Derivation of the FWI Hessian

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The regular FWI objective can be written as

$$\phi(m) = \sum_{k,l} \|P_r U_{k,l}(m) - d_{k,l}\|_F^2$$

where k indexes the frequency, l indexes the source position and

$$U_{k,l}(m) = (H(m,k))^{-1} q_{k,l} \tag{1}$$

is the solution of the Helmholtz equation. We drop the dependence on k, l for simplicity and we let $F(m) = [P_r U(m)]$ be the forward modeling operator, indexed implicitly over source and frequency.

The Jacobian of $U(m)$ in the direction δm is found by differentiating (1) in the direction δm. We write $H(m)U(m) = Q$. Differentiating this equation with respect to δm gives

$$\begin{aligned} H(m)DU(m)[\delta m] + DH(m)[\delta m]U(m) = 0 \\ DU(m)[\delta m] = H(m)^{-1}(-DH(m)[\delta m]U(m)) \end{aligned} \tag{2}$$

When $H(m) = A\mathrm{diag}([Bf(m)]^2) + L$, where $f(m)$ is a pointwise applied function and B is a constant matrix, then

$$D(H(m))[\delta m]y = A\mathrm{diag}(2[Bf(m)] \odot [Bdf(m)] \odot \delta m)y. \tag{3}$$

Let $M = \mathrm{diag}(2[Bf(m)] \odot [Bdf(m)])$, so $DH(m)[\delta m]y = AM\mathrm{diag}(\delta m)y = AM\mathrm{diag}(y)\delta m$

Therefore the Jacobian of $F(m)$ is simply $DF(m)[\delta m] = [P_r DU(m)[\delta m]]$, again, implicitly indexed over source and frequency.

The adjoint operator, $D(H(m))[\cdot]^*$, acting on a vector Z is

$$D(H(m))[\cdot]^* Z = \mathrm{diag}(\overline{y})\overline{M} A^H Z. \tag{4}$$

The adjoint of the mapping $DU(m)[\cdot]$ applied to the vector Z is therefore

$$DU(m)[\cdot]^* Z = DH(m)[\cdot]^*(-H(m)^{-H} Z) \tag{5}$$

The adjoint of the Jacobian of $F(m)$ acting on the vector $\tilde{Z}$ is simply the expression in (5) with $Z = P_r^T \tilde{Z}$.

Setting $\tilde{Z} = Pu(m) - d$, the gradient of ϕ is therefore

$$\nabla_m \phi = \overline{Mu(m)} \cdot (A^H(-H(m)^{-H}(P^T P u(m) - P^T d)))$$

We let $V_1 = \overline{M(m)u(m)}$, $V_2 = A^H(-H(m)^{-H})$, $V_3 = P^T P u(m) - P^T d$, so $\nabla_m \phi = V_1 \odot (V_2 V_3)$. Differentiating this relationship in the direction $[\delta m]$, we get that

$$\begin{aligned} DV_1(m)[\delta m] &= \overline{DM(m)[\delta m]u(m) + M(m)Du(m)[\delta m]} \\ DV_2(m)[\delta m] &= A^H(H(m)^{-H} DH(m)[\delta m] H(m)^{-H}) \\ DV_3(m)[\delta m] &= P^T P Du(m)[\delta m] \end{aligned}$$

Therefore the Hessian of ϕ, $\nabla_m^2 \phi[\delta m] = D(\nabla_m \phi)[\delta m]$, is

$$\nabla_m^2 \phi[\delta m] = DV_1(m)[\delta m] \odot (V_2 V3) + V_1 \odot (DV_2(m)[\delta m] V_3 + V_2 DV_3(m)[\delta m]) \tag{6}$$