(I ommitted the proof to $\lim_{n\to\infty} \{a_n\} = -\lim_{n\to\infty} \{-a_n\}$. It is trivial and uses the definition.)

(iv) A sequence of real numbers $\{a_n\}$ converges to $a \in \mathbb{R}$ if and only if $\liminf\{a_n\} = \limsup\{a_n\} = a$.

- $\liminf\{a_n\} = \limsup\{a_n\} = a \implies \{a_n\} \to a$
For any $\epsilon > 0$, there exists $N, M \in \mathbb{N}$ s.t.

$$\begin{cases} n \geq N \implies -\epsilon < a - \sup\{a_k | k \geq n\} \leq a - a_n \\ n \geq M \implies a - a_n \leq a - \inf\{a_k | k \geq n\} < \epsilon \end{cases}$$

So $\exists L = \max(N, M) \in \mathbb{N}$ s.t.

$$n \geq L \implies \begin{cases} -\epsilon < a - a_n \\ a - a_n < \epsilon \end{cases} \implies |a - a_n| < \epsilon.$$

By definition, $\{a_n\} \to a$

- $\{a_n\} \to a \implies \liminf\{a_n\} = \limsup\{a_n\} = a$
For any $\epsilon > 0$, there exists $N \in \mathbb{N}$ s.t.

$$\forall n \geq N, \begin{cases} -\epsilon < a - a_n \\ a - a_n < \epsilon \end{cases} \implies \forall n \geq N, \begin{cases} \inf\{a_k | k \geq n\} \leq a_n < a + \epsilon \\ a - \epsilon < a_n \implies a - \epsilon < \inf\{a_k | k \geq n\} \end{cases}.$$

which is equivalent to $|a - \inf\{a_k | k \geq n\}| < \epsilon$ for all $n \geq N$ and so $\liminf\{a_n\} = a$ by definition. Similar proof is done for $\limsup\{a_n\} = a$.

(v) $a_n \leq b_n \forall n \implies \limsup\{a_n\} \leq \liminf\{b_n\}$. (similar to book)

Consider the sequence $\{c_n\}$, where $c_n = \inf\{b_k | k \geq n\} - \sup\{a_k | k \geq n\}$ for all n.

By linearity of convergent sequences, $\{c_n\} \to c = \liminf\{b_n\} - \limsup\{a_n\}$. This means, $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t.

$$n \geq N \implies -\epsilon < c - c_n < \epsilon.$$

In particular, $0 \leq c_N < c + \epsilon$. Since $c \geq -\epsilon$ for any positive number ϵ, $c \geq 0$.

Ex 40.
Proven above in **Ex. 38**, $\liminf\{a_n\}$ and $\limsup\{a_n\}$ are the smallest and largest cluster points of $\{a_n\}$.
Shown above in **Ex. 39**, $\{a_n\} \to a \iff \liminf\{a_n\} = \limsup\{a_n\} = a$.
The proof is now trivial.
The sequence $\{a_n\}$ has only one cluster point if and only if $\liminf\{a_n\} = \limsup\{a_n\} = a$, which is equivalent to $\{a_n\} \to a$.

Ex 41. At every index n,

$$\inf\{a_k | k \geq n\} \leq \sup\{a_k | k \geq n\}$$

And so, by the linearity property of convergent sequences, $\lim_{n\to\infty} \inf\{a_k | k \geq n\} \leq \lim_{n\to\infty} \sup\{a_k | k \geq n\}$ or $\liminf\{a_n\} \leq \limsup\{a_n\}$.

Ex 43. Show that every real sequence has a monotone subsequence. Use this to provide another proof of the Bolzano-Weierstrass Theorem.

By contradiction, we assume that there exists a sequence $\{x_i\}_i$ such that it has no monotone subsequence. We devide the proof into two parts: a squence that has no increasing subsequence has a decreasing subsequence and vice versa. We only prove the first part since the other part are identical.

Note that we use the terminology similar to what is defined in the textbook: an increasing sequence has $a_n \leq a_{n+1}$. This is a bit misleading as a more correct term would be "non-decreasing" sequence.

First, assuming that the sequence $\{x_i\}$ has no increasing subsequence; that is: every construstion of such subsequences stops at some finite steps. More specifically,

- Starting at x_0, we construct the longest increasing subsequence $x_0, \ldots x_m$. This sequence is finite due to our assumption that $\{x_i\}$ has no increasing subsequence.

- We then have $x_m > x_j, \forall j > m$ or otherwise we can concatenate x_j to the subsequence found in the previous step to make a longer sequence.

- Let $a_m = x_m$, repeat this process with the rest of sequence $\{x_i\}_{i>m}$, the sequence $\{a_m\}$ is a decreasing sequence.

Ex 45. Let $\{a_n\}$ be a sequence of real numbers.

(i) The series $\sum_{k=1}^{\infty} a_k$ is summable if and only if for each $\epsilon > 0, \exists N \in \mathbb{N}$ s.t.

$$\left|\sum_{k=n}^{n+m} a_k\right| < \epsilon \forall n \geq N, m \in \mathbb{N}.$$

The series $\sum_{k=1}^{\infty} a_k$ is summable if and only if $\{s_n\}$ converges.

As such, for each $\epsilon > 0, \exists N \in \mathbb{N}$ s.t.

$$\begin{aligned} j > i-1 \geq N &\implies \epsilon > \left|\sum_{k=i}^{j} a_k\right| \\ \iff n \geq N, m \in \mathbb{N} &\implies \epsilon > \left|\sum_{k=n}^{n+m} a_k\right| \quad (i-1 = n, j = n+m) \end{aligned}$$

(ii) If the series $\sum_{k=1}^{\infty} |a_k|$ is summable, then $\sum_{k=1}^{\infty} a_k$ also is summable.

If the series $\sum_{k=1}^{\infty} |a_k|$ is summable, then the partial sum sequence $\{\sum_{k=1}^{n} |a_k|\}$ converges.

As such, $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t.

$$n, m \geq N \implies \epsilon > \left|\sum_{k=\min(m,n)}^{\max(m,n)} |a_k|\right| \geq \left|\sum_{k=\min(m,n)}^{\max(m,n)} a_k\right|.$$

The partial sum sequence $\{\sum_{k=1}^{n} a_k\}$ converges because it is Cauchy. As a result, the series $\sum_{k=1}^{\infty} a_k$ also is summable.

(iii) If each term a_k is nonnegative, then the series $\sum_{k=1}^{\infty} a_k$ is summable if and only if the sequence of partial sums is bounded.

Since $a_k > 0 \forall k \in \mathbb{N}$, $s_n = \sum_{k=1}^{n} a_k \leq \sum_{k=1}^{n+1} a_k = s_{n+1}$ for all n. In other words, the partial sum sequence is nondecreasing.

The series $\sum_{k=1}^{\infty} a_k$ is summable if and only if $\{s_n\}$ converges.

- If $\{s_n\}$ converges then it is bounded.
- If $\{s_n\}$ is bounded, then it converges to $s = \sup\{s_n | n \in \mathbb{N}\}$ (note that the suprimum exists thanks to the Completeness Axiom)

  For any $\epsilon > 0$, we have:

  +) $s_n \leq s < s + \epsilon$ for all n.

  +) Because $s - \epsilon$ is not an upperbound of $\{s_n | n \in \mathbb{N}\}$, $\exists N \in \mathbb{N}$ s.t $s_N > s - \epsilon$.
  And since the sequence $\{s_n\}$ is nondecreasing, $n \geq N \implies s_n > s - \epsilon$.

  By definition, $\{s_n\}$ converges to s.

Ex Extra. If $\{a_n\} \to a$, then:

- The limit is unique.

  We prove by contradiction. Assume $\{a_n\} \to a, \{a_n\} \to b$ and $a \neq b$.

  Let $d = |a - b|$ and $\epsilon = \frac{d}{2}$. By definition, there exists $N, M \in \mathbb{N}$ s.t.

$$\begin{cases} n \geq N \implies |a - a_n| < \epsilon \\ n \geq M \implies |b - a_n| < \epsilon \end{cases}$$

  So $\exists L = \max(N, M) \in \mathbb{N}$ s.t. $n \geq L$ implies both $|a - a_n|$ and $|b - a_n|$ are less than ϵ.

  By the triangle inequality, $d = |a - b| \leq |a - a_n| + |b - a_n| < 2\epsilon = 2 \times \frac{d}{2} = d$. In other words, $d < d$, which is a contradiction.

- The sequence is bounded.

  Choose any $\epsilon > 0$.

  By definition, $\exists N \in \mathbb{N}$ s.t.

  $$n \geq N \implies -\epsilon < a - a_n < \epsilon \iff a - \epsilon < a_n < a + \epsilon \implies |a_n| < |a| + \epsilon$$

  Denote $M_1 = \max\left[\{a_n | n \in \mathbb{N}, n < N\}\right]$, note that we can always find M_1 because this sequence is finite.

  We conclude that $\{a_n\}$ is bounded by $\max(|a| + \epsilon, M_1)$.

- $\forall c \in \mathbb{R}$, if $a_n \leq c \forall n$ then $a \leq c$.

Approach 1) Using only the definition.
For any $\epsilon > 0, \exists N \in \mathbb{N}$ s.t

$$n \geq N \implies |a - a_n| < \epsilon \implies a - \epsilon < a_n \leq c$$

Since $a - \epsilon < c$ is true for all $\epsilon > 0$, we conclude that $a \leq c$.

Approach 2) Using only the definition.
Prove by contradiction. Assume $a > c$, then set $\epsilon = a - c > 0$...

Approach 3) Consider the sequence $\{c_n\}$, where $c_n = c \forall n$ and use the monotonic property of convergent sequences. (Trivial)

1.6 Continuous Real-Valued Functions of Real Variable

1.6.1 Excercise

Ex 50. Show that a Lipschitz function is uniformly continuous but there are uniformly continuous functions that are not Lipschitz

We can prove that not all functions that are uniformly continuous are a Lipschitz function by using contradiction. Suppose we have function $f = \sqrt{x}$ uniformly continuous on $\{0, 1\}$ and is a Lipschitz function. Based on definition, there is a $c > 0$ for which

$$|\sqrt{x'} - \sqrt{x}| \leq c|x' - x|. \tag{2}$$

If we take $x = 0$, the equation become $|\sqrt{x'}| \leq c|x'|$. We can rewrite this as $|\sqrt{x'}|/|x'| \leq c$. However, if $x' \to 0$, we have $|\sqrt{x'}|/|x'| \to \infty$ which contradicts the inequality. Hence, $f = \sqrt{x}$ is not a Lipschitz function.

Ex 53. Show that a set E of real numbers is closed and bounded if and only if every open cover of E has a finite subcover.

(i) $(\Rightarrow)$ According to Heine-Borel theorem, if a set E of real numbers is closed and bounded, every open cover of E has a finite subcover.

(ii) $(\Leftarrow)$ We first prove that if every open cover of E has a finite subcover, then E is bounded. We form an open cover of E by defining a set $O_x = (x - 1, x + 1)$ for every $x \in E$. The collection $\{O_x : x \in E\}$ is an open cover for E. This collection must have a finite subcover $\{O_{x_1}, O_{x_2}, \ldots, O_{x_n}\}$. Since E is contained in a finite union of bounded sets, E must be bounded.

We now prove that E must be closed. Suppose E is not closed. Let $y \notin E$ be a point of closure of E. We form an open cover of E by defining a set $O_x = (x - r_x, x + r_x)$ where $r_x = |y - x|/2$ for every $x \in E$. The collection $\{O_x : x \in E\}$ must have a finite subcover $\{O_{x_1}, O_{x_2}, \ldots, O_{x_n}\}$. Let $r_{\min} = \min\{r_{x_1}, r_{x_2}, \ldots, r_{x_n}\}$. Since y is a point of closure of E, the open interval $(y - r_{\min}, y + r_{\min})$ must contain a point $x' \in E$. This means $|x' - y| < r_{\min}$. We now show that x' is not in the subcover.

$$\forall i : 1 \leq i \leq n, |x_i - x'| > |x_i - y| - |x' - y| > |x_i - y| - r_{\min} > 2r_{x_i} - r_{\min} > r_{x_i}$$

$\forall i : |x_i - x'| > r_{x_i} \Rightarrow \forall i : x' \notin O_{x_i} \Rightarrow x' \notin \bigcup_{1 \leq i \leq n} O_{x_i}$. This means the finite subcover fails to cover E. This contradiction implies that E is closed.

2 Lebesgue Measure

2.1 Introduction

2.1.1 Excercise

Ex 1. Prove that if A and B are two sets in $\mathcal{A}$ with $A \subseteq B$, then $m(A) \leq m(B)$. This property is called monotonicity.

Solution.
Let $C = B \setminus A$, then by countably additive property of m over countable disjoint set, then

$$m(A) = m(B) - m(C) \leq m(B)$$

where the last inequality is because m is a set function with values in $[0, \infty]$ □

Ex 2. Prove that if there is a set A in the collection $\mathcal{A}$ for which $m(A) < \infty$, then $m(\emptyset) = 0$

Solution.

$$m(A) = m(A \cup \emptyset) = m(A) + m(\emptyset)$$

$$m(\emptyset) = 0$$

□

Ex 3. Let $\{E_k\}_k$ be countable collection of sets in $\mathcal{A}$. Prove that $m(\bigcup_k E_k) \leq \sum_k m(E_k)$

Solution.
Let the set $A_1 = \emptyset$ and $A_i = \bigcup_{k<i} E_k$ with $i > 1$. Further define $X_i = E_i \setminus A_i$, then $\bigcup X_i = \bigcup E_i$ and $\{X_i\}$ are countable disjoint sets.

$$m(\bigcup_k E_k) = m(\bigcup_k X_k) = \sum_k m(X_k) \leq \sum_k m(E_k)$$

the last inequality is due to $X_k \subseteq E_k$. □

Ex 4. A set function c, defined on all subsets of $\mathbb{R}$, is defined as follows. Define $c(E)$ to be ∞ if E has infinitly many members and $c(E)$ to be equal the number of the elements in E if E is finite; define $c(\emptyset) = 0$. Show that c is a countably additive and translation invariant set function. This set function is called the counting measure.

Solution.
Sketch: The union of two finite, disjoint is finite and the elements in this union is equal the total number of element in two sets. If one of the two set are infinite, there union is infinite and the measure is infinite by definition.

Translation operation does not alter the number of elements in a set, so m is invariance under translation. □

2.2 Lebesgue Outer Measure

2.2.1 Excercise

Ex 5. By using properties of outer measure, prove that the interval $[0, 1]$ is not countable

Solution.
Let I be the interval of intest. By definition of outer measure on an interval, $m^*(I) = 1$.

Now suppose that I is countable, then there exists an enumeration of I, and due to the countably subadditive property of outer measure

$$1 = m^*(I) \leq \sum_{i \in I} m^*(\{i\}) = \sum^{\infty} 0 = 0,$$

a contradiction, so I is uncountable. □

Ex 6. Let A be the set of irrational numbers in the interval $[0, 1]$. Prove that $m^*(A) = 1$.

Solution.
Let $B = A^C \cap I$ the set of rational numbers in I, by finite subadditivity

$$1 = m^*(I) \leq m^*(A) + m^*(B) = m^*(A)$$

Furthermore, $A \subseteq I$ and by monotonicity of outer measure, then $m^*(A) \leq m^*(I) = 1$. As a result, $m^*(A) = 1$ □

Ex 7. A set of real numbers is said to be a G_δ set provided it is the intersection of a countable collection of open sets. Show that for any bounded set E, there is a G_δ set G for which

$$E \subseteq G \text{ and } m^*(G) = m^*(E)$$

Solution.
This exercise is inspired from the proof of theorem 11.

Since E is bounded, $m^*(E)$ is finite, then for each $\epsilon > 0$, there exists an open intervals $\{I_i\}_i$, let $O_\epsilon = \bigcup_i I_i$, then O_ϵ is open, $O_\epsilon \supseteq E$ and

$$m^*(O_\epsilon) \geq m^*(E) \geq m^*(O_\epsilon) - \epsilon$$

Define a new collection of open sets $\{O_n\}$ where each O_n is associated with an $\epsilon = 1/n$, then the intersection $\bigcap^\infty O_i$ is a G_δ set with the desired property. □

Ex 8. Let B be the set of rational numbers in the interval $I = [0, 1]$, and let $\{I_k\}^n$ be a finite colleciton of open intervals that cover B. Prove that $\sum^n m^*(I_k) \geq 1$

Solution.
By contradiction, assuming that there exists a finite collection of intervals that covers B, i.e. $B \subseteq \bigcup_i^n I_i$, and $\sum^n m^*(I_k) < 1$. These n intervals have a total of $2n$ endpoints, and together with the two endpoints of I, amount to at most $2n + 2$ endpoints. As a result, the set $I \setminus B$ can be expressed by a finite union of at most $n + 1$ intervals. Let $I \setminus B = \bigcup^{n+1} J_k$, with each J_k an interval or empty set. By finite subadditivity

$$m^*(I) \leq m^*(B) + \sum_k^{n+1} m^*(J_k) \leq \sum_k^n m^*(I_k) + \sum_k^{n+1} m^*(J_k) < 1 + \sum_k^{n+1} m^*(J_k)$$
$$0 < \sum_k^{n+1} m^*(J_k)$$

Therefore, among $n+1$ intervals, at least one of them has positive outer measure. Such intervals are non-degenerated since outer measure of intervals equal to their length. Finally, we can always choose an rational number in such intervals since the rational set are dense and this number is not in B, contradict with the definition of B. □

Ex 9. Prove that if $m^*(A) = 0$, then $m^*(A \cup B) = m^*(B)$

Solution.
By subadditivity

$$m^*(A \cup B) \leq m^*(A) + m^*(B) = m^*(B)$$

and monotonicity, with $B \subseteq B \cup A$

$$m^*(B) \leq m^*(B \cup A)$$

□

Ex 10. Let A and B be bounded sets for which there is an $\alpha > 0$ such that $|a - b| \geq \alpha$ for all $a \in A, b \in B$. Prove that $m^*(A \cup B) = m^*(A) + m^*(B)$

Solution.
Since A and B are bounded, $m^*(A \cup B)$ finite. For any $\epsilon > 0$, let $C_\epsilon = \{I_k\}$ a collection of open, bounded intervals that cover $A \cup B$ such that

$$m^*(A \cup B) \geq \sum_{C_\epsilon}^{\infty} l(I_k) - \epsilon \tag{3}$$

Next, we show that if $\epsilon < \alpha/2$, then no intervals in C_ϵ can contain some $a \in A$ and $b \in B$ at the same time. By contradiction, suppose there exists some $I \in C_\epsilon$ that satisfies such condition. then between a and b there is at least an interval with length α that contains no points from both A and B. Removing an interval with half the length $\alpha/2$ from I results in two smaller intervals I_1 and I_2 that cover all the points from $A \cup B$ that I covers. As a result,

$$C'_\epsilon := C_\epsilon \setminus \{I\} \cup \{I_1, I_2\}$$

is another collections of open intervals that cover $A \cup B$, to which

$$\sum_{C'} l(I_j) = \sum_{C} l(I_k) - l(I) + l(I_1) + l(I_2) = \sum_{C} l(I_k) - \alpha/2.$$

In word, we have found a new open cover of $A \cup B$ with total length $\alpha/2$ less than the original cover C_ϵ. However, from the definition of outer measure that is the infimum of all open cover,

$$\sum_{C'} l(I_j) \geq m^*(A \cup B),$$

this contradicts with (3). As a result, each I_k contains either points from A or B. Partition C into two collections C_A and C_B as follow

- $I_k \in C_A$ if I_k contains some point $a \in A$
- $I_k \in C_B$ otherwise

now we can write

$$m^*(A \cup B) \geq \sum_{C_\epsilon}^{\infty} l(I_k) - \epsilon = \sum_{C_A} l(I_a) + \sum_{C_B} l(I_b) - \epsilon \geq m^*(A) + m^*(B) - \epsilon$$

where the last inequality is because C_A and C_B cover A and B, respectively and outer measure is the infimum of such total lengths. The above inequality holds true for every $0 < \epsilon < \alpha/2$, so

$$m^*(A \cup B) \geq m^*(A) + m^*(B),$$

combine with countable subadditivity concludes the proof.

□

2.3 Introduction

2.3.1 Excercise

Ex 16 (iii). if E is measurable $\Rightarrow R \sim E$ is measurable.

For each $\epsilon > 0$, there is a open set O containing $R \sim E$ for which $m^*(O \sim (R \sim E)) < \epsilon$ Let $F = R \sim O$, then F contained in E and $m^*(E \sim F) < \epsilon$

Ex 17 First, suppose E is measurable. Then for each $\frac{\epsilon}{2} > 0$, there is open set O and closed set F for which $m^*(O \sim E) < \frac{\epsilon}{2}$ and $m^*(E \sim F) < \frac{\epsilon}{2}$

Because $O \sim F = (O \sim E) \cup (E \sim F) \Rightarrow m^*(O \sim F) \leq m^*(O \sim E) + m^*(E \sim F) < \epsilon$

Now suppose for each $\epsilon > 0$, there is a closed set F and open set O for which $F \subseteq E \subseteq O$ and $m^*(O \sim F) < \epsilon$, then $m^*(O \sim E) \leq m^*(O \sim F) < \epsilon \Rightarrow E$ is measurable.

Ex 18 See prob 17

Ex 19 If E is not measurable, by theorem 11 (i), there is $\epsilon_0 > 0$ for which for all open set O containing E, $m^*(O \sim E) \geq \epsilon_0$

$m^*(E) = inf\{\sum_{k=1}^{\infty} l(I_k) | A \subseteq \cup_{k=1}^{\infty} I_k\}$

$\Rightarrow$ there is countable of collection of open sets $\{I_k\}_{k=1}^{\infty}$ for which $m^*(E) > \sum_{k=1}^{\infty} l(I_k) - \epsilon$ for all $\epsilon > 0$

Let $O = A \subseteq \cup_{k=1}^{\infty} I_k \Rightarrow m^*(O) \leq \sum_{k=1}^{\infty} l(I_k)$

$\Rightarrow m^*(E) > \sum_{k=1}^{\infty} l(I_k) - \epsilon_0 \geq m^*(O) - \epsilon_0 \Rightarrow \epsilon_0 > m^*(O) - m^*(E)$

So $m^*(O \sim E) \geq \epsilon_0 > m^*(O) - m^*(E)$

Ex 21 Let A and B are measurable sets. By property (ii) of theorem 11, there is G_δ G_A, G_B for which $m^*(G_A \sim A) = 0, m^*(G_B \sim B) = 0$

Let $G = \cap_{k=1}^{\infty}(G_{Ak} \cup G_{Bk}) \Rightarrow G = G_A \cup G_B$
$0 \leq m^*(G \sim (A \cup B)) = m^*((G_A \cup G_B) \sim (A \cup B)) \leq m^*(G_A \sim (A \cup B)) + m^*(G_B \sim (A \cup B)) < m^*(G_A \sim A) + m^*(G_B \sim B) = 0$
So $m^*(G \sim (A \cup B)) = 0$, then $A \cup B$ is measurable.
Ex 22 Since $m^{**}(A) = inf(...)$, for $\epsilon > 0$, there is open set $O, A \subseteq O$ for which $m^{**}(A) \geq m^*(O) - \epsilon$
Since $m^*(A) = inf(...)$, for $\epsilon > 0$, there is $\{I_k\}_{k=1}^{\infty}$ (Let $O = \cup_{k=1}^{\infty} I_k \Rightarrow A \subseteq O$) for which
$m^*(A) \geq \sum_{k=1}^{\infty} l(I_k) - \epsilon \geq m^*(O) - \epsilon \geq m^{**}(A) - \epsilon$
Thus, $m^{**}(A) + \epsilon \geq m^*(A) \geq m^{**}(A) - \epsilon$ for each $\epsilon > 0 \Rightarrow m^{**}(A) = m^*(A)$

2.4 The Cantor set and The Cantor-Lebesgue function

2.4.1 Summary

To ease the definition of the Cantor set, define the following set functions for any real number $r \in \mathbb{R}$ and any disjoint union of intervals $S = \cup_n [a_n, b_n]$:

- $r + S = \cup_n [r + a_n, r + b_n]$,
- $r \times S = \cup_n [r \times a_n, r \times b_n]$.

Definition 6 (The Cantor Set)**.** *The Cantor set $\mathcal{C}$ can be defined by the following recursive sequence of sets*

$$\mathcal{C}_0 = [0,1] \tag{4}$$

$$\mathcal{C}_{n+1} = \frac{\mathcal{C}_n}{3} \cup \left(\frac{2}{3} + \frac{\mathcal{C}_n}{3}\right) \tag{5}$$

$$\mathcal{C} = \cap_{n=1}^{\infty} \mathcal{C}_n \tag{6}$$

The collection $\{\mathcal{C}_n\}_{n=0}^{\infty}$ has the properites:

(i) $\{\mathcal{C}_n\}_{n=0}^{\infty}$ is a descending sequence of closed sets.

(ii) For each n, $\mathcal{C}_n$ is a disjoint union of 2^n intervals, each of them with length $1/3^n$.

Theorem 5. *The Cantor set C is a closed, uncountable set of measure zero.*

A real-valued function f that is defined on a set of real numbers is said to be increasing provided $f(u) \leq f(v) \iff u \leq v$ and said to be strictly increasing, provided $f(u) < f(v) \forall u < v$.

Definition 7 (The Cantor function)**.** *Follow this link for a more rigorous definition (you'll need some function sequence convergence definitions)*

Proposition 4. *The Cantor-Lebesgue function φ is an increasing continous function that maps $[0,1]$ to $[0,1]$. Its derivative exists on the openset $\mathcal{O} = [0,1] \setminus \mathcal{C}$ and is 0.*

2.4.2 Excercise

Problems done: 38, 41, 42, 43, 44.

Ex 38.

Lemma 2 (Lipschitz image of open, bounded intervals)**.** *Let $f : [a,b] \to \mathbb{R}$ be Lipschitz with Lipschitz constant c, $I \subseteq [a,b]$ be any bounded open interval with length ϵ. $m^*(f(I))$ does not exceed $c\epsilon$.*

Proof.
Express I as $(x - \epsilon/2, x + \epsilon/2)$, denote $y = f(x)$.
We know that $\forall x_0 \in I, |x_0 - x| < \epsilon/2$. This implies that $\forall y_0 \in f(I), |y_0 - y| \leq c\,|x_0 - x| \leq c\epsilon/2$.
As such, $f(I)$ is fully contained in the open interval $(y - c\epsilon/2, y + c\epsilon/2)$ and thus $m^*(f(I)) \leq c\epsilon$.

Lemma 3 (Limit point)**.** *E is a closed set in $\mathbb{R}$. $\{x_n\}$ is any convergent sequence in E. The limit of $\{x_n\}$ is in E.*

Proof. We prove by contradiction. Assume $\{x_n\} \to x \notin E$, then $x \in \mathbb{R} \setminus E$.
E is closed, so $\mathbb{R} \setminus E$ is open, so there exists $\epsilon > 0$ satisfying $(x-\epsilon, x+\epsilon) \cap E = \{\emptyset\}$.
For such an ϵ, there exists $N \in \mathbb{N}$ s.t.

$$n \geq N \implies |x - x_n| < \epsilon \iff x_n \in (x-\epsilon, x+\epsilon)$$

which is a constradiction ($\{x_n\}$ is in E).
*This lemma is used to prove the following lemma.

Lemma 4 (Lipschitz image of closed, bounded sets)**.** *Let $f : [a,b] \to \mathbb{R}$ be Lipschitz. The image of every closed set $E \in [a,b]$ is closed.*

Proof.
Let y be any point of closure of $f(E)$. By definition, we can always choose $y_n \in (y-1/n, y+1/n) \cap f(E) \forall n \in \mathbb{N}$, we use this fact to create a sequence $\{y_n\}$ in $f(E)$ that converges to y.
Let $x_n = f^{-1}(y_n) \forall n$. We now show that $y \in f(E)$.
Firstly, $\{x_n\}$ is a bounded sequence, by the Bolzano-Weierstrass theorem, there exists a convergent subsequence $\{x_{n_k}\}$. This is a convergent sequence in E - a closed set in $\mathbb{R}$, so its limit x is in E (by the Limit point Lemma above).
Secondly, f is continous (Lipschitz functions are continous - trivially provable) so $\{x_{n_k}\} \to x \implies f(\{x_{n_k}\}) = y_{n_k} \to f(x)$.
Because limits are unique, $y = f(x)$ and so $y \in E$.
Since $f(E)$ contains all its points of closure, $f(E)$ is closed.

We use the 'Lipschitz image' lemmas, and theorem 11 of chapter 2.4 to solve this excercise.

- f map sets of measure zero to sets of measure zero.

  Let $Z \in (a,b)$ be any set of measure zero (extending this to sets of measure zero containing endpoints is trivial)

  From 'theorem 11', for any $\epsilon > 0$, there exists a disjoint countable union of open intervals $\cup_k I_k \subseteq (a,b)$ containing Z and has $\sum_k m(I_k) < \epsilon$.

  This implies $m^*(f(Z)) \leq m^*(\cup_k f(I_k)) \leq c\epsilon$ from the 'Lipschitz image of open, bounded intervals' lemma (equality happens at $c = 0$)

  As $m^*(f(Z)) \leq c\epsilon$ for all $\epsilon > 0$, $m^*(f(Z)) = 0$.

- f maps F_σ sets to F_σ sets.

  Let $E = \cup_k E_k \in [a,b]$ be any F_σ set.

  By the 'Lipschitz image of closed, bounded sets' lemma, $f(E_k)$ is closed for all k, as such $f(E) = \cup_k f(E_k)$ is an F_σ set.

- f maps measurable sets to measurable sets.

  Let $M \in [a,b]$ be a measurable set. From 'theorem 11', there exists an F_σ set E contained by M satisfying $m^*(M \setminus E) = 0$.

  We have

    $f(E)$ is an F_σ set (because E is an F_σ set), thus it is measurable.

    $f(M \setminus E)$ is a set of measure zero (since $m^*(M \setminus E) = 0$), which is also measurable.

  Because unions of 2 measurable sets are measurable and $f(M) = f(E) \cup f(M \setminus E)$, $f(M)$ is measurable.

Ex 41.
Pick any point $c \in \mathcal{C}$, fix any $\epsilon > 0$, let $I = (c-\epsilon, c+\epsilon)$
We pick any $N \in \mathbb{N}$ s.t. $1/3^N < \epsilon$.

At step N, the singular inverval containing c is a subset of the neighbour I, and as such, there is at least 1 point in $\mathcal{C}$ within this neighbour I (specifically, 1 end point of the interval containing c).

At step $N+1$, this inverval is splitted into 2, there are now 2 intevals contained in I, we can see there is at least 3 points in $\mathcal{C}$ within the neighbour I.

At step $N+2$, each inverval is splitted into 2 again, there are now 4 intevals contained in I, we can see there is at least 7 points in $\mathcal{C}$ within the neighbour I.

...

At step $N+k$, there is at least $\sum_{i=0}^{k} 2 \times 2^i - 1$ points in $\mathcal{C}$ within I.

Since $\left\{\sum_{i=0}^{k} 2 \times 2^i - 1\right\}_{k=0}^{\infty}$ converges to infinity, there are inifinite points in the Cantor set within any neighbour of any points in the Cantor set.

And because the Cantor set is closed, it is perfect.

Ex 42. Proof by contradiction:

Assume a perfect set X is countable, let $\{x_k\}_{k=1}^{\infty}$ be any of its enumeration.

Let F_0 be any closed and bounded subset of X. Define $F_{k+1} = \overline{F_k \setminus \{x_{k+1}\}}$.

We can observe that $\cap_{k=0}^{\infty} F_k$ is the intersection of

- descending ($F_{k+1} \subseteq F_k$, as it has 0 or 1 less element.)
- countable ($\{x_k\}_{k=1}^{\infty}$ is an enumeration of a countable set)
- non-empty ($F_k \setminus \{x_{k+1}\}$ cannot be empty, there are infinite points of X within any x_{k+1}'s neighbour because X is perfect.)
- and closed sets.
- of which F_0 is bounded.

Yet it's intersection is empty, since $\{x_k\}_{k=1}^{\infty}$ is an enumeration of a countable set. This contradicts the Nested Set Theorem.

Ex 43. I have proven **Ex 41.** and **Ex 42.**.

Ex 44.

We prove by contradiction.

Assumes the Cantor set is **not** nowhere dense in $\mathbb{R}$ (Assumption 1), there exists an open set $\mathcal{O}$ s.t. every open subset $U \cap \mathcal{C} \neq \{\emptyset\}$. This is equivalent to $\mathcal{C}^c \cap \mathcal{O} = \{\emptyset\}$, in other words, $\mathcal{O} \subseteq \mathcal{C}$. We now express $\mathcal{O}$ as a union of disjoint open intervals and pick any such interval $(a, b), b - a > 0$.

We now show that there cannot exist an open interval $(a, b) \subseteq \mathcal{C}$ also by contradiction.

Assume, a valid open inveral $(a, b) \subseteq \mathcal{C}$ does exist (Assumption 2). Thanks to how the Cantor set is constructed, this open interval must be a subset of $C_k \forall k$, which implies that the length of (a, b) cannot exceed C_k's total length for any k, in other words $b - a < (2/3)^k \forall k$.

But because $\left\{(2/3)^k\right\} \to 0$ as $k \to \infty$, we have $b - a < 0$, which contradicts assumption 2.

As such, there cannot exist $\mathcal{O} \subseteq \mathcal{C}$. Assumption 1 has created a contradiction.

3 Lebesgue Measurable Functions

3.1 Sequential Pointwise Limits And Simple Approximation

3.1.1 Excercise

Ex 14. Let f be a measurable function on E that is finite a.e. on E and $m(E) < \infty$. For each $\epsilon > 0$, show that there is a measurable set F contained in E such that f is bounded on F and $m(E \setminus F) < \epsilon$.

Solution.
Define:

$$E_n = \{x \in E \mid |f(x)| \leq n\}$$

Since f is a measurable function, each E_n is measurable. The restriction of f to E_n is bounded.

Since f is finite a.e. on E, we have:

$$E' = \{x \in E \mid |f(x)| = \infty\} \quad \text{and} \quad m(E') = 0.$$

We know that:

$$E \setminus E' = \bigcup_{n=1}^{\infty} E_n = \{x \in E \mid f(x) < \infty\}.$$

Thus,

$$m(E \setminus \bigcup_{n=1}^{\infty} E_n) = m(E') = 0.$$

We show that $\{E \setminus E_k\}_{k=1}^{\infty}$ is descending and:

$$m(E \setminus E_1) \leq m(E) < \infty.$$

Applying Theorem 15 (ii) / page 44,

$$\lim_{k\to\infty} m(E \setminus E_k) = m\left(\bigcap_{k=1}^{\infty}(E \setminus E_k)\right) = m\left(E \setminus \bigcup_{k=1}^{\infty} E_k\right) = 0.$$

Therefore, $\forall\ \epsilon > 0$, $\exists\ k$ such that if $n \geq k$, then:

$$m(E \setminus E_n) < \epsilon.$$

So, F is E_n. □

Ex 16. Let I be a closed, bounded interval and E a measurable subset of I. Let $\epsilon > 0$. Show that there is a step function h on I and a measurable subset F of I for which

$$h = \chi_E \text{ on } F \text{ and } m(I \setminus F) < \epsilon.$$

(Hint: Use Theorem 12 of Chapter 2.)

Solution.
Let I be a closed, bounded interval, and E a measurable subset of I. Following Theorem 12 (Chapter 2):
For any $\epsilon > 0$, there exists a finite collection of disjoint open intervals $\{I_k\}_{k=1}^{n}$ such that if $O = \bigcup_{k=1}^{n} I_k$, then

$$m(E \Delta O) < \epsilon.$$

Define the set F as:

$$F = I \setminus (E\Delta O) = I \cap ((E \cap O^c) \cup (O \cap E^c))^c.$$

Simplifying the complement:

$$F = I \cap ((E \cap O) \cup (E^c \cap O^c)) = (I \cap E \cap O) \cup (I \cap E^c \cap O^c).$$

The measure of the complement of F satisfies:

$$m(I \setminus F) = m(E\Delta O) < \epsilon.$$

Define the step function h as:

$$h(x) = \begin{cases} 1 & \text{for } x \in I \cap E \cap O, \\ 0 & \text{for } x \in I \cap E^c \cap O^c. \end{cases}$$

This ensures $h = \chi_E$ on F. □

Ex 17. Let I be a closed, bounded interval and ψ a simple function defined on I. Let $\epsilon > 0$. Show that there is a step function h on I and a measurable subset F of I for which

$$h = \psi \text{ on } F \text{ and } m(I \setminus F) < \epsilon.$$

(Hint: Use the fact that a simple function is a linear combination of characteristic functions and the preceding problem.)

Solution.
Let ψ be a simple function on a closed, bounded interval I, defined as $\psi = \sum_{k=1}^{n} a_k \chi_{E_k}$, where $E_k \subseteq I$ are measurable and a_k are constants.

For each k, there exists a step function h_k and a measurable set $F_k \subseteq I$ (**Problem 16**) such that:

$$h_k = \chi_{E_k} \text{ on } F_k \quad \text{and} \quad m(I \setminus F_k) < \frac{\epsilon}{n}.$$

Let $h = \sum_{k=1}^{n} a_k h_k$. Since each h_k is a step function, h is also a step function.
Define $F = \bigcap_{k=1}^{n} F_k$. On F, $h_k = \chi_{E_k}$ for all k, so $h = \psi$ on F.

$$m(I \setminus F) = m\left(\bigcup_{k=1}^{n} (I \setminus F_k)\right) \leq \sum_{k=1}^{n} m(I \setminus F_k) < \sum_{k=1}^{n} \frac{\epsilon}{n} = \epsilon.$$

□

Ex 18. Let I be a closed, bounded interval and f a bounded measurable function defined on I. Let $\epsilon > 0$. Show that there is a step function h on I and a measurable subset F of I for which

$$|h - f| < \epsilon \text{ on } F \text{ and } m(I \setminus F) < \epsilon.$$

Solution.

By the Simple Approximation Lemma, there exists a simple function ψ such that:

$$|\psi(x) - f(x)| < \epsilon$$

By (**Problem 17**), there exists a step function h and a measurable set $F \subseteq I$ such that:

$$h = \psi \text{ on } F \quad \text{and} \quad m(I \setminus F) < \epsilon.$$

□

Ex 22. (Dini's Theorem) Let $\{f_n\}$ be an increasing sequence of continuous functions on $[a, b]$ which converges pointwise on $[a, b]$ to the continuous function f on $[a, b]$. Show that the convergence is uniform on $[a, b]$.

(Hint: Let $\epsilon > 0$. For each natural number n, define

$$E_n = \{x \in [a, b] \mid f(x) - f_n(x) < \epsilon\}.$$

Show that $\{E_n\}$ is an open cover of $[a, b]$ and use the Heine-Borel Theorem.)

Solution.
For $\epsilon > 0$, let

$$E_n = \{x \in [a, b] \mid f(x) - f_n(x) < \epsilon\}.$$

Since f_n and f are continuous, $f - f_n$ is continuous. The set E_n is the preimage of the open interval $(-\infty, \epsilon)$, hence E_n is open. By pointwise convergence, for every $x \in [a, b]$, there exists $k \in \mathbb{N}$ such that $x \in E_k$. Thus,

$$[a, b] \subseteq \bigcup_{k \in \mathbb{N}} E_k,$$

making $\{E_k\}$ an open cover of $[a, b]$. Since $[a, b]$ is closed and bounded, there exists a finite subcover $\{E_{k_1}, E_{k_2}, \ldots, E_{k_m}\}$. Let $N = \max\{k_1, k_2, \ldots, k_m\}$.

Because $\{f_n\}$ is an increasing sequence, $E_n \subseteq E_{n+1}$. For $n \geq N$,

$$E_n = [a, b].$$

Therefore, with $n \geq N \ \forall \ x \in [a, b]$,

$$f(x) - f_n(x) < \epsilon.$$

This establishes uniform convergence □

4 Lebesgue Intergration

4.1 The General Lebesgue Integral

4.1.1 Excercise

Problems done: 28, 30, 31, 33, 34.

Ex 28.

By definition, $(f \cdot \chi_C)(x) = f(x)$ for all $x \in C$ and $(f \cdot \chi_C)(x) = 0$ for all $x \in A$ s.t. $A \cap C = \emptyset$.

From Corollary 18,

$$\begin{aligned}\int_E (f \cdot \chi_C) &= \int_C (f \cdot \chi_C) + \int_{E \setminus C} (f \cdot \chi_C) \\ &= \int_C f + \int_{E \setminus C} 0 \\ &= \int_C f\end{aligned}$$

Ex 30.

Note that you could use exercise 27 to speed up and simplify the proof, I didn't use it because I am stupid.

Assume on E unless specified.

(i) $\int_E \liminf f_n \leq \liminf \int_E f_n$

Let $h_n = f_n + g \forall n \in \mathbb{N}$. We now have $\{h_n\}$ - a sequence of non-negative a.e. ($|f_n| \leq g$), measurable everywhere functions on E.

Define $a_n = \inf\{h_k | k \geq n\}$, as $\liminf h$ always exists, $\{a_n\}$ is a convergent a.e sequence of non-negative a.e. functions. (Trivally, $\lim a_n = \liminf h_n$ a.e.)

Apply Fatou's Lemma to $\{a_n\}$,

$$\begin{aligned}\int_E \lim a_n &\leq \liminf \int_E a_n \\ \int_E \liminf h_n &\leq \liminf \int_E \inf\{h_k | k \geq n\} \\ \int_E \liminf (f_n + g) &\leq \liminf \int_E \inf\{f_k + g | k \geq n\} \\ \int_E \liminf f_n &\leq \liminf \int_E \inf\{f_k | k \geq n\} \leq \liminf \int_E f_n\end{aligned}$$

($\inf\{f_k + g | k \geq n\} \leq f_n \forall n \in \mathbb{N} \implies \int_E \inf\{f_k + g | k \geq n\} \leq \int_E f_n$ via monotonicity of intergration)

(ii) $\liminf \int_E f_n \leq \limsup \int_E f_n$

The sequence $\left\{\int_E f_n\right\}$ is a sequence in $\mathbb{R}^1$, so this inequality is true (see 1.5.2)

(iii) $\limsup \int_E f_n \leq \int_E \limsup f_n$

Let $p_n = g - f_n \forall n \in \mathbb{N}$. We now have $\{p_n\}$ - a sequence of non-negative a.e. ($|f_n| \leq g$), measurable everywhere functions on E.

Define $b_n = \inf\{h_k | k \geq n\}$, as $\liminf p$ always exists, $\{b_n\}$ is a convergent a.e. sequence of non-negative a.e. functions. (Trivally, $\lim b_n = \liminf p_n$ a.e.)

Apply Fatou's Lemma to $\{b_n\}$,

$$\int_E \lim b_n \leq \liminf \int_E b_n$$
$$\int_E \liminf p_n \leq \liminf \int_E \inf\{p_k|k \geq n\}$$
$$\int_E g - \int_E \limsup f_n \leq \int_E g - \limsup \int_E \sup\{f_n|k \geq n\} \quad (30*)$$
$$\limsup \int_E \sup\{f_n|k \geq n\} \leq \int_E \limsup f_n \quad (30**)$$
$$\limsup \int_E f_n \leq \int_E \limsup f_n$$

- (30*):
  - LHS:
  $$\int_E \liminf p_n = \int_E \liminf(g - f_n) = \int_E (g + \liminf(-f_n)) = \int_E (g - \limsup f_n) = \int_E g - \int_E \limsup f_n.$$
  - RHS:
  $$\begin{aligned}\liminf \int_E \inf\{p_k|k \geq n\} &= \liminf \int_E \inf\{g - f_n|k \geq n\} \\ &= \liminf \int_E (g + \inf\{-f_n|k \geq n\}) \\ &= \liminf \int_E (g - \sup\{f_n|k \geq n\}) \\ &= \int_E g + \liminf\left(-\int_E \sup\{f_n|k \geq n\}\right) \\ &= \int_E g - \limsup \int_E \sup\{f_n|k \geq n\}.\end{aligned}$$
- (30**): $f_n \leq \sup\{f_k + g|k \geq n\}\, \forall n \in \mathbb{N} \implies \int_E f_n \leq \int_E \sup\{f_k + g|k \geq n\}$ via monotonicity of intergration.

Q.E.D.

Ex 31. Let $f = g_1 + h_1 = g_2 + h_2$ be two decompositions of f, satisfying g_1, g_2 are both finite and integrable over E, and h_1, h_2 are both nonnegative on E.

Note that h_1, h_2 are both integrable via linearity of integration.

We now show that $\int_E g_1 + \int_E h_1 = \int_E g_2 + \int_E h_2$.

Since $g_1 + h_1 = g_2 + h_2$, this implies $g_1 - g_2 = h_2 - h_1$. We have:

- $\int_E \text{LHS} = \int_E (g_1 - g_2) = \int_E g_1 - \int_E g_2$ via linearity of integrations.
- $\text{RHS} = h_2 - h_1$,

  Define two sets $A = \{x \in E|\text{RHS}(x) \leq 0\}, B = \{x \in E|\text{RHS}(x) < 0\}$
  $$\int_A \text{RHS} = \int_A h_2 - \int_A h_1,$$
  $$\int_B \text{RHS} = \int_B h_2 - \int_B h_1,$$
  $$\implies \int_E \text{RHS} = \int_A \text{RHS} + \int_B \text{RHS} = \int_E h_2 - \int_E h_1.$$

Q.E.D.

Ex 33.

Theorem 6 (General Lebesgue Dominated Convergence Theorem)**.** *Given:*

- *$\{f_n\}$ is a sequence of measurable functions on E that converges pointwise a.e. on E to f.*
- *$\{g_n\}$ is a sequence of nonnegative measurable functions on E that converges pointwise a.e. on E to g.*
- *The sequence $\{g_n\}$ dominates $\{f_n\}$ on E ($|f_n| \leq g_n \forall n$ on E).*

We have:

$$lim_{n\to\infty} \int_E g_n = \int_E g < \infty \implies lim_{n\to\infty} \int_E f_n = \int_E f.$$

Since f is integrable over E, $f < \infty$ on E (and by extension, $|f|$ as well).

(i) $\lim_{n\to\infty} \int_E |f - f_n| = 0$ implies $\lim_{n\to\infty} \int_E |f_n| = \int_E |f|$.

- $f_n \to f$ a.e. on $E \implies |f_n| \to |f|$ a.e. on E. Also, $|f_n|$ is measurable.
  As such, $\{|f_n|\}$ is a sequence of measurable functions on E that converges pointwise a.e. on E to $|f|$.
- $\lim_{n\to\infty} \{|f| + |f - f_n|\} = |f| + \lim_{n\to\infty} \int_E |f - f_n| = |f|$.
  As such, $\{|f| + |f - f_n|\}$ is a sequence of nonnegative measurable functions on E that converges pointwise a.e. on E to $|f|$.
- Via the triangle inequality, $|f_n| \leq |f| + |f - f_n| \forall n \in \mathbb{N}$.
  As such, the sequence $\{|f| + |f - f_n|\}$ dominates $\{|f_n|\}$ on E.

From the General Lebesgue Dominated Convergence Theorem, since $\lim_{n\to\infty} \{|f| + |f - f_n|\} = |f| < \infty$, we have

$$\lim_{n\to\infty} \int_E |f_n| = \int_E |f|.$$

(ii) $\lim_{n\to\infty} \int_E |f_n| = \int_E |f|$ implies $\lim_{n\to\infty} \int_E |f - f_n| = 0$.

Define $a_n = |f| + |f_n| - |f - f_n|$

We have, $|f| + |f_n| - |f - f_n| \geq 0 + \forall n \in \mathbb{N}$. This implies $\{a_n\}$ is a sequence of nonnegative, measurable functions on E.

Also, trivally, $|f_n| \to |f|$ implies $\{a_n\} \to 2|f|$.

According to Fatou's Lemma:

$$\begin{aligned}
\int_E 2|f| &\leq \liminf \int_E a_n \\
&= \liminf \int_E (|f| + |f_n| - |f - f_n|) \\
&= \liminf \int_E |f| + \liminf \int_E |f_n| \, liminf - \int_E |f - f_n| \\
&= 2 \int_E |f| - \limsup \int_E |f - f_n|. \\
0 &= \limsup \int_E |f - f_n|
\end{aligned}$$

We also have $\liminf \int_E |f - f_n| \leq \limsup \int_E |f - f_n| = 0$ but because $|f - f_n| \geq 0 \forall n \in \mathbb{N}$ implies that $\int_E |f - f_n| \geq 0 \forall n \in \mathbb{N}$, We conclude that $\liminf \int_E |f - f_n| = \limsup \int_E |f - f_n| = 0$.

As such, $\lim_{n\to\infty} \int_E |f - f_n| = 0$.

Ex 34.

$$\begin{aligned}
\lim_{n\to\infty} \int_{-n}^{n} f &= \lim_{n\to\infty} \int_{[-n;n]} f \\
&= \lim_{n\to\infty} \int_{\mathbb{R}} (f \cdot \chi_{[-n;n]}) \\
&= \int_{\mathbb{R}} \lim_{n\to\infty} (f \cdot \chi_{[-n;n]}) \quad \text{(f is nonnegative)} \\
&= \int_{\mathbb{R}} f
\end{aligned}$$

(Trivially, the sequence $\lim_{n\to\infty}(f \cdot \chi_{[-n;n]})$ converges pointwise to f everywhere on $\mathbb{R}$ - proof via picking any random $x \in \mathbb{R}$.)