Theorem: Number of Elements in a Finite Union

Let $X_1, X_2, \dots, X_{n-1}, X_n \subseteq U$ let's set $N(S) = |S|$ for any set S, the following holds

$$\begin{aligned} N(X_1 \cup X_2 \cup \dots \cup X_{n-1} \cup X_n) = &\sum_{i=1}^{n} N(X_i) \\ &- \sum_{0 \le i_1 < i_2 \le n} N(X_{i_1} \cap X_{i_2}) \\ &+ \sum_{0 \le i_1 < i_2 < i_3 \le n} N(X_{i_1} \cap X_{i_2} \cap X_{i_3}) \\ &\vdots \\ &+ (-1)^{r+1} \sum_{0 \le i_1 < i_2 < \ldots < i_r \le n} N\left(X_{i_1} \cap X_{i_2} \cap \dots \cap X_{i_{r-1}} \cap X_{i_r}\right) \\ &\vdots \\ &+ (-1)^{n+1} N\left(\bigcap_{i=1}^{n} X_i\right) \end{aligned}$$

Proof

- *To prove that these two numbers are equal, we start with the fact that the left hand side simply counts the number elements in the set $\bigcup_{i=1}^{n} X_i$, to show that that the number on the right hand side is equal to that, we will also show that it counts the number of elements in the set $\bigcup_{i=1}^{n} X_i$. That is, let $a \in \bigcup_{i=1}^{n} X_i$, we know that the left hand side only counts this element once, let's show the right hand side only counts this element once.*
  - *Since $a \in \bigcup_{i=1}^{n} X_i$, then there exists $X_{\alpha_1}, X_{\alpha_2} \dots X_{\alpha_k}, X_{\beta_1}, X_{\beta_2} \dots X_{\beta_o}$ such that $a \in X_{\alpha_1} \cap X_{\alpha_2} \cap \dots \cap X_{\alpha_k}$ and $a \notin X_{\beta_1} \cap X_{\beta_2} \cap \dots \cap X_{\beta_o}$*
    - *If this is the case then the sum $\sum_{i=1}^{n} N(X_i)$ over counts a k times, this is because a counted for each X_{α_j}, as $a \in X_{\alpha_j}$*
    - *We consider the following sum $\sum_{0 \le i_1 < i_2 \le n} N(X_{i_1} \cap X_{i_2})$. We know that $a \in X_{\alpha_j} \cap X_{\alpha_l}$ for any $\alpha_j, \alpha_l \in \{\alpha_1, \alpha_2, \dots, \alpha_{k-1}, \alpha_k\}$. We notice that the number of ways to choose two elements from the set $\{\alpha_1, \alpha_2, \dots, \alpha_{k-1}, \alpha_k\}$ is simply $\binom{k}{2}$ thus this represents the number of times that a was over counted by this sum. Notice that we are subtracting this number.*
      - *Also note that we must choose two elements from $\{\alpha_1, \alpha_2, \dots, \alpha_{k-1}, \alpha_k\}$, as if we had chosen at least one from $\{\beta_1, \beta_2, \dots, \beta_{o-1}, \beta_o\}$, then a could not be in the above intersection.*
    - *Following the same logic, we see that if $a \in X_{\alpha_j} \cap X_{\alpha_l} \cap X_{\alpha_m}$ then the third summation will over count $\binom{k}{3}$ times.*
    - *Thus the i-th sum will over count $\binom{k}{i}$ times.*
- *Now to see how many times the right hand side counts a we will simply sum all of the ways it counts it (taking into account the subtractions and additions like in the original summation) to see that it looks like:*

$$\sum_{i=1}^{n} (-1)^{i+1} \binom{k}{i}$$

  - *Noting that when $i > k$ $\binom{k}{i} = 0$, so really, that is:*

$$\sum_{i=1}^{k} (-1)^{i+1} \binom{k}{i}$$

- *Recall that the binomial theorem tells us about a sum starting from 0, namely that*

$$(x+y)^k = \sum_{i=0}^{k} \binom{k}{i} x^i y^{k-i}$$

- *We attempt to manipulate this sum to answer our question about how many times the right hand side counts a. By letting $x = -1, y = 1$ we see that*

$$0 = \sum_{i=0}^{k} (-1)^i \binom{k}{i} = 1 + \sum_{i=1}^{k} (-1)^i \binom{k}{i} \Leftrightarrow 1 = \sum_{i=1}^{k} (-1)^{i+1} \binom{k}{i}$$

  Therefore the right hand side of only counts a once, as required.

∎