MAT237 - Multi-variable Calculus

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Chapter 1

Lecture 1 - Review

1.1 Sets & tuples

Definition 1 (Tuple)
A n tuple is an ordered list of n elements $(x_1, \ldots, x_n)$ notation

- *couple, a 2-tuple*
- *triple, a 3-tuple*

Fundamental Property

$$(x_1, \ldots, x_m) = (y_1, \ldots, y_m) \Leftrightarrow \forall i \in \{1, \ldots, m\}, x_i = y_i$$

Recall

$$\{1, 2, 3\} = \{3, 2, 1\}$$

But

$$(1, 2, 3) \neq (3, 2, 1)$$

In addition

$$(1, 2, 2, 3) \neq (1, 2, 3)$$

Also the comparison here doesn't even make sense since they are different sizes.

Definition 2 (Cartesian Product)
For sets A, B

$$A \times B = \big\{(a, b) : a \in A, b \in B\big\}$$

Note if we have $A = \emptyset$ or $B = \emptyset$ then $A \times B = \emptyset$

Example 1.1.1

$$A = \{\pi, e\} \text{ and } B = \left\{1, \sqrt{2}, \pi\right\}$$

$$A \times B = \left\{(\pi, 1), \left(\pi, \sqrt{2}\right), (\pi, \pi), \ldots\right\}$$

For multiple cartesian products we have

$$A_1 \times A_2, \ldots, A_n = \left\{(a_1, a_2, \ldots, a_m) : a_i \in A_i\right\}$$

Exercise 1.1.1
Is the following true ?

$$(A \times B) \times C = A \times (B \times C) = A \times B \times C$$

No, observe the tuples of $(A \times B) \times C$ are of the form

$$\big((a, b), c\big)$$

In the same way we observe that none of them are equal . Though in a functional type of sense, they are equal as they all still convey the same fundamental idea.

1.2 Functions

Definition 3 (Function)
A function is the data of two sets, A and B together with a "rule" that associates to each $x \in A$ a unique $f(x) \in B$.
We define a function like this

$$f : A \to B$$

Where A is the domain and B is the codomain.

Definition 4 (Image)
The image of $E \subseteq A$ by f is

$$f(E) = \left\{f(x) : x \in E\right\}$$

Definition 5 (Pre-Image)
The pre-image of $F \in B$ by f is

$$f^{-1}(F) = \left\{x \in A : f(x) \in F\right\}$$

Definition 6 (Graph)
The graph of f is

$$\Gamma f = \left\{(x, y) \in A \times B : y = f(x)\right\}$$

Definition 7 (Injective)
A function $f : A \to B$ is injective or one-to-one

$$\forall x_1, x_2 \in A, f(x_1) = f(x_2) \implies x_1 = x_2$$

We have the contrapositive

$$\forall x_1, x_2 \in A, x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$$

Definition 8 (Onto)
A function is surjective or onto if

$$\forall y \in B, \exists x \in A, y = f(x)$$

Definition 9 (Bijective)
$f : A \to B$ is bijective if it is injective and surjective.

$$\forall y \in B, \exists ! x \in A, y = f(x)$$

Definition 10 (Inverse)
$f : A \to B$ has an inverse if and only if there exists a function $g : B \to A$ such that

$$\forall x \in A, g \circ f(x) = x$$

and

$$\forall x \in B, f \circ g(x) = x$$

then we say that g is the inverse of f and $g = f^{-1}$

1.3 Function Questions

1.3.1 Visual Sets

1. not a function, observe that d maps to two different elements so it doesn't map to a unique element.

2. not a function, observe d is not being mapped to anything.

3. this is a function , it is injective as we can see no element in the codomain has two arrows leading to it, it is also surjective since for every element in the codomain there is an arrow leading to it. By definition it is bijective, and it's inverse is given by turning each arrow around.

4. It is a function, observe $f_4(c) = f_4(d)$ therefore it is not injective, though due to the same reasoning as the previous question it is surjective. Assume it's inverse exists then $g \circ f(c) = g \circ f(d)$ but then $c = d$ so a contradiction.

5. It is a function, it is injective, though not surjective nor bijective, the inverse does not exist.

6. It is a function, not injective nor surjective therefore not bijective and the inverse must not exist.

1.3.2 Defined sets

1. f_7 I believe this is a function, if we take an element from the codomain for example ae^b this must only have come from (a, b). It is not surjective

$$f(e, 0) = f(1, 1)$$

   It is surjective, let $k \in \mathbb{R}$ then we have $f(k, 0)$

2. f_8 I believe this is a function, take $f(j,k)$ this maps to the unique element (e^j, k^2). Not injective consider $f(0,1)$ and $f(0,-1)$. Not surjective, observe $e^x > 0$ therefore nothing maps to $(-1,p)$

1.3.3 Image and Inverse Image

1. $f\big(\{a,c,d\}\big)$ by definition

$$\big\{f(x) : x \in \{a,b,c\}\big\} = \{1,2\}$$

2. $f^{-1}\big(\{2,3,4\}\big)$ by definition

$$\big\{x \in \{a,b,c,d\} : f(x) \in \{2,3,4\}\big\} = \{c,d,b\}$$

3. By definition we have

$$\big\{(1,3),(2,5),(3,1),(4,5)\big\}$$

4. This first is a graph, by inspection There is a unique element in the codomain for every element in the domain.

5. The second is not, we observe $f(2) = j$ and $f(2) = k$ but $j \neq k$

1.3.4 Final Four

1. Not injective $f(a,b,z) = f(a,b,x), x \neq z$. It is surjective, it is not bijective so the inverse does not exist .

2. Initially, I thought it may be possible it it is not injective since we have x^2 though the e^x showed me that idea would not work. So I'll prove it's injective. Assume

$$f(a,b) = f(j,k) \Leftrightarrow \Big(e^a, \Big(a^2+1\Big)b\Big) = \Big(e^j, \Big(j^2+1\Big)k\Big)$$

we have $e^a = e^j \therefore a = j$, though we also have

$$\Big(a^2+1\Big)b = \Big(j^2+1\Big)k \Leftrightarrow \Big(a^2+1\Big)b = \Big(a^2+1\Big)k \Leftrightarrow b = k$$

therefore it's injective. We know $e^x > 0$ therefore $(a,k), a \leq 0$ is not mapped to. It's not bijective so the inverse does not exist.

3. I'll try the same idea as the last question,

$$(a+b,-a) = (j+k,-j)$$

therefore $a = j$ then we have $a + b = j + k \Leftrightarrow b = k$ so its injective, we will show it's surjective, let $(l, m) \in \mathbb{R}^2$ then take $x = -m$ and $y = m + l$ so $f(x, y) = \big(-m + m + l, -(-m)\big) = (l, m)$, since it is bijective then we know that the inverse exists, so we must find it. Observe we require

$$h^{-1}\big(h(x, y)\big) = (x, y) \Leftrightarrow h^{-1}(x + y, -x) = (x, y)$$

So we define $h^{-1}(j, k) = (j - k, j + y)$ let's verify, the two properties

4. I believe l is surjective and injective,

Proof.
We'll show it's surjective , let x, y bet two non-negative natural numbers we assume that $l(x) = l(y)$.
We observe that $l(x)$ and $l(y)$ are either both positive or negative, or else we get a contradiction.
Without loss of generality we assume they are both positive this implies that both x and y are even, then for contradiction we assume that $x \neq y$, then from our assumtion we have $\frac{x}{2} = \frac{y}{2} \Leftrightarrow x = y$, but then this is a contradiction, so then $x = y$
We'll now show it's surjective , let $k \in \mathbb{Z}$

- If $k \geq 0$. Then take $x = 2k$ then we have $l(2k) = k$ as required
- Otherwise $k \leq 0$. Then take $x = 1 - 2k$ and so $l(1 - 2k) = k$.

∎

Let n be a non-negative natural number and $y \in \mathbb{Z}$, we require $l^{-1}\big(l(n)\big) = n$ and $l\big(l^{-1}(y)\big) = y$. For the first part if n is even we have $l^{-1}\left(\frac{n}{2}\right) = n$ and so l^{-1} should just multiply by 2, in the other case we also undo the algebra. For y we observe that the algebra steps are also undone, as required.

Chapter 2

Lecture 2

2.1 Geometry in Higher Dimensions

Definition 11 (R^n)

$$R^n = \left\{(x_1, x_2, \dots x_{n-1}, x_n,) : x_i \in \mathbb{R}\right\}$$

Also note that it could be thought of as, though we get nesting couples.

$$\underbrace{\mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R} \times \mathbb{R}}_{n \textit{ times}}$$

- $\mathbb{R}^2$
  - We think of this as a plane, perhaps the x, y coordinate system
  - $(x, y) \in \mathbb{R}^2$
- $\mathbb{R}^3$
  - We can think of this as 3d, space the space we live in
  - $(x, y, z) \in \mathbb{R}^3$
- $\mathbb{R}^n$
  - This is n-dimensional space, hard to visualize, though it makes sense in an algebraic sense.
  - $(x_1, x_2, \dots, x_{n-1}, x_n) \in \mathbb{R}^n$
  - We denote an n-tuple of $\mathbb{R}^n$ like this $\vec{x}$
- $\vec{e_n} = (0, \dots, 0, 1, \dots)$ where 1 is at the n-th entry.

2.1.1 Operations on n-tuples

Let $a = (a_1, a_2, \ldots, a_{n-1}, a_n)$, $b = (b_1, b_2, \ldots, b_{n-1}, b_n)$ and $\lambda \in \mathbb{R}$

- Addition:
$$a + b = (a_1 + b_1, a_2 + b_2, \ldots, a_{n-1} + b_{n-1}, a_n + b_n)$$
- scalar multiplication
$$\lambda a = (\lambda a_1, \lambda a_2, \ldots, \lambda a_{n-1}, \lambda a_n)$$

Definition 12 (Dot Product)

$$a \cdot b = \underbrace{\sum_{i=0}^{n} a_i b_i}_{\chi} = b \cdot a$$

Note $\chi \in \mathbb{R}$

Definition 13 (Dot Product (Geometric))

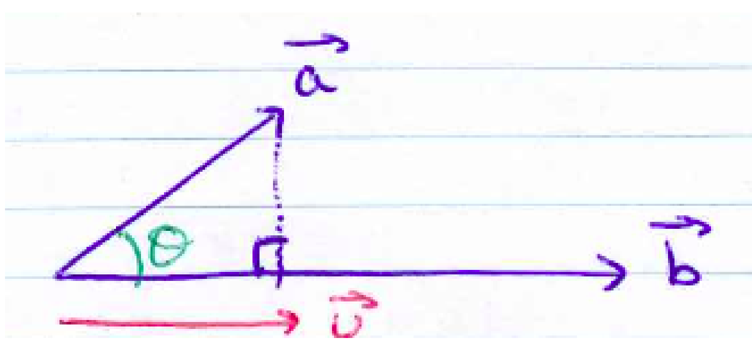
$$a \cdot b = \|a\|\|b\| \cos(\theta)$$

where θ *is the angle between* a *and* b

Properties of the Dot Product

- $(\lambda a) \cdot b = \lambda(a \cdot b)$
- $(a + b) \cdot c = a \cdot c + b \cdot c$
- $a \neq \vec{0} \implies a \cdot a > 0$ also $a \cdot a = 0 \implies a = \vec{0}$ additionally $\vec{0} \cdot a = 0$

Observe from the following image that



$$\|u\| = \cos(\theta)\|a\|$$

by trigonometry and so if we want to find what $\vec{u}$ is then we know it's in the direction of $\vec{b}$ so we scale $\vec{b}$ to be a unit vector and then multiply by $\|u\|$ that gives

$$\vec{u} = \|a\|\cos(\theta)\frac{\vec{b}}{\|b\|}$$

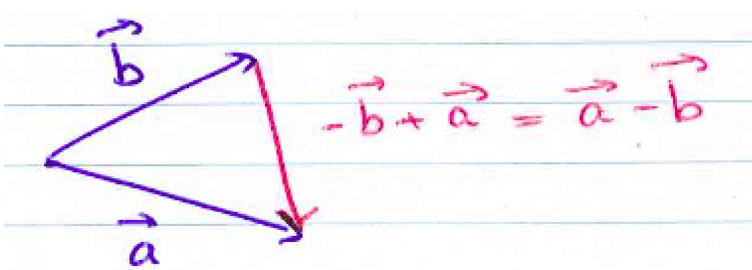
Then the projection of $\vec{a}$ on $span\vec{b}$ follows, observe

$$\vec{u} = \|a\|\cos(\theta)\frac{\vec{b}}{\|b\|}\left(\frac{\|b\|}{\|b\|}\right) = \|a\|\|b\|\cos(\theta)\frac{\vec{b}}{\|b\|^2} = \frac{a \cdot b}{\|b\|^2}\vec{b}$$

Definition 14 (Orthogonal Projection)
For two vectors $\vec{a}, \vec{b}$ the orthogonal projection is given by

$$\frac{a \cdot b}{\|b\|^2}\vec{b}$$

2.1.2 Law of Cosines

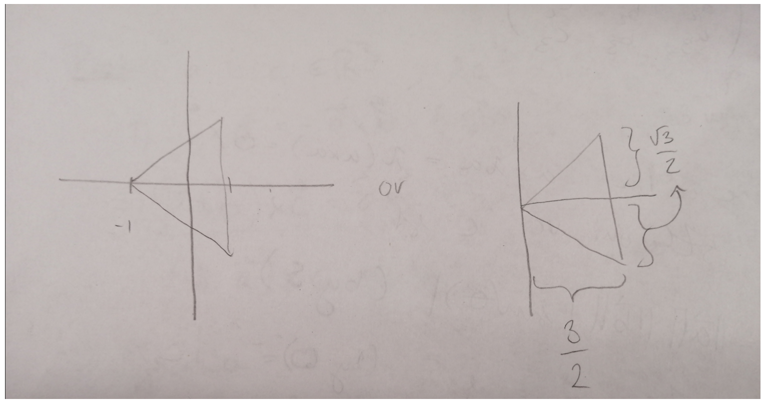


We have

$$\|a - b\|^2 = (a - b) \cdot (a - b) = \|a\|^2 + \|b\|^2 - 2(a \cdot b)$$

which matches the geometric intuition

Homework



Let $\vec{x} = \vec{AB}$ and $\vec{y} = \vec{AC}$,

- Algebraically we have $\vec{x} \cdot \vec{y} = \left(\frac{3}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{-\sqrt{3}}{2}\right) = \frac{9}{4} - \frac{3}{4} = \frac{6}{4} = \frac{3}{2}$
- Geometrically, we have $\|x\|\|y\| \cos(\theta)$, note $\|x\| = \|y\|$ and that $\|x\| = \sqrt{3}$ so then we have $3\cos(\theta) = \frac{3}{2} \Leftrightarrow \cos(\theta) = \frac{1}{2}$ so that is if $\theta = \frac{\pi}{6}$ by symetry, we know the other two angles are $\frac{5\pi}{12}$

Definition 15 (Orthogonality)
two vectors a, b are orthogonal if and only if $a \cdot b = 0$

- Observe that if we have $\|u\| = 1$ then u is a unit vector, and v be any vector, then $(u \cdot v)u$ is the projection of v onto the line defined by u. Since $(u \cdot v)$ is $\|u\|\|v\| \cos(\theta)$, where θ is the angle between the two vectors, and that $\|u\| = 1$ then $(u \cdot v)$ is the length of the adjacent side, then multiply a the unit vector which gives the projection.
- If we have $v = v_1 + v_2$ where v_1 is parrallel to u and v_2 is orthogonal to u then $(u \cdot v)u$ gives us the projection onto $\vec{u}$ which is just v_1
- $u \cdot v$ is the length of the projection of v onto u which is the same as $v_1 \cdot u$, but recall that v_1 and u are paralell, therefore the angle between them is either 0 or π, therefore we have $\|u\|\|v\| \cos(\theta) = \pm\|u\|\|v_1\|$
- Finally, we form a right triangle with hypotenuse given by v with a base which lies along u, then the base is given by $(u \cdot v)u$ using the same reasoning as in our first observation.

Definition 16 (Norm)
Let $a \in \mathbb{R}^n$

$$\|a\| = \sqrt{a \cdot a} = \sqrt{a_1^2 + a_2^2 + \cdots + a_{n-1}^2 + a_n^2}$$

geometrically this is the length of a, also $\|a - b\|$ is the distance between a and

2.1.3 Important properties of the norm

Prove the following.

1. $\|a\| \geq 0$
2. $\|a\| = 0 \implies a = \vec{0}$
3. $\|\lambda a\| = |\lambda|\|a\|$
4. $\|a + b\| \leq \|a\| + \|b\|$, (hint use 5)
5. $|a \cdot b| \leq \|a\|\|b\|$ (Cauchy-Schwarts inequality)
6. $a \cdot e_j = a_j$
7. $e_j \cdot e_j = 1$

8. for $i \neq j$, $e_j \cdot e_i = 0$

9. $a \cdot b = \frac{1}{4}\Big(\|a+b\|^2 - \|a-b\|^2\Big)$ (Polarization identity)

Proofs

1. $\|a\| \geq 0$

Proof.
$\forall x \in \mathbb{R}, x^2 \geq 0 \implies \sum_{i=0}^{n} a_i^2 \geq 0 \implies \sqrt{\sum_{i=0}^{n} a_i^2} \geq 0 \Leftrightarrow \|a\| \geq 0$ ■

2. $\|a\| = 0 \implies \vec{a} = \vec{0}$

Proof.
Assume $\|a\| = 0$, for contradiction we assume $\exists i \in \{1, \dots, n\}$ such that $a_i \neq 0$ then $\sqrt{\sum_{i=0}^{n} a_i^2} \neq 0 \Leftrightarrow \|a\| \neq 0$ thus we have a contradiction therefore $a = \vec{0}$. ■

3. $\|\lambda a\| = |\lambda| \|a\|$

Proof.
We commence ,

$$\begin{aligned}
\|\lambda a\| &= \sqrt{\sum_{i=0}^{n} (\lambda a_i)^2} \\
&= \sqrt{\lambda^2 \sum_{i=0}^{n} a_i^2} \\
&= \lambda \sqrt{\sum_{i=0}^{n} a_i^2} \\
&= \lambda \|a\|
\end{aligned}$$

■

4. $\|a+b\| \leq \|a\| + \|b\|$

5. $|a \cdot b| \leq \|a\| \|b\|$

Proof.
We will commense with left hand side

$$\begin{aligned}
\|a+b\| &= \big\|(a_1+b_1,\ldots,+a_n+b_n)\big\| \\
&= \sqrt{(a_1+b_1)^2+\ldots+(a_n+b_n)^2} \\
&= \sqrt{\left(a_1^2+2a_1b_1+b^2\right)+\ldots+\left(a_n^2+2a_nb_n+b_n^2\right)} \\
&= \sqrt{\sum_{i=1}^{n} a_i^2+\sum_{i=1}^{n} b_i^2+2(a\cdot b)} \\
&\leq \sqrt{\sum_{i=1}^{n} a_i^2+\sum_{i=1}^{n} b_i^2+2|a\cdot b|}
\end{aligned}$$

From 5, it follows that

$$\begin{aligned}
&\leq \sqrt{\sum_{i=1}^{n} a_i^2+\sum_{i=1}^{n} b_i^2+2\big(\|a\|\|b\|\big)} \\
&= \sqrt{\|a\|^2+\|b\|^2+2\|a\|\|b\|} \\
&= \sqrt{\big(\|a\|+\|b\|\big)^2} \\
&= \|a\|+\|b\|
\end{aligned}$$

Therefore

$$\|a+b\| \leq \|a\|+\|b\|$$

as required ■

Proof.
Let $f(t)=\|a+tb\|^2$, for $t\in\mathbb{R}, a,b\in\mathbb{R}^n$

$$\begin{aligned}
\|a+tb\|^2 &= (a+tb)^2 \\
&= (a\cdot a)+2(a\cdot tb)+\|b\|^2t^2
\end{aligned}$$

observe we have a quadratic polynomial whose leading coefficient is non-negative, and that $f(t)\geq 0$ since the norm of any vector is non-negative. Therefore it can have at most one solution, that is only if the inner part of the descriminant is less than or equal to 0

$$\sqrt{b^2-4ac}\leq 0$$

though in our case we have, using the fact that $\sqrt{x^2} = |x|$

$$\sqrt{4(a \cdot b)^2} - 4\|a\|^2\|b\|^2 \leq 0$$
$$4(a \cdot b)^2 - 4\|a\|^2\|b\|^2 \leq 0$$
$$(a \cdot b)^2 \leq \|a\|^2\|b\|^2$$
$$|a \cdot b| \leq \|a\|\|b\|$$

∎

6. We will prove $a \cdot e_j = a_j$

   Proof.
   we have $e_j = (0, \ldots, 1, \ldots, 0)$ where 1 is at the j-th entry then in the dot product we have

   $$a \cdot e_j = \sum_{i=1}^{n} a_i e_{j,i} = 0 + \ldots + a_j(1) + \ldots + 0 = a_j$$

   ∎

7. We will prove $e_j \cdot e_j = 1$

   $$e_j \cdot e_j = 0 \cdot 0 + \ldots + 1 \cdot 1 + 0 \cdot 0 \ldots + 0 \cdot 0 = 1$$

8. We will prove for $i \neq j, e_j \cdot e_i = 0$ We know that zero multiplied by anything is also 0, therefore the only chance that this sum is non-zero, is if e_j and e_i would have a non-zero value at the same entry, though by construction, it does not, therefore $e_j \cdot e_i = 0$.

Definition 17 (Subspace)
a subspace of euclidiean space $\mathbb{R}^n$ is a set V such that, if $a, b \in V$ then $c_1 a + c_2 b \in V$ for all $c_1, c_2 \in \mathbb{R}$. Observe, that $\vec{0} \in V$ always.

suppose A is $m \times n$ matrix

- If each of the n columns of A are linearly independent, then $\{Ax : x \in \mathbb{R}^n\}$ spans all of n dimensional space, we call the subspace consisting of all the linear combinations of A the column space or image of A.

- take $m = 3$ and $n = 2$, then we get a plane spanned by the two column vectors of A assuming they are linearly independent.

- If each of the m rows of A are linearly independent, then $\{x \in \mathbb{R}^n : Ax = 0\}$ is $(m - n)$ dimensional, since by the Rank Nullity Theorem, we know that the dimension of the null space added to the rank, gives n. Observe, that this is the sub space of vectors that are orthogonal to every row of A by the way that matrix vector multiplication is defined. Say we have $m = 2$ and $n = 3$ then the null space must be orthogonal to both of the row vectors who define a plane, and the null space is orthognal to every point on this plane.

Definition 18 (Cross Product)
Only in $\mathbb{R}^3$ we have a different way of multiplying two vectors. For a, b, the cross product (a vector) denoted as $a \times b$ id defined, algebraically by

$$a \times b = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$$

Geometrically, $a \times b = \vec{0}$ if a, b are linearly dependent , otherwise it is the unique vector that is orthogonal to both a and b with length given by

$$\|a \times b\|^2 = \|a\|^2\|b\|^2 - (a \cdot b)^2$$

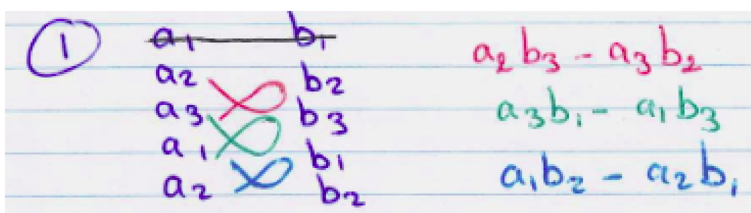
so that the parallelpiped formed by a, b and $a \times b$ is positively oriented.

We can verify that the definitions are the same algebraically, that $a \cdot (a \times b) = 0$ and $b \cdot (a \times b) = 0$

$$a \cdot (a \times b) = a_1a_2b_3 - a_1a_3b_2 + a_2a_3b_1 - a_2a_1b_3 + a_3a_1b_2 - a_3a_2b_1 = 0$$

$\|a \times b\|^2 = \|a\|^2\|b\|^2 - (a \cdot b)^2$, computations follow The determinant of $[a, b, a \times b]$ turns out to be the sum of squares...

To recall the definition for the cross product we can use the following diagram



2.1.4 Properties of the Cross Product

- $a \times b = -(b \times a)$
- $(cx + dy) \times z = (cx \times z) + (dy \times z)$ where $c, d \in \mathbb{R}$
- $\|a \times b\| = \|a\|\|b\| \sin(\theta)$ that is the length of $a \times b$ is equal to the area of the parallelogram generated by a and b
- The cross product is not associative, observe

  $$(i \times i) \times j = 0$$

  But

  $$i \times (i \times j) = -j$$

From Lecture

1. $b \times a = -(a \times b)$
2. $(\lambda a + b) \times c = \lambda(a \times b) + b \times c$
3. $a \times a = 0$
4. $\|a \times b\|^2 + (a \cdot b)^2 = \|a\|^2\|b\|^2$
5. $\|a \times b\| = \|a\|\|b\|\big|\sin(\theta)\big|$
6. $a \cdot (a \times b) = 0$ and $b \cdot (a \times b) = 0$
7. $i \times j = k, j \times k = i, k \times i = j$
8. $a \times (b \times c) = (a \cdot b)b - (a \cdot b)c$ also $(a \times b) \times c = (a \cdot c)b - (b - c)a$
9. $a \times (b \times c) + b \times (c \times a) + c \times (a \times b) = 0$
10. $(a \times b) \cdot c = det\left(\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\right)$

Proof Sketches

1. Expand the defn of $b \times a$ to get $x = (j - k, l - m, n - p)$, expanding def of $a \times b$ gies $k - j, m - l, p - n$ which is $-x$
2. Expand and factor out λ
3. yields $(j - j, k - k, o - o) = \vec{0}$
4.

Do this for practice

Chapter 3

Lecture 3

3.1 Visualizing Mult-Var

consider $f : \mathbb{R}^2 \to \mathbb{R}$

- We have the graph, that is

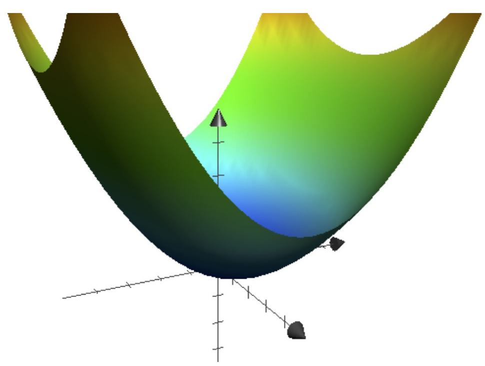
$$\Big\{(x, y, z) \in \mathbb{R}^3 : z = f(x, y)\Big\}$$

  Observe this yields a two-dimensional surface, with a given height z over a point (x, y)

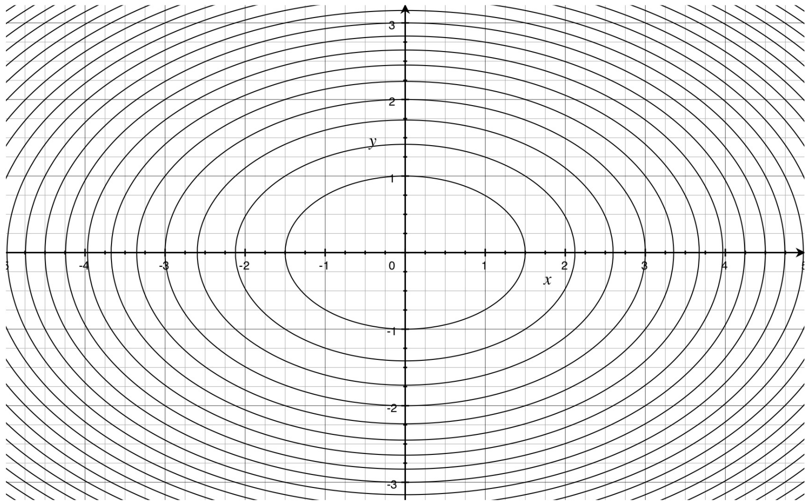
Definition 19 (Level Curve)
For $c \in \mathbb{R}$, the level curve (level set or contour plot) of $f : \mathbb{R}^2 \to R$ is

$$\big\{(x, y) \in \mathbb{R} : f(x, y) = c\big\}$$

Observe, we have the 3d graph of $f(x, y) = \frac{1}{9}x^2 + \frac{1}{4}y^2$



Then the level curve of the same function



Note that the values of c are chosen to be evenly spaced, thefeore we can observe when the curves are closer together we can see that the function changes faster.

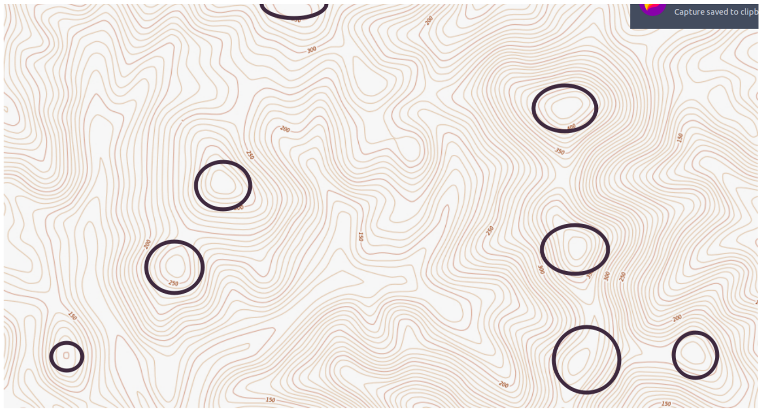
3.1.1 Level Curve Questions - Online

We'll match the graph to the level curve.

- for the first graph we note that it has flat sides, therefore if we were to slice through it, we will obtain squares, therefore this matches to the 2nd level curve
- The second has flat sides, with curved edges, therefore, though it's rotated $\frac{\pi}{2}$ from the previous graph, therefore this matches to the second last graph.
- The third graph is similar to the first though it has been pinched on the corners, therefore the contour plot, is the one that is similar ot the second, with the pinch, so we conclude that it is the last contour plot
- We see that the fourth graph is similar to the first, though it's sides are softer, so we conclude it matches to the first graph
- The last grpah looks like a cone, since it has no corners, therefore if we slice through it we should obtain circles, so the correct contour plot is the third.

3.1.2 Mountains and Valleys - Lecture

We note we are taking slices of of the mountains one at a time, and so just like slicing through a cone, if we observe concentric circles that decrease in radius it is most likely a mountain, if we can see that the elevation is increasing then it is for sure.



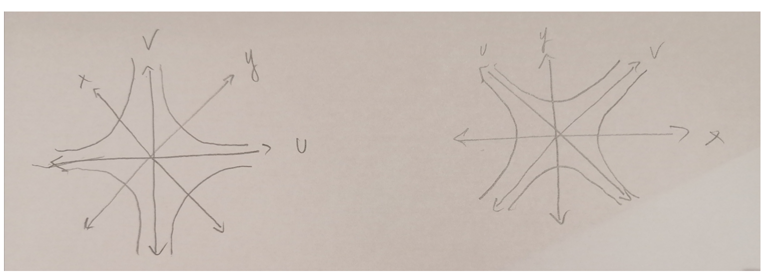
3.1.3 Level Curve Harder

Find some level curves of $f(x, y) = y^2 - x^2$

$$f(x, y) = y^2 - x^2 = (y + x)(y - x)$$

We then make a change of variable so

$$u = y - x \qquad\qquad v = y + x$$



3.1.4 3-Variable Functions

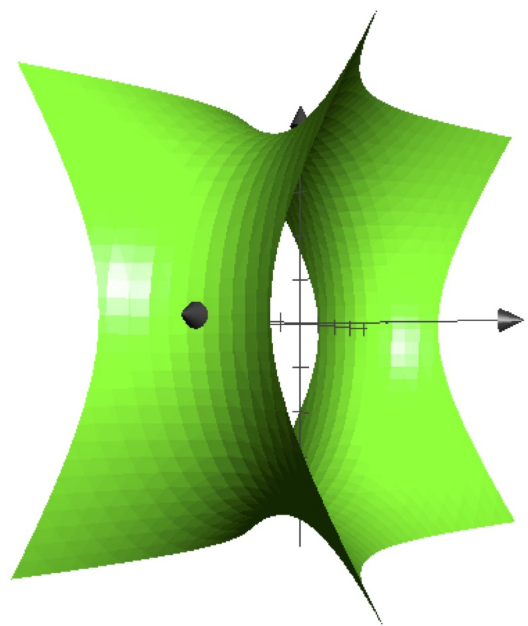
consider $f : \mathbb{R}^3 \to \mathbb{R}$ a graph can still make sense:

$$\left\{(x, y, z, w) \in \mathbb{R}^4 : w = f(x, y, z)\right\}$$

this s a 3 dimensional shape sitting inside 4-dimensional space and is very hard to visualize, therefore we can use a level set to attempt to describe it, that is

$$\Big\{(x, y, z) \in \mathbb{R}^3 : f(x, y, z) = c\Big\}$$

for different choices of c. We call these a level surface since they are 3-dimensional and let us visuzlize them easier, for the function $f(x, y, z) = \frac{1}{9}x^2 - \frac{1}{4}y^2 + \frac{1}{9}z^2$, the level surface for $c = -2$ is given below



Sometimes the level surfaces can be written in the form $z = f(x, y) - c$, and you can picture them as shifted graphs of a function of two variables. This situation is the simplest possible, so it may help you visualize what happens in higher dimensions, but it is rare.

What does this even mean?

3.1.5 4 or more

We can still define a graph for the function $f : \mathbb{R}^n \to \mathbb{R}$

$$\Big\{(x_1, x_2, \ldots, x_n, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} = f(x_1, x_2, \ldots, x_{n-1}, x_n)\Big\}$$

And also the level set

$$\big\{(x_1, x_2, dotsc, x_{n-1}, x_n) \in \mathbb{R}^n : f(x_1, x_2, \ldots, x_{n-1}, x_n) = c\big\}$$

Though it's impossible to imagine these for $n \geq 4$.

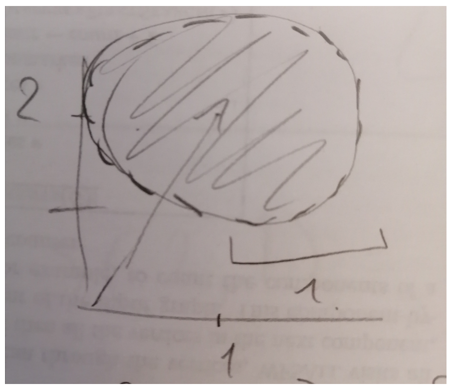
3.2 Topology

Definition 20 (Open Ball)
Let $a \in R^n$, $r \in \mathbb{R} > 0$, the open ball centered at a with radius r is

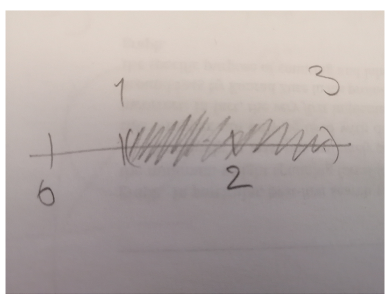
$$B(a,r) \coloneqq \left\{x \in \mathbb{R}^n : \|a - x\| \leq r\right\}$$

Example

- $B\big((1,2)1\big) \subseteq \mathbb{R}^2$



- $B(2,1) = (1,3) \subseteq \mathbb{R}$



- For these examples notice that there is no outer edge, like an onion with a thin layer peeled off.

Definition 21 (Closed Ball)
The closed ball centered at a, and radius r is

$$\bar{B}(a,r) \coloneqq \left\{x \in \mathbb{R}^n : \|a-x\| \leq r\right\}$$

Notice this differs from the open ball since it contains boundary points.

Definition 22 (Sphere)
The sphere with center a and radius r is

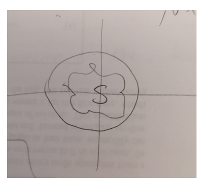
$$S(a,r) \coloneqq \left\{x \in \mathbb{R}^n : \|x-a\| = r\right\}$$

This is only the boundary points now.

Definition 23 (Bounded)
Let $S \subseteq \mathbb{R}^n$, S is bounded if there exists $r > 0$ such that $S \subseteq B\left(\vec{0}, r\right)$ formally that is

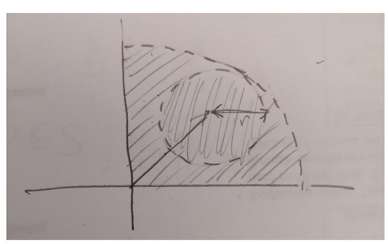
$$\exists r > 0, \forall x \in S, \|x\| \leq r$$

It's contained within an open ball.



Example We will prove $B(a,r) \subseteq \mathbb{R}^m$ is bounded.

Intuition, geometrically we have



Rough Work

$$\|a\| + r > \|x - a\| + \|a\| \tag{3.1}$$

From the triangle inequality

$$\|x - a\| + \|a\| \geq \|x\| = \|x\| \tag{3.2}$$

Proof.
The set that bounds $B(a, r)$ is $B\big(a, r + \|a\|\big)$ .Let $x \in B\left(\vec{0}, r\right)$ by definition we need to show that $\|x\| \leq r + \|a\|$ we have

$$\begin{aligned}
\|x\| &= \|x - a + a\| \\
&\leq \|x - a\| + \|a\| \\
&< r + \|a\|
\end{aligned}$$

∎

Definition 24 (Interior Point)
We say that $x \in \mathbb{R}^m$ is an interior point of S if

$$\exists \varepsilon > 0, B(x, \varepsilon) \subseteq S$$

This says that if you can have a ball around a point which is completely contained within S then it is on the interior.

Definition 25 (Interior)
The interior of S is

$$\mathring{S} \coloneqq \big\{x \in \mathbb{R}^n : \exists \varepsilon > 0, B(x, \varepsilon) \subseteq S\big\}$$

that is the set of all points that are interior to S

Definition 26 (Closure Point)
we say that $x \in R^n$ is a closure point of S if

$$\forall \varepsilon > 0, B(x, \varepsilon) \cap S \neq \emptyset$$

So points that lay on the boundary or are inside, must necessarily have this property

Definition 27 (Closure)
The closure of S is

$$\bar{S} := \left\{x \in \mathbb{R}^n : \forall \varepsilon > 0, B(x, \varepsilon) \cap S \neq \emptyset\right\}$$

that is, the set of all closure points of S

Theorem 1 (Connection between Closure and Interior)
For any S, we have

$$\mathring{S} \subseteq S \subseteq \bar{S}$$

Proof.
First we will prove $\mathring{S} \subseteq S$.

- Let $x \in \mathring{S}$ by definition

$$\exists \varepsilon_0 > 0 \text{ such that } B(x, \varepsilon_0) \subseteq S$$

  But note $x \in B(x, \varepsilon_0) \subseteq S$ therefore $x \in S$

We will prove $S \subseteq \bar{S}$

- let $x \in S$ we know for all $\varepsilon > 0$ that

$$x \in B(x, \varepsilon) \cap S$$

  so then the intersection of $B(x, \varepsilon) \cap S \neq 0$ so $x \in \bar{S}$

∎

Definition 28 (Boundary)
The boundary of S is
$$\partial S \coloneqq \bar{S} \setminus \mathring{S}$$

3.2.1 Slide Questions

1. $\mathring{S} = (0,3), \bar{S} = [0,3], \partial S = \{0,3\}$

2. $\mathring{S} = (0,3), \bar{S} = [0,3], \partial S = \{0,3\}$

3. $\mathring{S} = (0,3), \bar{S} = [0,3], \partial S = \{0,3\}$

4. $\mathring{S} = (0,3), \bar{S} = [0,3], \partial S = \{0,3\}$

5. $\mathring{S} = \emptyset, \bar{S} = \{0,3\}, \partial S = \{0,3\}$

6. $\mathring{S} = \mathbb{R} \setminus \{0\}, \bar{S} = \mathbb{R} = \{0\}$

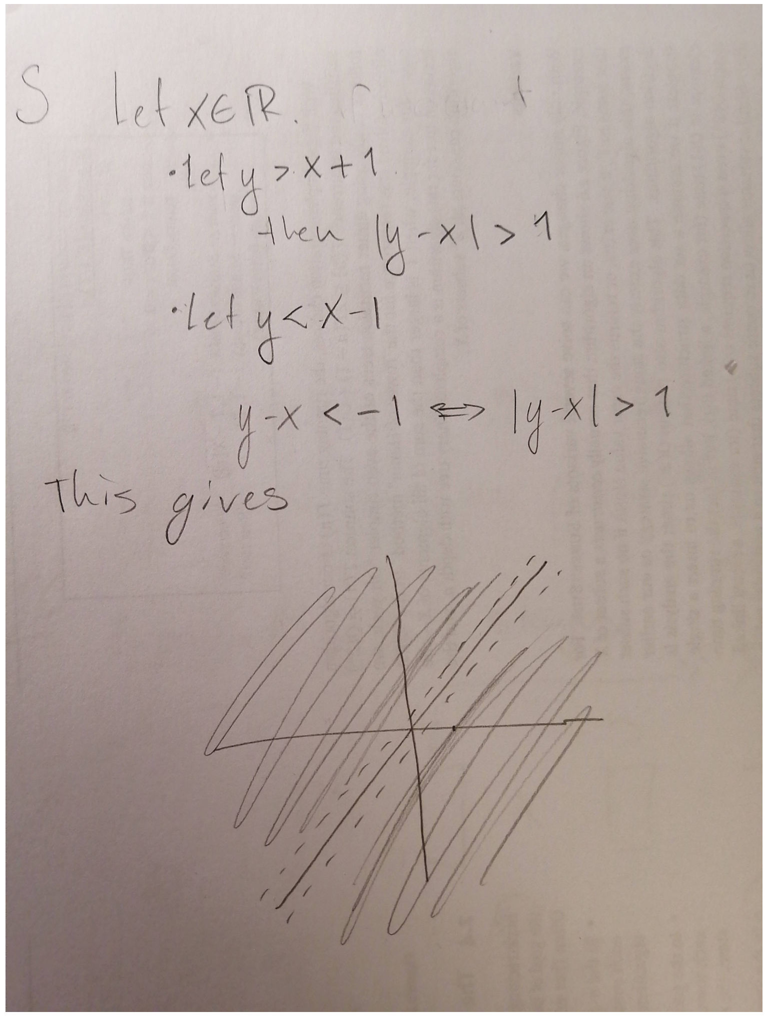
7. $\mathring{S} = (-\infty,0), \bar{S} = (-\infty,0], \partial S = \{0\}$

8. $\mathring{S} = (-\infty,0) \cup (3,\infty), \bar{S} = (n\infty,0] \cup [3,\infty), \partial S = \{0,3\}$

9. $\mathring{S} = (0,3) \times \{0\}, \bar{S} = [0,3] \times \{0\}, \partial S = \big\{(0,0),(3,0)\big\}$

10. $\mathring{S} = \emptyset, \bar{S} = [0,1], \partial S = \bar{S}$

11. First we go geometric



So $\mathring{S} = S, , \bar{S} = \big\{(x,y) \in \mathbb{R}^2 : |y-x| \geq 1\big\}, \partial S = \big\{(x,y) \in \mathbb{R}^2 : |y-x| = 1\big\}$

12. Note that we can find $x \in \mathbb{Q}$ as small as we want, so then we get gaps as small as we want, so we get $[0,1)$ with infinitely small holes in it. So $\mathring{S} = \emptyset, \bar{S} = [0,1), \partial S = [0,1)$

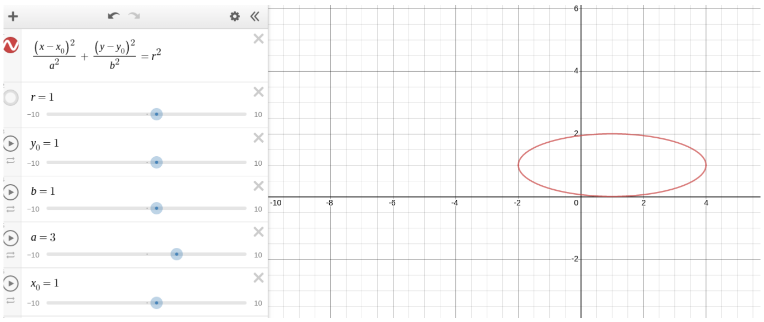
13. $\mathring{S} = B\Big(\vec{0},1\Big), \bar{S} = \bar{B}\Big(\vec{0},1\Big), \partial S = S\Big(\vec{0},1\Big)$

14. $\mathring{S} = \mathbb{R}^n, \bar{S} = \mathbb{R}^n, \partial S = \emptyset$

15. We know that no element is in $\emptyset$ so clearly no ball contained within $\emptyset$ so $\mathring{S} = \emptyset$, Also the intersection of anything and $\emptyset$ is also empty so $\bar{S} = \emptyset$, $\partial S = \emptyset$

3.3 Preliminary Questions

- We can see that that a effects the horizontal stretch and so b the vertical, y_0 and x_0 for the translation, r gives the general radius before any stretching occur, if r where 0 then we require the left hand side to be zero and so they only value that satisfies is (x_0, y_0)



- Keep thinking

- if there exists an $a \in \mathbb{R}$ such that $\vec{r1} = a\vec{r2}$

Chapter 4

Homework Week 1

1. We know that $T_A : R^7 \to R^6$ and that $null(A)$ is the set of elements from $\mathbb{R}^7$ that get mapped to $\vec{0}$
   (a) So $null(A) \subseteq \mathbb{R}^7$
   (b) elements of the null space are in $\mathbb{R}^7$ no chance to be a sub space in $\mathbb{R}^6$
   (c) The codomain is what it outputs, we already knwo it is a subset of the domain .
   (d) Yes, the null space is a sub space also note that since A represents a linear transformation that $T_A(0) = 0$ and so $A\vec{0} = 0$ and so $\vec{0} \in null(A)$.
2. If the rank is 4 and the nullspace has dimension 2 by the Rank Nullity Theorem we know there are 6 columns to A the column space is the linear combinations of the columns so if it is a sub space of $\mathbb{R}^5$ that means each column has 5 vertical entries so we can say this matrix is 5×6
3. Visual Inspection for the rest of the questions