Deduction: Quotient Remainder

$\forall n \in \mathbb{Z}, d \in \mathbb{N}^{\geq 1}, \exists! q, r \in \mathbb{Z}$ *such that* $n = dq + r$ & $0 \leq r < d$

Proof

- *Let* $n \in \mathbb{Z}, d \in \mathbb{N}^{\geq 1}$ *and set*
$$S = \{n - dk : n - dk \in \mathbb{N} \wedge k \in \mathbb{Z}\}$$
- *Claim* $|S| \geq 1$
  - *If* $n \geq 0$, *them* $n = n - d0 \in \mathbb{N}$, *so* $n \in S$
  - *Else* $n < 0$,$n - nd = n(1-d) \geq 0$
    * *Because* $d \geq 1 \Leftrightarrow 0 \geq 1 - d$, *and* $n < 0$
    * *So* $n - nd \in S$
- *By the principle of well ordering there is a least element* $r \in S$ *and therefore we have* $q \in \mathbb{Z}$ *such that* $r = n - dq \Leftrightarrow n = r + dq \quad (\alpha)$
- *One must show that* $r < d$
  - *But if* $r \geq d \quad (\beta)$, *then* $n - d(q+1) = n - dq - d \overset{\alpha}{=} r - d \overset{\beta}{\geq} 0$
  - *Then* $n - d(q+1) \in S$, *but* $n - d(q+1) < n - dq = r$, *so then* r *would not have been the smallest element in* S
  - *That is a contradiction, therefore* $r < d$

Uniqueness

- *Note without assumption on* $n \in \mathbb{Z}$ *by the above prove we get* $q_1, r_1, q_2, r_2 \in \mathbb{Z}$ *such that* $n = dq_1 + r_1$ *and* $n = dq_2 + r_2$, *then we obtain:*
$$r_1 - r_2 = d(q_1 - q_2)$$
  - *so* $q \mid (r_1 - r_2)$, *then since* $0 \leq r_1, r_2 < d$ *then we know that*
$$-d < r_1 - r_2 < d$$
  - *But since* $d \mid r_1 - r_2$ *then* $r_1 - r_2 = 0$ *and we get* $r_1 = r_2$, *if that's the case then* $d(q_1 - q_2) = 0$ *but* $d > 0$ *so similarly we have* $q_1 = q_2$

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