MAT237 - Multi-variable Calculus

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Chapter 1

Lecture 1 - Review

1.1 Sets & tuples

Definition 1 (Tuple)
A n tuple is an ordered list of n elements $(x_1, \dots, x_n)$ notation

- *couple, a 2-tuple*
- *triple, a 3-tuple*

Fundamental Property

$$(x_1, \dots, x_m) = (y_1, \dots, y_m) \Leftrightarrow \forall i \in \{1, \dots, m\}, x_i = y_i$$

Recall

$$\{1, 2, 3\} = \{3, 2, 1\}$$

But

$$(1, 2, 3) \neq (3, 2, 1)$$

In addition

$$(1, 2, 2, 3) \neq (1, 2, 3)$$

Also the comparison here doesn't even make sense since they are different sizes.

Definition 2 (Cartesian Product)
For sets A, B

$$A \times B = \big\{(a, b) : a \in A, b \in B\big\}$$

Note if we have $A = \emptyset$ or $B = \emptyset$ then $A \times B = \emptyset$

Example 1.1.1

$$A = \{\pi, e\} \text{ and } B = \left\{1, \sqrt{2}, \pi\right\}$$

$$A \times B = \left\{(\pi, 1), \left(\pi, \sqrt{2}\right), (\pi, \pi), \ldots\right\}$$

For multiple cartesian products we have

$$A_1 \times A_2, \ldots, A_n = \left\{(a_1, a_2, \ldots, a_m) : a_i \in A_i\right\}$$

Exercise 1.1.1
Is the following true ?

$$(A \times B) \times C = A \times (B \times C) = A \times B \times C$$

No, observe the tuples of $(A \times B) \times C$ are of the form

$$\left((a, b), c\right)$$

In the same way we observe that none of them are equal . Though in a functional type of sense, they are equal as they all still convey the same fundamental idea.

1.2 Functions

Definition 3 (Function)
A function is the data of two sets, A and B together with a "rule" that associates to each $x \in A$ a unique $f(x) \in B$.
We define a function like this

$$f : A \to B$$

Where A is the domain and B is the codomain.

Definition 4 (Image)
The image of $E \subseteq A$ by f is

$$f(E) = \left\{f(x) : x \in E\right\}$$

Definition 5 (Pre-Image)
The pre-image of $F \in B$ by f is

$$f^{-1}(F) = \left\{x \in A : f(x) \in F\right\}$$

Definition 6 (Graph)
The graph of f is

$$\Gamma f = \left\{(x, y) \in A \times B : y = f(x)\right\}$$

Definition 7 (Injective)
A function $f : A \to B$ is injective or one-to-one

$$\forall x_1, x_2 \in A, f(x_1) = f(x_2) \implies x_1 = x_2$$

We have the contrapositive

$$\forall x_1, x_2 \in A, x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$$

Definition 8 (Onto)
A function is surjective or onto if

$$\forall y \in B, \exists x \in A, y = f(x)$$

Definition 9 (Bijective)
$f : A \to B$ is bijective if it is injective and surjective.

$$\forall y \in B, \exists! x \in A, y = f(x)$$

Definition 10 (Inverse)
$f : A \to B$ has an inverse if and only if there exists a function $g : B \to A$ such that

$$\forall x \in A, g \circ f(x) = x$$

and

$$\forall x \in B, f \circ g(x) = x$$

then we say that g is the inverse of f and $g = f^{-1}$

1.3 Function Questions

1.3.1 Visual Sets

1. not a function, observe that d maps to two different elements so it doesn't map to a unique element.

2. not a function, observe d is not being mapped to anything.

3. this is a function , it is injective as we can see no element in the codomain has two arrows leading to it, it is also surjective since for every element in the codomain there is an arrow leading to it. By definition it is bijective, and it's inverse is given by turning each arrow around.

4. It is a function, observe $f_4(c) = f_4(d)$ therefore it is not injective, though due to the same reasoning as the previous question it is surjective. Assume it's inverse exists then $g \circ f(c) = g \circ f(d)$ but then $c = d$ so a contradiction.

5. It is a function, it is injective, though not surjective nor bijective, the inverse does not exist.

6. It is a function, not injective nor surjective therefore not bijective and the inverse must not exist.

1.3.2 Defined sets

1. f_7 I believe this is a function, if we take an element from the codomain for example ae^b this must only have come from (a, b). It is not surjective

   $$f(e, 0) = f(1, 1)$$

   It is surjective, let $k \in \mathbb{R}$ then we have $f(k, 0)$

2. f_8 I believe this is a function, take $f(j,k)$ this maps to the unique element (e^j, k^2). Not injective consider $f(0,1)$ and $f(0,-1)$. Not surjective, observe $e^x > 0$ therefore nothing maps to $(-1,p)$

1.3.3 Image and Inverse Image

1. $f\big(\{a,c,d\}\big)$ by definition
$$\big\{f(x) : x \in \{a,b,c\}\big\} = \{1,2\}$$

2. $f^{-1}\big(\{2,3,4\}\big)$ by definition
$$\big\{x \in \{a,b,c,d\} : f(x) \in \{2,3,4\}\big\} = \{c,d,b\}$$

3. By definition we have
$$\big\{(1,3),(2,5),(3,1),(4,5)\big\}$$

4. This first is a graph, by inspection There is a unique element in the codomain for every element in the domain.

5. The second is not, we observe $f(2) = j$ and $f(2) = k$ but $j \neq k$

1.3.4 Final Four

1. Not injective $f(a,b,z) = f(a,b,x), x \neq z$. It is surjective, it is not bijective so the inverse does not exist .

2. Initially, I thought it may be possible it it is not injective since we have x^2 though the e^x showed me that idea would not work. So I'll prove it's injective. Assume
$$f(a,b) = f(j,k) \Leftrightarrow \Big(e^a, \Big(a^2+1\Big)b\Big) = \Big(e^j, \Big(j^2+1\Big)k\Big)$$
we have $e^a = e^j \therefore a = j$, though we also have
$$\Big(a^2+1\Big)b = \Big(j^2+1\Big)k \Leftrightarrow \Big(a^2+1\Big)b = \Big(a^2+1\Big)k \Leftrightarrow b = k$$
therefore it's injective. We know $e^x > 0$ therefore $(a,k), a \leq 0$ is not mapped to. It's not bijective so the inverse does not exist.

3. I'll try the same idea as the last question,
$$(a+b,-a) = (j+k,-j)$$

therefore $a = j$ then we have $a + b = j + k \Leftrightarrow b = k$ so its injective, we will show it's surjective, let $(l, m) \in \mathbb{R}^2$ then take $x = -m$ and $y = m + l$ so $f(x, y) = \big(-m + m + l, -(-m)\big) = (l, m)$, since it is bijective then we know that the inverse exists, so we must find it. Observe we require

$$h^{-1}\big(h(x, y)\big) = (x, y) \Leftrightarrow h^{-1}(x + y, -x) = (x, y)$$

So we define $h^{-1}(j, k) = (j - k, j + y)$ let's verify, the two properties

4. I believe l is surjective and injective,

Proof.
We'll show it's surjective , let x, y bet two non-negative natural numbers we assume that $l(x) = l(y)$.
We observe that $l(x)$ and $l(y)$ are either both positive or negative, or else we get a contradiction.
Without loss of generality we assume they are both positive this implies that both x and y are even, then for contradiction we assume that $x \neq y$, then from our assumtion we have $\frac{x}{2} = \frac{y}{2} \Leftrightarrow x = y$, but then this is a contradiction, so then $x = y$
We'll now show it's surjective , let $k \in \mathbb{Z}$

- If $k \geq 0$. Then take $x = 2k$ then we have $l(2k) = k$ as required
- Otherwise $k \leq 0$. Then take $x = 1 - 2k$ and so $l(1 - 2k) = k$.

∎

Let n be a non-negative natural number and $y \in \mathbb{Z}$, we require $l^{-1}\big(l(n)\big) = n$ and $l\big(l^{-1}(y)\big) = y$. For the first part if n is even we have $l^{-1}\left(\frac{n}{2}\right) = n$ and so l^{-1} should just multiply by 2, in the other case we also undo the algebra. For y we observe that the algebra steps are also undone, as required.

1.4 Geometry in Higher Dimensions

Definition 11 (R^n)

$$R^n = \{(x_1, x_2, \ldots x_{n-1}, x_n,) : x_i \in \mathbb{R}\}$$

Also note that it could be thought of as, though we get nesting couples.

$$\underbrace{\mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R} \times \mathbb{R}}_{n\ times}$$

- $\mathbb{R}^2$
  - We think of this as a plane, perhaps the x, y coordinate system
  - $(x, y) \in \mathbb{R}^2$
- $\mathbb{R}^3$
  - We can think of this as 3d, space the space we live in
  - $(x, y, z) \in \mathbb{R}^3$
- $\mathbb{R}^n$
  - This is n-dimensional space, hard to visualize, though it makes sense in an algebraic sense.
  - $(x_1, x_2, \ldots, x_{n-1}, x_n) \in \mathbb{R}^n$
  - We denote an n-tuple of $\mathbb{R}^n$ like this $\vec{x}$
- $\vec{e_n} = (0, \ldots, 0, 1, \ldots)$ where 1 is at the n-th entry.

1.4.1 Operations on n-tuples

Let $a = (a_1, a_2, \ldots, a_{n-1}, a_n)$, $b = (b_1, b_2, \ldots, b_{n-1}, b_n)$ and $\lambda \in \mathbb{R}$

- Addition:
$$a + b = (a_1 + b_1, a_2 + b_2, \ldots, a_{n-1} + b_{n-1}, a_n + b_n)$$
- scalar multiplication
$$\lambda a = (\lambda a_1, \lambda a_2, \ldots, \lambda a_{n-1}, \lambda a_n)$$

Definition 12 (Dot Product)

$$a \cdot b = \underbrace{\sum_{i=0}^{n} a_i b_i}_{\chi} = b \cdot a$$

Note $\chi \in \mathbb{R}$

- $(\lambda a) \cdot b = \lambda(a \cdot b)$
- $(a + b) \cdot c = a \cdot c + b \cdot c$
- $a \neq \vec{0} \implies a \cdot a > 0$ also $a \cdot a = 0 \implies a = \vec{0}$ additionally $\vec{0} \cdot a = 0$

Definition 13 (Norm)
Let $a \in \mathbb{R}^n$

$$\|a\| = \sqrt{a \cdot a} = \sqrt{a_1^2 + a_2^2 + \cdots + a_{n-1}^2 + a_n^2}$$

geometrically this is the length of a*, also* $\|a - b\|$ *is the distance between* a *and*

1.4.2 Important properties of the norm

Prove the following.

1. $\|a\| \geq 0$
2. $\|a\| = 0 \implies a = \vec{0}$
3. $\|\lambda a\| = |\lambda| \|a\|$
4. $\|a + b\| \leq \|a\| + \|b\|$, (hint use 5)
5. $|a \cdot b| \leq \|a\| \|b\|$ (Cauchy-Schwarts inequality)
6. $a \cdot e_j = a_j$
7. $e_j \cdot e_j = 1$
8. for $i \neq j$, $e_j \cdot e_i = 0$
9. $a \cdot b = \frac{1}{4}\left(\|a + b\|^2 - \|a - b\|^2\right)$ (Polarization identity)

Proofs

1. $\|a\| \geq 0$

   Proof.
   $\forall x \in \mathbb{R}, x^2 \geq 0 \implies \sum_{i=0}^{n} a_i^2 \geq 0 \implies \sqrt{\sum_{i=0}^{n} a_i^2} \geq 0 \Leftrightarrow \|a\| \geq 0$ ■

2. $\|a\| = 0 \implies \vec{a} = \vec{0}$

   Proof.
   Assume $\|a\| = 0$, for contradiction we assume $\exists i \in \{1, \ldots, n\}$ such that $a_i \neq 0$ then $\sqrt{\sum_{i=0}^{n} a_i^2} \neq 0 \Leftrightarrow \|a\| \neq 0$ thus we have a contradiction therefore $a = \vec{0}$. ■

3. $\|\lambda a\| = |\lambda| \|a\|$

   Proof.
   We commence ,

$$\begin{aligned} \|\lambda a\| &= \sqrt{\sum_{i=0}^{n} (\lambda a_i)^2} \\ &= \sqrt{\lambda^2 \sum_{i=0}^{n} a_i^2} \\ &= \lambda \sqrt{\sum_{i=0}^{n} a_i^2} \\ &= \lambda \|a\| \end{aligned}$$

   ■

4. $\|a + b\| \leq \|a\| + \|b\|$

5. $|a \cdot b| \leq \|a\| \|b\|$

   Proof.
   Let $f(t) = \|a + tb\|^2$, for $t \in \mathbb{R}, a, b \in \mathbb{R}^n$

$$\begin{aligned} \|a + tb\|^2 &= (a + tb)^2 \\ &= (a \cdot a) + 2(a \cdot tb) + \|b\|^2 t^2 \end{aligned}$$

observe we have a quadratic polynomial whose leading coefficient is non-negative, and that $f(t) \geq 0$ since the norm of any vector is non-negative. Therefore it can have at most one solution, that is only if the inner part of the descriminant is less than or equal to 0

$$\sqrt{b^2 - 4ac} \leq 0$$

though in our case we have, using the fact that $\sqrt{x^2} = |x|$

$$\begin{gathered}
\sqrt{4(a \cdot b)^2} - 4\|a\|^2\|b\|^2 \leq 0 \\
4(a \cdot b)^2 - 4\|a\|^2\|b\|^2 \leq 0 \\
(a \cdot b)^2 \leq \|a\|^2\|b\|^2 \\
|a \cdot b| \leq \|a\|\|b\|
\end{gathered}$$

■