Final Value Theorem: (Poles in left) (all)

$$y_{ss} = \left[\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)\right]$$

Stability: All coefficients of characteristic polynomial MUST be positive.

Characteristic Eqn: Transfer function polynomial in denominator $= 0$ $(s^n + a_1 s^{n-1} \dots = 0)$

Effects of Zeros: - Zeros modify coefficients of the exponential terms whose shape is decided by poles
- Zeros increase overshoot and reduce rise time, but has very little influence on settling time.

$$H(s) = \frac{24(s+z)}{z(s+4)(s+6)}$$

- All coeffs positive
- stable if first col is +

Routh's Stab. Criterion:

s^n: 1 a_2 a_4 →

s^{n-1}: a_1 a_3 a_5 →

s^{n-2}: $\dfrac{-\det\begin{bmatrix}1 & a_2\\ a_1 & a_3\end{bmatrix}}{a_1}$ $\dfrac{-\det\begin{bmatrix}1 & a_4\\ a_1 & a_5\end{bmatrix}}{a_1}$

$\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ ↓ $ad-bc$

[redacted] in Routh's Stab. Criterion:

Roots

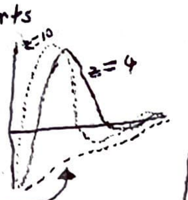
- Number of roots is equal to # of sign changes in col 1.

Poles and Zeros:

- LTI system is stable if all poles have negative real parts

$$\frac{24z}{(s^2+4s+24)(s+z)}$$

$$\frac{24}{(s^2+4s+24)}$$



Laplace Transforms

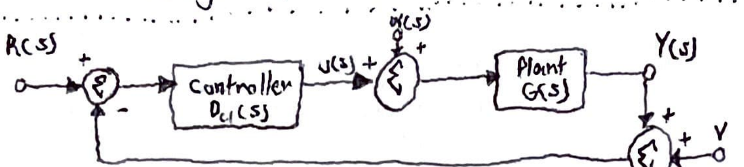
[redacted] **Fundamentals of Control Systems:**

Stability: All poles of the transfer function MUST be in LH of plane.

Tracking: Make the system's output follow reference input as close as possible.

Regulation: Keep the error small when the reference is at most a constant setpoint and disturbances are present.

Sensitivity: the relative change of transfer function with respect to the relative change of plant parameters



[Closed-Loop Sys.]

Matrix Multiply:

$$\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix} \times \begin{bmatrix}7 & 8\\ 9 & 10\\ 11 & 12\end{bmatrix}$$

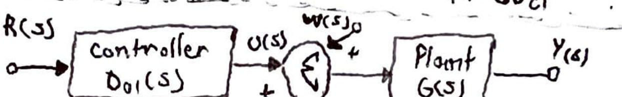
Row × Col 5

$$\begin{bmatrix}58 & \\ & \end{bmatrix}$$

- **Output:** $Y_{cl} = \dfrac{GD_{cl}}{1+GD_{cl}} R + \dfrac{G}{1+GD_{cl}} W - \dfrac{GD_{cl}}{1+GD_{cl}} V$

- **Error:** $E_{cl} = \dfrac{1}{1+GD_{cl}} R - \dfrac{G}{1+GD_{cl}} W + \dfrac{GD_{cl}}{1+GD_{cl}} V$

- **Transfer Function:** $T_{cl} = \dfrac{GD_{cl}}{1+GD_{cl}}$



[Open-Loop System] or optimize

- **Output:** $Y_{ol} = GD_{ol}R + GW$ **Error:** $E_{ol} = R - Y_{ol} = R - GD_{ol}R - GW$

- **Transfer Function:** $T_{ol} = \dfrac{Y_{ol}}{R} = GD_{ol}$

$G = \dfrac{(s+1)}{(s-1)(s+2)}$ ← Zeros, ← Poles

Open and Closed Loop Objectives:

Stability: - Open-loop system, poles of GD_{ol} includes poles of G and D_{ol}

[we can't turn unstable plant stable] * If any pole of G is in RH of plane, we need a zero of D_{ol} to cancel it out (Not robust to noises)

* If a zero of G in RH of plane shows poor response, cancel it out by a pole of D_{ol}, which induces stability issues

- Closed-loop system, poles determined by $1 + GD_{ol} = 0$

* Having a pole of G in RH plane does not prevent the design of a feedback controller that can make the system stable grant more freedom to control design

Tracking: Open Loop: - can be selected to cancel TF of plant (G) - can't more 0's than poles

Closed-loop: - Could be designed to make zero steady-state error

Regulation: Open-Loop has no disturbance response (to W and V) ∴ useless for regulation

$Y(s) = U(s)H(s)$

- Partial fraction expansion: Cover-Up Method

$$F(s) = \frac{C_1}{s-p_1} + \frac{C_2}{s-p_2} + \dots + \frac{C_n}{s-p_n}$$

$$f(t) = \sum_{i=1}^{n} C_i e^{p_i t}\, 1(t) \quad \text{where}$$

$$C_i = (s-p_i)F(s)\big|_{s=p_i}$$

- Abides by Final Value Theorem
- DC gain: The ratio of the system output to a unit step input $[(\frac{1}{s}) = U(s)]$ after all transients have decayed

System Type:

$$GD_{cl} = \frac{1}{s^n} K \frac{\Pi_{i=1}^{m_i}(s-z_i)}{\Pi_{i=1}^{n_i}(s-p_i)} = \frac{1}{s^n} P(s)$$

- If $n=0$, $\lim e(t) = \dfrac{1}{1+P(0)}$ at $k=0$, and $\to \infty$ for $k>0$

- If $n=1$, $GD_{cl} = \frac{1}{s}P(s)$, $\lim e(t) = \lim s \dfrac{s}{s+P(s)} \dfrac{1}{s^{k+1}}$: $k=0$, $\lim e(t) = 0$; $k=1$, $\lim e(t) = \dfrac{1}{P(0)}$; $k=2^{+}$, $\lim e(t) = \infty$

- If $n=2$, $GD_{cl} = \frac{1}{s^2}(P(s))$, $\lim e = \lim s \dfrac{s^2}{s^2+P(s)} \dfrac{1}{s^{k+1}}$: $k=0,1$, $\lim e(t) = 0$; $k=2$, $\lim e(t) = 1/P(0)$; $k=3^{+}$, $\lim e(t) = \infty$

Error $\displaystyle\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s \frac{s^n}{s^n + P(s)} \frac{1}{s^{k+1}}$

$$G(s)H(s) = \frac{K(s+z_1)(s+z_2)}{s^n(s+p_1)(s+p_2)} \quad \text{where } n = \text{type},\ k = \text{constant}$$

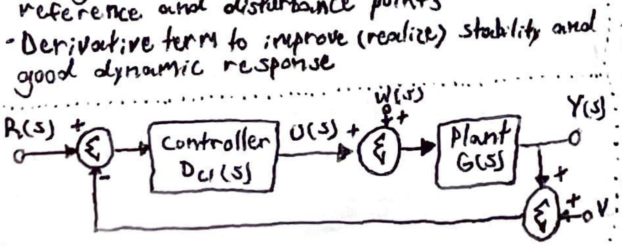
n = number of poles at origin (system type)

order of system = Highest TF exponent

Unity Feedback: Error $= \dfrac{1}{1+GD}$ ☆

PID Control: $D_{cl} = k_p + \dfrac{k_I}{s} + k_D s$

- Proportional term to close feedback loop
- Integral term to assure zero error to constant reference and disturbance points
- Derivative term to improve (realize) stability and good dynamic response



$D_{cl} = k_p$: * k_p is proportional gain * $u(t) = k_p e(t)$, $U(s) = k_p E(s)$

[Example] → * Second-order plant $G = \dfrac{A}{s^2 + a_1 s + a_2}$; Ch. eq. $= s^2 + a_1 s + a_2 + k_p A = 0$

Dampening ratio $= \zeta = \dfrac{a_1}{2\sqrt{a_2 + k_p A}}$ (larger k_p = lower damper, poor trans. rsp.)

$D_{cl} = k_I/s$ minimizes tracking error and improve response to disturbances!

* k_I is integral gain * $u(t) = k_I \int_{t_0}^{t} e(\tau)\, d\tau$, $U(s) = \dfrac{k_I}{s} E(s)$

* second-order plant char eq. $= s(s^2 + a_1 s + a_2) + k_I A = 0$ (Type 1)

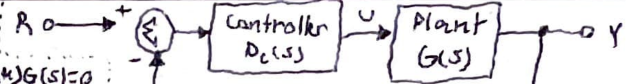
$D_{cl} = k_D s$ improves cl stability, speeds up transient response, and reduces overshoot!

* k_D is derivative gain * $u(t) = k_D \dot{e}(t)$, $U(s) = k_D s E(s)$ * $s^2 + (a_1 + k_D A)s + a_2 = 0$

* Represents anticipatory behavior, but amplifies noise.

* Usually augmented by proportional control, as it does not supply information on the desired end state (e.g. constant error).

Root Locus of Feedback System:



Name: Alex Gaskins

$TF_{cl} = \frac{Y(s)}{R(s)} = \frac{D_c(s,K)\,G(s)}{1+D_c(s)G(s)}$

- Characteristic equation is $1+D_c(s,K)G(s)=0$
- Reorganize to root locus form $1+KL(s)=0$ * Root locus is set of solutions s for $1+KL(s)=0$ when $K\in[0,\infty)$
- Roots on the locus are closed-loop poles of the system * Evaluate how changes in K will change closed-loop poles of the system, thus impacting the system dynamic response.
- K is unknown param that alters poles when changed

Derive Root Locus Form:

[Example]: Characteristic Equation:

$s^3+4s^2+Ks+1=0$

* Group all K terms:

$(s^3+4s^2+1)+Ks=0$

* Divide by non-K terms:

$\frac{(s^3+4s^2+1)+Ks=0}{(s^3+4s^2+1)}$ =

$1+K\frac{s}{s^3+4s^2+1}=0$

Rules for Drawing Root Locus: $[1+KG(s)=0 \rightarrow 1+K\frac{Q(s)}{P(s)}=0]$ (3 pole = 3 lines)

1.) There are n lines (loci) where n is the degree of Q or P, whichever greater.
2.) As K increases from 0 to ∞ the roots move from poles of G(s) to zeros of G(s).
3.) When roots are complex they occur in conjugate pairs $[\,{}^{x}_{x}\,]$
4.) At no time will the same root cross over its path
5.) The portion of the real axis to the left of an odd number of open loop poles and zeros are part of the loci.
6.) Lines leave (breakout) and enter (break in) the real axis a 90°.
7.) If there are not enough poles or zeros to make a pair, extra lines go to/from infinity.

State-Space Description: * Any finite dynamic system can be expressed as a set of first-order ODE's

- Single-input-single-output (SISO) System: $\dot{x}=Ax+Bu$ $y=Cx+Du$ where $x=[x_n\ x_{n+1}\ \dots]$ is sys. state for n^{th} order sys.

*A is n×n system matrix | B is n×1 input vector | C is 1×n output vector | D is a scalar called the direct transmission term

State-Variable Form:

$u-b\dot{p}=m\ddot{p}$ * Need to introduce additional state variables to convert 2nd order to first order

* Define $\dot{p}=v$, we have $\dot{v}+\frac{b}{m}v=\frac{u}{m}$ * u as input * y = p as output * $\begin{bmatrix}p\\v\end{bmatrix}$ sys. state

$\begin{bmatrix}\dot{p}\\\dot{v}\end{bmatrix}=\dot{x}=\begin{bmatrix}0&1\\0&-\frac{b}{m}\end{bmatrix}\begin{bmatrix}p\\v\end{bmatrix}+\begin{bmatrix}0\\\frac{1}{m}\end{bmatrix}u$ $=y=[1\ 0]\begin{bmatrix}p\\v\end{bmatrix}+[0]u$

Control Canonical Form:

$x_n^{(n)}+a_1x_n^{(n-1)}+a_2x_n^{(n-2)}+\dots+a_{n-1}x_n^{(1)}+a_nx_n=u$

$y=b_1x_n^{(n-1)}+b_2x_n^{(n-2)}+\dots+b_{n-1}x_n^{(1)}+b_nx_n$

$\dot{x}_2=x_1$ $\dot{x}_n=x_{n-1}$ | $\dot{x}_1+a_1x_1+a_2x_2+\dots+a_{n-1}x_{n-1}+a_nx_n=u$

$y=b_1x_1+b_2x_2+\dots+b_{n-1}x_{n-1}+b_nx_n$

$\begin{bmatrix}\dot{x}_1\\\dot{x}_2\\\vdots\\\dot{x}_n\end{bmatrix}=\begin{bmatrix}-a_1&-a_2&\dots&-a_n\\1&0&\dots&0\\0&\ddots&\ddots&\vdots\\0&0&1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}+\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}u$

$y=[b_1\ b_2\ \dots\ b_n]\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}+[0]u$

Controllable if
$C=[B\ AB\ \dots\ A^{n-1}B]$
obs. Matrix if det = 0
$\mathcal{O}=\begin{bmatrix}C\\CA\\\vdots\\CA^{n-1}\end{bmatrix}$

Observer Canonical Form [Dual of Control Canonical Form]:

$\begin{bmatrix}\dot{x}_1\\\dot{x}_2\\\vdots\\\dot{x}_n\end{bmatrix}=\begin{bmatrix}-a_1&1&\dots&0\\-a_2&0&\dots&0\\\vdots&&&1\\-a_n&0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}+\begin{bmatrix}b_1\\b_2\\\vdots\\b_n\end{bmatrix}u$

"a_n order or TF order = num of variables"

$x=[1\ 0\dots 0]\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}+[0]u$

"Start w/ first derivative"

Example: $x_2^{(2)}+7x_2^{(1)}+12x_2=u$ $y=x_2^{(1)}+2x_2$ $y=x_1+2x_2$

$\dot{x}_1=-7x_1-12x_2+u$; $\dot{x}_2=x_1$; $G(s)=\frac{Y(s)}{U(s)}=C(sI-A)^{-1}B+D=\frac{1s+2}{s^2+7s+12}$

Control Canonical Form:
$\begin{bmatrix}\dot{x}_1\\\dot{x}_2\end{bmatrix}=\begin{bmatrix}-7&-12\\1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}+\begin{bmatrix}1\\0\end{bmatrix}u$
$y=[1\ 2]\begin{bmatrix}x_1\\x_2\end{bmatrix}+[0]u$

Observer Canonical Form:
$\begin{bmatrix}\dot{x}_1\\\dot{x}_2\end{bmatrix}=\begin{bmatrix}-7&1\\-12&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}+\begin{bmatrix}1\\2\end{bmatrix}u$
$y=[1\ 0]\begin{bmatrix}x_1\\x_2\end{bmatrix}+[0]u$

Dynamic Response from the State Equations:

- ~~Single~~ SISO system $\dot{x}=Ax+Bu$, $y=Cx+Du$

- Laplace Transform:

$sX(s)-x(0)=AX(s)+BU(s)$; $X(s)=(sI-A)^{-1}BU(s)+(sI-A)^{-1}x(0)$

With $x(0)=0$, $Y(s)=CX(s)+DU(s)=C(sI-A)^{-1}BU(s)+DU(s)$

- Transfer Function:

$G(s)=\frac{Y(s)}{U(s)}=C(sI-A)^{-1}B+D$; $G(s)=\frac{\det\begin{bmatrix}sI-A&-B\\C&D\end{bmatrix}}{\det(sI-A)}$

Example:
$A=\begin{bmatrix}1&3\\0&2\end{bmatrix}$; $B=\begin{bmatrix}1\\0\end{bmatrix}$; $C=[1\ 0]$; $D=[0]$

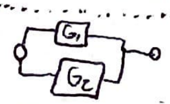
$TF(s)=C(sI-B)^{-1}+D=[1\ 0]\begin{bmatrix}s-1&-3\\0&s-2\end{bmatrix}^{-1}\begin{bmatrix}1\\0\end{bmatrix}$

$TF(s)=\frac{s-2}{s^2-3s+2}=\frac{1}{s-1}$

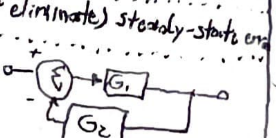
Misc:

- A state variable representation can't always be written in diagonal form
- The state-variables of a system comprise a set of variables that describe the future response of the system.
- A lead compensator can increase speed/stability of response in a system
- A lag compensator can reduce (but not eliminate) steady-state error

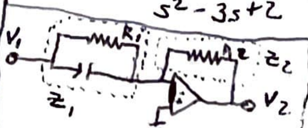
$\frac{Y(s)}{U(s)}=G_1\times G_2$



$\frac{Y(s)}{U(s)}=G_1+G_2$



$\frac{Y(s)}{U(s)}=\frac{G_1}{1+G_1G_2}$



$T(s)=\frac{-Z_2}{Z_1}$

$Z_1=\frac{R_1}{R_1sC+1}$ $Z_2=R_2$

$T(s)=-R_2[R_1sC+1]/R_1$

$\det\left(\begin{bmatrix}m&n\\p&q\end{bmatrix}\right)=mq-pn$;

$\begin{bmatrix}m&n\\p&q\end{bmatrix}^{-1}=\begin{bmatrix}q&-n\\-p&m\end{bmatrix}/mq-pn$

Model Canonical Form:

$\begin{bmatrix}\dot{z}_1\\\dot{z}_2\end{bmatrix}=\begin{bmatrix}-4&0\\0&-3\end{bmatrix}\begin{bmatrix}z_1\\z_2\end{bmatrix}+\begin{bmatrix}1\\1\end{bmatrix}u$; $G(s)=\frac{2}{s+4}+\frac{-1}{s+3}$

$y=[2\ -1]\begin{bmatrix}z_1\\z_2\end{bmatrix}+[0]u$

- Minimum phase: ~~All zeros~~ No zeros in the left-half of the plane.