Theorem: Differentiability implies Continuity

If $f(x)$ is differentiable at a then $f(x)$ is continuous at a

Proof

- *Because $f(x)$ is differentiable at a we know that the following limit exists*

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

- *We also note that the following limit exists as well:*

$$\lim_{x \to a} x - a = 0$$

- *By the product law for limits we obtain that*

$$\lim_{x \to a} \left[\frac{f(x) - f(a)}{x - a} \cdot (x - a) \right] = 0 \cdot f'(a) = 0$$

  And note that

$$\lim_{x \to a} \left[\frac{f(x) - f(a)}{x - a} \cdot (x - a) \right] = \lim_{x \to a} \left[f(x) - f(a) \right]$$

- *Therefore*

$$\lim_{x \to a} \left[f(x) - f(a) \right] = 0$$

- *Since* $\lim_{x \to a} f(a) = f(a)$ *we can use the sum rule to obtain*

$$\begin{aligned} \lim_{x \to a} f(a) + \lim_{x \to a} \left[f(x) - f(a) \right] &= \lim_{x \to a} \left[f(a) + f(x) - f(a) \right] \\ &= \lim_{x \to a} f(x) \end{aligned}$$

- *But recall that*

$$\lim_{x \to a} f(a) + \lim_{x \to a} \left[f(x) - f(a) \right] = f(a) + 0$$

- *Thus we know that*

$$\lim_{x \to a} f(x) = f(a)$$

  as required.

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