Exam 4

Friday, December 4, 2020 8:54 AM

"I pledge my honor I have abided by the Stevens Honor system."

-Alex Haskins

1. [10 pts] Assuming a product solution, $u(x,t) = X(x) \cdot T(t)$, use separation of variables to derive two ODEs for $X(x)$ and $T(t)$. *Do **NOT** solve the ODE's; just set up the equations.*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 5u + 2\frac{\partial u}{\partial x}$$

$$X''T + T''X = 5XT + 2X'T$$

$$\frac{Y''}{Y}X = \frac{5XY + 2X'Y - X''Y}{Y}$$

$$\frac{\frac{Y''}{Y}X}{X} = \frac{5X + 2X' - X''}{X}$$

$$\frac{Y''}{Y} = \frac{-X''}{X} + \frac{2X'}{X} + 5$$

2. [10 pts] Assuming a product solution, $u(x,y) = X(x) \cdot Y(y)$, use separation of variables to derive two ODEs for $X(x)$ and $Y(y)$. *Do **NOT** solve the ODE's; just set up the equations.*

$$\frac{\partial^2 u}{\partial y\, \partial x} - 4yu = 0$$

$$X'Y' - 4yXY = 0$$

$$X'Y' = 4yXY$$

$$\frac{X'}{X} = \frac{4yY}{Y'}$$

$$\frac{X'}{X} = -\lambda \; ; \quad \int \frac{X'}{X} = \int -\lambda$$

$$\ln(X) = -\lambda x$$

$$\boxed{X = ce^{-\lambda x}}$$

$$\boxed{X = ce^{-\lambda x}}$$

$$\frac{4_x Y}{Y'} = -\lambda$$

$$\frac{Y'}{\frac{1}{4}yY} = -\frac{1}{\lambda}$$

$$\frac{Y'}{yY} = -\frac{4}{\lambda}$$

$$\int \frac{Y'}{Y} = \int -\frac{4y}{\lambda}$$

$$\ln(Y) = -\frac{4y}{\lambda}$$

$$\boxed{Y = ce^{-\frac{2y^2}{\lambda}}}$$

3. [20 pts] Find *ALL* solutions (if any) to the boundary value problem,

$$y'' + y = 2\pi, \quad \text{for } 0 < x < \pi, \qquad y(0) = 0, \quad y(\pi) = 2\pi$$

$$y'' + y = 0$$

$y_p = 2\pi$ (constant)

$$m^2 + 1 = 0$$

$$m^2 = -1$$

$$m = \pm\sqrt{-1}$$

$$(\alpha \pm Bi)$$

$$y_c = e^0(c_1\cos(x) + c_2\sin(x))$$

$$y_c = c_1\cos(x) + c_2\sin(x)$$

$y_c = c_1\cos(x) + c_2\sin(x)$

$y = y_c + y_p$

$y = c_1\cos(x) + c_2\sin(x) + 2\pi$

$0 = c_1\cos(0) + \cancel{c_2\sin(0)} + 2\pi$

$c_1 = -2\pi$

$2\pi = c_1\cos(\pi) + \cancel{c_2\sin(\pi)} + 2\pi$

$c_1 = 0$

No solution

4. [20 pts] Find all eigenvalues (λ) and corresponding eigenfunctions for the eigenvalue problem,

$$y'' + (\lambda + 2)y = 0, \quad \text{for } 0 < x < 4, \quad y'(0) = 0, \quad y(4) = 0.$$

***Note*:** You only need to consider the *negative discriminant case* of the auxiliary equation. It is known that there are no eigenvalues for the cases of positive or zero discriminant.

$m^2 + (\lambda + 2) = 0$

$m^2 = -\lambda - 2$

$m = \sqrt{-x-2}$

$m = i\sqrt{\lambda+2} \qquad u = \sqrt{\lambda+2}$

$y = c_1\cos(ux) + c_2\sin(ux)$

$y = c_1\cos(\sqrt{\lambda+2}\,x) + c_2\sin(\sqrt{\lambda+2}\,x)$

Case $\lambda < -2$:

$\sqrt{\lambda+2} < 0$

$0 = c_1\cos(\sqrt{\lambda+2}\,n\pi) + \cancel{c_2\sin(\sqrt{\lambda+2}\,n\pi)}$

$$\sqrt{\lambda+2} = \frac{(2n-1)\pi}{2L}$$

$$y_n = \cos\left(\sqrt{\lambda+2}\,x\right)$$

$$\boxed{\begin{gathered}\lambda_n = \frac{(2n-1)^2\pi^2}{64} - 2 \\ X_n = \cos\left(\left(\frac{(2n-1)\pi}{8}+2\right)x\right)\end{gathered}}$$

5. [20 pts] Consider the function $f(x)$ defined on $(-2, 2)$, $f(x) = \begin{cases} 0, & -2 < x < -1 \\ -1, & -1 \le x < 0 \\ 1, & 0 \le x < 1 \\ 0, & 1 \le x < 2 \end{cases}$.

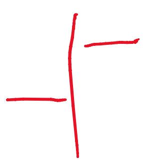
(a) Does $f(x)$ have Even symmetry, Odd symmetry, or neither?

(b) Find the general formula for the coefficients (a_n, b_n) in the trigonometric Fourier series for $f(x)$.

(c) From your result in (b), calculate the values of a_n and b_n for $n = 1, 2, 3$ and 4.

(d) What is the *Fundamental Period* of this Fourier Series?

A.)



Odd symmetry

B.)

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$a_n = \frac{1}{2}\left[\int_{-1}^{0} -\cos\left(\frac{n\pi x}{2}\right)dx + \int_{0}^{1}\cos\left(\frac{n\pi x}{2}\right)dx\right.$$

$$a_n = \frac{1}{2}\left[\frac{-\sin\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}}\right]_{-1}^{0} + \frac{1}{2}\left[\frac{\sin\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}}\right]_{0}^{1}$$

$$a_n = \frac{1}{2}\left[\frac{-2\sin\left(\frac{n\pi}{2}\right)}{n\pi}\right] + \frac{1}{2}\left[\frac{2\sin\left(\frac{n\pi}{2}\right)}{n\pi}\right]$$

$$a_n = 0$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{2}\int_{-1}^{0} -\sin\left(\frac{n\pi x}{2}\right)dx + \int_{0}^{1}\sin\left(\frac{n\pi x}{2}\right)dx$$

$$b_n = \frac{1}{2}\left[\frac{-2\cos\left(\frac{n\pi}{2}\right)-2}{n\pi}\right] + \frac{1}{2}\left[\frac{-2\cos\left(\frac{n\pi}{2}\right)-2}{n\pi}\right]$$

$$b_n = -\frac{\cos\left(\frac{n\pi}{2}\right)}{n\pi} + \frac{1}{n\pi} - \cos\left(\frac{n\pi}{2}\right) + \frac{1}{n\pi}$$

$$\frac{2}{n\pi} - \frac{2\cos\left(\frac{n\pi}{2}\right)}{n\pi}$$

$$\boxed{b_n = \frac{2}{n\pi}\left[1 - \cos\left(\frac{n\pi}{2}\right)\right]}$$

C.) $b_1 = \frac{2}{\pi}(1) = \frac{2}{\pi}$

$$b_2 = \frac{2}{2\pi}(1+1) = \frac{2}{\pi}$$

1 - 2 ~ .7 - 2

$$b_3 = \frac{2}{3\pi}\left[1-0\right] = \frac{2}{3\pi}$$

$$b_4 = \frac{2}{4\pi}\left[1-1\right] = 0$$

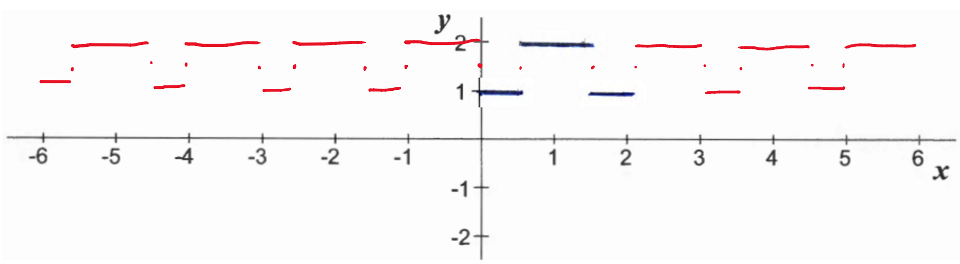
b.) Fundamental period is 4

6. [20 pts] Consider the function $f(x)$ defined on $(0, 2)$, as shown in the two graphs.

(a) On the interval $[-6, 6]$, graph the periodic function, $f_p(x)$, that represents the pointwise convergence of the *half-range Fourier* ***Cosine*** series for $f(x)$. Be sure to indicate the value of the limit at jump discontinuities

What is the *Fundamental Period* of $f_p(x)$?

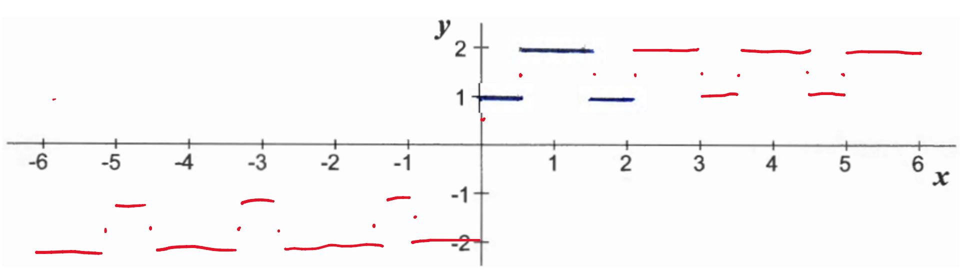
$$\frac{f(x^-) + f(x^+)}{2}$$



Converges at $y = 1.5$

(b) On the interval $[-6, 6]$, graph the periodic function, $f_p(x)$, that represents the pointwise convergence of the *half-range Fourier* ***Sine*** series for $f(x)$. Be sure to indicate the value of the limit at jump discontinuities

What is the *Fundamental Period* of $f_p(x)$?



Converges at $y = 1.5$ and $y = -1.5$