Homework 12-14

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"I pledge my honor I have abided by the Stevens Honor system."
- Alex [signature]

Problem 12.1. Consider the following functions

- $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(x) = x^2 - 3$
- $g : \mathbb{Z} \to \mathbb{Z}$ defined as $g(x) = x^3 + 3$
- $h : \mathbb{R} \to \mathbb{R}$ defined as $h(x) = x^3 + 3$
- $t : \mathbb{Z} \to \mathbb{N}$ defined as $t(x) = |x| + 1$.

a) Which of f, g, h, t are injective?

$f(1) = 1^2 - 3 = -2$

$f(-1) = (-1)^2 - 3 = -2$ $\quad 1 \neq -1$

$f(1) = f(-1)$

f is not injective

$g(x) = x^3 + 3$; $x_1, x_2 \in \mathbb{Z}$

$g(x_1) = g(x_2)$

$x_1^3 + 3 = x_2^3 + 3$

$x_1^3 = x_2^3$

$g(x_1) = g(x_2) \rightarrow x_1 = x_2$

$\boxed{g \text{ is injective}}$

Similarly for $h(x) = x^3 + 3$

$x_1, x_2 \in \mathbb{R}$

$x_1^3 + 3 = x_2^3 + 3$

$x_1^3 = x_2^3$

$x_1 = x_2$

$h(x_1) = h(x_2) \rightarrow x_1 = x_2$

$\boxed{h \text{ is injective}}$

$t(x) = |x| + 1$

Let $x_1, x_2 \in \mathbb{Z}$

$t(x_1) = t(x_2)$

$|x_1| + 1 = |x_2| + 1$

$|x_1| = |x_2|$

$x_1 = \pm x_2$

t is not injective

b) Which of f, g, h, t are surjective?

$x \in \mathbb{Z}$; $f(x) > -3$ for $\mathbb{Z} \to \mathbb{Z}$

i.e $x = -1$, $f(x) \neq -1$

f is not surjective

There exists no $x \in \mathbb{Z}$ where $g(x) = 1$ $\quad x = \sqrt[3]{y - 3} \notin \mathbb{Z}$

g is not surjective

g is not surjective

$y = x^3 + 3$ for $h(x)$

$x = \sqrt[3]{y-3}$

$h^{-1}(x) = (x-3)^{1/3}$

$\boxed{h \text{ is surjective}}$

$t: \mathbb{Z} \to \mathbb{N}$; $\forall x \in \mathbb{N}, \exists y \in \mathbb{Z}, t(y) = x$

$\boxed{t \text{ is surjective}}$ Any $\mathbb{Z} + 1$ can yield a natural number

c) Which of f, g, h, t are bijective?

$\therefore f$ is not bijective

$\therefore g$ is not bijective

$\boxed{\therefore h \text{ is bijective}}$

$\therefore t$ is not bijective

Problem 13.1. Prove that given arbitrary $N = 2^{2^n} + 1$ Boolean functions on n variables, say $f_1, f_2, \ldots, f_N$, two of them are logically equivalent, that is $f_i \equiv f_j$ for some $i \neq j$.

You can use the following fact: **There are 2^{2^n} different truth tables of Boolean functions on n variables.** We mentioned it briefly without a proof, and we will discuss later (when we start the combinatorics part) why this holds.

At least two functions will have the same truth table

There are 2^{2^n} different possible truth tables

So for $N = 2^{2^n} + 1$, for $2 \times 2 \times 2 \ldots n + 1$, it must have the same truth table as another function of 2^{2^n} because for $2^{2^n} + 1$ there can only be 2^{2^n} truth tables.

According to the pigeonhole principle $f_i = f_j$ where $i \neq j$

Problem 14.1.

a) Construct a bijection between even natural numbers and natural numbers.

$f: E^+ \to N$

<u>Injective:</u>

$f(x) = \frac{x}{2}$

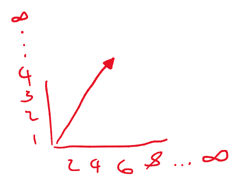
$\frac{x_1}{2} = \frac{x_2}{2}$

$x_1 = x_2$

<u>Surjective:</u>

As $E^+ \to \infty$

$N \to \infty$



b) Construct a bijection between integer numbers and even natural numbers.

$Z = \{0, 1, -1, 2, -2, 3, -3 \dots\}$

$2N = \{2, 4, 6, 8, 10, \dots\}$

<u>Example:</u>

$$f(x) = \begin{cases} 2x, & x > 0 \\ -4x + 4, & x \leq 0 \end{cases}$$

$2 \to 0$
$4 \to 1$
$6 \to -1$
$8 \to 2$
$10 \to -2$

For each integer there is a different $2N$ that corresponds to it.

(As $Z \to$, $2N \to$ for each individual value)

$f(x) = y; \; f^{-1}(y) = x$

$\therefore$ f is bijective

Problem 14.2. Prove that $P(\mathbb{N})$– the set of all subsets of natural numbers, is uncountable.

$A \subseteq N$ where A is represented as:

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A_n, $0.x_0x_1\ldots x_n\ldots$ (Assuming $P(n)$ is countable)

Where $x_n \in \{0,1\}$, and $x_n = 0$ if $n \in A$, and $x_n = 1$ if $n \notin A$

$f: N \to P(N)$ if $P(N)$ is countable for $n \in N$ by $f(n) = A_n$

$x = 0.x_1x_2\ldots x_n\ldots$ where $x_1 = 0$ if $x_{1,1} = 1$ and $x_1 = 1$ if $x_{1,1} = 0$

so $x_n = 0$ if $x_{n,n} = 1$ and $x_n = 1$ if $x_{n,n} = 0$

Then X differs from decimal numbers in the list above

So the decimal x corresponds to a subset X of N that is not on the list above.

This is a contradiction since f is assumed to be a bijection.

$\therefore P(N)$ is uncountable