Final

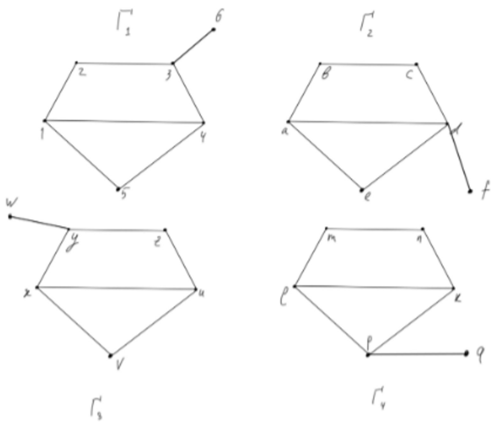
Thursday, May 13, 2021 10:41 PM

"I pledge my honor I have abided by the Stevens Honor system."

- Alex [illegible]

Problem 1. *(15 points)*

Consider the following 4 graphs:



(Γ_1 has vertices $1-6$; Γ_2 has vertices $a-f$; Γ_3 has verices $u-z$; and Γ_4 has vertices $k-q$)

a) Are graphs Γ_1 and Γ_2 isomorphic?

b) Are graphs Γ_1 and Γ_3 isomorphic?

c) Are graphs Γ_1 and Γ_4 isomorphic?

d) Are graphs Γ_2 and Γ_3 isomorphic?

e) Are graphs Γ_2 and Γ_4 isomorphic?

f) Are graphs Γ_3 and Γ_4 isomorphic?

Each correct answer is 1 point, the rest of the points are for explanations.

A.) Γ_1 and Γ_2 are not isomorphic, because the largest degree in Γ_2 is 4, but the largest degree in Γ_1 is 3.

B.) Γ_1 and Γ_3 are isomorphic, as Γ_3 is the same graph as Γ_1, it is just flipped vertically.

$u \to 1$

$z \to 2$

$y \to 3$

$x \to 4$

$v \to 5$
$w \to 6$

C.) Γ_1 and Γ_4 are not isomorphic, because 3 doesn't connect to 1, while p connects to both l and k where:

$$l \to 1 \quad \text{and} \quad h \to 4$$

D.) Γ_2 and Γ_3 are not isomorphic, because the largest degree in Γ_2 is 4, but the largest degree in Γ_3 is 3.

E.) Γ_2 and Γ_4 are not isomorphic, because the largest degree in Γ_2 is 4, but the largest degree in Γ_4 is 3.

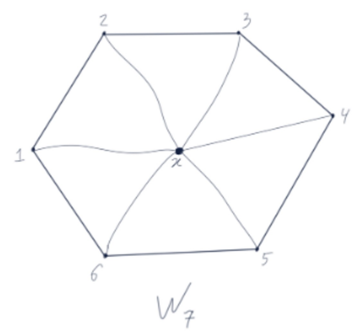
F.) Γ_3 and Γ_4 are not isomorphic, because y doesn't connect to u, while p connects to both l and k where:

$$l \to u \quad \text{and} \quad h \to x$$

Problem 2. *(15 points)*
Define the graph W_n (for $n \geq 3$) as follows:

1. It has n vertices: $n-1$ numbered vertices $1, 2, \ldots, n-1$, and x– a special vertex.
2. Vertices $1, 2, \ldots, n-1$ are connected in a cycle. That is there is an edge $\{1, 2\}$, there is an edge $\{2, 3\}, \ldots$, there is an edge $\{n-2, n-1\}$, and there is an edge $\{n-1, 1\}$ (closing the cycle).
3. Finally, the special vertex x is connected to all numbered vertices $1, 2, \ldots, n-1$.

W_7 is pictured below.



a) *(4 points)* Does W_7 have an Euler circuit?

b) *(4 points)* Does W_7 have a Hamiltonian circuit?

c) *(2 points)* Draw W_8 and W_6. Include the labels for vertices!

d) *(2 points)* How many edges does a graph W_n have (in terms of n)?

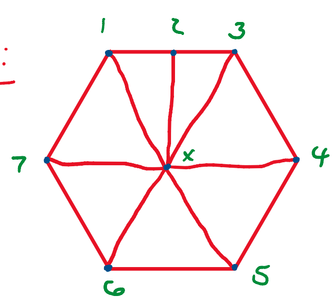
e) *(3 points)* Does an arbitrary graph W_n have Euler/Hamiltonian circuits? The answer might depend on n.

Explain your answers!

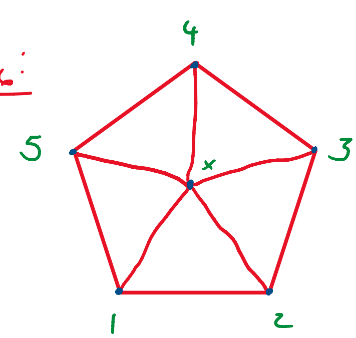
A.) No, because W_7 has verticies with odd degrees.

B.) Yes, $5 \to x \to 4 \to 3 \to 2 \to 1 \to 6 \to 5$

C.) W_7:



W_6:



D.) W_n has $(n-1) + (n-1) = 2(n-1)$ edges

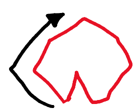
.) W_n has $(n-1)+(n-1) = 2(n-1)$ edges

[$n-1$ edges are adacent to x + $(n-1)$ edges on the longest cycle]

E.) W_n doesn't have an Euler circuit for $n>3$ where W_n has verticies of odd degrees

& W_3 has an Euler circuit.

W_n has a Hamiltonian Circuit for every n



Problem 3. *(10 points)*

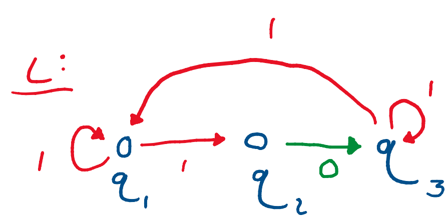
Define the language $L = \{\underbrace{11\ldots1}_{n \text{ ones}}0\underbrace{11\ldots1}_{n \text{ ones}} \mid n \geq 0\}$. In other words, L consists of all all words that have exactly one 0 exactly in the middle of that word, and the rest of the letters are 1's. Is L regular?

Explain your answer! If you state that the language is regular, write a regular expression (or draw an FSA) that defines that language. If you state that it is not regular, prove it.

3 points for the answer, the rest for the explanation.

$L = \{101, 11011, 1110111\ldots\}$

$L = (1)^* \, 0 \, (1)^*$



for $n=0$, $L = 101$

for $n=1$, $L = 11011$

for $n=2$, $L = 1110111$

L is regular, because it creates a standard pattern represented by $L = (1)^* 0 (1)^*$ for all input strings which satisfies the expression.

Problem 4. *(15 points)*

In this problem alphabet $\Sigma = \{a, b, c\}$. Write a regular expression that defines each of the following languages (different parts are not necessarily related)

1. *(2 points)* L_1 that consists of all words that do not contain c (i.e. that consist only of a, b),

$L_1 = \{\epsilon, a, b, ab, ba, abab \ldots\}$

All strings with a, b and nil (ϵ)

$$\boxed{(a+b)^*}$$

2. *(7 points)* L_2 that consists of all words that consist of at most two letters (e.g. $\epsilon, aaaa, abaab, acac, cbbbc$ are such words, while cab is not such a word).

$L_2 = \{\epsilon, a, b, c, ab, ac, bc \ldots\}$

All strings with two letters and nil (ϵ)

$$\boxed{(a+b)^* + (a+c)^* + (b+c)^*}$$

3. *(6 points)* L_3 all words that start with a, end with b, and have at least one letter c somewhere in the middle.

$L_3 = \{acb, accb, abacab \ldots\}$

Starts with a, c in middle, ends with b

$$\boxed{a(a+b)^* c^* (a+b)^* b}$$

In the next two problems, imagine you are a summer camp instructor for a group of 20 kids and you want to split them into two different teams (say Red team and Blue team). In all of the problems, we assume that each kid gets assigned to exactly one of those teams.

Problem 5. Combinatorics: basic selection processes. *(15 points)*

a) *(5 points)* How many ways are there to split 20 kids into two teams (of any size, including 0)?

b) *(5 points)* How many ways are there to split 20 kids into two teams of 10 kids?

c) *(5 points)* Assume you fix a team of 10 kids. How many ways are there select a captain, a vice captain and a cheerleader out of those 10 kids?

Each of the above problems corresponds to one of the basic selection processes (with correct parameters). Explain your choice of the selection process and parameters for full credit!

A.) 2 teams with $n=20$

team size up to 20

Team size 0 gives 1 choice

Team size 1 gives 20 choices

Team size 2 gives $\binom{20}{2}$ choices

so size n gives $\binom{20}{n}$ choices

$$\frac{1}{2}\sum_{n=0}^{20}\binom{n}{n} = \frac{1}{2}2^{20} = \boxed{2^{19} \text{ ways}}$$

B.) Out of 20, each team must be size 10

Different team possibilities, choose 10 each time out of a set of 20. (20 choose 10)

$$\boxed{\binom{20}{10} \text{ ways}}$$

C.)

Captain is first selected out of 10

then vice captain out of 9

Captain is first selected out of 10
then vice captain out of 9
then cheerleader out of 8

$10(9)(8) = \boxed{720 \text{ ways}}$

Problem 6: Combining multiple selection processes. *(10 points)*

a) *(5 points)* How many ways are there to split 20 kids into two teams in such a way that each team has at least 9 kids?

b) *(5 points)* How many ways are there to split 20 kids into two teams of 10, and then assign a captain, a vice captain, and a cheerleader for each of those two teams?

Each of those problems combines several selection processes. Explain your answers for full credit!

You can write your answers using combinatorial expressions (binomial coefficients, powers, factorials, etc) and combinations of combinatorial expressions (e.g. $100! + \binom{1000}{50}\binom{10}{3}^2$ is a valid expression).

A.) If red team has 9 then blue has 3 choices for size (9, 10 or 11 kids)

If red team has 10 then blue has 3 choices for size (9 or 10 kids)

If teams have 9 kids, red has $\binom{20}{9}$ choices and blue has $\binom{11}{9}$ choices.

Or one team has 9 and other 10

$\binom{20}{9}$ $\binom{11}{10}$ Remaining

9 and 9, $\binom{20}{9}\binom{11}{9}$ or 9 and 10, $\binom{20}{9}\binom{11}{10}$
and so on for sum of 20.

so $\binom{20}{9}\binom{11}{9} + \binom{20}{9}\binom{11}{10} + \binom{20}{9} + \binom{20}{10}$

$$\boxed{\binom{20}{9}\left(\binom{11}{9}+\binom{11}{10}+1\right)+\binom{20}{10} \text{ ways}}$$

B.) Each team has 10 so $\binom{20}{10}$

Captain is first selected out of 10

then vice captain out of 9

then cheerleader out of 8

For both teams $10\binom{20}{10}$

So total is $\left(10(9)(8)\binom{20}{10}+(10(9)(8)\binom{20}{10}\right)$

$$2\left(720\binom{20}{10}\right)$$

$$\boxed{1440\left(\binom{20}{10}\right) \text{ ways}}$$