1 Chapter 1

1.1 Complex Numbers

Verfication 1.1. commutativity

$$w + z = z + w \text{ and } wz = zw \text{ for all } w, z \in \mathbb{C}$$

We know that $w = \alpha + \beta i$ and $z = j + ki$ and so then

$$w + z = \alpha + \beta i + j + ki = (\alpha + j) + (\beta + k)\, i = (j + \alpha) + (k + \beta) = z + w$$

Verfication 1.2. associativity

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3) \text{ and } (z_1 z_2)\, z_3 = z_1\, (z_2 z_3) \text{ for all } z_1, z_2, z_3 \in \mathbb{C}$$

We know $z_1 = \alpha + \beta i, z_2 = j + ki, z_3 = l + mi$ and thus

$$\begin{aligned} (z_1 + z_2) + z_3 &= \alpha + j + (\beta + k)\, i + l + mi \\ &= (\alpha + j + l) + (\beta + k + m)\, i \\ &= z_1 + (z_2 + z_3) \end{aligned}$$

Now we'll do multiplication, we know

$$\begin{aligned} (z_1 z_2)\, z_3 &= (\alpha \cdot j - \beta \cdot k + (\alpha k + \beta j)\, i)\, (l + mi) \\ &= ajl - blk - (\alpha km + \beta jm) + (\alpha kl + \beta jl + ajm)\, i \end{aligned}$$

Now we also have

$$\begin{aligned} z_1\, (z_2 z_3) &= \alpha + \beta i\, (jl - km + (kl + jm)\, i) \\ &= ajl - akm - (\beta kl + \beta jm) + (\beta jl + akm + ajm)\, i \end{aligned}$$

And due to the associativity of $\mathbb{R}$ then we can say $(z_1 z_2)\, z_3 = z_1\, (z_2 z_3)$

Verfication 1.3. identities

$$z + 0 = z \text{ and } z1 = z \text{ for all } z \in \mathbb{C}$$

We know that

$$\begin{aligned} z + 0 &= \alpha + \beta i + 0 + 0i \\ &= (\alpha + 0) + (\beta + 0)\, i \\ &= \alpha + \beta i \\ &= z \end{aligned}$$

For multiplication we have

$$\begin{aligned}z1 &= \alpha + \beta i\,(1 + 0i)\\ &= \alpha - \beta 0 + (\beta + 0\alpha)\\ &= \alpha + \beta i\end{aligned}$$

Verfication 1.4. additive inverse
for every $z \in \mathbb{C}$ there is a unique $w \in \mathbb{C}$ such that $z + w = 0$

Let $z \in \mathbb{C}$ and so $z = \alpha + \beta i$ now we'll take $w = -\alpha + -\beta i$

$$\begin{aligned}z + w &= \alpha + \beta i + -\alpha + -\beta i\\ &= (\alpha - \alpha) + (\beta - \beta)\, i\\ &= 0 + 0i\\ &= 0\end{aligned}$$

To show that our choice of w was unique assume there is another solution namely $w = j + ki$ such that $j \neq \alpha, k \neq \beta$ but then their sum will yeild $x + yi$, where $x, y \neq 0$ and so we don't get 0 so we can say that our w is unique.

Verfication 1.5. multiplicative inverse Let $z \in \mathbb{C}$ so there exists some $\alpha, \beta \in \mathbb{R}$ so that $z = \alpha + \beta i$ let $w = \frac{\alpha}{\alpha^2+\beta^2} + \frac{-\beta}{\alpha^2+\beta^2} i$

$$\begin{aligned}zw &= (\alpha + \beta i)\left(\frac{\alpha}{\alpha^2+\beta^2} + \frac{-\beta}{\alpha^2+\beta^2}\right) i\\ &= \frac{\alpha^2}{\alpha^2+\beta^2} + \frac{\beta^2}{\alpha^2+\beta^2} + \left(\frac{\alpha^2}{\alpha^2+\beta^2} - \frac{\beta^2}{\alpha^2+\beta^2}\right)\\ &= 1 + 0i\\ &= 1\end{aligned}$$

Verfication 1.6. distributive property

$$\lambda\,(w + z) = \lambda w + \lambda z \text{ for all } \lambda, w, z \in \mathbb{C}$$

$$\begin{aligned}\lambda\,(w + z) &= \lambda\,(\alpha + \beta i + \delta + \varepsilon i)\\ &= \lambda\,(\alpha + \delta + (\beta + \varepsilon)\, i)\\ &= \lambda\alpha + \lambda\delta + (\lambda\beta + \lambda\varepsilon)\, i\\ &= \lambda\alpha + \lambda\beta i + \lambda\delta + \lambda\varepsilon i\\ &= \lambda\,(\alpha + \beta i) + \lambda\,(\delta + \varepsilon i)\\ &= \lambda w + \lambda z\end{aligned}$$