Homework 1-4

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"I pledge my honor I have abided by the Stevens Honor system."
- Alex Gaskins

Problem 1.1. 1. Provide an example of two (preferably "real-life") statements P and Q such that

" P OR (NOT Q") is a true statement, but "P AND (NOT Q)" is a false statement.

P = Running Q = Moving

I'm either running or not moving. → T

I'm running and not moving. → F

2. Are there any statements P and Q such that "P AND (NOT Q)" is a true statement, but "P OR (NOT Q)" is a false statement? Why?

P	Q	$P \wedge \neg Q$	$P \vee \neg Q$
T	T	F	T
T	F	T	T
F	T	F	F
F	F	F	T

In order for P AND (NOT Q) to be true, both P and ¬Q must be true, and thus P OR (NOT Q) also has to be true.

Problem 1.2. Construct the truth tables for the following Boolean functions. Are any of them logically equivalent?

1. $(ab) \vee \bar{a}$;

a	b	$\bar{a}$	ab	$(ab) \vee \bar{a}$
T	T	F	T	T
T	F	F	F	F
F	T	T	F	T
F	F	T	F	T

2. ab;

a	b	ab
T	T	T
T	F	F
F	T	F
F	F	F

$ab = \neg(\neg(ab))$

3. $\neg(\neg(ab))$;

a	b	$\neg(\neg(ab))$
T	T	T
T	F	F

T	T	T
T	F	F
F	T	F
F	F	F

4. $(a \vee \bar{b})(\bar{a} \vee b)$;

a	b	$\bar{a}$	$\bar{b}$	$(a \vee \bar{b})$	$(\bar{a} \vee b)$	$(a \vee \bar{b}) \wedge (\bar{a} \vee b)$
T	T	F	F	T	T	T
T	F	F	T	T	F	F
F	T	T	F	F	T	F
F	F	T	T	T	T	T

Problem 2.1. 1. Simplify the following Boolean formulas using standard elementary equivalences:

(a) $f = \neg(\bar{p} \vee q)$. $\equiv (\neg \bar{p}) \wedge \bar{q} \equiv \boxed{p\bar{q}}$

(b) $g = \neg(\bar{p} \vee q) \wedge q$. $\equiv p\bar{q} \wedge q \equiv p \wedge \underbrace{(q \wedge \bar{q})}_{\equiv F} \equiv p \wedge (F) \equiv \boxed{F}$

(c) $h = ((a \to b) \wedge a) \to b$. $((\bar{a} \vee b) \wedge a) \to b \equiv (\underbrace{\bar{a}a}_{\equiv F} \vee ba) \to b \equiv ba \to b \equiv \boxed{\neg a \vee \neg b \vee b}$

2. Indicate whether any of f, g, h are tautologies or contradictions.

A.)

p	q	$\bar{q}$	$p\bar{q}$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

can be true

$\boxed{\text{Tautology}}$

B.) $\boxed{\text{Contradiction}}$

c.) $\boxed{\text{Tautology}}$

Problem 3.1. Recall that $a \mid b = \neg(a \wedge b)$. Find the (full) DNF for $f(a, b) = a \mid b$.

$a | b = \neg(a \wedge b) = \bar{a} \vee \bar{b}$

a	b	$\bar{a}$	$\bar{b}$	$\bar{a} \vee \bar{b}$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

$$\boxed{f(a,b) = (a \wedge \bar{b}) \vee (\bar{a} \wedge b) \vee (\bar{a} \wedge \bar{b})}$$

Problem 4.1. 1. Prove that $\{\to, \neg\}$ is a complete set of operations.

$\neg x \to y \equiv x \vee y$ $\qquad$ $\neg(x \vee y) \equiv \bar{x}\,\bar{y}$

x	y	$x \vee y$
T	T	T
T	F	T
F	T	T
F	F	F

→ can be written as a statement with ∨ (OR). Thus, rewriting the operations as $(\vee, \neg)$, which is functionally complete. It can also be written with AND.

2. Prove that $\{\to\}$ is not a complete set of operations.

$a \to b = \bar{a} \vee b$

can't be rewritten using De Morgan's Laws, so not all three operations can be expressed.

Problem 4.2. Define a Boolean function $f = f(x_1, \ldots, x_n)$ to be in conjunctive normal form (CNF) if it is written as a conjunction of disjuncts of n variables or their negations (e.g. $(x_1 \vee x_2 \vee \bar{x_3}) \wedge (\bar{x_1} \vee x_2 \vee \bar{x_3})$ is a function of x_1, x_2, x_3 in CNF).

1. Find CNF of $a \to b$;

$a \to b \equiv \bar{a} \vee b$

a	b	$\bar{a}$	$\bar{a} \vee b$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

$f(a,b) = (a \vee b) \wedge (a \vee \bar{b}) \wedge (\bar{a} \vee b)$

$) \wedge (\bar{a} \vee \bar{b})$

$f(a,b) = (a \vee b) \wedge (a \vee b) \wedge (a \vee b)$

2. Find CNF of $a \oplus b \equiv (a \vee b)\neg(ab)$; $(a \vee b) \wedge \neg(a \wedge b) \equiv (a \vee b) \wedge (\bar{a} \vee \bar{b})$

a	b	$\bar{a}$	$\bar{b}$	$a \vee b$	$\bar{a} \vee \bar{b}$	$(a \vee b) \wedge (\bar{a} \vee \bar{b})$
T	T	F	F	T	F	F
T	F	F	T	T	T	T
F	T	T	F	T	T	T
F	F	T	T	F	T	F

$f(a,b) = (a \vee \bar{b}) \wedge (\bar{a} \vee b)$

3. Prove that any Boolean function is equivalent to a function in CNF.

Both DNF and CNF are commonly used in logic and computer science.

Any Boolean function can be rewritten as a CNF. However, not all are functionally complete.

For example:

$\neg x \rightarrow y \equiv x \vee y$

x	y	$x \vee y$
T	T	T
T	F	T
F	T	T
F	F	F

CNF: $f(x,y) = (x \vee y) \wedge (x \vee \bar{y}) \wedge (\bar{x} \vee y)$