Homework 15-19

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"I pledge my honor I have abided by the Stevens Honor system."

\- Alex Daskins

Problem 15.1. Define a sequence a_n in the following way: $a_0 = 1$, $a_1 = 3$, $a_n = 3a_{n-1} - 2a_{n-2}$ for $n \geq 2$ (e.g. $a_2 = 3a_1 - 2a_0 = 3 \cdot 3 - 2 \cdot 1 = 7$, $a_3 = 3a_2 - 2a_1 = 3 \cdot 7 - 2 \cdot 3 = 15$, etc).

a) Prove that $a_n < 3^n$ for all $n \geq 2$.

$a_n = 3a_{n-1} - 2a_{n-2}$

$n \geq 2$

$a_2 = 3a_1 - 2a_0$

$a_2 = 3(3) - 2(1)$

$a_2 < 3^2$

$a_3 = 3a_2 - 2a_1$

$a_3 = 3(3^2 - 2) - 2(3)$

$a_3 = 3^3 - 3(2)(2)$

$3^3 - 3(2)^2 < 3^3$

$a_3 < 3^3$

$a_4 = 3a_3 - 2a_2$

$a_4 = 3(3^3 - 3(2)^2) - 2(3^2 - 2)$

$a_4 = 3^4 - 3^2(2)^2 - 3^2(2) + 2^2$

$a_4 < 3^4$

$\boxed{\therefore a_n < 3^n}$

b) Prove that $a_n > 2^n$ for all $n \geq 1$.

$a_0 = 1$

$a_1 = 3 = 2^2 - 1 > 2^1$

$a_1 > 2^1$

$a_n = 3a_{n-1} - 2a_{n-2}$

$n \geq 2$

$a_2 = 3^2 - 2$

$a_2 = 7$

$a_3 = 3^3 - 3(2)^2$

$a_3 = 15$

$a_3 = 2^4 - 1 > 2^3$

$a_3 > 2^3$

$a_4 = 3^4 - 3^2(2)^2 - 3^2(2) + 2^2$

$a_4 = 31$

$a_4 = 2^5 - 1 > 2^4$

$a_2 = 3^2 - 2$

$a_2 = 7$

$a_2 = 2^3 - 1 > 2^2$

$a_2 > 2^2$

$a_4 = 31$

$a_4 = 2^5 - 1 > 2^4$

$a_4 > 2^4$

$$\boxed{\therefore a_n > 2^n}$$

c) Prove that $7 \mid a_{3k-1}$ for all natural k.

$a_0 = 1$

$a_1 = 3$

$a_n = 3a_{n-1} - 2a_{n-2}$

$a_2 = 3a_1 - 2a_0$

$a_2 = 3(3) - 2(1)$

$a_2 = 9 - 2$

$a_2 = 7$

$a_3 = 3a_2 - 2a_1$

$a_3 = 3(7) - 2(3)$

$a_3 = 15$

$a_4 = 3a_3 - 2a_2$

$a_4 = 3(15) - 2(7)$

$a_4 = 31$

$a_5 = 3a_4 - 2a_3$

$a_5 = 3(31) - 2(15)$

$a_5 = 63$

$a_6 = 3a_5 - 2a_4$

$a_6 = 3(63) - 2(31)$

$a_6 = 127$

$a_7 = 3a_6 - 2a_5$

$a_7 = 3(127) - 2(63)$

$a_7 = 255$

$a_8 = 3a_7 - 2a_6$

$a_8 = 3(255) - 2(127)$

$a_8 = 511$

$$\boxed{\therefore 7 \mid a_{3k-1} \ \forall \text{ natural } k}$$

Problem 16.1. In this problem you can use calculators to check your answers, but please refrain from using calculators to solve the problem. Show your work!

a) Compute gcd(361, 247) using Euclidean algorithm.

gcd(361, 247)

$$\left.\begin{aligned} 361 &= 247 + 114 \\ 247 &= 114(2) + 19 \\ 114 &= 19(6) \end{aligned}\right\} 19$$

b) Write gcd(361, 247) as a linear combination of 361 and 247.

$$19 = 3(247) - 2(361)$$

c) Use the expression above to solve the congruence $247x \equiv 266 \pmod{361}$

$$247x = 266 \pmod{361}$$

$$247(4) = 361(2) + 266$$

$$988 = 722 + 266$$

$$x = 4$$

d) Explain why 2021 can not be written as a linear combination of 361 and 247.

2021 is not a multiple of 19, so it cannot be written as a linear combination of 361 and 247.

e) What is the closest integer to 2021 that can be written as a linear combination of 361 and 247?

$$247(5) = 1235$$

$$2021 - 1235 = 786$$

$$786 = 361(2) + 64$$

$$247(5) + 361(2) = \boxed{1957}$$

$$2021 = 1957 + 64$$

Problem 18.1. Let $n = a_0 + 10a_1 + 10^2a_2 + 10^3a_3 + \ldots + a_k10^k$ for some $k \in \mathbb{N}$.

a) Compute 10^i modulo 11 (answer might depend on i).

$$10^i \pmod{11}$$

$$10 \equiv (-1) \pmod{11}$$

$$10^i \equiv (-1)^i \pmod{11}$$

$$= \begin{cases} -1 & \text{when } i \text{ is odd} \\ 1 & \text{when } i \text{ is even} \end{cases}$$

b) Using part a), simplify n modulo 11 (this is known as divisibility by 11 rule).

$$n = a_0 + 10a_1 + 10^2 a_2 + 10^3 a_3 + \ldots 10^k a_k$$

$$n \pmod{11} \equiv a_0 + (-1)a_1 + (-1)^2 a_2 + \ldots (-1)^k a_k$$

$$= \begin{cases} a_0 - a_1 + a_2 - a_3 + \ldots a_k & \text{if } k \text{ is even} \\ a_0 - a_1 + a_2 - a_3 + \ldots - a_k & \text{if } k \text{ is odd} \end{cases}$$

c) Note that if a_i are digits (i.e. integer numbers between 0 and 9), n is exactly a natural number with decimal expression $a_ka_{k-1}\ldots a_1a_0$. Use part b) to compute the remainder of dividing $n = 987654321$ by 11.

$n = 987654321$

Sum of alternate is $1-2+3-4+5-6+7-8+9$

$= 25 - 20 = 5$

And, 11 does not divide 5

So, remainder of 987654321 after dividing 11 is 5

Problem 19.1. Prove that 2019 can not be represented as a sum of three fourth powers of integers (i.e. as $x^4 + y^4 + z^4$ for integers x, y, z).

$1^4 = 1$

$2^4 = 16$

$3^4 = 81$

$4^4 = 256$

$5^4 = 625$

$6^4 = 1296$

$7^4 = 2401$

$6^4 + 5^4 + 4^4 = 1296 + 625 + 256 = 2177$

$6^4 + 5^4 + 3^4 = 1296 + 625 + 81 = 2002$

$2002 < 2019 < 2177$