Proposition: Closure of a Set is Itself and Boundary

$$\overline{S} = S \cup \partial S$$

Proof

Let $x \in \overline{S}$, *then*

- *If* $x \in S$, *then* $x \in S \cup \partial S$
- *If* $x \notin S$, *then we have to show that* $x \in \partial S \overset{\mathtt{D}}{=} \overline{S} \setminus \mathring{S}$, *we already know that* $x \in \overline{S}$ *so it remains to show that* $x \notin \mathring{S}$, *that is, we need to show that the following is false:*

  $$\exists \varepsilon \in \mathbb{R}^{>0} \text{ such that } B(x, \varepsilon) \subseteq S$$

  But $x \notin S$, *so it's impossible for* $B(x, \varepsilon) \subseteq S$ *to be true for any* ε. *Thus* $x \notin \mathring{S}$ *and we have* $x \in \partial S$

Let $x \in S \cup \partial S$, *we need to show that* $x \in \overline{S}$, *so* $\forall \varepsilon \in \mathbb{R}^{>0}, B(x, \varepsilon) \cap S \neq \varnothing$. *Let* $\varepsilon \in \mathbb{R}^{>0}$

- *If* $x \in S$ *then* $x \in B(x, \varepsilon)$ *and* $x \in S$ *so* $B(x, \varepsilon) \cap S \neq \varnothing$
- *If* $x \in \partial S \overset{\mathtt{D}}{=} \overline{S} \setminus \mathring{S}$, *then* $x \in \overline{S}$ *as needed.*

$\blacksquare$