Proposition: Little O and Differentiability Equivalence

A function f is differentiable at the point $\overline{x}$ if and only if there is a number $\alpha \in \mathbb{R}$ for any $h \in \mathbb{R}$ we have

$$f(\overline{x}+h) = f(\overline{x}) + h\alpha + l(h)$$

Where $l \in o(h)$

Proof

Observe the following sequence of logic

- *Definition of differentiability at $\overline{x}$, the following limit exists :*

$$f'(\overline{x}) = \lim_{h \to 0} \frac{f(\overline{x}+h) - f(\overline{x})}{h}$$

- *For any $m \in \mathbb{R}$ we know that $\lim_{h \to 0} m = m$, this in tandem with the sum law for limits, yields:*

$$\lim_{h \to 0} \frac{f(\overline{x}+h) - f(\overline{x}) - hf'(\overline{x})}{h} = 0$$

- *The above is true if and only if $f(\overline{x}+h) - f(\overline{x}) - hf'(\overline{x}) \in o(h)$*

Now, to the proofs

- $\Rightarrow$
  - *Assume that f is differentiable, by following the sequence of logic from top to bottom and letting $\alpha = f'(x)$ the proof is complete*
- $\Leftarrow$
  - *Assume that there is a number $\alpha \in \mathbb{R}$ and $l \in o(h)$, so that for any $h \in \mathbb{R}$ we have:*

    $$f(\overline{x}+h) = f(\overline{x}) + h\alpha + l(h) \Leftrightarrow l(h) = f(\overline{x}+h) - f(\overline{x}) - h\alpha$$

    thus $f(\overline{x}+h) - f(\overline{x}) - h\alpha \in o(h)$.
  - *Follow the logic in reverse order with $f'(\overline{x})$ replaced by α, this shows that the limit exists and equals α, therefore the proof is concluded.*

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