EE 575, Fall 2022
Introduction to Control Theory

Midterm Exam

Close book; ONE letter-size double-sided cheat sheet

You will be penalized by 5 points if you do NOT submit the cheat sheet.

Name: Alex Gaskins

Student ID: 10445143

Please Sign Pledge to Honor Code here:

Problem #	Maximum Points	Grading
1.1	0.5	
1.2	0.5	
1.3	0.5	
1.4	0.5	
1.5	0.5	2.5
1.6	0.5	
1.7	0.5	
1.8	0.5	
2.1	1	1
2.2	1	
3	4	3.25
4	5	4.5
5	5	4
6	5	4.25
7	5	4.75
Total	**30**	

24.25

Section 1: True or False (0.5 points each, 8 questions, 4 points total)

1.1. For a linear time-invariant system with the transfer function $H(s)$, for the input of e^{at}, the corresponding system output is $H(a) * e^{at}$.

(a) True
(b) False

Can be proved formulaically

1.2. On the root locus plot, the number of separate loci is equal to the number of zeros.

(a) True
(b) False

1.3. For a linear time-invariant system, if the system output is $y(t)$ with the input of $u(t)$, then the system output is $y(t-a)$ with the input of $u(t-a)$.

(a) True
(b) False

1.4. The system with the characteristic equation $s^7 + 8s^6 + 8s^5 - s^4 + 100s^3 + 100000s^2 + 9999s + 8976 = 0$ is stable.

(a) True
(b) False

1.5. Lead compensator approximates the PD control to improve the steady-state accuracy.

Not for

(a) True
(b) False

1.6. The transfer function $T(s) = \frac{(s+2)(s-1)}{s^2(s+10)(s^2+6s+25)}$ describes a minimum-phase system.

(a) True
(b) False

1.7. A state-space representation of a system can always be written in a strict diagonal form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

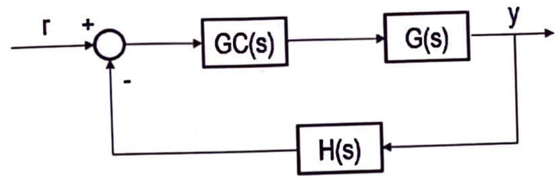
(a) True
(b) False

1.8. According to the final value theorem $\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s)$, for a system with the dynamic output of $Y(s) = \frac{3(s+2)}{s(s+4)(s-3)}$, the final steady-state output of the system is equal to $\frac{3*2}{4*(-3)} = -\frac{1}{2}$.

(a) True
(b) False

Section 2: Single Choice (1 point each, 2 questions, 2 points total)

2.1. Consider the system in the below figure with $GC(s) = 10$, $H(s) = 1$, and $G(s) = \frac{s+50}{s^2+60s+500}$. With a unit step input, the final output y_{ss} is



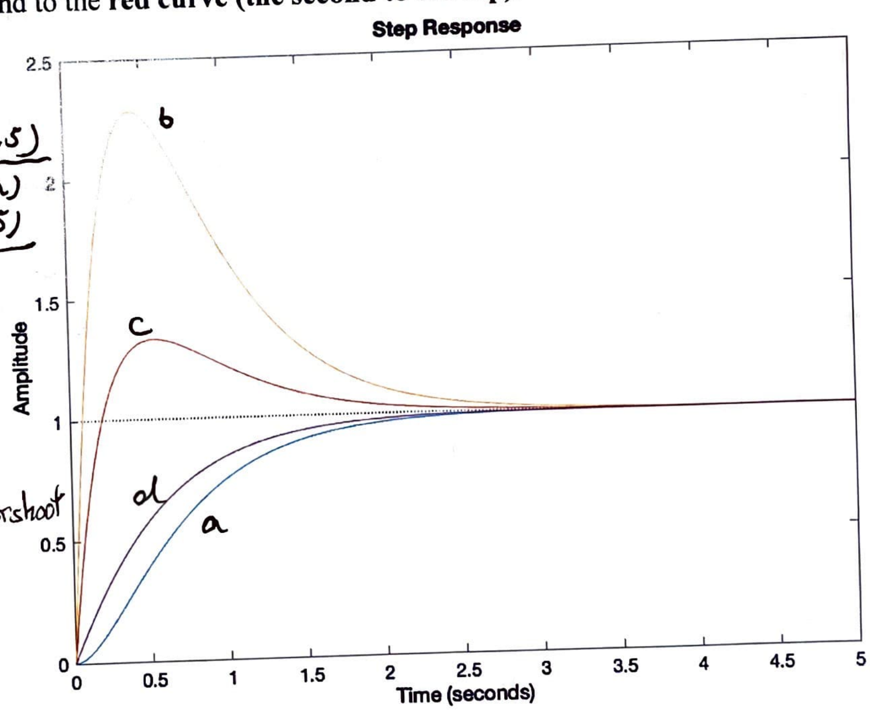
$$TF = \frac{GC(s)G(s)}{1+GC(s)G(s)(H(s))} = \frac{10s+500}{s^2+70s+1000}$$

(a) 100
(b) 1
(c) 50
(d) None of the above.

$\lim_{t \to 0}$

2.2 Below figure shows the time domain step responses of the following three transfer functions. Please choose which transfer function could correspond to the **red curve (the second to the top).**

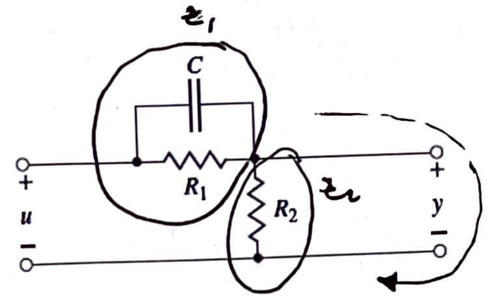
(a) ~~$G1(s) = \frac{8}{(s^2+6s+8)}$~~
(b) ~~$G2(s) = \frac{8(s+1)}{(s^2+6s+8)}$~~
(c) $G3(s) = \frac{16(s+0.5)}{(s^2+6s+8)} = \frac{8(s+0.5)}{0.5(m)}$
(d) ~~$G4(s) = \frac{1.6(s+5)}{(s^2+6s+8)}$~~ $\frac{8(s+5)}{5(m)}$



set equal to 0,
bigger s = bigger overshoot

Section 3: Computations

3. Transfer Function of Analog Circuit (4 points)
Calculate the transfer function T(s)=Y(s)/U(s) of the below system, and determine whether it is a lead or lag compensator.



$$T(s) = Y(s)/U(s) =$$

$z_1 \parallel z_2$ so $z_{\parallel} = \dfrac{z_2}{z_2 + z_1}$

$$z_1 = \frac{R_1}{R_1 sC + 1} \qquad z_2 = R_2$$

Voltage Division:

$$y = u \frac{z_2}{z_2 + z_1}$$

$$TF = \frac{Y}{u} = \boxed{\frac{R_2}{R_2 + sC + 1}}$$

check.

lead / lag?

4. Routh's Criteria for Stability Analysis (5 points)

The closed-loop transfer function of a system is

$$T(s) = \frac{18}{s^5 + 2s^4 + 5s^3 + 6s^2 + 11s + 2}$$

Use **Routh's Criteria** to determine the stability of the system. If it is unstable, how many closed-loop poles are located in the right half-plane?

s^5	1	5	11
s^4	2	6	2
s^3	4 2	10	0
s^2	1	2	0
s	2	0	
s^0	2		

$-\det\begin{bmatrix}1 & 11\\ 2 & 2\end{bmatrix}/2 = -\frac{[2-22]}{2} = 10$ ✓

$-\det\begin{bmatrix}1 & 5\\ 2 & 6\end{bmatrix}/2 = -\frac{[6-10]}{2} = \cancel{4}\ 2$

$-\det\begin{bmatrix}1 & 2\\ 2 & 0\end{bmatrix}/2 = 2$

$-\det\begin{bmatrix}2 & 6\\ 4 & 10\end{bmatrix}/4 = -\frac{[20-24]}{4} = 1$

$-\det\begin{bmatrix}2 & 2\\ 4 & 0\end{bmatrix}/4 = -\frac{[0-8]}{4} = 2$

$-\det\begin{bmatrix}4 & 10\\ 1 & 2\end{bmatrix}/1 = -[8-10] = 2$

$-\det\begin{bmatrix}4 & 0\\ 1 & 0\end{bmatrix}/1 = -\frac{[0-0]}{1} = 0$

First Column is all positive; system is stable.

$$GD = \frac{K(s+10)}{(4s^2+5s+1)(s+2)} = \frac{K(s+10)}{4s^3+8s^2+5s^2+10s+1s+2}$$

5. System Type and Stability (5 points)

Given a second-order system $G = \frac{1}{(s+1)(4s+1)}$, we add $D = \frac{K(s+10)}{s+2}$ in series with G(s) in a unity feedback structure.

a. Determine the system type, and the corresponding finite constant error.

b. What are the limits on K so that the system is stable? (Must be positive) > 0

A.) Characteristic Equation: $(s+2)(s+1)(4s+1)=0$

No poles at the origin; system is $\boxed{\text{Type } 0}$

$$\text{Error} = \frac{1}{1+GD} = \frac{1}{1+\left[\frac{K(s+10)}{4s^3+13s^2+11s+2}\right]} = \frac{4s^3+13s^2+11s+2}{4s^3+13s^2+[11+K]s+[10K+2]}$$

$\lim_{t\to\infty} e(t)$

B.)

$$\begin{array}{c|cc} s^3 & 4 & [11+K] \\ s^2 & 13 & [10K+2] \\ s & \frac{-27K+124}{13} & \end{array}$$

s^0 ?

$$-\det\begin{bmatrix} 4 & [11+K] \\ 13 & [10K+2] \end{bmatrix} \Big/ 13 = -\frac{\left[\begin{array}{c}[40K+8]-[13K+13K] \\ + \\ 27K-124\end{array}\right]}{13}$$

$4 \times [10K+2]$

$-$

$13 \times [10K+2]$

$$\frac{-27K+124}{13}$$

$$\frac{-27K+124}{13} > 0$$

$$-27K > -124$$

$$K < \frac{124}{27}$$

$$\boxed{0 < K < \frac{124}{27}}$$

$[P_1 P_2]^2 + [-P_1^2 P_2 - P_1 P_2^2]$

Control Canonical:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -P_1P_2 & -[P_1+P_2]\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}$$

$$y = \begin{bmatrix}z & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + [0]u$$

6. State-Space Representation, Controllability, Observability (5 points)

Consider the following transfer function $\frac{s+z}{(s+p_1)(s+p_2)}$,

a. Write down its observer canonical form.

b. If z, p_1, and p_2 are all negative, and z≠ p_1 and z≠ p_2, prove that the [redacted] canonical form in (a) is both controllable and observable.

observer

A.) $\frac{s+z}{(s+p_1)(s+p_2)} = \frac{1s + 1z}{(s+p_1)(s+p_2)} = \frac{\dot{x}_1 + z x_1 = y(t)}{\ddot{x}_1 + [P_1+P_2]\dot{x}_1 + P_1P_2 x_1 = u(t)}$

$s^2 + [P_1 + P_2]s + P_1P_2$

$x_2 = \dot{x}_1$

$u(t) = \dot{x}_2 + [P_1+P_2]x_2 + P_1P_2x_1 = u(t)$

$y(t) = x_2 + z x_1$

$\dot{x}_2 = -[P_1+P_2]x_2 - P_1P_2x_1 + u(t)$

$$y = \begin{bmatrix}z & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + [0]u$$

Observer:

B.) $$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & -P_1P_2\\ 1 & -[P_1+P_2]\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u$$

~~Control Canonical~~

Controllable: $C = [B \; AB \dots A^{n-1}B] \neq 0$

Observable: $\mathcal{O} = \begin{bmatrix}C\\ CA\\ CA^{n-1}\end{bmatrix} \neq 0$

$A = \begin{bmatrix}0 & -P_1P_2\\ 1 & [P_1+P_2]\end{bmatrix}$ $B = \begin{bmatrix}0\\ 1\end{bmatrix}$ $C = [B \; AB]$

$AB = \begin{matrix}0 + 0 + 0 - P_1P_2\\ 0 + 0 + 1 + P_1 + P_2\end{matrix} = \begin{bmatrix}-P_1P_2\\ 1+P_1+P_2\end{bmatrix}$

$C = \begin{bmatrix}0 & -P_1P_2\\ 1 & [1+P_1+P_2]\end{bmatrix} \neq 0$

$0 + P_1P_2 \neq 0$ ✓ ∴ Controllable

$\mathcal{O} = \begin{bmatrix}C\\ CA\end{bmatrix} \neq 0$ $CA = \begin{bmatrix}0 & -P_1P_2\\ 1 & [1+P_1+P_2]\end{bmatrix} \times \begin{bmatrix}0 & -P_1P_2\\ 1 & [P_1+P_2]\end{bmatrix} = \begin{bmatrix}-P_1P_2 & [P_1P_2]^2 + [-P_1^2P_2 - P_1P_2^2]\\ [1+P_1+P_2] & \end{bmatrix}$

$C = \begin{bmatrix}0 & -P_1P_2\\ 1 & [1+P_1+P_2]\end{bmatrix}$ $(-P_1P_2 + [P_1+P_2] - P_1P_2 - P_1^2P_2 - P_1P_2^2 + ([1+P_1+P_2] \times [P_1+P_2]))$

Not enough 0 products and terms to cancel out

∴ Observable

$$CA = C \doteq [z \;\; 1] \begin{bmatrix} 0 & -p_1p_2 \\ 1 & [p_1+p_2] \end{bmatrix} = \begin{bmatrix} z & 1 \\ -zp_1p_2+z[p_1+p_2] & \end{bmatrix}$$

$$[p_1+p_2-p_1p_2]$$

$$\begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} z & 1 \\ z & 1 \\ -zp_1p_2+z[p_1+p_2] & \end{bmatrix} \neq 0$$

[

7. **State-Space Transfer Function and Tracking Performance (5 points)**

Consider the single-input single-output system described via the following dynamic equations (with u as input and y as output)

$$\ddot{x}_1 + 2\dot{x}_1 + 4x_1 = Ku; \;\; y = x_1$$

a. Write down the system **state-space control canonical form** with $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ as state variables, where x_2 is defined as $x_2 = \dot{x}_1$.
b. Calculate the transfer function.
c. Determine the value of K resulting in a zero steady-state tracking error with a unit step input. The tracking error is defined as $e(t) = u(t) - y(t)$.

A.) $x_2 = \dot{x}_1$

$\dot{x}_2 + 2x_2 + 4x_1 = Ku$

$y = x_1$

$\dot{x}_2 = -2x_2 - 4x_1 + Ku$

$$\boxed{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ K \end{bmatrix} u}$$

$$\boxed{y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0]u}$$

B.) $TF = \dfrac{Y(s)}{U(s)} = \dfrac{\mathcal{L}(y)}{\mathcal{L}(u)}$ Inverse Laplace

$x_1^{(2)} + 2x_1^{(1)} + 4x_1 = KU$ $\qquad y = x_1$

$\approx s^2 + 2s + 4$ $\qquad \approx 1$

$$\boxed{TF = \frac{K}{s^2+2s+4}}$$

C.) $\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \left[s \dfrac{s^n}{s^n + P(s)} \dfrac{1}{s^{k+1}} \right]$

$\cancel{1 + \dfrac{K}{s^2+2s+4}}$

$Y(s) = \dfrac{K\left[\frac{1}{s}\right]}{U(s)\; s^2+2s+4}$ Unit-step

$\cancel{\lim_{s\to 0} s\left[\frac{1}{s}\frac{K}{(s^2+2s+4)}\right]}$

$\left[\frac{1}{s}(s^2+2s+4) - K\right] = E(s);\;\; \lim_{s\to 0} sE(s) = s\left[\frac{1}{s}(s^2+2s+4) - K\right] = 4 - K$

$4 - K = 0$

$$\boxed{K = 4}$$