Exam 3

Friday, November 13, 2020 8:43 AM

"I pledge my honor I have abided by the Stevens Honor system." -Alex Gaskins

1. [20 pts] Find $f(t) = \mathcal{L}^{-1}\left\{\frac{-s^2 - s + 10}{(s+1)(s^2+4)}\right\}$.

$$\frac{-s^2 - s + 10}{(s+1)(s^2+4)} = \frac{A}{s+1} + \frac{Bs + C}{s^2+4}$$

$Bs + C\,(s+1)$

$Bs^2 + Bs + Cs + C$

$$-s^2 - s + 10 = As^2 + 4A + Bs^2 + Bs + Cs + C$$

$A + B = -1$

$B + C = -1$

$4A + C = 10$

$B = -C - 1$

$A - C - 1 = -1$

$A = C$

$4C + C = 10$

$5C = 10$

$C = 2$

$B = -3$

$A = 2$

$$= \frac{2}{s+1} - \frac{3s + 2}{s^2+4}$$

$$L^{-1}\left(\frac{2}{s+1} - \frac{3s+2}{s^2+4}\right)$$

$\frac{s}{s^2+2^2} = \cos(2t)$

$\frac{1}{s^2+2^2} = \sin(2t)$

$$2L^{-1}\left(\frac{1}{s+1}\right) - 3L^{-1}\left(\frac{s}{s^2+4}\right) + 2L^{-1}\left(\frac{1}{2}\frac{2}{s^2+4}\right)$$

$4 = 2^2$ $\qquad$ $4 = 2^2$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$2e^{-t} \qquad -3\cos(2t) + 2\left(\frac{1}{2}\sin(2t)\right)$$

$$\boxed{y = 2e^{-t} - 3\cos(2t) + \sin(2t)}$$

2. [20 pts] Consider $f(t) = \begin{cases} t(4-t), & 0 \le t < 2 \\ 4, & 2 \le t < \infty \end{cases}$.

Find $F(s) = \mathcal{L}\{f(t)\}$ using step functions to represent $f(t)$ and applying the second translation theorem (Theorem 7.3.2). *You can use either the original version or the alternative version of the theorem.*

$$4t - t^2\,(u(t-0) - u(t-2)) + 4u(t-2)$$

$$4t - t^2\,(u(t)) - 4t + t^2 + 4\,(u(t-2))$$

$$F(s) = 4L\{t\} - L\{t^2\} + e^{-2s}L\{4\} - 4e^{-2s}L\{t\} + e^{-2s}L\{t^2\}$$

$$\boxed{F(s) = \frac{4}{s^2} - \frac{2}{s^2} + \frac{4e^{-2s}}{s} - \frac{4e^{-2s}}{s^2} + \frac{2e^{-2s}}{s^3}}$$

3. [15 pts] Find $g(t) = \mathcal{L}^{-1}\left\{\dfrac{3s+2}{s^2+4s+8}\right\}$.

$s^2+4s+8=0$ $\quad \left(\frac{b}{2a}\right)^2 = 4$ $\quad s^2+4s+4 = -8+4$ $\quad (s+2)^2+4=0$

$$\frac{3s+2}{s^2+4s+8} = 3\left(\frac{s+2}{(s+2)^2+4}\right) - 4\left(\frac{1}{(s+2)^2+4}\right)$$

$$3\,L^{-1}\left(\left(\frac{s+2}{(s+2)^2+4}\right)\right) - 4\,L^{-1}\left(\frac{1}{(s+2)^2+4}\right)$$

$\frac{s}{s^2+2^2} - \frac{(2)}{-(2)} \approx e^{-2t}$ $\qquad \frac{1}{2}\left(\frac{2}{s^2+2^2}\right) - \frac{(2)}{-(2)} \approx e^{-2t}$

$\frac{s}{s^2+2^2} = \cos(2t)$ $= e^{-2t}\cos(2t)$ $\qquad \frac{1}{2}\sin(2t)$ $= \frac{e^{-2t}}{2}\sin(2t)$

$$\boxed{y = 3e^{-2t}\cos(2t) - 2e^{-2t}\sin(2t)}$$

4. [25 pts] Solve the initial value problem (IVP) for $y(t)$.

$$L[y] = y'' - 3y' + 2y = 4e^{2t}, \quad y(0)=0, \quad y'(0)=1$$

$L(y'' - 3y' + 2y)$ $\quad 2[Ly]$ $\quad -3(s[Ly]-y(0))$

$s^2[Ly] - sy(0) - y'(0)$

$$s^2[Ly] - 0 - 1 - 3s[Ly] + 0 + 2[Ly]$$

$4L(e^{2t})$

$4\cdot\left(\frac{1}{s-2}\right)$

$\frac{4}{s-2}$

$s^2[Ly] - 3s[Ly] + 2[Ly] = \frac{4}{s-2} + 1$

$[Ly](s^2 - 3s + 2) = \frac{4}{s-2} + 1$

$\frac{\frac{4}{s-2}}{s^2-3s+2} + \frac{1}{s^2-3s+2}$

$[Ly] = \frac{s+2}{(s-2)(s^2-3s+2)}$

$y = L^{-1}\left(\frac{s+2}{(s-2)(s^2-3s+2)}\right)$

$s^2 - 3s + 2 = (s-2)(s-1)$

$\frac{s+2}{(s-2)(s-2)(s-1)} = \frac{A}{(s-2)^2} + \frac{B}{(s-2)} + \frac{C}{(s-1)}$

$(s-2)^2(s-1)$

$(s-2)(s-2)$
$s^2 - 4s + 4$

$s + 2 = As - A + Bs^2 - 3Bs + 2B + Cs^2 - 4Cs + 4C$

$B + C = 0$ $\quad B = -C$ $\quad B = -3$

$A - 3B - 4C = 1$ $\quad A = C + 1$

$2B + 4C - A = 2$ $\quad -2C + 4C - C - 1 = 2$
$C = 3$

$A = 4$

$\frac{4}{(s-2)^2} - \frac{3}{(s-2)} + \frac{3}{(s-1)}$

$4\,L\left(\frac{1}{(s-2)^2}\right) \approx \frac{1}{s^2} = e^{2t} \quad x\ t \approx 4te^{2t}$

$-3\,L\left(\frac{1}{s-2}\right) = e^{2t}$

$-3e^{2t}$

$3\,L\left(\frac{1}{s-1}\right) = e^{t}$

$- \quad t$

$-3e^{2t}$ $3e^{t}$

$$\boxed{y = 4t e^{2t} + 3e^{t} - 3e^{2t}}$$

5. [20 pts] Consider the function, $G(s) = \left(\frac{-1}{2s^3} + \frac{2}{s^2}\right) + \left(\frac{1}{2s^3} - \frac{4}{s}\right)e^{-4s}$.

 (a) Determine $g(t) = \mathcal{L}^{-1}\{G(s)\}$.

 (b) Express $g(t)$ in the form of a piecewise function.
 Obvious simplifications are required to receive full credit.

A.)

$$\mathcal{L}^{-1}\left(-\frac{1}{2s^3} + \frac{2}{s^2}\right) + \mathcal{L}^{-1}\left(\frac{1}{2s^3} - \frac{4}{s}\right)e^{-4s}$$

$$= -\frac{1}{2}\left(\frac{t^2}{2!}\right) + 2t + u(t-4)\left(\frac{1}{2}\,\frac{(t-4)^2}{2} - 4\right)$$

$$g(t) = -\frac{t^2}{4} + 2t + u(t-4)\left(\frac{1}{4}(t-4)^2 - 4\right)$$

B.)

$$g(t) = \begin{cases} -\frac{t^2}{4} + 2t + \frac{1}{4}(t-4)^2 - 4, & t \geq 4 \\ -\frac{t^2}{4} + 2t, & t < 4 \end{cases}$$