## Theorem: S Open iff It Shares Nothing with Boundary

$$S \text{ is open} \Leftrightarrow S \cap \partial S = \varnothing$$

## Proof

- $\Rightarrow$
  - *Assume that* $S$ *is open, that is* $S = \mathring{S}$ *so we have* $S \cap \partial S = \mathring{S} \cap \partial S \stackrel{\text{D}}{=} \mathring{S} \cap \left(\overline{S} \setminus \mathring{S}\right)$, *thus there can be no element in this set for it would have to be in* $\mathring{S}$ *but also not be in* $\mathring{S}$, *thus*
  $$S \cap \partial S = \varnothing$$
- $\Leftarrow$
  - *We will prove the contrapositive, that is if* $S$ *is not open, then* $S \cap \partial S \neq \varnothing$
  - *Since* $S$ *is not open, we have* $x \in S \setminus \mathring{S}$, *since* $S \subseteq \overline{S}$ *we know*
  $$x \in S \setminus \mathring{S} \subseteq \left(\overline{S} \setminus \mathring{S}\right) \stackrel{\text{D}}{=} \partial S$$
  - *So* $x \in S$ *and also* $x \in \partial S$, *therefore*
  $$S \cap \partial S \neq \varnothing$$

$\blacksquare$