Project 3

Technical requirements

- *Deadline:* [2024-01-29 Mon]
- *Format:* A single R file encoded as UTF-8.
- *Content:* The script must contain (a) your name and student's ID number in the first line of a script as a comment, (b) a definition of the function, and (c) two or more examples of the function's applications (you can use the following ones).
- *Packages:* Standard core packages coming with a typical R installation are the only packages allowed.
- *Instructor:* All files must be sent via email from your school account to the instructor's school account.

Project description

The task

This is a simple project, and it is about computing integrals of functions $f: R^2 \to R$ over rectangular areas. That is, given a non-negative function $f: R^2 \to R$ defined over a region $A = [x_0, x_1] \times [y_0, y_1]$, calculate the integral

$$\iint_A f(x, y) dx dy.$$

To this end, we need to define a function taking two main arguments representing a mathematical function (an integrand) and a region. The function can take additional arguments related to the algorithm used to calculate the integral. The function should return a value of the integral.

From technical point of view, we need to define a function `integrate3d()` taking two arguements:

- `f` a function, e.g., `f(x) {x ^ 2}`;
- `over` a list representing a region over which the function is integrated, e.g., `list(x = c(0,1), y = c(0, 1))`.

Algorithm

You can use any algorithm you like. I implemented a simple version of the Monte Carlo integration as an example. The algorithm is straightforward. Sample n points from the cuboid containing the graph of a given function from the uniform distribution. Next, calculate the volume of this cuboid. For each sampled point, decide if it is above or below the function's graph. The integral is a fraction of the total volume given by a fraction of points below the function's graph. One additional argument is required for this algorithm: a number of points sampled from the uniform distribution. In the following examples, it is the argument `n`.

Examples

Let's start with a function $f(x, y) = \cos(x)y$ over a region $[0, \pi/2] \times [0, 1]$. We can easily calculate this integral getting

$$\iint_{[0,\pi/2]\times[0,1]} \cos(x)y \, d(x,y) = \int_0^{\pi/2} \int_0^1 \cos(x)y \, dydx$$
$$= \int_0^{\pi/2} \cos(x) \left(\int_0^1 y \, dy \right) dx$$
$$= \int_0^{\pi/2} \cos(x) \, dx \cdot \int_0^1 y \, dy$$
$$= 1 \cdot \frac{1}{2} = \frac{1}{2}.$$

The function’s graph is shown below.

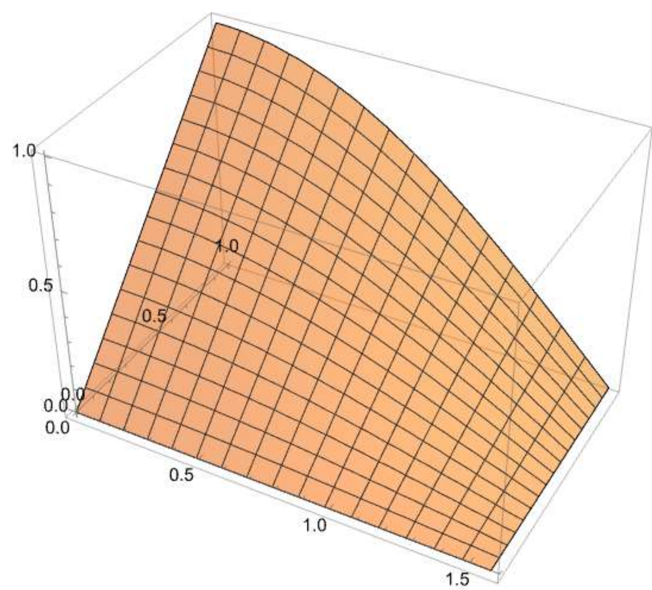

Figure 1: First example

We can apply the function `integrate3d()` to calculate the value of the integral numerically.

```
### Example 1 (low n)
integrate3d(
  f = function(x, y) {cos(x) * y},
  over = list(x = c(0, pi / 2), y = c(0, 1)),
  n = 10^2)

### Example 1 (high n)
integrate3d(
  f = function(x, y) {cos(x) * y},
  over = list(x = c(0, pi / 2), y = c(0, 1)),
  n = 10^5)
```

```
[1] 0.4868085
[1] 0.5007699
```

It is clear that for a low number of points sampled over the region, the result is not very accurate. However, when this number is more significant, the value is close to the actual value of the integral.

The second example is concerned with a function $f(x, y) = (\cos(x) + 2)(\sin(y) + 1)$. We want to integrate this function over the region $[0, \pi] \times [0, \pi]$. Again, we can easily calculate this integral to get

$$\begin{aligned}
\iint\limits_{[0,\pi]\times[0,\pi]} (\cos(x) + 2)(\sin(y) + 1)\ d(x, y) &= \int_0^\pi \int_0^\pi (\cos(x) + 2)(\sin(y) + 1)\ dydx \\
&= \int_0^\pi (\cos(x) + 2) \left(\int_0^\pi (\sin(y) + 1)\ dy \right) dx \\
&= \int_0^\pi (\cos(x) + 2)\ dx \cdot \int_0^\pi (\sin(y) + 1)\ dy \\
&= 2\pi \cdot (2 + \pi) \approx 32.3
\end{aligned}$$

The function's graph is shown below.

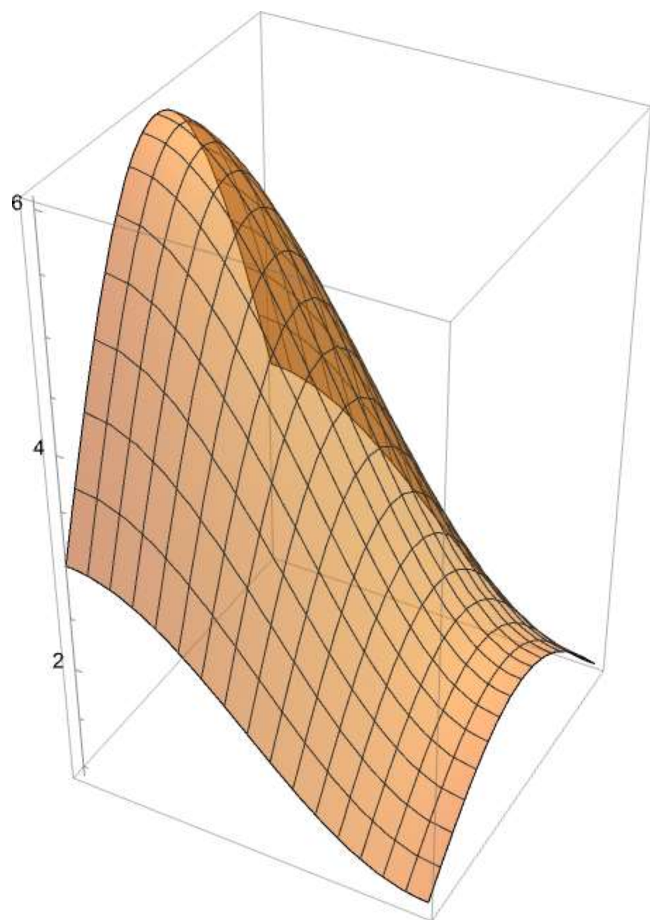

Figure 2: Second example

We can apply the function `integrate3d()` to calculate the value of the integral numerically.

```
### Example 2 (low n)
integrate3d(
```

```
  f = function(x, y) { (cos(x) + 2) * (sin(y) + 1)},
  over = list(x = c(0, pi), y = c(0, pi)),
  n = 10^2)

### Example 2 (high n)
integrate3d(
  f = function(x, y) { (cos(x) + 2) * (sin(y) + 1)},
  over = list(x = c(0, pi), y = c(0, pi)),
  n = 10^5)
```

```
[1] 33.16186
[1] 32.30677
```

Again, we can see that the number of points sampled over the region is crucial to get a good approximate result.

To recap the whole project. The task is to define a function calculating numerical integrals of functions of two variables over rectangular regions. The function must take at least two arguments: the mathematical function (an integrand) and a region of integration (possibly a list with two slots). You can use any algorithm, and depending on a particular choice, the function may take additional arguments.

Date: 2023-10-31 Tue 00:00
Author: Michał Ramsza
Created: 2023-11-01 Wed 14:40
Validate