MAT246 - Concepts in Abstract Mathematics

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Chapter 1

Lecture 1

1.1 Induction

Note 1

$$\mathbb{N} = \{1, 2, 3, \ldots\}$$

Definition 1 (The principle of mathmatical induction)
suppose $S \subseteq \mathbb{N}$
If

- $1 \in S$
- $k + 1 \in S$ *whenever* $k \in S$

Then

$$\boxed{S = \mathbb{N}}$$

The principle of mathmatical induction is simply saying if 1 is in S then $2, 3, \ldots$ is also in S

Example 1.1.1
Prove

$$\forall n \in \mathbb{N}, \underbrace{1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}}_{\chi}$$

Proof.
Let $S = \{n \in \mathbb{N} : \chi \text{ holds } \}$ At this point we don't know what S consists of but we must

show it is $\mathbb{N}$, then we can conclude that the formula holds for all natural numbers. We commence by verifying that $1 \in S$, we have

$$1^2 = \frac{1(1+1)(2+1)}{6}$$

both the right hand side and left hand side are equal to eachother, so the formula holds for 1.

We will now show if $k \in S$ then $k+1 \in S$. We assume that $k \in S$, that is :

$$1^2 + 2^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

We observe that if we add $k+1$ to both sides of the above equation we get the left hand side, of what we want to prove.

$$\begin{aligned}
1^2 + 2^2 + \ldots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\
&= \frac{(k+1)\big(k(2k+1) + 6(k+1)\big)}{6} \\
&= \frac{(k+1)\big(2k^2 + 7k + 6\big)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6} \\
&= \frac{(k+1)\big((k+1)+1\big)\big(2(k+1)+1\big)}{6}
\end{aligned}$$

After working out the right hand side it is the original formula with $k+1$ subbed in. Therefore we have shown that if $k \in S$ then $k+1 \in S$ as wanted, thus by the principle of mathmatical induction

$$S = \mathbb{N}$$

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Definition 2 (Extended principle of mathmatical induction)
This is the same as normal induction, though now we don't have to start with 1. If

- *Let* $n_0 \in \mathbb{N}, n_0 \in S$
- $k \in S \implies k+1 \in S$

Then

$$\boxed{S \supseteq \{n_0, n_0+1, \ldots\}}$$

Observe that S is only a subset of these numbers as these are the ones that are guarenteed to be in S, there may be others.

Example 1.1.2
Prove for all integers n greater than or equal to 7 that the following holds:

$$\underbrace{n! \geq 3^n}_{\chi}$$

Proof.
Let S be the set of all natural numbers that χ holds for. We verify that $7 \in S$

$$\underbrace{7!}_{5040} \geq \underbrace{3^7}_{2187}$$

therefore 7 satisfies χ and so $7 \in S$. Let $k \in \mathbb{N}$, we assume χ holds for k, that is

$$k! \geq 3^k$$

We will prove

$$(k+1)! \geq 3^{k+1}$$

We observe that $(k+1)! = (k+1)k!$, but recall that we assumed that $k! \geq 3^k$ so we have

$$k!(k+1) \geq 3^k(k+1)$$

Recall that $k \geq 7$

$$\begin{aligned} &\geq 3^k 8 \\ &\geq 3^{k+1} \end{aligned}$$

Therefore, we've shown that

$$(k+1)! \geq 3^{k+1}$$

as required, and so

$$S \supseteq \{7, 8, 9, \ldots\}$$

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