- Q1: Read the below paper and write a summary (a minimum of half a page) to describe what you have learned from it. (20 points)

J. Willems; S. Mitter, “Controllability, observability, pole allocation, and state reconstruction,” IEEE Transactions on Automatic Control Vol. 16, No. 6, Dec 1971

The article discusses the importance of state-space modeling in modern control systems, especially with its applications to dynamic systems and on linear feedback controllers and linear filters for linear stationary systems. The concepts of controllability, reachability, reconstructability, and observability attempt to show why these concepts are important in linear systems theory, where the concepts allow us to solve the existence problem of closed-loop regulation of a linear time-invariant finite-dimensional system. A useful concept that is later mentioned is that of dual systems. Dual dynamical systems have the intriguing property that control problems of one become observation problems for the other, and vice versa, thus allowing for optimality in cases of estimation. They continue to mention how problems concerning control face the question of controllability, problems of state reconstruction and filtering face the question of reconstructability and problems of output-feedback control will face both the questions of reconstructability and controllability. However, concluding internal properties from input-output properties, such as minimality and stability of the state-space realization, face the questions of reachability and observability.

Q2: Consider the system with the transfer function (55 points)

A.) $\frac{Y(s)}{V(s)} = \frac{b_0 s^n + b_1 s^{n-1} \dots + b_n}{s^n + a_1 s^{n-1} \dots a_n}$ $\quad \frac{Y(s)}{V(s)} = \frac{9}{s^2-9}$

$n=2 \quad a_1=0 \quad a_2=9 \quad b_0=0 \quad b_1=0 \quad b_2=9$

$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -a_2 \\ 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_2 - a_2 b_0 \\ b_1 - a_1 b_0 \end{bmatrix}$

$y(x) = [0 \ 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + b_0 v(x)$

$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 9 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 9 \\ 0 \end{bmatrix} u \quad y = [0 \ 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$A_0 = \begin{bmatrix} 0 & 9 \\ 1 & 0 \end{bmatrix} \quad B_0 = \begin{bmatrix} 9 \\ 0 \end{bmatrix} \quad C_0 = [0 \ 1]$

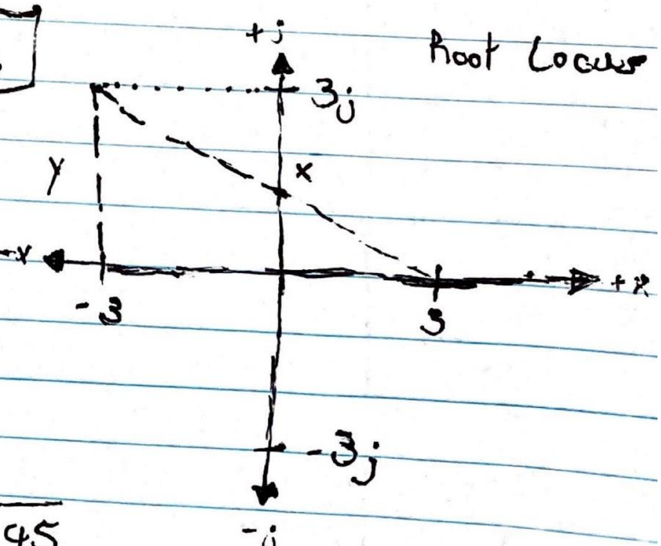
B.) $M = [B_0 : A_1 B_0]$ Controllable if $|M| \neq 0$

$A_0 B_0 = \begin{bmatrix} 0 & 9 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 9 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 9 \end{bmatrix}$

$M = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} = |M| = 9 \times 9 - 0 = 18$

$\boxed{|M| = 18 \ \therefore \text{Controllable}}$

Root Locus



C.) $s = -3 \pm 3j$

$G(s) = \frac{9}{s^2-9} = \frac{9}{(s+3)(s-3)}$

$x^2 = 3^2 + 6^2 \quad x = \sqrt{45}$

$K\Big|_{s=-3\pm 3j} = x \times y = 3\sqrt{45}$

$K = 20.12$

D.) $|N| = \begin{bmatrix} C \\ CA \end{bmatrix} \neq 0$ observable

$C \cdot A = [0\ 1] \begin{bmatrix} 0 & 9 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \end{bmatrix} [1\ 0]$

$|N| = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 0 - 1 = -1 \quad |N| \neq 0 \therefore$ observable

E.) $S = -12 \pm 12j$

$G = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$

Characteristic Polynomial: $|\lambda_1 - (A - G_0)| = 0$

$A - G_c \begin{bmatrix} 0 & 1 \\ 12 & 0 \end{bmatrix} - \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} [12\ 1]$

$= \begin{bmatrix} 0 & 1 \\ 12 & 0 \end{bmatrix} - \begin{bmatrix} 12g_1 & g_1 \\ 12g_2 & g_2 \end{bmatrix} = \begin{bmatrix} -12g_1 & 1-g_1 \\ 12-12g_2 & -g_2 \end{bmatrix}$

$[AI - CA - G_c] = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -12g_1 & 1-g_1 \\ 12-12g_2 & -g_2 \end{bmatrix}$

$= \begin{bmatrix} \lambda + 12g_1 & g_1 - 1 \\ 12g_2 - 16 & \lambda + g_2 \end{bmatrix} \quad \begin{matrix} \lambda + 12g_1(\lambda + g_2) - (12g_2 - 12)(g_2 - 1) \\ (\lambda + 12g_1)(\lambda + g_2) - (12g_2 - 12)(g_2 - 1) \end{matrix}$

$|\lambda_2 - (A - G_c)| = 0$

$\lambda^2 + \lambda(12g_1 + g_2) + 12g_1g_2 - 12g_1g_2 - 12 + 12g_1 + 12g_2 = 0$ i

$[\lambda - (-12 + g12)][A - (-12 - j12)] = 0$

$(\lambda + 12 - 12j)(\lambda + 12 + 12j) = 0$

$(\lambda + 12)^2 + 144 = 0 \approx \lambda^2 + 24\lambda + 288 = 0$ ii

$\lambda^2 + \lambda(12g_1 + g_2) + 12g_1 + 12g_2 - 12 = 0$ iii

E.) compare coefficients for i and ii

$$12g_1 + g_2 = 0$$

$$12g_1 + 12g_2 - 12 = 288$$

$$12(g_1 + g_2 - 1) = 288$$

$$g_1 + g_2 - 1 = 24$$

$$g_1 + g_2 = 25$$

$$12g_1 + g_2 = 24$$

$$-11g_1 = 1 \qquad g_1 = -\frac{1}{11} = -0.0909$$

$$g_2 = 24 + 1 + 0.0909 = 25.0909$$

$$G = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = 0 \qquad G = \begin{bmatrix} -0.0909 \\ 25.0909 \end{bmatrix}$$

Q3: For the below system (25 points)

$$\dot{x} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u; \quad y = [1 \quad 0 \quad 0]x + [0]u$$

Design a reduced-order estimator with y=x1 as the measurement, and the observer-error poles at -0.1 and -0.1

$$\dot{x} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \; 0 \; 0] x + [0] u$$

Characteristic Equation:

$$|sI - A_{bb} + K_e A_{ab}| = (s - M_1)(s - M_2) = (s + 0.1)^2$$

$$= s^2 + 0.2s + 0.01 = 0$$

$$K_e = \phi(A_{bb}) \begin{bmatrix} A_{ab} \\ A_{ab} A_{bb} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\phi(A_{bb}) = A_{bb}^2 + \hat{\alpha}_1 A_{bb} + \hat{\alpha}_2 I = A_{bb}^2 + 0.2 A_{bb} + 0.01 I$$

$$A_{aa} = 0 \quad A_{ab} = [-1 \; 0] \quad A_{ba} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A_{bb} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad B_a = 0 \quad B_b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$K_e = \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^2 + 0.2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 0.01 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} A_{ab} \\ A_{ab} A_{bb} \end{bmatrix}$$

$$A_{ab} = [-1 \; 0] \quad A_{bb} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad A_{ab} A_{bb} = [0 \; -1]$$

$$\begin{bmatrix} A_{ab} \\ A_{ab} A_{bb} \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$K_e = \left(\begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix} \right) \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$K_e = \begin{bmatrix} -0.01 \\ 0 \end{bmatrix}$$

$$\dot{n} = (A_{bb} - k_e A_{ab})n + [(A_{bb} - k_e A_{ab})k_e + A_{ba} - k_e A_{aa}]y + (B_b - k_e B_a)u$$

$$A_{bb} - k_e A_{ab} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} -0.01 \\ 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0.01 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -0.01 & 1 \\ 0 & 0 \end{bmatrix}$$

$$[A_{bb} - k_e A_{ab}]k_e = \begin{bmatrix} -0.01 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -0.01 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.0010 \\ 0 \end{bmatrix}$$

$$A_{ba} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\dot{n} = \begin{bmatrix} -0.01 & 1 \\ 0 & 0 \end{bmatrix} n + \begin{bmatrix} -0.9999 \\ 0 \end{bmatrix} y + \begin{bmatrix} a \\ 1 \end{bmatrix} n$$

$$\begin{bmatrix} n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} n_2 \\ n_3 \end{bmatrix} - k_e y$$