

# MAT223 - Linear Algebra

CLASSNOTES FOR SUMMER 2019

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# List of Definitions

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## *List of Theorems*

## *List of Procedures*



Linear transformations and such

### 1.1 Linear Transformations

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#### Definition 1 (Linear Transformation)

Let  $V$  and  $W$  be subspaces. A function  $\mathcal{T} : V \rightarrow W$  is called a linear transformation if for all  $\vec{u}, \vec{v} \in V$  and  $a \in \mathbb{R}$  it satisfies

1.  $\mathcal{T}(\vec{u} + \vec{v}) = \mathcal{T}(\vec{u}) + \mathcal{T}(\vec{v})$
2.  $\mathcal{T}(a\vec{u}) = a\mathcal{T}(\vec{u})$

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#### Example 1.1.1

We'll show that  $\mathcal{R}$  is a linear transformation where  $\mathcal{R}$  is a counter clockwise rotation of  $\frac{\pi}{2}$  radians

$$\mathcal{R} \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Let  $\vec{u}, \vec{v} \in \mathbb{R}^2$  we know that for some  $x_1, y_1, x_2, y_2 \in \mathbb{R}$  that

$$\vec{u} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \text{ and } \vec{v} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

$$\mathcal{R} \left( \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \right) + \mathcal{R} \left( \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \right) = \begin{bmatrix} -y_1 \\ x_1 \end{bmatrix} + \begin{bmatrix} -y_2 \\ x_2 \end{bmatrix}$$


$$= \begin{bmatrix} -(y_1 + y_2) \\ x_1 + x_2 \end{bmatrix}$$

Which is exactly equal to  $\mathcal{R}(\vec{u} + \vec{v})$  as required, then let  $\alpha \in \mathbb{R}$  and we know that

$$\mathcal{R}(\alpha \vec{u}) = \begin{bmatrix} -\alpha y_1 \\ \alpha x_1 \end{bmatrix}$$

But also that

$$\alpha \mathcal{R}(\vec{u}) = \begin{bmatrix} -\alpha y_1 \\ \alpha x_1 \end{bmatrix}$$

So then we've shown that  $\mathcal{R}(\alpha \vec{u}) = \alpha \mathcal{R}(\vec{u})$  but also that  $\mathcal{R}(\vec{u} + \vec{v}) = \mathcal{R}(\vec{u}) + \mathcal{R}(\vec{v})$  as req'd 

### Example 1.1.2


We'll show that  $\mathcal{T} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  where  $\mathcal{T} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+2 \\ y \end{bmatrix}$  is not a linear transformation.

Let  $\vec{j} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ,  $\vec{k} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$  we have that

$$\mathcal{T}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

But then we can see that

$$\mathcal{T}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) + \mathcal{T}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

Then we conclude that  $\mathcal{T}(\vec{j} + \vec{k}) \neq \mathcal{T}(\vec{j}) + \mathcal{T}(\vec{k})$  

### Example 1.1.3

We'll show that  $\mathcal{P}$  is a linear transformation <sup>1</sup>

<sup>1</sup> We'll show that it is closed under addition and multiplication

$$\mathcal{P}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \text{comp}_{\vec{u}} \begin{bmatrix} x \\ y \end{bmatrix}$$



Let  $\vec{j}, \vec{k} \in \mathbb{R}^2$  we know that

$$\text{comp}_{\vec{u}}\vec{j} = \left( \frac{\vec{u} \cdot \vec{j}}{\|\vec{u}\|^2} \right) \vec{u} \text{ and } \text{comp}_{\vec{u}}\vec{k} = \left( \frac{\vec{u} \cdot \vec{k}}{\|\vec{u}\|^2} \right) \vec{u}$$

And thus their product yields

$$\text{comp}_{\vec{u}}\vec{j} + \text{comp}_{\vec{u}}\vec{k} = \left( \frac{\vec{u} \cdot (\vec{j} + \vec{k})}{\|\vec{u}\|^2} \right) \vec{u}$$

Which is equal to

$$\text{comp}_{\vec{u}}(\vec{j} + \vec{k})$$

We must then show that it holds under multiplication let  $\alpha \in \mathbb{R}$  and we know that

$$\alpha \text{comp}_{\vec{u}}\vec{j} = \alpha \left( \frac{\vec{u} \cdot \vec{j}}{\|\vec{u}\|^2} \right) \vec{u} = \left( \frac{\vec{u} \cdot \alpha \vec{j}}{\|\vec{u}\|^2} \right) \vec{u} = \text{comp}_{\vec{u}}\alpha \vec{j} \quad \text{🔑}$$

#### Example 1.1.4

We'll show that  $W : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is not a linear transformation, where

$$\mathcal{W} \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} x^2 \\ y \end{bmatrix}$$

Let  $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ , and  $\alpha \in \mathbb{R}$  we know that

$$\mathcal{W} \left( \alpha \begin{bmatrix} x \\ y \end{bmatrix} \right) = \alpha^2 \begin{bmatrix} x^2 \\ y^2 \end{bmatrix} \neq \alpha \begin{bmatrix} x^2 \\ y^2 \end{bmatrix} = \alpha \mathcal{W} \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) \quad \text{🔑}$$

## 1.2 Image

### Definition 2 (Image)

Let  $L : V \rightarrow W$  be a transformation and let  $X \subseteq V$  be a set. The **image**

of the set  $X$  under  $L$ , denoted as  $L(X)$ , is the set

$$L(X) = \{\vec{x} \in W : \vec{x} = L(\vec{y}) \text{ for some } \vec{y} \in X\}$$

Let  $S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : 0 \leq x, y \leq 1 \right\}$  be a filled in unit square in the first quadrant. And let  $C = \{\vec{0}, \vec{e}_1, \vec{e}_2, \vec{e}_1 + \vec{e}_2\} \subseteq \mathbb{R}^2$  be the corners of the unit square

### Exercise 1.2.1

We'll find what  $\mathcal{R}(C)$  is, by the definition of image we have that

$$\begin{aligned} \mathcal{R}(C) &= \left\{ \mathcal{R}(\vec{0}), \mathcal{R}(\vec{e}_1), \mathcal{R}(\vec{e}_2), \mathcal{R}(\vec{e}_1 + \vec{e}_2) \right\} \\ &= \left\{ \vec{0}, \vec{e}_2, -\vec{e}_1, \vec{e}_2 - \vec{e}_1 \right\} \end{aligned}$$

### Exercise 1.2.2

We'll now find what  $\mathcal{W}(C)$ <sup>2</sup> is, again we will use the definition so we have

$$\mathcal{W}(C) = \left\{ \vec{0}, \vec{e}_1, \vec{e}_2, \vec{e}_1 + \vec{e}_2 \right\}$$

<sup>2</sup> Notice that it doesn't have to be a linear transformation

### Exercise 1.2.3

$$\mathcal{T}(C) = \left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}^3$$

<sup>3</sup> The square has been shifted right 2 units

### Exercise 1.2.4

We'll now operate on  $S$ , to find  $\mathcal{R}(S)$  we imagine all the vectors in  $\mathbb{R}^2$  that have been rotated  $\frac{\pi}{2}$  radians counter clockwise from the initial square, or we could also multiply by the rotation matrix, either way we get the set

$$\mathcal{R}(S) = \left\{ \begin{bmatrix} -y \\ x \end{bmatrix} : 0 \leq x, y \leq 1 \right\}$$

### Exercise 1.2.5

For  $\mathcal{T}(S)$  we can re-imagine how we determined  $\mathcal{T}(C)$  but for all the

points in the square, this gives us the full square shifted horizontally by two units, so we have

$$\mathcal{T}(S) = \left\{ \begin{bmatrix} x+2 \\ y \end{bmatrix} : 0 \leq x, y \leq 1 \right\}$$

### Exercise 1.2.6

As for  $\mathcal{P}(S)$  this is a bit more complicated, so we'll break it into two parts, the first is algebraically and the other will be visually.

Algebraically we know  $\text{proj}_{\vec{u}} \begin{bmatrix} x \\ y \end{bmatrix}$  will look like

$$\left( \frac{\vec{u} \cdot \begin{bmatrix} x \\ y \end{bmatrix}}{\|\vec{u}\|^2} \right) \vec{u}$$

But  $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$  so then

$$\text{proj}_{\vec{u}} \begin{bmatrix} x \\ y \end{bmatrix} = \left( \frac{2x+3y}{13} \right) \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

So then we can conclude that

$$\mathcal{P}(S) = \left\{ \frac{2x+3}{13} \begin{bmatrix} 2 \\ 3 \end{bmatrix} : 0 \leq x, y \leq 1 \right\}$$

### Exercise 1.2.7

Let  $\ell = \{t\vec{a} + (1-t)\vec{b} \text{ for some } t \in [0, 1]\}$  and  $\mathcal{A}$  be a linear transformation, we know that  $\mathcal{A}(\ell)$  represents all vectors that are in the range of the linear transformation, let  $\vec{u} \in \ell$ , then we know that  $\vec{u} = t\vec{a} + (1-t)\vec{b}$  for some  $t \in \mathbb{R}$  then we know

$$\begin{aligned} \mathcal{A}(\vec{u}) &= \mathcal{A}(t\vec{a} + (1-t)\vec{b}) \\ &= t\mathcal{A}(\vec{a}) + (1-t)\mathcal{A}(\vec{b}) \end{aligned}$$

And since we know that  $\mathcal{A}(\vec{a}), \mathcal{A}(\vec{b})$  are just two transformed vectors, then this defines a new line segment with endpoints  $\mathcal{A}(\vec{a})$  and  $\mathcal{A}(\vec{b})$ .

### Exercise 1.2.8

We'll now find the linear transformation that italicizes N, FIG below

To determine the linear transformation we start with the fact that if  $A$  is some matrix then  $A\vec{e}_i$  results in the  $i$ th column of  $A$ .

We then choose two points and see how they moved after the transformation, per the hint above we'll choose two vectors that reside on the  $x, y$  axis.

<sup>4</sup>, so our first vector will be  $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$  and our second is  $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ . Thus we know the following

<sup>4</sup>Our origin is the corner of the N

1.  $\mathcal{I}\left(\begin{bmatrix} 0 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ , but we know that applying a linear transformation is the same as just multiplying by some matrix so we know that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 3b \\ 3d \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

And so we conclude that  $b = \frac{4}{3}$  and  $d = \frac{1}{3}$  so we've determined the first column of the matrix

2.  $\mathcal{I}\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$  thus we know that  $2a = 2 \Leftrightarrow a = 1$  and that  $b = 0$  and we have our second column of the matrix.

So now we now that the matrix must look like

$$\begin{bmatrix} 1 & \frac{4}{3} \\ 0 & \frac{1}{3} \end{bmatrix}$$

### 1.2.1 From Transformation to Matrix

We defined  $\mathcal{P}$  as the  $\text{proj}_{\text{span}\vec{u}}\vec{x}$  where  $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$  and  $\mathcal{R}$  be a rotation ccw by  $\frac{\pi}{2}$  radians. We'll now find the matrices which define each transformation

**Example 1.2.1**

$$\begin{aligned}
 \mathcal{P}(\vec{x}) &= \left( \frac{\vec{u} \cdot \begin{bmatrix} x \\ y \end{bmatrix}}{\|\vec{u}\|^2} \right) \vec{u} \\
 &= \left( \frac{2x + 3y}{13} \right) \begin{bmatrix} 2 \\ 3 \end{bmatrix} \\
 &= \frac{1}{13} \begin{bmatrix} 4x + 6y \\ 6x + 9y \end{bmatrix}
 \end{aligned}$$

By observation we know the matrix must be

$$\frac{1}{13} \begin{bmatrix} 4 & 6 \\ 6 & 9 \end{bmatrix}$$

**Example 1.2.2**

We'll now find the matrix which defines the rotation  $\mathcal{R}$ , we start geometrically FIG below.

We could also use the fact that any matrix  $A$  times  $\vec{e}_i$  is equal to the  $i$ -th column of the matrix  $A$ . And we know that rotating  $\vec{e}_1$  ccw  $\frac{\pi}{2}$  moves it to  $\vec{e}_2$  and that  $\vec{e}_2$  rotated becomes  $-\vec{e}_1$  so then we can determine that the matrix must be

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

**1.2.2** *Composition of Transformations*

We know that  $f \circ g(x) = f(g(x))$  and so we can determine that  $\mathcal{P} \circ \mathcal{R} = \mathcal{P}(\mathcal{R}(\vec{x})) = \mathcal{P}(\mathcal{R}(\vec{x})) = \mathcal{P}(\mathcal{R}\vec{x}) = \mathcal{P}\mathcal{R}\vec{x}$ . So we can

determine

$$\mathcal{P} \circ \mathcal{R} = \frac{1}{13} \begin{bmatrix} 4 & 6 \\ 6 & 9 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \frac{1}{13} \begin{bmatrix} 6 & -4 \\ 9 & -6 \end{bmatrix}$$

And that

$$\mathcal{R} \circ \mathcal{P} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \frac{1}{13} \begin{bmatrix} 4 & 6 \\ 6 & 9 \end{bmatrix} = \frac{1}{13} \begin{bmatrix} -6 & -9 \\ 4 & 6 \end{bmatrix}$$

We notice that these matrices are certainly different and that  $\mathcal{P} \circ \mathcal{R}$  is first a projection onto  $\vec{u}$  and then a rotation, whereas  $\mathcal{R} \circ \mathcal{P}$  is first a rotation, then a projection onto  $\vec{u}$ .

### Definition 3 (Range)

The **range** of a linear transformation  $T : V \rightarrow W$  is the set of vectors that  $T$  can output. That is ,

$$\text{range}(T) = \{\vec{y} \in W : \vec{y} = T(\vec{x}) \text{ for some } x \in V\}$$

### Definition 4 (Null Space)

The **null space** or **kernel** of a linear transformation  $T : V \rightarrow W$  is the set of vectors that get mapped to zero under  $T$ . That is,

$$\text{null}(T) = \left\{ \vec{x} \in V : T(\vec{x}) = \vec{0} \right\}$$

### Exercise 1.2.9

Consider  $P : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  where  $\mathcal{P}$  is the projection onto the span of  $\vec{u}$ <sup>5</sup>, we'll determine the range and null space of  $\mathcal{P}$ .

<sup>5</sup> Remember  $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$


We know that the projection of any vector onto  $\vec{u}$  will be equal to some

scalar times  $\vec{u}$  so we know that the range will be

$$\alpha \vec{u}, \text{ for all } \alpha \in \mathbb{R}$$

As for the nullspace, we can think of what vectors will get mapped to zero under a projection onto  $\vec{u}$  with a bit of thought, we determine that it must be all vectors who are orthogonal to  $\vec{u}$  as their "shadow" will drop to the zero vector. We know that this will be all scalar multiples of a normal vector to  $\vec{u}$  so we can say  $\alpha \begin{bmatrix} -3 \\ 2 \end{bmatrix}$  for all  $\alpha \in \mathbb{R}$  or we could first take the image of  $\mathcal{P}$  then rotate all of those vectors, so we could say that  $\text{null}(\mathcal{P}) = \text{Image of the range } \mathcal{P} \text{ under } \mathcal{R}$

### Example 1.2.3

We let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear transformation. We'll show that the null space of  $T$  is a linear subspace and that the range of  $T$  is as well, so we'll show it is closed under addition and multiplication. 

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#### Proof

Let  $\vec{u}, \vec{x} \in \text{null}(T)$

$$T(\vec{u}) = 0 \text{ and } T(\vec{x}) = 0$$

Taking the sum of the above equations we get  $0 = T(\vec{u}) + T(\vec{x}) = T(\vec{x} + \vec{y})$  from the definition of linear transformation. But by the definition of null set, we can conclude that  $\vec{x} + \vec{y} \in \text{null}(T)$  because  $T(\vec{x} + \vec{y}) = 0$ .

Let  $\alpha \in \mathbb{R}$  and from above we know that  $T(\vec{x}) = 0 \Leftrightarrow \alpha T(\vec{x}) = \alpha 0 = 0$  and since  $T$  is a linear transformation we can say that  $T(\alpha \vec{x}) = 0$  so then we know that  $\alpha \vec{x} \in \text{null}(T)$  □

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#### Proof

Let  $\vec{j}, \vec{k} \in \text{range}(T)$  so we know that

$$\vec{j} = T(\vec{x}) \text{ and } \vec{k} = T(\vec{x}_1) \text{ for some } \vec{x}, \vec{x}_1 \in \mathbb{R}^n$$

Then we know that  $\vec{j} + \vec{k} = T(\vec{x} + \vec{x}_1)$  and thus we conclude that  $\vec{j} + \vec{k} \in \text{range}(T)$

Let  $\alpha \in \mathbb{R}$  and we know that  $\alpha\vec{j} = \alpha T(\vec{x}) = T(\alpha\vec{x})$  and so  $\alpha\vec{j} \in \text{range}(T)$  as required.  $\square$

### 1.3 Tutorial 5

1.  $\mathcal{B}$  is a basis for a subspace  $V$  if  $\mathcal{B}$  is linearly independent and  $\text{span}(\mathcal{B}) = V$ .
- 2.(a) We'll verify if the vectors in  $\mathcal{B}$  are in a basis. So for it to be a basis it must be linearly independent, and we'll assume that we are looking for a basis for the subspace  $\mathbb{R}^3$ . We'll start by looking at the linear combinations of each vector in the matrix that gives the zero vector to determine independence, so we have <sup>6</sup>

$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

<sup>6</sup> These are augmented matrices for the zero vector.

And thus we have one solution to  $\vec{0}$  and so these vectors are linearly independent, and they span  $\mathbb{R}^3$ .

We'll now focus our attention to  $\mathcal{C}$  using the same process as above we get

$$\begin{bmatrix} 1 & 0 & 2 \\ - & 1 & -2 \\ 1 & 1 & 3 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

And by the same reasoning as before  $\mathcal{C}$  is a basis for  $\mathbb{R}^3$ .

- (b) We'll start by finding what  $[\vec{v}]_{\mathcal{E}}$  is, but we don't have to do

$$\text{much as } \vec{v} = 4\vec{e}_1 - 4\vec{e}_2 + 2\vec{e}_3 \text{ so we know that } [\vec{v}]_{\mathcal{E}} = \begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix}.$$

Moving to  $[\vec{v}]_{\mathcal{B}}$  we know we are looking for some  $\alpha, \beta, \gamma \in \mathbb{R}$



that satisfy

$$\alpha \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix}$$

This relates to a system of equations, which is then stored in a matrix, so we have

$$\begin{bmatrix} 1 & 0 & 2 & 4 \\ -1 & 1 & -2 & -4 \\ 0 & 0 & 1 & 2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

And thus we conclude that  $\gamma = 2, \beta = 0, \alpha = 4 = 2\gamma = 0$  and so

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}.$$

Now we'll figure out  $[\vec{v}]_{\mathcal{C}}$  so again we apply the same idea to get the following matrix

$$\begin{bmatrix} 1 & 0 & 2 & 4 \\ -1 & 1 & -2 & -4 \\ 1 & 1 & 3 & 2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

And now we conclude that  $\gamma = -2, \beta = 0, \alpha = 8$  and thus we have

$$[\vec{v}]_{\mathcal{C}} = \begin{bmatrix} 8 \\ 0 \\ -2 \end{bmatrix}$$

(c) To determine  $[7\vec{v}]_{\mathcal{E}}$  we know we are looking for the solution to

$$\alpha_1 \vec{e}_1 + \beta_1 \vec{e}_2 + \gamma_1 \vec{e}_3 = 7\vec{v}$$

But previously we know that the coefficients should be when we were just looking for  $\vec{v}$  multiplying that equation by 7 on both sides tells us that

$$[7\vec{v}]_{\mathcal{E}} = \begin{bmatrix} 28 \\ -28 \\ 14 \end{bmatrix}$$

Then using the same process as above we can determine that

$$[7\vec{v}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 14 \end{bmatrix} \text{ and } [\vec{v}]_{\mathcal{C}} = \begin{bmatrix} 57 \\ 0 \\ -14 \end{bmatrix}$$

- (d) I would prefer to write my measurements of scalar multiples of  $\vec{v}$  in terms of the  $\mathcal{B}$  basis as I only have to do one calculation.

3. Get help with this one

4. I chose  $\mathcal{E}_3$  as then I could represent the vector  $\vec{v} = \begin{bmatrix} 1 \\ .12 \end{bmatrix}$  as

$$\left[ \begin{bmatrix} -11 \\ 12 \end{bmatrix} \right]_{\mathcal{E}_3}$$

## Bibliography

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