

Chapter 1

Complex Numbers

Verification 0.0.1. commutativity

$$w + z = z + w \text{ and } wz = zw \text{ for all } w, z \in \mathbb{C}$$

We know that $w = \alpha + \beta i$ and $z = j + ki$ and so then

$$w + z = \alpha + \beta i + j + ki = (\alpha + j) + (\beta + k)i = (j + \alpha) + (k + \beta)i = z + w$$

Verification 0.0.2. associativity

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3) \text{ and } (z_1 z_2) z_3 = z_1 (z_2 z_3) \text{ for all } z_1, z_2, z_3 \in \mathbb{C}$$

We know $z_1 = \alpha + \beta i, z_2 = j + ki, z_3 = l + mi$ and thus

$$\begin{aligned} (z_1 + z_2) + z_3 &= \alpha + j + (\beta + k)i + l + mi \\ &= (\alpha + j + l) + (\beta + k + m)i \\ &= z_1 + (z_2 + z_3) \end{aligned}$$

Now we'll do multiplication, we know

$$\begin{aligned} (z_1 z_2) z_3 &= (\alpha \cdot j - \beta \cdot k + (\alpha k + \beta j)i)(l + mi) \\ &= ajl - blk - (\alpha km + \beta jm) + (\alpha kl + \beta jl + ajm)i \end{aligned}$$

Now we also have

$$\begin{aligned} z_1(z_2 z_3) &= \alpha + \beta i(jl - km + (kl + jm)i) \\ &= ajl - akm - (\beta kl + \beta jm) + (\beta jl + akm + ajm)i \end{aligned}$$

And due to the associativity of \mathbb{R} then we can say $(z_1 z_2) z_3 = z_1 (z_2 z_3)$

Verification 0.0.3. identities

$$z + 0 = z \text{ and } z1 = z \text{ for all } z \in \mathbb{C}$$

We know that

$$\begin{aligned} z + 0 &= \alpha + \beta i + 0 + 0i \\ &= (\alpha + 0) + (\beta + 0)i \\ &= \alpha + \beta i \\ &= z \end{aligned}$$

For multiplication we have

$$\begin{aligned} z1 &= \alpha + \beta i(1 + 0i) \\ &= \alpha + \beta i + 0 + 0i \\ &= \alpha + \beta i \end{aligned}$$

Verification 0.0.4. additive inverse

for every $z \in \mathbb{C}$ there is a unique $w \in \mathbb{C}$ such that $z + w = 0$

Let $z \in \mathbb{C}$ and so $z = \alpha + \beta i$ now we'll take $w = -\alpha + -\beta i$

$$\begin{aligned} z + w &= \alpha + \beta i + -\alpha + -\beta i \\ &= (\alpha - \alpha) + (\beta - \beta)i \\ &= 0 + 0i \\ &= 0 \end{aligned}$$

To show that our choice of w was unique assume there is another solution namely $w = j + ki$ such that $j \neq \alpha, k \neq \beta$ but then their sum will yeild $x + yi$, where $x, y \neq 0$ and so we don't get 0 so we can say that our w is unique.

Verification 0.0.5. multiplicative inverse Let $z \in \mathbb{C}$ so there exists some $\alpha, \beta \in \mathbb{R}$ so that $z = \alpha + \beta i$ let $w = \frac{\alpha}{\alpha^2 + \beta^2} + \frac{-\beta}{\alpha^2 + \beta^2}i$

$$\begin{aligned} zw &= (\alpha + \beta i) \left(\frac{\alpha}{\alpha^2 + \beta^2} + \frac{-\beta}{\alpha^2 + \beta^2}i \right) \\ &= \frac{\alpha^2}{\alpha^2 + \beta^2} + \frac{\beta^2}{\alpha^2 + \beta^2} + \left(\frac{\alpha\beta}{\alpha^2 + \beta^2} - \frac{\alpha\beta}{\alpha^2 + \beta^2} \right)i \\ &= 1 + 0i \\ &= 1 \end{aligned}$$

Verification 0.0.6. distributive property

$$\lambda(w + z) = \lambda w + \lambda z \text{ for all } \lambda, w, z \in \mathbb{C}$$

$$\begin{aligned} \lambda(w + z) &= \lambda(\alpha + \beta i + \delta + \varepsilon i) \\ &= \lambda(\alpha + \delta + (\beta + \varepsilon)i) \\ &= \lambda\alpha + \lambda\delta + (\lambda\beta + \lambda\varepsilon)i \\ &= \lambda\alpha + \lambda\beta i + \lambda\delta + \lambda\varepsilon i \\ &= \lambda(\alpha + \beta i) + \lambda(\delta + \varepsilon i) \\ &= \lambda w + \lambda z \end{aligned}$$