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Tuesday, December 8, 2020 10:41 AM

2. The heat equation, $u_t = 4u_{xx}$ on $0 < x < 2\pi$, with boundary conditions, $u(0,t) = 0$ and $u(2\pi, t) = 0$, has the general solution,

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-n^2 t} \sin(nx/2)$$

For the initial temperature distribution, $u(x,0) = 4\sin(x/2) - \frac{\sin(2x)}{2} + \frac{\sin(3x)}{8}$,

(a) determine the coefficients A_n, $n = 1, 2, 3, \ldots$;

(b) write out the complete solution to the initial-boundary value problem.

A.) $u(x,t) = X(x)T(t)$

$u_t = XT'$, $u_{xx} = X''T$

$XT' = 4X''T$

$\frac{T'}{4T} = \frac{X''}{X} = \lambda$

$\underline{\lambda = 0:}$

$T' = 0 \longrightarrow T = A'$

$X'' = 0 \longrightarrow X = B'x + C'$

$XT = A'(B'x + C')$

$XT = A'B'x + A'C = Ax + B$

$A = A'B'$

$B = A'C$

$u = XT$

$u = Ax + B$

$u(0,t) \rightarrow B = 0$

$u(2\pi, t) = 2\pi A = 0 \longrightarrow A = 0$

$u(x,t) = 0$

$\underline{\lambda > 0:}$

$\lambda = n^2$

$T' = 4n^2 T \longrightarrow T = A' e^{4n^2 t}$

$\lambda > 0:$

$\lambda = n^2$

$T' = 4n^2 T \longrightarrow T = A' e^{4n^2 t}$

$X'' = n^2 X \longrightarrow X = B' e^{nx} + C' e^{-nx}$

$u = A' e^{4n^2 t} \left(B' e^{nx} + C' e^{-nx} \right)$

$u = e^{4n^2 t} \left(A e^{nx} + B e^{-nx} \right)$ $\quad A'B' = A$

$A'C' = B$

$u(0,t) = 0 \longrightarrow A + B = 0$

$u(2\pi, t) = 0 \longrightarrow A e^{2n\pi} + B e^{-2n\pi}$

$u(x,t) = 0$

$\lambda < 0:$

$\lambda = -n^2$

$T' = -4n^2 T \longrightarrow T = A' e^{-4n^2 t}$

$X'' = -n^2 X \longrightarrow X = B' \cos(nx) + C' \sin(nx)$

$u(x,t) = A' e^{-4n^2 t} \left(B' \cos(nx) + C' \sin(nx) \right)$

$u(x,t) = e^{-4n^2 t} \left(A \cos(nx) + B \sin(nx) \right)$ $\quad A'B' = A$

$A'C' = B$

$u(0,t) = 0 \longrightarrow A = 0$

$u(2\pi, t) = 0 \longrightarrow B \sin(2n\pi) = 0$

$B = 0, \quad u(x,t) = 0$

$B \neq 0$

$\sin(2n\pi) = 0 = \sin(m\pi)$

$2n\pi = m\pi$

$n = \frac{m}{2}$

$\sum^{\infty} \; -4(\frac{m}{2})^2 t$

$$u(x,t) = \sum_{m=1}^{\infty} A_m e^{-4\left(\frac{m}{2}\right)^2 t} \sin\left(\frac{m}{2}x\right)$$

$$u(x,t) = \sum_{m=1}^{\infty} A_m e^{-m^2 t} \sin\left(\frac{mx}{2}\right)$$

$$u(x,0) = 4\sin\left(\frac{x}{2}\right) - \frac{\sin(2x)}{2} + \frac{\sin(3x)}{2}$$

$$\sum_{m=1}^{\infty} A_m \sin\left(\frac{mx}{2}\right) = 4\sin\left(\frac{x}{2}\right) - \frac{\sin(2x)}{2} + \frac{\sin(3x)}{2}$$

$A_1 = 4$, $A_4 = -\frac{1}{2}$, $A_6 = \frac{1}{2}$

$A_n = 0 \quad A_n \neq 1, 4, 6$

B.)

$$u(x,t) = 4e^{-t}\sin\left(\frac{x}{2}\right) - \frac{1}{2}e^{-16t}\sin(2x) + \frac{1}{8}e^{-36}\sin(3x)$$

4. **IBVP for the Heat Equation.** Consider the following initial-boundary value problem modeling heat flow in a wire of length 1.

$$\text{(PDE)} \quad u_t = 2\,u_{xx}, \quad 0 < x < 1, \quad t > 0$$

$$\text{(BC)} \quad u(0,t) = 0, \quad u_x(1,t) = 0, \quad t > 0$$

$$\text{(IC)} \quad u(x,0) = f(x) = \begin{cases} 0, & 0 < x < 1/2 \\ 1, & 1/2 \le x < 1 \end{cases}$$

Separation of variables with $u(x,t) = X(x) \cdot T(t)$ yields the ODEs,

$$X''(x) + \lambda X(x) = 0, \quad 0 < x < 1; \quad X(0) = 0, \; X'(1) = 0 \quad \text{(EVP)}$$

$$T'(t) + 2\lambda T(t) = 0.$$

(a) Solve the eigenvalue problem (EVP) to determine the eigenvalues, λ_n, and corresponding solutions, $X_n(x)$. (There are no eigenvalues for $\lambda \le 0$.)

(b) For each λ_n, solve the first-order ODE for $T_n(t)$.

(c) Use $X_n(x)$ and $T_n(t)$ to form the general solution, $u(x,t)$, as an infinite series.

(d) Complete the solution to the IBVP for the initial condition, $u(x,0)$ described above.

A.)

$\lambda = 0 \qquad x'' = 0$

$\lambda > 0 \rightarrow \lambda = \alpha^2$

$x'' + \alpha^2 x = 0$

A.)

$\lambda = 0$

$x'' = 0$

$x(n) = A_n + B$

$x(0) = B = 0$

$x'(1) = A = 0$

$x(n) = 0$

$\lambda > 0 \rightarrow \lambda = x$

$x'' + \lambda^2 x = 0$

$m^2 + x^2 = 0$

$m = \pm \lambda_i$

$x(m) = c_1 \cos(\lambda_1 x) + c_2 \sin(\lambda_1 x)$

$x(0) = 0$

$c_1 = 0$

$x'(1) = \lambda_1 c_2 \cos(\lambda_1) = 0$

$c_2 = 0 \;, \quad c_2 \neq 0$

$\cos(x) = 0 \;; \quad \lambda_1 = \frac{(2n-1)\pi}{2}$

$\lambda = \lambda_1^2$

$$\boxed{\lambda_n = \frac{(2n-1)^2 \pi^2}{4}}$$

B.)

$$T' + \frac{2(2n-1)^2 \pi^2}{4} T = 0$$

$$T' + \frac{(2n-1)^2 \pi^2}{2} T = 0$$

$$\frac{T'}{T} = -\frac{(2n-1)^2 \pi^2}{2} dt$$

$$T_n = e^{-\left(\frac{(2n-1)^2 \pi^2}{2}\right) t}$$

$$u(x,t) = X(x) T(t)$$

$$u(x,t) = c_2 \sin\left(\frac{(2n-1)\pi}{2}\right) \times e^{-\left(\frac{(2n-1)^2 \pi^2}{2}\right) t}$$

C.)

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{(2n-1)\pi}{2}\right) \times e^{-\left(\frac{(2\pi-1)^2 \pi^2}{2}\right) t}$$

$$u(x,0) = f(x)$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{(2n-1)\pi n}{2}\right)$$

D.)

$$f(x) \longrightarrow g(t) = \begin{cases} 0, & 0 < t < \frac{\pi}{2} \\ 1, & \frac{\pi}{2} \leq t < \pi \end{cases}$$

$$g(t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{(2n-1)t}{2}\right)$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} g(t) \sin\left(\frac{(2n-1)t}{2}\right)$$

$$b_n = \frac{2}{\pi}\left[\int_{\pi/2}^{\pi} \sin\left(\frac{(2n-1)t}{2}\right)\right]$$

$$b_n = \frac{-4}{(2n-1)\pi}\left[0 - \cos\left(\frac{(2n-1)\pi}{4}\right)\right]$$

$$b_n = \frac{4}{(2n-1)\pi} \frac{(-1)^{n+1}}{\sqrt{2}} = \frac{(-1)^{n+1} 2\sqrt{2}}{(2n-1)\pi}$$

$$g(t) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2\sqrt{2}}{(2n-1)\pi} \sin\left(\frac{(2n-1)t}{2}\right)$$

$$t = n\pi$$

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$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\, 2\sqrt{2}}{(2n-1)\pi} \sin\left(\frac{(2n-1)\pi x}{2}\right)$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\, 2\sqrt{2}}{(2n-1)\pi} \sin\left(\frac{(2n-1)\pi x}{2}\right) e^{-\left(\frac{(2n-1)^2\pi^2}{2}\right)t}}$$