MAT223 - Linear Algebra

Classnotes for Summer 2019

by

Callum Cassidy-Nolan

Computer Science

University of Toronto

Table of Contents

List of Definitions

List of Theorems

List of Procedures

1 Week 9

Linear transformations and such

1.1 Linear Transformations

Definition 1 (Linear Transformation)

Let V and W be subspaces. A function $\mathcal{T} : V \to W$ is called a linear transformation if for all $\vec{u}, \vec{v} \in V$ and $a \in \mathbb{R}$ it satisfies

1. $\mathcal{T}(\vec{u} + \vec{v}) = \mathcal{T}(\vec{u}) + \mathcal{T}(\vec{v})$
2. $\mathcal{T}(a\vec{u}) = a\mathcal{T}(\vec{u})$

Example 1.1.1

We'll show that $\mathcal{R}$ is a linear transformation where $\mathcal{R}$ is a counter clockwise rotation of $\frac{\pi}{2}$ radians

$$\mathcal{R}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Let $\vec{u}, \vec{v} \in \mathbb{R}^2$ we know that for some $x_1, y_1, x_2, y_2 \in \mathbb{R}$ that

$$\vec{u} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \text{ and } \vec{v} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

$$\mathcal{R}\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) + \mathcal{R}\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = \begin{bmatrix} -y_1 \\ x_1 \end{bmatrix} + \begin{bmatrix} -y_2 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} -(y_1 + y_2) \\ x_1 + x_2 \end{bmatrix}$$

Which is exactly equal to $\mathcal{R}(\vec{u} + \vec{v})$ as required, then let $\alpha \in \mathbb{R}$ and we know that

$$\mathcal{R}(\alpha\vec{u}) = \begin{bmatrix} -\alpha y_1 \\ \alpha x_1 \end{bmatrix}$$

But also that

$$\alpha\mathcal{R}(\vec{u}) = \begin{bmatrix} -\alpha y_1 \\ \alpha x_1 \end{bmatrix}$$

So then we've shown that $\mathcal{R}(\alpha\vec{u}) = \alpha\mathcal{R}(\vec{u})$ but also that $\mathcal{R}(\vec{u} + \vec{v}) = \mathcal{R}(\vec{u}) + \mathcal{R}(\vec{v})$ as req'd

Example 1.1.2

We'll show that $\mathcal{T} : \mathbb{R}^2 \to \mathbb{R}^2$ where $\mathcal{T}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+2 \\ y \end{bmatrix}$ is not a linear transformation.

Let $\vec{j} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \vec{k} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ we have that

$$\mathcal{T}(\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

But then we can see that

$$\mathcal{T}(\begin{bmatrix} 0 \\ 0 \end{bmatrix}) + \mathcal{T}(\begin{bmatrix} 0 \\ 0 \end{bmatrix}) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

Then we conclude that $\mathcal{T}(\vec{j} + \vec{k}) \neq \mathcal{T}(\vec{j}) + \mathcal{T}(\vec{k})$

Example 1.1.3

We'll show that $\mathcal{P}$ is a linear transformation [1]

[1] We'll show that it is closed under addition and multiplication

$$\mathcal{P}(\begin{bmatrix} x \\ y \end{bmatrix}) = comp_{\vec{u}} \begin{bmatrix} x \\ y \end{bmatrix}$$

Let $\vec{j}, \vec{k} \in \mathbb{R}^2$ we know that

$$comp_{\vec{u}}\vec{j} = \left(\frac{\vec{u} \cdot \vec{j}}{\|\vec{u}\|^2}\right)\vec{u} \text{ and } comp_{\vec{u}}\vec{k} = \left(\frac{\vec{u} \cdot \vec{k}}{\|\vec{u}\|^2}\right)\vec{u}$$

And thus their product yields

$$comp_{\vec{u}}\vec{j} + comp_{\vec{u}}\vec{k} = \left(\frac{\vec{u} \cdot \left(\vec{j} + \vec{k}\right)}{\|\vec{u}\|^2}\right)\vec{u}$$

Which is equal to

$$comp_{\vec{u}}\left(\vec{j} + \vec{k}\right)$$

We must then show that it holds under multiplication let $\alpha \in \mathbb{R}$ and we know that

$$\alpha comp_{\vec{u}}\vec{j} = \alpha\left(\frac{\vec{u} \cdot \vec{j}}{\|\vec{u}\|^2}\right)\vec{u} = \left(\frac{\vec{u} \cdot \alpha\vec{j}}{\|\vec{u}\|^2}\right)\vec{u} = comp_{\vec{u}}\alpha\vec{j}$$

1.2 Image

Definition 2 (Image)

Let $L : V \to W$ be a transformation and let $X \subseteq V$ be a set. The ***image of the set X under L*** *, denoted as $L(X)$, is the set*

$$L(X) = \{\vec{x} \in W : \vec{x} = L(\vec{y}) \text{ for some } \vec{y} \in X\}$$

Bibliography

Index