Theorem: Boundary Point is Half In Half Out

$$x \in \partial S \Leftrightarrow \forall \varepsilon \in \mathbb{R}^{>0}, B\left(x, \varepsilon\right) \cap S \neq \varnothing \text{ and } B\left(x, \varepsilon\right) \cap S^{\complement} \neq \varnothing$$

Proof

$x \in \partial S \overset{\text{D}}{=} \overline{S} \setminus \mathring{S}$,

- $x \in \overline{S} \Leftrightarrow \forall \varepsilon \in \mathbb{R}^{>0}, B\left(x, \varepsilon\right) \cap S \neq \varnothing$
- $x \notin \mathring{S} \Leftrightarrow \neg\left(\exists \varepsilon \in \mathbb{R}^{>0}, B\left(x, \varepsilon\right) \subseteq S\right)$ *that is*

$$\forall \varepsilon \in \mathbb{R}^{>0}, \exists a \in B\left(x, \varepsilon\right), a \notin S \Leftrightarrow a \in B\left(x, \varepsilon\right) \cap S^{\complement}$$

if and only if $B\left(x, \varepsilon\right) \cap S^{\complement} \neq \varnothing$

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