Homework 9

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1. Draw the graph of the piecewise function, $g(t)$, and use the methods in 7.3.2 to find its Laplace transform $G(s) = \mathcal{L}\{g(t)\}$.

$$g(t) = \begin{cases} 5t^2, & 0 \le t < 1 \\ 5, & 1 \le t < 3 \\ -5(t-4), & 3 \le t < 4 \\ 0, & 4 \le t \end{cases}$$

$$5t^2(u(t) - u(t-1)) + 5(u(t-1) - u(t-3))$$

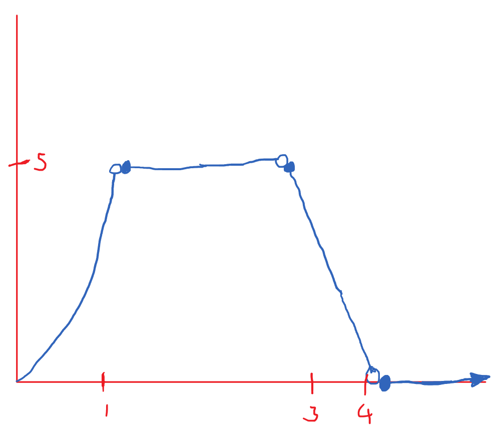
$$-(5t-20)(u(t-3) - u(t-4))$$

$$5t^2u(t) - 5t^2u(t-1) + 5u(t-1) - 5u(t-3) - (5t-20)u(t-3) + (5t-20)u(t-4)$$

$$\frac{5(2)}{s^3} = \frac{10}{s^3} \qquad \frac{10e^{-s}}{s^2} - \frac{10e^{-s}}{s^3} \qquad -5(t-3)+15 = -\frac{5e^{-3s}}{s}$$

$$-5(t-1) = -\frac{5(2)e^{-s}}{s^2}$$

$$\boxed{-\frac{10e^{-s}}{s^3} + \frac{10}{s^3} + \frac{5e^{-4}}{s^2} - \frac{5e^{-3s}}{s^2} - \frac{10e^{-s}}{s^2}}$$



2. Consider the function $f(t)$ expressed in terms of unit step functions,

$$f(t) = (4t - t^2)\,\mathcal{U}(t) + (t-2)^2\,\mathcal{U}(t-2) - 4\,\mathcal{U}(t-4).$$

Represent $f(t)$ in the usual format for a piecewise function and sketch its graph.

$$(4t - t^2)(t) - (4t - t^2)(t-2)$$

For $0 \le t < 2$

$$f(t) = (4t - t^2)$$

For $2 \le t < 4$

$$f(t) = 4t - t^2 + (t-2)^2$$

$$= 4t - t^2 + t^2 - 4t + 4$$

$$f(t) = 4$$

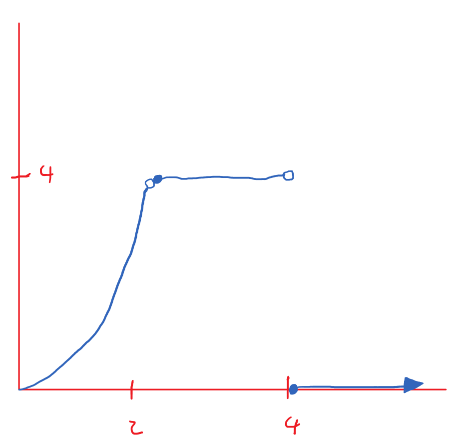
<u>For $t \ge 4$</u>

$$f(t) = (4t - t^2) + (t-2)^2 - 4$$

$$f(t) = 4t - t^2 + t^2 - 4t + 4 - 4$$

$$f(t) = 0$$

$$f(t) = \begin{cases} 4t - t^2, & 0 \le t < 2 \\ 4, & 2 \le t < 4 \\ 0, & t \ge 4 \end{cases}$$



3. For each of the transforms $G(s)$,

- express the inverse transform, $g(t) = \mathcal{L}^{-1}\{G(s)\}$, in terms of unit step functions;
- represent $g(t)$ in the usual format for a piecewise function and sketch its graph.

(a) $G(s) = \frac{-4}{s^3} + \frac{4}{s^2} + \left(\frac{4}{s^3} + \frac{4}{s^2}\right)e^{-2s}$

$L^{-1}\left(e^{-cs} F(s)\right) = u_c(t)\, u(t-c)$

$L^{-1}\left(\frac{-4}{s^3} + \frac{4}{s^2} + \left(\frac{4}{s^3} + \frac{4}{s^2}\right)e^{-2s}\right)$

$-2t^2 + 4t + L^{-1}\left(\left(\frac{4}{s^3} + \frac{4}{s^2}\right)e^{-2s}\right)$

$L^{-1}\left(\frac{4}{s^3} + \frac{4}{s^2}\right)$

$= 2t^2 + 4t$

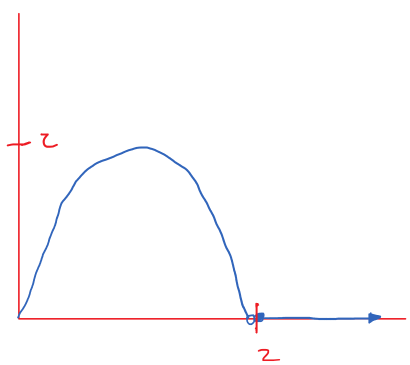
$L^{-1}\left(\left(\frac{4}{s^3} + \frac{4}{s^2}\right)e^{-2s}\right)$

$= u_2(t)\left[2(t-2)^2 + 4(t-2)\right]$

$-2t^2 + 4t + 2t^2 - 4t$
$= 0$

$L^{-1}(G(s)) = -2t^2 + 4t + u_2(t)\left[2(t-2)^2 + 4(t-2)\right]$

$$f(t) = \begin{cases} -2t^2 + 4t, & t < 2 \\ 0, & t \geq 2 \end{cases}$$



(b) $G(s) = \frac{2\pi}{s^2 + \pi^2/4}e^{-s} - \frac{2\pi}{s^2 + \pi^2/4}e^{-5s}$

$\frac{2\pi}{s^2 + (\frac{\pi}{2})^2} = \frac{4(\frac{\pi}{2})}{s^2 + (\frac{\pi}{2})^2}$

$L_y\left(\frac{4(\frac{\pi}{2})}{s^2 + (\frac{\pi}{2})^2}\right) = 4\sin\left(\frac{\pi}{2}t\right)$

$L_y(e^{-s}) = U(t-1)$

$(e^{-5s}) = U(t-5)$

$\frac{4(\frac{\pi}{2})}{\;} e^{-s} - \frac{4(\frac{\pi}{2})}{\;} e^{-5s}$

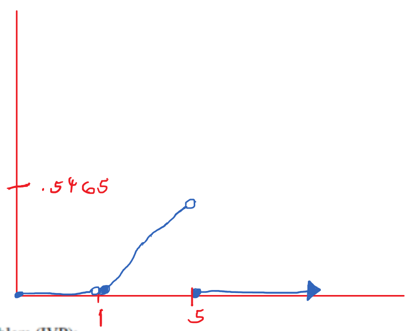
$L_y(\quad)$

$$\frac{4(\frac{\pi}{2})}{s^2+(\frac{\pi}{2})^2}e^{-s} - \frac{4(\frac{\pi}{2})}{s^2+(\frac{\pi}{2})^2}e^{-5s}$$

$$L_y(e^{-5s}) = U(t-5)$$

$$g(t) = 4\sin\left(\frac{\pi}{2}t\right)U(t-1) - 4\sin\left(\frac{\pi}{2}t\right)U(t-5)$$

$$g(t) = \begin{cases} 0, & t<1 \\ 4\sin(\frac{\pi}{2}t), & 1\le t<5 \\ 0, & t\ge 5 \end{cases}$$



4. Use Laplace Transforms to solve the following initial value problem (IVP):

$$y''(t) + y(t) = f(t), \quad y(0) = 0,\ y'(0) = 0, \quad \text{where } f(t) = \begin{cases} 1, & 0 \le t < \pi \\ -1, & \pi \le t < 2\pi \\ 0, & 2\pi \le t \end{cases}$$

$$s^2[L_y] - s(\cancel{y(0)}) - \cancel{y'(0)} + [L_y] = f(t)$$

$$[L_y](s^2+1) = f(t)$$

$$f(t) = U(t) - U(t-\pi) - U(t-\pi) + (t-2\pi)$$

$$f(t) = U(t) - 2U(t-\pi) + U(t-2\pi)$$

$$F(s) = e^{-s} - 2e^{-\pi s} + e^{-2\pi s}$$

$$[L_y](s^2+1) = e^{-s} - 2e^{-\pi s} + e^{-2\pi s}$$

$$y = L^{-1}\left(\frac{e^{-s} - 2e^{-\pi s} + e^{-2\pi s}}{s^2+1}\right)$$

$$\boxed{y = \sin(t-1)\,u(t-1) - 2\sin(t-\pi)\,u(t-\pi) + \sin(t-2\pi)\,u(t-2\pi)}$$