Definition: Number of K-Combinations of a Set

Let X be a set where $|X| = n \in \mathbb{N}$, the number of k-combinations of X is equal to

$$^{n}C_k \overset{\texttt{N}}{=} \binom{n}{k} \overset{\texttt{D}}{=} \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 1} = \frac{n!}{(n-k)! \cdot k!}$$

Proof

- *Recall that the number of k permutations on n elements is*

$$\frac{n!}{(n-k)!}$$

- *Consider $x_1, x_2, \ldots, x_{k-1}, x_k \in X$, then there are $k!$ different permutations of these elements, but each of these permutations only corresponds to a single k-combination.*

- *This means that for each possible choice of $x_1, x_2, \ldots, x_{k-1}, x_k$ we are overcounting $k!$ times, that is:*

$$\frac{n!}{(n-k)!} = \sum_{\{x_1, x_2, \ldots, x_{k-1}, x_k\} \subseteq X} k! \Leftrightarrow \frac{n!}{k!(n-k)!} = \sum_{\{x_1, x_2, \ldots, x_{k-1}, x_k\} \subseteq X} 1$$

- *And $\sum_{\{x_1, x_2, \ldots, x_{k-1}, x_k\} \subseteq X} 1$ is the number of k-combinations of X thus*

$$^{n}C_k = \frac{n!}{k!(n-k)!}$$

  as required.

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