Homework 1

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Name: Alex Gaskins Collaborators: John Oliveira, Trevor Dameika

1. Consider the first-order ODE, $\frac{dy}{dt} = y^2 - 1$.

   (a) Identify the equilibrium points and determine their stability by sketching the one-dimensional phase line for this *autonomous* first-order ODE. (See solutions from Tuesday's recitation for example of phase line analysis.)

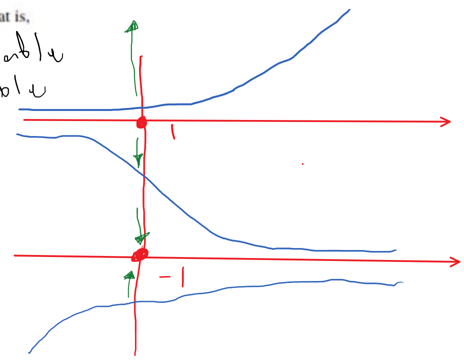
   (b) Verify that $y = \frac{1 - Ce^{2t}}{1 + Ce^{2t}}$ is a family of solutions to the ODE (C is any constant).

   (c) Find the (unique) solution, $y = \phi(t)$, with initial condition $y(0) = 3$ and determine its existence interval (a, b).

   (d) Evaluate the limit of the solution in (c) as t approaches the end points of the existence interval. That is, evaluate $\lim_{t\to a^+} \phi(t)$ and $\lim_{t\to b^-} \phi(t)$.

1 is unstable

-1 is stable



a.)

$$y^2 - 1 = 0$$
$$y^2 = 1$$
$$y = \sqrt{1} = \pm 1$$

b.)

$$y = \frac{1 - Ce^{2t}}{1 + Ce^{2t}}$$
$$y' = -\frac{4ce^{2t}}{(ce^{2t}+1)^2}$$

$$-\frac{4ce^{2t}}{(ce^{2t}+1)^2} = \left(\frac{1-ce^{2t}}{ce^{2t}+1}\right)^2 - 1$$

$$-4ce^{2t} = (1-ce^{2t})^2 - (ce^{2t}+1)^2$$
$$(1-ce^{2t})(1-ce^{2t}) - (ce^{2t}+1)^2$$
$$1 - 2ce^{2t} + ce^{4t} - (ce^{2t}+1)^2$$
$$1 - 2ce^{2t} + ce^{4t} - 1 - 2ce^{2t} - ce^{4t}$$
$$-4ce^{2t} = -4ce^{2t} \checkmark$$

c.)

$$3 = \frac{1 - c}{1 + c}$$

$$3 + 3c = 1 - c$$
$$4c = -2$$
$$c = -\frac{1}{2}$$

Existence Interval:

$(-1, 1)$

d.)

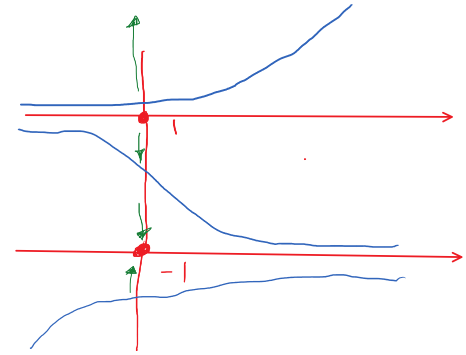
$$0 = \frac{1-\frac{1}{2}e^{2t}}{1+\frac{1}{2}e^{2t}} \qquad e^{2t} = 2$$

$$0 = 1 - \frac{1}{2}e^{2t} \qquad 2t = \ln(2)$$

$$\frac{1}{2}e^{2t} = 1 \qquad t = \frac{\ln(2)}{2}$$

$$\lim_{t \to \frac{\ln(2)}{2}^{+}} \left(\frac{1-\frac{1}{2}e^{2t}}{1+\frac{1}{2}e^{2t}}\right)^{>1} = -\infty$$

$$\lim_{t \to \frac{\ln(2)}{2}^{-}} \left(\frac{1-\frac{1}{2}e^{2t}}{1+\frac{1}{2}e^{2t}}\right)^{<1} = \infty$$



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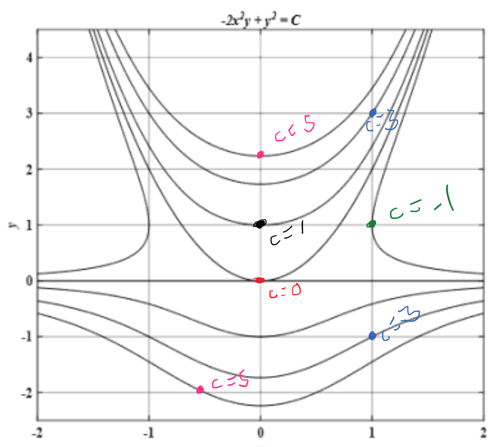
2. Consider the first-order ODE, $(x^2 - y)\frac{dy}{dx} + 2xy = 0$.

(a) Verify that the expression $-2x^2y + y^2 = C$ is an *implicit* solution to the differential equation, for any constant C.

(b) The figure below (see page 2) shows a sample of level curves, $-2x^2y + y^2 = C$. Label each of the curves with the appropriate value of C.

(c) Determine the value of C for the solution curve satisfying $y(-1/2) = -2$ and express the solution in explicit form, $y = \phi(x)$. In the figure below, identify the curve that represents this solution.



A.)

$$-2x^2y + y^2 = C$$

$$-2x^2\frac{dy}{dx} - 4xy + 2y\frac{dy}{dx} = 0$$

$$-2\left(\left(x^2\frac{dy}{dx} - y\frac{dy}{dx}\right) + 2xy\right) = 0$$

$$\left(x^2\frac{dy}{dx} - y\frac{dy}{dx}\right) + 2xy = 0$$

$$\boxed{\frac{dy}{dx}(x^2 - y) + 2xy = 0}$$

B.)

$(0,0)$

$c = -2(0)^2(0) + (0)^2 = 0$

$(0,1)$

$c = -2(0)^2(-1) + (-1)^2 = 1$

$(1,3)$

$c = -2(1)^2(3) + (3)^2 = 3$

$(1,1)$

$c = -2(1)^2(1) + (1)^2 = -1$

$(1,-1)$

$c = -2(1)^2(-1) + (-1)^2 = 3$

c.)

$$-2\left(-\tfrac{1}{2}\right)^2(-2) + (-2)^2 = C$$

$$1 + 4 = C = 5$$

$$-2x^2y + y^2 = 5$$

$$y^2 - 2x^2y - 5 = 0$$

$$\frac{2x^2 \pm \sqrt{(2x^2)^2 - 4(1)(-5)}}{2(1)}$$

$2(1)$

$$\boxed{y = \phi(x) = x^2 \pm \sqrt{x^4 + 5}}$$

$(1, -1)$

$c = -2(1)^2(-1) + (-1)^2 = 3$

$(-\frac{1}{2}, -2)$

$c = -2(-\frac{1}{2})^2(-2) + (-2)^2 = 5$

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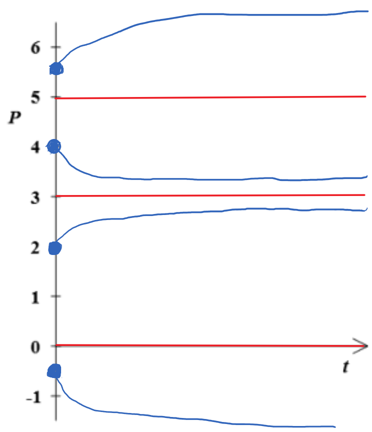
3. Consider the *autonomous* first order ODE, $P'(t) = P(P^2 - 8P + 15)$, for modeling population dynamics where $P(t) \geq 0$ is the population at time t.

   (a) Draw the one-dimensional phase portrait (phase line analysis) on the vertical (P) axis provided in the figure below. Identify the equilibrium solutions (constant solutions) and the direction of the solution (increasing or decreasing) in each region of the phase line, for increasing t. Classify each equilibrium solution as either *asympotically stable*, *unstable* or *semi-stable*.

   (b) In the same figure, draw the equilibrium solutions from part (a) as functions over t, for $t \geq 0$. Then sketch approximate solution curves $P(t)$ satisfying initial conditions $P(0) = -0.5$, $P(0) = 2$, $P(0) = 4$ and $P(0) = 5.5$.

   (c) Determine the value(s) of P at which the solutions have an *inflection point*. This can be calculated directly from the ODE.

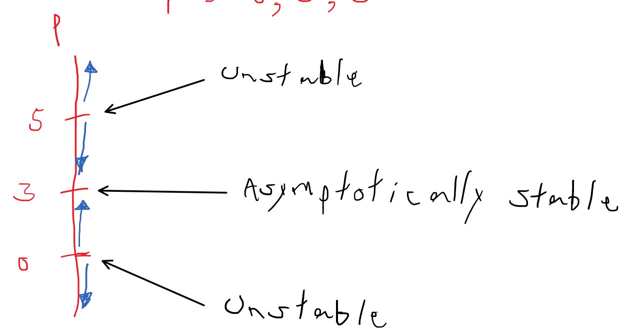


A.) $P'(t) = P(P^2 - 8P + 15)$

$P(P^2 - 8P + 15) = 0$

$P(P-3)(P-5) = 0$

$P = 0, 3, 5$



C.) $P'(t) = P(P^2 - 8P + 15)$

$P'(t) = P^3 - 8P^2 + 15P$

$P''(t) = 3P^2 - 16P + 15$

$3P^2 - 16P + 15 = 0$

$$\frac{16 \pm \sqrt{256 - 4(3)(15)}}{6}$$

$$\frac{16 \pm \sqrt{76}}{6} \approx \frac{8 \pm \sqrt{19}}{3}$$

$$\boxed{P = \frac{8 \pm \sqrt{19}}{3}}$$