Upwind Stabilization of the Advection Term

Consider a triangle $T \in \mathcal{T}_h$ and a fixed edge $E \subset T$. Let T' denote the adjacent triangle across E as illustrated in Fig. 4.1, where we also allow for the case of $T' = \emptyset$. According to (4.14b), the associated entry in the system of equations (4.13) is

$$u_T C^n_{ET} \overset{(4.14g)}{=} u_T \sum_{\{E' \subset T; E' \notin \mathcal{E}_\mathrm{N}\}} c^n_{E'} H^n_{T,E',E} \quad \text{for} \quad E \in \mathcal{E}_\Omega \cup \mathcal{E}_\mathrm{D} .$$

If E is an inflow edge, then $c^n_{ET} := c^n_E \sigma_{ET} < 0$ holds and we have to weight $u_{T'}$ in upwind direction in dependency of the local Péclet number (cf. Rem 4.12). Vice versa, if $c^n_{ET} > 0$ we have to take account of the weighting between u_T and $u_{T'}$ in opposite direction due to symmetry reasons. Hence, the above term is approximated by

$$\Big(\alpha\,(\,\mathrm{sign}(c^n_{ET})\mathrm{Pe}^n_T)\,u_T + (1 - \alpha\,(\,\mathrm{sign}(c^n_{ET})\mathrm{Pe}^n_T)\,u_{T'})\Big)\,C^n_{ET}.$$

We consider the following *upwind formulas* (**KA2003**):

(i)

$$\alpha(z) = \frac{1}{2}\,(\mathrm{sign}(z) + 1) \;\; \in \{0, 1\} \qquad \text{(full upwinding)}\,,$$

(ii)

$$\alpha(z) = 1 - \frac{1}{z}\left(1 - \frac{z}{e^z - 1}\right) \qquad \text{(exponential upwinding)}\,,$$

(iii)

$$\alpha(z) = \begin{Bmatrix} (1-\tau)/2, \; z < 0 \\ (1+\tau)/2, \; z \geq 0 \end{Bmatrix}, \quad \tau(z) := \max\left\{0, 1 - \frac{2}{|z|}\right\} .$$