Exam 2

Monday, April 12, 2021 9:44 AM

"I pledge my honor I have abided by the Stevens Honor system."

- Alex [signature]

Problem 1. *(20 points)*

Let

- $f : \mathbb{N} \to \mathbb{N}$ be defined as $f(x) = x^2$,
- $g : \mathbb{Z} \to \mathbb{Z}$ be defined as $g(x) = -x + 3$
- $h : \mathbb{Z} \to \mathbb{Z}$ be defined as $\begin{cases} h(x) = x/2 \text{ if } x \text{ is even} \\ h(x) = 0 \text{ if } x \text{ is odd} \end{cases}$

Select all of the properties that f, g, h satisfy.
Explain your answers! Correct answers with no explanations will not be awarded full points.

	Injective	Surjective	Bijective
f	✓	×	×
g	✓	✓	✓
h	×	✓	×

F:

$f(x) = x^2$
$f(1) = 1$
$f(2) = 4$
$f(3) = 9$
$f(4) = 16$

It is injective, for every x there is one unique y value that corresponds to it (for $N \to N$)

It is not surjective because it skips numbers in the set of N.

$(x^2 = y \; ; \; \pm\sqrt{y} = x)$

g:

$g(x) = -x + 3$
$g(0) = 3$ $\quad g(-1) = 4$
$g(1) = 2$ $\quad g(-2) = 5$
$g(2) = 1$ $\quad g(-3) = 6$
$g(3) = 0$
$g(4) = -1$

It is injective for every x has a unique y value corresponding to is

It is surjective on $Z \to Z$ because every y has at least one x and every number is represented.

$(-x + 3 = y \; ; \; x = 3 - y)$

h:

$h(x) = \frac{x}{2}$ if even

h:

$h(x) = \frac{x}{2}$ if even

$h(x) = 0$ if odd

$h(2) = 1$ $\quad h(3) = 0$

$h(4) = 2$ $\quad h(5) = 0$

$h(6) = 3$

$h(8) = 4$

It is not injective for every x is not paired with a unique y (0 pairs with all odd x's)

It is surjective because every y has a corresponding x on $\mathbb{Z} \to \mathbb{Z}$

$\left(\frac{x}{2} = y;\ x = 2y \text{ or } 0 = x\right)$

Problem 2. *(10 points)*

Assume that we know that $\ln(1) = 0$ and $\ln(n+1) - \ln(n) > \frac{1}{n+1}$ for $n \geq 1$. Prove that $\ln(n) > \frac{1}{2} + \ldots + \frac{1}{n}$ for natural $n \geq 2$.

$\ln(1) = 0 \quad \ln(n+1) - \ln(n) > \frac{1}{n+1}$ for $n \geq 1$

$\cancel{\ln(2)} - \ln(1) > \frac{1}{2}$ for $n=1$

$\cancel{\ln}(3) - \cancel{\ln}(2) > \frac{1}{3}$ for $n=2$

$\cancel{\ln}(4) - \ln\cancel{(3)} > \frac{1}{4}$ for $n=3$

so $\ln(n-1+1) - \ln(n-1) > \frac{1}{(n-1)+1}$ for $n = n-1$

$\ln(n) - \ln(1) > \frac{1}{n} + \frac{1}{(n-1)} + \ldots \frac{1}{4} + \frac{1}{3} + \frac{1}{2}$

$\therefore \ln(n) > \frac{1}{2} + \frac{1}{3} + \ldots \frac{1}{n}$ where $\ln(1) = 0$

Problem 3. *(20 points)*

Which of the following have integer solutions x (or α, β)?

1. $19x \equiv 3 \pmod{40}$
2. $15x \equiv 7 \pmod{40}$
3. $15\alpha + 40\beta = 25$
4. $15\alpha + 40\beta = 19$

In cases where a solution exists, find one (show your work). In cases where a solution doesn't exist, explain why.

3 points for each correct answer with explanation, the rest of points for finding solutions

1.) $\gcd(19, 40) = 1$

(used calculator to check)

$19(17) = 3 \bmod 40$

$\boxed{x = 17}$

2.) $x = 5(8) + \frac{40}{5}(0) = 5(8)$

$x = 5(8) + 8 = 13(8)$

$x = 5(8) + 8(2) = 21(8)$

$x = 5(8) + 8(3) = 29(8)$

$x = 5(8 + 8(4) = 37(8)$

$\boxed{x = 5(8), 13(8), 21(8), 29(8), 37(8) \ldots}$

3.) $15\alpha + 40\beta = 25$

$\boxed{\alpha = 1 \quad \beta = \frac{1}{4}}$

$15(1) + \frac{40}{4} = 25$ ✓

4.) $15\alpha + 40\beta = 19$

$\boxed{\alpha = 1 \quad \beta = \frac{1}{10}}$

$15(1) + 40/10 = 19$ ✓

Problem 4. *(20 points)*

Which of the following systems have an integer solution by Chinese Remainder Theorem?

1. $\begin{cases} n \equiv 3 \pmod{29} \\ n \equiv 9 \pmod{13} \end{cases}$ gcd(13, 29) = 1 ✓

2. $\begin{cases} n \equiv 7 \pmod{15} \\ n \equiv 1 \pmod{35} \end{cases}$ gcd(15, 35) = 5 × gcd ≠ 1 so CRT does not work

3. $\begin{cases} n \equiv 2 \pmod{6} \\ n \equiv 3 \pmod{7} \\ n \equiv 7 \pmod{9} \end{cases}$ gcd(6, 9) = 3 × gcd ≠ 1 so CRT does not work

In cases where Chinese Remainder Theorem is applicable, find a solution (show your work). In cases where it's not, explain why.

You don't have to simplify your answer: you can have it as a product/sum of number, e.g. something like $19 \cdot 23 - 17 \cdot 21 \cdot 7$ is acceptable as an answer.

9 points for correct answers with explanations, 11 points for finding solutions.

1. $n \equiv 3 \bmod 29$

$n \equiv 9 \bmod 13$

$n = 13(2) + 29(3)$

$\boxed{n = 113}$

$m_1 = 29$ $m_2 = 13$

so $29 \cdot 13$ where $a = 3$ and $b = 9$

only 1 has a solution