Theorem: A Set is Open iff Each Point is an Interior Point

$$S \text{ is open } \Leftrightarrow \forall x \in S, \exists \varepsilon \in \mathbb{R}^{>0}, B(x, \varepsilon) \subseteq S$$

Proof

- $\Rightarrow$
  - *Let* $x \in S$*, then* $x \in \mathring{S}$ *by assumption, so we have* $\varepsilon_\circ \in \mathbb{R}^{>0}, B(x, \varepsilon_\circ) \subseteq S$*. Take* $\varepsilon = \varepsilon_\circ$
- $\Leftarrow$
  - *We already know that* $\mathring{S} \subseteq S$ *therefore we must show that* $S \subseteq \mathring{S}$
    * *Let* $x \in S$*, thus by assumption, we have* $\varepsilon \in \mathbb{R}^{>0}$ *such that* $B(x, \varepsilon) \subseteq S$
    * *Since* $\mathring{S} \stackrel{\mathtt{D}}{=} \left\{x \in \mathbb{R}^n : \exists \varepsilon \in \mathbb{R}^{>0}, B(x, \varepsilon) \subseteq S\right\}$*, then* $x \in \mathring{S}$

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