Theorem: Number of K-Combinations of a Set

Let X be a set where $|X| = n \in \mathbb{N}$, the number of k-combinations of X is equal to

$$^{n}C_k \overset{\texttt{N}}{=} \binom{n}{k} \overset{\texttt{D}}{=} \frac{n\,(n-1)\cdots(n-k+1)}{k\,(k-1)\cdots 1} = \frac{n!}{(n-k)!\cdot k!}$$

Proof

- *Recall that the number of k permutations on n elements is*
$$\frac{n!}{(n-k)!}$$
- *Consider $x_1, x_2, \ldots, x_{k-1}, x_k \in X$, then there are $k!$ different permutations of these elements, but each of these permutations only corresponds to a single k-combination.*
- *This means that for each possible choice of $x_1, x_2, \ldots, x_{k-1}, x_k$ we are overcounting $k!$ times, that is:*
$$\frac{n!}{(n-k)!} = \sum_{\{x_1,x_2,\ldots,x_{k-1},x_k\}\subseteq X} k! \Leftrightarrow \frac{n!}{k!\,(n-k)!} = \sum_{\{x_1,x_2,\ldots,x_{k-1},x_k\}\subseteq X} 1$$
- *And $\sum_{\{x_1,x_2,\ldots,x_{k-1},x_k\}\subseteq X} 1$ is the number of k-combinations of X thus*
$$^{n}C_k = \frac{n!}{k!\,(n-k)!}$$
as required.

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