MAT247 - Probability with Computer Applications

Callum Cassidy-Nolan

September 16, 2019

Contents

List of Definitions

List of Theorems

Chapter 1

Lecture 1

1.1 Stats vs Probability

Statistics

- Reverse Probability
- Making observations and then based on probability we describe the original model
- Critique whether a model is correct

Probability

- How likely based on a theoretical situation, for example a coin toss has a 50/50 chance of one of the outcomes.
- We know all the parameters.

1.2 To do Well

- Understand the formulas
- Make sure to the hard questions
- Office hours in HS 386

1.3 Useful Terminology

Definition 1 (Random Experiment)
A process of gathering data or observations. We can perform the experiment multiple times so long as the conditions aren't changed and the outcome of each experiment is random, we don't know what the result will be, though we know the set of possible outcomes.

Examples

- Rolling a die, and observing the number on the top face.
- Rolling n dice and observing the resulting pair of numbers.
- Drawing 3 cards from a deck of cards
- Asking a professor how old they are

Definition 2 (Sample Space)
This is the set of all possible outcomes/results from a random experiment , we denote this set as

$$\Omega \text{ or } S$$

The sample space depends on the outcome of interest.

Note 1
If our outcome of interest is say, whether after flipping a coin it is heads or tails, we don't care how many times it turned in the air before landing on the ground.

Examples

1. $S = \{n \in \mathbb{N} : 1 \leq n \leq 20\}$. All the faces of the dice.
2. $\Omega_1 = \{ \text{ True }, \text{ False } \}$
3. $\Omega_2 = \mathbb{R}^{\geq 0}$. If they are precise in their measurement, though more likely to any hour between 0 and 10.

Definition 3 (Event)
Any subset of our sample space of interest.

- *Simple Event: One with exactly one outcome*
- *Compound Event: One with multiple outcomes*

Simple Event

- flipping a coin, and observing whether it has landed on it's edge (that's possible!)

Compound Event

- flipping 3 coins, and observing whether at least one has landed on it's edge.

Definition 4 (Complement)
The complement of an event A is the event consisting of outcome that are nto in A. We denote this as A^c .
For example: *the complement of our previous example, that would be if no, coin has landed on it's edge.*

Definition 5 (Union)
The union of two events A and B is the event consisting of outcomes in A or B or both. Denoted as $A \cup B$

- *The union of events is usually said as A or B*
- *Observe $AUA^c = \Omega$*

Definition 6 (Intersection)
The intersection of two events A and B is the event consisting of outcomes in both A and B, written as $A \cap C$

- *The intersection of events is described by A and B*
- *We see $A \cap A^c = \emptyset$*

Definition 7 (Disjoint)
events A and B are disjoint, if they have no overlapping events , that is $A \cap B = \emptyset$

1.4 Event Laws

- Commutative Law: $A \cup B = B \cup A$
- Associative Law:

$$(A \cup B) \cup C = A \cup (B \cup C)$$
$$(A \cap B) \cap C = A \cap (B \cap C)$$

- Distributive Law: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (This one will be useful for proving $P(A \cup B \cup C) = P(A) + P(B) + \ldots + P(A \cap B \cap C)$)

1.5 Mutually Exclusive

Definition 8 (Mutually Exclusive)
Two events are mutually exclusive if the events cannot both occur. That is if event A occurs, event B cannot occur. In a Venn Diagram A and B being disjoint means that $A \cap B = \emptyset$.

Example

- The events $A = \{$ Roll and even # $\}$ and $B = \{$ Roll and odd # $\}$ from rolling a die are mutually exclusive, since a number is either even or odd, not both.

1.6 Independence

Definition 9 (Independent)
Two events A and B are independent if the occurance of one event does not affect the occurance of the other in any way.

Example

- The events
  - $C = \{$ Roll an even # on the first toss $\}$ and

– $D = \{$ Roll an odd # on the second toss$\}$

are independent of each other.

mutually exclusive events must be dependent recall that two mutually exclusive events cannot both occur, therefore if one of the events is true, it forces the other to be false, and so the occurance of one does affect the occurance of the other.

Example 1.6.1
Are all dependent events mutually exclusive ? recall, if A is mutually exclusive of B, that is

$$A \text{ occurs } \implies P(B) = 0$$

and if A is dependent on B

Should this be swapped?

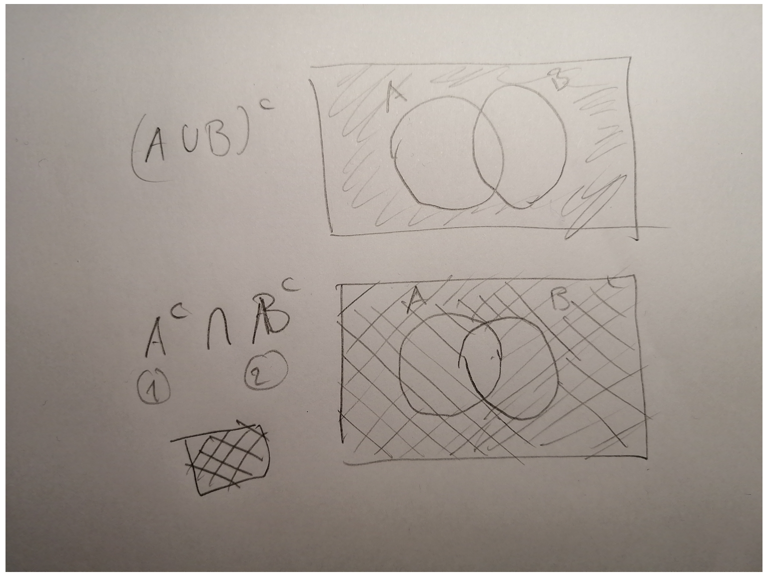
$$A \text{ occurs } \implies P(B \text{ given } A \text{ ocurred }) \neq P(B)$$

Example *: Let A be the event that the person next to me is sick, and B the event that I am sick, note the outcome of interest is sickness. We can see that if indeed the person next to me is sick then the probability of me being sick shall increase. Though they are not mutually exclusive of eachother, if I am sick, it is possible that the person next to me is as well.*

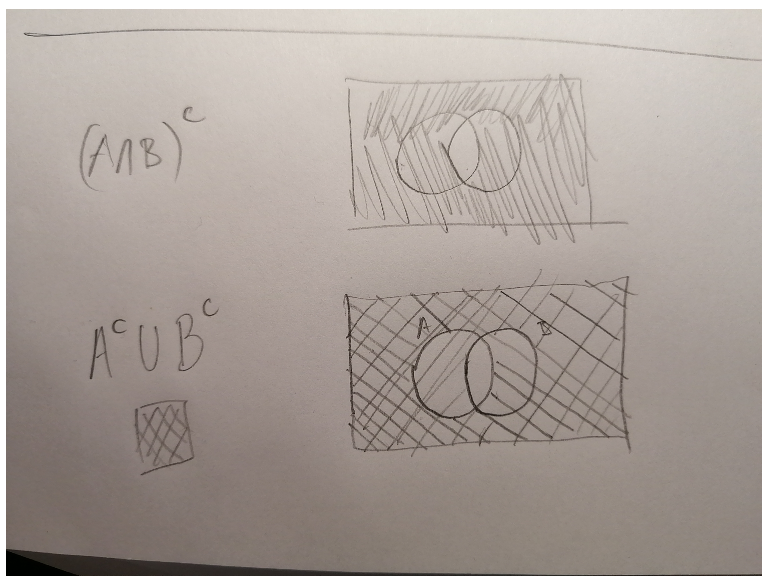
1.7 DeMorgan's Laws

From Venn Diagrams, some results follow, between union and intersection of events.

- $(A \cup B)^c = A^c \cap B^c$, saying Not in either A nor B is the same as not in A and not in B



- $(A \cap B)^c = A^c \cup B^c$, that is not both and and B, is not in A, or not in B or not in either A or B



The general version we have for set events $\{A_1, A_2, \ldots, A_n\}$

- $\left(\bigcup_{i=1}^n A_i\right)^c = \bigcap_{i=1}^n A_i^c$
- $\left(\bigcap_{i=1}^n A_i\right)^c = \bigcup_{i=1}^n A_i^c$

1.8 Sample Space Examples

1. Flipping a coin 3 times in a row

$$\Omega = \{HHH, HHT, HTH, HTT, TTT, TTH, THT, THH\}$$

observe that getting two heads, means you also got one tail, therefore $P(\text{ Two H }) = P(\text{ One T }) = \frac{3}{8}$

2. Flipping a fair coin 3 times and measuring the number of tails

$$\Omega = \{0, 1, 2, 3\}$$

Not all of these are equally as likely, as there are multiple ways of getting just 1 tail, ex, HHT and HTH

3. Number Selection of an individual's Lotto649 ticket

$$\Omega = \big\{(a, b, c, d, e, f) : 1 \leq a, b, c, d, e, f \leq 49 \in \mathbb{Z}\big\}$$

note $n(\Omega) = 6^{49}$ since there are 49 options for each position

1.9 Event Examples

1. Tossing two different dice. The event of tossing a double

$$D = \big\{(1, 1), (2, 2,), \ldots, (6, 6)\big\}$$

2. Tossing such that the sum of the two dice is 8

$$S = \big\{(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)\big\}$$

observe that the first index of the tuple is the first dice and the second position is for the second dice, therefore $(2, 6) \neq (6, 2)$

3. Tossing doubles or tossing a sum of 8

$$C = A \cup B$$

4. Tossing doubles and tossing a sum of 8

$$D = A \cap B = \big\{(4, 4)\big\}$$

1.10 Exclusive & independence Examples

a) If we know that the course subject is comp sci, then we can assume that the percntage of males will be higher, if we do not know the course subject, then we have no idea, then we don't know if the percentage of males will be higher. Therefore Dependent.

b) The fact that the coin is baised doesn't matter, every time the coin is tossed, it is still the same probability, therefore getting a head on the first toss tells us nothing about what will happen on the second toss. Independent

c) If we have someone who doesn't like hiking, then we know that the probability of them climbing mount everest, is lower than just a normal random individual climbing mount everest,therefore by definition it is dependent, though note it is not mutually exclusive since even if they don't like hiking, maybe they want to conquer their fear, so they climb the mountain.

d) dependent , $P($ Red Card $) = \frac{1}{2}$ and $P($ Red Card if QoH $) = 1$ therefore by definition we know it is dependent . Note if G occurs then , $P($ H given G $) > \frac{1}{52}$ since one card has been removed

1.11 Intro to Probability

Definition 10 (Probability)
In a random experiment with sample space Ω, the probability of an event A, denoted $P(A)$ is a function that assigns to event A a numerical value that measures the chance that event A will occur.

1.11.1 Axioms of Probability

1. $P(A) \geq 0$, negative probability doesn't make sense

2. $P(\Omega) = 1$, that is the probability for anything to happen must be 100

3. For a set of disjoint events (that is they are mutually exclusive) $A_1, A_2, \ldots, A_{n-1}, A_n$ in Ω

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$$

this says that probability of any of them happening, is the sum of each probability individually. note this one is very useful for proving things

1.11.2 probability proof

We'll show the complement relationship

Proof.
Recall $A \cup A^c = \Omega$, and $A \cap A^c = \emptyset$, it follows that

$$P(\Omega) = P(A \cup A^c)$$
$$1 = P(A) + P(A^c)$$
$$P(A) = 1 - P(A^c)$$

■

Example If there is a 30% chance it will rain tommorow, we can conclude that there is a 70% chance that it will not.

1.12 Inculsion/Exclusion Principle

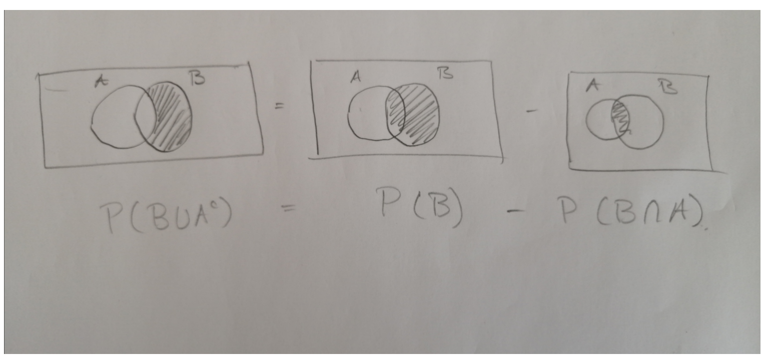
We will prove

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

intuitively, this makes sense as we must subtract what we have double counted.

Before we continue we will prove a lemma, that is

$$P(B \cup A^c) = P(B) - P(B \cap A)$$



Before we continue with the proof, observe the following from above,

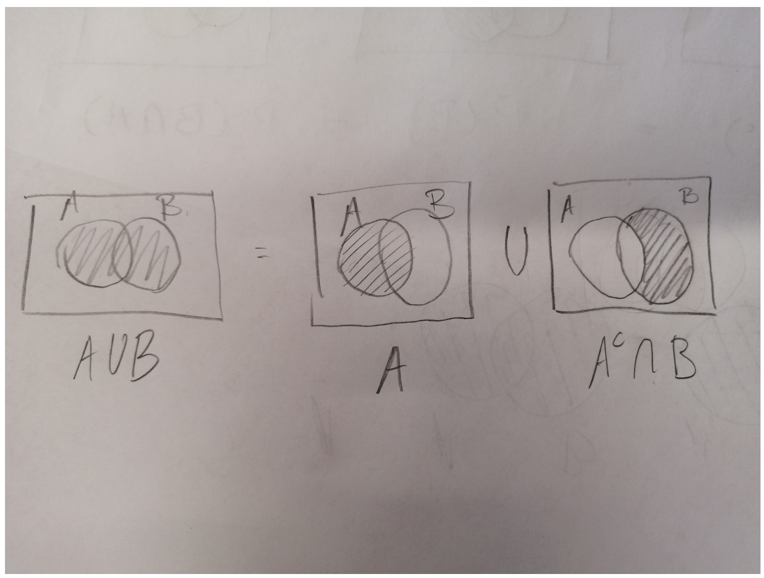
$$P\big((B \cup A^c) \cup (P \cap A)\big) = P(B)$$

Though note, that $(B \cup A^c)$ and $(B \cap A)$ are disjoint, not proof but visually we can observe it. By axiom 3, we know

$$P(B) = P\big((B \cup A^c)\big) + P(B \cap A) \Leftrightarrow P(B \cup A^c) = P(B) - P(B \cap A)$$

We will now continue to the main proof

Proof.
Observe $A \cup B = A \cup (B \cap A^c)$

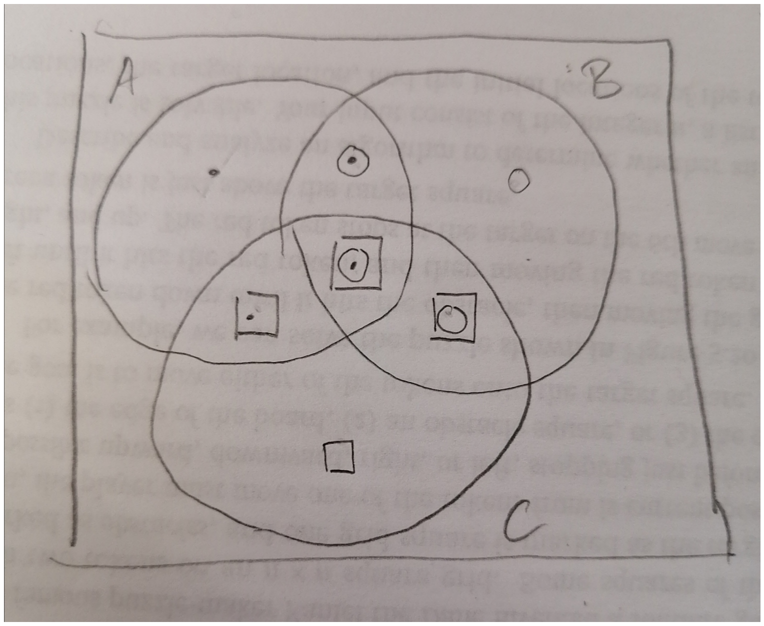


Therefore we know that A and $A^c \cap B$ are disjoint, it follows that

$$\begin{aligned} P(A \cup B) &= P\big(A \cup (B \cap A^c)\big) \\ &= P(A) + P(A \cap A^c) && \text{(By axiom 3)} \\ &= P(A) + P(B) - P(A \cap B) && \text{(By our lemma)} \end{aligned}$$

■

1.12.1 Triple Example



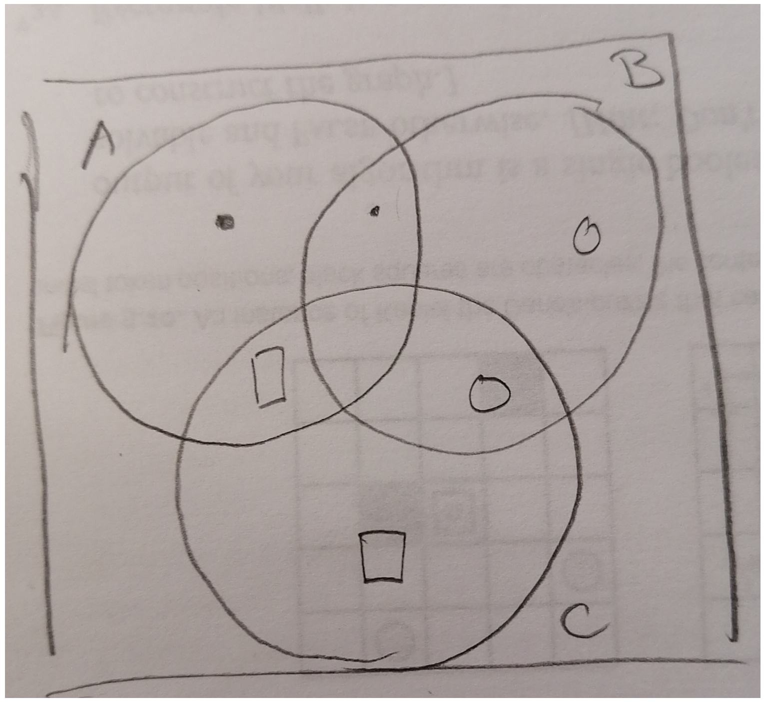
In this example we want to find $P(A \cup B \cup C)$, observe if we find $P(A) + P(B) + P(C)$ then clearly we have over counted, but by how much? Let's find out,

- From the dots and circles we can see that $A \cup B$ has been double couned.

- From the squares and circles, $B \cup C$ has been double counted

- From the dots and squares we see that $A \cup C$ has been double counted

- Therefore we must subtract both circles from $A \cup B$, both squares from $B \cup C$ and both dots from $C \cup A$



After doing so, we see that we have removed all counts of $A \cup B \cup C$ therefore we must add it back, giving us the formula

$$P(A \cup B \cup C) = P(A) + P(B) + P(B) - P(A \cup B) - P(B \cup C) - P(C \cup A) + P(A \cup B \cup C)$$

Then for n events we get

Definition 11 (Inclusion Exclusion Principle)

$$
\begin{aligned}
P\left(\bigcup_{i=1}^{n} A_i\right) &= \sum_{i=0}^{n} P(A_i) \\
&- \sum_{1\le i<j\le n}^{n} P\big(A_i \cap A_j\big) \\
&+ \sum_{1\le i<j<k\le n} P\big(A_i \cup A_j \cap A_k\big) \\
&\vdots \\
&+ (-1)^{n-1} P\left(\bigcup_{i=1}^{n} A_i\right)
\end{aligned}
$$

The left hand side is the probability of the union of n events

- On the first line of the right hand side we count the probability of being in each set invidually and add it up, though if the sets A_i are not disjoint, the some elements have been counted more than once.
- therefore on the second line we must subtract anything that was double counted, that is some of the $P\big(A_i \cap A_j\big)$ may be 0, though if not then we are in fact removing double counts. Since $A_j \cap A_i = A_i \cap A_j$, to avoid duplications we decide to consider only pairs (A_i, A_j) with $i < j$.
- On the previous line we also may have double subtracted though, consider an event who is in $A_i \cup A_j \cup A_k$ it would have been counted for in $A_i \cup A_j$ in $A_i \cup A_k$ and $A_j \cup A_k$ therefore we add back any elements that had been subtracted more than once
- The process contines until the end.

1.12.2 Inclusion Exclusion Examples

In our class of 200, suppose 87 students had seen Bandersnatch and 45 students had seen Bird Box, while 78 students had watched neither of these two films. Determine the number of students who...

Watched Both movies

- Let A be the event of watching Bandersnatch, and B the event of watching Bird Box we want to find the number of people, we need $n(A \cap B)$ though recall we don't have a formula for that, though we do have one for $n(A \cup B)$, that is

$$n(A \cup B) = n(A) + n(B) - n(A \cap B) \Leftrightarrow n(A \cap B) = n(A) + n(B) - n(A \cup B)$$

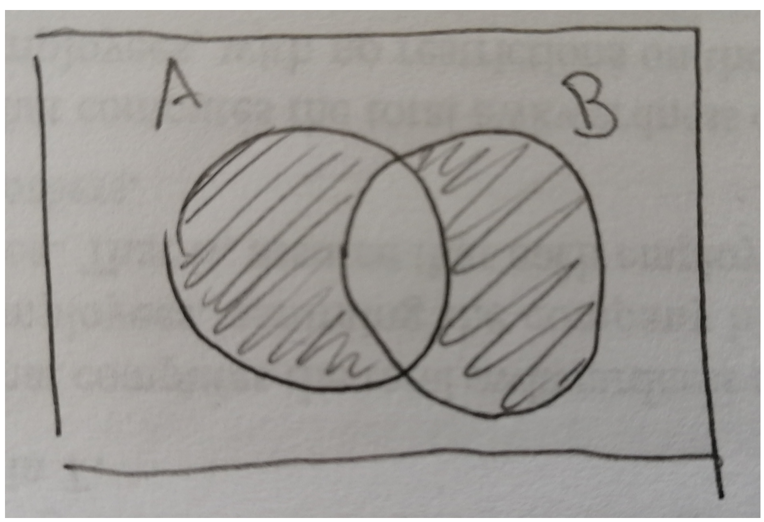
We also note that

$$n(A \cup B) = 200 - n\big((A \cup B)^c\big) = 200 - 78 = 122$$

thus we can continue we have

$$\begin{aligned} n(A \cap B) &= n(A) + n(B) - n(A \cup B) \\ &= 87 + 45 - 122 \\ &= 132 - 122 \\ &= 10 \end{aligned}$$

Watched only one of the two movies We want to find the two lunes of A and B on the Venn Diagram, like so



We can write this as $n\big((A^c \cap B) \cup (A \cup B^c)\big)$ we observe that they are disjoint therefore we have $n\big((A^c \cap B) \cup (A \cup B^c)\big) = n(A^c \cap B) + n(A \cap B^c)$

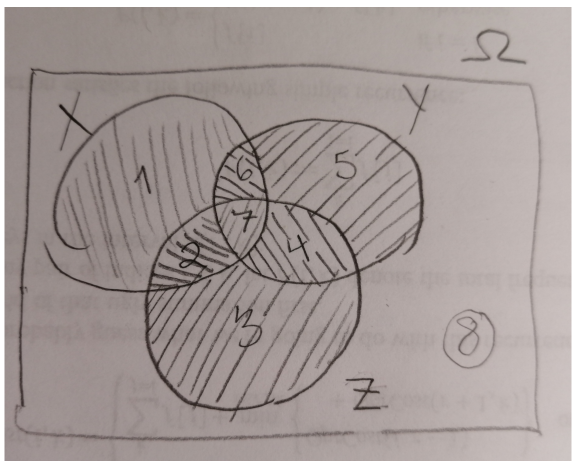
- Note that $n(A^c \cap B) = n(B) - n(A \cap B)$ and $n(A \cup B^c) = n(A) - n(A \cap B)$, so we continue with the calculation to get

$$\begin{aligned} n(A^c \cup B) + n(A \cup B^c) &= n(B) - n(A \cap B) + n(A) - n(A \cap B) \\ &= 45 - 10 + 87 - 10 \\ &= 112 \end{aligned}$$

Did not watch Bandersnatch By the completment relation ship we have

$$n(A^c) = 200 - n(A) = 200 - 87 = 113$$

1.12.3 Disjoint Events on a Venn Diagram

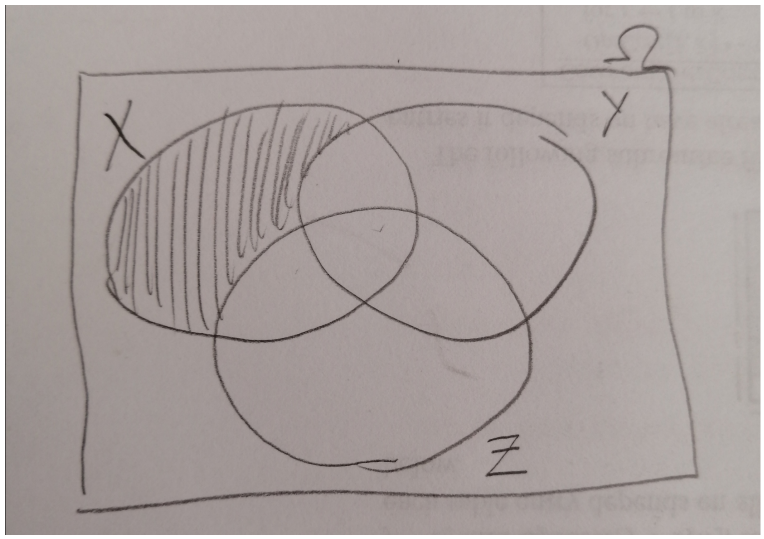


Two events are disjoint if their intersection is $\emptyset$ the intersection between any of $1, 2, \ldots, 7$ are all $\emptyset$ so the disjoint events are

1. $(Y \cup Z)^c \cap X$
2. $(X \cap Y) \cap Y^c$
3. $(X \cup Y)^c \cap Z$
4. $(Z \cap Y) \cap X^c$
5. $(Z \cup X)^c \cap Y$
6. $(X \cap Y) \cap Z^c$
7. $(X \cap Y) \cap Z$

8. $(X \cup Y \cup Z)^c$

We will now sketch the region $(Y^c \cap Z^c) \cap X$, for the left part, we know that Y^c is everything not in Y and that Z^c is everything not in Z so then $Y^c \cap Z^c$ is everything not in Z nor in Y then we require that it is also in X which yields. Formally, by demorgan's laws we know $(Y^c \cap Z^c) = (Z \cup Y)^c$, which is faster



1.12.4 Denoting Events & More Calculation

Question In a class of 60 students, 13 could not roll their tongue, 17 had attached earlobes, and 10 could roll their tongues and had attached earlobes. A student is randomly selected from the class. Let T denote the event that the student can roll their tongue, and E denote the event that they have attached earlobes. Symbolically denote the following events and identify the number of students in each.

- The student can roll their tongue

$$n(T) = 60 - n(T^c) = 60 - 13 = 47$$

- The student can neither roll his or her tongue nor has attached earlobes.

$$n(T^c \cap E^c) = n\big((T \cup E)^c\big)$$

By DeMorgan's Laws, then

$$n\big((T \cup C)^c\big) = 60 - n(T \cup C) = 60 - 10 = 50$$

- The student has attached earlobes but cannot roll his or her tongue. Symbolically, we have $n(E \cap T^c)$. Observe that $E = (E \cap T^c) \cup (E \cap T) \Leftrightarrow n(E) = n(E \cap T^c) + n(E \cap T)$, therefore

$$n(E \cap T^c) = n(E) - n(E \cap T) = 17 - 10 = 7$$

- The student can roll his or her tongue or has attached earlobes, but not both.
  - Symbolically:

    $$n\big((E \cap T^c) \cup (E^c \cap T)\big) = n(E \cap T^c) + n(E^c \cap T)$$

    since they are disjoint .
  - We must find what $n(E^c \cap T)$ is, similarly from the previous question we have

    $$n(E^c \cap T) = n(T) - n(T \cap E) = 47 - 10 = 37$$

  - thus we have

    $$n(E \cap T^c) + n(E^c \cap T) = 7 + 37 = 44$$

1.12.5 Shows Question

Question In a class of 300 students, 147 students have watched Stranger Things (S), 111 students have watched Game of Thrones (G), and 59 students have watched The Haunting of Hill House (H). There are 68 students have watched both Stranger Things and Game of Thrones, 31 students have watched both Game of Thrones and The Haunting of Hill House, while 10 students have watched all three shows and 79 students have watched none.

What we know

$n(\Omega) = 300$	$n(S) = 147$	$n(G) = 111$	$n(H) = 59$
$n(S \cap G) = 68$	$n(G \cap H) = 31$	$n(S \cap G \cap H) = 10$	$n\big((S \cap G \cap H)^c\big) = 79$

a) How many students have watched only The Haunting of Hill House and Game of Thrones.

- Symbolically:

  $$n\big((G \cap H) \cap (G \cap S \cap H)^c\big)$$

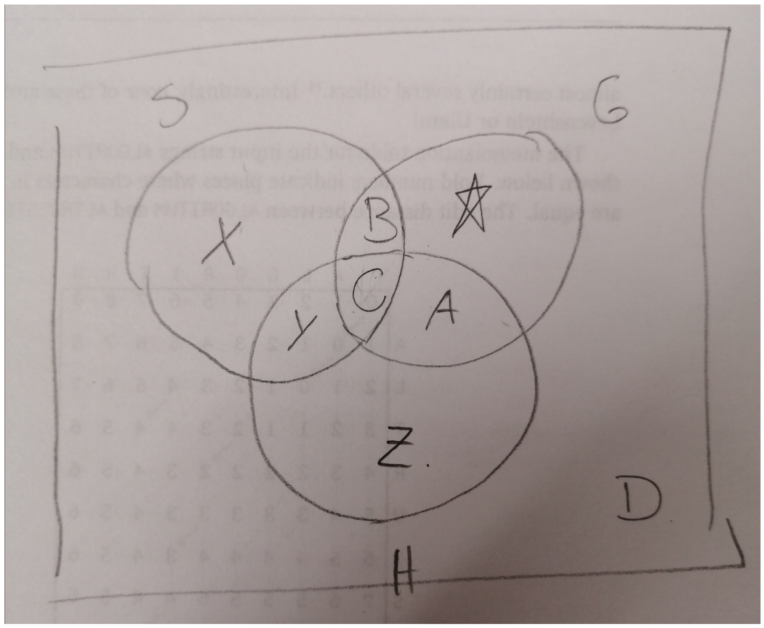
- Observe

$$\begin{aligned} n(G \cap H) &= n\Big(\big((G \cap H) \cap (G \cap S \cap H)^c\big) \cup (G \cap S \cap H)\Big) \\ &= n\big((G \cap H) \cap (G \cap S \cap H)^c\big) + n(G \cap H \cap S) \qquad \text{(Axiom 3)} \end{aligned}$$

$\therefore$

$$\begin{aligned} n\big((G\cap H)\cap(G\cap S\cap H)^c\big) &= n(G\cap H) - n(G\cap H\cap S) \\ &= 31 - 10 \\ &= 21 \end{aligned}$$

b) Illustrate with a Venn diagram and complete the missing numbers. You may need to solve a simple system of linear equations.



Let $k =$ the star , we know

$$k \cup (S\cap G)\cup(G\cap H) = G$$

and that k is disjoint from any other set so we get

$$n(k) + n\big((S\cap H)\cup(G\cap H)\big) = n(G)$$

And so by the inclusion exclusion principle we have

$$n(k) + n(S\cap G) + n(G\cap H) - n(S\cap G\cap H) = n(G)$$

Finish this

so then we can isolate for $n(k)$ to find the answer

c) If a student were randomly selected, what is the probability that the student wants to watch none of the three shows?

- From the previous question we know $n(X, Y, Z, \dots A)$ that is the total number of students in each disjoint event therefore the sum of each gives $n(S \cup G \cup H) = l$ and so the probability is $\frac{300-l}{300} \cdot 100$

d)

finish

1.13 Suggested Exercises

1.13.1 2.8

Info:

$n(\Omega) = 50$	$n(D) = 25$
$n(D \cap C) = 7$	$n\big((D \cup C)^c\big) = 18$

a) $C^c \cap D$, we note

$$(C^c \cap D) \cup (C \cap D) = D$$

therefore by axiom 3 we have

$$n(C^c \cap D) + n(C \cap D) = D$$

therefore

$$n(C^c \cap D) = 25 - 7 = 18$$

b) C , we have

$$C^c = (D \cap C)^c \cup (C^c \cap D)$$

Note the two sets on the right hand side are disjoint so by axiom 3 and by the complement relationship we have

$$\begin{aligned} n(C^c) &= n\big((D \cup C)^c\big) + n(C^c \cap D) && \text{(By axiom 3)} \\ 50 - n(C) &= 18 + 18 && \text{(Complement)} \\ n(C) &= 14 \end{aligned}$$

c) $D^c \cup C$ we have

$$(D^c \cap C) \cup (D \cap C) = C$$

and so

$$n(D^c \cap c) + n(D \cap C) = n(C)$$
$$n(D^c \cap C) = 17 - 4 = 7$$

d) $(D^c \cap C) \cup (D \cap C^c)$, then since they are disjoint

$$n\big((D^c \cap C) \cup (D \cap C^c)\big) = n(D^c \cap C) + n(D \cap C^c) = 18 + 7 = 25$$

1.13.2 2.9

a) sample space, rename to 1, 2, 3, 4

$$\{\{1,2\},\{1,3\}\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}$$

b) A contains 3 different outcomes, that is the bm with each of the other 3 in pairs

c) Every event is a choice of two, if we have a gup, but no gold or neon, then we have bm and gup, which there is one way to choose.

1.13.3 2.10

a) The set containing each pair of possible choices for applicants, it's size is given by $\binom{5}{3}$

b) Let J be the event that Jim is chosen, and D the event Don is chosen then we know we are looking for $J \cup D$ we know

$$n(J \cup D) = n(J) + n(D) - n(J \cap D)$$

We know that $n(J) = 4$ since he can be paired with the 4 others and the same for $n(D)$ and there is only 1 way to get $n(J \cap D)$ so we have

$$n(J \cup D) = 4 + 4 - 1 = 7$$

c) We remove the J and D case we get 6
Formally $J \cup D \setminus J \cap D$ or $(J \cap D^c) \cup (J^c \cap D) = X$ we know

$$X \cup J \cap D = J \cup D$$

therefore since they are disjoint we have

$$n(X) = n(J \cup D) - n(J \cap D) = 7 - 1 = 6$$

d) $A \cap B^c$ that is there is at least one male selected but not exactly one

finish

e)

1.13.4 2.12

Let A be the event that the student drank alcohol, and T be the event that a student used tobacco.

a) We must find $P(A \cap T)$, by re-arranging the inclusion/exclusion principle

$$P(A \cap T) = P(A) + P(T) - P(A \cup T) = .84 + .33 - .86 = .31$$

therefore 31%.

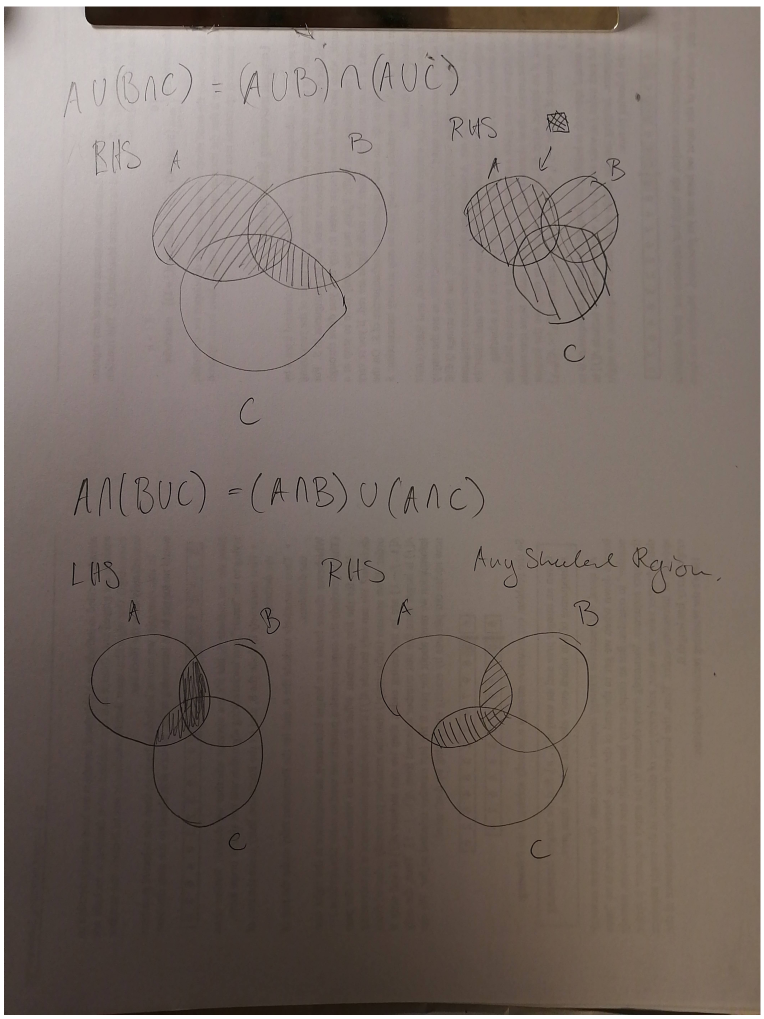
b) $(A \cup T)^c$ recall $P\big((A \cup T)\big) = .86$ therefore $P\big((A \cup T)^c\big) = 1 - .86 = .14$ so 14%

c) $X = (A^c \cap T) \cup (A \cap T^c)$ recall from 2.10 part c we have

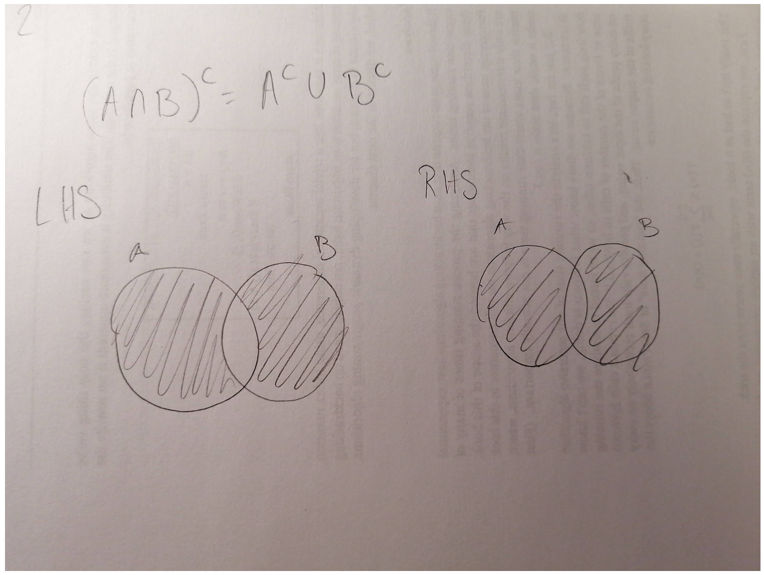
$$P(X) = P(A \cup T) - P(A \cap T) = .86 - .31 = .55$$

so 55%

1.13.5 2.15



1.13.6 2.16



1.13.7 2.20

a) Recall
$$(A \cap B^c) \cup (A \cap B) = A$$
therefore
$$P(A) = P(A \cap B^c) + P(A \cap B) = .30 + .10 = .4$$
thus the probability is 40%

b) Recall
$$A \cup B = \big((A^c \cap B) \cup (A \cap B^c)\big) \cup (A \cap B)$$
therefore
$$P(A \cup B) = P(A^c \cap B) + P(A \cap B^c) + P(A \cap B) = .30 + .40 + .10 = .80$$
thus from the complement relationship $P\big((A \cup B)^c\big) = .20$, remember our goal is to find $P(B^c)$, note $B^c = (A \cup B)^c \cup (A \cap B^c)$ thus
$$P(B^c) = P\big((A \cup B)^c\big) + P(A \cap B^c) = .80 - .40 = .40$$
So the probability is 40%

c) We need $A^c \cup B^c$ by DeMorgan's law we have $(A \cap B)^c$ and by the complement relationship it follows that $P\big((A \cap B)^c\big) = 1 - P(A \cap B) = 1 - .1 = .90$ therefore the probability is 90%

d) We are looking for $A^c \cap B^c$ by DeMorgan we have $(A \cup B)^c$ by the complement relationship we have $P\big((A \cup B)^c\big) = 1 - P(A \cup B) = 1 - .80 = .2$ so the probability is 20%

1.13.8 2.21

Before we continue, remember that you can only have one blood type

a) One in every two people gave O^+ so 50% chance.

b) Giving type $O^+(X)$ and $O^-(Y)$ are disjoint events, and giving O is $X \cup Y$ therefore we can add the probability, that is $\frac{1}{2} + \frac{1}{11} = \frac{13}{22} \approx 59\%$

c) Using the same reasoning from the last question we can take the sum that is

$$\frac{1}{4} + \frac{1}{20} = \frac{6}{20} = \frac{30}{100} = 30\%$$

d) Same as above

e) Let X be the event of the first person showing up and having type A^+ or O^+, we are looking for X^c so if they dont have those two they have one of O^- or A^- in that case we add the probability again.

1.13.9 2.23

a) We need $P(B \cap S^c) = P(B) - P(B \cap S) = .8 - .2 = .6$ so 6%

b) $P(S \cup B) = P(B) + P(S) - P(S \cap B) = .10 + .8 - .2 = .16 = 16\%$

c) $P\big((S \cup B) \setminus (S \cap B)\big) = P(S \cup B) - P(S \cap B) = .16 - .2 = .14 = 14\%$

d) $P\big((A \cup B)^c\big) = 1 - P(A \cup B) = 1 - .16 = .84 = .84\%$

1.13.10 2.24

a) Histogram?

b) Again?

1.13.11 2.27

We will prove

$$P(A \cap B) \leq P(A)$$

recall $A = (A \cap B^c) \cup (A \cap B)$ therefore

$$\begin{aligned} P(A \cup B) &\leq P(A \cup B) + P(A \cap B^c) && \text{(Axiom 1)} \\ &= P(A) && \text{(Axiom 3)} \end{aligned}$$

1.13.12 2.28

Recall from the Inculsion/Exclusion Principle and re-arranging we have

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

We note that any event $X \subseteq \Omega$ therefore $P(X) \leq 1$ and so $P(A \cup B) \leq 1 \Leftrightarrow -P(A \cup B) \geq -1$ thus we have

$$P(A \cap B) = P(A) + P(B) - P(A \cup B) \geq P(A) + P(B) - 1$$

so we have

$$P(A \cup B) \geq P(A) + P(B) - 1$$

1.13.13 2.29

We will prove

$$P(A \cap B^c) = P(A) - P(A \cap B)$$

Note that $(A \cap B^c) \cup (A \cap B) = A$, let L be the left hand side and R be the right hand side. We will prove that in fact $L = R$

- Let $x \in L$
  - **Case 1**: $x \in (A \cap B^c)$ then $x \in A$.
  - **Case 2**: $x \in (A \cap B)$ then $x \in A$ as required.
- Therefore $L \subseteq R$
- Let $y \in R$ then $y \in A$ and so $y \in (A \cap B^c)$, in addition $y \in A \cap B$
- so $R \cup L$ therefore

$$\boxed{R = L}$$

Then we note that $(A \cap B^c)$ and $(A \cap B)$ are disjoint (do we have to prove this) and so by axiom 3, we have

$$P(A \cap B^c) + P(A \cap B) = P(A)$$

and we conclude

$$\boxed{P(A \cap B^c) = P(A) - P(A \cap B)}$$