Homework 8-11

Sunday, February 28, 2021 2:02 PM

"I pledge my honor I have abided by the Stevens Honor system."

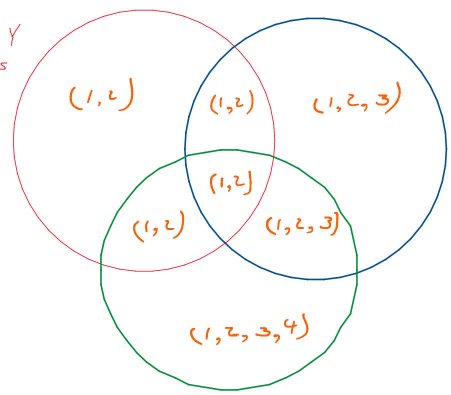
- Alex [signature]

Problem 8.1. a) Do there exist sets A, B, C such that $\emptyset \neq A \cap B$, $\emptyset \neq A \cap C$, $\emptyset \neq B \cap C$, $\emptyset = A \cap B \cap C$? If yes, draw a Venn diagram of such sets, if no– explain why.

No, if each set can relate to one another, then they must all have at least one thing in common. This relates to the idea of transitivity. For this to be true, at least one of the sets must be empty.

b) Do there exist sets X, Y, Z such that $X \subset Y$, $Y \subset Z$, $X \neq \emptyset$, $X \cap Z = \emptyset$? If yes, draw a Venn diagram of such sets, if no– explain why.

No, if $X \neq \emptyset$, and X is a subset of Y and Y is a subset of Z, $X \cap Z = \emptyset$ is false.



$x = (1,2)$

$y = (1,2,3)$

$z = (1,2,3,4)$

Problem 10.1. We say that a binary relation R on X is serial if every $x \in X$ is R-related to some element y (that is $\forall x \in X, \exists y \in X, R(x, y)$).

a) Does there exist a binary relation S on $\mathbb{Z}$ that is not reflexive, symmetric, transitive and serial?

Yes, if $S = [(1,2),(2,3)]$ and a set $(1,3) \nexists (1,3)$ because it's not in S, so it's not reflexive. $(1,2) \nexists (2,1)$, so it's not symmetric. For $(1,2),(2,3)$, $(1,3)$ is not a part of the set, so it is not transitive. It is not serial because for $3 \in \mathbb{Z}$ $(3,y) \in S$ does not exist.

b) Does there exist a binary relation T on $\mathbb{Z}$ that is reflexive, not symmetric and not transitive?

Yes, say $T = ((i,i) : i \in \mathbb{Z}) \cup (0,1),(1,2)$, which directly satisfies the reflexive property. Similar to part a, $(0,1)$ can't relate to $(1,0)$ because $(1,0)$ doesn't exist in T, which means it is not symmetric. Similarly, $(0,2) \notin T$ for $T = (0,1),(1,2)$, which means it is not transitive.

Explain your answers. If you claim that such a relation exists– provide an example. If you claim that there is no such relation, prove it.

Problem 10.2. Recall that we say integer n is divisible by d if there is another integer k such that $n = kd$. We denote it as $d \mid n$ (reads "d divides n").

a) Let R_1 be the relation between integer numbers defined as aR_1b if $3 \mid a^2 - b^2$. Is R_1 an equivalence relation?

$a^2 - a^2 = 0 = n$; $n = k3$

$\frac{3}{a^2 - a^2} \approx \frac{3}{0}$ Makes it reflexive

If $a^2 - b^2 = 3k$,

$|k| = k$ for $b^2 - a^2$

Divides the same amount of times so $aRb \approx bRa$

$\left.\begin{aligned} a^2 - b^2 &= 3k_1 \\ b^2 - c^2 &= 3k_2 \end{aligned}\right\} 3(k_1 + k_2) = a^2 - c^2$

It is transitive where k_1 and $k_2 \in \mathbb{Z}$ (integers)

... is satisfied by

$a^2 - a^2 = 0 = n$; $n = 4 \cdot 0$

$\frac{3}{a^2 - a^2} \approx \frac{3}{0}$ Makes it reflexive

aRa

$|n| = n$ for $b^2 - a^2$

Divides the same amount of times so $aRb \approx bRa$

So it is reflexive

$b^2 - c^2 = 3n_2$

It is transitive where n_1 and $n_2 \in \mathbb{Z}$ (integers)

$3/a^2 - c^2$ is satisfied by the fact that $3/a^2 - b^2$ and $3/b^2 - c^2$ are satisfied, so aRc.

Yes it is an equivalence relation

b) Let R_2 be the equivalence relation between integer numbers defined as aR_2b if $3 \mid a^2 + b^2$ or $a = b$. Is R_1 an equivalence relation?

If $a = b$ means aR_2b, $a = a$ is also a relation

So it is reflexive

$a^2 + b^2 = b^2 + a^2$

So it is symmetric

If $a = b$ and $b = c$ then $a = c$

So it is transitive

Yes it is an equivalence relation

Problem 11.1. Define a relation on natural numbers in the following way: $a \leq_2 b$ if either $a = b$ or the number of 2's in the decimal expression of a is strictly less than the number of 2's in the decimal expression of b.

E.g. $2023 \leq_2 2022$, $1000 \leq_2 2$

a) Is $\leq_2$ a partial order?

$a \leq_2 a$ is reflexive because $a = a$

$a \leq_2 b$ = $b \leq_2 a$ if $a = b$, so it is anti-symmetric

If $a \leq b$ and $b \leq c$ then $a \leq c$ so it is transitive
$(22 \leq 222,\ 222 \leq 2222,\ 22 \leq 2222)$

$\therefore$ It is a partial order

b) Is ≤ 2 a total order?

For any case either $a = b$, $a \leq_2 b$ or $b \leq_2 a$ which means it's always comparable

$\therefore$ It is a total order

Explain your answers.