Theorem: Differentiability implies Continuity

If $f(x)$ is differentiable at a then $f(x)$ is continuous at a

Proof

- *Because $f(x)$ is differentiable at a we know that the following limit exists*
$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$
- *We also note that the following limit exists as well:*
$$\lim_{x\to a} x - a = 0$$
- *By the product law for limits we obtain that*
$$\lim_{x\to a} \left[\frac{f(x) - f(a)}{x - a} \cdot (x - a)\right] = 0 \cdot f'(a) = 0$$
And note that
$$\lim_{x\to a} \left[\frac{f(x) - f(a)}{x - a} \cdot (x - a)\right] = \lim_{x\to a} [f(x) - f(a)]$$
- *Therefore*
$$\lim_{x\to a} [f(x) - f(a)] = 0$$
- *Since* $\lim_{x\to a} f(a) = f(a)$ *we can use the sum rule to obtain*
$$\begin{aligned} \lim_{x\to a} f(a) + \lim_{x\to a} [f(x) - f(a)] &= \lim_{x\to a} [f(a) + f(x) - f(a)] \\ &= \lim_{x\to a} f(x) \end{aligned}$$
- *But recall that*
$$\lim_{x\to a} f(a) + \lim_{x\to a} [f(x) - f(a)] = f(a) + 0$$
- *Thus we know that*
$$\lim_{x\to a} f(x) = f(a)$$
as required.

$\blacksquare$