Proposition: Little O and Differentiability Equivalence

A function f is differentiable at the point \overline{x} if and only if there is a number $\alpha \in \mathbb{R}$ for any $h \in \mathbb{R}$ we have

$$f\left(\overline{x}+h\right)=f\left(\overline{x}\right)+h\alpha+l\left(h\right)$$

Where $l \in o(h)$

Proof -

Observe the following sequence of logic

• Definition of differentiability at \overline{x} , the following limit exists:

$$f'(\overline{x}) = \lim_{h \to 0} \frac{f(\overline{x} + h) - f(\overline{x})}{h}$$

• For any $m \in \mathbb{R}$ we know that $\lim_{h\to 0} m = m$, this in tandem with the sum law for limits, yields:

$$\lim_{h\to 0}\frac{f\left(\overline{x}+h\right)-f\left(\overline{x}\right)-hf'\left(\overline{x}\right)}{h}=0$$

• The above is true if and only if $f(\overline{x} + h) - f(\overline{x}) - hf'(\overline{x}) \in o(h)$

Now, to the proofs

- ⇒
- Assume that f is differentiable, by following the sequence of logic from top to bottom and letting $\alpha = f'(x)$ the proof is complete
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 - Assume that there is a number $\alpha \in \mathbb{R}$ and $l \in o(h)$, so that for any $h \in \mathbb{R}$ we have:

$$f\left(\overline{x}+h\right)=f\left(\overline{x}\right)+h\alpha+l\left(h\right)\Leftrightarrow l\left(h\right)=f\left(\overline{x}+h\right)-f\left(\overline{x}\right)-h\alpha$$

thus
$$f(\overline{x} + h) - f(\overline{x}) - h\alpha \in o(h)$$
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– Follow the logic in reverse order with $f'(\overline{x})$ replaced by α , this shows that the limit exists and equals α , therefore the proof is concluded.