Homework 6

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Name: Alex Gaskins Collaborators: John Oliveira, Trevor Dameika

1. A mass-spring system is suspended vertically with a mass of 1 kg, a damping force that is 3 times the instantaneous velocity, and a spring constant of 2 N/m. The system is driven by a periodic external force, $f(t) = 4\cos(2t)$ N. Let $y(t)$ denote the vertical displacement of the mass from its equilibrium position oriented so that y is increasing in the downward direction. (*i.e.*, $y > 0$ corresponds to the spring being stretched.) At $t = 0$, the mass is released at a position 1 m above the equilibrium point with a downward velocity of 2 m/sec.

   (a) Set up the initial value problem (IVP) being described in this problem.

   (b) Solve the IVP for $y(t)$.

A.) $x'' + 3x' + 2x = 4\cos(2t)$

B.) $m^2 + 3m + 2 = 0$

$m = -2$

$m = -1$

$x_c = c_1 e^{-2t} + c_2 e^{-t}$

x_p guess $= A\cos(2t) + B\sin(2t)$

$x_p' = -2A\sin(2t) + 2B\cos(2t)$

$x_p'' = -4A\cos(2t) - 4B\sin(2t)$

$-4A\cos(2t) - 4B\sin(2t) - 6A\sin(2t) + 6B\cos(2t) + 2A\cos(2t) + 2B\sin(2t) = 4\cos(2t)$

$-6A\sin(2t) + 6B\cos(2t) - 2A\cos(2t) - 2B\sin(2t) = 4\cos(2t)$

$6B - 2A = 4$

$-2B - 6A = 0$

$-20A = 4$

$A = -\frac{1}{5}$

$B = \frac{3}{5}$

$x(t) = x_c + x_p$

$x(t) = x_c + x_p$

$x(t) = c_1 e^{-2t} + c_2 e^{-t} - \frac{1}{5}\cos(2t) + \frac{3}{5}\sin(2t)$

$x'(t) = -2c_1 e^{-2t} - c_2 e^{-t} + \frac{2}{5}\sin(2t) + \frac{6}{5}\cos(2t)$

$x(0) = -1 \quad x'(0) = 2$

$-1 = c_1 + c_2 - \frac{1}{5}$

$-\frac{4}{5} = c_1 + c_2$

$2 = -2c_1 - c_2 + \frac{6}{5}$

$c_1 = 0$

$c_2 = -\frac{4}{5}$

$$\boxed{x(t) = -\frac{4}{5}e^{-t} - \frac{1}{5}\cos(2t) + \frac{3}{5}\sin(2t)}$$

$y = x^m$

$y' = mx^{m-1}$

$y'' = m^2 x^{m-2} - mx^{m-2}$

2. Solve the initial value problem. Describe the behavior of the solution as $x \to \infty$.

$$x^2 y''(x) + 3xy'(x) + 5y(x) = 0, \quad y(1) = 1, \quad y'(1) = -1.$$

$x^2(m^2 x^{m-2} - mx^{m-2}) + 3x(mx^{m-1}) + 5x^m =$

$m^2 - m + 3m + 5 = 0$

$m^2 + 2m + 5 = 0$

$-1 \pm 2i$

$y_c = x^{-1}(c_1\cos(2\ln(x)) + c_1\sin(2\ln(x)))$

$$y_c' = -\frac{1}{x^2}\left(c_1\cos(2\ln(x)) + c_2\sin(2\ln(x))\right) + x^{-1}\left(-2c_1\sin(2\ln(x)) + 2c_2\cos(2\ln(x))\right)$$

$$c_1 = 1$$

$$c_2 = 0$$

$$\boxed{y = x^{-1}\left(\cos(2\ln(x))\right)}$$

As $x \to \infty$, the solution approaches 0.

3. Find the general solution to the ODE, $L[y] = x^2y''(x) + xy'(x) - y(x) = \frac{3\ln x}{x^2}$.

$$m^2 - 1 = 0$$

$$m = \pm 1$$

$$y_c = c_1 x + c_2 x^{-1}$$

$$y_1 = x$$

$$y_2 = x^{-1}$$

$$W = \begin{vmatrix} x & x^{-1} \\ 1 & -x^{-2} \end{vmatrix} = -x^{-1} - x^{-1} = -2x^{-1}$$

$$W_1 = \begin{vmatrix} 0 & x^{-1} \\ \frac{3\ln(x)}{x^4} & -x^{-2} \end{vmatrix} = \frac{3\ln(x)}{x^5}$$

$$W_2 = \begin{vmatrix} x & 0 \\ 1 & \frac{3\ln(x)}{x^4} \end{vmatrix} = \frac{3\ln(x)}{x^3}$$

$$u_2' = \frac{W_2}{W} = \frac{\left(\frac{3\ln(x)}{x^3}\right)}{\left(-\frac{2}{x}\right)} = -\frac{3\ln(x)}{2x^2}$$

$$u_2 = \int u_2' = -\frac{3}{2}x^{-1}\left(-\ln(x)-1\right)$$

$$u_1' = \frac{W_1}{W} = \frac{\left(-\frac{3\ln(x)}{x^5}\right)}{\left(-\frac{2}{x}\right)} = \frac{3}{2}x^{-4}\ln(x)$$

$$u_1 = \int u_1' = -\frac{\ln(x)}{2x^3} - \frac{1}{6x^3}$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$y_p = \left(-\frac{\ln(x)}{2x^3} - \frac{1}{6x^3}\right)x + \left(-\frac{3}{2}x^{-1}\left(-\ln(x)-1\right)\right)x^{-1}$$

$$y_p = 3x^{-2}\left(3\ln(x) + 4\right)$$

$$y = y_c + y_p$$

$$\boxed{y = c_1 x + c_2 x^{-1} + 3x^{-2}\left(3\ln(x) + 4\right)}$$