Exam 2

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"I pledge to have abided by the stevens honor system"

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1. [10 pts] In this problem you are asked to derive the auxiliary equations used for linear ODEs.

(a) Consider the *constant coefficient* linear ODE, $L[y] = ay'' + by' + cy = 0$, with constants a, b, c. Assuming a solution of the form $y = e^{mx}$, derive the auxiliary equation for determining the values of m by evaluating $L[e^{mx}] = 0$.

$y = e^{mx} \quad y' = me^{mx} \quad y'' = m^2 e^{mx}$

$L[e^{mx}] = a(m^2 e^{mx}) + b(me^{mx}) + c(e^{mx}) = 0$

$e^{mx}(am^2 + bm + c) = 0$

$\boxed{am^2 + bm + c = 0}$

(b) Consider the *variable coefficient* linear ODE, $L[y] = ax^2y'' + bxy' + cy = 0$, with constants a, b, c. Assuming a solution of the form $y = x^m$, derive the auxiliary equation for determining the values of m by evaluating $L[x^m] = 0$.

$y = x^m \quad y' = mx^{m-1} \quad y'' = (m^2 - m)x^{m-2}$

$L[x^m] = \overbrace{ax^2(m^2 - m)x^{m-2}}^{x^{2+m-2}} + \overbrace{bx(mx^{m-1})}^{x^{1+m-1}} + c(x^m) = 0$

$ax^m(m^2 - m) + bmx^m + cx^m = 0$

$x^m(am^2 - am + bm + c) = 0$

$\boxed{am^2 - am + bm + c = 0}$

2. [20 pts] Consider the differential equation, $L[y] = y'' + 3y' - 4y = 12\cos t$.

(a) Find the general solution of the homogeneous equation $L[y] = 0$.

$m^2 + 3m - 4 = 0$

$(m + 4)(m - 1) = 0$

$m = -4 \quad m = 1$

$\boxed{y_c = c_1 e^{-4t} + c_1 e^{t}}$

(b) Find a particular solution of the equation, $L[y] = y'' + 3y' - 4y = \cos t$.

$y_c = \underset{y_1}{c_1 e^{t}} + \underset{y_2}{c_2 e^{-4t}}$

$W = \begin{vmatrix} e^{t} & e^{-4t} \\ e^{t} & -4e^{-4t} \end{vmatrix} = -4e^{-3t} - e^{-3t} = -5e^{-3t}$

$W = \begin{vmatrix} & \\ e^t & -4e^{-4t} \end{vmatrix} = -1e$

$W_1 = \begin{vmatrix} 0 & e^{-4t} \\ \cos(t) & -4e^{-4t} \end{vmatrix} = 0 - e^{-4t}\cos(t) = -e^{-4t}\cos(t)$

$W_2 = \begin{vmatrix} e^t & 0 \\ e^t & \cos(t) \end{vmatrix} = e^t\cos(t) - 0 = e^t\cos(t)$

$u_1 = \int \frac{W_1}{W} = \int \frac{1}{5} e^{-t}\cos(t)$

$u_1 = \frac{e^{-t}}{10}(\sin(t) - \cos(t))$

$u_2 = \int \frac{W_2}{W} = \int -\frac{1}{5} e^{4t}\cos(t)$

$u_2 = -\frac{e^{4t}}{85}(\sin(t) + 4\cos(t))$

$y_p = u_1 y_1 + u_2 y_2 = \frac{e^{-t}}{10}(\sin(t) - \cos(t))(e^t) + -\frac{e^{4t}}{85}(\sin(t) + 4\cos(t))(e^{-4t})$

$y = y_c + y_p$

$y = c_1 e^t + c_2 e^{-4t} + \frac{e^{-t}}{10}(\sin(t) - \cos(t))(e^t) + -\frac{e^{4t}}{85}(\sin(t) + 4\cos(t))(e^{-4t})$

$\boxed{y = c_1 e^t + c_2 e^{-4t} + \frac{3\sin(t) - 5\cos(t)}{34}}$

(c) Give the general solution to the equation, $L[y] = y'' + 3y' - 4y = 12\cos t$.

$y_c = c_1 e^t + c_2 e^{-4t}$

$y_p = 12\left(\frac{3\sin(t) - 5\cos(t)}{34}\right)$

$y_p = \frac{36\sin(t) - 60\cos(t)}{34}$

$y_p = \frac{18\sin(t) - 30\cos(t)}{17}$

$y = y_c + y_p$

$\boxed{y = c_1 e^t + c_2 e^{-4t} + \frac{18\sin(t) - 30\cos(t)}{17}}$

3. [20 pts] Consider the differential equation, $L[y] = x^2y'' + 5xy' + 4y = 0$, for $x > 0$.

(a) Find two linearly independent solutions to the equation $L[y] = 0$.

$a(m^2 - m) + bm + c = 0$

$x^{-2} \quad c \quad x^{-2}\ln(x)$

(a) Find two linearly independent solutions to the equation $L[y] = 0$.

$a(m^2 - m) + bm + c = 0$

$m^2 - m + 5m + 4 = 0$

$m^2 + 4m + 4 = 0$

$(m+2)(m+2) = 0$

$m = -2$ multiplicity of 2

$$\boxed{y_c = c_1 x^{-2} + c_2 x^{-2} \ln(x)}$$

(b) Verify that the two solutions in (a) are *linearly independent* on the interval $0 < x < \infty$.

$y_1 = x^{-2}$ $\quad y_2 = x^{-2} \ln(x)$

$$W = \begin{vmatrix} x^{-2} & x^{-2}\ln(x) \\ -2x^{-3} & x^{-3} - 2x^{-3}\ln(x) \end{vmatrix} = x^{-5}$$

$\boxed{x^{-5} \neq 0 \text{ : thus they are linearly independent.}}$

(c) Find the unique solution to the initial value problem,

$$L[y] = x^2y'' + 5xy' + 4y = 0, \quad \text{for } x > 0, \quad y(1) = 2,\ y'(1) = 2.$$

$y_c = c_1 x^{-2} + c_2 x^{-2} \ln(x)$

$2 = c_1 + c_2(0)$

$c_1 = 2$

$y_c' = -2c_1 x^{-3} + c_2 x^{-3} - 2c_2 x^{-3} \ln(x)$

$2 = -2c_1 + c_2$

$c_2 = 6$

$$\boxed{y_c = 2x^{-2} + 6x^{-2} \ln(x)}$$

4. [25 pts] Use the method of *variation of parameters* to find the general solution to,

$$L[y] = y'' - 4y = 4xe^{-2x}.$$

$e^{mx}(m^2 - 4) = 0$

$m = 2 \quad m = -2$

$y_c = \underset{y_1}{c_1 e^{2x}} + \underset{y_2}{c_1 e^{-2x}}$

$$W = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix} = -2 - 2 = -4$$

$$W_1 = \begin{vmatrix} 0 & e^{-2x} \\ 4xe^{-2x} & -2e^{-2x} \end{vmatrix} = 0 - 4xe^{-4x} = -4xe^{-4x}$$

$$W_2 = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & 4xe^{-2x} \end{vmatrix} = 4x - 0 = 4x$$

$$u_1 = \int \frac{W_1}{W} = \int xe^{-4x}dx = \frac{-4xe^{-4x} - e^{-4x}}{16}$$

$$u_2 = \int \frac{W_2}{W} = \int -x\,dx = -\frac{x^2}{2}$$

$$y_p = u_1y_1 + u_2y_2 = \left(\frac{-4xe^{-4x} - e^{-4x}}{16}\right)e^{2x} - \left(\frac{x^2}{2}\right)e^{-2x}$$

$$y_p = -\frac{ex^{-2x}}{4} - \frac{ex^2e^{-2x}}{2}$$

$$\boxed{y = c_1e^{2x} + c_2e^{-2x} - \frac{ex^{-2x}}{4} - \frac{ex^2e^{-2x}}{2}}$$

5. [25 pts] Five differential equations (a-e) are listed below. For each differential equation, choose from the list of functions (A-Z) the appropriate form of the particular solution, $y_p(x)$, if you were to solve for y_p using the method of *undetermined coefficients*.

(a) $\dfrac{d^2y}{dx^2} + y = 9\cos x$ K

(b) $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} = 12xe^{-x}$ G

(c) $\dfrac{d^2y}{dx^2} - 4y = 7e^{2x}$ H

(d) $\dfrac{d^2y}{dx^2} - y = 6x$ B

(e) $\dfrac{d^2y}{dx^2} + y = 3x\sin(2x)$ Z

A. Ax **B.** $Ax + B$ **C.** $Ax^2 + Bx$ **D.** $Ax^2 + Bx + C$

E. Ae^x **F.** Axe^x **G.** $(Ax + B)e^x$

H. Ae^{2x} **I.** Axe^{2x} **J.** $(Ax + B)e^{2x}$

K. $A\cos x + B\sin x$ **L.** $Ax\cos x + Bx\sin x$ **M.** $(Ax + B)\cos x + (Cx + D)\sin x$

N. $A\cos(2x) + B\sin(2x)$ **O.** $Ax\cos(2x) + Bx\sin(2x)$ **P.** $(Ax + B)\cos(2x) + (Cx + D)\sin(2x)$

Y. NA - undetermined coefficients is not applicable

Z. None of the above; the method applies but the form of y_p is not in the list