Proposition: Little O and Differentiability Equivalence

A function f is differentiable at the point $\overline{x}$ if and only if there is a number $\alpha \in \mathbb{R}$ for any $h \in \mathbb{R}$ we have

$$f\left(\overline{x}+h\right)=f\left(\overline{x}\right)+h\alpha+l\left(h\right)$$

Where $l \in o\left(h\right)$

Proof

Observe the following sequence of logic

- *Definition of differentiability at $\overline{x}$, the following limit exists :*

$$f'\left(\overline{x}\right)=\lim_{h\to 0}\frac{f\left(\overline{x}+h\right)-f\left(\overline{x}\right)}{h}$$

- *For any $m \in \mathbb{R}$ we know that $\lim_{h\to 0} m = m$, this in tandem with the sum law for limits, yields:*

$$\lim_{h\to 0}\frac{f\left(\overline{x}+h\right)-f\left(\overline{x}\right)-hf'\left(\overline{x}\right)}{h}=0$$

- *The above is true if and only if $f\left(\overline{x}+h\right)-f\left(\overline{x}\right)-hf'\left(\overline{x}\right) \in o\left(h\right)$*

Now, to the proofs

- $\Rightarrow$
  - *Assume that f is differentiable, by following the sequence of logic from top to bottom and letting $\alpha = f'\left(x\right)$ the proof is complete*
- $\Leftarrow$
  - *Assume that there is a number $\alpha \in \mathbb{R}$ and $l \in o\left(h\right)$, so that for any $h \in \mathbb{R}$ we have:*

    $$f\left(\overline{x}+h\right)=f\left(\overline{x}\right)+h\alpha+l\left(h\right) \Leftrightarrow l\left(h\right)=f\left(\overline{x}+h\right)-f\left(\overline{x}\right)-h\alpha$$

    thus $f\left(\overline{x}+h\right)-f\left(\overline{x}\right)-h\alpha \in o\left(h\right)$.
  - *Follow the logic in reverse order with $f'\left(\overline{x}\right)$ replaced by α, this shows that the limit exists and equals α, therefore the proof is concluded.*

$\blacksquare$