Chapter 1

Complex Numbers

Verfication 0.0.1. commutativity

$$w + z = z + w \text{ and } wz = zw \text{ for all } w, z \in \mathbb{C}$$

We know that $w = \alpha + \beta i$ and $z = j + ki$ and so then

$$w + z = \alpha + \beta i + j + ki = (\alpha + j) + (\beta + k)i = (j + \alpha) + (k + \beta) = z + w$$

also from the commutivity of the Real Numbers it follows that

$$w \cdot z = (\alpha + \beta i) \cdot (j + ki) = (\alpha j - \beta k) + (\alpha k + \beta j)i = (j\alpha - k\beta) + (k\alpha + j\beta) = z \cdot w;$$

Verfication 0.0.2. associativity

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3) \text{ and } (z_1 z_2)z_3 = z_1(z_2 z_3) \text{ for all } z_1, z_2, z_3 \in \mathbb{C}$$

We know $z_1 = \alpha + \beta i, z_2 = j + ki, z_3 = l + mi$ and thus

$$\begin{aligned}(z_1 + z_2) + z_3 &= \alpha + j + (\beta + k)i + l + mi \\ &= (\alpha + j + l) + (\beta + k + m)i \\ &= z_1 + (z_2 + z_3)\end{aligned}$$

Now we'll do multiplication, we know

$$\begin{aligned}(z_1 z_2)z_3 &= \big(\alpha \cdot j - \beta \cdot k + (\alpha k + \beta j)i\big)(l + mi) \\ &= ajl - blk - (\alpha km + \beta jm) + (\alpha kl + \beta jl + ajm)i\end{aligned}$$

Now we also have

$$\begin{aligned}z_1(z_2 z_3) &= \alpha + \beta i\big(jl - km + (kl + jm)i\big) \\ &= ajl - akm - (\beta kl + \beta jm) + (\beta jl + akm + ajm)i\end{aligned}$$

And due to the associativity of $\mathbb{R}$ then we can say $(z_1 z_2)z_3 = z_1(z_2 z_3)$

Verfication 0.0.3. identities

$$z + 0 = z \text{ and } z1 = z \text{ for all } z \in \mathbb{C}$$

We know that

$$\begin{aligned} z + 0 &= \alpha + \beta i + 0 + 0i \\ &= (\alpha + 0) + (\beta + 0)i \\ &= \alpha + \beta i \\ &= z \end{aligned}$$

For multiplication we have

$$\begin{aligned} z1 &= \alpha + \beta i(1 + 0i) \\ &= \alpha - \beta 0 + (\beta + 0\alpha) \\ &= \alpha + \beta i \end{aligned}$$

Verfication 0.0.4. additive inverse
for every $z \in \mathbb{C}$ there is a unique $w \in \mathbb{C}$ such that $z + w = 0$

Let $z \in \mathbb{C}$ and so $z = \alpha + \beta i$ now we'll take $w = -\alpha + -\beta i$

$$\begin{aligned} z + w &= \alpha + \beta i + -\alpha + -\beta i \\ &= (\alpha - \alpha) + (\beta - \beta)i \\ &= 0 + 0i \\ &= 0 \end{aligned}$$

To show that our choice of w was unique assume there is another solution namely $w = j + ki$ such that $j \neq \alpha, k \neq \beta$ but then their sum will yeild $x + yi$, where $x, y \neq 0$ and so we don't get 0 so we can say that our w is unique.

Verfication 0.0.5. multiplicative inverse Let $z \in \mathbb{C}$ so there exists some $\alpha, \beta \in \mathbb{R}$ so that $z = \alpha + \beta i$ let $w = \frac{\alpha}{\alpha^2+\beta^2} + \frac{-\beta}{\alpha^2+\beta^2} i$

$$\begin{aligned} zw &= (\alpha + \beta i)\left(\frac{\alpha}{\alpha^2 + \beta^2} + \frac{-\beta}{\alpha^2 + \beta^2} i\right) \\ &= \frac{\alpha^2}{\alpha^2 + \beta^2} + \frac{\beta^2}{\alpha^2 + \beta^2} + \left(\frac{\alpha\beta}{\alpha^2 + \beta^2} - \frac{\alpha\beta}{\alpha^2 + \beta^2}\right) i \\ &= 1 + 0i \\ &= 1 \end{aligned}$$

Verfication 0.0.6. distributive property

$$\lambda(w+z) = \lambda w + \lambda z \text{ for all } \lambda, w, z \in \mathbb{C}$$

$$\begin{aligned}
\lambda(w+z) &= \lambda(\alpha + \beta i + \delta + \varepsilon i) \\
&= \lambda\big(\alpha + \delta + (\beta + \varepsilon)i\big) \\
&= \lambda\alpha + \lambda\delta + (\lambda\beta + \lambda\varepsilon)i \\
&= \lambda\alpha + \lambda\beta i + \lambda\delta + \lambda\varepsilon i \\
&= \lambda(\alpha + \beta i) + \lambda(\delta + \varepsilon i) \\
&= \lambda w + \lambda z
\end{aligned}$$

Verfication 0.0.7. $\mathbb{F}^n$ is a vector space over $\mathbb{F}$ we know

$$\mathbb{F}^n = \big\{(x_1, x_2, \ldots, x_{n-1}, x_n) : x_j \in \mathbb{F} \text{ for } j = 1, 2, \ldots, n-1, n\big\}$$

Let $u, v \in \mathbb{F}^n$

- commutivity: we will show

  $$u + v = v + u$$

  we have

  $$(u_1, u_2, \ldots, u_{n-1}, u_n) + (v_1, v_2, \ldots, v_{n-1}, v_n) = (u_1 + v_1, \ldots, u_n + v_n)$$

  But since we have both commutivity for $\mathbb{R}$ and $\mathbb{C}$ it follows that

  $$\begin{aligned}
  &= (v_1 + u_1, \ldots, v_n + u_n) \\
  &= v + u
  \end{aligned}$$

- associativity also follows similarly from the associativity of $\mathbb{R}$ and $\mathbb{C}$

- the additive identity is $0 = (0, 0, \ldots, 0, 0)$ we will prove $u + 0 = u$. As we have the additive identity for both $\mathbb{R}$ and $\mathbb{C}$

  $$\begin{aligned}
  (u_1, u_2, \ldots, u_{n-1}, u_n) + 0 &= (0 + u_1, 0 + u_2, \ldots, 0 + u_{n-1}, 0 + u_n) \\
  &= u
  \end{aligned}$$

- additive inverse: we have found an addtive inverse for the Real Numbers and complex numbers so if we are dealing with $\mathcal{F}$ we know there exists a $w \in \mathbb{F}^n$ where w is a list of additive inverses for each $u_1, u_2, \ldots, u_{n-1}, u_n$

  $$u + w = (u_1 + w_1, u_2 + w_2, \ldots, u_{n-1} + w_{n-1}, u_n + w_n) = 0$$

- we know that the multiplicative identity of multiplying by 1, holds in the Real Numbers and complex numbers so we know

$$1(u) = (1 \cdot u_1, 1 \cdot u_2, \ldots, 1 \cdot u_{n-1}, 1 \cdot u_n) = (u_1, u_2, \ldots, u_{n-1}, u_n) = u$$

- distributive property

$$\begin{aligned} a(u+v) &= \big(a \cdot (u_1 + v_1), a \cdot (u_2 + v_2), \ldots, a \cdot (u_{n-1} + v_{n-1}), a \cdot (u_n + v_n)\big) \\ &= (a \cdot u_1 + a \cdot v_1 a \cdot u_2 + a \cdot v_2 \ldots a \cdot u_{n-1} + a \cdot v_{n-1} a \cdot u_n + a \cdot v_n) \end{aligned}$$