Homework 3

Tuesday, September 22, 2020 5:43 PM

Name: Alex Gaskins Collaborators: John Oliveira, Trevor Dameika

1. Solve the initial value problem, $(1/x + 2y^2x)\,dx + (2yx^2 - \cos y)\,dy = 0$, with $y(1) = \pi$.

$M = (1/x + 2y^2x)$ $N = (2yx^2 - \cos(y))$

$M' = 4xy$ $N' = 4xy$

$M = \frac{dF}{dx}$

$\int (2yx^2 - \cos(y)) + h(x) = f(x,y)$

$y^2x^2 - \sin(y) + h(x) = f(x,y)$

$\frac{\partial}{\partial x}\left(y^2x^2 - \sin(y) + h(x)\right) = \frac{dF}{dx}$

$2y^2x + h'(x) = \frac{dF}{dx}$

$\cancel{2y^2x} + h'(x) = \frac{1}{x} + \cancel{2y^2x}$

$h'(x) = \frac{1}{x}$

$y^2x^2 - \sin(y) + \ln|x| = f(x,y) = C$

$y(1) = \pi$

$(\pi)^2(1)^2 - \sin(\pi) + \ln|1| = C$

$C = \pi^2$

$$\boxed{y^2x^2 - \sin(y) + \ln|x| = \pi^2}$$

2. For the first-order ODE, $(2y^2 - 3xy)\,dx + (4xy - 3x^2)\,dy = 0$, find an *integrating factor* of the form $\mu(x, y) = x^n y^m$ and solve the transformed equation.

$(\underbrace{2x^n y^{m+2} - 3x^{n+1}y^{m+1}}_{M})dx + (\underbrace{4x^{n+1}y^{m+1} - 3x^{n+2}y^m}_{N})dy = 0$

$\frac{\partial M}{\partial y} = 2(m+2)x^n y^{m+1} - 3(m+1)x^{n+1}y^m$

$\partial N = 4(n+1)x^n y^{m+1} - 3(n+2)x^{n+1}y^m$

a)

$$\frac{\partial N}{\partial x} = 4(n+1)x^n y^{m+1} - 3(n+2)x^{n+1}y^m$$

$$2(m+2) = 4(n+1)$$

$$3(m+1) = 3(n+2)$$

$$m = 2 \qquad ; \qquad M = xy^2$$
$$n = 1$$

$$(2xy^2 - 3x^2y^3)\,dx + (4x^2y^3 - 3x^3y^2)\,dy = 0$$

$$\frac{\partial}{\partial x} = 2xy^4 - 3x^2y^3$$

$$F = \int 2xy^4 - 3x^2y^3\,dx$$

$$F = x^2y^4 - x^3y^3 + h(y)$$

$$\frac{\partial}{\partial y} = \cancel{4x^2y^3} - \cancel{3x^3y^2} + h'(y) = \cancel{4x^2y^3} - \cancel{3x^3y^2}$$

$$h'(y) = 0$$

$$\boxed{x^2y^4 - x^3y^3 = C}$$

3. Consider the initial value problem: $2x^2y'' + 6xy' + 2y = 0, \quad y(1) = 2,\ y'(1) = -1.$

(a) Verify that the set of functions, $\{x^{-1}, x^{-1}\ln(x)\}$ form a *fundamental set of solutions* on $(0, \infty)$.

(b) Determine the unique solution to the initial value problem.

$$y_1(x) = x^{-1}; \quad y_1'(x) = -x^{-2}; \quad y_1''(x) = 2x^{-3}$$

$$2x^2(2x^{-3}) + 6x(-x^{-2}) + 2(x^{-1}) = 0$$

$$4x^{-1} - 6x^{-1} + 2x^{-1} = 0 \quad \checkmark$$

$$y_2(x) = x^{-1}\ln(x); \quad y_2'(x) = x^{-2}(1 - \ln(x)); \quad y_2'' = x^{-3}(2\ln(x) - 3)$$

$$2x^2(x^{-3}(2\ln(x) - 3)) + 6x(x^{-2}(1 - \ln(x))) + 2(x^{-1}\ln(x)) = 0$$

$$4x^{-1}\ln(x) - 6x^{-1} + 6x^{-1} - 6x^{-1}\ln(x) + 2x^{-1}\ln(x) = 0 \quad \checkmark$$

$$4x^{-1}\ln(x) - 6x^{-1} + 6x^{-1} - 6x^{-1}\ln(x) + 2x^{-1}\ln(x) = 0 \checkmark$$

$$W = \begin{vmatrix} x^{-1} & x^{-1}\ln(x) \\ -x^{-2} & -x^{-2}(\ln(x)-1) \end{vmatrix}$$

$$-x^{-2}\left(x^{-1}\ln(x)\right) + x^{-1}\left(-x^{-2}\left(\ln(x)-1\right)\right)$$

$$x^{-3} = x^{-3}$$

<u>Solutions:</u> $\left\{ x^{-1},\ x^{-1}(\ln(x)) \right\}$

$$y = c_1 x^{-1} + c_2 x^{-1}(\ln(x)) \ ; \quad y(1) = 2$$

$$(2) = c_1(1)^{-1} + c_2(1)^{-1}(\ln(1))$$

$$2 = c_1 + c_2(0)$$

$$c_1 = 2$$

$$y' = -c_1 x^{-2} - c_2 x^{-2}(\ln(x)-1) ; y'(1) = -1$$

$$(-1) = -c_1(1)^{-2} - c_2(1)^{-2}(\ln(1)-1)$$

$$-1 = -c_1 + c_2$$

$$c_2 = 1$$

$$\boxed{y = 2x^{-1} + x^{-1}(\ln(x))}$$