MAT246 - Concepts in Abstract Mathematics

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Chapter 1

Lecture 1

1.1 Induction

Note 1

$$\mathbb{N} = \{1, 2, 3, \ldots\}$$

Definition 1 (The principle of mathmatical induction)
suppose $S \subseteq \mathbb{N}$
If

- $1 \in S$
- $k + 1 \in S$ *whenever* $k \in S$

Then

$$\boxed{S = \mathbb{N}}$$

The principle of mathmatical induction is simply saying if 1 is in S then $2, 3, \ldots$ is also in S

Example 1.1.1
Prove

$$\forall n \in \mathbb{N}, \underbrace{1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}}_{\chi}$$

Proof.
Let $S = \{n \in \mathbb{N} : \chi \text{ holds } \}$ At this point we don't know what S consists of but we must

show it is $\mathbb{N}$, then we can conclude that the formula holds for all natural numbers. We commence by verifying that $1 \in S$, we have

$$1^2 = \frac{1(1+1)(2+1)}{6}$$

both the right hand side and left hand side are equal to eachother, so the formula holds for 1.

We will now show if $k \in S$ then $k+1 \in S$. We assume that $k \in S$, that is :

$$1^2 + 2^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

We observe that if we add $k+1$ to both sides of the above equation we get the left hand side, of what we want to prove.

$$\begin{aligned}
1^2 + 2^2 + \ldots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\
&= \frac{(k+1)\big(k(2k+1) + 6(k+1)\big)}{6} \\
&= \frac{(k+1)\big(2k^2 + 7k + 6\big)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6} \\
&= \frac{(k+1)\big((k+1)+1\big)\big(2(k+1)+1\big)}{6}
\end{aligned}$$

After working out the right hand side it is the original formula with $k+1$ subbed in. Therefore we have shown that if $k \in S$ then $k+1 \in S$ as wanted, thus by the principle of mathmatical induction

$$S = \mathbb{N}$$

. ■

Definition 2 (Extended principle of mathmatical induction)
This is the same as normal induction, though now we don't have to start with 1. If

- *Let* $n_0 \in \mathbb{N}, n_0 \in S$
- $k \in S \implies k+1 \in S$

Then

$$\boxed{S \supseteq \{n_0, n_0+1, \ldots\}}$$

Observe that S *is only a subset of these numbers as these are the ones that are guarenteed to be in* S*, there may be others.*

Example 1.1.2
Prove for all integers n *greater than or equal to 7 that the following holds:*

$$\underbrace{n! \geq 3^n}_{} \chi$$

Proof.
Let S be the set of all natural numbers that χ holds for. We verify that $7 \in S$

$$\underbrace{7!}_{5040} \geq \underbrace{3^7}_{2187}$$

therefore 7 satisfies χ and so $7 \in S$. Let $k \in \mathbb{N}$, we assume χ holds for k, that is

$$k! \geq 3^k$$

We will prove

$$(k+1)! \geq 3^{k+1}$$

We observe that $(k+1)! = (k+1)k!$, but recall that we assumed that $k! \geq 3^k$ so we have

$$k!(k+1) \geq 3^k(k+1)$$

Recall that $k \geq 7$

$$\begin{aligned} &\geq 3^k 8 \\ &\geq 3^{k+1} \end{aligned}$$

Therefore, we've shown that

$$(k+1)! \geq 3^{k+1}$$

as required, and so

$$S \supseteq \{7, 8, 9, \ldots\}$$

■

Chapter 2

Lecture 2

Definition 3 (Well Ordering Principle)
Every subset of $\mathbb{N}$ *other than* $\emptyset$ *has a smallest element.*

2.1 Proof of Induction

Remark 2.1.1
We accepted the Principle of Mathematical Induction, though we should prove it.

Recall, the Principle of Mathematical Induction, suppose $S \subseteq N$, if

- $1 \in S$
- $k + 1 \in S$ whnever $k \in S$

then

$$S = \mathbb{N}$$

We'll prove the statement

Proof.
Let $T = \{n \in \mathbb{N} : n \notin S\}$. suppose that $T \neq \emptyset$, therefore by the Well Ordering Principle we know that T has a smallest element, let n_0 be that element. Note that $n_0 \in \mathbb{N}, n_0 \neq 1$ since $1 \in S \therefore 1 \notin T$, therefore $n_0 \geq 2$.
since $n_0 \geq 2$ we know $n_0 - 1 \in \mathbb{N}$ and that $n_0 - 1 \notin T$ since n_0 is the smallest element in T.

$$n_0 - 1 \notin T \implies n_0 - 1 \in S$$

But by property 2, of S we know that if $n_0 \in S$ then $n_0 \in S$, though this is a contradiction as $n_0 \notin S$
Therefore $T = \emptyset$ and $S = \mathbb{N}$ ■

2.2 Division

Definition 4 (Divides)
for $a, b \in \mathbb{N}$ *we say that* a *divides* b *if there exists a* $c \in \mathbb{N}$ *such that*

$$b = ca$$

And we say

$$a \mid b$$

Remark 2.2.1
$2 \cdot 3.5 = 7$, *though our definition is only for natural numbers, since no* $c \in \mathbb{N}$ *gives* $2 \cdot c = 7$

Definition 5 (Prime)
$p \in \mathbb{N}$ *is* <u>*prime*</u> *if the only divisor of* p *are* 1 *and* p *and* $p \neq 1$

Example

- 7 is prime, since the only divisor is 1 and 7
- 10 is not prime, 2 and 5 divide 10

Theorem 1 (Product of Primes)
for all $n \in \mathbb{N}, n \neq 1$ n *can be written as a product of primes*

Example

- $42 = 2 \cdot 3 \cdot 7$
- $12 = 3 \cdot 2^2$

Definition 6 (Complete Induction)
Let $S \subseteq \mathbb{N}$

- *if* $n_0 \in S$
  - *and* $k+1 \in S$ *when* $n_0, n_0+1, \ldots, k \in S$

  Then
$$S \supseteq \{n_0, n_0+1, \ldots\}$$

We will prove the product of primes theorem

Proof.
Let $S = \{n \in \mathbb{N} : \text{ theorem holds for } n\}$ we will prove

$$S = \mathbb{N}$$

- 2, is prime therefore it is a product of primes and so the Base Case holds.
- We assume if $2, 3, \ldots, k \in S$ then $k+1 \in S$
  - **Case 1**: $k+1$ is prime, then we are done like the base case
  - **Case 2**: $k+1$ is not prime, then there exists an $m \in \mathbb{N}$ such that $1 < m < k+1$ and $m \mid k+1$ by definition this means

    $$k+1 = c \cdot m, \text{ for some } c \in \mathbb{N}$$

    observe that $1 < c < k+1$ since if $c = 1$,$c = k+1$ or if larger we get a contradiction.
    Therefore we can use the Induction Hypothesis on c and m to write thpem both as a product of primes, multiplying them together gives us a new product of primes equal to $k+1$ as required.

Therefore by the principle of complete induction we can say that

$$S \supseteq \{2, 3, \ldots\}$$

though we want to show that $S = \{1, 2, 3, \ldots\}$ observe that 1 is not a product of primes as it is not prime and also not composite, therefore $1 \notin S$ so $S = \{2, 3, \ldots\}$ ■

The intuition behind this proof comes from the fact that if we take a number say 24 it is either prime or not, in this case it is not, and we can write it as $24 = 6 \cdot 4$ then by an inductive argument, we already know that 6 and 4 are already product of primes so we are done. We will show next that in fact this is a unique product.