Homework 2

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1. Consider the first-order ODE, $\dfrac{dy}{dx} = \dfrac{3x^2 + 2x - 4}{2y - 4}$.

   (a) Find a one parameter family of solutions to the ODE. Express the family of solutions in ***implicit*** form.

   (b) Find the unique solution satisfying the initial condition $y(2) = 0$.
   Express this solution in ***explicit*** form, $y = \phi(x)$, and determine its *existence interval*.

a.) $(2y - 4)\,dy = (3x^2 + 2x - 4)\,dx$

$y^2 - 4y = x^3 + x^2 - 4x + C$

$y = 2 \pm (x^3 + x^2 - 4x + C)$

b.) $0 = (2)^3 + (2)^2 - 4(2) + C$

$C = -4$

$y^2 - 4y - x^3 - x^2 + 4x - 4 = 0$

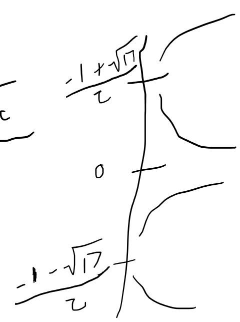
$\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x^3 + x^2 - 4x = 0$

$x(x^2 + x - 4) = 0$

$x = 0 \quad x = \dfrac{-1 \pm \sqrt{17}}{2}$

$I_E: \left(\dfrac{-1 - \sqrt{17}}{2}, 0\right)$



2. A mathematical model for a falling chain in the absence of resistive forces is given by the ODE,

$$xv\frac{dv}{dx} + v^2 = gx.$$

Here $x > 0$ is the length of the chain hanging over the edge of the platform and v is the velocity of the chain. (x is in units of feet (ft), v is in ft/s, and $g = 32\,\text{ft/s}^2$ is the gravitational acceleration.)

   (a) Find a one-parameter family of solutions using the fact that the ODE is a Bernoulli equation.

   (b) With 3 feet of the chain hanging over the edge, the chain is falling at a rate of 2 ft/sec. Determine the speed of the falling chain at the point when its length is 6 feet.

a.) $\dfrac{dv}{dx} + \dfrac{1}{x}v = \dfrac{g}{v}$ $\quad u = v^{1-n}$ $\quad n = -1$

$u = v^2$ $\quad v = u^{1/2}$ $\quad dv/du = \frac{1}{2}u^{-1/2}$

$\dfrac{\frac{1}{2}u^{-1/2}\frac{du}{dx} + \frac{1}{x}u^{\frac{1}{2}} = g^{-1/2}}{\frac{1}{2}u^{-1/2}}$ $\quad \dfrac{dv}{dx} = \dfrac{dv}{du}\dfrac{du}{dx} = \frac{1}{2}u^{-1/2}\dfrac{du}{dx}$

$\dfrac{du}{dx} + \dfrac{2}{x}u = 2g \quad \mu \quad \mu\dfrac{2}{x} = \dfrac{d\mu}{dx}$ $\quad 2\ln(x) = \ln(\mu)$

$\mu = x^2$

$-\frac{1}{2}u^{-2}$ $\quad du$

$\frac{du}{dx} + \frac{2}{x}u = 2g\,M$

$M\frac{du}{dx} + M\frac{2}{x}u = M2g \longrightarrow M\frac{2}{x} = \frac{dM}{dx}$ $\qquad 2\ln(x) = \ln(M)$

$M = x^2$

$\int \frac{2}{x}\,dx = \int \frac{1}{M}\,du$

$\int d\,Mu = \int 2Mg\,dx$

$x^2 u = \frac{2}{3}x^3 g + C \qquad v = \pm\sqrt{\frac{2}{3}xg + \frac{C}{x^2}}$

b.) $v = \sqrt{\frac{2}{3}xg + \frac{C}{x^2}}$ $\qquad v = \sqrt{\frac{2}{3}(6)g - \frac{540}{(6)^2}}$

$2 = \sqrt{2g + \frac{C}{9}}$ $\qquad v = \sqrt{4g - 15}$ $\qquad v = 10.63\ ft/s$

$9(4 - 2g) = C$

$C = -540$ $\qquad v = \sqrt{113}$

3. Consider the following example of the *logistic* equation. This equation is used as a simple model for the growth rate of a single species population, $P(t)$, that includes competition for limited resources.

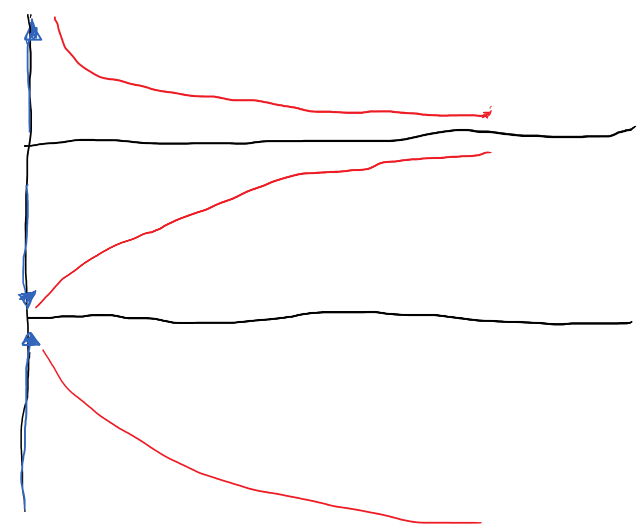
$$\frac{dP}{dt} = \frac{1}{2}P(4 - P)$$

(a) Use phase line analysis to identify the *asymptotically stable equilibrium solutions*.

(b) Use the method for Bernoulli equations to find the general solution for $P(t)$. Determine $\lim_{t\to+\infty} P(t)$ for positive initial conditions, $P(0) > 0$, and compare this with your phase line analysis in part (a).

a.) $0 = \frac{1}{2}P(4 - P)$

$P = 0 \quad P = 4$



b.) $\frac{dP}{dt} - 2P = -\frac{P^2}{2}$

$u = P^{1-n}$ $\qquad \frac{dP}{du} = -u^{-2}$

$= P^{1-2} = P^{-1}$

$\frac{dP}{dt} = \frac{dP}{du}\frac{du}{dt} = -u^{-2}\frac{du}{dt}$

$$\frac{dp}{dt} = \frac{dp}{du}\,\frac{du}{dt} = -u \quad dt$$

$$-u^{-2}\frac{du}{dt} - 2u^{-1} = -\frac{1}{2}u^{-2}$$

$$\frac{du}{dt} + 2u = \frac{1}{2} \longrightarrow M\left(\frac{du}{dt} + 2u\right) = \frac{1}{2}\cdot M$$

$$M\frac{du}{dt} + 2Mu = \frac{1}{2}M$$

$$2M = \frac{2M}{dt} \qquad \int 2\,dt = \int \frac{dM}{M} \qquad 2t = \ln M \qquad M = e^{2t}$$

$$\int d\,Mu = \int \frac{1}{2}M\,dt$$

$$Mu = \int \frac{1}{2}M\,dt$$

$$e^{2t}u = \int \frac{1}{2}e^{2t}\,dt$$

$$e^{2t}u = \frac{1}{4}e^{2t} + c \quad = \quad u = \frac{1}{4} + \frac{c}{e^{2t}}$$

$$p^{-1} = \frac{1}{4} + ce^{-2t} \qquad \lim_{t\to\infty}\frac{1}{\frac{1}{4}+ce^{-2t}} = 4$$

$$p = \frac{1}{\frac{1}{4} + ce^{-2t}}$$