Theorem: Binomial Theorem

Let $x, y \in \mathbb{R}$ such that $x + y \neq 0$ then for any $n \in \mathbb{N}$

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i$$

Proof

Consider the product

$$(x_1 + y_1)(x_2 + y_2) \dots (x_n + y_n)$$

The result is a sum of 2^n terms wherein, each term is the product of n factors. Each of these terms will contain as a factor either x_i or y_i for each $i \in [n]$ for example:

$$(x_1 + y_1)(x_2 + y_2) = x_1 x_2 + x_1 y_2 + x_2 y_1 + y_1 y_2$$

For a given term, we will have n factors, if k of them are x_i's then $n-k$ of the rest are y_i's. The number of different ways we can choose k x_i's from the set $\{x_1, x_2, \dots, x_n\}$ is simply $\binom{n}{k}$ and so this corresponds precisely to the number of terms with k x_i's. Thus by letting $x_i = x$, $y_i = y$ for all $i \in \{1, \dots, n\}$, we have

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

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