MAT223 - Linear Algebra

Classnotes for Summer 2019

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Linear transformations and such

1.1 Linear Transformations

Definition 1 (Linear Transformation)

Let V and W be subspaces. A function $\mathcal{T} : V \to W$ is called a linear transformation if for all $\vec{u}, \vec{v} \in V$ and $a \in \mathbb{R}$ it satisfies

1. $\mathcal{T}(\vec{u} + \vec{v}) = \mathcal{T}(\vec{u}) + \mathcal{T}(\vec{v})$
2. $\mathcal{T}(a\vec{u}) = a\mathcal{T}(\vec{u})$

Example 1.1.1

We'll show that $\mathcal{R}$ is a linear transformation where $\mathcal{R}$ is a counter clockwise rotation of $\frac{\pi}{2}$ radians

$$\mathcal{R}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Let $\vec{u}, \vec{v} \in \mathbb{R}^2$ we know that for some $x_1, y_1, x_2, y_2 \in \mathbb{R}$ that

$$\vec{u} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \text{ and } \vec{v} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

$$\mathcal{R}\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) + \mathcal{R}\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = \begin{bmatrix} -y_1 \\ x_1 \end{bmatrix} + \begin{bmatrix} -y_2 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} -(y_1+y_2) \\ x_1+x_2 \end{bmatrix}$$

Which is exactly equal to $\mathcal{R}(\vec{u}+\vec{v})$ as required, then let $\alpha \in \mathbb{R}$ and we know that

$$\mathcal{R}(\alpha\vec{u}) = \begin{bmatrix} -\alpha y_1 \\ \alpha x_1 \end{bmatrix}$$

But also that

$$\alpha\mathcal{R}(\vec{u}) = \begin{bmatrix} -\alpha y_1 \\ \alpha x_1 \end{bmatrix}$$

So then we've shown that $\mathcal{R}(\alpha\vec{u}) = \alpha\mathcal{R}(\vec{u})$ but also that $\mathcal{R}(\vec{u}+\vec{v}) = \mathcal{R}(\vec{u}) + \mathcal{R}(\vec{v})$ as req'd

Example 1.1.2

We'll show that $\mathcal{T} : \mathbb{R}^2 \to \mathbb{R}^2$ where $\mathcal{T}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+2 \\ y \end{bmatrix}$ is not a linear transformation.

Let $\vec{j} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \vec{k} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ we have that

$$\mathcal{T}(\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

But then we can see that

$$\mathcal{T}(\begin{bmatrix} 0 \\ 0 \end{bmatrix}) + \mathcal{T}(\begin{bmatrix} 0 \\ 0 \end{bmatrix}) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

Then we conclude that $\mathcal{T}(\vec{j}+\vec{k}) \neq \mathcal{T}(\vec{j}) + \mathcal{T}(\vec{k})$

Example 1.1.3

We'll show that $\mathcal{P}$ is a linear transformation [1]

$$\mathcal{P}(\begin{bmatrix} x \\ y \end{bmatrix}) = comp_{\vec{u}}\begin{bmatrix} x \\ y \end{bmatrix}$$

[1] We'll show that it is closed under addition and multiplication

Let $\vec{j}, \vec{k} \in \mathbb{R}^2$ we know that

$$comp_{\vec{u}}\vec{j} = \left(\frac{\vec{u}\cdot\vec{j}}{||\vec{u}||^2}\right)\vec{u} \text{ and } comp_{\vec{u}}\vec{k} = \left(\frac{\vec{u}\cdot\vec{k}}{||\vec{u}||^2}\right)\vec{u}$$

And thus their product yields

$$comp_{\vec{u}}\vec{j} + comp_{\vec{u}}\vec{k} = \left(\frac{\vec{u}\cdot\left(\vec{j}+\vec{k}\right)}{||\vec{u}||^2}\right)\vec{u}$$

Which is equal to

$$comp_{\vec{u}}\left(\vec{j}+\vec{k}\right)$$

We must then show that it holds under multiplication let $\alpha \in \mathbb{R}$ and we know that

$$\alpha comp_{\vec{u}}\vec{j} = \alpha\left(\frac{\vec{u}\cdot\vec{j}}{||\vec{u}||^2}\right)\vec{u} = \left(\frac{\vec{u}\cdot\alpha\vec{j}}{||\vec{u}||^2}\right)\vec{u} = comp_{\vec{u}}\alpha\vec{j}$$

Example 1.1.4

We'll show that $W : \mathbb{R}^2 \to \mathbb{R}^2$ is not a linear transformation, where

$$\mathcal{W}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x^2 \\ y \end{bmatrix}$$

Let $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, and $\alpha \in \mathbb{R}$ we know that

$$\mathcal{W}\left(\alpha\begin{bmatrix} x \\ y \end{bmatrix}\right) = \alpha^2\begin{bmatrix} x^2 \\ y^2 \end{bmatrix} \neq \alpha\begin{bmatrix} x^2 \\ y^2 \end{bmatrix} = \alpha\mathcal{W}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$$

1.2 *Image*

Definition 2 (Image)

Let $L : V \to W$ be a transformation and let $X \subseteq V$ be a set. The ***image***

of the set X ***under L*** *, denoted as* $L(X)$*, is the set*

$$L(X) = \{\vec{x} \in W : \vec{x} = L(\vec{y}) \text{ for some } \vec{y} \in X\}$$

Let $S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : 0 \leq x, y \leq 1 \right\}$ be a filled in unit square in the first quadrant. And let $C = \left\{ \vec{0}, \vec{e_1}, \vec{e_2}, \vec{e_1} + \vec{e_2} \right\} \subseteq \mathbb{R}^2$ be the corners of the unit square

Exercise 1.2.1

We'll find what $\mathcal{R}(C)$ *is, by the definition of image we have that*

$$\begin{aligned} \mathcal{R}(C) &= \left\{ \mathcal{R}\left(\vec{0}\right), \mathcal{R}(\vec{e_1}), \mathcal{R}(\vec{e_2}), \mathcal{R}(\vec{e_1} + \vec{e_2}) \right\} \\ &= \left\{ \vec{0}, \vec{e_2}, -\vec{e_1}, \vec{e_2} - \vec{e_1} \right\} \end{aligned}$$

Exercise 1.2.2

We'll now find what $\mathcal{W}(C)$ [2] *is, again we will use the definition so we have*

$$\mathcal{W}(C) = \left\{ \vec{0}, \vec{e_1}, \vec{e_2}, \vec{e_1} + \vec{e_2} \right\}$$

[2] Notice that it doesn't have to be a linear transformation

Exercise 1.2.3

$$\mathcal{T}(C) = \left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$$ [3]

[3] The square has been shifted right 2 units

Exercise 1.2.4

We'll now operate on S*, to find* $\mathcal{R}(S)$ *we imagine all the vectors in* $\mathbb{R}^2$ *that have been rotated* $\frac{\pi}{2}$ *radians counter clockwise from the intial square, or we could also multiply by the rotation matrix, either way we get the set*

$$\mathcal{R}(S) = \left\{ \begin{bmatrix} -y \\ x \end{bmatrix} : 0 \leq x, y \leq 1 \right\}$$

Exercise 1.2.5

For $\mathcal{T}(S)$ *we can re-imagine how we determined* $\mathcal{T}(C)$ *but for all the* *points in the square, this gives us the full square shifted horizontally by two units, so we have*

$$\mathcal{T}(S) = \left\{ \begin{bmatrix} x+2 \\ y \end{bmatrix} : 0 \leq x, y \leq 1 \right\}$$

Exercise 1.2.6

As for $\mathcal{P}(S)$ this is a bit more complicated, so we'll break it into to parts, the first is algebreically and the other will be vizually.

Algebraically we know $\text{proj}_{\vec{u}} \begin{bmatrix} x \\ y \end{bmatrix}$ will look like

$$\left(\frac{\vec{u} \cdot \begin{bmatrix} x \\ y \end{bmatrix}}{\|\vec{u}\|^2} \right) \vec{u}$$

But $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ so then

$$\text{proj}_{\vec{u}} \begin{bmatrix} x \\ y \end{bmatrix} = \left(\frac{2x+3y}{13} \right) \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

So then we can conclude that

$$\mathcal{P}(S) = \left\{ \frac{2x+3}{13} \begin{bmatrix} 2 \\ 3 \end{bmatrix} : 0 \leq x, y \leq 1 \right\}$$

Exercise 1.2.7

Let $\ell = \left\{ t\vec{a} + (1-t)\vec{b} \text{ for some } t \in [0,1] \right\}$ and $\mathcal{A}$ be a linear transformation, we know that $\mathcal{A}(\ell)$ represents all vectors that are in the range of the linear transformation, let $\vec{u} \in \ell$, then we know that $\vec{u} = t\vec{a} + (1-t)\vec{b}$ for some $t \in \mathbb{R}$ then we know

$$\begin{aligned} \mathcal{A}(\vec{u}) &= \mathcal{A}\left(t\vec{a} + (1-t)\vec{b}\right) \\ &= t\mathcal{A}(\vec{a}) + (1-t)\mathcal{A}\left(\vec{b}\right) \end{aligned}$$

And since we know that $\mathcal{A}(\vec{a}), \mathcal{A}\left(\vec{b}\right)$ *are just two transformed vectors, then this defines a new line segment with endpoints* $\mathcal{A}(\vec{a})$ *and* $\mathcal{A}\left(\vec{b}\right)$.

Exercise 1.2.8

We'll now find the linear transformation that italicizes N, FIG below

To determine the linear transformation we start with the fact that if A *is some matrix then* $A\vec{e_i}$ *results in the ith column of* A.

We then choose two points and see how they moved after the transormation, per the hint above we'll choose two vectors that reside on the x, y axis. [4], *so our first vector will be* $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$ *and our second is* $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$. *Thus we know the following*

[4] Our origin is the corner of the N

1. $\mathcal{I}\left(\begin{bmatrix} 0 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, *but we know that applying a linear transformation is the same as just multiplying by some matrix so we know that*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 3b \\ 3d \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

   And so we conclude that $b = \frac{4}{3}$ *and* $d = \frac{1}{3}$ *so we've determined the first column of the matrix*

2. $\mathcal{I}\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ *thus we know that* $2a = 2 \Leftrightarrow a = 1$ *and that* $b = 0$ *and we have our second column of the matrix.*

So now we now that the matrix must look like

$$\begin{bmatrix} 1 & \frac{4}{3} \\ 0 & \frac{1}{3} \end{bmatrix}$$

1.2.1 *From Transformation to Matrix*

We defined $\mathcal{P}$ as the $proj_{span\vec{u}}\vec{x}$ where $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\mathcal{R}$ be a rotation ccw by $\frac{\pi}{2}$ radians. We'll now find the matricies which define each transformation

Example 1.2.1

$$\begin{aligned}
\mathcal{P}(\vec{x}) &= \left(\frac{\vec{u} \cdot \begin{bmatrix} x \\ y \end{bmatrix}}{||\vec{u}||^2} \right) \vec{u} \\
&= \left(\frac{2x+3y}{13} \right) \begin{bmatrix} 2 \\ 3 \end{bmatrix} \\
&= \frac{1}{13} \begin{bmatrix} 4x+6y \\ 6x+9y \end{bmatrix}
\end{aligned}$$

By observation we know the matrix must be

$$\frac{1}{13} \begin{bmatrix} 4 & 6 \\ 6 & 9 \end{bmatrix}$$

Example 1.2.2

We'll now find the matrix which defines the rotation $\mathcal{R}$, we start geometrically FIG below.

We could also use the fact that any matrix A times $\vec{e_i}$ is equal to the i-th column of the matrix A. And we know that rotating $\vec{e_1}$ ccw $\frac{\pi}{2}$ moves it to $\vec{e_2}$ and that $\vec{e_2}$ rotated becomes $-\vec{e_1}$ so then we can determine that the matrix must be

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

1.2.2 *Composition of Transformations*

We know that $f \circ g(x) = f(g(x))$ and so we can determine that $\mathcal{P} \circ \mathcal{R} = \mathcal{P}(\mathcal{R}(\vec{x})) = \mathcal{P}(\mathcal{R}(\vec{x})) = \mathcal{P}(\mathcal{R}\vec{x}) = \mathcal{P}\mathcal{R}\vec{x}$. So we can determine

$$\mathcal{P} \circ \mathcal{R} = \frac{1}{13} \begin{bmatrix} 4 & 6 \\ 6 & 9 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \frac{1}{13} \begin{bmatrix} 6 & -4 \\ 9 & -6 \end{bmatrix}$$

And that

$$\mathcal{R} \circ \mathcal{P} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \frac{1}{13} \begin{bmatrix} 4 & 6 \\ 6 & 9 \end{bmatrix} = \frac{1}{13} \begin{bmatrix} -6 & -9 \\ 4 & 6 \end{bmatrix}$$

We notice that these matricies are certainly different and that $\mathcal{P} \circ \mathcal{R}$ is first a projection onto $\vec{u}$ and then a rotation, whereas $\mathcal{R} \circ \mathcal{P}$ is first a rotation, then a projection onto $\vec{u}$.

Definition 3 (Range)

*The **range** of a linear transformatoin $T : V \to W$ is the set of vectors that T can output. That is ,*

$$range(T) = \{\vec{y} \in W : \vec{y} = T(\vec{x}) \text{ for some } x \in V\}$$

Definition 4 (Null Space)

*The **null space** or **kernel** of a linear transformation $T : V \to W$ is the set of vectors that get mapped to zero under T. That is,*

$$null(T) = \left\{\vec{x} \in V : T(\vec{x}) = \vec{0}\right\}$$

Exercise 1.2.9

Consider $P : \mathbb{R}^2 \to \mathbb{R}^2$ where $\mathcal{P}$ is the projection onto the span$\vec{u}$ [5], we'll determine the range and null space of $\mathcal{P}$.

[5] Remember $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

We know that the projection of any vector onto $\vec{u}$ will be equal to some *scalar times $\vec{u}$ so we know that the range will be*

$$\alpha\vec{u}, \text{ for all } a \in \mathbb{R}$$

As for the nullspace, we can think of what vectors will get mapped to zero under a projection onto $\vec{u}$ with a bit of thought, we determine that it must be all vectors who are orthogonal to $\vec{u}$ as their "shadow" will drop to the zero vector. We know that this will be all scalar multiples of a normal vector to $\vec{u}$ so we can say $\alpha \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ for all $\alpha \in \mathbb{R}$ or we could first take the image of $\mathcal{P}$ then rotate all of those vectors, so we could say that $null(\mathcal{P}) =$ Image of the rangeP under $\mathcal{R}$

Example 1.2.3

We let $T : R^n \to R^m$ be a linear transformation. We'll show that the null space of T is a linear subspace and that the range of T is as well, so we'll show it is closed under addition and multiplication.

Proof

Let $\vec{u}, \vec{x} \in null(T)$

$$T(\vec{u}) = 0 \text{ and } T(\vec{x}) = 0$$

Taking the sum of the above equations we get $0 = T(\vec{u}) + T(\vec{x}) = \mathcal{T}(\vec{x} + \vec{y})$ from the definition of linear transformation. But by the definition of null set, we can conclude that $\vec{x} + \vec{y} \in null(T)$ because $T(\vec{x} + \vec{y}) = 0$.

Let $\alpha \in \mathbb{R}$ and from above we know that $T(\vec{x}) = 0 \Leftrightarrow \alpha T(\vec{x}) = \alpha 0 = 0$ and since T is a linear transformationwe can say that $T(\alpha\vec{x}) = 0$ so then we know that $\alpha\vec{x} \in null(T)$ □

Proof

Let $\vec{j}, \vec{k} \in range(T)$ so we know that

$$\vec{j} = T(\vec{x}) \text{ and } \vec{k} = T(\vec{x}_1) \text{ for some } \vec{x}, \vec{x}_1 \in R^n$$

Then we know that $\vec{j}+\vec{k} = T(\vec{x}+\vec{x}_1)$ and thus we conclude that $\vec{j}+\vec{k} \in range(T)$

Let $\alpha \in \mathbb{R}$ and we know that $\alpha\vec{j} = \alpha T(\vec{x}) = T(\alpha\vec{x})$ and so $\alpha\vec{j} \in range(T)$ as required. $\square$

1.3 *Tutorial 5*

1. $\mathcal{B}$ is a basis for a subspace V if $\mathcal{B}$ is linearly independent and $span(\mathcal{B}) = V$.

2.(a) We'll verify if the fectors in $\mathcal{B}$ are in a basis. So for it to be a basis it must be linearly independent, and we'll assume that we are looking for a basis for the subspace $\mathbb{R}^3$. We'll start by looking at the linear combinations of each vector in the matrix that gives the zero vector to determine independendence, so we have [6]

$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

[6] These are augmented matricies for the zero vector.

And thus we have one solution to $\vec{0}$ and so these vectors are linearly independent, and they span $\mathbb{R}^3$.

We'll now focus our attention to $\mathcal{C}$ using the same process as above we get

$$\begin{bmatrix} 1 & 0 & 2 \\ - & 1 & -2 \\ 1 & 1 & 3 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

And by the same reasoning as before $\mathcal{C}$ is a basis for $\mathbb{R}^3$.

(b) We'll start by finding what $[\vec{v}]_{\mathcal{E}}$ is, but we don't have to do much as $\vec{v} = 4\vec{e}_1 - 4\vec{e}_2 + 2\vec{e}_3$ so we know that $[\vec{v}]_{\mathcal{E}} = \begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix}$.

Moving to $[\vec{v}]_{\mathcal{B}}$ we know we are looking for some $\alpha, \beta, \gamma \in \mathbb{R}$

that satisfy

$$\alpha \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix}$$

This relates to a system of equations, which is then stored in a matrix, so we have

$$\begin{bmatrix} 1 & 0 & 2 & 4 \\ -1 & 1 & -2 & -4 \\ 0 & 0 & 1 & 2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

And thus we conclude that $\gamma = 2, \beta = 0, \alpha = 4 = 2\gamma = 0$ and so $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$.

Now we'll figure out $[\vec{v}]_{\mathcal{C}}$ so again we apply the same idea to get the following matrix

$$\begin{bmatrix} 1 & 0 & 2 & 4 \\ -1 & 1 & -2 & -4 \\ 1 & 1 & 3 & 2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

And now we conclude that $\gamma = -2, \beta = 0, \alpha = 8$ and thus we have

$$[\vec{v}]_{\mathcal{C}} = \begin{bmatrix} 8 \\ 0 \\ -2 \end{bmatrix}$$

(c) To determine $[7\vec{v}]_{\mathcal{E}}$ we know we are looking for the solution to

$$\alpha_1 \vec{e_1} + \beta_1 \vec{e_2} + \gamma_1 \vec{e_3} = 7\vec{v}$$

But previously we know that the coeficients should be when we were just looking for $\vec{v}$ multiplying that equation by 7 on both sides tells us that

$$[7\vec{v}]_{\mathcal{E}} = \begin{bmatrix} 28 \\ -28 \\ 14 \end{bmatrix}$$

Then using the same process as above we can determine that

$$[7\vec{v}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 14 \end{bmatrix} \text{ and } [\vec{v}]_{\mathcal{C}} = \begin{bmatrix} 57 \\ 0 \\ -14 \end{bmatrix}$$

(d) I would prefer to write my measurements of scalar multiples of $\vec{v}$ in terms of the $\mathcal{B}$ basis as I only have to do one calculation.

3. Get help with this one

4. I chose $\mathcal{E}_3$ as then I could represent the vector $\vec{v} = \begin{bmatrix} 1 \\ .12 \end{bmatrix}$ as $\left[\begin{bmatrix} -11 \\ 12 \end{bmatrix}\right]_{\mathcal{E}_3}$

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