Part 0. Referee Instructions

To referee, follow this procedure

0.1. **Read this rubic** *first* so you are sure you understand the problem

0.2. If you find an error in this rubric, **tell me immediately** so I can fix it for everyone.

0.3. Read the author's solution on its own, without referring to the rubric directly. Ask yourself "Did they communicate what they are doing, and why?" and "Is this complete, or is there something they forgot to mention?" You are a human, not a computer; however, you should *not* have to extrapolate their intent — do not think for them! If you know a solution is correct, but not for the reasons stated, then make a note of it. **Make comments, as a human, about what is and isn't clear.**

0.4. Read each solution a second time. This time, follow the rubric to assign a **numerical score to each problem**.

0.5. Write the **numerical scores and your Referee Code** on the author's submission.

0.6. I will personally review all of these comments and scores. Good commentary and correct grades are worth 4 points to the Referee.

0.7. Sometimes, people find solutions that are completely unlike what I was expecting. If you find that the solution does not fit the rubric at all, please make a note of where it stops making sense to you, and do not assign a number.

Part 1. Perform row-reduction to solve this system. (Hint! The final solution is $(x, y, z) = (4, 6, 8)$. You still need to do the work, though.)

$$\begin{cases} 4x + 24y + 0z = 160 \\ 13x + 47y - 6z = 286 \\ 12x + 78y + 1z = 524 \end{cases}$$

Convert to augmented matrix and row-reduce. Here's one way.

$$\begin{pmatrix} 4 & 24 & 0 & 160 \\ 13 & 47 & -6 & 286 \\ 12 & 78 & 1 & 524 \end{pmatrix}, \qquad \text{divide row1 by 4}$$

$$\begin{pmatrix} 1 & 6 & 0 & 40 \\ 13 & 47 & -6 & 286 \\ 12 & 78 & 1 & 524 \end{pmatrix} \qquad \text{subtract row3 from row2}$$

$$\begin{pmatrix} 1 & 6 & 0 & 40 \\ 1 & -31 & -7 & -238 \\ 12 & 78 & 1 & 524 \end{pmatrix} \qquad \text{replace row3 with row3-12*row1}$$

$$\begin{pmatrix} 1 & 6 & 0 & 40 \\ 1 & -31 & -7 & -238 \\ 0 & 6 & 1 & 44 \end{pmatrix} \qquad \text{subtract row1 from row2}$$

$$\begin{pmatrix} 1 & 6 & 0 & 40 \\ 0 & -37 & -7 & -278 \\ 0 & 6 & 1 & 44 \end{pmatrix} \qquad \text{multiply row2 by 6, and row3 by 37}$$

$$\begin{pmatrix} 1 & 6 & 0 & 40 \\ 0 & -222 & -42 & -1668 \\ 0 & 222 & 37 & 1628 \end{pmatrix} \qquad \text{add row2 to row3}$$

$$\begin{pmatrix} 1 & 6 & 0 & 40 \\ 0 & -222 & -42 & -1668 \\ 0 & 0 & -5 & -40 \end{pmatrix} \qquad \text{add row2 to row3}$$

$$\begin{pmatrix} 1 & 6 & 0 & 40 \\ 0 & -222 & -42 & -1668 \\ 0 & 0 & -5 & -40 \end{pmatrix} \qquad \text{divide row3 by } -5$$

$$\begin{pmatrix} 1 & 6 & 0 & 40 \\ 0 & -222 & -42 & -1668 \\ 0 & 0 & 1 & 8 \end{pmatrix} \qquad \text{add 42 row3 to row2}$$

$$\begin{pmatrix} 1 & 6 & 0 & 40 \\ 0 & -222 & 0 & -1332 \\ 0 & 0 & 1 & 8 \end{pmatrix} \qquad \text{divide row2 by -222}$$

$$\begin{pmatrix} 1 & 6 & 0 & 40 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 8 \end{pmatrix} \qquad \text{subtract 6row2 from row1}$$

$$\begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 8 \end{pmatrix}.$$

Therefore, the solution $(x, y, z) = (4, 6, 8)$.

- This problem is worth 5 points.
- Subtract 2 points they did something other than row-reduction.
- Subtract 1 point they did not end with

$$\begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 8 \end{pmatrix}$$

- Subtract 1 point for each step of row-reduction where something went wrong with the arithmetic. Referee should check if something went wrong by verifying that $(4, 6, 8)$ satisfies each of the equations.
- If they caught an arithmetic error, and noted it, but were unable to fix it, credit back 1 point. (can only be used once.)

Part 2. Write down all of the 3×4 reduced row-echelon form matrices that have rank 2. (Use $*$ to indicate entries that could be any number.)

Rank 2 means we have to choose the position of 2 pivots. There are only 4 columns, so there are $\binom{4}{2} = 6$ possible forms.

$$\begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & * & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & * & * & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- This problem is worth 2 points.
- Subtract 0.5 point for each matrix that is not in RREF.
- Subtract 0.25 point for each matrix that is missing $*$'s where they should be.
- Subtract 0.5 point for each missing case.

Part 3. Suppose that A is a 4×4 matrix for which the following procedure yields I. First, swap ρ_2 and ρ_4. Second, replace ρ_1 with $\rho_1 + 5\rho_3$. Third, multiply ρ_3 by 6, and divide ρ_1 by 5. Finally, replace ρ_4 with $\rho_4 - 3\rho_1$. Recall that these row-operations can be un-done in the correct order, to change I back to A. Find the matrix A.

The first row-operation is

$$E_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

The second row-operation is

$$E_2 = \begin{pmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The third row-operation is

$$E_3 = \begin{pmatrix} \frac{1}{5} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The fourth row-operation is

$$E_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{pmatrix}$$

And, we have

$$E_4 E_3 E_2 E_1 A = I$$

Therefore,

$$A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} I$$

Hence, undoing each of the above operations in the correct order, we have

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & -5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{6} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ +3 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 0 & -\frac{5}{6} & 0 \\ 3 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{6} & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

- This problem is worth 3 points.
- Subtract 1 point for not converting the row-operations to elementary matrices.
- Subtract 0.5 points for each incorrect elementary matrix.
- Subtract 1 point for not un-doing each row-operations to find its respective inverse. (Note! It is fair to jump straight to this, without writing down the original row-ops).
- Subtract 0.5 points for each incorrect inverse.
- Subtract 1 point for not multiplying the inverses in the correct order
- Subtract 0.5 points (total) for arithmetic errors in matrix multiplication.

Part 4.

Suppose that A is a 3×3 matrix and that $E_1, E_2, \ldots, E_k$ are elementary matrices such that

$$E_k \cdots E_2 E_1 A = R = \begin{pmatrix} 1 & 0 & \frac{1}{5} \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad E_k \cdots E_2 E_1 = \begin{pmatrix} 4 & 2 & 8 \\ 8 & 1 & 3 \\ 0 & 1 & 1 \end{pmatrix}.$$

4.1. Find the solution of the system $A\vec{x} = \vec{0}$ in parametric form.

The solution to $A\vec{x} = \vec{0}$ is the same as the solution to $R\vec{x} = \vec{0}$, since row operations don't change the 0s. Therefore, using x_3 as a free variable, the solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{5} \\ 3 \\ 1 \end{pmatrix} t$$

This is a line through the origin. By rescaling t, it could be written without fractions as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 15 \\ 5 \end{pmatrix} t$$

- This sub-problem is worth 2 points.
- Subtract 1 point for not writing $Rx = 0$ or equivalent, or for not recognizing that $Ax = 0$ and $Rx = 0$ have the same solution.

- Subtract 1 point for completely incorrect parametric form.
- Subtract 0.5 points for $\pm$ mistakes in the parametric form.

4.2. Write the solution to the inhomogeneous system

$$A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -3 \\ 3 \end{pmatrix},$$

or explain precisely why no solution exists.

The solution to $A\vec{x} = \vec{b}$ is the same as the solution to $R\vec{x} = \vec{c}$ where $\vec{c} = E_k \cdots E_2 E_1 \vec{b}$. That vector is

$$\vec{c} = E_k \cdots E_2 E_1 \vec{b} = \begin{pmatrix} 4 & 2 & 8 \\ 8 & 1 & 3 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ -3 \\ 3 \end{pmatrix} = \begin{pmatrix} 20 \\ 10 \\ 0 \end{pmatrix}$$

So, we are solving the system

$$\begin{cases} x_1 + 0x_2 + \frac{1}{5}x_3 = 20 \\ x_2 - 3x_3 = 10 \\ 0 = 0 \end{cases}$$

The solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 20 \\ 10 \\ 0 \end{pmatrix} + \begin{pmatrix} -\frac{1}{5} \\ 3 \\ 1 \end{pmatrix} t$$

- This sub-problem is worth 2 points.
- Subtract 1 point for not finding $c = (20, 10, 0)$ at all.
- Subtract 0.5 point for arithmetic mistake in finding c.
- Subtract 1 point for not writing $Rx = c$ or equivalent, or for not recognizing that $Ax = b$ and $Rx = c$ have the same solution.
- Subtract 1 point for completely incorrect parametric form.
- Subtract 0.5 points for $\pm$ arithmetic mistakes in the parametric form.

4.3. Write the solution to the inhomogeneous system

$$A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 39 \\ -9 \end{pmatrix},$$

or explain precisely why no solution exists.

The solution to $A\vec{x} = \vec{b}$ is the same as the solution to $R\vec{x} = \vec{c}$ where $\vec{c} = E_k \cdots E_2 E_1 \vec{b}$. That vector is

$$\vec{c} = E_k \cdots E_2 E_1 \vec{b} = \begin{pmatrix} 4 & 2 & 8 \\ 8 & 1 & 3 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 39 \\ -9 \end{pmatrix} = \begin{pmatrix} 10 \\ 20 \\ 30 \end{pmatrix}$$

Note that $R\vec{x} = \vec{c}$ has the row $0 = 30$. The system in inconsistent – there is not solution.

- This sub-problem is worth 1 point.
- Subtract 0.5 point for not finding $c = (10, 20, 30)$ at all.
- Subtract 0.25 point for arithmetic mistake in finding c.
- Subtract 0.75 point for not recognizing the inconsistent system.

Part 5. The following pictures show two lines, corresponding to a system of two linear equations in two variables, $[A|b]$. For each picture, answer these questions:

- Is the system homogeneous or inhomogeneous?
- How many solutions does the system have?
- What is the rank of the 2×2 matrix A?
- What is the rank of the 2×3 augmented matrix $[A|b]$?

(removed images for brevity)
The first image is two identical lines through the origin. This would arise from a system of the form

$$\begin{pmatrix} 1 & * & |0 \\ 0 & 0 & |0 \end{pmatrix}$$

This is homogeneous (the solution contains $(0,0)$), and the solution is a line (1-dimensional, with infinitely many solutions). A has rank 1, and $[A|b]$ also has rank 1, as seen above.

The second image is parallel identical lines. This would arise from a system of the form

$$\begin{pmatrix} 1 & * & |0 \\ 0 & 0 & |1 \end{pmatrix}$$

This is inhomogeneous (the solution does not contain $(0,0)$), and the solution is a empty. There are no solutions. A has rank 1, and $[A|b]$ has rank 2, as seen above.

The third image is two identical lines that miss the origin. This would arise from a system of the form

$$\begin{pmatrix} 1 & * & |* \\ 0 & 0 & |0 \end{pmatrix}$$

This is inhomogeneous (the solution does not contain $(0,0)$), and the solution is a line (1-dimensional, with infinitely many solutions). A has rank 1, and $[A|b]$ also has rank 1, as seen above.

The fourth image is two lines that cross at a point that is not the origin. This would arise from a system of the form

$$\begin{pmatrix} 1 & 0 & |* \\ 0 & 1 & |* \end{pmatrix}$$

This is inhomogeneous (the solution does not contain $(0,0)$), and the solution is a single point (0-dimensional, with exactly one solution). A has rank 2, and $[A|b]$ also has rank 2, as seen above.

- This sub-problem is worth 5 points.
- Subtract 0.5 point for each incorrect homogeneous/inhomogenous
- Subtract 0.5 point each incorrect statement about the number of solutions. In this problem, it is acceptable to say "1-dimensional" or "infinitely many" interchangeably. Also, one may say "0-dimensional" or "exactly one" interchangeably.
- Subtract 0.5 point each incorrect rank statement.
- Add back 1 point if someone mis-stated the ranks, but write an example system or wrote a parametrization of the solution correctly. This can be used only once.