Homework 12 Name: Alex Gaskins Collaborators: John Oliveira, Trevor Dameika

Sunday, November 29, 2020 5:49 PM

1. Consider the function $f(x) = 1 - (x-1)^2$ defined on the interval $0 < x < 2$.

   (a) Derive a general expression for the coefficients in the Fourier Cosine series for $f(x)$. Then write out the Fourier series through the *first four nonzero terms*.

   (b) Graph the extension of $f(x)$ on the interval $(-6, 6)$ that represents the pointwise convergence of the Cosine series in (a). At jump discontinuities, identify the value to which the series converges.

A.)

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

$$a_0 = \frac{2}{L}\int f(x)\,dx$$

$$a_n = \frac{2}{L}\int f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$L = 2$

$$a_0 = \frac{2}{2}\int_0^2 \left(1-(x-1)^2\right)dx$$

$$a_0 = \left[x - \frac{(x-1)^3}{3}\right]_0^3$$

$$a_0 = \frac{4}{3}$$

$$a_n = \frac{2}{2}\int_0^2 (1-(x-1)^2\cos\left(\frac{n\pi x}{2}\right)dx$$

$$a_n = -\frac{8}{n^2\pi^2}\left[\cos(n\pi)+1\right]$$

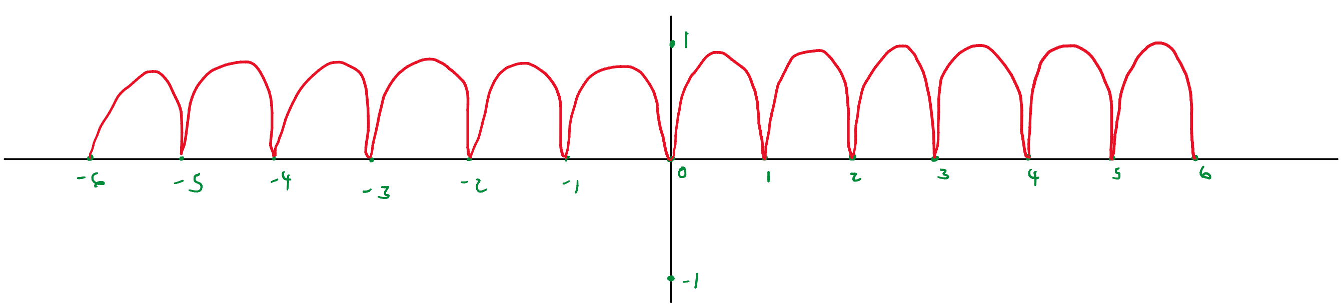
$$a_n = -\frac{8((-1)^2+1)}{n^2\pi^2}$$

$$f(x) = \frac{2}{3} + \sum_{n=1}^{\infty} \frac{8((-1)^2+1)}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right)$$

$$f(x) = \frac{2}{3} - \frac{8}{\pi^2}\left[\frac{2}{2^2}\cos(\pi x) + \frac{2}{4^2}\cos(2\pi x) + \frac{2}{6^2}\cos(3\pi x)\right]$$

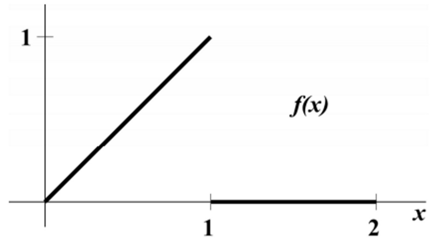
$$\boxed{f(x) = \frac{2}{3} - \frac{4}{\pi^2}\left[\cos(\pi x) + \frac{\cos(2\pi x)}{4} + \frac{\cos(3\pi x)}{6}\right]}$$

B.)



Converges at all points

2. Consider the function $f(x)$ defined on $0 < x < 2$ (*see graph*).



(a) Derive a general expression for the coefficients in the Fourier Sine series, $f(x) \sim \sum_{n=1}^{\infty} b_n \sin(n\pi x/2)$.

(b) Graph the extension of $f(x)$ on the interval $(-6, 6)$ that represents the pointwise convergence of the Sine series. At jump discontinuities, identify the value to which the series converges.

A.)

$$f(x) = \begin{cases} x, & 0 \le x < 1 \\ 0, & 1 < x \le 2 \end{cases}$$

$$b_n = \frac{2}{\pi}\left[\int_0^1 x \sin\left(\frac{n\pi x}{2}\right) dx + \int_1^2 0\, dx\right]$$

$$b_n = \frac{2}{\pi}\int_0^1 x \sin\left(\frac{n\pi x}{2}\right) dx$$

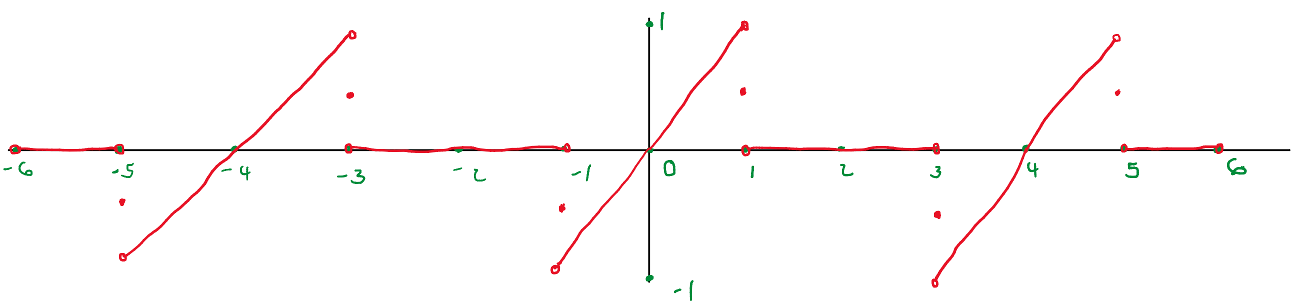
$$b_n = \frac{2}{\pi}\left[-\frac{2x}{n\pi}\cos\left(\frac{n\pi x}{2}\right) + \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi x}{2}\right)\right]_0^1$$

$$\boxed{f(x) = \sum_{n=1}^{\infty} \frac{4}{n\pi^2}\left[\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right) - \cos\left(\frac{n\pi}{2}\right)\right]\sin\left(\frac{n\pi x}{2}\right)}$$

B.)

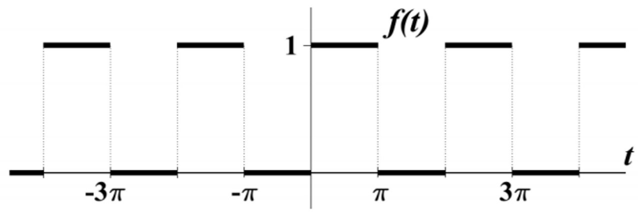
$$f(1) = \frac{f(1^-) + f(1^+)}{2} \qquad f(-1) = \frac{f(-1^-) + f(-1^+)}{2}$$

$$f(1) = \frac{1}{2} \qquad f(-1) = -\frac{1}{2}$$



3. **Linear Oscillator with Periodic Forcing.**

Consider the second-order differential equation, $L[y] = y'' + 2y = f(t)$, where $f(t)$ is the periodic square wave shown in the figure.



The Fourier trigonometric series for $f(t)$ is given by,

$$f(t) \sim \frac{1}{2} + \sum_{k=1}^{\infty} \frac{2}{(2k-1)\pi} \underbrace{\sin((2k-1)t)}_{f_k(t)} = \frac{1}{2} + \frac{2}{\pi}\left(\sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \cdots\right).$$

(a) Derive a Fourier series representation for a *particular solution* to $L[y] = f(t)$, as follows:

- For $k = 0, 1, 2, \ldots$, find a particular solution $y_k(t)$ satisfying $y_k'' + 2y_k = f_k(t)$, where $f_k(t)$ is the k^{th} term in the Fourier series for $f(t)$.
- Sum over all $y_k(t)$ to get the solution $y_p(t)$.

(b) Write out the *first four terms* in the Fourier series for $y_p(t)$.

A.) $y_0 = y'' + 2y = \frac{1}{2}$

at $k = 0$

$s^2 = (2k-1)^2$

$$y_k = \frac{f_k(t)}{s^2+2} \longrightarrow y_k = \frac{2}{(2k-1)\pi}\left(\frac{\sin(2k-1)t}{2-(2k-1)^2}\right)$$

$y_k = (s^2+2)y = \frac{1}{2}$

$s \to 0$

$$\frac{\left(\frac{1}{2}\right)}{0+2} = \frac{1}{4}$$

$y_{p_k} = y_0 + y_k$

$$\boxed{y_{p_k} = \frac{1}{4} + \frac{2}{(2k-1)\pi}\left(\frac{\sin(2k-1)t}{2-(2k-1)^2}\right)}$$

B.)

$$y_p = \frac{1}{4} + \sum_{k=1}^{3} \frac{2}{(2k-1)\pi}\left(\frac{\sin(2k-1)t}{2-(2k-1)^2}\right)$$

$n = 1$

$$\frac{2}{(2(1)-1)\pi}\left(\frac{\sin(2(1)-1)t}{2-(2(1)-1)^2}\right) = \frac{2}{\pi}\sin(t)$$

$n = 2$

$$\frac{2}{(2(2)-1)\pi}\left(\frac{\sin(2(2)-1)t}{2-(2(2)-1)^2}\right) = -\frac{2}{21\pi}\sin(3t)$$

$n = 3$

$$\frac{2}{(2(3)-1)\pi}\left(\frac{\sin(2(3)-1)t}{2-(2(3)-1)^2}\right) = -\frac{2}{115\pi}\sin(5t)$$

$$\boxed{y_p = \frac{1}{4} + \frac{2}{\pi}\sin(t) - \frac{2}{21\pi}\sin(3t) - \frac{2}{115\pi}\sin(5t)}$$