Homework 8

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Name: Alex Gaskins

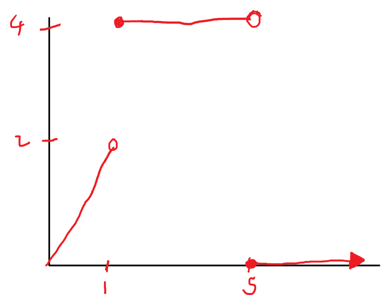
Collaborators: John Oliveira, Trevor Dameika

1. **Laplace Transforms from the definition:** $F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt$

Sketch the graph of the function $f(t)$ and calculate its Laplace Transform directly from the definition as an improper integral. *Be sure to simplify (clean up) your final answer.*

$$f(t) = \begin{cases} 2t, & 0 \le t < 2 \\ 4, & 2 \le t < 5 \\ 0, & 5 \le t < \infty \end{cases}$$

$$L\{2t\} = 2\int_0^2 te^{-st}\,dt$$



$$uv - \int v\,du$$

$$u = t \quad dv = e^{-st}$$

$$du = 1 \quad v = \frac{e^{st}}{s}$$

$$\frac{te^{st}}{s} - \int \frac{e^{st}}{s}$$

$$\left[2\left(\frac{te^{st}}{s} - \frac{e^{st}}{s^2}\right)\right]_0^2 \; ; \; \frac{2}{s^2}\left(\left[2se^{2s} - e^{2s}\right] - \left[-1\right]\right)$$

$$= \frac{2}{s^2}\left(e^{2s}(2s-1)+1\right)$$

$$L\{4\} = \int_2^5 4e^{-st}\,dt$$

$$4\int -\frac{e^u}{s}\,du$$

$$u = -st$$

$$= \left[-\frac{4}{s}e^{-st}\right]_2^5$$

$$-\frac{4}{s}\left(e^{-5s} - e^{-2s}\right)$$

$$L\{0\} = \int_5^\infty 0\,dt$$

$$= 0$$

$$\boxed{\frac{2}{s^2}\left(e^{2s}(2s-1)+1\right) - \frac{4}{s}\left(e^{-5s} - e^{-2s}\right)}$$

2. Determine the inverse Laplace transforms.

(a) $f(t) = \mathcal{L}^{-1}\left\{\frac{3}{s^5}\right\}$

$$L^{-1}\left(\frac{3}{s^5}\right)$$

$$L^{-1}\left(\frac{n!}{s^{n+1}}\right) = t^n$$

$$\frac{1}{8}L^{-1}\left(\frac{24}{s^5}\right)$$

$$= t^4$$

$$\boxed{y = \frac{t^4}{8}}$$

$\mathcal{L}\left(\frac{\quad}{s^{n+1}}\right) = \cdot$

$n = 4$

$$\boxed{y = \frac{t^4}{8}}$$

(b) $g(t) = \mathcal{L}^{-1}\left\{\frac{-3s+4}{s^2+9}\right\}$

$$\frac{-3s}{s^2+9} + \frac{4}{s^2+9}$$

$$\frac{4}{3}\mathcal{L}^{-1}\left(\frac{3}{s^2+3^2}\right) = \frac{4}{3}\sin(3t)$$

$$-3\mathcal{L}^{-1}\left(\frac{s}{s^2+3^2}\right) = -3\cos(3t)$$

$$\boxed{y = \frac{4}{3}\sin(3t) - 3\cos(3t)}$$

(c) $h(t) = \mathcal{L}^{-1}\left\{\frac{-3s+4}{s^2+4s+20}\right\}$

$s^2 + 4s = -20$

$s^2 + 4s + \left(\frac{4}{2}\right)^2 = -20 + \left(\frac{4}{2}\right)^2$

$s^2 + 4s + 4 = -16$

$(s+2)^2 = -16$

$(s+2)^2 + 16$

$$\frac{-3(s+2)}{(s+2)^2+16} + \frac{10}{(s+2)^2+16}$$

$$-3\mathcal{L}^{-1}\left(\frac{s+2}{(s+2)^2+16}\right) \approx \frac{s}{s^2+4^2} = -3\cos(4t)$$

$$10\mathcal{L}^{-1}\left(\frac{1}{(s+2)^2+16}\right) \approx \frac{1}{4}\,\frac{4}{s^2+4^2} = \frac{5}{2}\sin(4t)$$

$$\boxed{y = \frac{5}{2}\sin(4t) - 3\cos(4t)}$$

3. Determine the inverse Laplace transforms.

(a) $f(t) = \mathcal{L}^{-1}\left\{\frac{3s^2+8s+9}{(s+2)^3}\right\}$

$$\frac{3s^2+8s+9}{(s+2)^3} = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{(s+2)^3}$$

$$3s^2 + 8s + 9 = A(s+2)^2 + B(s+2) + C$$

$\underline{s = -2:}$

$5 = C$

$$3s^2 + 8s + 4 = As^2 + 4As + 4A + Bs + 2B$$

$$A = 3$$

$$4A + B = 8$$

$$4A + 2B = 4$$

$$B = -4$$

$$3\,\mathcal{L}^{-1}\left(\frac{1}{s+2}\right) = 3e^{-2t}$$

$$-4\,\mathcal{L}^{-1}\left(\frac{1}{(s+2)^2}\right) = -4te^{-2t}$$

$$\mathcal{L}^{-1}\left(\frac{3}{s+2} - \frac{4}{(s+2)^2} + \frac{5}{(s+2)^3}\right)$$

$$5\,\mathcal{L}^{-1}\left(\frac{1}{(s+2)^3}\right) = \frac{5}{2}t^2e^{-2t}$$

$$\boxed{y = 3e^{-2t} - 4te^{-2t} + \frac{5}{2}t^2e^{-2t}}$$

(b) $y(t) = \mathcal{L}^{-1}\left\{\dfrac{s^2 + 11s + 20}{(s^2 + 4s + 8)(s+1)^2}\right\}.$

$$\frac{s^2 + 11s + 20}{(s^2+4s+8)(s+1)^2} = \frac{Bs + A}{s^2 + 4s + 8} + \frac{C}{s+1} + \frac{D}{(s+1)^2}$$

$$s^2 + 11s + 20 = Bs + A(s+1)^2 + C(s^2+4s+8)(s+1) + D(s^2+4s+8)$$

$$s = -1$$

$$10 = 5D$$

$$D = 2$$

$$-s^2 + 3s + 4 = Bs^3 + 2Bs^2 + Bs + As^2 + 2As + A + Cs^3 + 5Cs^2 + 12Cs + 8C$$

$$B + C = 0 \qquad C = -B$$

$$2B + A + 5C = -1$$

$$B + 2A + 12C = 3$$

$$A + 8C = 4 \qquad A = 8B + 4$$

$$2B + (8B + 4) - 5B = -1$$

$$(-1) + 2A + 12(1) = 3$$

$$5B = -5$$

$$2A = -8$$

$$B = -1$$

$(-1) + 2A + 10(1) =$

$2A = -8$

$A = -4$

$5B = -5$

$B = -1$

$C = 1$

$$\mathcal{L}^{-1}\left(\frac{-s-4}{s^2+4s+8} + \frac{1}{s+1} + \frac{2}{(s+1)^2}\right)$$

$$\mathcal{L}^{-1}\left(\frac{-s-4}{s^2+4s+8}\right) =$$

$s^2 + 4s = -8$

$s^2 + 4s + \left(\frac{4}{2}\right)^2 = -8 + \left(\frac{4}{2}\right)^2$

$s^2 + 4s + 4 = -4$

$-s - 4 = -(s+2) - 2$

$(s+2)^2 = -4$

$(s+2)^2 + 4$

$$= \frac{-(s+2)-2}{(s+2)^2+4}$$

$$= \frac{-(s+2)}{(s+2)^2+4} - \frac{2}{(s+2)^2+4}$$

$-\mathcal{L}^{-1}\left(\frac{s+2}{(s+2)^2+4}\right) \approx \frac{s}{s^2+4} = e^{-2t}$

$\frac{s}{s^2+2^2} = \cos(2t)$

$= -e^{-2t}\cos(2t)$

$-2\mathcal{L}^{-1}\left(\frac{1}{(s+2)^2+4}\right) \approx \frac{1}{2}\frac{2}{s^2+2^2} = \frac{1}{2}\sin(2t)$

$= -e^{-2t}\sin(2t)$

$\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) = e^{-t}$

$2\mathcal{L}^{-1}\left(\frac{1}{(s+1)^2}\right) = 2t\,e^{-t}$

$$\boxed{y = -e^{-2t}\cos(2t) - e^{-2t}\sin(2t) + e^{-t} + 2t\,e^{-t}}$$

4. Use the Laplace transform to solve the following initial value problem for $y(t)$.

$$y''(t) + 2y'(t) + 10y = 9e^{-t}, \quad y(0) = 7, \; y'(0) = -1.$$

$\mathcal{L}y[y'' + 2y' + 10y] = \mathcal{L}y[9e^{-t}]$

$s^2[\mathcal{L}y] - s\,y(0) - y'(0) + 2(s[\mathcal{L}y] - y(0)) + 10[\mathcal{L}y] = \frac{9}{s+1}$

$s^2[\mathcal{L}y] - 7s + 1 + 2s[\mathcal{L}y] - 14 + 10[\mathcal{L}y] = \frac{9}{s+1}$

$(s+1)\quad(\mathcal{L}y(s^2 + 2s + 10) = 9(s+1)^{-1} + 7s + 13)$

$$(s+1)\quad \left(L_y(s^2+2s+10) = 9(s+1)^{-1} + 7s + 13 \right)$$

$$L_y(s+1)(s^2+2s+10) = 9 + 7s^2 + 7s + 13s + 13$$

$$L_y = \frac{7s^2+20s+22}{(s+1)(s^2+2s+10)}$$

$$y = L^{-1}\left(\frac{7s^2+20s+22}{(s+1)(s^2+2s+10)}\right)$$

$$\frac{7s^2+20s+22}{(s+1)(s^2+2s+10)} = \frac{A}{s+1} + \frac{Cs+B}{s^2+2s+10}$$

$$7s^2+20s+22 = As^2 + 2As + 10A + Cs^2 + Cs + Bs + B$$

$$A + C = 7 \qquad C = 7 - A$$

$$2A + C + B = 20 \qquad B = 22 - 10A$$

$$10A + B = 22 \qquad 2A + (7-A) + (22-10A) = 20$$

$$-9A = -9$$

$$A = 1$$

$$B = 12$$

$$C = 6$$

$$L^{-1}\left(\frac{1}{s+1} + \frac{6s+12}{s^2+2s+10}\right)$$

$$L^{-1}\left(\frac{1}{s+1}\right) = e^{-t}$$

$$L^{-1}\left(\frac{6s+12}{s^2+2s+10}\right) \qquad s^2+2s = -10$$

$$s^2+2s+1 = -9$$

$$(s+1)^2+9$$

$$6(s+1)+6$$

$$6L^{-1}\left(\frac{s+1}{(s+1)^2+9}\right) \approx \frac{s}{s^2+3^2} = 6\cos(3t)\,(e^{-t})$$

$$6L^{-1}\left(\frac{1}{\ldots+9}\right) \approx \frac{1}{3}\,\frac{3}{s^2+3^2} = 2\sin(3t)\,(e^{-t})$$

$$6\,\mathcal{L}^{-1}\left(\frac{1}{(s+1)^2+9}\right) \approx \frac{1}{3}\,\frac{3}{s^2+3^2} = 2\sin(3t)\left(e^{-t}\right)$$

$$\boxed{y = e^{-t} + 6e^{-t}\cos(3t) + 2e^{-t}\sin(3t)}$$