

## Definition: Binomial Theorem

Let  $x, y \in \mathbb{R}$  such that  $x + y \neq 0$  then for any  $n \in \mathbb{N}$

$$(x + y)^n = \sum_{i=0}^n \binom{n}{i} x^{n-i} y^i$$

*Proof.*

Consider the product

$$(x_1 + y_1)(x_2 + y_2) \dots (x_n + y_n)$$

The result is a sum of  $2^n$  terms wherein, each term is the product of  $n$  factors. Each of these terms will contain as a factor either  $x_i$  or  $y_i$  for each  $i \in [n]$  for example:

$$(x_1 + y_1)(x_2 + y_2) = x_1x_2 + x_1y_2 + x_2y_1 + y_1y_2$$

For a given term, we will have  $n$  factors, if  $k$  of them are  $x_i$ 's then  $n - k$  of the rest are  $y_i$ 's. The number of different ways we can choose  $k$   $x_i$ 's from the set  $\{x_1, x_2, \dots, x_n\}$  is simply  $\binom{n}{k}$  and so this corresponds precisely to the number of terms with  $k$   $x_i$ 's. Thus by letting  $x_i = x$ ,  $y_i = y$  for all  $i \in \{1, \dots, n\}$ , we have

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$$

