Property: Constant in Limit

Assume that the following limit exists $\lim_{x \to a} f(x)$ *and define* $\mathcal{L}$ *to be it's value, then for any* $c \in \mathbb{R}$

$$\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$$

Proof

- *If* $c = 0$ *then using the fact that the limit of a constant is that constant itself, we have:*

$$\lim_{x \to a} [0f(x)] = \lim_{x \to a} 0 = 0 = 0 \lim_{x \to a} f(x)$$

- *If* $c \neq 0$*, we must prove that for any* $\varepsilon_c \in \mathbb{R}^{>0}$ *there exists* δ_c *such that for all* $x_c \in \operatorname{dom}(f)$

$$|x_c - a| \leq \delta_c \Rightarrow |cf(x_c) - c\mathcal{L}| \leq \varepsilon_c$$

- *Notice that if we were to let* ε *in the orignal definition be equal to* $\frac{\varepsilon_c}{|c|}$ *then we could multiply the equation after the implication on both sides by* $|c|$ *(so that we can absorb it into the absolute value).*

- *Let* $\varepsilon_c \in \mathbb{R}^{>0}$ *since the original limit holds for any epsilon, bind* ε *to* $\frac{\varepsilon_c}{|c|}$ *and we get* δ *such that for all* $x \in \operatorname{dom}(f)$*, the following holds:*

$$|x - a| \leq \delta \Rightarrow |f(x) - \mathcal{L}| \leq \frac{\varepsilon_c}{|c|} \tag{α}$$

- *Take* $\delta_c = \delta$*, let* $x_c \in \operatorname{dom}(f)$ *and bind* x *in the original definition to* x_c*, and assume that* $|x_c - a| \leq \delta_c$*, because of our choice for* δ_c *we satisfy* α*'s hypothesis with* x *replaced by* x_c *and we get*

$$|f(x_c) - \mathcal{L}| \leq \frac{\varepsilon_c}{c} \Leftrightarrow |c| \, |f(x) - \mathcal{L}| \leq \varepsilon_c$$

- *Since for any* $a, b \in \mathbb{R}$ *we have* $|ab| = |a| \, |b|$ *we can conclude with distributivity in* $\mathbb{R}$ *that*

$$|cf(x_c) - c\mathcal{L}| \leq \varepsilon_c$$

As required.

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