Theorem: Fundamental Theorem of Calculus II

If f is continuous on $[a,b]$ then:

$$\int_a^b f(t)\ dt = F(b) - F(a)$$

where F is any antiderivative of f.

Proof

Let F be an antiderivative of f, and define

$$g(x) \overset{\mathtt{D}}{=} \int_a^x f(t)\ dt$$

- *By FTC part I, g is continuous on $[a,b]$ and differentiable on (a,b) with $g'(x) = f(x)$ for every $x \in (a,b)$*
- *Also we define a function h*
$$h(x) = g(x) - F(x)$$
- *We can see that h is continuous on $[a,b]$ and differentiable on (a,b) as it is a difference of two functions with those same properites. Further, for any $x \in (a,b)$ we get*
$$h'(x) = g'(x) - F'(x)$$
- *By FTC part I, and the fact that $F'(x) = f(x)$ by definition of F we obtain that*
$$h'(x) = g'(x) - F'(x) = f(x) - f(x) = 0$$
this in tandem with the fact that h is continuous satifies the hypothesis for h to be constant on $[a,b]$ therefore $h(a) = h(b)$
- *This yields the following*

$$\begin{gathered}
h(b) = h(a) \\
\Updownarrow \text{ (Def.)} \\
g(b) - F(b) = g(a) - F(a) \\
\Updownarrow \text{ (Alg.)} \\
g(b) = g(a) + F(b) - F(a) \\
\Updownarrow \text{ (Def.)} \\
\int_a^b f(t)\ dt = \int_a^a f(t)\ dt + (F(b) - F(a)) \\
\Updownarrow \text{ (Integral Property)} \\
\int_a^b f(t)\ dt = 0 + F(b) - F(a) \\
\Updownarrow \text{ (Simp.)} \\
\int_a^b f(t)\ dt = F(b) - F(a)
\end{gathered}$$

$\blacksquare$