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1. **The IBVP for the Heat Equation.**

Consider the following initial-boundary value problem (IBVP) modeling heat flow in a wire.

$$\text{(PDE)} \qquad \frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}, \qquad \text{for } 0 < x < 2\pi, \quad t > 0$$

$$\text{(BC)} \qquad u_x(0,t) = 0,\ u(2\pi,t) = 0, \quad t > 0$$

$$\text{(IC)} \qquad u(x,0) = f(x) = \begin{cases} 1, & 0 < x < \pi \\ 0, & \pi \le x < 2\pi \end{cases}$$

Assuming a product solution, $u(x,t) = X(x)\cdot T(t)$, separation of variables leads to an eigenvalue problem for $X(x)$ and a first-order ODE for $T(t)$.

$X''(x) + \lambda X(x) = 0,\ \ X'(0) = 0,\ X(2\pi) = 0 \quad$ (EVP)

$T'(t) + 2\lambda T(t) = 0$

The eigenvalues are $\lambda_n = \left(\frac{2n-1}{4}\right)^2$ with solutions, $X_n(x) = A_n \cos\left(\frac{(2n-1)x}{4}\right)$, for $n = 1, 2, 3, \ldots$

Complete the solution to the IBVP as follows:

- For each eigenvalue, λ_n, determine $T_n(t)$.
- Form the general solution, $u(x,t)$, as an infinite series.
- Use the initial condition, $u(x,0) = f(x)$, to determine the values of the coefficients in the general solution. This will require the appropriate Fourier Series for $f(x)$.

$$\lambda_n = \left(\frac{2n-1}{4}\right)^2$$

$$T' + 2\left(\left(\frac{2n-1}{4}\right)^2\right) T = 0$$

$$\frac{T'}{T} = -2\left(\left(\frac{2n-1}{4}\right)^2\right) dt$$

$$\ln(T) = -2\left(\left(\frac{2n-1}{4}\right)^2\right) t$$

$$T_n = e^{\left(-2\left(\frac{2n-1}{4}\right)^2\right) t}$$

$$u(x,t) = T(t)\cdot X(x)$$

$$u(x,t) = \left(e^{\left(-2\left(\frac{2n-1}{4}\right)^2\right) t}\right)\left(A_n \cos\left(\left(\frac{2n-1}{4}\right)x\right)\right)$$

$$u(x,t) = A_n \cos\left(\left(\frac{2n-1}{4}\right)x\right) e^{\left(-2\left(\frac{2n-1}{4}\right)^2\right)t}$$

$$A_n = \frac{2}{2\pi} \int_0^{2\pi} f(x) \cos\left(\frac{n\pi x}{2}\right) dx$$

$$A_n = \frac{2}{2\pi} \left(\int_0^{\pi} \cos\left(\frac{n\pi x}{2}\right) dx \right)$$

$$A_n = \frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right)$$

$$u(x,0) = \frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right) \cos\left(\left(\frac{2n-1}{4}\right)x\right)$$

$$u(x,t) = \frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right) \cos\left(\left(\frac{2n-1}{4}\right)x\right) e^{\left(-2\left(\frac{2n-1}{4}\right)^2\right)t}$$

2. **The IBVP for the Wave Equation.**

Solve the following initial-boundary value problem.

$$\text{(PDE)} \qquad u_{tt} = 16\,u_{xx}\,, \qquad \text{for } 0 < x < 2, \quad t > 0$$

$$\text{(BC)} \qquad u(0,t) = 0, \;\; u(2,t) = 0, \qquad t > 0$$

$$\text{(IC)} \qquad u(x,0) = \frac{\sin(\pi x/2)}{4} - \frac{\sin(3\pi x/2)}{16}, \qquad 0 < x < 2$$

$$\text{(IC)} \qquad u_t(x,0) = \frac{\sin(\pi x/2)}{4} - \frac{\sin(5\pi x/2)}{20}, \qquad 0 < x < 2$$

Assuming a product solution, $u(x,t) = X(x)\cdot T(t)$, separation of variables leads to an eigenvalue problem for $X(x)$ and a second-order ODE for $T(t)$.

$X''(x) + \lambda X(x) = 0, \;\; X(0) = 0, \; X(2) = 0 \quad \text{(EVP)}$

$T''(t) + 16\lambda T(t) = 0$

The eigenvalues are $\lambda_n = (n\pi/2)^2$ with solutions, $X_n(x) = C_n \sin(n\pi x/2)$, for $n = 1, 2, 3, \ldots$

Complete the solution to the IBVP as follows:

- For each eigenvalue, λ_n, determine $T_n(t)$.
- Form the general solution, $u(x,t)$, as an infinite series.
- Use the initial conditions, $u(x,0)$ and $u_t(x,0)$, to determine the values of the coefficients in the general solution.

$T'' + cT = 0$ $\qquad b = \left(\frac{a(n\pi^2)}{L^2}\right)$ $\qquad c = \sqrt{b}$

$T'' + 4\frac{n\pi}{2}T = 0$ $\qquad b = 16\left(\frac{n\pi}{L}\right)^2$ $\qquad c = 4\frac{n\pi}{2}$

For wave equation

$T_n(t) = A_n \cos\left(\frac{4n\pi}{2}t\right) + B_n \sin\left(\frac{4n\pi}{2}t\right)$

$$u(x,t) = \sum_{n=1}^{\infty}\left(A_n \cos\left(\frac{an\pi}{L}t\right) + B_n \sin\left(\frac{an\pi}{L}t\right)\right)\sin\left(\frac{n\pi}{L}x\right)$$

$$u(x,t) = \sum_{n=1}^{\infty}\left(A_n \cos(2\pi n t) + B_n \sin(2\pi n t)\right)\sin\left(\frac{n\pi}{2}\right)$$

$$u(x,0) = \sum_{n=1}^{\infty}\left(A_n \cos(0) + B_n \sin(0)\right)\sin\left(\frac{n\pi}{L}x\right)$$

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{2}x\right)$$

$n=1$

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{2} x\right)$$

[Initial conditions]

$$u_t(x,0) = \sum_{n=1}^{\infty} \left[0 + B_n (2n\pi)\right] \sin\left(\frac{\pi x}{2}\right)$$

$$= \frac{1}{4} \sin\left(\frac{\pi x}{2}\right) - \frac{1}{20} \sin\left(\frac{5\pi}{2} x\right)$$

$$= 2\pi B_1 \sin\left(\frac{\pi x}{2}\right) + 4\pi B_2 \sin(\pi x) + 6\pi B_3 \sin\left(\frac{3\pi x}{2}\right) + 8\pi B_4 \sin(2\pi x) + 10\pi B_5 \sin\left(\frac{5\pi x}{2}\right)$$

$$2\pi B_1 = \frac{1}{4}$$

$$B_2 = 0$$

$$B_3 = 0$$

$$B_4 = 0$$

$$10\pi B_5 = \frac{1}{20}$$

$$u(x,t) = \frac{1}{4} \cos(2\pi t) \sin\left(\frac{\pi x}{2}\right) - \frac{1}{16} \cos(6\pi t) \sin\left(\frac{3\pi x}{2}\right) + \frac{1}{3\pi} \sin(2\pi t) \sin\left(\frac{\pi x}{2}\right) - \frac{1}{200\pi} \sin(10\pi t) \sin\left(\frac{5\pi}{2} x\right)$$