MAT223 - Linear Algebra

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Contents

List of Definitions

List of Theorems

Chapter 1

Week 9

Linear transformations and such

1.1 Linear Transformations

Definition 1 (Linear Transformation). Let V and W be subspaces. A function $\mathcal{T} : V \to W$ is called a linear transformation if for all $\vec{u}, \vec{v} \in V$ and $a \in \mathbb{R}$ it satisfies

1. $\mathcal{T}(\vec{u} + \vec{v}) = \mathcal{T}(\vec{u}) + \mathcal{T}(\vec{v})$

2. $\mathcal{T}(a\vec{u}) = a\mathcal{T}(\vec{u})$

Theorem 1 (Unique Matrix for LT's). For any linear transformation $L : V \to W$ there is a matrix A such that $L(\vec{x}) = A\vec{x}$ for all $\vec{x} \in V$

Example 1.1.1. We'll show that $\mathcal{R}$ is a linear transformation where $\mathcal{R}$ is a counter clockwise rotation of $\frac{\pi}{2}$ radians

$$\mathcal{R}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Let $\vec{u}, \vec{v} \in \mathbb{R}^2$ we know that for some $x_1, y_1, x_2, y_2 \in \mathbb{R}$ that

$$\vec{u} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \text{ and } \vec{v} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

$$\mathcal{R}\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) + \mathcal{R}\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = \begin{bmatrix} -y_1 \\ x_1 \end{bmatrix} + \begin{bmatrix} -y_2 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} -(y_1 + y_2) \\ x_1 + x_2 \end{bmatrix}$$

Which is exactly equal to $\mathcal{R}(\vec{u} + \vec{v})$ as required, then let $\alpha \in \mathbb{R}$ and we know that

$$\mathcal{R}(\alpha\vec{u}) = \begin{bmatrix} -\alpha y_1 \\ \alpha x_1 \end{bmatrix}$$

But also that

$$\alpha\mathcal{R}(\vec{u}) = \begin{bmatrix} -\alpha y_1 \\ \alpha x_1 \end{bmatrix}$$

So then we've shown that $\mathcal{R}(\alpha\vec{u}) = \alpha\mathcal{R}(\vec{u})$ but also that $\mathcal{R}(\vec{u} + \vec{v}) = \mathcal{R}(\vec{u}) + \mathcal{R}(\vec{v})$ as req'd

Example 1.1.2. We'll show that $\mathcal{T} : \mathbb{R}^2 \to \mathbb{R}^2$ where $\mathcal{T}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 2 \\ y \end{bmatrix}$ is not a linear transformation.

Let $\vec{j} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \vec{k} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ we have that

$$\mathcal{T}(\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

But then we can see that

$$\mathcal{T}(\begin{bmatrix} 0 \\ 0 \end{bmatrix}) + \mathcal{T}(\begin{bmatrix} 0 \\ 0 \end{bmatrix}) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

Then we conclude that $\mathcal{T}(\vec{j} + \vec{k}) \neq \mathcal{T}(\vec{j}) + \mathcal{T}(\vec{k})$

Example 1.1.3. We'll show that $\mathcal{P}$ is a linear transformation n

Note 1.1.1. We'll show that it is closed under addition and multiplication

$$\mathcal{P}(\begin{bmatrix} x \\ y \end{bmatrix}) = comp_{\vec{u}} \begin{bmatrix} x \\ y \end{bmatrix}$$

Let $\vec{j}, \vec{k} \in \mathbb{R}^2$ we know that

$$comp_{\vec{u}}\vec{j} = \left(\frac{\vec{u} \cdot \vec{j}}{\|\vec{u}\|^2}\right)\vec{u} \text{ and } comp_{\vec{u}}\vec{k} = \left(\frac{\vec{u} \cdot \vec{k}}{\|\vec{u}\|^2}\right)\vec{u}$$

And thus their product yields

$$comp_{\vec{u}}\vec{j} + comp_{\vec{u}}\vec{k} = \left(\frac{\vec{u} \cdot \left(\vec{j} + \vec{k}\right)}{\|\vec{u}\|^2}\right)\vec{u}$$

Which is equal to

$$comp_{\vec{u}}\left(\vec{j} + \vec{k}\right)$$

We must then show that it holds under multiplication let $\alpha \in \mathbb{R}$ and we know that

$$\alpha\, comp_{\vec{u}}\vec{j} = \alpha\left(\frac{\vec{u} \cdot \vec{j}}{\|\vec{u}\|^2}\right)\vec{u} = \left(\frac{\vec{u} \cdot \alpha\vec{j}}{\|\vec{u}\|^2}\right)\vec{u} = comp_{\vec{u}}\alpha\vec{j}$$

Example 1.1.4. We'll show that $W : \mathbb{R}^2 \to \mathbb{R}^2$ is not a linear transformation, where

$$\mathcal{W}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x^2 \\ y \end{bmatrix}$$

Let $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, and $\alpha \in \mathbb{R}$ we know that

$$\mathcal{W}\left(\alpha\begin{bmatrix} x \\ y \end{bmatrix}\right) = \alpha^2\begin{bmatrix} x^2 \\ y^2 \end{bmatrix} \neq \alpha\begin{bmatrix} x^2 \\ y^2 \end{bmatrix} = \alpha\mathcal{W}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$$

1.2 Image

Definition 2 (Image). Let $L : V \to W$ be a transformation and let $X \subseteq V$ be a set. The **image of the set X under L** , denoted as $L(X)$, is the set

$$L(X) = \left\{\vec{x} \in W : \vec{x} = L(\vec{y}) \text{ for some } \vec{y} \in X\right\}$$

Let $S = \left\{\begin{bmatrix} x \\ y \end{bmatrix} : 0 \leq x, y \leq 1\right\}$ be a filled in unit square in the first quadrant. And let

$C = \left\{\vec{0}, \vec{e_1}, \vec{e_2}, \vec{e_1} + \vec{e_2}\right\} \subseteq \mathbb{R}^2$ be the corners of the unit square

Exercise 1.2.1. We'll find what $\mathcal{R}(C)$ is, by the definition of image we have that

$$\mathcal{R}(C) = \left\{ \mathcal{R}\left(\vec{0}\right), \mathcal{R}(\vec{e_1}), \mathcal{R}(\vec{e_2}), \mathcal{R}(\vec{e_1} + \vec{e_2}) \right\}$$
$$= \left\{ \vec{0}, \vec{e_2}, -\vec{e_1}, \vec{e_2} - \vec{e_1} \right\}$$

Exercise 1.2.2. We'll now find what $\mathcal{W}(C)$ (Notice that it doesn't have to be a linear transformation) is, again we will use the definition so we have

$$\mathcal{W}(C) = \left\{ \vec{0}, \vec{e_1}, \vec{e_2}, \vec{e_1} + \vec{e_2} \right\}$$

Exercise 1.2.3. $\mathcal{T}(C) = \left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$ (The square has been shifted right 2 units)

Exercise 1.2.4. We'll now operate on S, to find $\mathcal{R}(S)$ we imagine all the vectors in $\mathbb{R}^2$ that have been rotated $\frac{\pi}{2}$ radians counter clockwise from the intial square, or we could also multiply by the rotation matrix, either way we get the set

$$\mathcal{R}(S) = \left\{ \begin{bmatrix} -y \\ x \end{bmatrix} : 0 \leq x, y \leq 1 \right\}$$

Exercise 1.2.5. For $\mathcal{T}(S)$ we can re-imagine how we determined $\mathcal{T}(C)$ but for all the points in the square, this gives us the full square shifted horizontally by two units, so we have

$$\mathcal{T}(S) = \left\{ \begin{bmatrix} x + 2 \\ y \end{bmatrix} : 0 \leq x, y \leq 1 \right\}$$

Exercise 1.2.6. As for $\mathcal{P}(S)$ this is a bit more complicated, so we'll break it into to parts, the first is algebreically and the other will be vizually.

Algebraically we know $proj_{\vec{u}} \begin{bmatrix} x \\ y \end{bmatrix}$ will look like

$$\left(\frac{\vec{u} \cdot \begin{bmatrix} x \\ y \end{bmatrix}}{\|\vec{u}\|^2} \right) \vec{u}$$

But $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ so then

$$proj_{\vec{u}} \begin{bmatrix} x \\ y \end{bmatrix} = \left(\frac{2x + 3y}{13} \right) \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

So then we can conclude that

$$P(S) = \left\{ \frac{2x + 3}{13} \begin{bmatrix} 2 \\ 3 \end{bmatrix} : 0 \le x, y \le 1 \right\}$$

Exercise 1.2.7. Let $\ell = \left\{ t\vec{a} + (1 - t)\vec{b} \text{ for some } t \in [0, 1] \right\}$ and $\mathcal{A}$ be a linear transformation, we know that $\mathcal{A}(\ell)$ represents all vectors that are in the range of the linear transformation, let $\vec{u} \in \ell$, then we know that $\vec{u} = t\vec{a} + (1 - t)\vec{b}$ for some $t \in \mathbb{R}$ then we know

$$\mathcal{A}(\vec{u}) = \mathcal{A}\left(t\vec{a} + (1 - t)\vec{b} \right)$$
$$= t\mathcal{A}(\vec{a}) + (1 - t)\mathcal{A}\left(\vec{b} \right)$$

And since we know that $\mathcal{A}(\vec{a}), \mathcal{A}\left(\vec{b} \right)$ are just two transformed vectors, then this defines a new line segment with endpoints $\mathcal{A}(\vec{a})$ and $\mathcal{A}\left(\vec{b} \right)$.

Exercise 1.2.8. We'll now find the linear transformation that italicizes N, FIG below

To determine the linear transformation we start with the fact that if A is some matrix then $A\vec{e_i}$ results in the ith column of A.

We then choose two points and see how they moved after the transormation, per the hint above we'll choose two vectors that reside on the x, y axis. (Our origin is the corner of the N), so our first vector will be $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$ and our second is $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$. Thus we know the following

1. $\mathcal{I}\left(\begin{bmatrix} 0 \\ 3 \end{bmatrix} \right) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, but we know that applying a linear transformation is the same as just multiplying by some matrix so we know that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 3b \\ 3d \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

And so we conclude that $b = \frac{4}{3}$ and $d = \frac{1}{3}$ so we've determined the first column of the matrix

2. $\mathcal{I}\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ thus we know that $2a = 2 \Leftrightarrow a = 1$ and that $b = 0$ and we have our

   second column of the matrix.

So now we now that the matrix must look like

$$\begin{bmatrix} 1 & \frac{4}{3} \\ 0 & \frac{1}{3} \end{bmatrix}$$

1.2.1   From Transformation to Matrix

We defined $\mathcal{P}$ as the $proj_{span\vec{u}}\vec{x}$ where $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\mathcal{R}$ be a rotation ccw by $\frac{\pi}{2}$ radians.

We'll now find the matricies which define each transformation

Example 1.2.1.

$$\mathcal{P}(\vec{x}) = \left(\frac{\vec{u} \cdot \begin{bmatrix} x \\ y \end{bmatrix}}{\|\vec{u}\|^2}\right)\vec{u}$$

$$= \left(\frac{2x + 3y}{13}\right)\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$= \frac{1}{13}\begin{bmatrix} 4x + 6y \\ 6x + 9y \end{bmatrix}$$

By observation we know the matrix must be

$$\frac{1}{13}\begin{bmatrix} 4 & 6 \\ 6 & 9 \end{bmatrix}$$

Example 1.2.2. We'll now find the matrix which defines the rotation $\mathcal{R}$, we start geometrically FIG below.

   We could also use the fact that any matrix A times $\vec{e_i}$ is equal to the i-th column of the matrix A. And we know that rotating $\vec{e_1}$ ccw $\frac{\pi}{2}$ moves it to $\vec{e_2}$ and that $\vec{e_2}$ rotated becomes $-\vec{e_1}$ so then we can determine that the matrix must be

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

1.2.2 Composition of Transformations

We know that $f \circ g(x) = f(g(x))$ and so we can determine that $\mathcal{P} \circ \mathcal{R} = \mathcal{P}(\mathcal{R}(\vec{x})) = \mathcal{P}(\mathcal{R}(\vec{x})) = \mathcal{P}(\mathcal{R}\vec{x}) = \mathcal{P}\mathcal{R}\vec{x}$. So we can determine

$$\mathcal{P} \circ \mathcal{R} = \frac{1}{13} \begin{bmatrix} 4 & 6 \\ 6 & 9 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \frac{1}{13} \begin{bmatrix} 6 & -4 \\ 9 & -6 \end{bmatrix}$$

And that

$$\mathcal{R} \circ \mathcal{P} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \frac{1}{13} \begin{bmatrix} 4 & 6 \\ 6 & 9 \end{bmatrix} = \frac{1}{13} \begin{bmatrix} -6 & -9 \\ 4 & 6 \end{bmatrix}$$

We notice that these matricies are certainly different and that $\mathcal{P} \circ \mathcal{R}$ is first a projection onto $\vec{u}$ and then a rotation, whereas $\mathcal{R} \circ \mathcal{P}$ is first a rotation, then a projection onto $\vec{u}$.

Definition 3 (Range). The **range** of a linear transformatoin $T : V \to W$ is the set of vectors that T can output. That is ,

$$range(T) = \{\vec{y} \in W : \vec{y} = T(\vec{x}) \text{ for some } x \in V\}$$

Definition 4 (Null Space). The **null space** or **kernel** of a linear transformation $T : V \to W$ is the set of vectors that get mapped to zero under T. That is,

$$null(T) = \{\vec{x} \in V : T(\vec{x}) = \vec{0}\}$$

Exercise 1.2.9. Consider $P : \mathbb{R}^2 \to \mathbb{R}^2$ where $\mathcal{P}$ is the projection onto the *span$\vec{u}$* (Remember $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$), we'll determine the range and null space of $\mathcal{P}$.

We know that the projection of any vector onto $\vec{u}$ will be equal to some scalar times $\vec{u}$ so we know that the range will be

$$\alpha\vec{u}, \text{ for all } a \in \mathbb{R}$$

As for the nullspace, we can think of what vectors will get mapped to zero under a projection onto $\vec{u}$ with a bit of thought, we determine that it must be all vectors who are orthogonal to $\vec{u}$ as their "shadow" will drop to the zero vector. We know that this will be all scalar multiples of a normal vector to $\vec{u}$ so we can say $\alpha \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ for all $\alpha \in \mathbb{R}$ or we could first take the image of $\mathcal{P}$ then rotate all of those vectors, so we could say that $null(\mathcal{P}) = $ Image of the *range$P$* under $\mathcal{R}$

Example 1.2.3. We let $T : R^n \to R^m$ be a linear transformation. We'll show that the null space of T is a linear subspace and that the range of T is as well, so we'll show it is closed under addition and multiplication.

Proof. Let $\vec{u}, \vec{x} \in null(T)$

$$T(\vec{u}) = 0 \text{ and } T(\vec{x}) = 0$$

Taking the sum of the above equations we get $0 = T(\vec{u}) + T(\vec{x}) = \mathcal{T}(\vec{x} + \vec{y})$ from the definition of linear transformation. But by the definition of null set, we can conclude that $\vec{x} + \vec{y} \in null(T)$ because $T(\vec{x} + \vec{y}) = 0$.

Let $\alpha \in \mathbb{R}$ and from above we know that $T(\vec{x}) = 0 \Leftrightarrow \alpha T(\vec{x}) = \alpha 0 = 0$ and since T is a linear transformationwe can say that $T(\alpha \vec{x}) = 0$ so then we know that $\alpha \vec{x} \in null(T)$ □

Proof. Let $\vec{j}, \vec{k} \in range(T)$ so we know that

$$\vec{j} = T(\vec{x}) \text{ and } \vec{k} = T(\vec{x_1}) \text{ for some } \vec{x}, \vec{x_1} \in R^n$$

Then we know that $\vec{j} + \vec{k} = T(\vec{x} + \vec{x_1})$ and thus we conclude that $\vec{j} + \vec{k} \in range(T)$

Let $\alpha \in \mathbb{R}$ and we know that $\alpha \vec{j} = \alpha T(\vec{x}) = T(\alpha \vec{x})$ and so $\alpha \vec{j} \in range(T)$ as required. □

1.3 Tutorial 5

1. $\mathcal{B}$ is a basis for a subspace V if $\mathcal{B}$ is linearly independent and $span(\mathcal{B}) = V$.

2. (a) We'll verify if the fectors in $\mathcal{B}$ are in a basis. So for it to be a basis it must be linearly independent, and we'll assume that we are looking for a basis for the subspace $\mathbb{R}^3$. We'll start by looking at the linear combinations of each vector in the matrix that gives the zero vector to determine independendence, so we have (These are augmented matricies for the zero vector.)

$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

And thus we have one solution to $\vec{0}$ and so these vectors are linearly independent, and they span $\mathbb{R}^3$.

We'll now focus our attention to $\mathcal{C}$ using the same process as above we get

$$\begin{bmatrix} 1 & 0 & 2 \\ - & 1 & -2 \\ 1 & 1 & 3 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

And by the same reasoning as before $\mathcal{C}$ is a basis for $\mathbb{R}^3$.

(b) We'll start by finding what $[\vec{v}]_{\mathcal{E}}$ is, but we don't have to do much as $\vec{v} = 4\vec{e_1} - 4\vec{e_2} + 2\vec{e_3}$ so we know that $[\vec{v}]_{\mathcal{E}} = \begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix}$.

Moving to $[\vec{v}]_{\mathcal{B}}$ we know we are looking for some $\alpha, \beta, \gamma \in \mathbb{R}$ that satisfy

$$\alpha \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix}$$

This relates to a system of equations, which is then stored in a matrix, so we have

$$\begin{bmatrix} 1 & 0 & 2 & 4 \\ -1 & 1 & -2 & -4 \\ 0 & 0 & 1 & 2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

And thus we conclude that $\gamma = 2, \beta = 0, \alpha = 4 = 2\gamma = 0$ and so $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$.

Now we'll figure out $[\vec{v}]_{\mathcal{C}}$ so again we apply the same idea to get the following matrix

$$\begin{bmatrix} 1 & 0 & 2 & 4 \\ -1 & 1 & -2 & -4 \\ 1 & 1 & 3 & 2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

And now we conclude that $\gamma = -2, \beta = 0, \alpha = 8$ and thus we have

$$[\vec{v}]_{\mathcal{C}} = \begin{bmatrix} 8 \\ 0 \\ -2 \end{bmatrix}$$

(c) To determine $[7\vec{v}]_{\mathcal{E}}$ we know we are looking for the solution to

$$\alpha_1 \vec{e_1} + \beta_1 \vec{e_2} + \gamma_1 \vec{e_3} = 7\vec{v}$$

But previously we know that the coeficients should be when we were just looking for $\vec{v}$ multiplying that equation by 7 on both sides tells us that

$$[7\vec{v}]_{\mathcal{E}} = \begin{bmatrix} 28 \\ -28 \\ 14 \end{bmatrix}$$

Then using the same process as above we can determine that

$$[7\vec{v}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 14 \end{bmatrix} \text{ and } [\vec{v}]_{\mathcal{C}} = \begin{bmatrix} 57 \\ 0 \\ -14 \end{bmatrix}$$

(d) I would prefer to write my measurements of scalar multiples of $\vec{v}$ in terms of the $\mathcal{B}$ basis as I only have to do one calculation.

3. Get help with this one

4. I chose $\mathcal{E}_3$ as then I could represent the vector $\vec{v} = \begin{bmatrix} 1 \\ .12 \end{bmatrix}$ as $\left[\begin{bmatrix} -11 \\ 12 \end{bmatrix} \right]_{\mathcal{E}_3}$

1.3.1   Video Notes

1.3.2   Extra Textbook Questions

Exercise 1.3.1. A1

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$$
\begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 3 & 1 \\ 1 & 2 & 1 & 6 \\ 2 & 5 & 1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & 2 & 2 & 3 \\ 0 & 5 & 3 & -5 \end{bmatrix} \rightsquigarrow
$$

$$
\begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -4 & 1 \\ 0 & 0 & -12 & -10 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & -13 \end{bmatrix}
$$

We can stop here because we see that in the bottom row we have $0 = -13$ which is certainly false so the system is inconsistent. So we can conclude that $\begin{bmatrix} 3 \\ 1 \\ 6 \\ 1 \end{bmatrix}$ is not in the range of L.

- We'll figure out if $\begin{bmatrix} 3 \\ -5 \\ 1 \\ 5 \end{bmatrix}$ is in the range of L, so we're looking for some vector in $\mathbb{R}^4$

  that becomes that vector under L. Represented as an augmented matrix we get

  $$\begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 3 & -5 \\ 1 & 2 & 1 & 1 \\ 2 & 5 & 1 & 5 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 3 & -5 \\ -2 & -4 & -2 & -2 \\ 0 & 1 & -1 & 3 \end{bmatrix} \rightsquigarrow$$

  $$\begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 3 & -5 \\ 0 & -4 & -4 & 4 \\ 0 & 1 & -1 & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 3 & -5 \\ 0 & 0 & -8 & 16 \\ 0 & 1 & -1 & 3 \end{bmatrix} \rightsquigarrow$$

  $$\begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 3 & -5 \\ 0 & 0 & -4 & 8 \\ 0 & 0 & -8 & 16 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 3 & -5 \\ 0 & 0 & -4 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mid \cdot -\frac{1}{4} \rightsquigarrow \begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 3 & -5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

  And thus we know there is a solution so $\begin{bmatrix} 3 \\ -5 \\ 1 \\ 5 \end{bmatrix}$ is in the range of L.

- A2

  1. We'll start with the range which we know looks like

     $$\left\{ \vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} 2x_1 \\ -x_2 + 2x_3 \end{bmatrix} \right\}$$

     We know that $2x_1$ and $-x_2 + 2x_3$ will span over all of $\mathbb{R}$ and so we know that the range looks like $span\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ and a basis for the range is the set

     $$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

As for the null space we know that x component will be zero only if $x1 = 0$ and that the y component will be zero if and only if $-x2 + 2x_3 = 0 \Leftrightarrow x_2 = 2x_3$ and we let $t \in \mathbb{R}$ and let $x_3 = t$ and our solutions to the zero vector look like

$$\left\{ \vec{x} \in \mathbb{R}^3 : \vec{x} = \begin{bmatrix} 0 \\ 2t \\ t \end{bmatrix} \right\}$$

$$\left\{ \vec{x} \in \mathbb{R}^3 : \vec{x} = t \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}$$

Thus we conclude that a basis for the null space is $\left\{ \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}$

2. We know that the range looks like

$$\left\{ \vec{x} = \begin{bmatrix} x_4 \\ x_3 \\ 0 \\ x_2 \\ x_1 + x_2 - x_3 \end{bmatrix} \right\}$$

$$\left\{ \vec{x} \in \mathbb{R}^5 : \vec{x} = x_3 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_1 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \text{ for some} \right\}$$

And thus a basis for the range is

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

To show they are linearly independent we look at linear the linear combinations that give the zero vector, which would but each scalar has to be 0 so only the trivial answer exists so this set is linearly independent.

- A3

  1. Let's define this transformation as

  $$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = (x - y)\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

  So then the matrix we are looking for is

  $$\begin{bmatrix} 1 & -1 \\ 2 & -2 \\ 3 & -3 \end{bmatrix}$$

  2. We know that this transformation should be

  $$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = (2x + y)\vec{e_3}$$

  And we use $\vec{e_3}$ so we get the range. And thus we know our matrix should be

  $$\begin{bmatrix} 2 & 1 \\ 2 & 1 \\ 2 & 1 \end{bmatrix}$$

- A6

  1. We'll start by finding a basis for the rowspace, the row space is

  $$span\left\{\begin{bmatrix} 1 \\ 2 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}\right\}$$

  A basis for this span is the set of lineary independent that span the row space so let's row reduce the rowspace

  $$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ -2 & 5 & 8 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 7 & 10 \end{bmatrix} \rightsquigarrow$$

  $$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -7 & -14 \\ 0 & 0 & -4 \end{bmatrix} \begin{matrix} | \cdot -\frac{1}{7} \\ | \cdot -\frac{1}{4} \end{matrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

And thus we know the only solution to the zero vector is a trivial one and so this set is linearly independent a basis could be $\{\vec{e_1}, \vec{e_2}, \vec{e_3}\}$. We'll now look at the column space which is the span of each of the columns let's look at the linear combinations of the columns which give the zero vector

$$
\begin{bmatrix} 1 & 2 & 8 \\ 1 & 1 & 5 \\ 1 & 0 & -2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 8 \\ 0 & -1 & -3 \\ 0 & -2 & -10 \end{bmatrix}
$$

$$
\rightsquigarrow \begin{bmatrix} 1 & 2 & 8 \\ 0 & 2 & 6 \\ 0 & 0 & -4 \end{bmatrix} \begin{matrix} \\ | \cdot \frac{1}{2} \\ | \cdot -\frac{1}{4} \end{matrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 8 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}
$$

And thus we know that these vectors are linearly independent so a basis which is a subset of the colspace is

$$
\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ 5 \\ -2 \end{bmatrix} \right\}
$$

Now we'll find the null space which we know are all vectors that get mapped to $\vec{0}$ after being multiplied by this matrix so we're looking for vectors in $\mathbb{R}^3$ that satisfy

$$
\begin{bmatrix} 1 & 2 & 8 \\ 1 & 1 & 5 \\ 1 & 0 & -2 \end{bmatrix} \vec{x} = \vec{0}
$$

But remember this simply represents the linear combinations of the columns of the matrix so we're just looking at the combinations of the columns of the matrix that give the zero vector, which we know is only the trivial one, thus $\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and the null space is $\left\{ \vec{0} \right\}$, so a basis for this is the empty set. As for the rank-nullity theorem, we know that the rank is 3 and the nullity is 0 so the number of columns should be 3 as it is.

2. Same process as above

3. Same process as above

- A8

1. We know that row reduction doesn't change the row space as we are just adding multiples of rows and adding them so we can say that the row space is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

If it were to span a subspace it would have to to span $\mathbb{R}^5$ but we know it doesn't becase we have less than or equal to 3 linearly independent vectors in that set. A basis for the rowspace of A is the set of row vectors which have a pivot since if only these vectors were in a set together then they would only yield a trivial solution to the zero vector which we know would be the set

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

2. If the column space is a basis then it must be a basis for $\mathbb{R}^4$ as these vectors are four dimensional.

1.3.3   Extra Handout
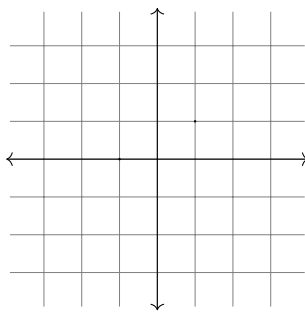
Images and Sets under transformations.

1. (a) We know that $\mathcal{M}(P)$ is equal to

$$\left\{ \vec{x} \in \mathbb{R}^2 : \vec{x} = M\vec{v} \text{ for some } \vec{v} \in P \right\}$$

But that the only elements of P are $\vec{e_1}, \vec{e_2}$ so the image is the set

$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right\}$$

Drawing this we have

(b) We know that the convec linear combinations of P look like

$$\left\{\vec{x} \in \mathbb{R}^2 : \vec{x} = t\vec{e_1} + (1-t)\vec{e_2} = \begin{bmatrix} t \\ (1-t) \end{bmatrix} \text{ for some } t \in [0,1] \right\}$$

And so vectors in the range of $\mathcal{M}$ should be the set

$$\left\{\vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} t \\ (1-t) \end{bmatrix} \right\}$$

which are the vectors

$$\begin{bmatrix} 2t-1 \\ t \end{bmatrix} = (2t-1)\vec{e_1} + t\vec{e_2} \text{ for some } t \in [0,1]$$
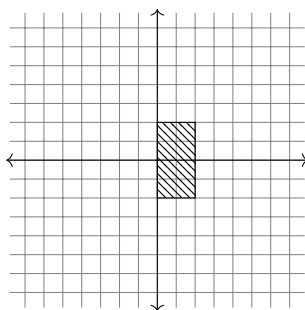
We know that the span of a vector in X will look like $\begin{bmatrix} \alpha \\ 0 \end{bmatrix}$ for some $\alpha \in \mathbb{R}$ and so the range will end up looking like

$$\left\{\vec{x} \in \mathbb{R} : \vec{x} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ 0 \end{bmatrix} \right\}$$

And equivalently

$$\left\{ \vec{x} \in \mathbb{R} : \vec{x} = \begin{bmatrix} \alpha \\ \alpha \end{bmatrix} \right\}$$

which is exactly $span\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ Drawing this we have
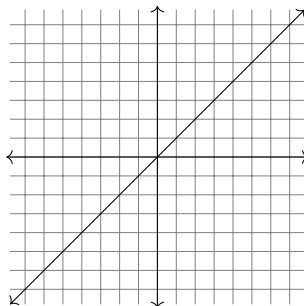
2. Let $\ell = \left\{ a\vec{v} \text{ for some } a \in [0, \beta] \text{ for some } \beta \in \mathbb{R} \right\}$ then we'll look at the image of $\mathcal{M}$ which will look like

$$\left\{ \vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \alpha \begin{bmatrix} x \\ y \end{bmatrix} \right\}$$

Which equivalenlty is

$$\left\{ \vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} \alpha x - \alpha y \\ \alpha x \end{bmatrix} \right\}$$

which gives us the vectors

$$\alpha \begin{bmatrix} x - y \\ x \end{bmatrix}$$

Which by definition is a line segment.
Alternatively we know that a line segment looks like $\alpha v + \vec{u}$ and that the image of this under some linear transformation $\mathcal{J}$ is the set of vectors $\left\{ \mathcal{J}(\alpha \vec{v} + \vec{u}) \right\} = \left\{ \alpha \mathcal{J}(\vec{v}) + \mathcal{J}(\vec{u}) \right\}$ which is also a line segment.

3. Q3

   (a) $\mathcal{A}(\mathbb{R}^2)$ is the set

$$\left\{ \vec{v} \in \mathbb{R}^2 : \vec{v} = \begin{bmatrix} x - 2y \\ -2x + 4y + 2 \end{bmatrix} \text{ for some } \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \right\}$$

(b) $\mathcal{A}(B)$ is the set

$$\left\{ \vec{v} \in \mathbb{R}^2 : \vec{v} = \begin{bmatrix} x - 2y \\ -2x + 4y + 2 \end{bmatrix} \text{ for some } \begin{bmatrix} x \\ y \end{bmatrix} \in B \right\}$$

4. Q4

(a) We know that the the xaxis is defined by the set X

$$\left\{ \vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} \alpha \\ 0 \end{bmatrix} \text{ for some } \alpha \in \mathbb{R} \right\}$$

And so the image under $\mathcal{Q}$ is the set

$$\left\{ \vec{v} \in \mathbb{R}^2 : \vec{v} = \begin{bmatrix} x \\ y^2 \end{bmatrix} \text{ for some } \begin{bmatrix} x \\ y \end{bmatrix} \in X \right\}$$

But we know the y of any vector in X is 0 so this just redfines the X axis and we know that $\mathbb{R}$ is closed under addition and multiplication so this is certianly a subspace and a subset of $\mathbb{R}^2$ for example take an element in X and it's guarnteed to look like $\begin{bmatrix} \alpha \\ 0 \end{bmatrix} \subseteq \mathbb{R}^2$ as required.

(b) Let's call the line segment ℓ and we know that the image of ℓ under $\mathcal{Q}$ is the set

$$\left\{ \vec{v} \in \mathbb{R}^2 : \vec{v} = \begin{bmatrix} 3t \\ (2+t)^2 \end{bmatrix} \text{ for some } t \in [0,1] \right\}$$

We know that $(2+t)^2 = 4 + 4t + t^2$ and so the vectors in the image are of the form

$$\begin{bmatrix} 0 \\ 4 \end{bmatrix} + t \begin{bmatrix} 3 \\ 4+t \end{bmatrix}$$

Which defines a line that has been shifted, nevertheless it ends as $t \in [0,1]$ and thus it is a segment as required.

5. A line segment is a line that is described by $\alpha \vec{v}$ for some $a \in [0, \beta]$ for some $\beta \in \mathbb{R}$ we know if we look at $\ell = \{\alpha \vec{v} \text{ for some } \alpha \in \mathbb{R}\}$ where $\vec{v} \neq \vec{0}$ and $\vec{v} \neq \vec{e_1}$ and $\vec{v} \neq \vec{e_2}$ that the image will result in a vshape which cannot be described as a single line segment

6. Q5

(a)

$$\left\{ \vec{v} \in \mathbb{R}^2 : \vec{v} = \begin{bmatrix} e^y \cos x \\ e^y \sin x \end{bmatrix} \text{ for some } \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \right\}$$

Chapter 2

Week 10

2.1 Lecture 1

Null space & Range

Definition 5 (Induced Transformation)**.** Let M be an $n \times m$ matrix. We say M **induces** a linear transformation $T_M : \mathbb{R}^m \to \mathbb{R}^M$ defined by

$$[T_M \vec{v}]_{\mathcal{E}'} = M[\vec{v}]_{\mathcal{E}}$$

Notice that $\mathcal{E}'$ and $\mathcal{E}$ are standard basis

Example 2.1.1. For example from the above definition if we have the vector $\vec{v} = \left[\begin{bmatrix} 1 \\ 2 \end{bmatrix} \right]_{\mathcal{B}}$ and we wanted to determine what $T_M(\vec{v})$ is we must first translate $\left[\begin{bmatrix} 1 \\ 2 \end{bmatrix} \right]_{\mathcal{B}}$ to the standard basis. One method to do so is to find the matrix X that translates from one basis to another. Then plug that new box of numbers into $M[\vec{v}]_{\mathcal{E}}$ and we get our solution to $T_M(\vec{v})$

Example 2.1.2. The difference between $M\vec{v}$ and $M[\vec{v}]_{\mathcal{E}}$ is the level of rigor? on the right we have something precise but on the left we don't. Also we can remember that $[\vec{v}]_{\mathcal{E}}$ represents a box of numbers, the coordinates in the basis, where as $\vec{v}$ is assumed to be in the $\mathcal{E}$ basis but this could alos mean a vector just floating in space without any type of basis involved.

Example 2.1.3. To determine what $[T_M \vec{e_1}]_{\mathcal{E}'}$ use the fact that it's equal to $M[\vec{e_1}]_{\mathcal{E}}$ and we know that this corresponds to the first column in M as $[\vec{e_1}]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Example 2.1.4. We start by knowing that each column must be in the range of T_M by inputting $\vec{e_i}$ into the equation. But we showed last time that the range is a subspace which means that if we find two elements in it, we know the sum and product of the vectors is also in the range, so we know that the span of each column is a susbset of the range. We then want to see if the span of the columns is actually eqaual to the range, so we must show the other direction now. So let $\vec{x_1}, \vec{x_2} \in range(T_M)$ so we know that $\vec{x} = T_M(\vec{v})$ for some $\vec{v} \in \mathbb{R}^m \Leftrightarrow [\vec{x}]_{\mathcal{E}'} = \left[T_M(\vec{v}) \right]_{\mathcal{E}'} \Leftrightarrow [\vec{x}]_{\mathcal{E}'} = M[\vec{v}]_{\mathcal{E}}$ and we recall that M is only a list of vectors and that $[\vec{v}]_{\mathcal{E}}$ is but a box of numbers and so using matrix vector multiplication we know that we just get a linear combination of the vectors as columns in M and so showed that $\vec{x}$ looks like a vector in the span of the columns of M as we wanted.

Definition 6 (Fundamental Subspaces)**.** Associated with any matrix M are three fundamental subspaces: the **row space** of M, denoted $row(M)$, is the span of the rows of M the **column space** of M, denoted $col(M)$, is the span of the columns of M; and the **null space** of M, denoted $null(M)$, is the set of solutions to $M\vec{x} = 0$

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Example 2.1.5. We know that the row space is $span\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ or

$$\left\{ \begin{bmatrix} \alpha \\ \beta \\ 0 \end{bmatrix} \in \mathbb{R}^3, \text{ for some } \alpha, \beta \in \mathbb{R} \right\}$$

Example 2.1.6. The column space of A is $span\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$ or equivalently $\mathbb{R}^2$, we can tell instantly that the column space is not the same as the row space as one is in $\mathbb{R}^3$ where the othe is $\mathbb{R}^2$.

Example 2.1.7. We are looking for all the vectors that are perpendicular to $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

so we require the the vectors $\vec{x} \in \mathbb{R}^3$ such that

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \text{ and } \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$x + y = 0$$
$$x = -y$$

And so we know all solutions are

$$\alpha \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \text{ for all } \alpha, \beta \in \mathbb{R}$$

Example 2.1.8. For the null space of A we know we are looking for all vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$

so that

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} y + \begin{bmatrix} 0 \\ 0 \end{bmatrix} z = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So we determine that $x = y = 0$ but $z \in \mathbb{R}$ so we can think of this as the z axis coming out from the x y plane.

Example 2.1.9. The range of $T_A : \mathbb{R}^3 \rightarrow \mathbb{R}^2$

$$\left\{ \vec{u} \in \mathbb{R}^2 : T_A(\vec{x}) = \vec{u}, \text{ for some } \vec{x} \in \mathbb{R}^3 \right\}$$

$$\left\{ \vec{u} \in \mathbb{R}^2 : \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \vec{u}, \text{ for some } \vec{x} \in \mathbb{R}^3 \right\}$$

Which we know is saying that we are looking for the vectors $\vec{u}$ such that they are linear combinations of the columns of the matrix A so equivalently we have that $\vec{u} \in span\{\vec{e_1}, \vec{e_2}\}$

We'll now look for the $null(T_A)$ we know that the nullspace is defined by

$$\left\{ \vec{x} \in \mathbb{R}^3 : T_A(\vec{x}) = 0 \right\}$$

So we're really looking for the vectors which satisfy $A\vec{x} = 0$, which is equivalent to $null(A)$

Let $B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$ $C = rref(B) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$

Example 2.1.10. The row space of B is equal to $span\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ and the row reducing

involves the operations of addition and multiplication of the rows together, let $\vec{x}, \vec{u}$ represnt the first and second vectors in the span, we know that in the row reduced form (considering rows to be vectors) we have that the first row

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \vec{u} - (\vec{x} - \vec{u})$$

and the second row

$$\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \vec{x} - \vec{u}$$

Thus we know that the rows in the row reduced form are just linear combinations of the original rows, so we can say that

$$row(B) = row(C)$$

Example 2.1.11. As for $null(B) = \{\vec{x} : B\vec{x} = 0\}$ the process of finding soltions to the zero vector will be identical to just row reducing, where it's an augmented matrix with 0's on the right side, so due to the same reasoning as before we know that they will have the same solutions as row operations don't change the number of solutions so we can say that

$$null(B) = null(C)$$

Example 2.1.12. Now if we are asked to compute the null space of B then we can make things quite a bit easier on ourselves by using the row reduced matrix so well have

$$\left\{ \vec{x} \in \mathbb{R}^3 : B\vec{x} = 0 \right\}$$

So we're looking at the solutions to

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

Let $z = t$ for some $t \in \mathbb{R}$ then we determine $y = -2t$ and $x = t$ in general form we get

$$t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \text{ for all } t \in \mathbb{R}$$

Let $P = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}$ and $Q = rref(P) = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$

Example 2.1.13. $col(P) = span\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\} = span\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ and $col(Q) = span\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$
We notice these are different so then row reduction can change the columns space unlike the row space.

Definition 7 (Rank). For a linear transformation $T : V \to W$, the **rank** of T, denoted $rank(T)$, is the dimension of the range of T
   For an $n \times m$ matrix M, the **rank** of M, denoted $rank(M)$, is the number of pivots in $rref(M)$

Let $\mathcal{P}$ be the projection onto $span\{\vec{u}\}$ where $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, and let $\mathcal{R}$ be a rotation counterclockwise by $\frac{\pi}{2}$ radians.

Example 2.1.14. $range(\mathcal{P})$ is equal to $span\vec{u}$ because we know that every vector you choose will become some a scalar of $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ thus the dimension is 1, so it's rank is 1.

   As for the rank of $\mathcal{R}$ we know that iwe we consider every vector in $\mathbb{R}^2$ and rotate them all $\frac{\pi}{2}$ radians then we still have all of $\mathbb{R}^2$ so we know that the rank is still 2.

Example 2.1.15. We'll now find the rank of the matricies that go along with $\mathcal{P}$ and $\mathcal{R}$ first we know that the matricies are

$$\mathcal{P} = \frac{1}{13} \begin{bmatrix} 4 & 6 \\ 6 & 9 \end{bmatrix} \qquad\qquad \mathcal{R} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Starting with $\mathcal{P}$

$$\begin{bmatrix} 4 & 6 \\ 6 & 9 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 24 & 36 \\ 24 & 36 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 24 & 36 \\ 0 & 0 \end{bmatrix}$$

And so we know that there is one pivot so the rank of this matrix is 1

As for $\mathcal{R}$ by swapping row one and row two we see that there must be two pivots and so the rank is 2.

2.2   Lecture 2

Let M be a matrix and recall that

$$row(M) = span \text{ rows of M} \qquad\qquad col(M) = span \text{ cols of M}$$

And that the following properties hold

$$row(M) = row\big(rref(M)\big) \qquad\qquad col(M) \neq col\big(rref(M)\big)$$

Also we know that
$$rank(M) = \text{ pivots of } rref(M)$$

Example 2.2.1. If M looked like this after row reduction we know it's rank is two (two pivots)

$$\begin{bmatrix} a & b & c \\ d & e & f \\ j & i & k \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 0 \end{bmatrix}$$

Corollary 2. $rank(M) = $ # linearly independent columns of $rref(M)$

$$\begin{aligned} &= dim\Big(col\big(rref(M)\big)\Big) \\ &= dim\big(col(M)\big) \\ &= \text{ # linearly independent columns of } M \\ &= dim\Big(row\big(rref(M)\big)\Big) \\ &= dim\big(row(M)\big) \end{aligned}$$

$\Big)$ *Solutions aren't lost*

$\Big)$ *pivots are both horizontal and vertical*

Definition 8 (Transpose of a Matrix)**.** The transpose of a matrix M is the matrix M^T whose rows are the columns of M

Example 2.2.2. If

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Leftrightarrow M^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

Or

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \Leftrightarrow M^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Corollary 3.

$$rank(M) = rank\left(M^T\right)$$

because $dim\big(col(M)\big) = dim\big(row(M)\big)$ and we know that $col(M) = row\left(M^T\right)$

Find the rank of the following matricies

Exercise 2.2.1. $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ we can see that there is one linearly independent vector so the dimension is 1 and rank is 1

Exercise 2.2.2. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ we can see that there are 2 linearly independent vectors so dimension is 2 and rank is 2

Exercise 2.2.3. $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ We know that the dimension of the column space is 2 and thus rank is 2

Exercise 2.2.4. $\begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}$ the rank is 1, there is only one column

Let M be a 3×4 matrix whose rank is 3 does this mean that all of the columns of M are linearly independent ?

No! if the matrix is 3×4 then we know that there are four columns, and if the rank is 3 we know 3 of the are linearly independent but we don't know about the last one, so we can't be sure.

What if M was 4×3 then we could be sure that every column is linearly independent as we know that it must have 3 pivots and so it must be true.

Theorem 4 (Rank-nullity)**.** The **nullity** of a matrix is the dimension of the null space. The rank-nullity theorem for a matrix A states

$$rank(A) + nullity(A) = \# \text{ of columns of } A$$

Example 2.2.3. If we have

$$\begin{bmatrix} . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & . & . & . \\ 0 & 1 & . & . & . \\ 0 & 0 & . & . & . \end{bmatrix}$$

And let's assume that the last 3 columns are free variable columns so then we know that the solution to the zero vector will have three paramers $t_1, t_2, t_3 \in \mathbb{R}$ and so we know that the dimension of the nullspace will be 3 .

Example 2.2.4. Let $\vec{u}, \vec{v} \in \mathbb{R}^9$ be linearly independent and $\vec{w} = 2\vec{u} - \vec{v}$

$$A = \begin{bmatrix} \vec{v} \mid \vec{u} \mid \vec{w} \end{bmatrix}$$

We instantly know that the dimension of the column space of A is 2 and so rank is 2 thus nullity is 1 by R-N-T

As for A^T we know that A has two pivot columns and so A^T also has 2, and so by R-N-T it has nullity of 7.

Theorem 5 (Rank-nullity for linear transformations)**.** Let $T : V \to W$ be a linear transformation thus there is a matrix A such that $T = T_A$ so

$$T(\vec{x}) = A\vec{x} \text{ for all } \vec{x} \in V$$

We know from theorem 4 that

$$\begin{aligned} \# \text{ columns of } A &= rank(A) + nullity(A) \\ &= dim\big(col(A)\big) + dim\big(null(A)\big) \\ &= dim\big(range(T)\big) + dim\big(null(T)\big) \end{aligned}$$

So we can finally say that

$$dim\big(range(T)\big) + dim\big(null(T)\big) = \# \text{ columns of } A$$

2.2.1   Inverse Matrix

Definition 9 (Inverse Transformation)**.** If it exsits, the inverse transformation of $T : R^n \to R^m$ is

$$T^{-1} : R^m \to R^n$$

That satisfies the following properties

$$T\Big(T^{-1}(\vec{x})\Big) = \vec{x} \text{ for all } \vec{x} \in \mathbb{R}^n$$
$$T^{-1}\big(T(\vec{y})\big) = \vec{y} \text{ for all } \vec{y} \in \mathbb{R}^m$$

If $T = T_A$ and $T^{-1} = T_B$ then

$$\vec{x} = T\left(T^{-1}(\vec{x})\right)$$
$$= T(B\vec{x})$$
$$= AB\vec{x}$$

But for this to be true for all $\vec{x} \in \mathbb{R}^m$ then we require

$$AB = I$$

And if A is $n \times m$ and B is $m \times n$ then we know that I is $n \times n$
Similarly $T^{-1}\left(T(\vec{y})\right) = \vec{y} \Leftrightarrow BA = I$
Thus we say B is the inverse Matirx of A denoted $B = A$

Definition 10 (Inverse Matrix). We say B is the inverse matrix of A denoted $B = A^{-1}$ if

$$AB = BA = I$$

2.3   Tutorial 6

1. We say the function $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear if we can say

$$T(\vec{x} + \alpha\vec{y}) = T(\vec{x}) + \alpha T(\vec{y}) \text{ for all } \vec{x}, \vec{y} \in \mathbb{R}n$$

2. (a) *Proof.* We'll show that $\mathcal{A}$ is linear so let $\vec{u}, \vec{v} \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$ we know that

$$T(\vec{u} + \alpha\vec{v}) = T\left(\begin{bmatrix} x_1 + \alpha x_2 \\ y1 + \alpha y_2 \end{bmatrix}\right) = \begin{bmatrix} y_1 + \alpha y_2 \\ 0 \\ x_1 + \alpha x_2 \end{bmatrix} = T(\vec{u}) + \alpha T(\vec{v})$$

$\square$

(b) We will show that $\mathcal{B}$ is not a linear transformation we know

$$\mathcal{B}\left(\begin{bmatrix} -2 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \neq \mathcal{B}\left(\begin{bmatrix} -2 \\ 0 \end{bmatrix}\right) + \mathcal{B}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

(c) We will show that $\mathcal{C}$ is a linear transformation it follows that

$$\mathcal{C}(\vec{u} + \alpha\vec{v}) = \vec{0} = \vec{0} + \alpha\vec{0} = \mathcal{C}(\vec{u}) + \alpha\mathcal{C}(\vec{v})$$

(d) We will show that $\mathcal{D}$ is not a linear transformation

$$\mathcal{D}(\vec{u} + \alpha\vec{v}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \mathcal{D}(\vec{u}) + \alpha\mathcal{D}(\vec{u})$$

3. We will now compute the null space, range and rank of each of the linear transformations

   (a) We'll start by comuputing the range which we can we can say is

   $$\left\{ \vec{v} \in \mathbb{R}^3 : \vec{v} = s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ for some } t, s \in \mathbb{R} \right\}$$

   we know that that null space is only $\vec{0}$, and that the rank must be 2 as a basis for the range is

   $$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

   (b) We'll start with the range
   $$\{x \in \mathbb{R} : x \geq 0\}$$

   And the null space is
   $$\{x \in \mathbb{R} : x < 0\}$$

   So then the rank of $\mathcal{B}$ is 0

   (c) For $\mathcal{C}$ we know the range is

   $$\left\{ \vec{0} \right\}$$

   And then null space is

   $$\left\{ \vec{x} \in \mathbb{R}^2 \right\}$$

   And then rank is 0

   (d) The range of $\mathcal{D}$ is $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ and the null space is $\emptyset$ since $\mathcal{D}$ is not a subspace then there is no basis for it, so the rank doesn't exist.

4. (a) $\mathcal{X}$: the tranformation invoked by the identity matrix.

   (b) $\mathcal{Y}$: cannot exist as we know $range(Y) \subseteq \mathbb{R}^2$ and so the dimension of $\mathcal{Y} \leq 2$

(c) $\mathcal{Z} : comp_{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} \vec{x}$ which defines a line in 3d space thus has dimension 1 so rank is 1

(d) $\mathcal{W}$: the identity transformation

(e) Impossible?

Chapter 3

Week 11

3.1 Inverses

3.1.1 56

We start by applying some row operations to some identity matrices

$$
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = E_1
$$

$$
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} = E_2
$$

Applying E_1 and E_2 to A we can see...

$$
\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 \cdot 1 + 7 & 2 \cdot 2 + 8 & 2 \cdot 3 + 9 \end{bmatrix}
$$

Which we note is the same thing as adding twice the first row to the third. Then we know that E_2A would be subtracting two of row 1 from row 3.

If we apply E_1 then E_2 then we just added and subtraced the first row which does nothing so $E_1 E_2 = I = E_2 E_1$

3.1.2 57

We know that matrix A represents adding twice row 2 to row1 and adding -3 row 1 to row 3, by observation these operations will turn D into the identity matrix, now let's look at

it the other way.

We can see that matrix D represents adding 3 of row 1 to row 3 then adding -2 row 2 to row 1 which we can tell are the inverse operations as what A does so we know this will yield the identity matrix. So we can say A and D are inverses of each other.

We'll now prove why the algorithm to determine the inverse matrix actually works, for example if we have the matrix

$$\begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix}$$

Then we would set something up like this

$$\left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{array}\right]$$

Where on the right hand side we have the matrix that will hold an of the ro operations we do to transform this matrix on the left into the identity matrix, thus once we're done the matrix on the right will represent all the row operations that would tern it into the identity matrix.

So it's just a way of keeping track of two different pieces of data visually.

multiply row 2 by $\frac{1}{2}$, so now the matrix on the right represents this operation

$$\left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{2} \end{array}\right]$$

add -4 of row 2 to row 1

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & -2 \\ 0 & 1 & 0 & -\frac{1}{2} \end{array}\right]$$

And then we know the matrix on the right represents applying the two row operations that turn the matrix on the left into the identity matrix upon multiplication by the matrix. Thus we've found the inverse matrix for the left matrix.

Theorem 6 (Inverse Existance)**.** A matrix has an inverse if and only if it is square and can be "row reduced" to the identity matrix.

We know you can reach the identity matrix if you can reach row reduced echolelon form which means you must have a pivot for each column, so we also require that the columns must be linearly independent.

Remark 3.1.1. If it's not square it's not one to one and onto, you are losing a dimesion.

What does this mean?

Remark 3.1.2. Equivalent to theorem 6 we can require the rows be linearly independent as we know that $dim\big(row(A)\big) = dim\big(col(A)\big) = rank(A)$

Why do we know this again?

59

By definition $A^{-1}A = I$ and so since we know that A has an inverse, then then we know there we know it must have been able to be reduced to the I with row operations, thus $rank(A) = I$

Example 3.1.1. Let C be the matrix whose left columns are A and rightmost column is $\vec{b}$ we know that this represents the following

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{b}$$

so then we can say that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}\vec{b}$$

by multiplying by the inverse of A to both sides.

60

We know that if we have two square matricies X, Y that $(XY)^{-1}$ is the matrix A such that $A(XY) = I$ and we require that $(XY)A = I$ in this case we know we just need the inverses next to eachother and this will cause a chain reaction of identity matrices so we can see that $(XY)^{-1} = Y^{-1}X^{-1}$

61

We know that $\vec{e_1} = \frac{1}{2}\vec{b_1} + \frac{1}{2}\vec{b_2}$ and so $[\vec{e_1}]_{\mathcal{B}} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$ and then $\vec{e_2} = \frac{1}{2}\vec{b_1} - \frac{1}{2}\vec{b_2}$ so we know that $[\vec{e_2}]_{\mathcal{B}} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$.

We know that $X = \begin{bmatrix} \vec{b_1} & \vec{b_2} \end{bmatrix}$ let's assume we have a vector such that $[v]_{\mathcal{B}} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ so we know that X converts to standard basis.

$$X \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \alpha\vec{b_1} + \beta\vec{b_2} = [\vec{v}]_{\mathcal{E}}$$

And thus we know $X[\vec{e_2}]_{\mathcal{B}} = [\vec{e_2}]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $X[31]_{\mathcal{B}} = [\vec{e_1}]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

62

- We know that X^{-1} should exists since the vectors in $\mathcal{B}$ form a basis so each is linearly independent thus row reduction will lead to reduced row echelon form and since X is square we know that X^{-1} must exist.

- We know that if X is the matrix that converts from $\mathcal{B}$ to $\mathcal{E}$ and so X^{-1} must convert
  $\mathcal{E} \to \mathcal{B}$

explain why this is true

$$X^{-1}[\vec{v}]_{\mathcal{E}} = [\vec{v}]_{\mathcal{B}}$$

3.2   Lecture 2 Week 11

Recall: we say M induces the linear transformation

$$T : \mathbb{R}^n \to \mathbb{R}^n \qquad (\alpha)$$

Or M is the matrix corresponding to T if

$$\left[T(\vec{v})\right]_{\mathcal{E}} = M[\vec{v}]_{\mathcal{E}}$$

Where $\mathcal{E}$ is the standard basis for $\mathbb{R}^n$ and $T_M = T$
If we don't sepcify the basis, then $\mathcal{E}$ is assumed. So we can replace α by $T(\vec{v}) = M\vec{v}$
    Some properties follow

Prove this!

1. We say T_M and T_N are inverses $\Leftrightarrow$ M and N are inverses

2. T_M is invertible $\Leftrightarrow$ M is invertaible in which case

$$(T_M)^{-1} = T_{M^{-1}}$$

3. We define the folowing notation
$$M = [T]_{\mathcal{E}}$$

   Means that M is the matrix for T in the $\mathcal{E}$ basis such that α holds

4. If $\mathcal{B}$ is a nother basis for $\mathbb{R}^n$ and $A = [T]_{\mathcal{B}}$ then A is the matrix for T in the basis $\mathcal{B}$
   so we know
$$\left[T(\vec{v})\right]_{\mathcal{B}} = A[\vec{v}]_{\mathcal{B}} \text{ for all } \vec{v} \in \mathbb{R}^n$$

Example 3.2.1. Let $n = 2$ $\mathcal{B} = \left\{\vec{b_1}, \vec{b_2}\right\}$ and $\left[\vec{b_1}\right]_{\mathcal{E}}$ and $\left[\vec{b_2}\right]_{\mathcal{E}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ so this means

$$\vec{b_1} = 1\vec{e_1} + 1\vec{e_2} \qquad\qquad \vec{b_2} = 0\vec{e_1}1 + 1\vec{e_2}2$$

And sayign $b_1 = \begin{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{bmatrix}_{\mathcal{E}}$ is the vector whose $\mathcal{E}$ coordinaqtes are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and similarly b_2

coordinates are $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ when we are looking at the $\mathcal{E}$ basis.

Example 3.2.2. If $[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ then we can say

$$T\left(\begin{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{bmatrix}_{\mathcal{E}}\right) = T\left(\vec{b1}\right) = T\left(\begin{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix}\right)$$

Remember here that $\begin{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{bmatrix}_{\mathcal{E}}$ is the vector in $\mathcal{E}$ cordinates And the result of this will be

$$\begin{bmatrix} A\begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}_{B} = \vec{b_1} + 3\vec{b_2} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}_{\mathcal{E}}$$

Remark 3.2.1. say we have $[\vec{v}]_{\mathcal{E}}$ and we are trying to find what is $\left[T(\vec{v})\right]_{\mathcal{E}}$ and all we know is A which is the matrix which which induces the transformation T for vectors written in the $\mathcal{B}$ basis, then the process could look like this.

$$[\vec{v}]_{\mathcal{E}} \implies [\vec{v}]_{\mathcal{B}} \implies A[\vec{v}]_{\mathcal{B}} = \left[T(\vec{v})\right]_{\mathcal{B}} \implies \text{ convert to } \mathcal{E} \implies \left[T(\vec{v})\right]_{\mathcal{E}}$$