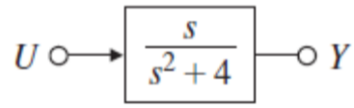
- Q1: Read the below paper and write a summary (a minimum of half a page) to describe what you have learned from it. (20 points)

A.E. Bryson, "Optimal control-1950 to 1985," IEEE Control Systems Magazine, Vol. 16, No. 3, June 1996

To enhance optimality in control systems, mathematics had to be enhanced as well. The invention of advanced calculus by Issac Newton and his colleagues was a breakthrough that allowed for further analysis to be performed in the area of optimality in control systems. Feedback in systems could be given more attention. Not only could an output be analyzed, but it could provide a specific response in different conditions. It is often said that this spark in ventures regarding optimal control was greatly due to the invention of calculus of variations. Advancements in calculus of variations allowed for Laplace transform to be directed at control theory. Over one hundred years after the invention of these advanced operations, analytical methods based on Laplace/Fourier transforms and complex variables were developed for predicting stability and performance of closed-loop control systems. Nonlinear instances were greatly influenced by the use of these various improvements made on the existing principles developed through the use of calculus. Using these methods, relevant control systems and their corresponding constraints were able to be modeled, manipulated and experimented upon as nonlinear programming became a process commonly used in control theory. This allowed the use of professional NLP codes that reliably handled inequality constraints on the controls and states. Optimal trajectory problems can be solved using NLP codes by parameterizing the control histories or the output histories using the concept of inverse OC. Eventually, concepts like these were brought into computer systems following the World War II era.

- Q2: For the below system: (35 points)
  - (a) Write the standard control canonical form as $\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{B}u$ and $y = \boldsymbol{Cx}$
  - (b) Design a control law of the form $u = -[K_1\ K_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ which will place the closed-loop poles at $s = -2 \pm 2j$



$$\frac{Y(s)}{U(s)} = \frac{s}{s^2+4}\ ; \quad \frac{Y(s)}{X(s)} \cdot \frac{X(s)}{U(s)} = s\left[\frac{1}{s^2+4}\right]$$

$$\frac{Y(s)}{X(s)} = s \qquad \frac{X(s)}{U(s)} = \frac{1}{s^2+4}$$

$$Y(s) = X(s)[s]; \quad U(s) = X(s)[(s^2+4)] = s^2X(s) + 4X(s)$$

Inverse Laplace for F(t):

$$Y(s) = x'(t) \qquad x''(t) + 4x(t) = u(t)$$

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}\begin{bmatrix} x(t) \\ x'(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(t)$$

$$y = [0\ 1]\begin{bmatrix} x(t) \\ x'(t) \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = [0\ 1]$$

$$A - BK = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}[K_1\ K_2] = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ K_1 & K_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4-K_1 & -K_2 \end{bmatrix}$$

Characteristic Equation:

$$\begin{bmatrix} s & -(1) \\ 4+K_1 & s+K_2 \end{bmatrix} = 0\ ; \quad s^2 + K_2 s + K_1 + 4 = 0$$

For closed-loop:

$$(s - s_1)(s - s_2) = 0$$

$$(s - (-2-2j))(s - (-2+2j)) = (s+2)^2 + 4 = s^2 + 4s + 8 = 0$$

We get $K_2 = 4$ from the acquired expression

$$K_1 + 4 = 8; \quad K_1 = 4$$

$$K = [K_1\ K_2] = [4\ 4]$$

- Q3: For the system described below $\ddot{\theta} = \theta + u, \ddot{p} = -0.5\theta - u$ (45 points)
  - (a) With the state defined as $\boldsymbol{x} = [\theta\ \dot{\theta}\ p\ \dot{p}]^T$, find the feedback gain $\boldsymbol{K}$ that places the closed-loop poles as $s = -1, ; -1; -1 \pm j$.
  - (b) Compute $\boldsymbol{N_x}$ and N_u for zero steady-state error to a constant command input on p, and compare step response of the this close-loop system with the one in (a) via Matlab figures.
  - (c) use LQR to calculate the feedback gain $\boldsymbol{K}$ that leads to the optimal control with ρ=1
  - Note: in (b) and (c) use $\boldsymbol{C} = [0\ 0\ \ 1\ \ 0]$ and $d = 0$, i.e., $y = p$.

$\ddot{\theta} = \theta + u, \ddot{p} = -0.5\theta - u$ $\qquad \dot{x} = Ax + bu$

A.) $x = [\theta\ \dot{\theta}\ p\ \dot{p}]^T = \begin{bmatrix} \theta \\ \dot{\theta} \\ p \\ \dot{p} \end{bmatrix}$ $\quad s = -1; -1; -1 \pm j \quad \dot{x}_1 = x_2 \quad \dot{x}_3 = x_4$

$\ddot{\theta} = \theta + u$

$$\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{p} \\ \ddot{p} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -0.5 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \\ p \\ \dot{p} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} u$$

$\ddot{p} = -0.5\theta + (-u)$

characteristic equation: $\quad \det[sI - (A - Bk)] = 0$

$$(s-(-1))(s-(-1))(s-(-1+j))(s-(-1-j)) = 0$$
$$(s+1)^2(s+1-j)(s+1+j) = s^4 + 4s^3 + 7s^2 + 6s + 2 = 0$$

$$A - Bk = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -0.5 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ k_1 & k_2 & k_3 & k_4 \\ 0 & 0 & 0 & 0 \\ -k_1 & -k_2 & -k_3 & -k_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1-k_1 & -k_2 & -k_3 & -k_4 \\ 0 & 0 & 0 & 1 \\ -0.5+k_1 & k_2 & k_3 & k_4 \end{bmatrix}$$

$$sI = \begin{bmatrix} s & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & s \end{bmatrix} \qquad sI - (A - Bk) = \begin{bmatrix} s & -1 & 0 & 0 \\ -1+k_1 & s+k_2 & k_3 & k_4 \\ 0 & 0 & s & -1 \\ 0.5-k_1 & -k_2 & -k_3 & s-k_4 \end{bmatrix}$$

$$\det[sI - (A - Bk)] = s^4 + (k_2 + (-k_4))s^3 + (k_1 + (-k_3) - 1)s^2 + (k_4 - 0.5k_4)s + k_3 - 0.5k_3$$

$k_2 - k_4 = 4 \quad k_1 - k_3 - 1 = 7 \quad 0.5k_4 = 6 \quad 0.5k_3 = 2$

$k_4 = 12 \quad k_2 = 16 \quad k_3 = 4 \quad k_1 = 12$

$$k = [12\ \ 16\ \ 4\ \ 12]$$

B.)

$N_x = [0 \quad 0 \quad 1 \quad 0]$

$N_u = 0$

C.)

$\rho = 1 \quad K = LQR(A, B, Q_2, R) = [5.8 \quad 7.5 \quad 1 \quad 4.4]$

MatLab: