Theorem: Fundamental Theorem of Calculus II

If f is continuous on $[a, b]$ then:

$$\int_a^b f(t)\ dt = F(b) - F(a)$$

where F is any antiderivative of f.

Proof

Let F be an antiderivative of f, and define

$$g(x) \stackrel{\text{D}}{=} \int_a^x f(t)\ dt$$

- *By FTC part I, g is continuous on $[a, b]$ and differentiable on (a, b) with $g'(x) = f(x)$ for every $x \in (a, b)$*
- *Also we define a function h*

  $$h(x) = g(x) - F(x)$$

- *We can see that h is continuous on $[a, b]$ and differentiable on (a, b) as it is a difference of two functions with those same properites. Further, for any $x \in (a, b)$ we get*

  $$h'(x) = g'(x) - F'(x)$$

- *By FTC part I, and the fact that $F'(x) = f(x)$ by definition of F we obtain that*

  $$h'(x) = g'(x) - F'(x) = f(x) - f(x) = 0$$

  this in tandem with the fact that h is continuous satifies the hypothesis for h to be constant on $[a, b]$ therefore $h(a) = h(b)$

- *This yields the following*

$$\begin{gathered}
h(b) = h(a) \\
\Updownarrow \text{(Def.)} \\
g(b) - F(b) = g(a) - F(a) \\
\Updownarrow \text{(Alg.)} \\
g(b) = g(a) + F(b) - F(a) \\
\Updownarrow \text{(Def.)} \\
\int_a^b f(t)\ dt = \int_a^a f(t)\ dt + (F(b) - F(a)) \\
\Updownarrow \text{(Integral Property)} \\
\int_a^b f(t)\ dt = 0 + F(b) - F(a) \\
\Updownarrow \text{(Simp.)} \\
\int_a^b f(t)\ dt = F(b) - F(a)
\end{gathered}$$

$\blacksquare$