Final Exam

Thursday, December 17, 2020 7:55 AM

"I pledge my honor I have abided by the Stevens Honor system." -Alex Baskins

1. (12 pts.) Find a general solution of the differential equation.

$$L[y] = y'' - 4y' + 4y = 12xe^{2x}.$$

$y = e^{mx}$

$(m-2)^2 = 0$

$m = 2$

$y_c = c_1 e^{2x} + c_2 x e^{2x}$

$W = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix} = e^{4x}$

$W_1 = \begin{vmatrix} 0 & xe^{2x} \\ 12xe^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix} = -12x^2 e^{4x}$

$W_2 = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & 12xe^{2x} \end{vmatrix} = 12xe^{4x}$

$\int u_1' = \int \frac{W_1}{W} = \int -12x^2\,dx = -4x^3 = u_1$

$\int u_2' = \int \frac{W_2}{W} = \int 12x\,dx = 6x^2 = u_2$

$y_p = u_1 y_1 + u_2 y_2$

$y_p = -4x^3 e^{2x} + 6x^3 e^{2x}$

$y_p = 2x^3 e^{2x}$

$y = y_c + y_p$

$$\boxed{y = c_1 e^{2x} + c_2 x e^{2x} + 2x^3 e^{2x}}$$

2. (12 pts) Find a general solution of

$$xy'' - 2y' = 9x^2, \quad x > 0.$$

$y = x^m \quad y' = mx^{m-1} \quad y'' = m(m-1)x^{m-2}$

$x(m(m-1)x^{m-2}) - 2(mx^{m-1}) = 0$

$(mx^{m-1})\ m(m-1) - 2m\ (x^{m}) = 0$

$y_c = c_1 x^3 + c_2$

$W = \begin{vmatrix} x^3 & 1 \\ 3x^2 & 0 \end{vmatrix} = -3x^2$

$y_1 = y'' - \dfrac{2y'}{x} = 9x$

$u_1 = \int -\dfrac{y_2(g(x))}{W}\,dx = \int -\dfrac{1(9x)}{-3x^2}dx = \int \dfrac{3}{x}\,dx = 3\ln|x|$

$u_2 = \int \dfrac{y_1(g(x))}{W}\,dx = \int \dfrac{x^3(9x)}{-3x^2}\,dx = \int -3x^2 = -x^3$

$y_p = u_1 y_1 + u_2 y_2 = 3x^3 \ln|x| - x^3$

$y = y_c + y_p$

$$\boxed{y = c_1 x^3 + c_2 + 3x^3 \ln|x| - x^3}$$

3 (a) (6 pts.) Find the inverse Laplace transform, $y(t) = \mathscr{L}^{-1}\{Y(s)\}$ for

$$Y(s) = \frac{5s^2 + 19}{(s^2 + 2s + 5)(s - 1)}.$$

$\dfrac{5s^2 + 19}{(s^2+2s+5)(s-1)} = \dfrac{As + B}{s^2 + 2s + 5} + \dfrac{C}{s-1}$

$5s^2 + 19 = As + B\ (s-1) + Cs^2 + 2Cs + 5C$

$5s^2 + 19 = As^2 - As + Bs - B + Cs^2 + 2Cs + 5C$

$$\left.\begin{aligned} As^2 + Cs^2 &= 5s^2 \\ Bs - As + 2Cs &= 0 \\ 5C - B &= 19 \end{aligned}\right\} \quad \begin{aligned} A + C &= 5 \\ B - A + 2C &= 0 \\ 5C - B &= 19 \end{aligned}$$

$A = 5 - C$

$B = 5C - 19$

$5C - 19 - 5 + C + 2C = 0$

$8C = 24$

$C = 3$

$A = 2$

$B = -4$

$$\mathcal{L}^{-1}\left(\frac{2s-4}{s^2+2s+5} + \frac{3}{s-1}\right)$$

$s^2 + 2 + 1 = -5 + 1$

$(s+1)^2 + 4$

$$\left(\frac{b}{2a}\right) = \frac{2}{2} = 1$$

$$\frac{2(s+1) - 6}{(s+1)^2+4} + \frac{3}{s-1} = 2\frac{s+1}{(s+1)^2+4} - \frac{6}{(s+1)^2+4} + \frac{3}{s-1}$$

$$\left(2 \cdot \frac{s+1}{(s+1)^2+4}\right) = 2e^{-t}\cos(2t)$$

$$\left(-6\,\frac{1}{(s+1)^2+4}\right) = -3e^{-t}\sin(2t)$$

$$\left(3\,\frac{1}{s-1}\right) = 3e^{t}$$

$$\boxed{y = 2e^{-t}\cos(2t) - 3e^{-t}\sin(2t) + 3e^{t}}$$

3 (b) (4 pts.) Find the inverse Laplace transform, $y(t) = \mathcal{L}^{-1}\{Y(s)\}$ for

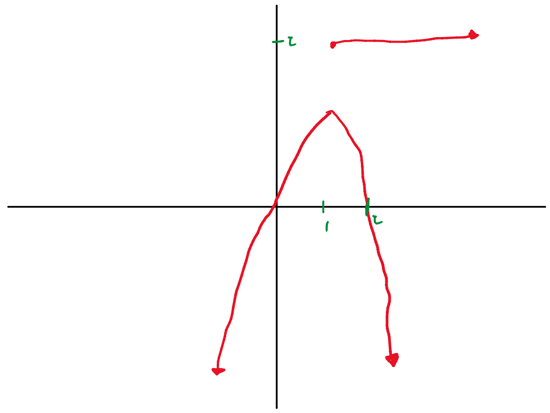
$$Y(s) = \frac{5s^2+19}{(s^2+2s+5)(s-1)}e^{-s}.$$ Indicates a shift

$$\boxed{y = 2e^{-(t-1)}\cos(2t) - 3e^{-(t-1)}\sin(2t) + 3e^{(t-1)}}$$

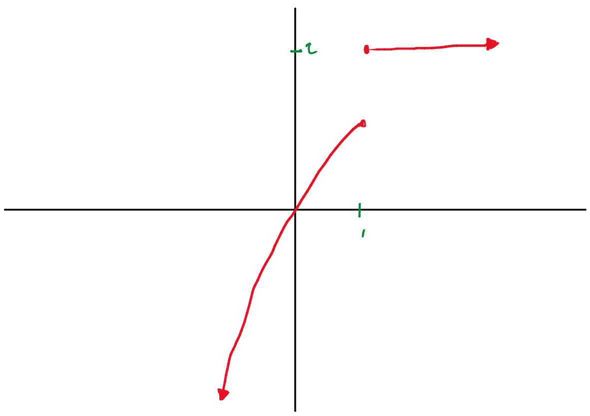
4. (10 pts.) Consider the piecewise function, $f(t)$,

$$f(t) = \begin{cases} 2t - t^2, & 0 \le t < 1 \\ 2, & 1 \le t < \infty \end{cases}$$

(i) Sketch the graph of $f(t)$.



(ii) Use *unit step functions* to express $f(t)$ on a single line.



(iii) Find the Laplace transform, $F(s) = \mathcal{L}\{f(t)\}$.

$$2t - t^2 \left(u(t-0) - u(t-1)\right) + 2\left(u(t-1)\right)$$

$$(2t - t^2)\, u(t-0) + (-2t + t^2 + 2)\, u(t-1)$$

$$F(s) = 2L(t) - L(t^2) - 2e^{-s} L(t) + e^{-s} L(t^2) + e^{-s} L(2)$$

$$\boxed{F(s) = \frac{2}{s^2} - \frac{2}{s^3} - \frac{2e^{-s}}{s^2} + \frac{e^{-s}}{s^3} + \frac{2e^{-s}}{s}}$$

5. (10 pts.) Consider the second order IVP

$$L[y] = y'' + 3y' + 2y = e^{-2t}, \quad y(0) = 1,\ y'(0) = -2.$$

Use the method of *Laplace transforms* to find the unique solution, $y(t)$.

$$L(y'') + 3L(y') + 2L(y) = L(e^{-2t})$$

$$\left(s^2[Ly] - s\,y(0) - y'(0)\right) + 3\left(s[Ly] - y(0)\right) + 2[Ly] = \frac{1}{s+2}$$

$$s^2[Ly] - s + 2 + 3s[Ly] - 3 + 2[Ly] = \frac{1}{s+2}$$

$$(s^2+3s+2)[\mathcal{L}y] = s+1 + \frac{1}{s+2}$$

$$\mathcal{L}_y = \frac{s+1}{(s+1)(s+2)} + \frac{1}{(s+1)(s+2)^2}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s+2}\right) + \mathcal{L}^{-1}\left(\frac{1}{(s+1)(s+2)^2}\right) = \mathcal{L}^{-1}\left(\frac{1}{s+1}\right) + \mathcal{L}^{-1}\left(\frac{1}{s+2}\right) + \mathcal{L}^{-1}\left(\frac{1}{(s+2)^2}\right) + e^{-2t}$$

$$y = e^{-2t} + e^{-t} - e^{-2t} - te^{-2t}$$

$$\boxed{y = e^{-t} - te^{-2t}}$$

6 (10 pts.) Find all solutions (if any) to the boundary value problem

$$y'' + y = x, \quad y(0) = 0, \quad y(2\pi) = 2\pi.$$

$$m^2 + 1 = 0$$

$$m = \pm i$$

$$y_c = c_1 \cos(x) + c_2 \sin(x)$$

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix} = 1$$

$$u_1 = \int \frac{-\sin(x)(x)}{1} = x\cos(x) - \sin(x)$$

$$u_2 = \int \frac{\cos(x)(x)}{1} = x\sin(x) + \cos(x)$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$y_p = x\cos^2(x) + x\sin^2(x)$$

$$y_p = x\left(\cos^2(x) + \sin^2(x)\right)^1 = x$$

$$y = c_1\cos(x) + c_2\sin(x) + x$$

$$0 = c_1\cos(0) + c_2\sin(0) + 0$$

$$c_1 = 0$$

$$2\pi = c_2\sin(2\pi) + 2\pi$$

$$c_2 = 0$$

$$2\pi = 2\pi$$

$$\boxed{y = x + c_2\sin(x)}$$

7 (12 pts.) Consider the following initial-boundary value problem for the heat equation. obtain a solution by using the procedure outlined below.

$$\begin{aligned}
&\text{PDE: } u_t = 5u_{xx}, && 0 < x < 2,\ t > 0\\
&\text{BC: } u_x(0,t) = 0, && t > 0\\
&\text{BC: } u_x(2,t) = 0, && t > 0\\
&\text{IC: } u(x,0) = 2\cos(\pi x) + 5\cos\left(\tfrac{3}{2}\pi x\right), && 0 < x < 2
\end{aligned}$$

A four step solution method is outlined below. The results for the first two steps are given. Use these results to complete the solution - i.e. solve parts 7 (c) and 7 (d).

7 (a) The method of separation of variables, $u(x,t) = X(x) \cdot T(t)$, obtains two ordinary differential equations for $X(x)$ and $T(t)$.

Solution: The ODEs are $X'' + \lambda X = 0$ and $T' + 5\lambda T = 0$.

$$X'' + \lambda X = 0$$
$$T' + 5\lambda T = 0$$

7 (b) By applying the boundary conditions one obtains an eigenvalue problem. The solution of the eigenvalue problem is given below.

Solution:

$$\lambda_n = \left(\frac{n\pi}{2}\right)^2, \qquad n = 0,1,2,3,\cdots$$
$$X_n = c_n\cos\left(\frac{n\pi}{2}x\right), \quad n = 0,1,2,3,\cdots$$

$$\lambda_n = \left(\frac{n\pi}{2}\right)^2 \qquad X_n = c_n\cos\left(\frac{n\pi}{2}x\right)$$

7 (c) For each eigenvalue, solve the corresponding differential equation for $T(t)$. For each eigenvalue, give the corresponding solution to this wave equation. .

$$T' + 5T\lambda_n = 0$$
$$\frac{T'}{T} = -5\lambda_n$$

$$\ln(T) = -5\lambda_n t + \ln(c) \qquad \lambda_n = \left(\frac{n\pi}{2}\right)^2$$

$$\boxed{T = e^{-5\left(\frac{n\pi}{2}\right)^2 t}}$$

7 (d) Set up the formal series solution, u(x,t), to the initial-boundary value problem. Apply the initial conditions to determine the coefficients of the series. Write the complete solution to the initial-boundary value problem.

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-5\left(\frac{n\pi}{2}\right)^2 t} \cos\left(\frac{n\pi x}{2}\right)$$

$$u(x,0) = 2\cos(\pi x) + 5\cos\left(\frac{3}{2}\pi x\right)$$

$$a_n = \frac{2}{2}\int_0^2 \left[2\cos(\pi x) + 5\cos\left(\frac{3\pi}{2}x\right)\right]\cos\left(\frac{n\pi x}{2}\right)dx$$

$$a_n = 2\int_0^2 \cos(\pi x)\cos\left(\frac{n\pi x}{2}\right)dx + 5\int_0^2 \cos\left(\frac{3\pi}{2}x\right)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$a_n = \frac{4n\sin(n\pi)}{\pi(n^2-4)} - \frac{10n\sin(\pi n)}{\pi(n^2-9)}$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \left[\frac{4n\sin(n\pi)}{\pi(n^2-4)} - \frac{10n\sin(\pi n)}{\pi(n^2-9)} \right] e^{-5\left(\frac{n\pi}{2}\right)^2 t} \cos\left(\frac{n\pi x}{2}\right)}$$

8. Consider the function.

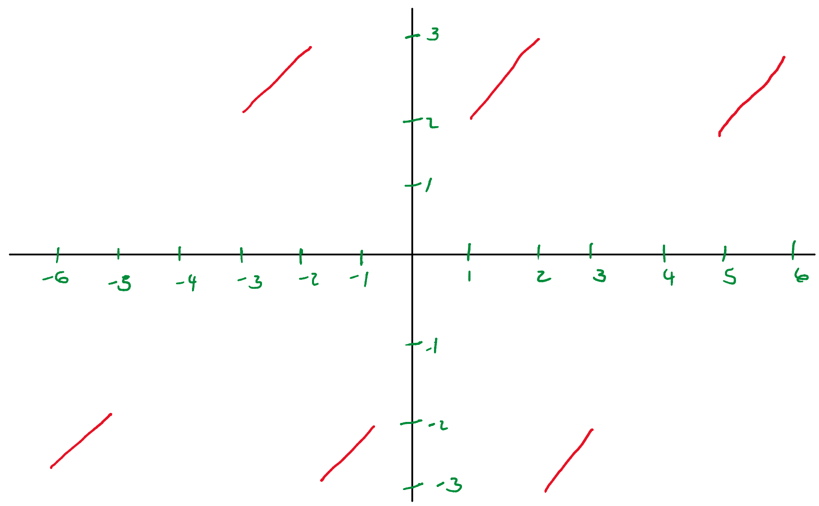
$$f(x) = \begin{cases} 0, & 0 < x < 1 \\ 1+x, & 1 \le x < 2 \end{cases}$$

8(a) (6 pts) Sketch the graph, on the interval [-6,6], of the function to which the half range Fourier sine series of f converges. What is the period of this function?

$$f(x) = \begin{cases} -(1-x) \ ; & -2 < x < -1 \\ 0 \ ; & -1 < x < 0 \\ 0 \ ; & 0 < x < 1 \\ 1+x \ ; & 1 < x < 2 \end{cases}$$

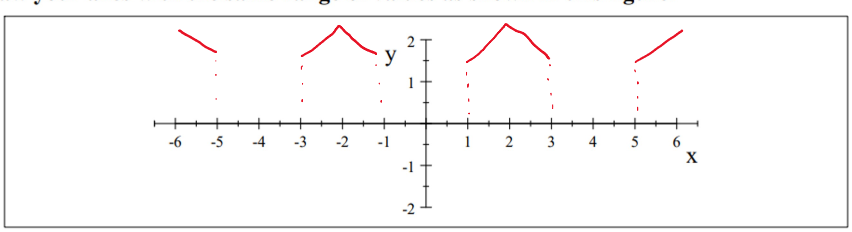
$2-(-2) = 4$

Period = 4



8(b) (6 pts) Sketch the graph, on the interval [-6,6], of the function to which the half range Fourier cosine series of f converges. What is the period of this function?

Draw your axes with the same range of values as shown in this figure.



$$f(x) = \begin{cases} -(1-x) \ ; & -2 < x < -1 \\ 0 \ ; & -1 < x < 0 \\ 0 \ ; & 0 < x < 1 \\ 1+x \ ; & 1 < x < 2 \end{cases}$$

$2-(-2) = 4$

Period = 4

9 (12 pts) Consider the eigenvalue problem

$$y'' + 4y' + (4 + \lambda)y = 0$$

$$y(0) = y(2) = 0.$$

Find all eigenvalues and the corresponding eigenfunctions for the case of complex roots of the auxiliary equation.

$$m^2 + 4m + (4+\lambda) = 0$$

$$m = \frac{-4 \pm \sqrt{4^2 - 4(1)(4+\lambda)}}{2(1)}$$

$$m = -4 \pm \sqrt{16 - 16 - 4\lambda}$$

$$m = \frac{-4 \pm \sqrt{-4\lambda}}{2} \qquad \theta^2 = \lambda$$

$$m = -2 \pm i\theta$$

$$y = e^{-2x}(c_1 \cos(\theta x) + c_2 \sin(\theta x))$$

$$y(0) = 0$$

$$e^0 (c_1 \cos(0) + c_2 \sin(0)) = 0$$

$$c_1 = 0$$

$$y(2) = 0$$

$$e^{-4} \quad c_2 \sin(2\theta) = 0 = \sin(n\pi)$$

$$\sin(2\theta) = \sin(n\pi)$$

$$2\theta = 2\pi$$

$$\theta = \frac{n\pi}{2}$$

$$\boxed{\lambda_n = \frac{n^2\pi^2}{4} \qquad y_n = k_n e^{-2x} \sin\left(\frac{n\pi x}{2}\right)}$$