1. **Boundary Value Problem with Nonhomogeneous ODE.**
For each choice of $g(x)$ listed below, find *all solutions* to the following boundary value problem.

$$4y'' + \pi^2 y = g(x) \quad \text{for } 0 < x < 2, \quad y'(0) = 0, \; y'(2) = 0.$$

(a) $g(x) = 0$ (b) $g(x) = x$

A.) $4y'' + \pi^2 y = 0$

$4m^2 + \pi^2 = 0$

$m = \sqrt{-\frac{\pi^2}{4}}$

$m = \pm \frac{\pi}{2} i$

$y = c_1 \cos\left(\frac{\pi}{2} t\right) + c_2 \sin\left(\frac{\pi}{2} t\right)$

$y' = \frac{\pi}{2} c_2 \cos\left(\frac{\pi}{2} t\right) - \frac{\pi}{2} c_1 \sin\left(\frac{\pi}{2} t\right)$

$0 = \frac{\pi}{2} c_2 \cos(0) - \cancel{\frac{\pi}{2} c_1 \sin(0)}$

$c_2 = 0$

$0 = \frac{\pi}{2} c_2 \cos(\pi) - \cancel{\frac{\pi}{2} c_1 \sin(\pi)}$

$c_2 = 0$

$\boxed{y = c_1 \cos\left(\frac{\pi}{2} t\right)}$

B.)

$y_c = c_1 \cos\left(\frac{\pi}{2} t\right) + c_2 \sin\left(\frac{\pi}{2} t\right)$

For x:

$y_p = Ax + B$

$y_1' = A$

$y_p'' = 0$

$\pi^2 (Ax + B) = x$

$\pi^2 A x + \pi^2 B = x$

$A = \frac{1}{\pi^2}$

$B = 0$

$y_p = \frac{1}{\pi^2} x + 0$

$y_p = \frac{x}{\pi^2}$

$y = c_1 \cos\left(\frac{\pi}{2} t\right) + c_2 \sin\left(\frac{\pi}{2} t\right) + \frac{x}{\pi^2}$

$y' = \frac{\pi}{2} c_2 \cos\left(\frac{\pi}{2} t\right) - \frac{\pi}{2} c_1 \sin\left(\frac{\pi}{2} t\right) + \frac{1}{\pi^2}$

$0 = \frac{\pi}{2} c_2 \cos(0) - \cancel{\frac{\pi}{2} c_1 \sin(0)} + \frac{1}{\pi^2}$

$\frac{\pi}{2} c_2 = -\frac{1}{\pi^2}$

$c_2 = -\frac{2}{\pi^3}$

$\boxed{y = c_1 \cos\left(\frac{\pi}{2} t\right) - \frac{2}{\pi^3} \sin\left(\frac{\pi}{2} t\right) + \frac{x}{\pi^2}}$

2. **Eigenvalue Problem with first derivative in linear operator.**

Find the eigenvalues and eigenfunctions for the boundary value problem,

$$y'' + 4y' + \lambda y = 0 \quad \text{on} \quad 0 < x < \pi, \qquad y'(0) = 0,\ y'(\pi) = 0.$$

$$m^2 + 4m + \lambda = 0$$

$$m = \frac{-4 \pm \sqrt{16 - 4\lambda}}{2}$$

$$m = -2 \pm \sqrt{4 - \lambda}$$

Case 1:

$\lambda = 4$

$-2 \pm \sqrt{4-4}$

$-2 \pm \sqrt{0}$

$m = -2$

$$y = c_1 e^{-2t} + c_2 t e^{-2t}$$

$$y' = -2c_1 e^{-2t} + c_2 e^{-2t} - 2c_2 t e^{-2t}$$

$$0 = -2c_1 + c_2$$

$$c_1 = \tfrac{1}{2} c_2$$

$$0 = -2\left(\tfrac{1}{2} c_2\right) e^{-2\pi} + c_2 e^{-2\pi} - 2c_2 \pi e^{-2\pi}$$

$$0 = \cancel{-c_2 + c_2} - 2c_2\pi$$

$$c_2 = 0$$

$$c_1 = 0$$

$\boxed{y = 0}$

Case 2:

$\lambda < 4$

$m = -2 \pm \sqrt{4 - \lambda}$

$$y = c_1 e^{(-2+\sqrt{4-\lambda})} + c_2 e^{(-2-\sqrt{4-\lambda})}$$

$$y' = c_1\left(-2+\sqrt{4-\lambda}\right) e^{(-2+\sqrt{4-\lambda})t} + c_2\left(-2-\sqrt{4-\lambda}\right) e^{(-2-\sqrt{4-\lambda})t}$$

$$0 = c_1\left(-2+\sqrt{4-\lambda}\right) + c_2\left(-2-\sqrt{4-\lambda}\right)$$

$$c_1 = c_2\left(\frac{2+\sqrt{4-\lambda}}{-2+\sqrt{4-\lambda}}\right)$$

$$0 = c_1\left(-2+\sqrt{4-\lambda}\right) e^{(-2+\sqrt{4-\lambda})\pi} + c_2\left(-2-\sqrt{4-\lambda}\right) e^{(-2-\sqrt{4-\lambda})\pi}$$

$$0 = c_2\left(2+\sqrt{4-\lambda}\right) e^{(-2+\sqrt{4-\lambda})\pi} + c_2\left(-2-\sqrt{4-\lambda}\right) e^{(-2-\sqrt{4-\lambda})\pi}$$

$$0 = c_2\left(2+\sqrt{4-\lambda}\right)\left[e^{(-2+\sqrt{4-\lambda})\pi} - e^{(-2-\sqrt{4-\lambda})\pi}\right]$$

$$c_2 = 0$$

$$c_1 = 0$$

$\boxed{y = 0}$

Case 3:

$\lambda > 4$

$$y = e^{-2t}\left(c_1 \cos\left(\sqrt{\lambda-4}\, t\right) + c_2 \sin\left(\sqrt{\lambda-4}\, t\right)\right)$$

$\lambda > 4$

$4 - \lambda < 0$

$m = -2 \pm i\sqrt{\lambda - 4}$

$$y = e^{-2t}\left(c_1 \cos\left(\sqrt{\lambda-4}\,t\right) + c_2 \sin\left(\sqrt{\lambda-4}\,t\right)\right)$$

$$y = -2e^{-2t}\left(c_1 \cos\left(\sqrt{\lambda-4}\,t\right) + c_2 \sin\left(\sqrt{\lambda-4}\,t\right)\right) + e^{-2t}\left(-c_1\sqrt{\lambda-4}\sin\left(\sqrt{\lambda-4}\,t\right) + c_2\sqrt{\lambda-4}\cos\left(\sqrt{\lambda-4}\,t\right)\right)$$

$$0 = -2c_1 + c_2\sqrt{\lambda-4}$$

$$c_1 = \frac{1}{2}c_2\sqrt{\lambda-4}$$

$$0 = -2e^{-2\pi}\left(c_1\cos\left(\sqrt{\lambda-4}\,\pi\right) + c_2\sin\left(\sqrt{\lambda-4}\,\pi\right)\right) + e^{-2t}\left(-c_1\sqrt{\lambda-4}\right.$$

$$-c_2\lambda\sin\left(\sqrt{\lambda-4}\,\pi\right) = 0$$

$c_2 = 0$

$c_1 = 0$

$y = 0$

trivial solution

$$\sin\left(\sqrt{\lambda-4}\,\pi\right);$$

$$\sqrt{\lambda-4}\,\pi = n\pi$$

$$\lambda = n^2 + 4$$

Eigenvalue

$$y = e^{-2t}\left(c_1\cos\left(\sqrt{\lambda-4}\,t\right) + c_2\sin\left(\sqrt{\lambda-4}\,t\right)\right)$$

$$y = e^{-2t}\left(c_1\cos\left(\sqrt{n^2+4-4}\,t\right) + c_2\sin\left(\sqrt{n^2+4-4}\,t\right)\right)$$

$$\boxed{y_n = e^{-2t}\left(c_1\cos(nt) + c_2\sin(nt)\right)}$$

Eigenfunction for $\lambda_n = n^2 + 4$

3. **Fourier Trigonometric Series.**

Consider the function $f(x)$ defined on $(-\pi, \pi)$, $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ 1, & 0 \le x < \pi/2 \\ 0, & \pi/2 \le x < \pi \end{cases}$ and its Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\Big(a_n\cos(nx) + b_n\sin(nx)\Big).$$

(a) Derive expressions for a_0, a_n and b_n, for $n = 1, 2, \ldots$.

(b) Write out the terms of the Fourier series through $n = 5$.

(c) Graph the periodic extension of $f(x)$ on the interval $(-3\pi, 3\pi)$ that represents the pointwise convergence of the Fourier series in part (b). At jump discontinuities, identify the value to which the series converges.

A.)

$$a_0 = \frac{1}{L}\int_0^{\pi/2} 1\,dx$$

$$a_0 = \frac{1}{\pi}\left[\frac{\pi}{2}\right]$$

$$a_0 = \frac{1}{2}$$

$$a_n = \frac{1}{\pi}\int_0^{\pi/2}\cos\left(\frac{n\pi x}{\pi}\right)dx$$

$$a_n = \frac{1}{\pi}\left[\frac{\sin(nx)}{n}\right]_0^{\pi/2}$$

$$a_n = \frac{\sin\left(\frac{\pi}{2}n\right)}{\pi n}$$

$$b_n = \frac{1}{\pi}\int_0^{\frac{\pi}{2}}\sin\left(\frac{n\pi x}{\pi}\right)dx$$

$$b_n = \frac{1}{\pi}\left[\frac{-\cos(nx)}{\pi n}\right]_0^{\pi/2}$$

$$b_n = \frac{-\cos\left(\frac{\pi}{2}n\right)}{\pi n} - -\frac{1}{\pi n}$$

$$b_n = \frac{1}{\pi n}$$

B.)

$$f(t) = \frac{1}{4} + \sum \frac{\sin\left(\frac{\pi}{2}n\right)}{\pi n}\cos(nx) + \frac{1}{\pi n}\sin(nx)$$

$n = 1$

$$\frac{1}{4} + \frac{\sin\left(\frac{\pi}{2}\right)}{\pi}\cos(x) + \frac{1}{\pi}\sin(x) = \frac{1}{4} + \frac{\cos(x)}{\pi} + \frac{\sin(x)}{\pi}$$

$n=2$

$$\frac{1}{4} + \cancel{\frac{\sin(\pi)}{2\pi}} \cos(2x) + \frac{\sin(2x)}{2\pi} = \frac{1}{4} + \frac{\sin(2x)}{2\pi}$$

$n=3$

$$\frac{1}{4} + \frac{\sin\left(\frac{3\pi}{2}\right)}{3\pi} \cos(3x) + \frac{\sin(3x)}{3\pi} = \frac{1}{4} - \frac{\cos(3x)}{3\pi} + \frac{\sin(3x)}{3\pi}$$

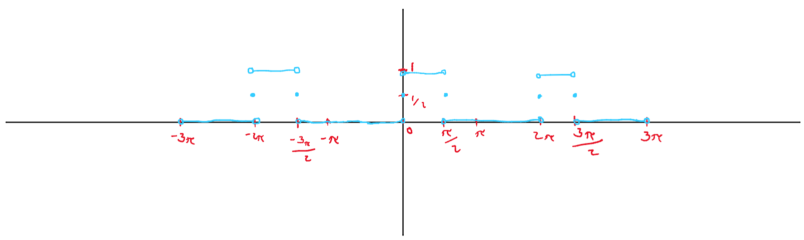
$n=4$

$$\frac{1}{4} + \frac{\sin(2\pi)}{4\pi} \cos(4x) + \frac{\sin(4x)}{4\pi} = \frac{1}{4} + \frac{\sin(4x)}{4\pi}$$

$n=5$

$$\frac{1}{4} + \frac{\sin\left(\frac{5\pi}{2}\right)}{5\pi} \cos(5x) + \frac{\sin(5x)}{5\pi} = \frac{1}{4} + \frac{\cos(5x)}{5\pi} + \frac{\sin(5x)}{5\pi}$$

c.)



Converges at $\frac{(\lim[1] + \lim[0])}{2} = \frac{1}{2}$

4. **Eigenvalue Problem for Cauchy-Euler Equation.**

Find the eigenvalues and eigenfunctions for the boundary value problem,

$$4x^2y'' + 4xy' + \lambda y = 0, \text{ on } 1 < x < 4, \qquad y(1) = 0, \ \ y'(4) = 0.$$

$y = x^m \quad y' = mx^{m+1} \quad y'' = m^2x^{m+2}$

$$4x^2\left(m^2x^{m+2}\right) + 4x\left(mx^{m+1}\right) + \lambda\left(x^m\right) = 0$$

$$4m^2 + 4m - 4m + \lambda = 0$$

$$4m^2 + \lambda = 0$$

$$m = \sqrt{\frac{-\lambda}{4}} = \pm i\frac{\sqrt{\lambda}}{2}$$

$$y = c_1\cos\left(\frac{\sqrt{\lambda}}{2}\ln(x)\right) + c_2\sin\left(\frac{\sqrt{\lambda}}{2}\ln(x)\right)$$

$y(1) = 0$

$$c_1\cos(0) + c_2\sin(0) = 0$$

$$c_1 = 0$$

$$y' = c_2\frac{\sqrt{\lambda}}{2x}\cos\left(\frac{\sqrt{\lambda}}{2}\ln(x)\right)$$

$y'(4) = 0$

$$0 = c_2\frac{\sqrt{\lambda}}{8}\cos\left(\frac{\sqrt{\lambda}}{2}\ln(4)\right)$$

$$\cos\left(\frac{\sqrt{\lambda}}{2}\ln(4)\right) = \cos\left(\sqrt{\lambda}\ln(2)\right)$$

$c_2 \neq 0$:

$$\cos\left(\sqrt{\lambda}\ln(2)\right)$$

$$\downarrow$$

$$\sqrt{\lambda}\ln(2) = (2n+1)\frac{\pi}{2}$$

$$\sqrt{\lambda} = \frac{(2n+1)\frac{\pi}{2}}{\ln(2)}$$

Eigen value $\longrightarrow$ $$\lambda_n = \frac{(2n+1)^2\frac{\pi^2}{4}}{(\ln(2))^2}$$

$$y = \sin\left(\frac{\sqrt{\lambda}}{2}\ln(x)\right)$$

$$y(x) = \sin\left(\frac{(2n+1)\frac{\pi}{2}}{2\ln(2)}\ln(x)\right)$$

$$\uparrow$$

Eigen function