MAT223 - Linear Algebra

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Contents

List of Definitions

List of Theorems

Chapter 1

Week 9

Linear transformations and such

1.1 Linear Transformations

Definition 1 (Linear Transformation)**.** Let V and W be subspaces. A function $\mathcal{T} : V \to W$ is called a linear transformation if for all $\vec{u}, \vec{v} \in V$ and $a \in \mathbb{R}$ it satisfies

1. $\mathcal{T}(\vec{u} + \vec{v}) = \mathcal{T}(\vec{u}) + \mathcal{T}(\vec{v})$

2. $\mathcal{T}(a\vec{u}) = a\mathcal{T}(\vec{u})$

Theorem 1 (Unique Matrix for LT's)**.** For any linear transformation $L : V \to W$ there is a matrix A such that $L(\vec{x}) = A\vec{x}$ for all $\vec{x} \in V$

Example 1.1.1. We'll show that $\mathcal{R}$ is a linear transformation where $\mathcal{R}$ is a counter clockwise rotation of $\frac{\pi}{2}$ radians

$$\mathcal{R}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Let $\vec{u}, \vec{v} \in \mathbb{R}^2$ we know that for some $x_1, y_1, x_2, y_2 \in \mathbb{R}$ that

$$\vec{u} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \text{ and } \vec{v} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

$$\mathcal{R}\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) + \mathcal{R}\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = \begin{bmatrix} -y_1 \\ x_1 \end{bmatrix} + \begin{bmatrix} -y_2 \\ x_2 \end{bmatrix}$$
$$= \begin{bmatrix} -(y_1 + y_2) \\ x_1 + x_2 \end{bmatrix}$$

Which is exactly equal to $\mathcal{R}(\vec{u} + \vec{v})$ as required, then let $\alpha \in \mathbb{R}$ and we know that

$$\mathcal{R}(\alpha\vec{u}) = \begin{bmatrix} -\alpha y_1 \\ \alpha x_1 \end{bmatrix}$$

But also that

$$\alpha\mathcal{R}(\vec{u}) = \begin{bmatrix} -\alpha y_1 \\ \alpha x_1 \end{bmatrix}$$

So then we've shown that $\mathcal{R}(\alpha\vec{u}) = \alpha\mathcal{R}(\vec{u})$ but also that $\mathcal{R}(\vec{u} + \vec{v}) = \mathcal{R}(\vec{u}) + \mathcal{R}(\vec{v})$ as req'd

Example 1.1.2. We'll show that $\mathcal{T} : \mathbb{R}^2 \to \mathbb{R}^2$ where $\mathcal{T}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+2 \\ y \end{bmatrix}$ is not a linear transformation.

Let $\vec{j} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \vec{k} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ we have that

$$\mathcal{T}(\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

But then we can see that

$$\mathcal{T}(\begin{bmatrix} 0 \\ 0 \end{bmatrix}) + \mathcal{T}(\begin{bmatrix} 0 \\ 0 \end{bmatrix}) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

Then we conclude that $\mathcal{T}(\vec{j} + \vec{k}) \neq \mathcal{T}(\vec{j}) + \mathcal{T}(\vec{k})$

Example 1.1.3. We'll show that $\mathcal{P}$ is a linear transformation n

Note 1.1.1. We'll show that it is closed under addition and multiplication

$$\mathcal{P}(\begin{bmatrix} x \\ y \end{bmatrix}) = comp_{\vec{u}}\begin{bmatrix} x \\ y \end{bmatrix}$$

Let $\vec{j}, \vec{k} \in \mathbb{R}^2$ we know that

$$comp_{\vec{u}}\vec{j} = \left(\frac{\vec{u} \cdot \vec{j}}{\|\vec{u}\|^2}\right)\vec{u} \text{ and } comp_{\vec{u}}\vec{k} = \left(\frac{\vec{u} \cdot \vec{k}}{\|\vec{u}\|^2}\right)\vec{u}$$

And thus their product yields

$$comp_{\vec{u}}\vec{j} + comp_{\vec{u}}\vec{k} = \left(\frac{\vec{u} \cdot \left(\vec{j} + \vec{k}\right)}{\|\vec{u}\|^2}\right)\vec{u}$$

Which is equal to

$$comp_{\vec{u}}\left(\vec{j} + \vec{k}\right)$$

We must then show that it holds under multiplication let $\alpha \in \mathbb{R}$ and we know that

$$\alpha comp_{\vec{u}}\vec{j} = \alpha\left(\frac{\vec{u} \cdot \vec{j}}{\|\vec{u}\|^2}\right)\vec{u} = \left(\frac{\vec{u} \cdot \alpha\vec{j}}{\|\vec{u}\|^2}\right)\vec{u} = comp_{\vec{u}}\alpha\vec{j}$$

Example 1.1.4. We'll show that $W : \mathbb{R}^2 \to \mathbb{R}^2$ is not a linear transformation, where

$$\mathcal{W}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x^2 \\ y \end{bmatrix}$$

Let $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, and $\alpha \in \mathbb{R}$ we know that

$$\mathcal{W}\left(\alpha\begin{bmatrix} x \\ y \end{bmatrix}\right) = \alpha^2\begin{bmatrix} x^2 \\ y^2 \end{bmatrix} \neq \alpha\begin{bmatrix} x^2 \\ y^2 \end{bmatrix} = \alpha\mathcal{W}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$$

1.2 Image

Definition 2 (Image)**.** Let $L : V \to W$ be a transformation and let $X \subseteq V$ be a set. The **image of the set X under L** , denoted as $L(X)$, is the set

$$L(X) = \left\{\vec{x} \in W : \vec{x} = L(\vec{y}) \text{ for some } \vec{y} \in X\right\}$$

Let $S = \left\{\begin{bmatrix} x \\ y \end{bmatrix} : 0 \leq x, y \leq 1\right\}$ be a filled in unit square in the first quadrant. And let $C = \left\{\vec{0}, \vec{e_1}, \vec{e_2}, \vec{e_1} + \vec{e_2}\right\} \subseteq \mathbb{R}^2$ be the corners of the unit square

Exercise 1.2.1. We'll find what $\mathcal{R}(C)$ is, by the definition of image we have that

$$\begin{aligned}\mathcal{R}(C) &= \left\{\mathcal{R}\left(\vec{0}\right), \mathcal{R}(\vec{e_1}), \mathcal{R}(\vec{e_2}), \mathcal{R}(\vec{e_1}+\vec{e_2})\right\}\\ &= \left\{\vec{0}, \vec{e_2}, -\vec{e_1}, \vec{e_2}-\vec{e_1}\right\}\end{aligned}$$

Exercise 1.2.2. We'll now find what $\mathcal{W}(C)$ (Notice that it doesn't have to be a linear transformation) is, again we will use the definition so we have

$$\mathcal{W}(C) = \left\{\vec{0}, \vec{e_1}, \vec{e_2}, \vec{e_1}+\vec{e_2}\right\}$$

Exercise 1.2.3. $\mathcal{T}(C) = \left\{\begin{bmatrix}2\\0\end{bmatrix}, \begin{bmatrix}3\\0\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}1\\3\end{bmatrix}\right\}$ (The square has been shifted right 2 units)

Exercise 1.2.4. We'll now operate on S, to find $\mathcal{R}(S)$ we imagine all the vectors in $\mathbb{R}^2$ that have been rotated $\frac{\pi}{2}$ radians counter clockwise from the intial square, or we could also multiply by the rotation matrix, either way we get the set

$$\mathcal{R}(S) = \left\{\begin{bmatrix}-y\\x\end{bmatrix} : 0 \leq x, y \leq 1\right\}$$

Exercise 1.2.5. For $\mathcal{T}(S)$ we can re-imagine how we determined $\mathcal{T}(C)$ but for all the points in the square, this gives us the full square shifted horizontally by two units, so we have

$$\mathcal{T}(S) = \left\{\begin{bmatrix}x+2\\y\end{bmatrix} : 0 \leq x, y \leq 1\right\}$$

Exercise 1.2.6. As for $\mathcal{P}(S)$ this is a bit more complicated, so we'll break it into to parts, the first is algebreically and the other will be vizually.

Algebraically we know $proj_{\vec{u}}\begin{bmatrix}x\\y\end{bmatrix}$ will look like

$$\left(\frac{\vec{u}\cdot\begin{bmatrix}x\\y\end{bmatrix}}{\|\vec{u}\|^2}\right)\vec{u}$$

But $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ so then

$$proj_{\vec{u}} \begin{bmatrix} x \\ y \end{bmatrix} = \left(\frac{2x + 3y}{13} \right) \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

So then we can conclude that

$$\mathcal{P}(S) = \left\{ \frac{2x+3}{13} \begin{bmatrix} 2 \\ 3 \end{bmatrix} : 0 \leq x, y \leq 1 \right\}$$

Exercise 1.2.7. Let $\ell = \left\{ t\vec{a} + (1-t)\vec{b} \text{ for some } t \in [0,1] \right\}$ and $\mathcal{A}$ be a linear transformation, we know that $\mathcal{A}(\ell)$ represents all vectors that are in the range of the linear transformation, let $\vec{u} \in \ell$, then we know that $\vec{u} = t\vec{a} + (1-t)\vec{b}$ for some $t \in \mathbb{R}$ then we know

$$\begin{aligned} \mathcal{A}(\vec{u}) &= \mathcal{A}\left(t\vec{a} + (1-t)\vec{b} \right) \\ &= t\mathcal{A}(\vec{a}) + (1-t)\mathcal{A}\left(\vec{b}\right) \end{aligned}$$

And since we know that $\mathcal{A}(\vec{a}), \mathcal{A}\left(\vec{b}\right)$ are just two transformed vectors, then this defines a new line segment with endpoints $\mathcal{A}(\vec{a})$ and $\mathcal{A}\left(\vec{b}\right)$.

Exercise 1.2.8. We'll now find the linear transformation that italicizes N, FIG below

To determine the linear transformation we start with the fact that if A is some matrix then $A\vec{e_i}$ results in the ith column of A.

We then choose two points and see how they moved after the transormation, per the hint above we'll choose two vectors that reside on the x, y axis. (Our origin is the corner of the N), so our first vector will be $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$ and our second is $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$. Thus we know the following

1. $\mathcal{I}\left(\begin{bmatrix} 0 \\ 3 \end{bmatrix} \right) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, but we know that applying a linear transformation is the same as just multiplying by some matrix so we know that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 3b \\ 3d \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

And so we conclude that $b = \frac{4}{3}$ and $d = \frac{1}{3}$ so we've determined the first column of the matrix

2. $\mathcal{I}\left(\begin{bmatrix}2\\0\end{bmatrix}\right) = \begin{bmatrix}2\\0\end{bmatrix}$ thus we know that $2a = 2 \Leftrightarrow a = 1$ and that $b = 0$ and we have our second column of the matrix.

So now we now that the matrix must look like

$$\begin{bmatrix}1 & \frac{4}{3}\\0 & \frac{1}{3}\end{bmatrix}$$

1.2.1 From Transformation to Matrix

We defined $\mathcal{P}$ as the $proj_{span\vec{u}}\vec{x}$ where $\vec{u} = \begin{bmatrix}2\\3\end{bmatrix}$ and $\mathcal{R}$ be a rotation ccw by $\frac{\pi}{2}$ radians. We'll now find the matricies which define each transformation

Example 1.2.1.

$$\begin{aligned}\mathcal{P}(\vec{x}) &= \left(\frac{\vec{u}\cdot\begin{bmatrix}x\\y\end{bmatrix}}{\|\vec{u}\|^2}\right)\vec{u}\\ &= \left(\frac{2x+3y}{13}\right)\begin{bmatrix}2\\3\end{bmatrix}\\ &= \frac{1}{13}\begin{bmatrix}4x+6y\\6x+9y\end{bmatrix}\end{aligned}$$

By observation we know the matrix must be

$$\frac{1}{13}\begin{bmatrix}4 & 6\\6 & 9\end{bmatrix}$$

Example 1.2.2. We'll now find the matrix which defines the rotation $\mathcal{R}$, we start geometrically FIG below.

We could also use the fact that any matrix A times $\vec{e_i}$ is equal to the i-th column of the matrix A. And we know that rotating $\vec{e_1}$ ccw $\frac{\pi}{2}$ moves it to $\vec{e_2}$ and that $\vec{e_2}$ rotated becomes $-\vec{e_1}$ so then we can determine that the matrix must be

$$\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$$

1.2.2 Composition of Transformations

We know that $f \circ g(x) = f\big(g(x)\big)$ and so we can determine that $\mathcal{P} \circ \mathcal{R} = \mathcal{P}\big(\mathcal{R}(\vec{x})\big) = \mathcal{P}\big(\mathcal{R}(\vec{x})\big) = \mathcal{P}(\mathcal{R}\vec{x}) = \mathcal{P}\mathcal{R}\vec{x}$. So we can determine

$$\mathcal{P} \circ \mathcal{R} = \frac{1}{13}\begin{bmatrix} 4 & 6 \\ 6 & 9 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \frac{1}{13}\begin{bmatrix} 6 & -4 \\ 9 & -6 \end{bmatrix}$$

And that

$$\mathcal{R} \circ \mathcal{P} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}\frac{1}{13}\begin{bmatrix} 4 & 6 \\ 6 & 9 \end{bmatrix} = \frac{1}{13}\begin{bmatrix} -6 & -9 \\ 4 & 6 \end{bmatrix}$$

We notice that these matricies are certainly different and that $\mathcal{P} \circ \mathcal{R}$ is first a projection onto $\vec{u}$ and then a rotation, whereas $\mathcal{R} \circ \mathcal{P}$ is first a rotation, then a projection onto $\vec{u}$.

Definition 3 (Range)**.** The **range** of a linear transformatoin $T : V \to W$ is the set of vectors that T can output. That is ,

$$range(T) = \big\{\vec{y} \in W : \vec{y} = T(\vec{x}) \text{ for some } x \in V\big\}$$

Definition 4 (Null Space)**.** The **null space** or **kernel** of a linear transformation $T : V \to W$ is the set of vectors that get mapped to zero under T. That is,

$$null(T) = \Big\{\vec{x} \in V : T(\vec{x}) = \vec{0}\Big\}$$

Exercise 1.2.9. Consider $P : \mathbb{R}^2 \to \mathbb{R}^2$ where $\mathcal{P}$ is the projection onto the $span\vec{u}$ (Remember $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$), we'll determine the range and null space of $\mathcal{P}$.

We know that the projection of any vector onto $\vec{u}$ will be equal to some scalar times $\vec{u}$ so we know that the range will be

$$\alpha\vec{u}, \text{ for all } a \in \mathbb{R}$$

As for the nullspace, we can think of what vectors will get mapped to zero under a projection onto $\vec{u}$ with a bit of thought, we determine that it must be all vectors who are orthogonal to $\vec{u}$ as their "shadow" will drop to the zero vector. We know that this will be all scalar multiples of a normal vector to $\vec{u}$ so we can say $\alpha\begin{bmatrix} -3 \\ 2 \end{bmatrix}$ for all $\alpha \in \mathbb{R}$ or we could first take the image of $\mathcal{P}$ then rotate all of those vectors, so we could say that $null(\mathcal{P}) =$ Image of the $rangeP$ under $\mathcal{R}$

Example 1.2.3. We let $T : R^n \to R^m$ be a linear transformation. We'll show that the null space of T is a linear subspace and that the range of T is as well, so we'll show it is closed under addition and multiplication.

Proof. Let $\vec{u}, \vec{x} \in null(T)$

$$T(\vec{u}) = 0 \text{ and } T(\vec{x}) = 0$$

Taking the sum of the above equations we get $0 = T(\vec{u}) + T(\vec{x}) = \mathcal{T}(\vec{x} + \vec{y})$ from the definition of linear transformation. But by the definition of null set, we can conclude that $\vec{x} + \vec{y} \in null(T)$ because $T(\vec{x} + \vec{y}) = 0$.

Let $\alpha \in \mathbb{R}$ and from above we know that $T(\vec{x}) = 0 \Leftrightarrow \alpha T(\vec{x}) = \alpha 0 = 0$ and since T is a linear transformationwe can say that $T(\alpha\vec{x}) = 0$ so then we know that $\alpha\vec{x} \in null(T)$ □

Proof. Let $\vec{j}, \vec{k} \in range(T)$ so we know that

$$\vec{j} = T(\vec{x}) \text{ and } \vec{k} = T(\vec{x_1}) \text{ for some } \vec{x}, \vec{x_1} \in R^n$$

Then we know that $\vec{j} + \vec{k} = T(\vec{x} + \vec{x_1})$ and thus we conclude that $\vec{j} + \vec{k} \in range(T)$

Let $\alpha \in \mathbb{R}$ and we know that $\alpha\vec{j} = \alpha T(\vec{x}) = T(\alpha\vec{x})$ and so $\alpha\vec{j} \in range(T)$ as required. □

1.3 Tutorial 5

1. $\mathcal{B}$ is a basis for a subspace V if $\mathcal{B}$ is linearly independent and $span(\mathcal{B}) = V$.

2. (a) We'll verify if the fectors in $\mathcal{B}$ are in a basis. So for it to be a basis it must be linearly independent, and we'll assume that we are looking for a basis for the subspace $\mathbb{R}^3$. We'll start by looking at the linear combinations of each vector in the matrix that gives the zero vector to determine independendence, so we have (These are augmented matricies for the zero vector.)

$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

And thus we have one solution to $\vec{0}$ and so these vectors are linearly independent, and they span $\mathbb{R}^3$.

We'll now focus our attention to $\mathcal{C}$ using the same process as above we get

$$\begin{bmatrix} 1 & 0 & 2 \\ - & 1 & -2 \\ 1 & 1 & 3 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

And by the same reasoning as before $\mathcal{C}$ is a basis for $\mathbb{R}^3$.

(b) We'll start by finding what $[\vec{v}]_{\mathcal{E}}$ is, but we don't have to do much as $\vec{v} = 4\vec{e_1} - 4\vec{e_2} + 2\vec{e_3}$ so we know that $[\vec{v}]_{\mathcal{E}} = \begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix}$.

Moving to $[\vec{v}]_{\mathcal{B}}$ we know we are looking for some $\alpha, \beta, \gamma \in \mathbb{R}$ that satisfy

$$\alpha \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix}$$

This relates to a system of equations, which is then stored in a matrix, so we have

$$\begin{bmatrix} 1 & 0 & 2 & 4 \\ -1 & 1 & -2 & -4 \\ 0 & 0 & 1 & 2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

And thus we conclude that $\gamma = 2, \beta = 0, \alpha = 4 = 2\gamma = 0$ and so $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$.

Now we'll figure out $[\vec{v}]_{\mathcal{C}}$ so again we apply the same idea to get the following matrix

$$\begin{bmatrix} 1 & 0 & 2 & 4 \\ -1 & 1 & -2 & -4 \\ 1 & 1 & 3 & 2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

And now we conclude that $\gamma = -2, \beta = 0, \alpha = 8$ and thus we have

$$[\vec{v}]_{\mathcal{C}} = \begin{bmatrix} 8 \\ 0 \\ -2 \end{bmatrix}$$

(c) To determine $[7\vec{v}]_{\mathcal{E}}$ we know we are looking for the solution to

$$\alpha_1 \vec{e_1} + \beta_1 \vec{e_2} + \gamma_1 \vec{e_3} = 7\vec{v}$$

But previously we know that the coeficients should be when we were just looking for $\vec{v}$ multiplying that equation by 7 on both sides tells us that

$$[7\vec{v}]_{\mathcal{E}} = \begin{bmatrix} 28 \\ -28 \\ 14 \end{bmatrix}$$

Then using the same process as above we can determine that

$$[7\vec{v}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 14 \end{bmatrix} \text{ and } [\vec{v}]_{\mathcal{C}} = \begin{bmatrix} 57 \\ 0 \\ -14 \end{bmatrix}$$

(d) I would prefer to write my measurements of scalar multiples of $\vec{v}$ in terms of the $\mathcal{B}$ basis as I only have to do one calculation.

3. Get help with this one

4. I chose $\mathcal{E}_3$ as then I could represent the vector $\vec{v} = \begin{bmatrix} 1 \\ .12 \end{bmatrix}$ as $\left[\begin{bmatrix} -11 \\ 12 \end{bmatrix}\right]_{\mathcal{E}_3}$

1.3.1 Video Notes

Finding entries of the matrix A below

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3x - z \\ -2x + 4y \\ 6y - 3z \end{bmatrix}$$

We know that A must be 3×3 and so

$$A = \begin{bmatrix} 3 & 0 & -1 \\ -2 & 4 & 0 \\ 0 & 6 & -3 \end{bmatrix}$$

Standard matrix of a linear transformation

Let $T : \mathbb{R}^2 \to R^m \vec{v_1} = T(\vec{e_1}), \vec{v_2} = T(\vec{e_2})$. We will now find the matrix A_T that is the matrix which induces T.
Let $\vec{x} \in \mathbb{R}^2$ and so for some $x_1, x_2 \in \mathbb{R}$ we know that

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \cdot \vec{e_1} + x_2 \cdot \vec{e_2}$$

Then

$$\begin{aligned}
T(\vec{x}) &= T(x_1\vec{e_1} + x_2\vec{e_2}) \\
&= x_1T(\vec{e_1}) + x_2T(\vec{e_2}) \\
&= x_1\vec{v_1} + x_2\vec{v_2} \\
&= \begin{bmatrix} \vdots & \vdots \\ \vec{v_1} & \vec{v_2} \\ \vdots & \vdots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
&= \begin{bmatrix} \vdots & \vdots \\ \vec{v_1} & \vec{v_2} \\ \vdots & \vdots \end{bmatrix} \vec{x}
\end{aligned}$$

More generally we can say that matrix which induces the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is the matrix

$$A_T = \begin{bmatrix} \vdots & & \vdots \\ T(\vec{e_1}) & \dots & T(\vec{e_n}) \\ \vdots & & \vdots \end{bmatrix}$$

For example let $T : \mathbb{R}^n \to \mathbb{R}^m$ and we know

$$T(\vec{e_1}) = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \qquad T(\vec{e_2}) = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$$

Then we know that A_T is

$$\begin{bmatrix} 2 & 4 \\ 3 & 3 \\ 4 & 2 \end{bmatrix}$$

1.3.2 Extra Textbook Questions

Exercise 1.3.1. A1

- $$\begin{bmatrix}1&0&-1&3\\0&1&3&1\\1&2&1&6\\2&5&1&1\end{bmatrix}\begin{matrix}-1\quad -2\\ \\ +\\ +\end{matrix} \rightsquigarrow \begin{bmatrix}1&0&-1&3\\0&1&3&1\\0&2&2&3\\0&5&3&-5\end{bmatrix}\begin{matrix} \\ -2\quad -5\\ +\\ +\end{matrix} \rightsquigarrow$$
$$\begin{bmatrix}1&0&-1&3\\0&1&3&1\\0&0&-4&1\\0&0&-12&-10\end{bmatrix}\begin{matrix} \\ \\ -3\\ +\end{matrix} \rightsquigarrow \begin{bmatrix}1&0&-1&3\\0&1&3&1\\0&0&-4&1\\0&0&0&-13\end{bmatrix}$$

  We can stop here because we see that in the bottom row we have $0 = -13$ which is certainly false so the system is inconsistent. So we can conclude that $\begin{bmatrix}3\\1\\6\\1\end{bmatrix}$ is not in the range of L.

- We'll figure out if $\begin{bmatrix}3\\-5\\1\\5\end{bmatrix}$ is in the range of L, so we're looking for some vector in $\mathbb{R}^4$ that becomes that vector under L. Represented as an augmented matrix we get

$$\begin{bmatrix}1&0&-1&3\\0&1&3&-5\\1&2&1&1\\2&5&1&5\end{bmatrix}\begin{matrix} \\ \\ -2\\ +\end{matrix} \rightsquigarrow \begin{bmatrix}1&0&-1&3\\0&1&3&-5\\-2&-4&-2&-2\\0&1&-1&3\end{bmatrix}\begin{matrix}2\\ \\ +\\ \\ \end{matrix} \rightsquigarrow$$
$$\begin{bmatrix}1&0&-1&3\\0&1&3&-5\\0&-4&-4&4\\0&1&-1&3\end{bmatrix}\begin{matrix} \\ \\ +\\ 4\end{matrix} \rightsquigarrow \begin{bmatrix}1&0&-1&3\\0&1&3&-5\\0&0&-8&16\\0&1&-1&3\end{bmatrix}\begin{matrix} \\ -1\\ \\ +\end{matrix} \rightsquigarrow$$
$$\begin{bmatrix}1&0&-1&3\\0&1&3&-5\\0&0&-4&8\\0&0&-8&16\end{bmatrix}\begin{matrix} \\ \\ -2\\ +\end{matrix} \rightsquigarrow \begin{bmatrix}1&0&-1&3\\0&1&3&-5\\0&0&-4&8\\0&0&0&0\end{bmatrix}\begin{matrix} \\ \\ \mid \cdot -\frac{1}{4}\\ \\ \end{matrix} \rightsquigarrow \begin{bmatrix}1&0&-1&3\\0&1&3&-5\\0&0&1&-2\\0&0&0&0\end{bmatrix}$$

And thus we know there is a solution so $\begin{bmatrix} 3 \\ -5 \\ 1 \\ 5 \end{bmatrix}$ is in the range of L.

- A2

1. We'll start with the range which we know looks like

$$\left\{ \vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} 2x_1 \\ -x_2 + 2x_3 \end{bmatrix} \right\}$$

We know that $2x_1$ and $-x_2 + 2x_3$ will span over all of $\mathbb{R}$ and so we know that the range looks like $span\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ and a basis for the range is the set $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

As for the null space we know that x component will be zero only if $x1 = 0$ and that the y component will be zero if and only if $-x2 + 2x_3 = 0 \Leftrightarrow x_2 = 2x_3$ and we let $t \in \mathbb{R}$ and let $x_3 = t$ and our solutions to the zero vector look like

$$\left\{ \vec{x} \in \mathbb{R}^3 : \vec{x} = \begin{bmatrix} 0 \\ 2t \\ t \end{bmatrix} \right\}$$

$$\left\{ \vec{x} \in \mathbb{R}^3 : \vec{x} = t \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}$$

Thus we conclude that a basis for the null space is $\left\{ \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}$

2. We know that the range looks like

$$\left\{\vec{x} = \begin{bmatrix} x_4 \\ x_3 \\ 0 \\ x_2 \\ x_1 + x_2 - x_3 \end{bmatrix}\right\}$$

$$\left\{\vec{x} \in \mathbb{R}^5 : \vec{x} = x_3 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_1 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \text{ for some } \right\}$$

And thus a basis for the range is

$$\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}$$

To show they are linearly independent we look at linear the linear combinations that give the zero vector, which would but each scalar has to be 0 so only the trivial answer exists so this set is linearly independent.

- A3

1. Let's define this transformation as

$$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = (x - y) \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

So then the matrix we are looking for is

$$\begin{bmatrix} 1 & -1 \\ 2 & -2 \\ 3 & -3 \end{bmatrix}$$

2. We know that this transformation should be

$$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = (2x + y)\vec{e_3}$$

And we use $\vec{e_3}$ so we get the range. And thus we know our matrix should be

$$\begin{bmatrix} 2 & 1 \\ 2 & 1 \\ 2 & 1 \end{bmatrix}$$

- A6

1. We'll start by finding a basis for the rowspace, the row space is

$$span\left\{ \begin{bmatrix} 1 \\ 2 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \right\}$$

A basis for this span is the set of lineary independent that span the row space so let's row reduce the rowspace

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ -2 & 5 & 8 \end{bmatrix} \begin{array}{l} 2 \\ \\ + \end{array} \rightsquigarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 7 & 10 \end{bmatrix} \begin{array}{l} \\ -7 \\ + \end{array} \rightsquigarrow$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -7 & -14 \\ 0 & 0 & -4 \end{bmatrix} \begin{array}{l} \\ \mid \cdot -\frac{1}{7} \\ \mid \cdot -\frac{1}{4} \end{array} \rightsquigarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

And thus we know the only solution to the zero vector is a trivial one and so this set is linearly independent a basis could be $\{\vec{e_1}, \vec{e_2}, \vec{e_3}\}$. We'll now look at the column space which is the span of each of the columns let's look at the linear combinations of the columns which give the zero vector

$$\begin{bmatrix} 1 & 2 & 8 \\ 1 & 1 & 5 \\ 1 & 0 & -2 \end{bmatrix} \begin{array}{l} -1 \quad -1 \\ + \\ \quad\quad + \end{array} \rightsquigarrow \begin{bmatrix} 1 & 2 & 8 \\ 0 & -1 & -3 \\ 0 & -2 & -10 \end{bmatrix} \begin{array}{l} \\ -2 \\ + \end{array}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 2 & 8 \\ 0 & 2 & 6 \\ 0 & 0 & -4 \end{bmatrix} \begin{array}{l} \\ \mid \cdot \frac{1}{2} \\ \mid \cdot -\frac{1}{4} \end{array} \rightsquigarrow \begin{bmatrix} 1 & 2 & 8 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

And thus we know that these vectors are linearly independent so a basis which is a subset of the colspace is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ 5 \\ -2 \end{bmatrix} \right\}$$

Now we'll find the null space which we know are all vectors that get mapped to $\vec{0}$ after being multiplied by this matrix so we're looking for vectors in $\mathbb{R}^3$ that satisfy

$$\begin{bmatrix} 1 & 2 & 8 \\ 1 & 1 & 5 \\ 1 & 0 & -2 \end{bmatrix} \vec{x} = \vec{0}$$

But remember this simply represents the linear combinations of the columns of the matrix so we're just looking at the combinations of the columns of the matrix that give the zero vector, which we know is only the trivial one, thus $\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and the null space is $\left\{\vec{0}\right\}$, so a basis for this is the empty set. As for the rank-nullity theorem, we know that the rank is 3 and the nullity is 0 so the number of columns should be 3 as it is.

2. Same process as above

3. Same process as above

- A8

1. We know that row reduction doesn't change the row space as we are just adding multiples of rows and adding them so we can say that the row space is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

If it were to span a subspace it would have to to span $\mathbb{R}^5$ but we know it doesn't becase we have less than or equal to 3 linearly independent vectors in that set. A basis for the rowspace of A is the set of row vectors which have a pivot since if only these vectors were in a set together then they would only yield a trivial solution to the zero vector which we know would be the set

$$\left\{ \begin{bmatrix}1\\0\\2\\0\\3\end{bmatrix}, \begin{bmatrix}0\\1\\-1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\\1\end{bmatrix} \right\}$$

2. If the column space is a basis then it must be a basis for $\mathbb{R}^4$ as these vectors are four dimensional. We are looking for a basis for the column space of A, we know that if we have the set of the vectors who correspond to the pivots in reduced row echelon form of A then we know that when they are row reduced we will only get the trivial solution to the zero vector and so they are linearly indepenednt. Thus a basis for the column space is

$$\left\{ \begin{bmatrix}1\\2\\1\\1\end{bmatrix}, \begin{bmatrix}1\\3\\1\\2\end{bmatrix}, \begin{bmatrix}1\\2\\3\\-1\end{bmatrix} \right\}$$

– We'll now find the solutions to

$$A\vec{x} = \vec{0}$$

We know that the matrix A can be thought of as a augmented matrix where we are solving for the zero vector and so row reducing gives us solutions for the linear combinations of the columns which yield the zero vector. Thus from the reduced row echelon form of A we can say that (where $\vec{x} = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}$).
That $x_5 = t$ for some $t \in \mathbb{R}$, $x_4 = -t$ $x_3 = s$ for some $s \in \mathbb{R}$ $x_2 = s - t$ and $x_1 = -(3t + 2s)$ as a general solution this is.

1.3.3 Extra Handout

Images and Sets under transformations.

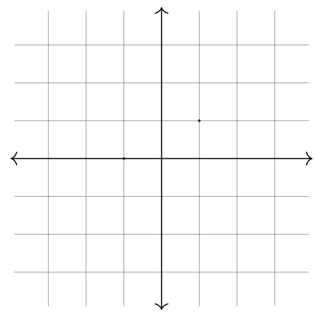
- 1. We know that $\mathcal{M}(P)$ is equal to

$$\left\{ \vec{x} \in \mathbb{R}^2 : \vec{x} = M\vec{v} \text{ for some } \vec{v} \in P \right\}$$

But that the only elements of P are $\vec{e_1}, \vec{e_2}$ so the image is the set

$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right\}$$

Drawing this we have



2. We have

$$\mathcal{M}(S) = \mathcal{M}\big(t\vec{e_1} + (1-t)\vec{e_2}\big)$$

Due to the linearity of M we know

$$= tM(\vec{e_1}) + (1-t)\mathcal{M}(\vec{e_2})$$

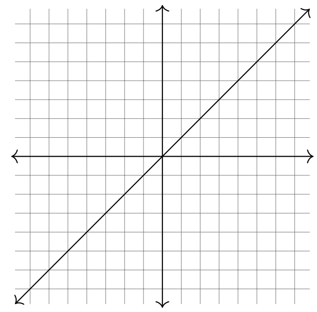
And thus this is the convex linear combinations of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$ We know that the span of a vector in X will look like $\begin{bmatrix} \alpha \\ 0 \end{bmatrix}$ for some $\alpha \in \mathbb{R}$ and so the range will end up looking like

$$\left\{ \vec{x} \in \mathbb{R} : \vec{x} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ 0 \end{bmatrix} \right\}$$

And equivalently

$$\left\{ \vec{x} \in \mathbb{R} : \vec{x} = \begin{bmatrix} \alpha \\ \alpha \end{bmatrix} \right\}$$

which is exactly $\mathit{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ Drawing this we have



- Let $\ell = \{a\vec{v} \text{ for some } a \in [0,\beta] \text{ for some } \beta \in \mathbb{R}\}$ then we'll look at the image of $\mathcal{M}$ which will look like
$$\left\{\vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \alpha \begin{bmatrix} x \\ y \end{bmatrix}\right\}$$
Which equivalenlty is
$$\left\{\vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} \alpha x - \alpha y \\ \alpha x \end{bmatrix}\right\}$$
which gives us the vectors
$$\alpha \begin{bmatrix} x - y \\ x \end{bmatrix}$$
Which by definition is a line segment.
Alternatively we know that a line segment looks like $\alpha v + \vec{u}$ and that the image of this under some linear transformation $\mathcal{J}$ is the set of vectors $\{\mathcal{J}(\alpha\vec{v} + \vec{u})\} = \{\alpha\mathcal{J}(\vec{v}) + \mathcal{J}(\vec{u})\}$ which is also a line segment.

- Q3

  1. $\mathcal{A}(\mathbb{R}^2)$ is the set
$$\left\{\vec{v} \in \mathbb{R}^2 : \vec{v} = \begin{bmatrix} x - 2y \\ -2x + 4y + 2 \end{bmatrix} \text{ for some } \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2\right\}$$
  But equivalenty this is
$$\vec{v} = x\begin{bmatrix} 1 \\ -2 \end{bmatrix} + y\begin{bmatrix} -2 \\ 4 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

but notice that $\begin{bmatrix} 1 \\ -2 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} -2 \\ 4 \end{bmatrix}$ so then we have

$$\vec{v} = s\begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

2. $\mathcal{A}(B)$ is the set

$$\left\{\vec{v} \in \mathbb{R}^2 : \vec{v} = x\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}\right\}$$

$$\left\{\vec{v} \in \mathbb{R}^2 : \vec{v} = \left(x - \frac{1}{2y}\right)\begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}\right\}$$

recall that $|x|, |y| \leq 1$ thus by definition we have $-1 \leq x, y \leq 1$ and thus

$$-\frac{3}{2} \leq x - \frac{1}{2}y \leq \frac{1}{2}$$

and so we can rewrite this set as

$$\left\{\vec{v} \in \mathbb{R}^2 : \vec{v} = s\begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} \text{ for some } s \in \left[-\frac{3}{2}, \frac{1}{2}\right]\right\}$$

- Q4

1. We know that the the xaxis is defined by the set X

$$\left\{\vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} \alpha \\ 0 \end{bmatrix} \text{ for some } \alpha \in \mathbb{R}\right\}$$

And so the image under $\mathcal{Q}$ is the set

$$\left\{\vec{v} \in \mathbb{R}^2 : \vec{v} = \begin{bmatrix} x \\ y^2 \end{bmatrix} \text{ for some } \begin{bmatrix} x \\ y \end{bmatrix} \in X\right\}$$

But we know the y of any vector in X is 0 so this just redfines the X axis and we know that $\mathbb{R}$ is closed under addition and multiplication so this is certianly a subspace and a subset of $\mathbb{R}^2$ for example take an element in X and it's guarnteed to look like $\begin{bmatrix} \alpha \\ 0 \end{bmatrix} \subseteq \mathbb{R}^2$ as required.

2. Let's call the line segment ℓ and we know that the image of ℓ under $\mathcal{Q}$ is the set

$$\left\{ \vec{v} \in \mathbb{R}^2 : \vec{v} = \mathcal{Q}\left(\begin{bmatrix} 3t \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right) \right\}$$

But since $\mathcal{Q}$ is not linear then we can't proceed to break up the sum, so we have to add what's inside the function so we have

$$\left\{ \vec{v} \in \mathbb{R}^2 : \vec{v} = \mathcal{Q}\left(\begin{bmatrix} 3t \\ 2 \end{bmatrix} \right) \right\}$$

And we know $\mathcal{Q}\left(\begin{bmatrix} 3t \\ 2 \end{bmatrix} \right) = \begin{bmatrix} 3t \\ 4 \end{bmatrix}$ which gives the line segment

$$t \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

- Take the line segment
$$\alpha(\vec{e_1} + \vec{e_2}) \text{ for all } \alpha \in [-1, 1]$$
We know when you square the y values you get a parabola, proof that this is not a line segment, is that if you choose two points on it, there exists a point that is not part of that line.

- Q5

- $$\left\{ \vec{v} \in \mathbb{R}^2 : \vec{v} = \begin{bmatrix} e^y \cos x \\ e^y \sin x \end{bmatrix} \text{ for some } \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \right\}$$

Chapter 2

Week 10

2.1 Lecture 1

Null space & Range

Definition 5 (Induced Transformation)**.** Let M be an $n \times m$ matrix. We say M **induces** a linear transformation $T_M : \mathbb{R}^m \to \mathbb{R}^M$ defined by

$$[T_M \vec{v}]_{\mathcal{E}'} = M[\vec{v}]_{\mathcal{E}}$$

Notice that $\mathcal{E}'$ and $\mathcal{E}$ are standard basis

Example 2.1.1. For example from the above definition if we have the vector $\vec{v} = \begin{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{bmatrix}_{\mathcal{B}}$ and we wanted to determine what $T_M(\vec{v})$ is we must first translate $\begin{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{bmatrix}_{\mathcal{B}}$ to the standard basis. One method to do so is to find the matrix X that translates from one basis to another. Then plug that new box of numbers into $M[\vec{v}]_{\mathcal{E}}$ and we get our solution to $T_M(\vec{v})$

Example 2.1.2. The difference between $M\vec{v}$ and $M[\vec{v}]_{\mathcal{E}}$ is the level of rigor? on the right we have something precise but on the left we don't. Also we can remember that $[\vec{v}]_{\mathcal{E}}$ represents a box of numbers, the coordinates in the basis, where as $\vec{v}$ is assumed to be in the $\mathcal{E}$ basis but this could alos mean a vector just floating in space without any type of basis involved.

Example 2.1.3. To determine what $[T_M \vec{e_1}]_{\mathcal{E}'}$ use the fact that it's equal to $M[\vec{e_1}]_{\mathcal{E}}$ and we know that this corresponds to the first column in M as $[\vec{e_1}]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Example 2.1.4. We start by knowing that each column must be in the range of T_M by inputting $\vec{e_i}$ into the equation. But we showed last time that the range is a subspace which means that if we find two elements in it, we know the sum and product of the vectors is also in the range, so we know that the span of each column is a susbset of the range. We then want to see if the span of the columns is actually eqaual to the range, so we must show the other direction now. So let $\vec{x_1}, \vec{x_2} \in range(T_M)$ so we know that $\vec{x} = T_M(\vec{v})$ for some $\vec{v} \in \mathbb{R}^m \Leftrightarrow [\vec{x}]_{\mathcal{E}'} = \left[T_M(\vec{v})\right]_{\mathcal{E}'} \Leftrightarrow [\vec{x}]_{\mathcal{E}'} = M[\vec{v}]_{\mathcal{E}}$ and we recall that M is only a list of vectors and that $[\vec{v}]_{\mathcal{E}}$ is but a box of numbers and so using matrix vector multiplication we know that we just get a linear combination of the vectors as columns in M and so we showed that $\vec{x}$ looks like a vector in the span of the columns of M as we wanted.

Definition 6 (Fundamental Subspaces)**.** Associated with any matrix M are three fundamental subspaces: the **row space** of M, denoted $row(M)$, is the span of the rows of M the **column space** of M, denoted $col(M)$, is the span of the columns of M; and the **null space** of M, denoted $null(M)$, is the set of solutions to $M\vec{x} = 0$

$$\text{Let } A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Example 2.1.5. We know that the row space is $span\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$ or

$$\left\{\begin{bmatrix}\alpha\\\beta\\0\end{bmatrix} \in \mathbb{R}^3, \text{ for some } \alpha, \beta \in \mathbb{R}\right\}$$

Example 2.1.6. The column space of A is $span\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}\right\}$ or equivalently $\mathbb{R}^2$, we can tell instantly that the column space is not the same as the row space as one is in $\mathbb{R}^3$ where the othe is $\mathbb{R}^2$.

Example 2.1.7. We are looking for all the vectors that are perpendicular to $\begin{bmatrix}1\\0\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\\0\end{bmatrix}$

so we require the the vectors $\vec{x} \in \mathbb{R}^3$ such that

$$\begin{bmatrix}1\\0\\0\end{bmatrix} \cdot \begin{bmatrix}x\\y\\z\end{bmatrix} = 0 \text{ and } \begin{bmatrix}0\\1\\0\end{bmatrix} \cdot \begin{bmatrix}x\\y\\z\end{bmatrix} = 0$$

$$x + y = 0$$

$$x = -y$$

And so we know all solutions are

$$\alpha \begin{bmatrix}-1\\1\\0\end{bmatrix} + \beta \begin{bmatrix}0\\0\\1\end{bmatrix}, \text{ for all } \alpha, \beta \in \mathbb{R}$$

Example 2.1.8. For the null space of A we know we are looking for all vectors $\begin{bmatrix}x\\y\\z\end{bmatrix} \in \mathbb{R}^3$ so that

$$A \begin{bmatrix}x\\y\\z\end{bmatrix} = 0$$

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\end{bmatrix} \begin{bmatrix}x\\y\\z\end{bmatrix} = 0$$

$$\begin{bmatrix}1\\0\end{bmatrix} x + \begin{bmatrix}0\\1\end{bmatrix} y + \begin{bmatrix}0\\0\end{bmatrix} z = \begin{bmatrix}0\\0\end{bmatrix}$$

So we determine that $x = y = 0$ but $z \in \mathbb{R}$ so we can think of this as the z axis coming out from the x y plane.

Example 2.1.9. The range of $T_A : \mathbb{R}^3 \to \mathbb{R}^2$

$$\left\{\vec{u} \in \mathbb{R}^2 : T_A(\vec{x}) = \vec{u}, \text{ for some } \vec{x} \in \mathbb{R}^3\right\}$$

$$\left\{\vec{u} \in \mathbb{R}^2 : \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\end{bmatrix} = \vec{u}, \text{ for some } \vec{x} \in \mathbb{R}^3\right\}$$

Which we know is saying that we are looking for the vectors $\vec{u}$ such that they are linear combinations of the columns of the matrix A so equivalently we have that $\vec{u} \in span\{\vec{e_1}, \vec{e_2}\}$

We'll now look for the $null(T_A)$ we know that the nullspace is defined by

$$\left\{\vec{x} \in \mathbb{R}^3 : T_A(\vec{x}) = 0\right\}$$

So we're really looking for the vectors which satisfy $A\vec{x} = 0$, which is equivalent to $null(A)$

Let $B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$ $C = rref(B) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$

Example 2.1.10. The row space of B is equal to $span\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$ and the row reducing involves the operations of addition and multiplication of the rows together, let $\vec{x}, \vec{u}$ represnt the first and second vectors in the span, we know that in the row reduced form (considering rows to be vectors) we have that the first row

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \vec{u} - (\vec{x} - \vec{u})$$

and the second row

$$\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \vec{x} - \vec{u}$$

Thus we know that the rows in the row reduced form are just linear combinations of the original rows, so we can say that

$$row(B) = row(C)$$

Example 2.1.11. As for $null(B) = \{\vec{x} : B\vec{x} = 0\}$ the process of finding soltions to the zero vector will be identical to just row reducing, where it's an augmented matrix with 0's on the right side, so due to the same reasoning as before we know that they will have the same solutions as row operations don't change the number of solutions so we can say that

$$null(B) = null(C)$$

Example 2.1.12. Now if we are asked to compute the null space of B then we can make things quite a bit easier on ourselves by using the row reduced matrix so well have

$$\left\{\vec{x} \in \mathbb{R}^3 : B\vec{x} = 0\right\}$$

So we're looking at the solutions to

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

Let $z = t$ for some $t \in \mathbb{R}$ then we determine $y = -2t$ and $x = t$ in general form we get

$$t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \text{ for all } t \in \mathbb{R}$$

Let $P = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}$ and $Q = rref(P) = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$

Example 2.1.13. $col(P) = span\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\} = span\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ and $col(Q) = span\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$
We notice these are different so then row reduction can change the columns space unlike the row space.

Definition 7 (Rank)**.** For a linear transformation $T : V \to W$, the **rank** of T, denoted $rank(T)$, is the dimension of the range of T

For an $n \times m$ matrix M, the **rank** of M, denoted $rank(M)$, is the number of pivots in $rref(M)$

Let $\mathcal{P}$ be the projection onto $span\{\vec{u}\}$ where $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, and let $\mathcal{R}$ be a rotation counter-clockwise by $\frac{\pi}{2}$ radians.

Example 2.1.14. $range(\mathcal{P})$ is equal to $span\vec{u}$ because we know that every vector you choose will become some a scalar of $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ thus the dimension is 1, so it's rank is 1.

As for the rank of $\mathcal{R}$ we know that iwe we consider every vector in $\mathbb{R}^2$ and rotate them all $\frac{\pi}{2}$ radians then we still have all of $\mathbb{R}^2$ so we know that the rank is still 2.

Example 2.1.15. We'll now find the rank of the matricies that go along with $\mathcal{P}$ and $\mathcal{R}$ first we know that the matricies are

$$\mathcal{P} = \frac{1}{13} \begin{bmatrix} 4 & 6 \\ 6 & 9 \end{bmatrix} \qquad \mathcal{R} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Starting with $\mathcal{P}$

$$\begin{bmatrix} 4 & 6 \\ 6 & 9 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 24 & 36 \\ 24 & 36 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 24 & 36 \\ 0 & 0 \end{bmatrix}$$

And so we know that there is one pivot so the rank of this matrix is 1

As for $\mathcal{R}$ by swapping row one and row two we see that there must be two pivots and so the rank is 2.

2.2 Lecture 2

Let M be a matrix and recall that

$$row(M) = span \text{ rows of M} \qquad col(M) = span \text{ cols of M}$$

And that the following properties hold

$$row(M) = row\big(rref(M)\big) \qquad col(M) \neq col\big(rref(M)\big)$$

Also we know that

$$rank(M) = \text{ pivots of } rref(M)$$

Example 2.2.1. If M looked like this after row reduction we know it's rank is two (two pivots)

$$\begin{bmatrix} a & b & c \\ d & e & f \\ j & i & k \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 0 \end{bmatrix}$$

Corollary 2. $rank(M) = \text{ \# linearly independent columns of } rref(M)$

$$\begin{aligned}
&= dim\Big(col\big(rref(M)\big)\Big) \\
&= dim\big(col(M)\big) \qquad \textit{Solutions aren't lost} \\
&= \text{ \# linearly independent columns of } M \\
&= dim\Big(row\big(rref(M)\big)\Big) \\
&= dim\big(row(M)\big) \qquad \textit{pivots are both horizontal and vertical}
\end{aligned}$$

Definition 8 (Transpose of a Matrix)**.** The transpose of a matrix M is the matrix M^T whose rows are the columns of M

Example 2.2.2. If

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Leftrightarrow M^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

Or

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \Leftrightarrow M^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Corollary 3.

$$rank(M) = rank\Big(M^T\Big)$$

because $dim\big(col(M)\big) = dim\big(row(M)\big)$ and we know that $col(M) = row\Big(M^T\Big)$

Find the rank of the following matricies

Exercise 2.2.1. $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ we can see that there is one linearly independent vector so the dimension is 1 and rank is 1

Exercise 2.2.2. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ we can see that there are 2 linearly independent vectors so dimension is 2 and rank is 2

Exercise 2.2.3. $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ We know that the dimension of the column space is 2 and thus rank is 2

Exercise 2.2.4. $\begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}$ the rank is 1, there is only one column

Let M be a 3×4 matrix whose rank is 3 does this mean that all of the columns of M are linearly independent ?

No! if the matrix is 3×4 then we know that there are four columns, and if the rank is 3 we know 3 of the are linearly independent but we don't know about the last one, so we can't be sure.

What if M was 4×3 then we could be sure that every column is linearly independent as we know that it must have 3 pivots and so it must be true.

Theorem 4 (Rank-nullity)**.** The **nullity** of a matrix is the dimension of the null space. The rank-nullity theorem for a matrix A states

$$rank(A) + nullity(A) = \# \text{ of columns of } A$$

Example 2.2.3. If we have

$$\begin{bmatrix} . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & . & . & . \\ 0 & 1 & . & . & . \\ 0 & 0 & . & . & . \end{bmatrix}$$

And let's assume that the last 3 columns are free variable columns so then we know that the solution to the zero vector will have three paramers $t_1, t_2, t_3 \in \mathbb{R}$ and so we know that the dimension of the nullspace will be 3 .

Example 2.2.4. Let $\vec{u}, \vec{v} \in \mathbb{R}^9$ be linearly independent and $\vec{w} = 2\vec{u} - \vec{v}$

$$A = \left[\vec{v} \mid \vec{u} \mid \vec{w}\right]$$

We instantly know that the dimension of the column space of A is 2 and so rank is 2 thus nullity is 1 by R-N-T

As for A^T we know that A has two pivot columns and so A^T also has 2, and so by R-N-T it has nullity of 7.

Theorem 5 (Rank-nullity for linear transformations)**.** Let $T : V \to W$ be a linear transformation thus there is a matrix A such that $T = T_A$ so

$$T(\vec{x}) = A\vec{x} \text{ for all } \vec{x} \in V$$

We know from theorem 4 that

$$\begin{aligned} \#\text{ columns of } A &= rank(A) + nullity(A) \\ &= dim\big(col(A)\big) + dim\big(null(A)\big) \\ &= dim\big(range(T)\big) + dim\big(null(T)\big) \end{aligned}$$

So we can finally say that

$$dim\big(range(T)\big) + dim\big(null(T)\big) = \#\text{ columns of } A$$

2.2.1 Inverse Matrix

Definition 9 (Inverse Transformation)**.** If it exsits, the inverse transformation of $T : R^n \to R^m$ is

$$T^{-1} : R^m \to R^n$$

That satisfies the following properties

$$T\Big(T^{-1}(\vec{x})\Big) = \vec{x} \text{ for all } \vec{x} \in \mathbb{R}^n$$
$$T^{-1}\big(T(\vec{y})\big) = \vec{y} \text{ for all } \vec{y} \in \mathbb{R}^m$$

If $T = T_A$ and $T^{-1} = T_B$ then

$$\begin{aligned}\vec{x} &= T\Big(T^{-1}(\vec{x})\Big)\\ &= T(B\vec{x})\\ &= AB\vec{x}\end{aligned}$$

But for this to be true for all $\vec{x} \in \mathbb{R}^m$ then we require

$$AB = I$$

And if A is $n \times m$ and B is $m \times n$ then we know that I is $n \times n$
Similarly $T^{-1}\big(T(\vec{y})\big) = \vec{y} \Leftrightarrow BA = I$
Thus we say B is the inverse Matirx of A denoted $B = A$

Definition 10 (Inverse Matrix)**.** We say B is the inverse matrix of A denoted $B = A^{-1}$ if

$$AB = BA = I$$

2.3 Tutorial 6

1. We say the function $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear if we can say

$$T(\vec{x} + \alpha\vec{y}) = T(\vec{x}) + \alpha T(\vec{y}) \text{ for all } \vec{x}, \vec{y} \in \mathbb{R}n$$

2. (a) *Proof.* We'll show that $\mathcal{A}$ is linear so let $\vec{u}, \vec{v} \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$ we know that

$$T(\vec{u} + \alpha\vec{v}) = T\left(\begin{bmatrix} x_1 + \alpha x_2 \\ y1 + \alpha y_2 \end{bmatrix}\right) = \begin{bmatrix} y_1 + \alpha y_2 \\ 0 \\ x_1 + \alpha x_2 \end{bmatrix} = T(\vec{u}) + \alpha T(\vec{v})$$

□

(b) We will show that $\mathcal{B}$ is not a linear transformation we know

$$\mathcal{B}\left(\begin{bmatrix} -2 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \neq \mathcal{B}\left(\begin{bmatrix} -2 \\ 0 \end{bmatrix}\right) + \mathcal{B}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

(c) We will show that $\mathcal{C}$ is a linear transformation it follows that

$$\mathcal{C}(\vec{u} + \alpha\vec{v}) = \vec{0} = \vec{0} + \alpha\vec{0} = \mathcal{C}(\vec{u}) + \alpha\mathcal{C}(\vec{v})$$

(d) We will show that $\mathcal{D}$ is not a linear transformation

$$\mathcal{D}(\vec{u}+\alpha\vec{v}) = \begin{bmatrix}1\\1\end{bmatrix} \neq \begin{bmatrix}1\\1\end{bmatrix} + \alpha\begin{bmatrix}1\\1\end{bmatrix} = \mathcal{D}(\vec{u}) + \alpha\mathcal{D}(\vec{u})$$

3. We will now compute the null space, range and rank of each of the linear transformations

(a) We'll start by comupting the range which we can we can say is

$$\left\{\vec{v}\in\mathbb{R}^3 : \vec{v} = s\begin{bmatrix}1\\0\\0\end{bmatrix} + t\begin{bmatrix}0\\0\\1\end{bmatrix} \text{ for some } t, s\in\mathbb{R}\right\}$$

we know that that null space is only $\vec{0}$, and that the rank must be 2 as a basis for the range is

$$\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$$

(b) We'll start with the range

$$\{x\in\mathbb{R} : x\geq 0\}$$

And the null space is

$$\{x\in\mathbb{R} : x < 0\}$$

So then the rank of $\mathcal{B}$ is 0

(c) For $\mathcal{C}$ we know the range is

$$\left\{\vec{0}\right\}$$

And then null space is

$$\left\{\vec{x}\in\mathbb{R}^2\right\}$$

And then rank is 0

(d) The range of $\mathcal{D}$ is $\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$ and the null space is $\emptyset$ since $\mathcal{D}$ is not a subspace then there is no basis for it, so the rank doesn't exist.

4. (a) $\mathcal{X}$: the tranformation invoked by the identity matrix.

(b) $\mathcal{Y}$: cannot exist as we know $\mathit{range}(Y)\subseteq\mathbb{R}^2$ and so the dimension of $\mathcal{Y}\leq 2$

(c) $\mathcal{Z}$: *comp* $\begin{bmatrix}1\\1\\1\end{bmatrix}$ $\vec{x}$ which defines a line in 3d space thus has dimension 1 so rank is 1

(d) $\mathcal{W}$: the identity transformation

(e) Impossible?

2.4 Extra Textbook Questions

- Q1

  a) We can see that $\begin{bmatrix}3\\2\end{bmatrix}$ and $\begin{bmatrix}-4\\5\end{bmatrix}$ are linearly independent and so we know that we can use row operations to get this turn this matrix into the identity matrix.

$$\begin{bmatrix}3 & -4 & 1 & 0\\2 & 5 & 0 & 1\end{bmatrix}\begin{array}{l}+\\-1\end{array} \rightsquigarrow \begin{bmatrix}1 & -9 & 1 & -1\\2 & 5 & 0 & 1\end{bmatrix}\begin{array}{l}-2\\+\end{array}$$
$$\rightsquigarrow \begin{bmatrix}1 & -9 & 1 & -1\\0 & 23 & -2 & 3\end{bmatrix} \mid \cdot \tfrac{1}{23} \rightsquigarrow \begin{bmatrix}1 & -9 & 1 & -1\\0 & 1 & -\frac{2}{23} & \frac{3}{23}\end{bmatrix}$$
$$\rightsquigarrow \begin{bmatrix}1 & 0 & 1-\frac{18}{23} & -1+\frac{27}{23}\\0 & 1 & -\frac{2}{23} & \frac{3}{23}\end{bmatrix}$$

  And so the inverse is

$$\frac{1}{23}\begin{bmatrix}5 & 4\\-2 & 3\end{bmatrix}$$

  b) We can see that these are 3 linearly independent vectors and so it can be reduced to 3 pivots, so it is invertible

  c) We know that the third column is 2 of the first column plus the second column thus we know that after row reduction we will only get two pivots and thus we can't get the identity matrix as that is 3. why again?

  d) If we swap all the rows of this matrix and then reduce we can clearly see we can get the indentity matrix so we know it has an inverse

  e) Row reduce and see if it it has 4 pivots.

  f) Reduce upwards and we get the identity matrix, it's invertible.

- A2

a) Row reduce B to I which we know must be possible as the columns are linearly independent and so $\vec{s} = B^{-1}\begin{bmatrix}1\\1\\1\end{bmatrix}$

a) $\vec{x} = B^{-1}\begin{bmatrix}-1\\0\\1\end{bmatrix}$

a) Same as above.

- A4

a) If the rotation is counter clockwise by $\frac{\pi}{6}$ then then inverse must be a clockwise rotation by $\frac{\pi}{6}$.

a) This subtracts 3 times the y component of the vector from the x component a given input vector, so to invert this we add 3 times the y component to the x component to cancel it back out, so the matrix that represents the operation is

$$\begin{bmatrix}1 & 3\\0 & 1\end{bmatrix}$$

a) This matrix represents multiplying the x component of the vector by 5, the inverse is multiplying the x component of the vector by $\frac{1}{5}$ which is represented by the matrix

$$\begin{bmatrix}\frac{1}{5} & 0\\0 & 1\end{bmatrix}$$

a) For some vector in $\vec{v} \in \mathbb{R}^3$ we know that it looks like $\begin{bmatrix}x\\y\\z\end{bmatrix}$, after being operated on by this matrix, we multiply y by -1 and so we get $\begin{bmatrix}x\\-y\\z\end{bmatrix}$ so we have to multiply by the same matrix to get back so the inverse is

$$\begin{bmatrix}1 & 0 & 0\\0 & -1 & 0\\0 & 0 & 1\end{bmatrix}$$

- A1

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{array}{l} \leftarrow + \\ -5 \\ \end{array} \rightsquigarrow \begin{bmatrix} 1 & -5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{array}{l} \\ \leftarrow \\ \leftarrow \end{array}$$

$$\rightsquigarrow \begin{bmatrix} 1 & -5 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{array}{l} \quad 4 \\ \mid \cdot 6 \\ \mid \cdot -1 \leftarrow + \end{array} \rightsquigarrow \begin{bmatrix} 1 & -5 & 0 \\ 0 & 0 & 6 \\ 4 & -21 & 0 \end{bmatrix}$$

Thus we know

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = B$$

Where B is the matrix which does all the row operations at once.

- A3

  a) add -4 row 2 to row 3
  b) multiply row 1 and row 3 by -1 (not elementary)
  c) multiply row 1 by 3 and add row 3 (not elementary)
  d) swap row 1 and 3
  e) d) then swap row 1 and 2 (not elementary)
  f) do nothing

is this elementary?

- A4

  a) We'll start by describing the elementary row operations to get to A from I.
     - swap row 2 and 3
     - add 3 row 3 to row 1
     - multipy row 2 by 2
     - add 2 row 2 to row 1

     To undo this we do
     - add -2 row 2 to row 1
     - multiply row 2 by $\frac{1}{2}$
     - add -3 row 3 to row 1
     - swap row 2 and 3

These steps represented as a matrix look like

$$\begin{bmatrix}1 & 0 & 0\\0 & 0 & 1\\0 & 1 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & -3\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & \frac{1}{2} & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & -2 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

Which is the matrix

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{array}{l} \leftarrow\!\urcorner\, + \\ \lrcorner\, -2 \\ \end{array} \rightsquigarrow \begin{bmatrix}1 & -2 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{array}{l} \leftarrow\!\urcorner\, + \\ \mid \cdot \frac{1}{2} \\ \lrcorner\, -3 \end{array} \begin{array}{l} \\ \leftarrow\!\urcorner \\ \leftarrow\!\lrcorner \end{array} \rightsquigarrow \begin{bmatrix}1 & -2 & -3\\0 & 0 & 1\\0 & \frac{1}{2} & 0\end{bmatrix}$$

A is the product of the inverse operations as matricies ofthe steps to get to I from A, we can use the list I provided earlier. The rest of the questions are the same.

Chapter 3

Week 11

3.1 Inverses

3.1.1 56

We start by applying some row operations to some identity matrices

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\begin{array}{l}\urcorner^{2}\\ \\ \hookleftarrow_{+}\end{array} \rightsquigarrow \begin{bmatrix}1&0&0\\0&1&0\\2&0&1\end{bmatrix} = E_1$$

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{array}{l}\urcorner^{-2}\\ \\ \hookleftarrow_{+}\end{array} \begin{bmatrix}1&0&0\\0&1&0\\-2&0&1\end{bmatrix} = E_2$$

Applying E_1 and E_2 to A we can see...

$$\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\2&0&1\end{bmatrix} = \begin{bmatrix}1&2&3\\4&5&6\\2\cdot 1+7&2\cdot 2+8&2\cdot 3+9\end{bmatrix}$$

Which we note is the same thing as adding twice the first row to the third. Then we know that E_2A would be subtracting two of row 1 from row 3.

If we apply E_1 then E_2 then we just added and subtraced the first row which does nothing so $E_1E_2 = I = E_2E_1$

3.1.2 57

We know that matrix A represents adding twice row 2 to row1 and adding -3 row 1 to row 3, by observation these operations will turn D into the identity matrix, now let's look at

it the other way.
We can see that matrix D represents adding 3 of row 1 to row 3 then adding -2 row 2 to row 1 which we can tell are the inverse operations as what A does so we know this will yield the identity matrix. So we can say A and D are inverses of each other.

We'll now prove why the algorithm to determine the inverse matrix actually works, for example if we have the matrix

$$\begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix}$$

Then we would set something up like this

$$\left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{array}\right]$$

Where on the right hand side we have the matrix that will hold an of the ro operations we do to transform this matrix on the left into the identity matrix, thus once we're done the matrix on the right will represent all the row operations that would tern it into the identity matrix.

So it's just a way of keeping track of two different pieces of data visually.

multiply row 2 by $\frac{1}{2}$, so now the matrix on the right represents this operation

$$\left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{2} \end{array}\right]$$

add -4 of row 2 to row 1

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & -2 \\ 0 & 1 & 0 & -\frac{1}{2} \end{array}\right]$$

And then we know the matrix on the right represents applying the two row operations that turn the matrix on the left into the identity matrix upon multiplication by the matrix. Thus we've found the inverse matrix for the left matrix.

Theorem 6 (Inverse Existance)**.** A matrix has an inverse if and only if it is square and can be "row reduced" to the identity matrix.
We know you can reach the identity matrix if you can reach row reduced echolelon form which means you must have a pivot for each column, so we also require that the columns must be linearly independent.

What does this mean?

Remark 3.1.1. If it's not square it's not one to one and onto, you are losing a dimesion.

Remark 3.1.2. Equivalent to theorem 6 we can require the rows be linearly independent as we know that $dim\big(row(A)\big) = dim\big(col(A)\big) = rank(A)$

Why do we know this again?

3.1.3 59

By definition $A^{-1}A = I$ and so since we know that A has an inverse, then then we know there we know it must have been able to be reduced to the I with row operations, thus $rank(A) = I$

Example 3.1.1. Let C be the matrix whose left columns are A and rightmost column is $\vec{b}$ we know that this represents the following

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{b}$$

so then we can say that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}\vec{b}$$

by multiplying by the inverse of A to both sides.

3.1.4 60

We know that if we have two square matricies X, Y that $(XY)^{-1}$ is the matrix A such that $A(XY) = I$ and we require that $(XY)A = I$ in this case we know we just need the inverses next to eachother and this will cause a chain reaction of identity matrices so we can see that $(XY)^{-1} = Y^{-1}X^{-1}$

3.1.5 61

We know that $\vec{e_1} = \frac{1}{2}\vec{b_1} + \frac{1}{2}\vec{b_2}$ and so $[\vec{e_1}]_\mathcal{B} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$ and then $\vec{e_2} = \frac{1}{2}\vec{b_1} - \frac{1}{2}\vec{b_2}$ so we know that $[\vec{e_2}]_\mathcal{B} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$.

We know that $X = \begin{bmatrix} \vec{b_1} & \vec{b_2} \end{bmatrix}$ let's assume we have a vector such that $[v]_\mathcal{B} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ so we know that X converts to standard basis.

$$X \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \alpha\vec{b_1} + \beta\vec{b_2} = [\vec{v}]_\mathcal{E}$$

And thus we know $X[\vec{e_2}]_\mathcal{B} = [\vec{e_2}]_\mathcal{E} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $X[31]_\mathcal{B} = [\vec{e_1}]_\mathcal{E} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

3.1.6 62

- We know that X^{-1} should exists since the vectors in $\mathcal{B}$ form a basis so each is linearly independent thus row reduction will lead to reduced row echelon form and since X is square we know that X^{-1} must exist.

- We know that if X is the matrix that converts from $\mathcal{B}$ to $\mathcal{E}$ and so X^{-1} must convert $\mathcal{E} \to \mathcal{B}$

explain why this is true

$$X^{-1}[\vec{v}]_{\mathcal{E}} = [\vec{v}]_{\mathcal{B}}$$

3.2 Lecture 2 Week 11

Recall: we say M induces the linear transformation

$$T : \mathbb{R}^n \to \mathbb{R}^n \tag{α}$$

Or M is the matrix corresponding to T if

$$\big[T(\vec{v})\big]_{\mathcal{E}} = M[\vec{v}]_{\mathcal{E}}$$

Where $\mathcal{E}$ is the standard basis for $\mathbb{R}^n$ and $T_M = T$
If we don't sepcify the basis, then $\mathcal{E}$ is assumed. So we can replace α by $T(\vec{v}) = M\vec{v}$

Some properties follow

Prove this!

1. We say T_M and T_N are inverses $\Leftrightarrow$ M and N are inverses

2. T_M is invertible $\Leftrightarrow$ M is invertaible in which case

$$(T_M)^{-1} = T_{M^{-1}}$$

3. We define the folowing notation

$$M = [T]_{\mathcal{E}}$$

Means that M is the matrix for T in the $\mathcal{E}$ basis such that α holds

4. If $\mathcal{B}$ is a nother basis for $\mathbb{R}^n$ and $A = [T]_{\mathcal{B}}$ then A is the matrix for T in the basis $\mathcal{B}$ so we know

$$\big[T(\vec{v})\big]_{\mathcal{B}} = A[\vec{v}]_{\mathcal{B}} \text{ for all } \vec{v} \in \mathbb{R}^n$$

Example 3.2.1. Let $n = 2$ $\mathcal{B} = \left\{\vec{b_1}, \vec{b_2}\right\}$ and $\left[\vec{b_1}\right]_{\mathcal{E}}$ and $\left[\vec{b_2}\right]_{\mathcal{E}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ so this means

$$\vec{b_1} = 1\vec{e_1} + 1\vec{e_2} \qquad \vec{b_2} = 0\vec{e_1}1 + 1\vec{e_2}2$$

And sayign $b_1 = \left[\begin{bmatrix}1\\1\end{bmatrix}\right]_{\mathcal{E}}$ is the vector whose $\mathcal{E}$ coordinaqtes are $\begin{bmatrix}1\\1\end{bmatrix}$ and similarly b_2 coordinates are $\begin{bmatrix}0\\1\end{bmatrix}$ when we are looking at the $\mathcal{E}$ basis.

Example 3.2.2. If $[T]_{\mathcal{B}} = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$ then we can say

$$T\left(\left[\begin{bmatrix}1\\1\end{bmatrix}\right]_{\mathcal{E}}\right) = T\left(\vec{b1}\right) = T\left(\left[\begin{bmatrix}1\\0\end{bmatrix}\right]\right)$$

Remember here that $\left[\begin{bmatrix}1\\1\end{bmatrix}\right]_{\mathcal{E}}$ is the vector in $\mathcal{E}$ cordinates And the result of this will be

$$\left[A\begin{bmatrix}1\\0\end{bmatrix}\right]_{\mathcal{B}} = \begin{bmatrix}1\\3\end{bmatrix}_B = \vec{b_1} + 3\vec{b_2} = \begin{bmatrix}1\\4\end{bmatrix}_{\mathcal{E}}$$

Remark 3.2.1. say we have $[\vec{v}]_{\mathcal{E}}$ and we are trying to find what is $\big[T(\vec{v})\big]_{\mathcal{E}}$ and all we know is A which is the matrix which which induces the transformation T for vectors written in the $\mathcal{B}$ basis, then the process could look like this.

$$[\vec{v}]_{\mathcal{E}} \implies [\vec{v}]_{\mathcal{B}} \implies A[\vec{v}]_{\mathcal{B}} = \big[T(\vec{v})\big]_{\mathcal{B}} \implies \text{ convert to } \mathcal{E} \implies \big[T(\vec{v})\big]_{\mathcal{E}}$$

Remark 3.2.2. There has recently been a source of confusion relating to change of basis, so here is what you have to remember.

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix}\alpha\\ \beta\end{bmatrix} \qquad \text{(a box of numbers)}$$

$$\left[\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\right]_{\mathcal{B}} = \alpha\vec{b_1} + \beta\vec{b_2} = \vec{v}$$

Example 3.2.3. Alternatively if our goal was to find $\left[T\left(\vec{b_1}\right)\right]_{\mathcal{E}}$ then we could have done this.

$$\begin{aligned}\left[T\left(\vec{b_1}\right)\right]_{\mathcal{B}} &= A\left[\vec{b_1}\right]_{\mathcal{B}} \\ &= A\begin{bmatrix}1\\0\end{bmatrix} \\ &= \begin{bmatrix}1\\3\end{bmatrix} && \text{(a box of numbers for the } \mathcal{B} \text{ basis)}\end{aligned}$$

So then we know that

$$T\left(\vec{b_1}\right) = \vec{b_1} + 3\vec{b_2} = \begin{bmatrix}1\\4\end{bmatrix}_{\mathcal{E}}$$

3.2.1 63

- We know by definition that $[\vec{c_1}]_{\mathcal{C}} = \begin{bmatrix}1\\0\end{bmatrix}$ and $[\vec{c_2}]_{\mathcal{C}} = \begin{bmatrix}0\\1\end{bmatrix}$

- we know the following is true

$$T\left(\begin{bmatrix}2\\1\end{bmatrix}_{\mathcal{E}}\right) = T(\vec{c_1}) = 2\vec{c_1} \qquad T\left(\begin{bmatrix}5\\3\end{bmatrix}_{\mathcal{E}}\right) = T(\vec{c_2}) = \vec{c_2}$$

- We already know that $T\vec{c_1} = 2\vec{c_1}$ and $T\vec{c_2} = \vec{c_2}$ so then we know

$$[2\vec{c_1}]_{\mathcal{C}} = \begin{bmatrix}2\\0\end{bmatrix} \qquad [\vec{c_2}]_{\mathcal{C}} = \begin{bmatrix}0\\1\end{bmatrix}$$

- We know that $\begin{bmatrix}\alpha\\\beta\end{bmatrix}_{\mathcal{C}} = \alpha\vec{c_1} + \beta\vec{c_2}$ so then

$$T(\alpha\vec{c_1} + \beta\vec{c_2}) = \alpha T(\vec{c_1}) + \beta T\left(\vec{\vec{c_2}}\right) = \alpha 2\vec{c_1} + \beta\vec{c_2} = \begin{bmatrix}2\alpha\\\beta\end{bmatrix}$$

- Let X be the matrix which induces the transformation T on elements of the $\mathcal{C}$ basis we know the following

$$\left[T(\vec{c_1})\right]_{\mathcal{C}} = X[\vec{c_1}]_{\mathcal{C}}$$

But we have $\left[T(\vec{c_1})\right]_{\mathcal{C}} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ we also know that $\left[T(\vec{c_2})\right]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and thus

$$X = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

And here we know that $X = [T]_{\mathcal{C}}$

- We know that we want to find $[T]_{\mathcal{E}}$, remember that's the matrix K such that the following properties holds
$$\left[T(\vec{v})\right]_{\mathcal{E}} = K[\vec{v}]_{\mathcal{E}}$$
Here's the plan, we know letting $\vec{v} = \vec{e_1}$ or $\vec{e_2}$ will give us the columns of K but we must determine what $T(\vec{v})$ is then convert that into the $\mathcal{E}$ basis. Though know what $[T]_{\mathcal{C}}$ is which is useful in the following formula
$$\left[T(\vec{v})\right]_{\mathcal{C}} = [T]_{\mathcal{C}}[\vec{v}]_{\mathcal{C}}$$
Ok, so let's start by converting $\vec{e_1}$ to the other basis, by observation we can tell that $\vec{e_1} = 3\vec{c_1} - \vec{c_2} \Leftrightarrow [\vec{e_1}]_{\mathcal{C}} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and also that $\vec{e_2} = -5\vec{c_1} + 2\vec{c_2} \Leftrightarrow [\vec{e_2}]_{\mathcal{C}} = \begin{bmatrix} -5 \\ 2 \end{bmatrix}$ and now we know
$$\begin{aligned}\left[T(\vec{e_1})\right]_{\mathcal{C}} &= \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -5 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} -10 \\ 2 \end{bmatrix}\end{aligned}$$
And also
$$\begin{aligned}\left[T(\vec{e_2})\right]_{\mathcal{C}} &= \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 6 \\ -1 \end{bmatrix}\end{aligned}$$
And thus we know that $T(\vec{e_2}) = \begin{bmatrix} 7 \\ 3 \end{bmatrix}_{\mathcal{E}}$ and $T(\vec{e_1}) = \begin{bmatrix} -10 \\ -4 \end{bmatrix}_{\mathcal{E}}$ and thus
$$K = \begin{bmatrix} -10 & 7 \\ -4 & 3 \end{bmatrix}$$
Alternatively we already know that A^{-1} columns are $[\vec{e_1}]_{\mathcal{C}}$ and $[\vec{e_2}]_{\mathcal{C}}$ so we could have gone from there then multiplied by $[T]_{\mathcal{C}}$ to get what $T(\vec{c_1})$ and $T(\vec{c_2})$ were.

Tutorial 7

- Q1
  - If f is invertible this means that there exists a function f^{-1} such that

$$f \circ f^{-1} = I$$
$$f^{-1} \circ f = I$$
$$f\Big(f^{-1}(\vec{x})\Big) = \vec{x} \text{ for all } \vec{x} \text{ in f's range}$$
$$f^{-1}\big(f(\vec{x})\big) = \vec{x} \text{ for all } \vec{x} \text{ in f's domain}$$

  - We say that A is invertible if there exists a matrix A^{-1} such that

$$AA^{-1} = I = A^{-1}A$$

- Q2
  - We know that a is incorrect since matrix multiplication isn't commutative that is for any two matricies A, B
$$AB \neq BA$$
  - b is correct they multiplied by A^{-1} from the left to both right hand side and left hand side .
  - This is incorrect division by a matrix is not defined $\frac{1}{A}$
  - d also fails due to the same reason as c.
  - division is not defined for vectors and so $\frac{1}{\vec{b}}$ doesn't make sense.
- Q3
  - R
    * We know it is invertible as the inverse function exists, namely the transformation R^{-1}, a rotation counter clockwise by $\frac{\pi}{6}$
    * If we imagine all vectors in $\mathbb{R}^2$ that have undergone a rotation of $\frac{\pi}{6}$ we still have $\mathbb{R}^2$ and thus $rank(R) = 2$
  - D
    * We know that it is invertible as the inverse transformation exists namely anothether reflection about the line $y = 4x$
    * If we imagine the image of $\mathbb{R}^2$ under D we can see that everything above and below the line get swapped, this still spans all of $\mathbb{R}^2$ and so the $rank(R) = 2$

- P
  - We know that P does not have an inverse, for example multiples of a normal vector for the line $y = 4x$ will all get mapped to the zero vector thus the $P^{-1}\left(\vec{0}\right)$ doesn't have a unique value associated with it.
  - We know that the range looks like scalar multiples of the vector $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ thus a basis is the set $\left\{\begin{bmatrix} 1 \\ 4 \end{bmatrix}\right\}$ and so $rank(P) = 1$
- S
- We know that S is invertible as there exists S^{-1} namely halving the length of every vector.
- Doubling every vector still will span $\mathbb{R}^2$ and thus we know $rank(S) = 2$

• Q4

- I believe that a transformation $T : V \to W$ has an inverse if and only if $rank(V) = rank(W)$

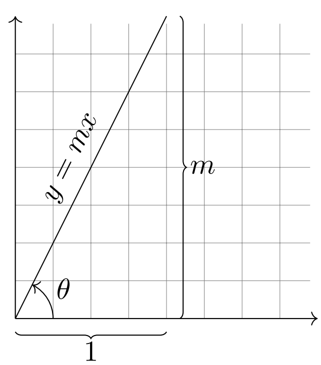
  Proof. For contradiction let's assume that T has an inverse but $rank(V) \neq rank(W)$. Let $n = rank(V)$ and $k = rank(W)$
  - Case 1 $n < k$
    - We know that a basis for V has n elements
    $$\{\vec{v_1}, \ldots, \vec{v_n}\}$$
    and thus the range of T looks like $\left\{\vec{x} \in W : \vec{x} = T(\vec{y}) \text{ for some } y \in V\right\}$ and thus the range is
    $$\left\{T(\alpha_1\vec{v_1} + \cdots + \alpha_5\vec{v_5}) \text{ for some } \alpha_1, \ldots, \alpha_n \in \mathbb{R}\right\}$$
    which will be
    $$spanT(\vec{v_1}), \ldots, T(\vec{v_n})$$
    and so the dimension of the range is less than or equal to n so there will be vectors in W that are not reached as it's dimension is greater than n and so the inverse of these vectors won't be defined, so the inverse must not exist which is a contradiction.
  - Case $n > k$.
    - Making the same argument as above shows but looking at the range of T^{-1} will give the same contradiction.

□

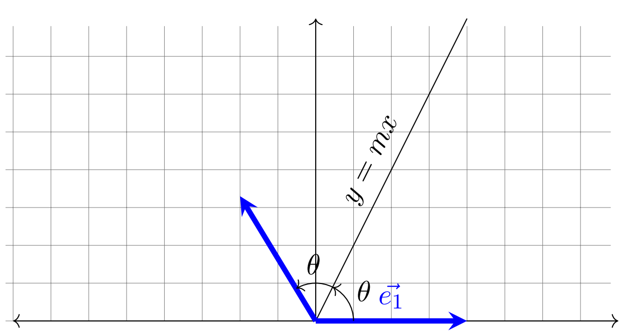
- We will determine the matrix which rotates a vector by θ. We know that when $\vec{e_1}$ is rotated by θ that this situation is the same as the unit circle, in which we determined the x, y coordinates to be $\cos\theta, \sin\theta$ respectively as the hypotenuse is 1, thus we know that the first row of the matrix we are looking for is $\begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix}$. We'll determine the second row now by seeing what happens to $\vec{e_2}$ which gives the angle $\frac{\pi}{2}+\theta$ we know that sine of this angle is still the same thing but cosine of this angle is $-\cos\theta$ and so the generic rotation matrix of θ radians is given by

$$\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}$$

- We'll now observer what happens when $\vec{e_1}$ and $\vec{e_2}$ are reflected across the line $y = mx$ for some $m \in \mathbb{R}$



Thus we know the hypotenuse is $\sqrt{m^2+1^2}$ and so $\sin\theta = \frac{m}{\sqrt{m^2+1}}$ and $\cos\theta = \frac{1}{\sqrt{m^2+1}}$ Observe, that when we reflect over the line



As we can see this is a counter clockwise rotation of 2θ and thus we know that the vfirst column of the matrix must be

$$\begin{bmatrix} \cos 2\theta \\ \sin 2\theta \end{bmatrix}$$

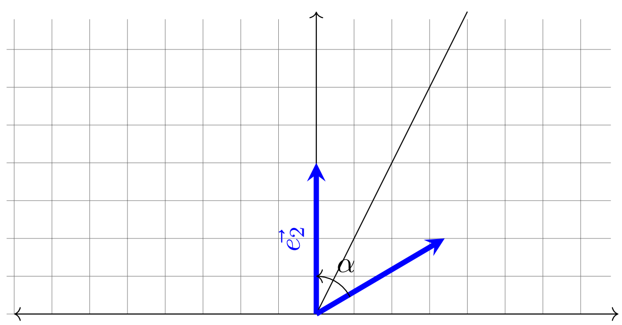
we know that $\sin(2\theta) = \sin(\theta + \theta) = 2\sin(\theta)\cos(\theta)$ and that $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$ or equivalently

$$\begin{bmatrix} \cos^2(\theta) - \sin^2(\theta) \\ 2\sin(\theta)\cos(\theta) \end{bmatrix}$$

But we know from above that $\sin(\theta) = \frac{m}{\sqrt{m^2+1}}$ and $\cos(\theta) = \frac{1}{\sqrt{m^2+1}}$ so we can say that $\cos^2(\theta) - \sin^2(\theta) = \frac{1-m^2}{m^2+1}$ and that $2\sin(\theta)\cos(\theta) = \frac{2m}{m^2+1}$ and so so the first column is

$$\begin{bmatrix} \frac{1-m^2}{m^2+1} \\ \frac{2m}{m^2+1} \end{bmatrix}$$

Now we'll see what happens to $\vec{e_2}$



But this time notice that $\alpha = 2 \cdot \left(\frac{\pi}{2} - \theta\right) = \pi - 2\theta$ and thus

$$\begin{aligned} \sin(\pi - 2\theta) &= \sin(\pi)\cos(2\theta) - \cos(\pi)\sin(2\theta) \\ &= 2\sin(\theta)\cos(\theta) \end{aligned}$$

as for cosine

$$\begin{aligned} \cos(\pi - 2\theta) &= \cos(\pi)\cos(2\theta) + \sin(\pi)\sin(2\theta) \\ &= \sin^2(\theta) - \cos^2(\theta) \end{aligned}$$

And thus our total matrix is
$$\begin{bmatrix} \frac{1-m^2}{m^2+1} & \frac{-2m}{m^2+1} \\ \frac{2m}{m^2+1} & \frac{m^2-1}{m^2+1} \end{bmatrix}$$

Chapter 4

Week 12

4.1 Lecture 1 Week 12

Remember from last week, if we have that $X = [T]_{\mathcal{C}}$ that that means for all $\vec{v}$

$$\big[T(\vec{v})\big]_{\mathcal{C}} = X[\vec{v}]_{\mathcal{C}}$$

Specificially if $\vec{v} = \alpha\vec{c_1} + \beta\vec{c_2}$ then we know

$$\big[T(\alpha\vec{c_1} + \beta\vec{c_2})\big]_{\mathcal{C}} = X\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$$

And so we recall that

$$[\vec{v}]_{\mathcal{C}} \xrightarrow{X} \big[T(\vec{v})\big]_{\mathcal{C}}$$

also if $Y = [T]_{\mathcal{E}}$ then we know

$$[\vec{v}]_{\mathcal{E}} \xrightarrow{Y} \big[T(\vec{v})\big]_{\mathcal{E}}$$

Thus if we only know what X is and we want to find out what what $\big[T(\vec{v})\big]_{\mathcal{E}}$ then our process to do so would look like

$$[\vec{v}]_{\mathcal{E}} \xrightarrow{A^{-1}} [\vec{v}]_{\mathcal{C}} \xrightarrow{X} \big[T(\vec{v})\big]_{\mathcal{C}} \xrightarrow{A} \big[T(\vec{v})\big]_{\mathcal{E}}$$

Thus we've determined the following

$$AXA^{-1}[\vec{v}]_{\mathcal{E}} = \big[T(\vec{v})\big]_{\mathcal{E}}$$

Thus we conclude $AXA^{-1} = [T]_{\mathcal{E}}$

Definition 11 (Similar Matricies)**.** We say that two matricies A and B are **similar** denoted $A \sim B$. If A and B represent the same linear transformation but in possibly different bases. Equivalently , $A \sim B$, if there exists an invertible matrix X so that

$$A = XBX^{-1}$$

Definition 12 (Unit n-cube)**.** The **unit n-cube** is the n-dimensional cube with sides given by the standard basis vectors and lower-left corner located at the origin. That is

$$C_n = \left\{ \vec{x} \in \mathbb{R}^n : \vec{x} = \sum_{i=1}^{n} \alpha_i \vec{e_i} \text{ for some } \alpha_1, \ldots, \alpha_n \in [0,1] \right\} = [0,1]^n$$

4.1.1 64

- $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$ we can see that $T(\vec{e_2}) = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$ thus a matrix for T is

$$T = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$$

- To find the volume of the image of the unit-square under T we will subtract area from a square with length $\frac{3}{2}$ after some observation we recognize that the area we are subtracting is 6 squares of side length $\frac{1}{2}$ which is a total volume of $\frac{6}{4}$ then our answer is

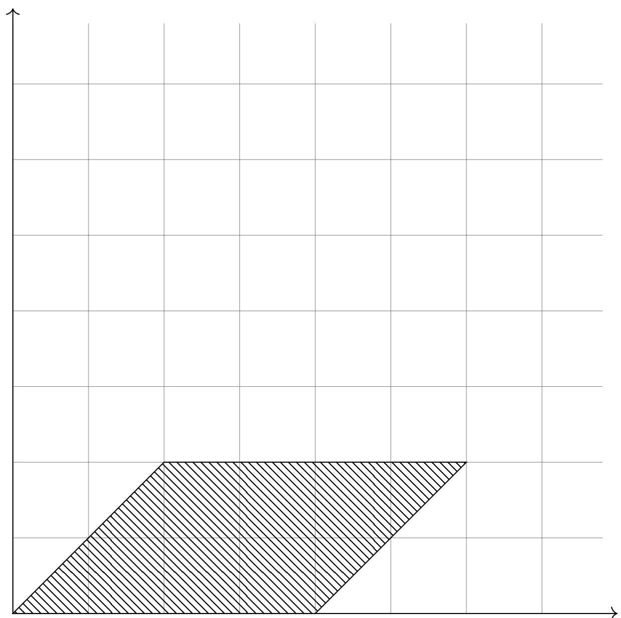
$$\left(\frac{3}{2}\right)^2 - \frac{6}{4} = \frac{3}{4}$$

Definition 13 (Determinant)**.** The **determinant** of a linear transformation $X : \mathbb{R}^n \to \mathbb{R}^n$ is the oriented volume of the image of the unit n-cube . The determinant of a square matrix is the determinant of it's induced transformation

Example 4.1.1. We know that the determinant of T should be positive since we can move $T(\vec{e_1})$ and $T(\vec{e_2})$ are positively oriented $\left\{ \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \right\}$ so indeed the determinant is $\frac{3}{4}$

4.1.2 65

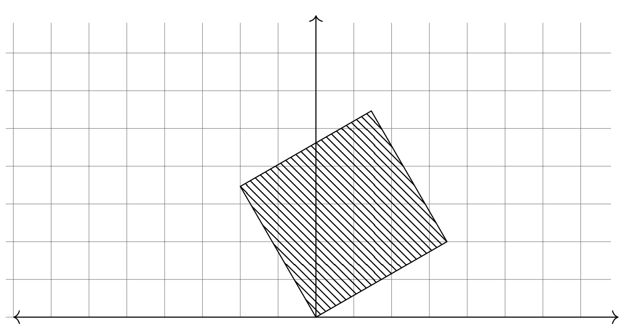
We already know what C_2 looks like lets see what $A(C_2)$ looks like.



We can see that the determinant should be positive and that the area is 1 so $det(A) = 1$

4.1.3 66

We will get



We observe that this doesn't modify the cube in any way so $det(R) = 1$

4.1.4 67

We can see that this just switches $\vec{e_2}$ and $\vec{e_1}$ so $det(F) = -1$

Theorem 7 (Volume Theorem I)**.** For a square matrix M, $det(M)$ is the oriented volume of the parallelpiped (n-dimensional parallelogram) given by the column vectors of M

Theorem 8 (Volume Theorem II)**.** For a square matrix M, $det(M)$ is the oriented volume of the parallelpiped (n-dimensional parallelogram) given by the row vectors of M.

Why???

Example 4.1.2. Thus we know that $det(A) = det\big(A^{-1}\big)$ for example

$$A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \qquad\qquad A^{-1} = \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}$$

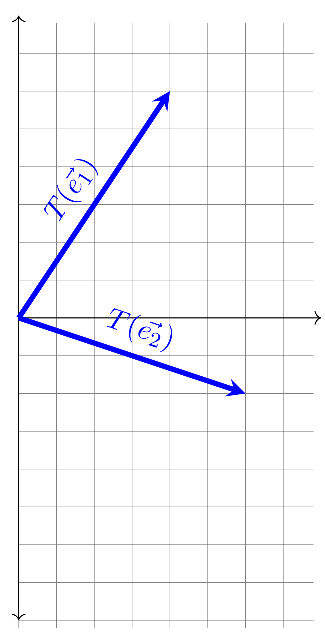
4.1.5 68

Well the columns of the matrix M are $T(\vec{e_i})$ and so if we look at linear combinations of those vectors where the coefficients are elements of the set $[0, 1]$

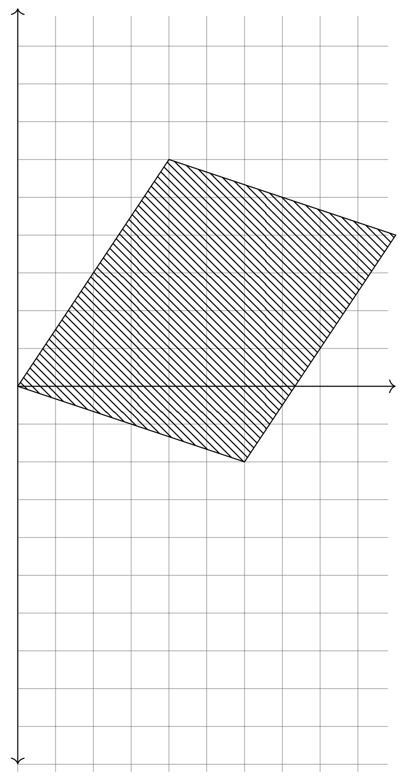
Remark 4.1.1. Let $\vec{v} \in C_n$ then there exists $\alpha_1, \alpha_2, \ldots, \alpha_{k-1}\alpha_k \in [0, 1]$ such that $\vec{v} = \alpha_1\vec{e}_1 + \alpha_2\vec{e}_2 + \cdots + \alpha_{k-1}\vec{e}_{k-1} + \alpha_k\vec{e}_k$ so then

$$\begin{aligned} T(\vec{v}) &= T(\alpha_1\vec{e}_1 + \alpha_2\vec{e}_2 + \cdots + \alpha_{k-1}\vec{e}_{k-1} + \alpha_k\vec{e}_k) \\ &= \alpha_1 T(\vec{e_1}) + \alpha_2 T(\vec{e_2}) + \cdots \alpha_{k-1} T(\vec{e_{k-1}}) + \alpha_k T(\vec{e_k}) \end{aligned}$$

And if $k = 2$ then visually we would have



Then adding the coefficients we get the parallelogram defined by $T(\vec{e_1}), T(\vec{e_2})$ which would look like



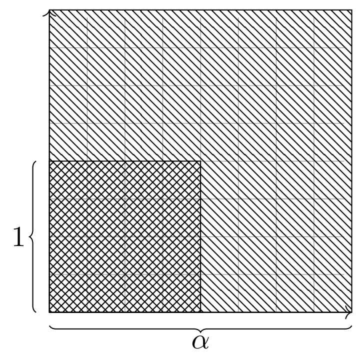
Remark 4.1.2. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ we'll denote the volume of a translated unit-cube like this

$$V\left(T\left(C_n + \vec{b}\right)\right) \text{ for some } \vec{b} \in \mathbb{R}^n$$

Due to linearity we know

$$V\left(T(C_n) + T\left(\vec{b}\right)\right) = V\big(T(C_n)\big) = det(T)$$

Remark 4.1.3. Let's consider $V\big(T(\alpha C_n)\big)$ we know this either shrinks or stretches C_n



From our diagram we observe that the increase or decreasing in volume must be α^2 and for the n th dimension a^nthus it follows that

$$\begin{aligned}V\big(\alpha T(c_n)\big) &= a^n V\big(T(C_n)\big)\\ &= a^n\, det(T)\\ &= V(\alpha C_n)\, det(T)\end{aligned}$$

Thus we conclude

$$V\left(T\left(\alpha C_n + \vec{b}\right)\right) = det(T)V\left(\alpha C_n + \vec{b}\right)$$

We'll now show that it holds for any arbitrary shape X

Proof. Let X represent some arbitrary shape (set of vectors), we know that we can represent any shape as the sum of many or infinitely many squares of varying sizes, which some of which could also be infinitely small. So X consists of elements that look like $\alpha C_n + \vec{b}$ for some $\alpha \in \mathbb{R}, \vec{b} \in \mathbb{R}^n$.
Let $T : R^n \to \mathbb{R}^n$ be a linear transformation and then the image of X under T are the vectors $\vec{v} \in \mathbb{R}^n$ that satisfy the following

$$\begin{aligned}\vec{v} &= T(\vec{y}) \text{ for some } \vec{y} \in R^n\\ &= T\left(\alpha C_n + \vec{b}\right)\\ &= \alpha T(C_n) + T\left(\vec{b}\right)\end{aligned}$$

We must now find the volume of the image the which is the sum of each square we get so we have

complete this

$$\sum_{i=0}^{\infty} \alpha_i T(C_n) + T\left(\vec{b_i}\right)$$

□

4.2 Week 12 lecture 2

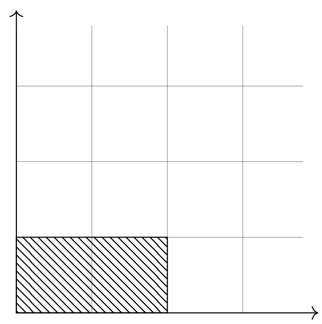
Recall previously that if we have a transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ and that

$$V\left(T\left(\alpha C_n + \left\{\vec{b}\right\}\right)\right) = V\left(\alpha C_n + \left\{\vec{b}\right\}\right)\left|det(T)\right|$$

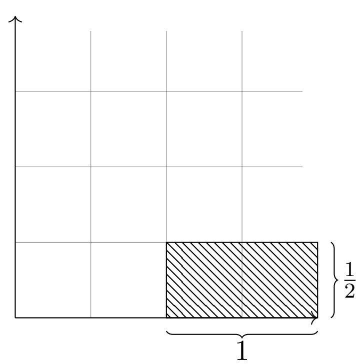
And so we know for any shape A

$$V\big(T(A)\big) = \big|det(T)\big|V(A)$$

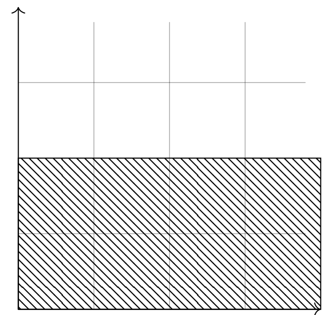
- <u>69</u>
  - $R_1 = R_2 = R_3 = R_4 = \frac{1}{4}$ and $R = 1$
  - $M(R_1)$ has area $\frac{1}{2}$, observe.



$M(R_2)$ has the same area of $M(R_1)$ we see,



And then $M(R)$ has area 2



- Recall $V\big(T(R_2)\big) = \big|det(T)\big| V(R_2)$ and thus $2 \cdot \frac{1}{4} = \frac{1}{2}$, we know that we are just translating $T(R)$ by the vector $T(\vec{e_2})$ and so the oriented volume is still the same as the non-translated, 2.
- We start by proving the following claim. For any two linear transformations S, T
$$V\big(S \circ T(R)\big) = det(S) \cdot det(T) \cdot V(R)$$

Proof.

$$\begin{aligned} V\big(S \circ T(R)\big) &= det(S) \cdot V\big(T(R)\big) \\ &= det(S) \cdot det(T) \cdot V(R) \end{aligned}$$

□

Corollary 9. for any two square matricies A, B

$$det(AB) = det(A)det(B)$$

As A, B must induce a transformation

- <u>70</u>
  - $E_f = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ which is the transformation f such that

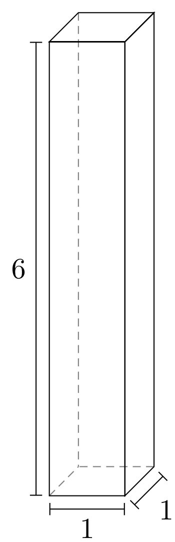
$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} y \\ x \\ z \end{bmatrix}$$

As we see this $f(\vec{e_1}) = \vec{e_2}$ and $f(\vec{e_2}) = \vec{e_1}$ and so it's negatively oriented as we can get to the standard basis by crossing once. Thus $det(f) = -1 = det\big(E_f\big)$

- $E_m = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{bmatrix}$

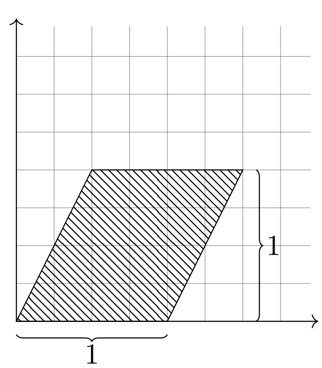
$$M\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 6z \end{bmatrix}$$

  and so the unit cube becomes



  And thus $det(M) = det(E_m) = 6$

- $E_A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and so $E_A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + 2y \\ y \\ z \end{bmatrix}$ so $E_A\vec{e_1} = \vec{e_1}, E_A\vec{e_2} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, E_A = \vec{e_3}$ and so let we let the z axis point out of the page, thus the base of the base looks like



And so the area of the base is 1, so $det(E_A) = 1$

Notice that if we have the matrix $\begin{bmatrix}1&\alpha&0\\0&1&0\\0&0&1\end{bmatrix}$ and so the middle column represents

$$\alpha\vec{e_1} + \vec{e_2}$$

and that $\alpha\vec{e_1}$ gives the shift to the parallelogram which we know doesn't actually change the area of the parallelogram

- $E_f E_m = \begin{bmatrix}0&1&0\\1&0&0\\0&0&6\end{bmatrix}$ which takes a vector $\begin{bmatrix}x\\y\\z\end{bmatrix}$ to $\begin{bmatrix}y\\x\\6z\end{bmatrix}$ which is the same as E_M though $\vec{e_1}$ and $\vec{e_2}$ are now swapped thus it is negatively oriented so $det(f \circ m) = -6$
- $det(4I_{3\times 3}) = 4^3 = 64$ since $\vec{e_1} \to 4\vec{e_1}, \dots$
- We know that W must look like

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{array}{l} \\ \\ \mid\cdot 6 \mid \cdot 6\end{array} \rightsquigarrow \begin{bmatrix}0&1&0\\1&0&0\\0&0&36\end{bmatrix}\begin{array}{l}+\\ \quad 2\\ \end{array} \rightsquigarrow \begin{bmatrix}2&1&0\\1&0&0\\0&0&36\end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix}1&0&0\\2&1&0\\0&0&36\end{bmatrix}$$

So we know the base has area 1 with height 36 so $det(W) = 36$

4.3 Tutorial 75

- Q1
  $rank(A)$ is the number of pivots in reduced row echelon form of A, the rank of T is $dim\big(range(T)\big)$
- Q2

  a)

$$\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}$$

b)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

c) We know that the number of pivots in a matrix cannot exceed the number of rows of a matrix so this is impossible

d)

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We know that $\left\{\vec{0}\right\}$ is linearly dependent since for any $\gamma \in \mathbb{R}, \vec{0} = \gamma\vec{0}$ thus a non-trivial linear combination that yields the zero vector.

- We'll simultaneously show a and b.
  If $\mathcal{L}$ is not one-to-one then we have for some $\vec{x}, \vec{y} \in R^n$ such that $\vec{x} \neq \vec{y}$ that

  $$\mathcal{L}(\vec{x}) = \mathcal{L}(\vec{y})$$
  $$\mathcal{L}(\vec{x} - \vec{y}) = 0$$

  But we recall that $\vec{x} \neq \vec{y}$ and so the null space is non-trivial and since the null space is a subspace we conclude that $dim\Big(null\big(\mathcal{L}(x)\big)\Big) \geq 1$ thus from the Rank Nullity Theorem we know that $rank(\mathcal{L}) + nullity(\mathcal{L}) = n$ and so $rank(\mathcal{L}) + 1 \geq n$ thus we conclude that

  $$\boxed{rank(\mathcal{L}) < n}$$

  By the contrapositive of the statement we have just proven that $rank(\mathcal{L}) \geq n \implies \mathcal{L}$ is one-to-one but we know that $rank\big(row(A)\big) = rank\big(col(A)\big)$ so then $rank(\mathcal{L}) \leq \min(m, n)$ so the our implication changes to

  $$rank(\mathcal{L}) = n \implies \mathcal{L} \text{ is one-to-one}$$

  This is actually a bi-implication so we have to prove the other direction. But our conclusion is that $\mathcal{L}$ is one-to-one if it is full rank

- We'll show c and d now
  Let $T : X \to Y$ we know that a function is onto if there exists an element in X for every element in Y so that it maps to it equivalently the range of T is Y or or we can say $T(X) = Y$ so we know

  $$image(\mathcal{L}) = R^m$$
  $$rank(\mathcal{L}) = m$$

So we've shown that if $\mathcal{L}$ is onto then $rank(\mathcal{L}) = m$
We'll show d now, remember for any $a, b \in \mathbb{R}$ one of the following is true

$$a = b \vee a > b \vee a < b$$

so if $rank(\mathcal{L}) \neq m \implies \big(rank(\mathcal{L}) < m\big) \vee \big(rank(\mathcal{L}) > m\big)$ though we know that $rank(\mathcal{L}) > m$ cannot be true so we know

$$\text{onto } \Leftrightarrow rank(\mathcal{L}) = m$$

and so then

$$\text{not-onto } \Leftrightarrow rank(\mathcal{L}) < m$$

And finally we conclude if a function is both onto and one-to-one (bijective) if $rank(\mathcal{L}) = m \wedge rank(\mathcal{L}) = n$ equivalently that is $m = n$

- Q4

  a) Let $\mathcal{T}$ be $proj_{\mathcal{P}}\vec{x}$ for all $\vec{x} \in \mathbb{R}^3$ for some plane $\mathcal{P}$ and $\mathcal{S}$ be $proj_{\mathcal{Q}}\vec{x}$ for some plane $\mathcal{Q}$ for all $\vec{x} \in \mathbb{R}^3$ for example $S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, which removes the z component of every vector and drops it onto the $x - y$ plane. So $S = T =$ the projection onto the $x - y$ plane.

  b) If we first project onto the $x - y$ plane, then the $x - z$ plane, then we get the projection onto the $x-$ axis, so we have

  $$\mathcal{T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \mathcal{S} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

  And then applying both gives

  $$\mathcal{T} \circ \mathcal{S} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

  c) impossible, we know that $rank(S) = 2$ so by definition we know that

  $$dim\Bigg(\Big\{\mathcal{S}\big(\mathcal{T}(\vec{x})\big) \text{ for some } \vec{x} \in R^3\Big\}\Bigg)$$

  Recall that $range(\mathcal{T}) \subseteq \mathbb{R}^3$ and so equivalently we have

  $$dim\Bigg(\Big\{\mathcal{S}(\vec{v}) \text{ for some } \vec{v} \in \mathbb{R}^3\Big\}\Bigg)$$

  which is equal to 2 which is not 3

d) Let $\mathcal{T}(\vec{x})$ be the projection of $\vec{x}$ onto the $x-y$ axis, let $\mathcal{S}(\vec{x})$ be the projection onto the z axis, then a projection onto the $x-y$ then onto the z axis only yields the 0 vector and so $rank(S \circ T) = 0$ as required.

- Q5
  - We say two matricies A, B are similar if there exists a matrix X such that $B = XAX^{-1}$
  - We know that

$$B = XAX^{-1}$$

We assume that X^{-1} exists, so then it's columns and rows are linearly independent which means that we won't have a row of zero's also we know that A doubles the second row and triples the third when it multiplies against a matrix and so we know the bottom right value of XAX^{-1} is not zero, so such a matrix doesn't exist.

4.4 Extra Homework Problems Determinants

- A1

a) $det(A) = \begin{vmatrix} 1 & 2 & 4 \\ 3 & 1 & 0 \\ -1 & 3 & 2 \end{vmatrix} = 1(2) - 2(6) + 4(9) =$

Chapter 5

Week 13

5.1 Week 13 Lecture 1

Recall: If $\mathcal{E}$ is an elementary matrix obtained from $\mathcal{I}_{n\times n}$ by the row opeation

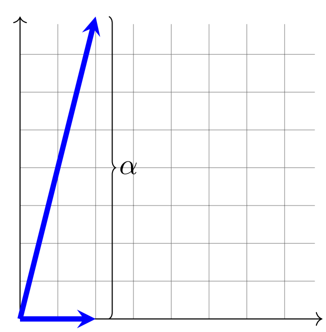
$$R_i \to R_i + \alpha R_j \text{ and } i \neq j$$

Thus we know that $det(\mathcal{E}) = 1$

Proof. We know $\mathcal{E}$ looks like

$$\begin{bmatrix} 1 & \dots & \alpha \\ \vdots & 1 & \vdots \\ 0 & \dots & 1 \end{bmatrix}$$

We know this represents



We know that this just gives the parallelogram a slant which doesn't change the area. $\square$

Example 5.1.1. Then we know

$$det\left(\begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}\right) = 1$$

Example 5.1.2.

$$det\left(\begin{bmatrix} 1 & 0 & 0 \\ \beta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 1$$

5.1.1 Upper and Lower triangular matricies

Let's say we have

$$X = \begin{bmatrix} 2 & 1 & 3 & 6 \\ 0 & 3 & 2 & 5 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad J = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

observe that we can obtain X from elementary row operations applied to J. We know that each row operation relates to an elementary matrix and all of them applied at once is the multiplication of each of the elementary matricies say $\mathcal{E}_x$ so we know that

$$X = \mathcal{E}_x J$$

And so

$$\begin{aligned} det(X) &= det(\mathcal{E}_x J) \\ &= det(\mathcal{E}_x) \cdot det(J) \end{aligned}$$

Recall that we have shown that the the determinant of an elementary matrix is 1 and that E_x is the product of only elementary matricies and so $det(\mathcal{E}_x) = 1$ thus we know

$$det(X) = 1 \cdot det(J)$$

And recall that J is triangular so it's determinant is just the product of the diagonal and so $det(J) = 1 \cdot 2 \cdot 3 \cdot 1 \cdot -1 = -6$

Note that if a matrix is upper or lower triangular matrix is the product of the diagonal entries.

Example 5.1.3.

$$det\left(\begin{bmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 5 & 6 & 3 \end{bmatrix}\right) = 6$$

5.1.2 Value of the determinant

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ if

$$det(T) \neq 0$$

then we know that the parallelogram generated has area. We know this because if a parallelogram has no area that means $T(\vec{e_1}), T(\vec{e_2})$ are linearly dependent and so they define a line which clearly has no area. By the contrapositive we know that it has area thehn $T(\vec{e_1}), T(\vec{e_2})$ are linearly independent so we have

$$det(T) = 0 \implies T(\vec{e_1}) = \alpha T(\vec{e_2}) \text{ for some } \alpha \in \mathbb{R}$$

Equivalently we know if A invokes $\mathcal{T}$ such that $A = [T]_{\mathcal{E}}$ we have

$$det(A) \neq 0 \implies \text{ columns of } A \text{ are linearly independent}$$

then also

$$det(A) = 0 \implies \text{ the cols of } A \text{ are linearly dependent, we lost at least one dimension}$$

in particular

- If $rref(A)$ has a row of zeros then we know that at least one column is dependent thus we know $\boxed{det(A) = 0}$
- $F_A : \mathbb{R}^2 \to \mathbb{R}^2$ is the projection onto $span\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$ then we have

$$\boxed{det(A) = 0}$$

  since all vectors in the range of A are

$$\alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ for all } \alpha \in \mathbb{R}$$

- If A is invertible then it's columns are linearly independent thus

$$\boxed{det(A) \neq 0}$$

- If A is not invertible then $rref(A)$ has a row of zeros so then

$$\boxed{det(A) = 0}$$

- If A has a trivial null space then we know the only solution to the zero vector is a trivial one, so then by definition the columns are linearly independent so

$$\boxed{det(A) \neq 0}$$

Thus from the lemmas above we have

Theorem 10 (determinant properties).

$$\begin{aligned} det(A) = 0 &\Leftrightarrow \text{ columns of } A \text{ are linearly dependent} && \text{(Important!)} \\ &\Leftrightarrow rref(A) \text{ has a row of zeros} \\ &\Leftrightarrow A \text{ is not invertible} \\ &\Leftrightarrow A \text{ has a non trivial null space} \end{aligned}$$

5.1.3 inverse determinant

Let A be a matrix that is invertible that is A^{-1} exists we know

$$\begin{aligned} AA^{-1} &= I \\ det\left(AA^{-1}\right) &= det(I) \\ det(A)det\left(A^{-1}\right) &= 1 \end{aligned}$$

We know that division is allowed since $det(A)$ and $det\left(A^{-1}\right)$ are non-zero Thus we know

$$\boxed{det(A) = \frac{1}{det\left(A^{-1}\right)}} \qquad \boxed{det\left(A^{-1}\right) = \frac{1}{det(A)}}$$

5.2 Eigen

Let $\mathcal{T} : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that stretches in the $y = -3x$ direction by factor of 2 as a vector equation

$$\ell = t \begin{bmatrix} 1 \\ -3 \end{bmatrix} \text{ for all } t \in \mathbb{R}$$

We we consider the vector $\vec{b_1} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ we know after the stretch that since it is parallel to the stretch

$$T\left(\vec{b_1}\right) = 2\vec{b_1}$$

If we consider a normal vector to the line $y = -3$ we can see that this will not get stretched when T is applied. Let $\vec{b_2} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ so we have

$$T\left(\vec{b_2}\right) = \vec{b_2}$$

Now let $\mathcal{B} = \left\{\vec{b_1}, \vec{b_2}\right\}$ thus we we know $[T]_{\mathcal{B}}$ that is the matrix which applies T to boxes of numbers in the $\mathcal{B}$ basis moreover we know

$$\left[T\left(\vec{b_1}\right)\right]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \qquad \left[T\left(\vec{b_2}\right)\right]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

In this case $\mathcal{B}$ is called an **Eigenbasis** and $\vec{b_1}, \vec{b_2}$ are **eigen vectors** for T since we know

$$T\left(\vec{b_1}\right) = 2\vec{b_1} \qquad T\left(\vec{b_2}\right) = \vec{b_2}$$

Then 2 is the eigen value corresponding to $\vec{b_1}$ and 1 is the eigen value for $\vec{b_2}$

Definition 14 (Eigen Vector)**.** Let X be linear transformation or a matrix, we say $\vec{v}$ is an eigen vector for X if $\vec{v} \neq \vec{0}$ and

$$\boxed{X\vec{v} = \lambda\vec{v}} \text{ for some } \lambda \in \mathbb{R}$$

And we say λ is the eigen value corresponding to the eigen vector $\vec{v}$

Example 5.2.1.

$$A = \frac{1}{10}\begin{bmatrix} 11 & -3 \\ -3 & 9 \end{bmatrix}$$

A is the matrix corresponding to the transformation that stretches in the $y = -3x$ direction by a factor of 2. We have

$$\vec{b_1} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \qquad \vec{v_2} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Thus we know

$$A\vec{b_1} = 2\vec{b_1} \qquad A\vec{b_2} = \vec{b_2}$$

it is not self-evident that this it true, we know this because it induces

Remark 5.2.1. given the eigen vectors it's trivial to find the eigen values by multiplying by A.

Remark 5.2.2. We observe that for any linear transformation T there is often such a basis and

$$[T]_{\mathcal{B}}$$

and we have

$$\begin{bmatrix} \vdots & & \vdots \\ T(\vec{e_1}) & \dots & T(\vec{e_n}) \\ \vdots & & \vdots \end{bmatrix} = \begin{bmatrix} \vdots & & \vdots \\ \alpha\vec{e_1} & \dots & \beta\vec{e_n} \\ \vdots & & \vdots \end{bmatrix}$$

Example 5.2.2. for example, let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be the counter clockwise rotation by $\frac{\pi}{2}$. We know that every non-zero vector gets rotated we know they are not linear combinations of eachother because they are orthogonal, so we know that eigen values, and eigen vectors must not exists.

5.2.1 From eigen vectors to eigen values

This is useful if you know what a transformation does

1. find the eigen values (the vectors that don't change direction under the transformation)
2. determine their eigen values visually or by the definition of the transformation, this increases in difficulty in higher dimensions.

5.2.2 From eigen values to eigen vectors

1. first find the eigen values
2. find their corresponding eigen vectors

Example 5.2.3.

$$A = \frac{1}{10}\begin{bmatrix} 11 & -3 \\ -3 & 19 \end{bmatrix}$$

as before.
We first find possible eigen values that is $\lambda \in \mathbb{R}$ such that

$$\begin{aligned} A\vec{v} &= \lambda\vec{v} \\ &= (\lambda I)\vec{v} \\ \underbrace{(A - \lambda I)}_{\mathcal{E}_x}\vec{v} &= 0 \end{aligned}$$

But by definition we know that $\vec{v} \neq \vec{0}$ as we are looking for eigen vectors but then we know the only way we can get solutions are if if the rows are linearly dependent equivalently that is $\mathcal{E}_x$ has a non-trivial null space, to determine this we know this relates to the determinant as if the determinant of $\mathcal{E}_x = 0$ then we know that the columns are linearly independent and the solution depends on λ .
We have

$$\begin{aligned} \mathcal{E}_x &= A - \lambda I \\ &= \begin{bmatrix} \frac{11}{1}0 - \lambda & -\frac{3}{10} \\ -\frac{3}{10} & \frac{19}{10} - \lambda \end{bmatrix} \end{aligned}$$

thus the determinant of $\mathcal{E}_x$ is

$$\left(\frac{11}{10} - \lambda\right)\left(\frac{19}{10} - \lambda\right) - \left(\frac{9}{10}\right)$$

eventually we know that this boils down to (we previously determined the eigen values to be $1, -2$)

$$(1 - \lambda)(2 + \lambda)$$

Now that we know what the eigen values are we'll determine the eigen vectors for each λ. For when $\lambda = 1$ we are looking for vectors that satisfy

$$\begin{aligned} A\vec{v} &= \vec{v} \\ &= I\vec{v} \end{aligned}$$

Then we're searching for solutions to

$$\underbrace{(A - I)}_{\mathcal{E}_1} \vec{v} = 0$$

that is linear combinations of the columns of

$$\mathcal{E}_1 = \frac{1}{10}\begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix}$$

But notice that -3 of the first column is the second column or that the same equailty holds between row one and row two so if we row reduce we get

$$rref(\mathcal{E}_1) = \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$$

thus we have a free variable for x_2 so we let $t \in \mathbb{R}$ and $x_2 = t$ then $x_1 = 3t$ so we can say that the solutions to the original system are

$$\begin{bmatrix} 3t \\ t \end{bmatrix} = t\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

And so the null space of $\mathcal{E}_1$ is $span\left\{\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right\}$ thus we know that all these vectors stay unchanged under the transformation so we ca take our eigen vector as $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Remark 5.2.3. If $null(\mathcal{E}_\lambda)$ is trivial that is $null(\mathcal{E}_\lambda) = \left\{\vec{0}\right\}$ then we know the λ we determined must be incorrect as if the columns of $\mathcal{E}_\lambda$ are linearly dependent as we solved for when the determinant of $\mathcal{E}_\lambda = 0$ we know that the nullity of $\mathcal{E}_\lambda$ must be greater than 1 which is not 0

We will now solve for our second eigen value $\lambda = 2$ that yields

$$\mathcal{E}_2 = \frac{1}{10}\begin{bmatrix} -9 & -3 \\ -3 & -1 \end{bmatrix}$$

We note the first column equals 3 of the second and thus we have

$$null(\mathcal{E}_2) = span\left\{\begin{bmatrix} 1 \\ -3 \end{bmatrix}\right\}$$

And these are the vectors that double in length under the transformation one possible eigen vector could be $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$.

5.2.3 Diagonalization

Let $\mathcal{B}$ be a eigen basis recall the two ways to find out how a vector looks in a certain basis after a transformation

$$[\vec{v}]_\mathcal{E} \xrightarrow{A} \left[T_A(\vec{v})\right]_\mathcal{E}$$

Let P be the matrix that converts from $\mathcal{B} \to \mathcal{E}$ and its inverse P^{-1} that converts from $\mathcal{E} \to \mathcal{B}$ we recall the alternate method of doing so, which is

$$[\vec{v}]_\mathcal{E} \xrightarrow{P^{-1}} [\vec{v}]_\mathcal{B} \xrightarrow{[T_A]_\mathcal{B}} \left[T_A(\vec{v})\right]_\mathcal{B} \xrightarrow{P} \left[T_A(\vec{v})\right]_\mathcal{E}$$

We know that A is that matrix which induces T in the $\mathcal{E}$ basis and also that. We also recall that $P = \begin{bmatrix} \vdots & & \vdots \\ \left[\vec{b_1}\right]_{\mathcal{E}} & \dots & \left[\vec{b_2}\right]_{\mathcal{E}} \\ \vdots & & \vdots \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}$

$$A = P[T_A]_{\mathcal{B}}P^{-1}$$

But recall that $[T_A]_{\mathcal{B}}$ has columns that describes what happens to $\vec{b_1}$ and $\vec{b_2}$ in the $\mathcal{B}$ basis and thus is diagonal and we have that

$$P^{-1}AP = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \Leftrightarrow P^{-1}AP = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

We note that A is similar to a diagonal matrix and thus we can say A is diagonalizable.

5.3 Week 13 Lecture 2

Definition 15 (Characteristic Polynomial)**.** for a matrix A the **characteristic polynomial** of A is

$$char(A) = det(A - \lambda I)$$

in the variable λ

Example 5.3.1.

$$C = \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix}$$

We will find an invertible matrix P and a diagonalizable matrix D such that

$$P^{-1}CP = D$$

Recall one way we are are guarenteed to find this is the method discussed in the last lecture, if we are to find it this way then we know that D will diagonal as it's columns are the vectors in the eigen basis that are getting scaled, also recall that the entries will be eigen values and columns with be eigen vectors.
We start by finding the eigen values for C we can do this because we know there is a transformation T_C for it. We start by looking for the possible values of λ that satisfy the following equation where $\vec{x} \neq \vec{0}$

$$C\vec{x} = \lambda\vec{x}$$

Recall we can rewrite a scalar multiplied by a matrix as λI so now we solve

$$(C - \lambda I)\vec{x} = 0$$

we know this statment is true if and only if the matrix $C - \lambda I$ has linearly dependent columns which is case if and only if the determinant of $C - \lambda I$ is zero that is if and only if if characteristic polynomial for C has real roots. So we proceed by finding if it does.

$$
\begin{aligned}
det\left(\begin{bmatrix} -1-\lambda & 2 \\ 1 & -\lambda \end{bmatrix}\right) &= (\lambda - 1)(\lambda + 2) \\
&\implies \lambda = 1 \text{ or } \lambda = -2
\end{aligned}
$$

Now we will determine the eigen vectors

- For $\lambda = 1$ we have

  $$\mathcal{E}_1 = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}$$

  Now we will find the vectors that satisfy

  $$\mathcal{E}_1 \vec{x} = 0$$

  that is the null space or the linear combinations of the columns of $\mathcal{E}_1$ that yield the zero vector we have

  $$
  \begin{aligned}
  \alpha \begin{bmatrix} -2 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ -1 \end{bmatrix} &= \alpha \begin{bmatrix} -2 \\ 1 \end{bmatrix} - \beta \begin{bmatrix} -2 \\ 1 \end{bmatrix} \\
  &= (\alpha - \beta) \begin{bmatrix} -2 \\ 1 \end{bmatrix}
  \end{aligned}
  $$

  Which is only true if $\alpha = \beta$ and so the vectors that satisfy $\mathcal{E}_1 \vec{x} = 0$ are $span\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$

- For when $\lambda = -2$ we have

  $$\mathcal{E}_{-2} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$$

  then we have that all the eigen vectors $null(\mathcal{E}_{-2}) = span\left\{\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right\}$. recall these are the vectors that double scale by -2 under the transformation so an eigen vector is $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

  We know that an eigen basis for C is

  $$\mathcal{B} = \left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix}\right\}$$

And then we have

$$[T_C]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$$

Let P be the matrix who's columns are $\left[\vec{b_1}\right]_{\mathcal{E}}, \left[\vec{b_2}\right]_{\mathcal{E}}$ then we know this matrix converts from $\mathcal{B} \to \mathcal{E}$ and we have

$$D = [T_C]_{\mathcal{E}} = PDP^{-1} \Leftrightarrow P^{-1}CP = D$$

Definition 16 (Diagonizable)**.** A matrix is diagonalizable if it is similar to a diagonal matrix

Example 5.3.2. Let R be the rotation counter clockwise by $\frac{\pi}{2}$ in $\mathbb{R}^2$ we know that it has no eigen vectors thus the matrix corresponding to R is not diagonalizable for example if

$$[R]_{\mathcal{E}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

then

$$\begin{aligned} char(R) &= det\left(\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}\right) \\ &= \lambda^2 + 1 \end{aligned}$$

which has no real roots, thus R has no eigen values so it's not diagonalizable

Theorem 11 (n distinct eigen values)**.** If an $n \times n$ matrix has n unique

$$\lambda_1, \lambda_2, \ldots \lambda_{n-1}, \lambda_n$$

and

$$\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{n-1}, \vec{v}_n\}$$

where $\vec{v}_i$ is an eigen vectors with eigen values λ_i then it is a basis (linearly independent set) for $\mathbb{R}^n$ moreover there will be only one choice for each $\vec{v}_i$ up to a scalar multiple. prove this

Proof.
Assume that A has n distinct eigen values

$$\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, \lambda_n$$

but that the following set is linearly dependent

$$\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{n-1}, \vec{v}_n\}$$

then there exists a $k \in \mathbb{Z}$ such that

$$X = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{k-2}, \vec{v}_{k-1}\} \text{ and } Y = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{k-1}, \vec{v}_k\}$$

And X is linearly independent,and Y is linearly dependent.

$$\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \dots + \alpha_{k-1}\vec{v}_{k-1} + \alpha_k\vec{v}_k = 0 \tag{κ}$$

where there exists at least one $\alpha_i \neq 0$, then left multiplication by A in the above equation yields

$$\alpha_1\lambda_1\vec{v}_1 + \alpha_2\lambda_2\lambda_3 + \dots + \alpha_{k-1}\lambda_{k-1}\vec{v}_{k-1} + \alpha_k\lambda_k\vec{v}_k = 0 \tag{P}$$

But also multplying (κ) by λ_k we have

$$\alpha_1\lambda_k\vec{v}_1 + \alpha_2\lambda_k\vec{v}_2 + \dots + \alpha_{k-1}\lambda_k\vec{v}_{k-1} + \alpha_k\lambda_k\vec{v}_k = 0 \tag{Q}$$

And subtracting equation Q from P we have

$$\alpha_1(\lambda_1 - \lambda_k)\vec{v}_1 + \alpha_2(\lambda_2(\lambda_3 + \dots + \alpha_{k-1}(\lambda_{k-1} - \lambda_k)\vec{v}_{k-1} + \alpha_k(\lambda_k - \lambda_k)\vec{v}_k = 0$$

But recall that we assumed that each λ_i was distinct which means every $\alpha_i = 0$, but recall one was non-zero so we arrive at a contradiction thus we conclude that the following set is linearly independent

$$\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{n-1}, \vec{v}_n\}$$

□

Example 5.3.3. Let A be an $n \times n$ matrix and suppose its eigen basis is, which spans $\mathbb{R}^n$

$$B = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{n-1}, \vec{v}_n\}$$

and let $\lambda_1, \lambda_2, \dots, \lambda_{n-1}, \lambda_n$ be the eigen values respectively

- Is A diagonalizable? that is does there exists a invertible matrix P such that for some diagonalizable D

  $$P^{-1}AP = D$$

  we know that P is the matrix which converts from $\mathcal{B} \to \mathcal{E}$ and so it's rows are eigen vectors and since they form a basis we know that they are linearly independent thus we only get the trivial solution to the zero vector and so we know we can row reduce to the reduced row echelon form which means we can determine what the inverse matrix P^{-1} must be and so we know that

  $$A = PDP^{-1} \Leftrightarrow P^{-1}AP = D$$

  for some diagonal matrix D.

- If one of the eigen vectors is the zero vector then D for one of the eigen values but recall a matrix is diagonal if and only if it is upper triangular and lower triangular, so we still get that A is diagonalizable

- If the eigen vectors didn't form a basis for $\mathbb{R}^n$, then we know by the contrapositive of then there are not n unique eigen vectors so then when we write P as the the eigen vectors written in the standard basis then one of the columns will be the same as another and after we reduce to reduced row echelon form we'll have a row of zeros so we can reduced it to the identity matrix thus P^{-1} must not exist. And A is not diagonalizable .

We will prove if $P^{-1}AP = D$ for some matrix P and diagonal matrix D then there is an eigen basis for $\mathbb{R}^n$

Proof.
We assume that

$$P^{-1}AP = D$$

thus we know that P^{-1} exists by then P can be row reduced to the identity matrix by being multiplied by some set of elemetary matricies being multiplied together, let $\mathcal{E}_x = \mathcal{E}_1\mathcal{E}_2\cdots\mathcal{E}_{n-1}\mathcal{E}_n$ thus we have

$$\mathcal{E}_xP = I \Leftrightarrow \mathcal{E}_x = P^{-1}$$

And if P could be row reduce to the identity matrix that means that the columns where linearly independent recall that the rows of P are eigen vectors in the standard basis, let $\vec{v}_{,1}\,\vec{v}_{,2}\ldots\vec{v}_{,n-1}\,\vec{v}_{,n}$ be the columns of P then we have our eigen basis

$$\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{n-1}, \vec{v}_n\}$$

it spans $\mathbb{R}^n$ as we have a set of n linearly independent vectors $\square$

Textbook problems §6.1 A1-A2, B2, D1, D3

5.4 Textbook problems 6.1 A1-A2, B2, D1, D3

- A2
  - $A\vec{v_1}$ results in a vector whos second row is non-zero so it's not an eigen vector
  - $A\vec{v_2}$ yields the zero vector which is $0\vec{v_2}$ and so it is an eigen vector
  - $A\vec{v_3} = \begin{bmatrix} -6 \\ -6 \\ 6 \end{bmatrix} = -6\vec{v_3}$ so it is an eigen vector

- $A\vec{v_4}$ we can tell that the top two entries of the resulting vector will be the same, though to be an eigen vector for $\vec{v_4}$ we require that one must be positive and the other negative so $\vec{v_4}$ is not an eigen vector.
- $A\vec{v_5} = 4\vec{v_5}$ so $\vec{v_5}$ is an eigen vector.

- A3
  - We will commence by finding the eigen values for the following matrix.

$$A = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix}$$

that is non-zero vectors $\vec{v}$ that satisfy

$$A\vec{v} = \lambda\vec{v} \text{ for some } \lambda \in \mathbb{R}$$

equivalently we solve

$$(A - \lambda I)\vec{v} = 0$$

though the only way this will produce solutions is if $det(A - \lambda I) = 0$ we know that

$$A - \lambda I = \begin{bmatrix} -\lambda & 1 \\ -6 & 5-\lambda \end{bmatrix} \Leftrightarrow det(A - \lambda I) = (-\lambda)(5-\lambda) + 6$$

expanding right hand side we get

$$\lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3)$$

and so we have that $\lambda = 3$ OR $\lambda = 2$

* We'll now find the eigen space of $\lambda = 3$, so we're finding the non-zero vectors $\vec{v}$ that satisfy

$$A\vec{v} = 3\vec{v} \Leftrightarrow A\begin{bmatrix} x \\ y \end{bmatrix} = 3\begin{bmatrix} x \\ y \end{bmatrix}$$

and so we have that

$$x\begin{bmatrix} 0 \\ -6 \end{bmatrix} + y\begin{bmatrix} 1 \\ 5 \end{bmatrix} = 3\begin{bmatrix} x \\ y \end{bmatrix}$$

thus we know $y = 3x$ so we have

$$x\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right)$$

and so the eigen space is

$$span\left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right\}$$

– the eigen space for $\lambda = 2$ are the vectors that satisfy

$$A\vec{v} = 2\vec{v} \Leftrightarrow (A - 2I)\vec{v} = 0$$

equivalently the linear combinations of the columns of $A - 2I$ that yields the zero vector, we know

$$A - 2I = \begin{bmatrix} -2 & 1 \\ -6 & 3 \end{bmatrix}$$

So let's find the linear combinations of the columns that yield the zero vector. We observe that the second column is -2 times the first, and so we see that

$$1\begin{bmatrix} -2 \\ -6 \end{bmatrix} + -2\begin{bmatrix} 1 \\ 3 \end{bmatrix} = 0 \Leftrightarrow \alpha\left(\begin{bmatrix} -2 \\ -6 \end{bmatrix} + -2\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right) = 0$$

And thus we get the solutions

$$\alpha\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

And so the eigen space is

$$span\left\{\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right\}$$

Chapter 6

Week 14

Theorem 12 (Forced Diagonalization)**.** Recall A is an $n \times n$ matrix with n-distinct eigen values $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}\lambda_n$, then $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{n-1}, \vec{v}_n\}$ is an eigen basis for $\mathbb{R}^n$ and are linearly independent , where $\vec{v}_i$ is the eigen vector corresponding to λ_i If this is true then A is diagonalizable since it has n distinct eigen values
Moreover, there will be only one choce for each $\vec{v}_i$ (up to a scalar multiple)

Example 6.0.1. Suppose $char(A) = -\lambda(2-\lambda)(-3-\lambda)$.

- Is A diagonalizable?
  - We know that there are 3 distinct eigen values and that the matrix must be 3×3 as the characteristic polynomial has 3 λ 's, so it is diagonalizable
- is A invertible? , where

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

  recall that

$$A = PDP^{-1} \Leftrightarrow A^{-1} = PD^{-1}P^{-1}$$

  But we know that D^{-1} must not exists as it cannot be row reduced to the identity matrix so then A^{-1} doesn't exist either.
  alternatively we know that

$$\begin{aligned} det(A) &= det\Big(P^{-1}DP\Big) \\ &= det\Big(P^{-1}\Big)det(D)det(P) \end{aligned}$$

But recall

$$det(P) = \frac{1}{det\big(P^{-1}\big)} \Leftrightarrow det(P)det\Big(P^{-1}\Big) = 1$$

and so we have that

$$\begin{aligned} det(A) &= det(D) \\ &= 0 \end{aligned}$$

Thus we conclude that A doesn't have an inverse (one of our eigen values is zero)

- Can we determine what $dim\big(null(a)\big)$ is? observe that $A = A - 0I$ and recall that 0 is an eigen value and then we know

$$A\vec{x} = 0 \text{ for some } \vec{x} \neq \vec{0}$$

So then $null(A)$ is non-trivial and so we know that $dim\big(null(A)\big) >= 1$ but we know by n distinct eigen values theorem that

Why??

$$dim\big(null(A)\big) = 1 \qquad\qquad dim\big(null(A+3I)\big) = 1$$

when we analyze the characteristic polynomial we know we have found all the values that tell us that the determinant is zeros and so if if we choose an number for λ that is not one of the eigen values we found then we know it must make the determinant non-zero, and so the columns of $A - kI$ where k is not an eigen value are linearly independent. Thus we know

$$dim\big(null(A-3I)\big) = 0$$

since 3 is not an eigen value

Definition 17 (Eigen Space)**.** Let A be a matrix with the eigen values that follow

$$\lambda_1, \lambda_2, \ldots, \lambda_{m-1}, \lambda_m$$

The eigen space of A corresponding to λ_i is

$$null(A - \lambda_i) \Leftrightarrow (A - \lambda_i I)\vec{v} = 0$$

that is the space spanned by all the eigen vectors that have the eigen value λ_i

Definition 18 (Geometric Multiplicity)**.** the geometric multiplicity of λ_i is the dimension of its eigen space

Definition 19 (Algebraic Multiplicity)**.** the algebraic multiplicity of λ_i is the number of times $(\lambda - \lambda_i)$ occurs as a factor of $char(A)$

Example 6.0.2.

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

thus we have

$$char(A) = (3-\lambda)(3-\lambda) = (3-\lambda)^2$$

and we say A has algebraic multiplicity 2 for $\lambda = 3$.

$$(A - 3I) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

And so $null(A - 3I)$ are the vectors that get mapped to zero under

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \vec{x} = \vec{0}$$

observe that any vector times the zero matrix yields the zero vector so $null(A - 3I) = \mathbb{R}^2$ thus the dimension of this eigen space is 2, and the geometric multiplicity is 2

Example 6.0.3.

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} \implies char(A) = (3-\lambda)^2$$

thus A only has one eigen value namely $\lambda = 3$ with algebraic multiplicity 2.

$$A - 3I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

And so $null(A - 3I) = \left\{ \vec{v} \in \mathbb{R}^2 : \vec{v} = \begin{bmatrix} \alpha \\ 0 \end{bmatrix} \text{ for some } \alpha \in \mathbb{R} \right\}$ the dimension is 1 so the geometric multiplicity of $\lambda = 3$ is 1

- Is A diagonalizable ?
  - We know that all the eigen vectors of A are $span\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} \implies$ they are not an eigen basis for $\mathbb{R}^2 \implies$ that the matrix that convert from $\mathcal{E} \to \mathcal{B}$ must not exist thus A is not diagonalizable .
    in fact we require that the geometric multiplicity be 2 $\implies$ we get two linearly independent vectors for our eigen basis .

Explain this again.

Theorem 13 (Diagonalizable and Multiplicities)**.** A matrix is diagonalizable if and only if the algebraic multiplicity and geometric multiplicity are the same

$$\boxed{\text{diagonalizable} \ \Leftrightarrow \ \text{algebraic multiplicity} \ = \ \text{geometric multiplicity}}$$

Theorem 14 (Algebraic and Geometric connection to roots and eigen basis)**.** for an $n \times n$ matrix A whose characteristic polynomial is a degree n polynomial

- It has n roots and is diagonalizable or if it has less then n it is not diagonalizable. For example, if we had for some matrix A that

  $$char(A) = \left(\lambda^2 + 1\right)\lambda$$

  then we know that A is not diagonalizable.

- If the sum of the geometric multiplicities taken over each λ_i is $n \implies$ that taking bases for each eigen space each and finding that eigen space's dimension aka (geometric multiplicity) call it n_i and summing each n_i yields n. Let k be the number of distinct eigen values, then we have

  $$\sum_{i=0}^{k} n_i = n$$

  Why is this true?

  in addition we get n eigen vectors all of whom are linearly independent , so we can append them all to a new set and get an eigen basis that spans $\mathbb{R}^n$

- The geometric multiplicity is greater than or equal to 1, recall this comes from the fact that every eigen value we find will give us a matrix whos columns are linearly dependent and so the dimension of the null space must be greater than or equal to 1 as we are guarenteed at least one degree of freedom in our solutions. Though suprisingly we also have that

  Why is this?

  $$\text{geometric multiplicity} \ \leq \ \text{algebraic multiplicity}$$

  Explain please

  We can now chain these together, we have

  $$\begin{aligned} \text{diagonalizable} \ &\Leftrightarrow \sum \ \text{geometric multiplicity} \ = n \\ &\Leftrightarrow \sum \ \text{algebraic multiplicity} \ = n \\ &\text{and} \ \ \text{geometric multiplicity} \ = \ \text{algebraic multiplicity} \end{aligned}$$

Example 6.0.4.

$$char(A) = \left(\lambda^2 + 1\right)\lambda$$

then we only have one eigen value with algebraic multiplicity of 1 for $\lambda = 0$ so then A is not diagonalizable.

Example 6.0.5. How does this theorem imply

$$\text{n distinct eigen values} \implies \text{diagonalizable}$$

Proof.
If we have n distinct eigen values then each one must have degree 1, and so then the sum of all the algebraic multiplicities for each λ_i is n... □ complete this

Example 6.0.6.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

thus we know

$$char(A) = (1-\lambda)(2-\lambda)^2$$

- the algebraic multiplicity for $\lambda = 1$ is 1
- the algebraic multiplicity for $\lambda = 2$ is 2

then the sum of all the algebraic multiplicies is 3 so A is diagonalizable. We'll now double check our theorem which says that the sum of the geometric multiplicities must be equal to the sum of the algebraic multiplicities.

- For $\lambda = 1$ we know the following

  $$1 \leq \text{ geometric multiplicity } \leq \text{ algebraic multiplicity}$$

  But recall that the algebraic multiplicity for $\lambda = 1$ was one, so then this forces the geometric multiplicity to be 1.

- For $\lambda = 2$ we'll verify the geometric multiplicity, that is the dimension of $(A - 2I)$ which is the number of linearly dependent columns of $(A - 2I)$ or equivalently the number of free columns of reduced row echelon form of A as each one adds a new parameter which leads to an independent vector for each one in the solution set.

  $$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

  and so the dimension of the null space of $(A - 2I)$ is 1 as we only get one parameter in the solution set. Thus the geometric multiplicity is 1, thus the sum of the geometric multiplicities does not sum to 3, only 2 so A is not diagonalizable is this a contradiction ??

Example 6.0.7.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

we have

$$char(A) = (1-\lambda)(2-\lambda)^2$$

We know that from the previous example that the geometric multiplicity of $\lambda = 1$ is forced to be one so A is diagonalizable if and only if $\lambda = 2$ has geometric multiplicity 2. We row reduce $(A - 2I)$ to get

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which has 2 free columns and so the dimension of it's null space is 2 so it's geometric multiplicity is 2 so the sum of all the geometric multiplicities over each λ_i gives n so A is diagonalizable. Also recall that the geometric multiplicity is dimension of the eigen basis associated with each λ_i are all the vectors that get multiplied by λ_i after A being applied to it, and so we have that there are two basis vectors for the eigen space for $\lambda = 2$ and 1 basis vector for the eigen space for $\lambda = 1$ if we create a matrix P where each column is a basis vector, we'll have

$$P = \begin{bmatrix} \vdots & \vdots & \vdots \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ \vdots & \vdots & \vdots \end{bmatrix}$$

We know from the n distinct eigen vectors theorem that actually together they form a basis for $\mathbb{R}^3$ and so P has an inverse, then

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

where does this equation come from??

6.1 Application of Diagonalization

suppose that $A = PDP^{-1}$ where

$$P = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -2 & 0 \end{bmatrix}, D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 6.1.1. We will compute A^{100}, observe

$$A^{100} = PDP^{-1}PDP^{-1}\dots PDP^{-1}PDP^{-1} = PD^{100}P^{-1} = P\begin{bmatrix} 2^{100} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}P^{-1}$$

Example 6.1.2. We'll now compute $A^{100}\vec{x}$ where $B = \{\vec{v}_1, \vec{v}_2\vec{v}_3\}$ ($\vec{v}_i$ is the i'th column of P), and we know $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$

- recall that $P^{-1} : \mathcal{E} \to \mathcal{B}$
- There exists a transformation T such that A induces it in the standard basis. That is

$$A^{100}\vec{x} = T_A^{100}(\vec{x})$$

then we have

$$\left[A^{100}\vec{x}\right]_{\mathcal{B}} = \left[T_A^{100}(\vec{x})\right]_{\mathcal{B}}$$

but recall we know that matrix that induces T_A in the $\mathcal{B}$ basis, so we have

$$\begin{aligned} \left[T_A^{100}(\vec{x})\right]_{\mathcal{B}} &= \left[T_A^{100}\right]_{\mathcal{B}}[\vec{x}]_{\mathcal{B}} \\ &= \begin{bmatrix} 2^{100} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} 2^{100} \\ 3 \\ 4 \end{bmatrix} \end{aligned}$$

We must then find what $\left[A^{100}\vec{x}\right]_{\mathcal{E}}$ but we have P so we have

$$\left[A^{100}\vec{x}\right]_{\mathcal{B}} = P\begin{bmatrix} 2^{100} \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2^{100} - 1 \\ 2^{100} + 7 \\ 2^{100} - 6 \end{bmatrix}$$

6.2 Final Tutorial

- Q1
  - $T\vec{v} = \lambda\vec{v}$ for some $\lambda \in \mathbb{R}$ and $\vec{v} \neq \vec{0}$

  - If $\vec{v}$ is an eigen vector for the linear transformation T then it's image under T is simply a scaled version of itself.

- Q2

  - before we proceed observe that $\vec{v}_6$ is the zero vector, thus by definition it is not an eigen vector we also have that $\vec{v}_5$ upon left multiplication by any matrix is simply the first column of that matrix, we also see that this vector's y component is 0, and that none of the first columns of any of the matricies we are dealing with, share that properties thus we conclude that $\vec{v}_5$ is not an eigen vector. Finally the C matrix has the following properties, that is

$$\begin{bmatrix} -3 & 3 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -3\alpha + 3\beta \\ -3\alpha + 3\beta \end{bmatrix} = 3(\alpha - \beta) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

    so any vector multiplied by is in the set $span \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ thus $\vec{v_4}$ is the only vector which is an eigen vector for this matrix.
    The rest can be completed by left multiplication by a matrix onto each of the vectors that we have not verified.

- Q3

  - The eigen vectors for $\mathcal{P}$ are not ones that do not lay on the x-axis and y-axis as they drop down to the x-axis. So for all the vectors that lay on the x-axis clearlytheir image is the same as the original vector, so for any vector of the form $\begin{bmatrix} \alpha \\ 0 \end{bmatrix}$, it's eigen value is $\lambda = 1$. Recall that any vector whose tip lays upon the y-axis drops down to zero, so all of these vectors $\begin{bmatrix} 0 \\ \alpha \end{bmatrix}$ must have an eigen value of $\lambda = 0$.
  - As for $\mathcal{R}$ we have already observed that every non-zero vector changes direction, exactly by $\frac{\pi}{2}$ counter clockwise and so there must not exist any eigen values. Alternatively, view that any vectors moves to the next quadrant thus either it's x or y component changes from positive to negative, and so it cannot be an eigen vector.

- Q4

  - We have

$$T(\vec{v}) = 2\vec{v}$$

we suppose that $7\vec{v}$ is an eigen vector with eigen value 14, that is

$$T(7\vec{v}) = 14(7\vec{v})$$

But then

$$T(7\vec{v}) = 7T(\vec{v}) = 14\vec{v}$$

clearly $14 \cdot 7 \neq 14$ and so $7\vec{v}$ is is not an eigen vector for T with eigen value 14.

- Q5
  - Recall that the composition of linear transformation is once more a linear transformation, and so we have

    $$T^{100}(\vec{w}) = T^{100}(\vec{v_1} + \vec{v_2}) = T^{100}(\vec{v_1}) + T^{100}(\vec{v_2})$$

    Observe that $T(\vec{v_1}) = \vec{v_1}$ and $T(\vec{v_2}) = \frac{1}{2}\vec{v}$, thus applying the linear transformation T 100 times to $\vec{v_1}$ results in $\vec{v_1}$ and applying T to $\vec{v_2}$ 100 times results in $\frac{1}{2^{100}}\vec{v_2}$. We will now approximate $T^{100}(\vec{w}) = \vec{v_1} + \frac{1}{2^{100}}\vec{v_2}$, since their lengths are constrained within some finite value and that $\frac{1}{2^{100}}$ is much smaller compared to that finite value, then we know that $\frac{1}{2^{100}}\vec{v_2}$ is not very significant and thus we conclude $T^{100}(\vec{w}) \approx \vec{v_1}$
  - We commence by supposing that $\vec{v_1}$ and $\vec{v_2}$ do not form a basis, that is one is a linear combination of the other, so we have

    $$\vec{v_1} = \alpha\vec{v_2} \text{ for some } \alpha \in \mathbb{R}$$

    thus we know

    $$T(\vec{v_1}) = T(\alpha\vec{v_2}) \Leftrightarrow \vec{v_1} = \alpha\frac{1}{2}\vec{v_2}$$

    but we then we have a contradiction so they must be linearly independent and thus form a basis.
  - Yes, observe that $\vec{v_1}$ and $\vec{v_2}$ are linearly independent and thus span $\mathbb{R}^2$ then any vector $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ can be written as a linear combination of $\vec{v_1}$ and $\vec{v_2}$ then we know $T^{100}(j\vec{v_1} + k\vec{v_2}) \approx j\vec{v_1}$
  - first we observe

    $$\alpha^{100} \approx 0 \text{ and } \beta^{100} \geq 1 \text{ for some } \alpha \in [0,1), \beta \in [1,\infty)$$

    thus if α, β are eigen values and $\vec{v_\alpha}$ and $\vec{v_\beta}$ are corresponding eigen vectors, then we know that they are linearly independent and form a basis (same proof as above) so then for any vector in $\mathbb{R}^2$ it can be written as a linear combination of them and then $S^{100}(\vec{w}) \approx \beta^{100}\vec{v_2}$