LINEAR ALGEBRA
线性代数

仝辉
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注意事项

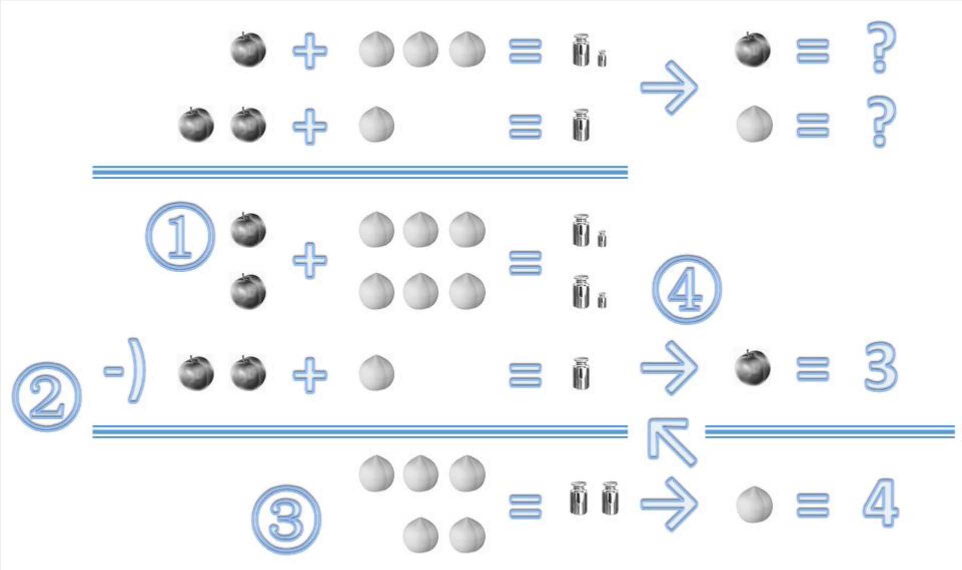
- 课程QQ群号码 764940233，自愿实名加入，须提供姓名和10位学号。
- 有任何关于课程学习的问题，都可以通过QQ群或者Email联系。
  - *Email地址：tonghui@bupt.edu.cn*
- 课堂认真听讲，课后及时独立完成作业。
  - *请尽量用 **数学作业纸** 写作业，交作业时 **不要交作业本**。*
- 不要拖延学习进度，以免“欠账太多还不上”。

注意事项

- 如果上课迟到，直接进入教室找到座位坐下听讲即可。
  - *注意低调。*
- 课程总评成绩，由期末卷面成绩与平时成绩加权平均后得到。
  - *平时成绩包括出勤成绩、作业成绩和课堂测验等。*
- 线性代数词汇中英对照表
  - *https://mp.weixin.qq.com/s/Vd6aJTtxtvnzyULvZJ-yDQ*

Example
例题

- How to solve $\begin{cases} x + 3y = 15 \\ 2x + y = 10 \end{cases}$?
- $2x + 6y = 30$
- $5y = 20$
- $y = 4$
- $2x = 6$
- $x = 3$
- Gaussian eliminination
  - *高斯消元法*



Definition
定义

- **Definition 1.1.1** (Linear Equation)
  - *A **linear equation** [线性方程] in n unknows is an equation of the form*

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

  - *Where $a_i(i = 1,2,\cdots,n)$ and b are numbers and $x_i(i = 1,2,\cdots,n)$ are called **variables** [变量].*

- **Definition 1.1.2** (Linear Systems)
  - *A linear system of m equations in n unknowns is a system of the form*

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

  - *We call this system $m \times n$ **linear system** [线性代数方程组], and read as "m by n linear system".*

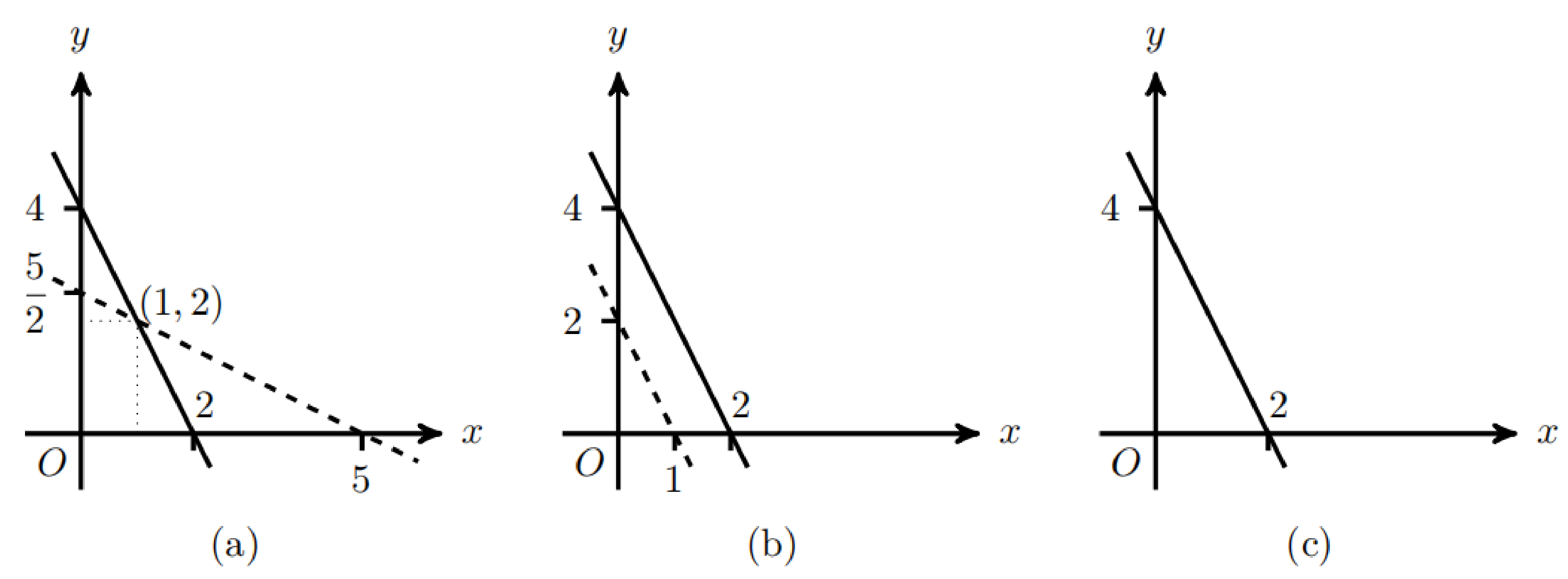
Definition
定义

- **Definition 1.1.3** (Solution of Linear System)
  - *The **solution** [解] of an $m \times n$ linear system is an ordered n-tuple of numbers*
    $$(x_1, x_2, \cdots, x_n),$$
  - *which satisfies all equations of the $m \times n$ linear system.*
- **The process of solving linear system** [线性代数方程组的求解过程]
- If there is at least one solution of an $m \times n$ linear system, we say the linear system is **consistent** [相容].
  - *Otherwise, we say the linear system is **inconsistent** [不相容].*
- If there is more than one solution, we call the set of all solutions the **solution set** [解集].

Example 1.1.1 (Consistency of $m \times n$ linear system)

Consider the consistency of the following 2×2 linear systems. If they are consistent, find solution(s) of them.

- (a) $\begin{cases} 2x + y = 4 \\ x + 2y = 5 \end{cases}$
- (b) $\begin{cases} 2x + y = 4 \\ x + \frac{y}{2} = 1 \end{cases}$
- (c) $\begin{cases} 2x + y = 4 \\ 4x + 2y = 8 \end{cases}$



- (a) is consistent, with unique solution $(1,2)$; (b) is inconsistent; (c) is consistent, with infinite number of solutions $\{(\alpha, -2\alpha + 4) | \alpha \ is\ real\ number.\}$

- Moreover, based on our knowledge of geometry, we know that only three possible relation positions for two lines on the xOy plane: intersecting, parallel, or coincident. So, we can claim that the consistency of all 2×2 linear systems must be one of the following three cases.
  - *(1) consistent, with unique solution;*
  - *(2) consistent, with infinite number of solutions;*
  - *(3) inconsistent.*
- Readers can try to consider the problem of the consistency of a 3×3 linear systems.

Equivalent Linear Systems
等价(同解)方程组

- **Definition 1.1.4** (Equivalent Linear Systems) Two linear systems are said to be **equivalent** [等价] if they have the same solution or solution set.
- **Theorem 1.1.1** (Properties of Equivalence) Let A, B, C be the three linear systems, then
  - *(1) If A is equivalent to B and B is equivalent to C, then A is equivalent to C;*
  - *(2) If A is equivalent to C and B is equivalent to C, then A is equivalent to B.*
- **Example 1.1.2** (Equivalent Linear Systems) Show that the following two linear systems are equivalent.

$$(1)\begin{cases} 2x_1 + x_2 - x_3 = 5 \\ x_2 = 1 \\ -x_3 = 2 \end{cases} \qquad (2)\begin{cases} 2x_1 + x_2 - x_3 = 5 \\ -2x_1 + x_3 = -4 \\ -2x_1 - x_2 = -3 \end{cases}$$

Theorem 1.1.2 [定理1.1.2]
Operations to Obtain Equivalent Linear Systems

- There are three basic operations involved in equivalently changing a linear system into another linear system,
  - *(1) interchanging the written order of two equations of a linear system does not change the solution set;*
  - *(2) multiplying both sides of an equation of a linear system by a nonzero real number does not change the solution set;*
  - *(3) replacing an equation of a linear system by a product of other equation added with the equation does not change the solution set.*
- Operations listed in Theorem 1.1.2 are generally used to derive a linear system, which is easy to be solved, from a linear system, equivalently.

Strict Triangular Form of Linear Systems

- **Definition 1.1.5** (Strict Triangular System) A linear system is said to be in **strict triangular form** [严格三角形式] if and only if in the kth equation the coefficients of the previous k-1 variables are all zero and the coefficient of the kth variable is nonzero (k=1,2,…,n).
- **Example 1.1.3**
  - $\begin{cases} 2x_1 + x_2 - x_3 = 5 \\ x_2 = 1 \\ -x_3 = 2 \end{cases}$ *is strict triangular system,* $\begin{cases} 2x_1 + x_2 - x_3 = 5 \\ -2x_1 + x_3 = -4 \\ -2x_1 - x_2 = -3 \end{cases}$ *is not.*
- To obtain the solution or solution set of an $n \times n$ strict triangular linear system is almost trivial, since we can use a process which is called **back substitution** [回代法].

Solving Linear Systems
—— <u>Strict Triangular Form</u> or <u>NOT</u>

- **Example 1.1.4** (Back Substitution) Solve the linear system

$$\begin{cases} 2x_1 + x_2 - x_3 = 5 \\ x_2 = 1 \\ -x_3 = 2 \end{cases}$$

  By the process of back substitution, we obtain the solution $x_3 = 2$, $x_2 = 1$, $x_1 = 1$.

- **Example 1.1.5** Solve the linear system

$$\begin{cases} 2x_1 + x_2 - x_3 = 5 \\ -2x_1 + x_3 = -4 \\ -2x_1 - x_2 = -3 \end{cases}$$

  – *Since this system is not a strict triangular linear system, so we need to use the process in Example 1.1.2 to change it into a strict triangular linear system, equivalently.*

  – *Then, by the process of back substitution, we can derive the solution.*