Bayesian Machine Learning

03/05/21 - François HU

Outline

1 **Bayesian statistics**
- define a probabilistic model
- apply bayesian inference
- conjugate priors

2 **Latent variable models**
- define latent variable and apply them to simplify probabilistic model
- cluster data with latent models like GMM
- train probabilistic models with EM-algorithm

3 **Variational Inference**
- apply variational inference for probabilistic models
- understand variational interpretation of LDA
- application of LDA to text mining

4 **Markov Chain Monte Carlo**
- train / do inference almost any probabilistic model with MCMC
- pros and cons of MCMC / VI

5 **Extensions and oral presentations**

PREREQUISITE

THEORY

1. Notions of **probability & statistics**
2. **Statistical Learning** :
   supervised **&** unsupervised learning
3. **Information theory** :
   Entropy, KL-divergence, …
4. **Monte Carlo & Markov Chain**

APPLICATION

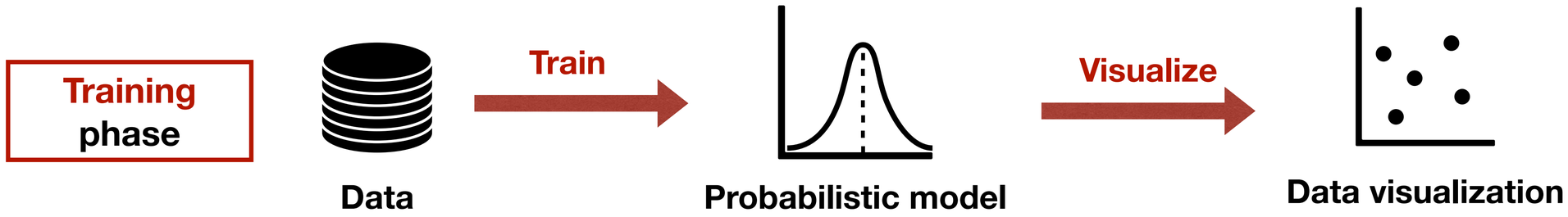
Python (or at least R)

ALGORITHM

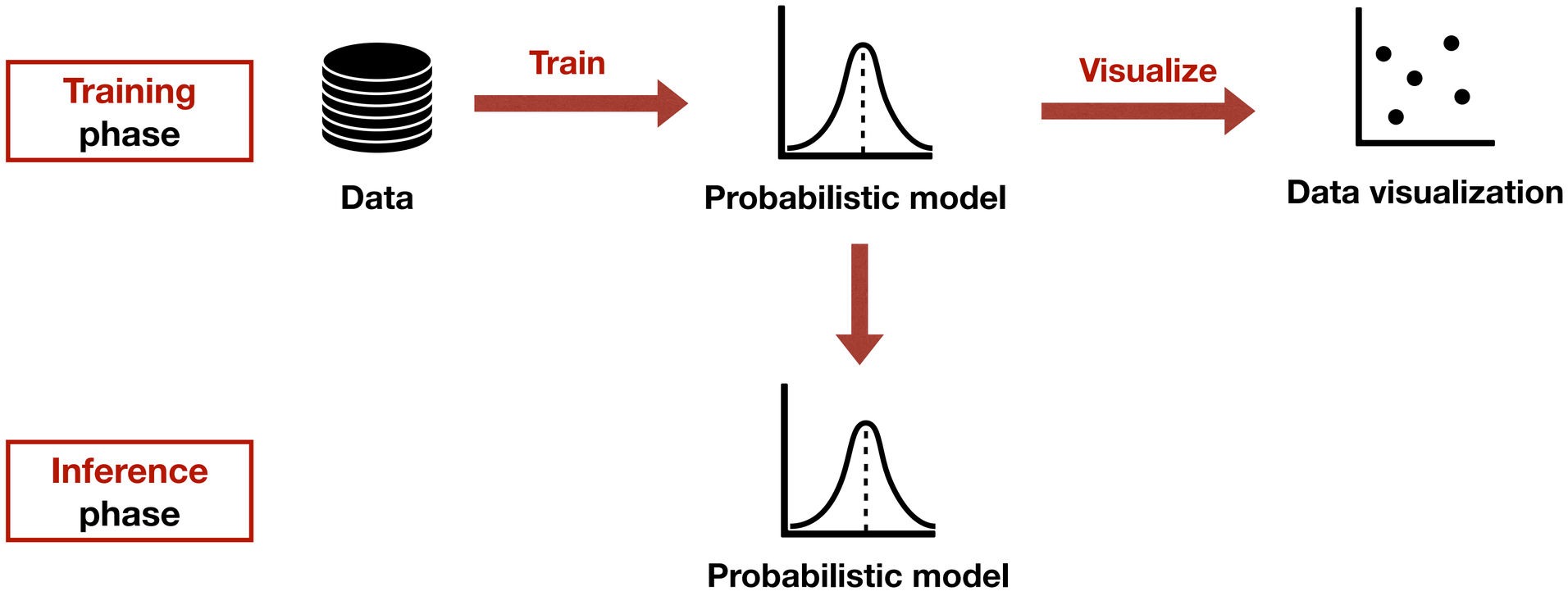
Some « classical » supervised & unsupervised models

0 Gentle introduction to statistical learning

Simplified statistical learning process



Simplified statistical learning process



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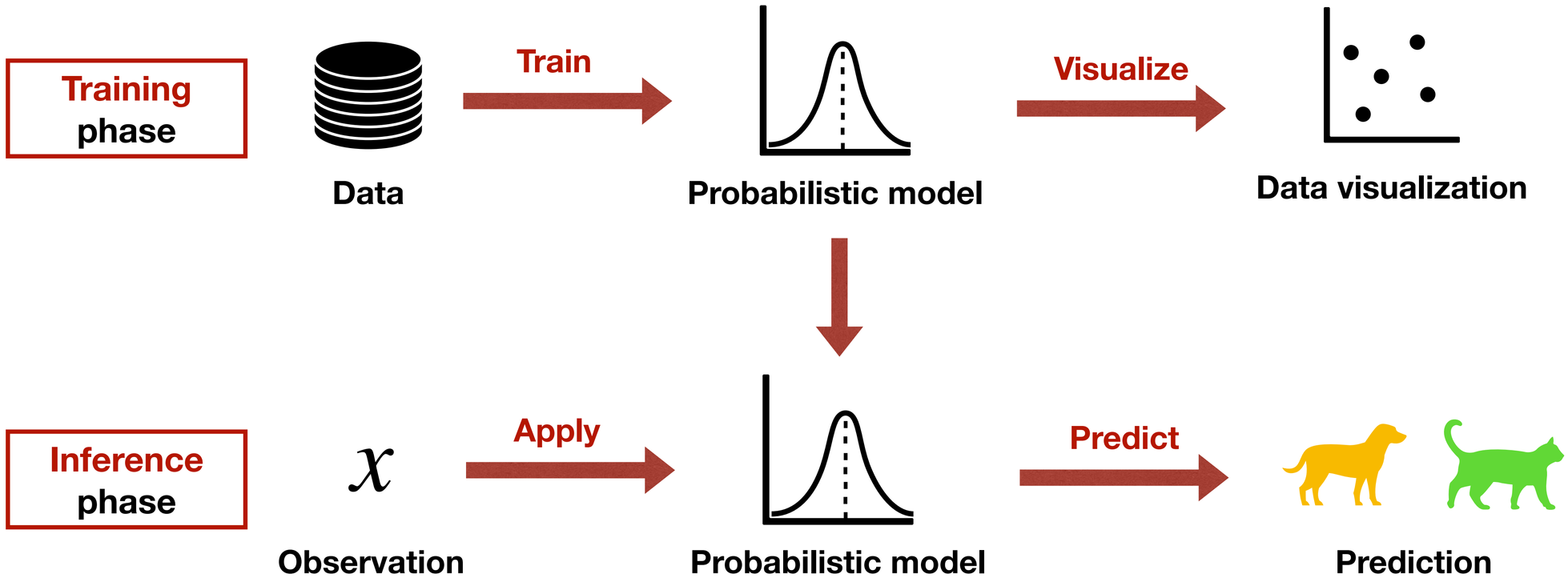
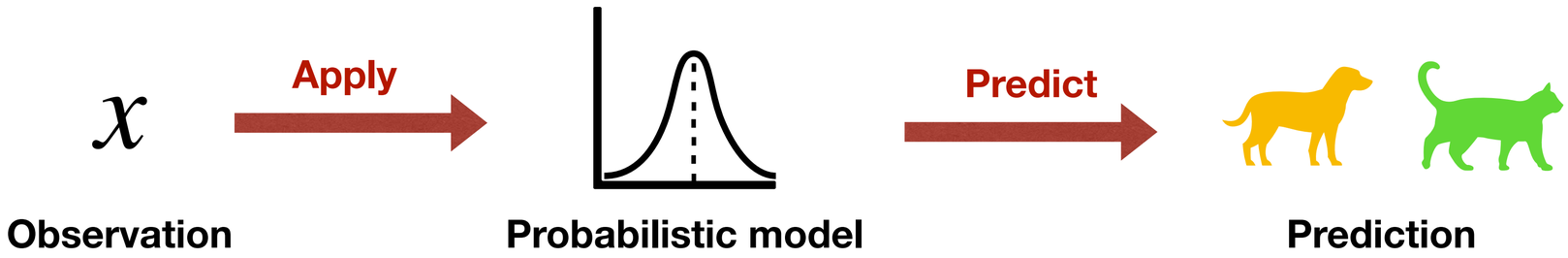


Inference phase



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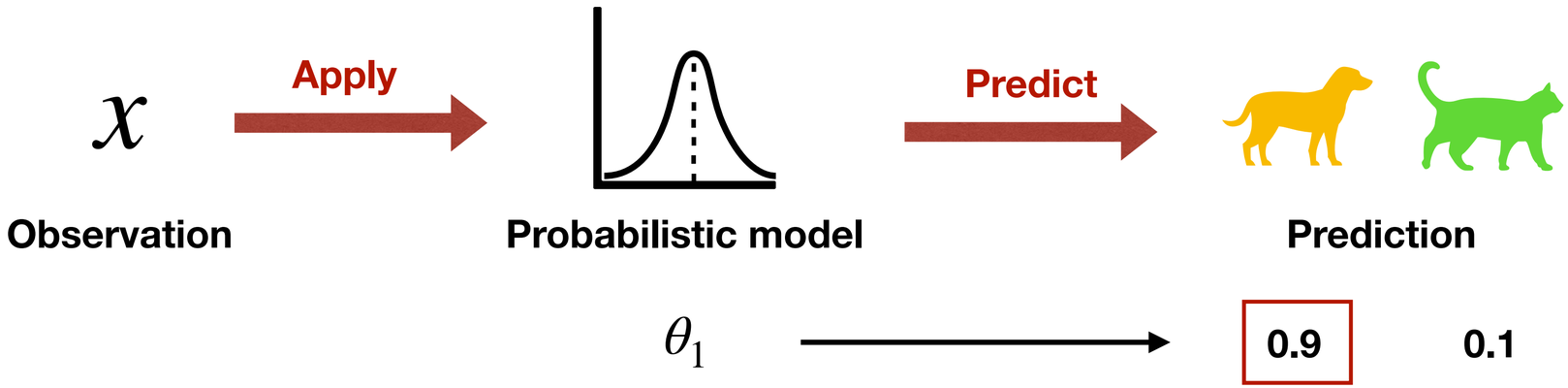

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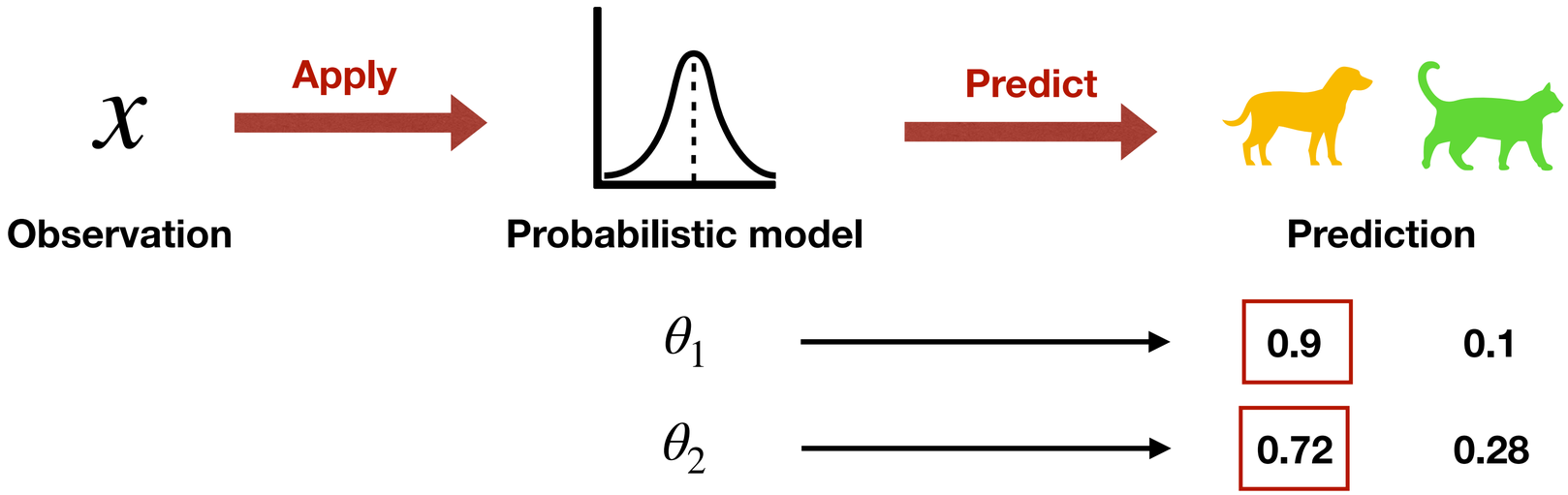

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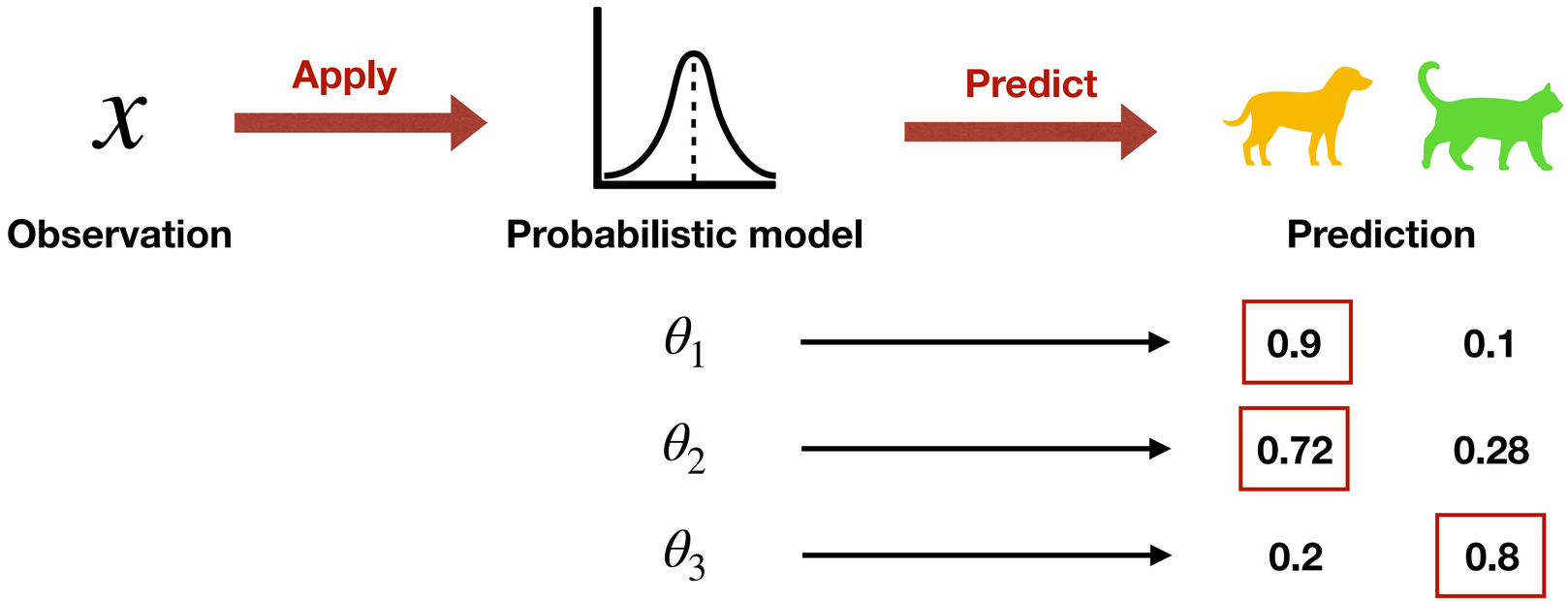

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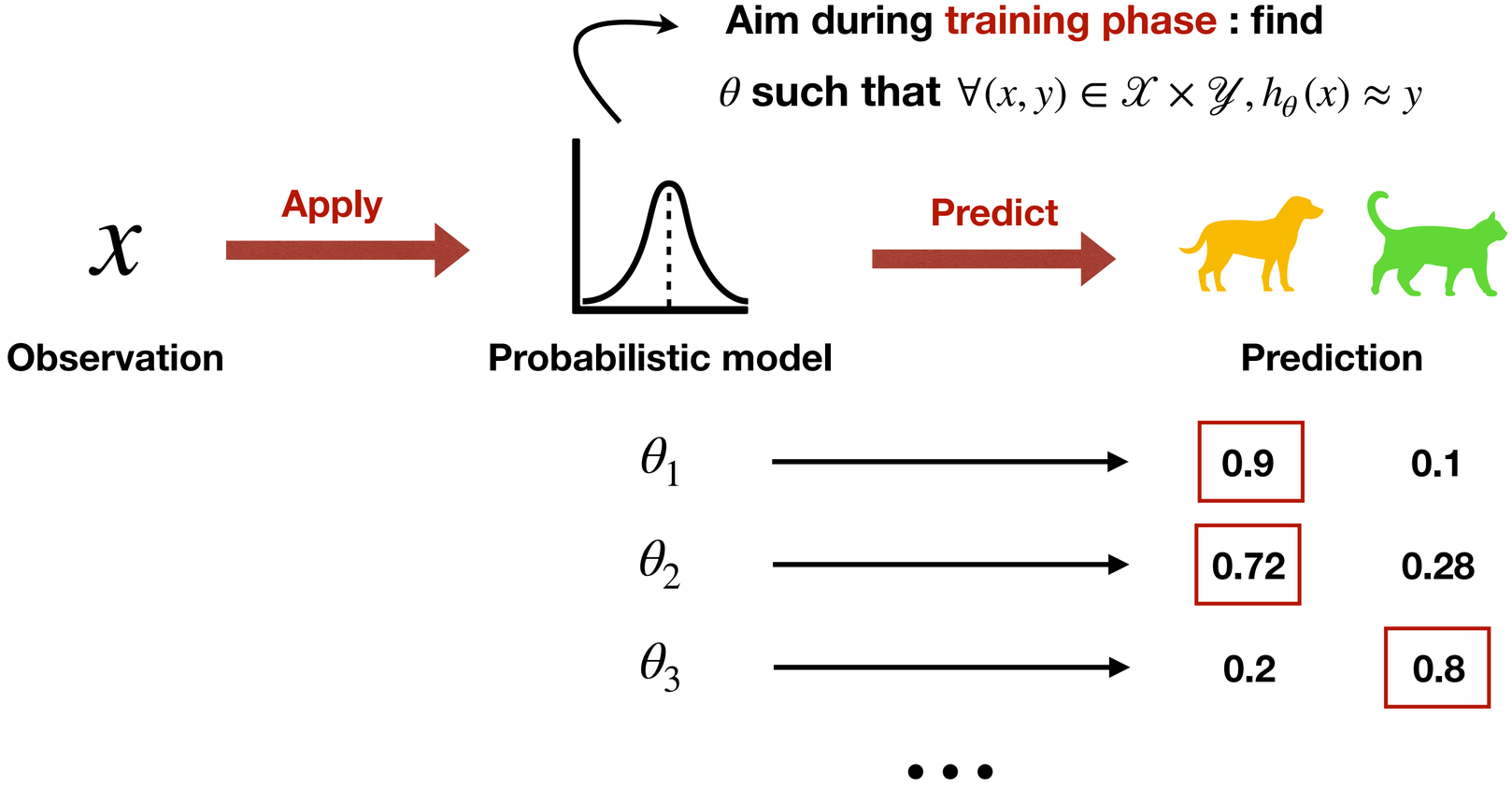
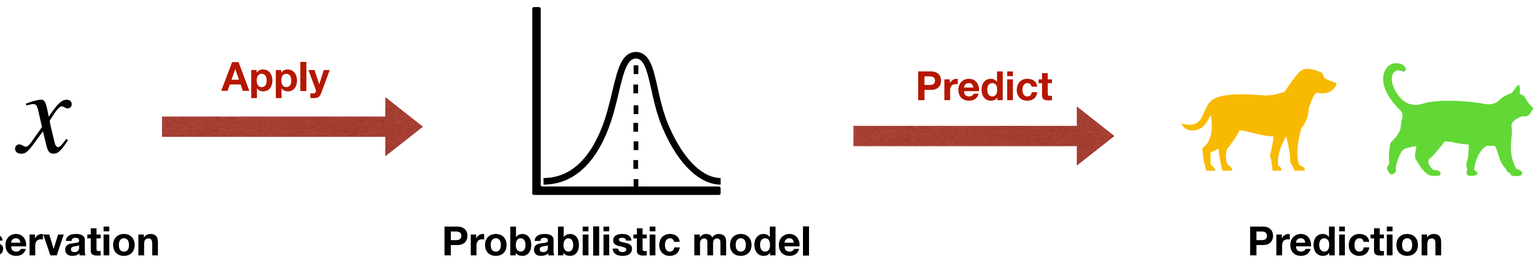


Inference phase

Aim during training phase : find

θ such that $\forall (x, y) \in \mathcal{X} \times \mathcal{Y}, h_\theta(x) \approx y$

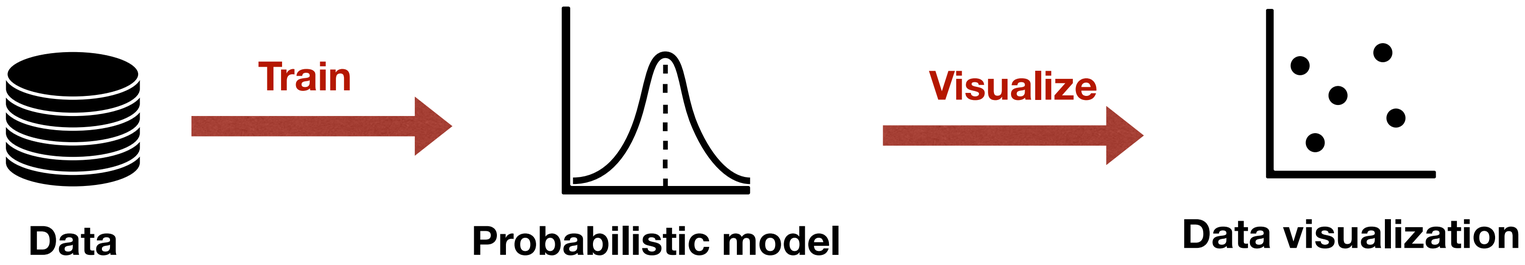


- « Classical » frequentist point of view
- Parameter θ is fixed and the data X is random

θ_1	**0.9**	0.1
θ_2	**0.72**	0.28
θ_3	0.2	**0.8**

• • •

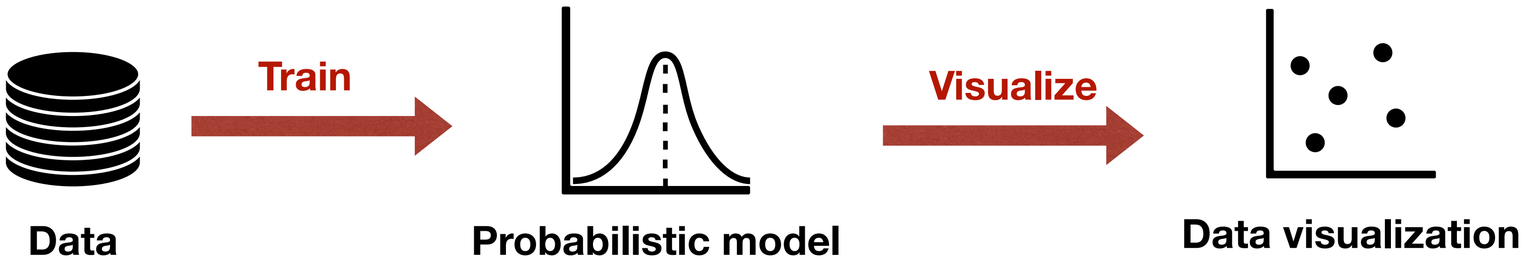
Training phase



Aim during training phase : find θ such that $\forall (x, y) \in \mathcal{X} \times \mathcal{Y}, h_\theta(x) \approx y$

Usually in frequentist statistics we use the MLE : Maximum Likelihood Estimation $\hat{\theta} = \arg\max_\theta P(X \mid \theta)$

Training phase



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- Only works well if we have big data : $|X| \gg |\theta|$
- Cannot start with a « belief » hence not practical nor flexible
- Cannot express uncertainty of estimated model parameters and predictions

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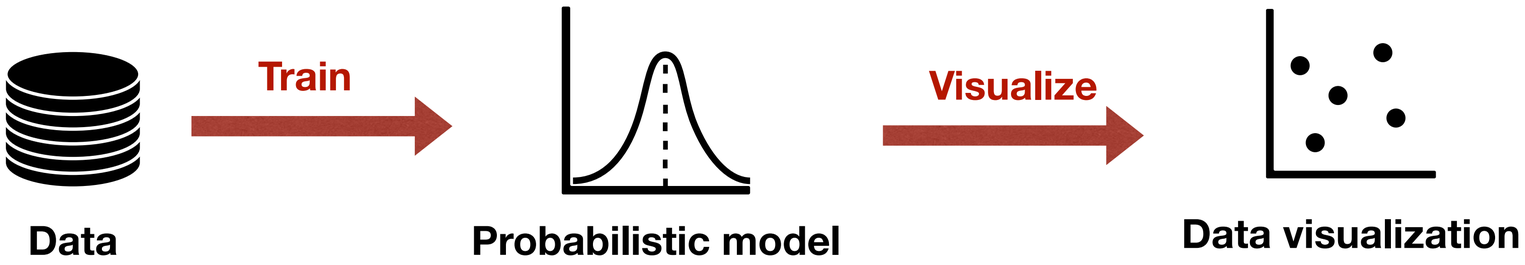

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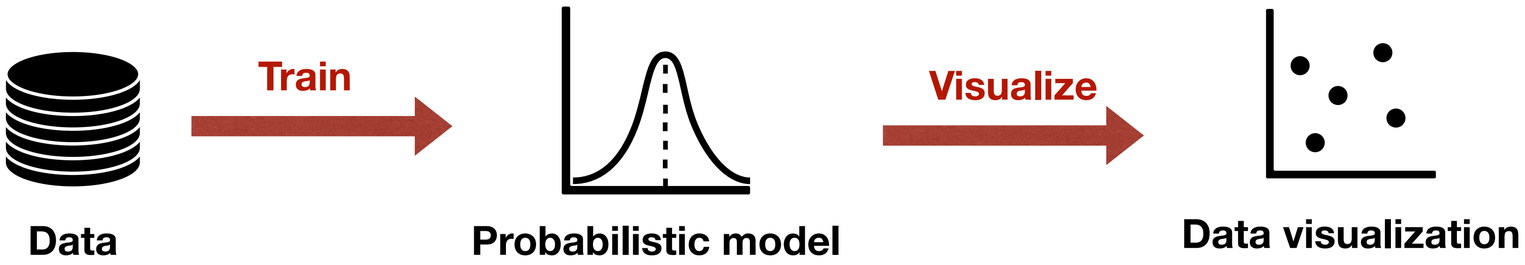

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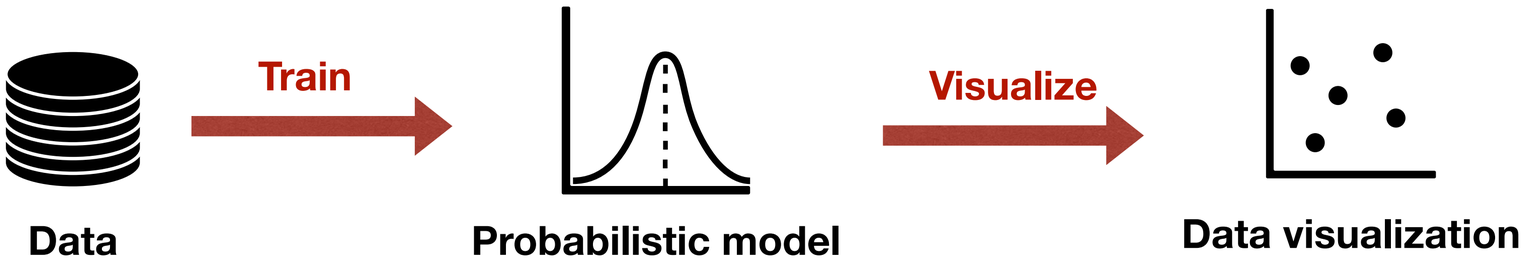

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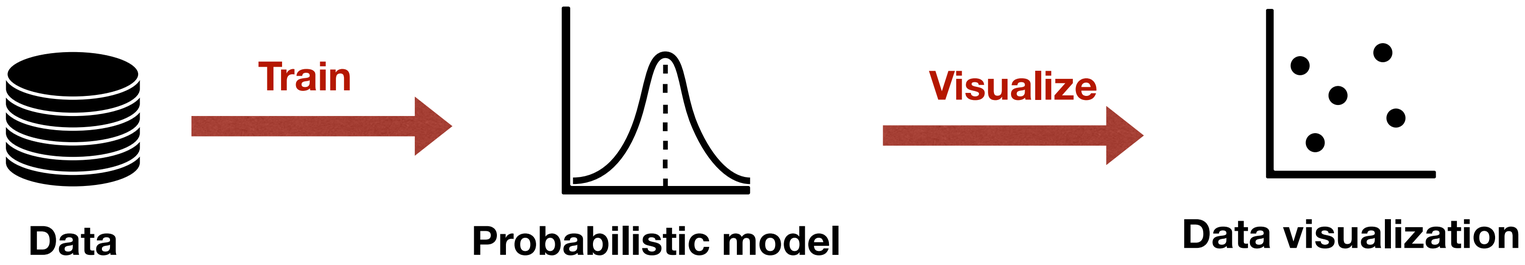

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Bayesian statistics

1 Introduction to bayesian statistics

1. Introduction to bayesian statistics

Probability & statistics : small review

Probability (non axiomatic definition) of an event : relative **frequency** of an event in an infinite trials

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Probability & statistics : small review

Probability (non axiomatic definition) of an event : relative **frequency** of an event in an infinite trials

Example : a deck of 52 playing cards

$$P(\ \text{  }\ ) = \frac{1}{52} \qquad P(\ \text{  }\ ) = \frac{1}{4}$$

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Random variable

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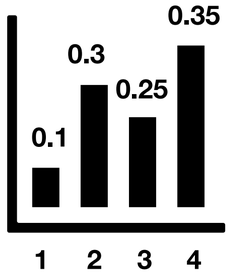
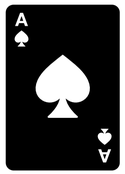
Random variable

- **Discrete** variable



Example



Probability Mass Function (PMF)

$$P(X) = \begin{cases} 0.1 \text{ if } X = 1 \\ 0.3 \text{ if } X = 2 \\ 0.25 \text{ if } X = 3 \\ 0.35 \text{ if } X = 4 \end{cases}$$

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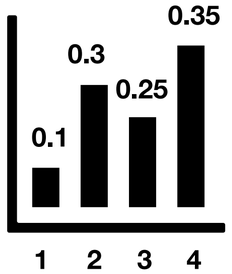

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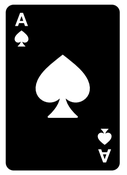

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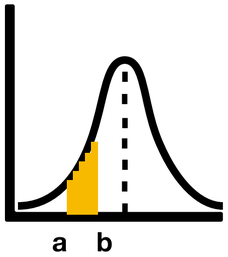
- **Continuous** variable



Example

Probability Density function (PDF)

$$P(X \in [a, b]) = \int_a^b p(s)ds$$

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Random variable - **Discrete** variable - **Continuous** variable

Independence

1. Introduction to bayesian statistics

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Independence : two random variables X and Y are **independent** if $P(X, Y) = P(X)P(Y)$

joint probability **marginals**

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joint probability **marginals**

Example :

dependency : **one** deck of 52 playing cards

$$P(X_1 = \text{A♥}, X_2 = \text{A♥}) = 0$$

$$P(X_1 = \text{A♥}) \cdot P(X_2 = \text{A♥}) = \frac{1}{52^2}$$

independency : **two** dices

$$P(X_1 = \text{⚅}, X_2 = \text{⚅}) = \frac{1}{6^2}$$

$$P(X_1 = \text{⚅}) \cdot P(X_2 = \text{⚅}) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{6^2}$$

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Conditional probability

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Conditional probability : probability of X **given that Y happened**

$$P(X \mid Y) = \frac{P(X, Y)}{P(Y)}$$

joint probability

conditional

marginal

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joint probability

marginal

Example : **one** deck of 52 playing cards **conditional**

$$P(\text{A♥} \mid \text{A♣}) = \frac{P(\text{A♥}, \text{A♣})}{P(\text{A♣})}$$

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↳ **Rules :**

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→ **Rules :**

→ **Chain rule** $P(X_1, X_2) = P(X_1 \mid X_2) \times P(X_2)$

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Rules :

Chain rule $P(X_1, X_2, X_3) = P(X_1 \mid X_2, X_3) \times P(X_2 \mid X_3) \times P(X_3)$

1. Introduction to bayesian statistics

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→ **Rules :**

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Rules :

Chain rule $P(X_1, \ldots, X_n) = \prod_{k=1,\ldots,n} P(X_k \mid X_1, \ldots, X_{k-1})$

Sum rule $P(X) = \sum_{Y \in \mathcal{Y}} P(X, Y)$ $P(X) = \int_{Y \in \mathcal{Y}} P(X, Y) \cdot dY$

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Bayes theorem :

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Rules : **Chain rule** **Sum rule**

Bayes theorem :

θ Parameters

X Observations (data)

Likelihood **Prior**

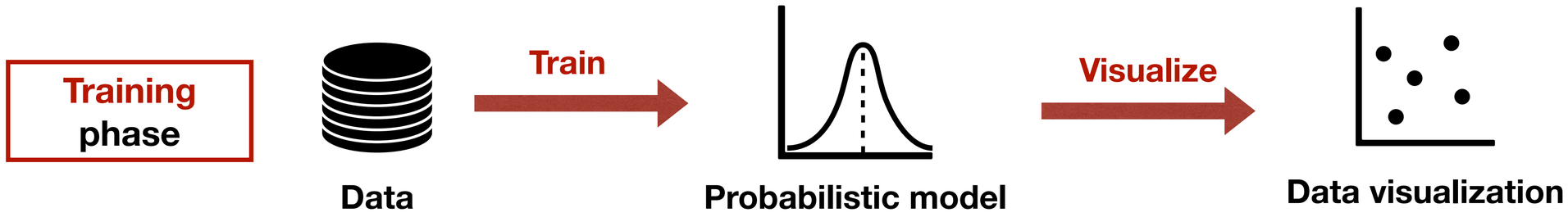
$$P(\theta \mid X) = \frac{P(X, \theta)}{P(X)} = \frac{P(X \mid \theta) \times P(\theta)}{P(X)}$$

Posterior

Evidence

1. Introduction to bayesian statistics

Frequentist VS Bayesian point of view



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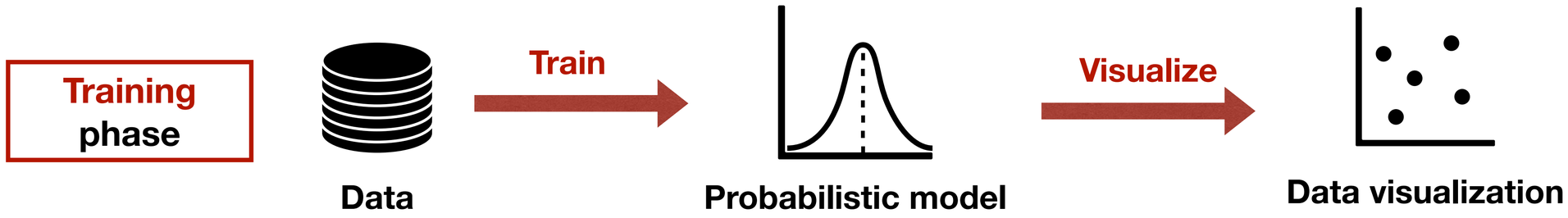
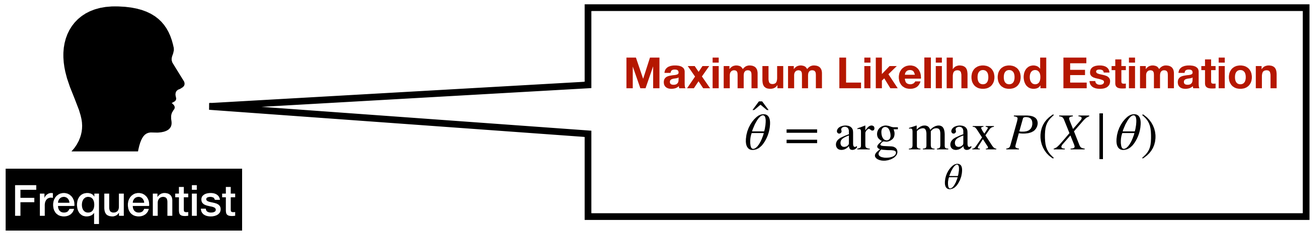
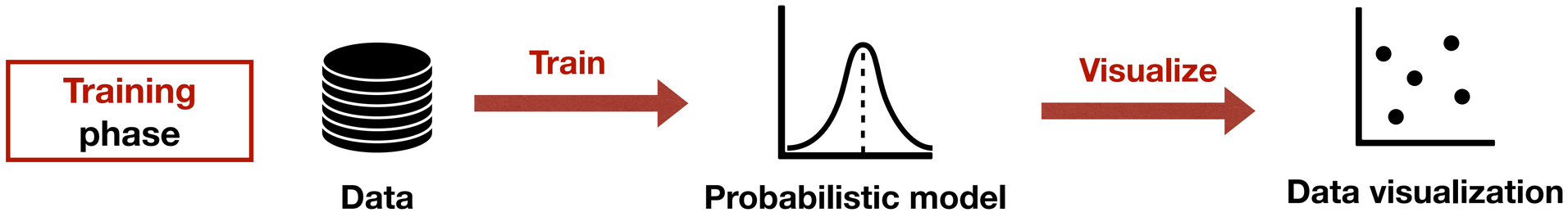


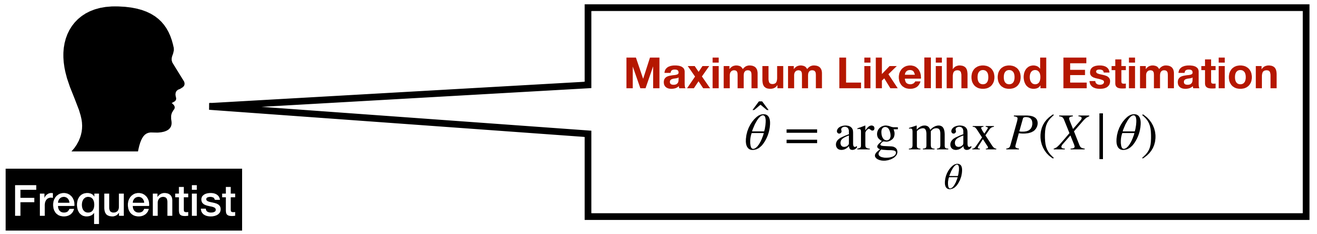


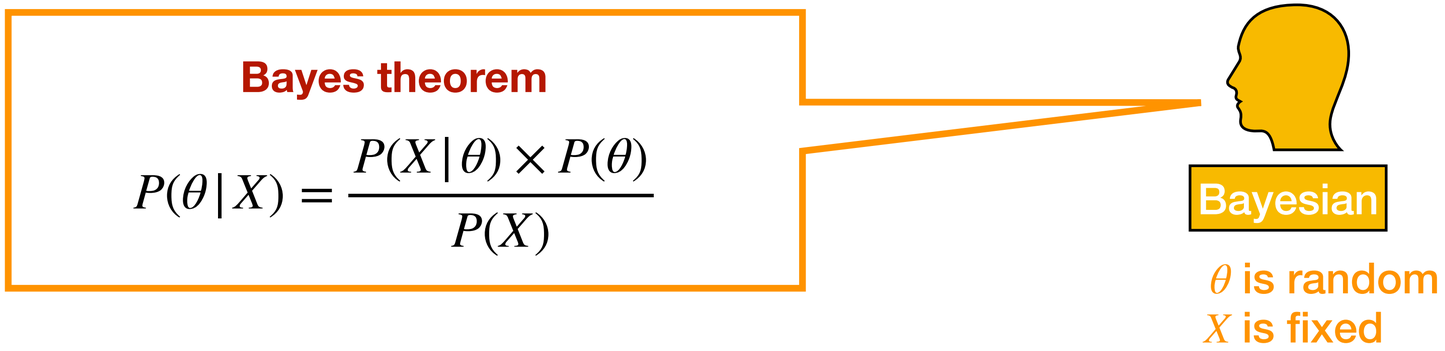
θ is fixed
X is random

1. Introduction to bayesian statistics

Frequentist VS Bayesian point of view







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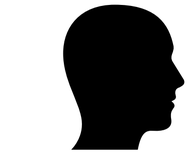


Frequentist

Maximum Likelihood Estimation

$$\hat{\theta} = \arg\max_{\theta} P(X \mid \theta)$$

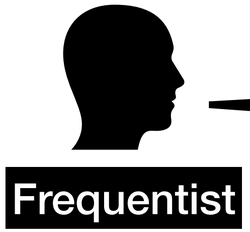
Bayes theorem

$$P(\theta \mid X) = \frac{P(X \mid \theta) \times P(\theta)}{P(X)}$$

Bayesian

1. Introduction to bayesian statistics

Frequentist VS Bayesian point of view



Frequentist

Maximum Likelihood Estimation

$$\hat{\theta} = \arg\max_{\theta} P(X \mid \theta)$$

Problems :

- Only works well if we have big data : $|X| \gg |\theta|$
- Cannot start with a « belief » hence not practical nor flexible
- Cannot express uncertainty of estimated model parameters and predictions

Bayes theorem

$$P(\theta \mid X) = \frac{P(X \mid \theta) \times P(\theta)}{P(X)}$$

Bayesian

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Frequentist VS Bayesian point of view

Frequentist

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Bayesian

1. Introduction to bayesian statistics

Bayesian point of view : classification

Bayes theorem

$$P(\theta \mid X) = \frac{P(X \mid \theta) \times P(\theta)}{P(X)}$$

Training phase

$$P(\theta \mid X_{train}, y_{train}) = \frac{P(y_{train} \mid X_{train}, \theta) \times P(\theta)}{P(y_{train} \mid X_{train})}$$

1. Introduction to bayesian statistics

Bayesian point of view : **training**

Bayes theorem

$$P(\theta \mid X) = \frac{P(X \mid \theta) \times P(\theta)}{P(X)}$$

Training phase

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Can regularize your model when training on your data

1. Introduction to bayesian statistics

Bayesian point of view : inference

Bayes theorem

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Training phase

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Can regularize your model when training on your data

Inference phase

$$P(y_{new} \mid X_{new}, X_{train}, y_{train}) = \int P(y_{new} \mid X_{train}, \theta) \times P(\theta \mid X_{train}, y_{train}) d\theta$$

1. Introduction to bayesian statistics

Bayesian point of view : online learning

Bayes theorem

$$P(\theta|X) = \frac{P(X|\theta) \times P(\theta)}{P(X)}$$

Training phase

$$P(\theta|X_{train}, y_{train}) = \frac{P(y_{train}|X_{train}, \theta) \times P(\theta)}{P(y_{train}|X_{train})}$$

Can regularize your model when training on your data

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$$P(y_{new}|X_{new}, X_{train}, y_{train}) = \int P(y_{new}|X_{train}, \theta) \times P(\theta|X_{train}, y_{train}) d\theta$$

Online learning

$$P_{new}(\theta) = P(\theta|x_{new}) = \frac{P(x_{new}|\theta) \times P_{old}(\theta)}{P(x_{new})}$$

New prior

Posterior

2 Probabilistic models

2. Probabilistic model

Probabilistic Graphical Model (PGM)

Probabilistic graphical models : analysis using **diagrammatic representations** of probability distributions

2. Probabilistic model

Probabilistic Graphical Model (PGM)

Probabilistic graphical models : analysis using **diagrammatic representations** of probability distributions

- **Nodes :** random variables
- **Links :** probabilistic relationships

2. Probabilistic model

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Bayesian networks
(**Directed** graphical models)

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Model : joint probability over all variables $P(X_1, \cdots, X_N) = \ldots$

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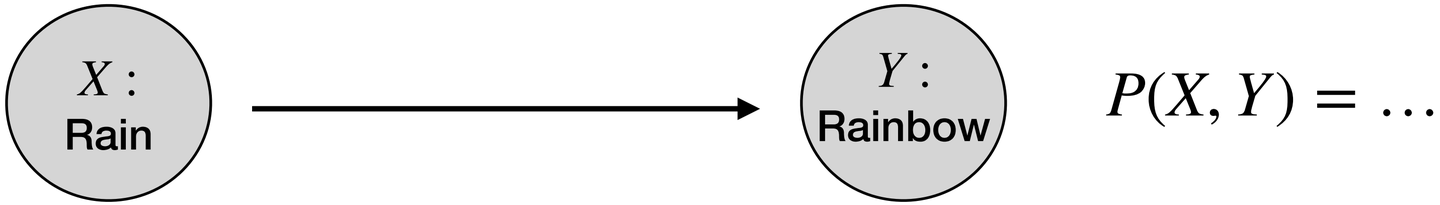
Example :



$P(X, Y) = \ldots$

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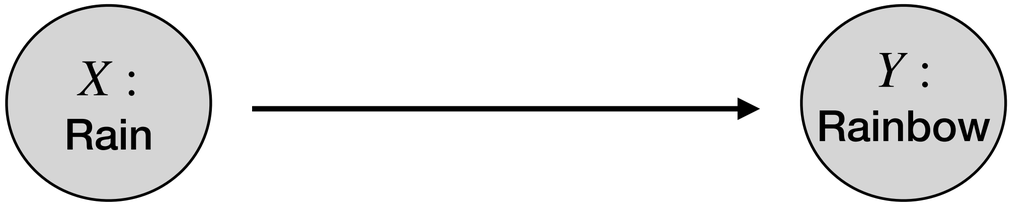

$$P(X, Y) = P(Y \mid X) \times P(X)$$

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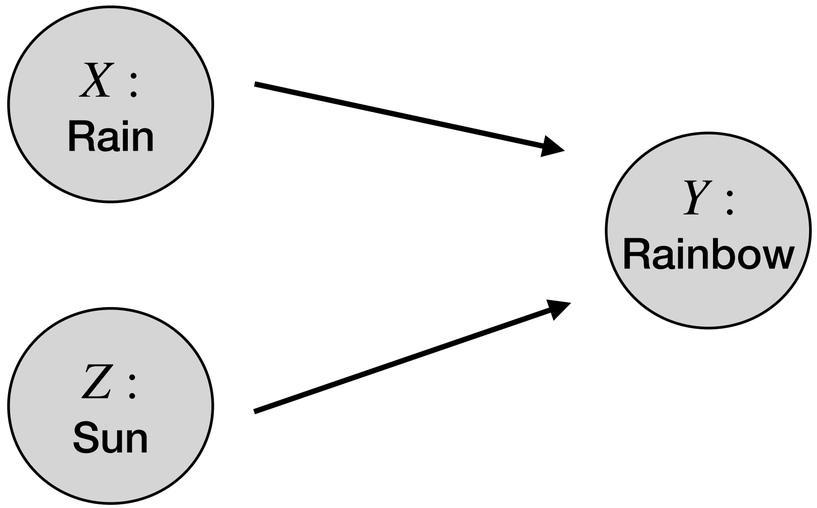

$P(X, Y, Z) = \ldots$

Features

X , Z conditionally independent given Y

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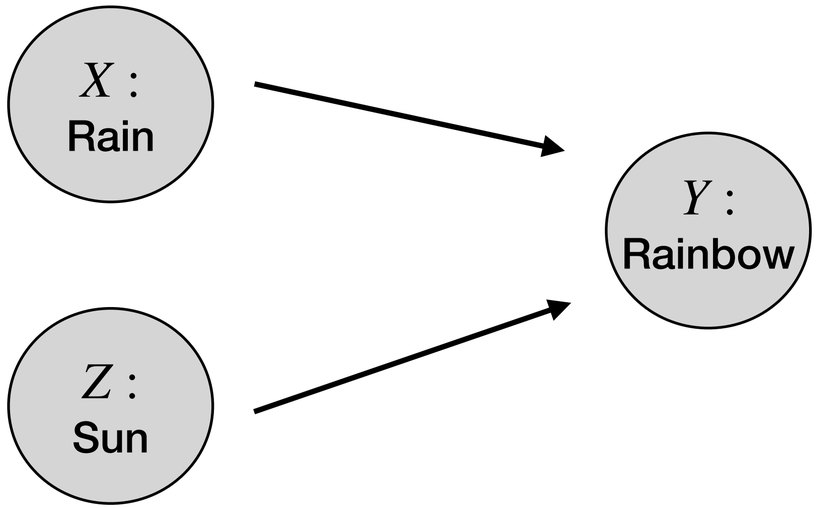

$$P(X, Y, Z) = P(Y \mid X, Z) \times P(X, Z)$$
$$= P(Y \mid X, Z) \times P(X) \times P(Z)$$

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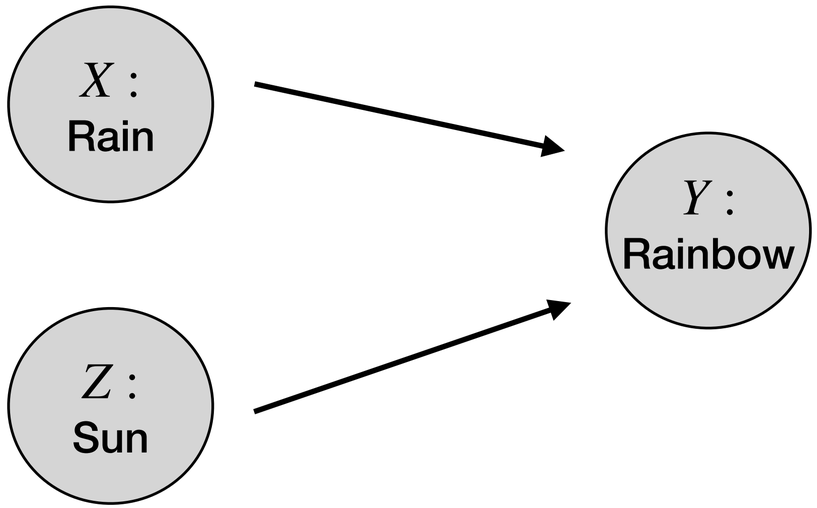
Model : joint probability over all variables

$$P(X_1, \cdots, X_N) = \prod_{i=1,\cdots,N} P(\ X_i \mid parents(X_i)\ )$$

Example :



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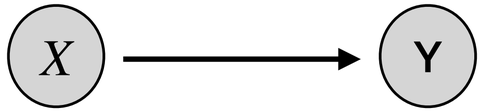
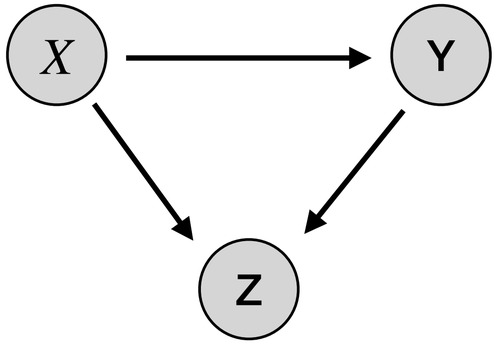


$P(X, Y) = P(Y \mid X) \times P(X)$



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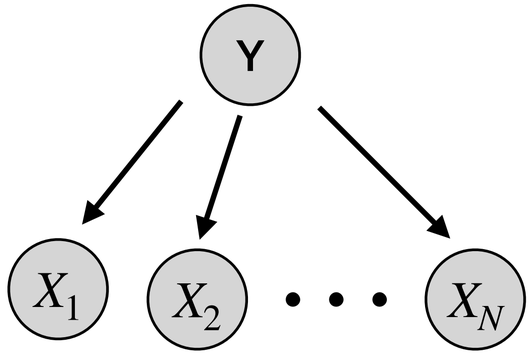

$P(Y, X_1, \cdots, X_N) =$

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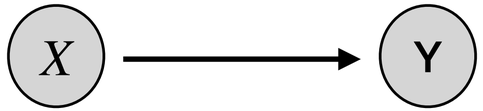

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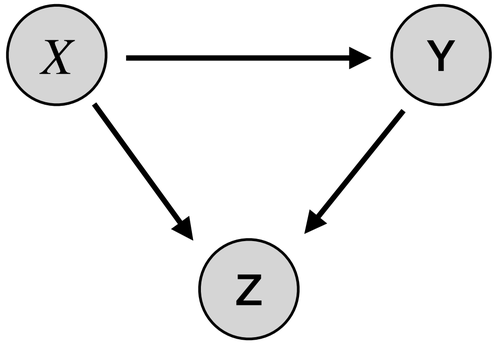
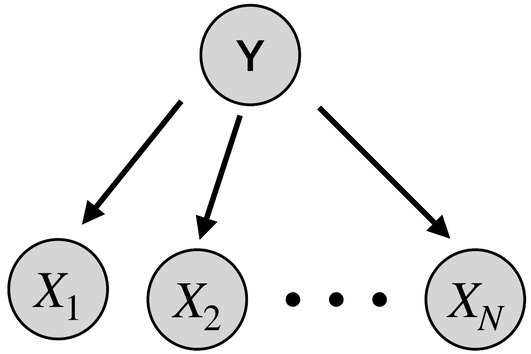


$P(X, Y, Z) = P(Z|X, Y) \times P(X, Y)$
$= P(Z|X, Y) \times P(Y|X) \times P(X)$



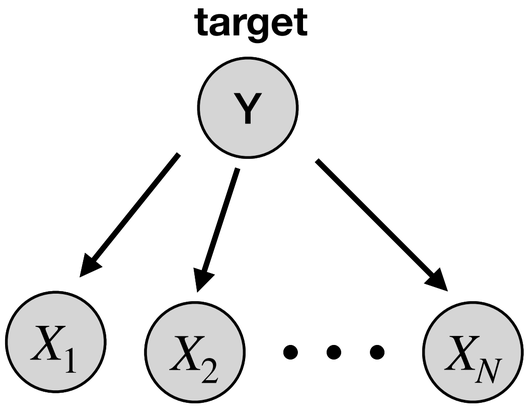
$P(Y, X_1, \cdots, X_N) = P(Y) \prod_{i=1\ldots,N} P(X_i|Y)$

2. Probabilistic model

Plates and examples of probabilistic model

Naive Bayes Classifier

$$P(Y, X_1, \cdots, X_N) = P(Y) \prod_{i=1\ldots,N} P(X_i \mid Y)$$



Features

X_i , X_j conditionally independent given Y

2. Probabilistic model

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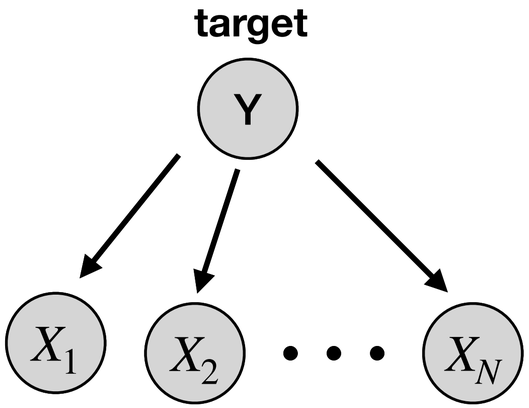

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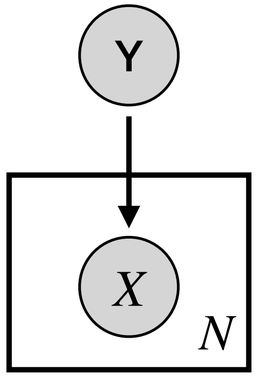
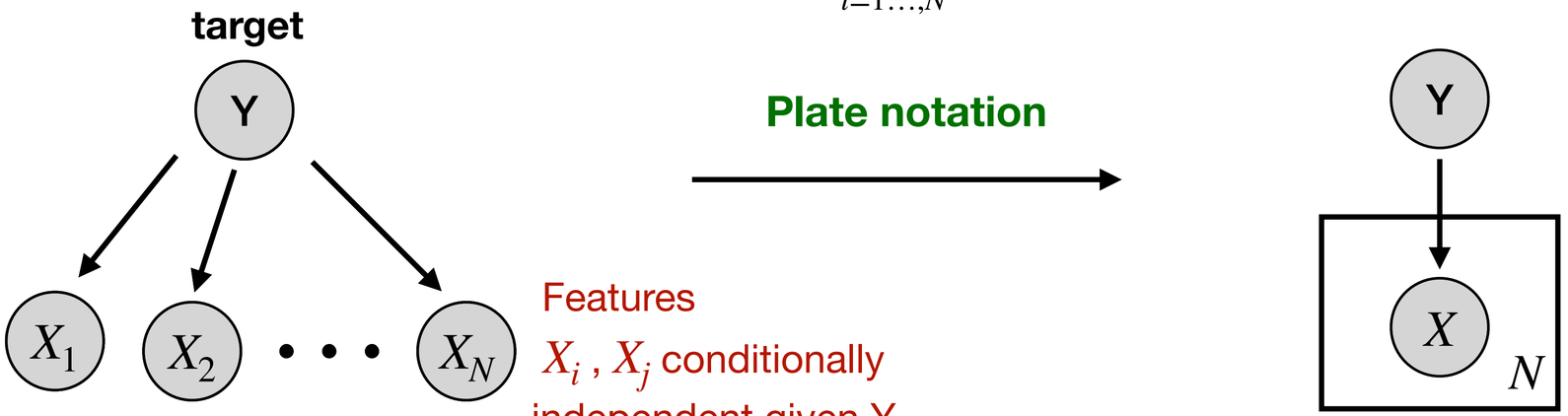
Plate notation



2. Probabilistic model

Plates and examples of probabilistic model

Naive Bayes Classifier $P(Y, X_1, \cdots, X_N) = P(Y) \prod_{i=1\ldots,N} P(X_i \mid Y)$



Reminder : Frequentist linear regression $x_i \in \mathbb{R}^d \quad y_1 \in \mathbb{R}$

Scalar notation :

$$D = \{(x_1, y_1), \ldots, (x_n, y_n)\}$$

$$y_i = x_i^T \theta + \epsilon_i$$

Matrix notation :

$$\mathbf{X} = (x_1, \ldots, x_n) \text{ and } \boldsymbol{y} = (y_1, \ldots, y_n)$$

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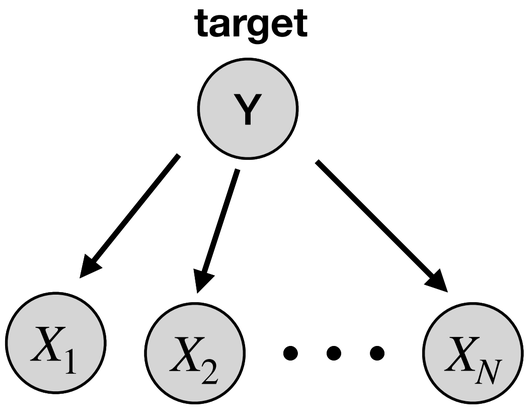
2. Probabilistic model

Plates and examples of probabilistic model

Naive Bayes Classifier $P(Y, X_1, \cdots, X_N) = P(Y) \prod_{i=1\ldots,N} P(X_i \mid Y)$



Features
X_i , X_j conditionally independent given Y

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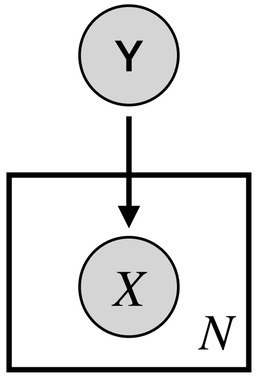

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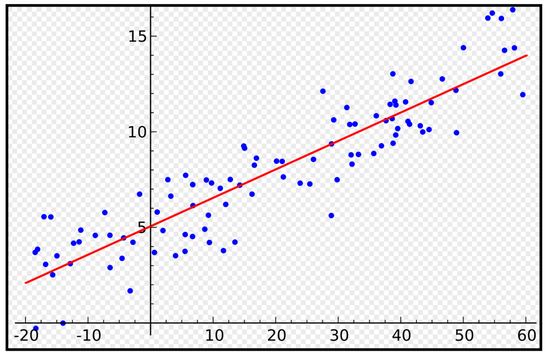


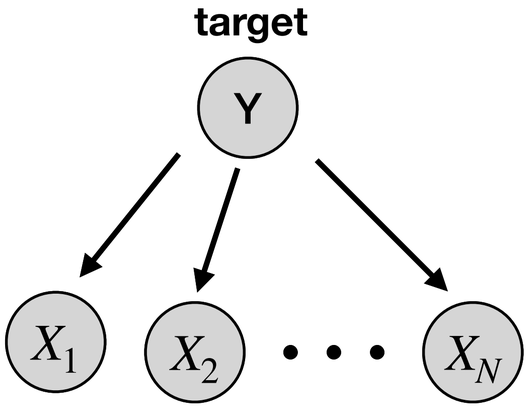
$$\min_{\theta} \|\theta^t X - y\|^2$$

2. Probabilistic model

Plates and examples of probabilistic model

Naive Bayes Classifier

$$P(Y, X_1, \cdots, X_N) = P(Y) \prod_{i=1\ldots,N} P(X_i \,|\, Y)$$



Features X_i , X_j conditionally independent given Y

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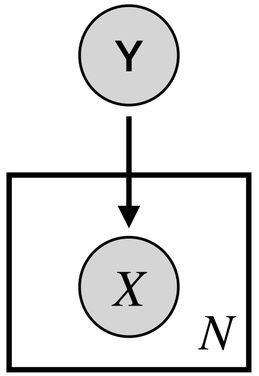
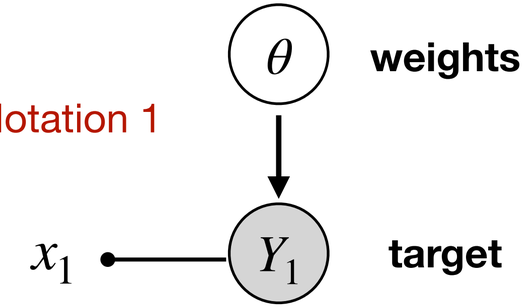


Bayesian Linear regression

(x_1, y_1) with $x_1 \in \mathbb{R}^d$, $y_1 \in \mathbb{R}$

Notation 1



$$P(\theta, y_1 \,|\, x_1) = P(y_1 \,|\, \theta, x_1) \times P(\theta)$$

$$P(y_1 \,|\, \theta, x_1) = \ldots$$

$$P(\theta) = \ldots$$

Legend :

Fixed variable — Hidden (latent) r.v. — Observed r.v.

2. Probabilistic model

Plates and examples of probabilistic model

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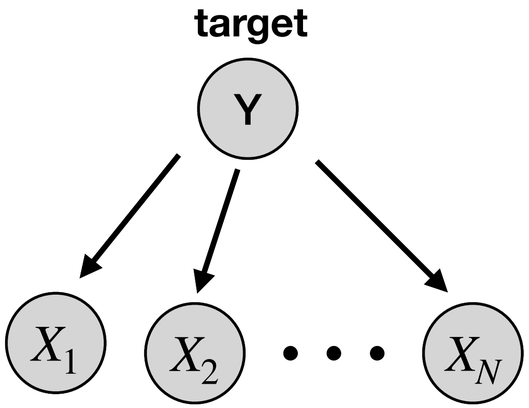

Features X_i , X_j conditionally independent given Y

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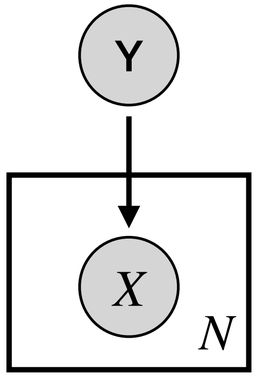

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(x_1, y_1) with $x_1 \in \mathbb{R}^d$, $y_1 \in \mathbb{R}$

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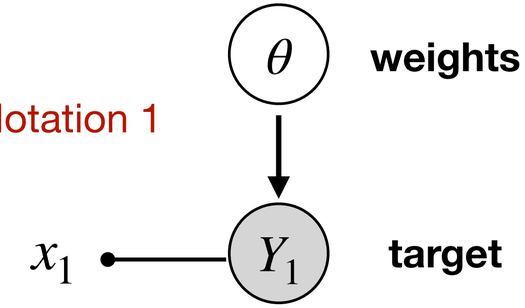

$$P(\theta, y_1 \mid x_1) = P(y_1 \mid \theta, x_1) \times P(\theta)$$

$$P(y_1 \mid \theta, x_1) = \mathcal{N}(y_1 \mid \theta^T x_1, \sigma^2)$$

$$P(\theta) = \mathcal{N}(\theta \mid 0, \gamma^2)$$

Legend :

Fixed variable

Hidden (latent) r.v.

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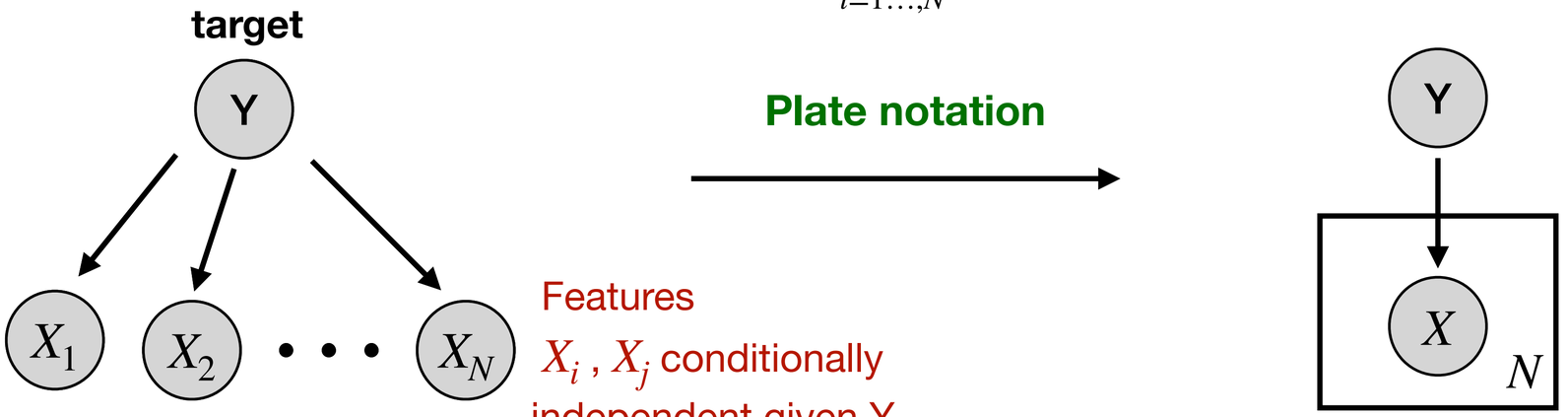
2. Probabilistic model

Plates and examples of probabilistic model

Naive Bayes Classifier $P(Y, X_1, \cdots, X_N) = P(Y) \prod_{i=1\ldots,N} P(X_i \mid Y)$



Bayesian Linear regression $D = \{(x_1, y_1), \ldots, (x_n, y_n)\}$ with $x_i \in \mathbb{R}^d$, $y_i \in \mathbb{R}$

$$P(\theta, y_i \mid x_i) = P(y_i \mid \theta, x_i) \times P(\theta)$$
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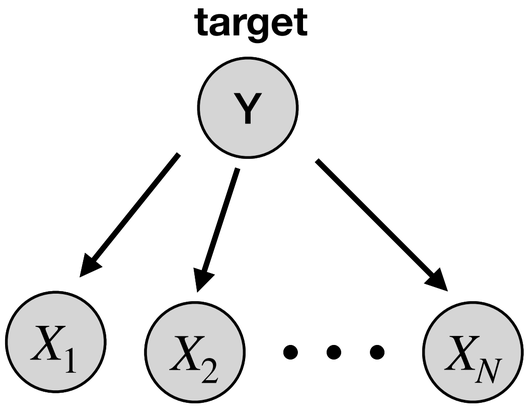
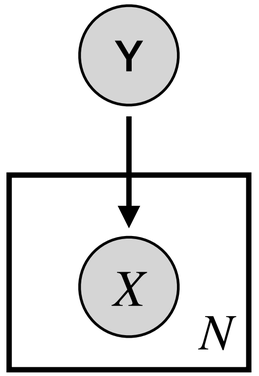
Plates and examples of probabilistic model

Naive Bayes Classifier $P(Y, X_1, \cdots, X_N) = P(Y) \prod_{i=1\ldots,N} P(X_i \mid Y)$



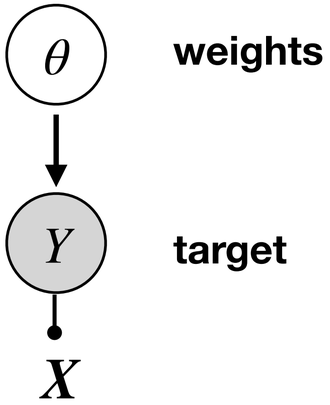
Features X_i , X_j conditionally independent given Y

Plate notation



Bayesian Linear regression $\mathbf{X} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_n)$ with $x_i \in \mathbb{R}^d$, $y_i \in \mathbb{R}$

Notation 3



$$P(\theta, y \mid x) = P(y \mid \theta, x) \times P(\theta)$$

$$P(y \mid \theta, x) = \ldots$$

$$P(\theta) = \ldots$$

Legend :

Fixed variable

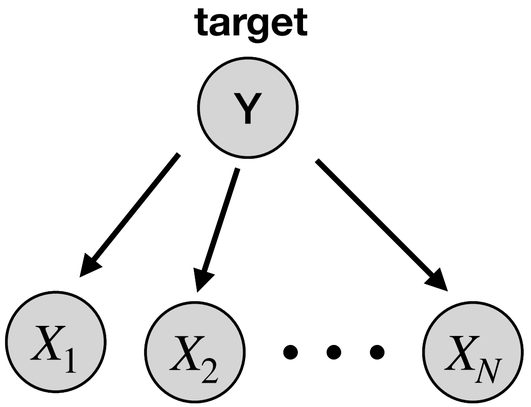
Hidden (latent) r.v.

Observed r.v.

2. Probabilistic model

Plates and examples of probabilistic model

Naive Bayes Classifier $P(Y, X_1, \cdots, X_N) = P(Y) \prod_{i=1\ldots,N} P(X_i \mid Y)$



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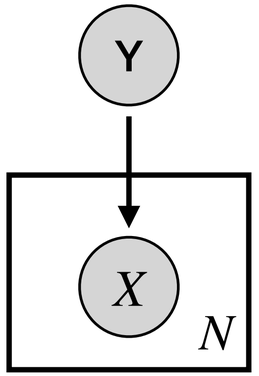

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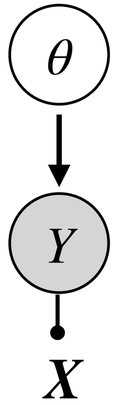


$$P(\theta, y \mid x) = P(y \mid \theta, x) \times P(\theta)$$

$$P(y \mid \theta, x) = \mathcal{N}(y \mid \theta^T X,\ \sigma^2 I_n)$$

$$P(\theta) = \mathcal{N}(\theta \mid 0,\ \gamma^2 I_n)$$

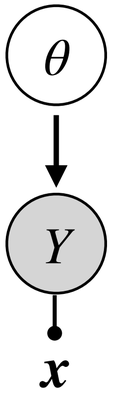
Legend :

Fixed variable — Hidden (latent) r.v. — Observed r.v.

2. Probabilistic model

Linear regression

Bayesian Linear regression



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Objective :

Frequentist linear regression

Objective : $\min_{\theta} \mathscr{L}(\theta) = \min_{\theta} \|\theta^t X - y\|^2$

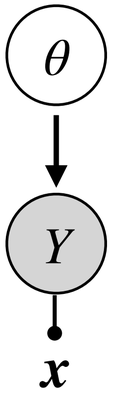
$$\hat{\theta} = \arg\min_{\theta} \mathscr{L}(\theta)$$

$$\hat{\theta} = (X^T X)^{-1} X^T y$$

2. Probabilistic model

Linear regression

Bayesian Linear regression



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Frequentist linear regression

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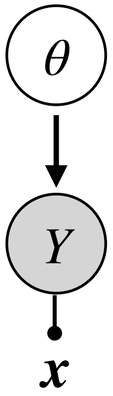
Bayesian Linear regression



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Theorem :

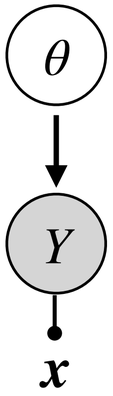
There exists $\lambda \in \mathbb{R}$ such that : $\arg\max_\theta P(\theta \mid X, y) = \arg\min_\theta \left\{ \|\theta^t X - y\|^2 + \lambda \|\theta\|^2 \right\}$

So by adding a normal prior on the weight we turned this problem into a L_2 **regularised problem**

2. Probabilistic model

Linear regression

Bayesian Linear regression



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So by adding a normal prior on the weight we turned this problem into a L_2 **regularised problem**

Proof : see the whiteboard in class **or** left as an exercice

3 Analytical Inference

3. Analytical Inference

Reminder of posterior distribution

Posterior distribution

Likelihood **Prior**

$$P(\theta \mid X) = \frac{P(X, \theta)}{P(X)} = \frac{P(X \mid \theta) \times P(\theta)}{P(X)}$$

Posterior

Evidence

3. Analytical Inference

Reminder of posterior distribution

Posterior distribution

Fixed by model

Fixed by us

Likelihood **Prior**

$$P(\theta \mid X) = \frac{P(X, \theta)}{P(X)} = \frac{P(X \mid \theta) \times P(\theta)}{P(X)}$$

Posterior

Evidence

Fixed by data

3. Analytical Inference

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Posterior

Evidence

HARD TO COMPUTE

Fixed by data

$$P(X) = \int_{\theta} P(X \mid \theta) \cdot P(\theta) \cdot d\theta$$

3. Analytical Inference

Maximum a posteriori (MAP) : definition & remarks

Posterior distribution

Fixed by **model** — Likelihood

Prior — Fixed by **us**

$$P(\theta \mid X) = \frac{P(X,\theta)}{P(X)} = \frac{P(X \mid \theta) \times P(\theta)}{P(X)}$$

Posterior

Evidence

HARD TO COMPUTE

Fixed by **data**

$$P(X) = \int_{\theta} P(X \mid \theta) \cdot P(\theta) \cdot d\theta$$

Remarks

- We have to avoid computing the evidence
- Naive approach : maximum a posteriori ,

$$\hat{\theta}_{MAP} = \arg\max_{\theta} \left\{ \frac{P(\theta \mid X) \cdot P(\theta)}{P(X)} \right\}$$

$$= \arg\max_{\theta} P(X \mid \theta) \cdot P(\theta)$$

- This maximization can be done with numerical optimization problem

3. Analytical Inference

Maximum a posteriori (MAP) : limitations

Posterior distribution

Fixed by model

Fixed by us

Likelihood Prior

$$P(\theta|X) = \frac{P(X,\theta)}{P(X)} = \frac{P(X|\theta) \times P(\theta)}{P(X)}$$

Posterior

Evidence

HARD TO COMPUTE

Fixed by data

$$P(X) = \int_{\theta} P(X|\theta) \cdot P(\theta) \cdot d\theta$$

Remarks

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$$= \arg\max_{\theta} P(X|\theta) \cdot P(\theta)$$

- This maximization can be done with numerical optimization problem

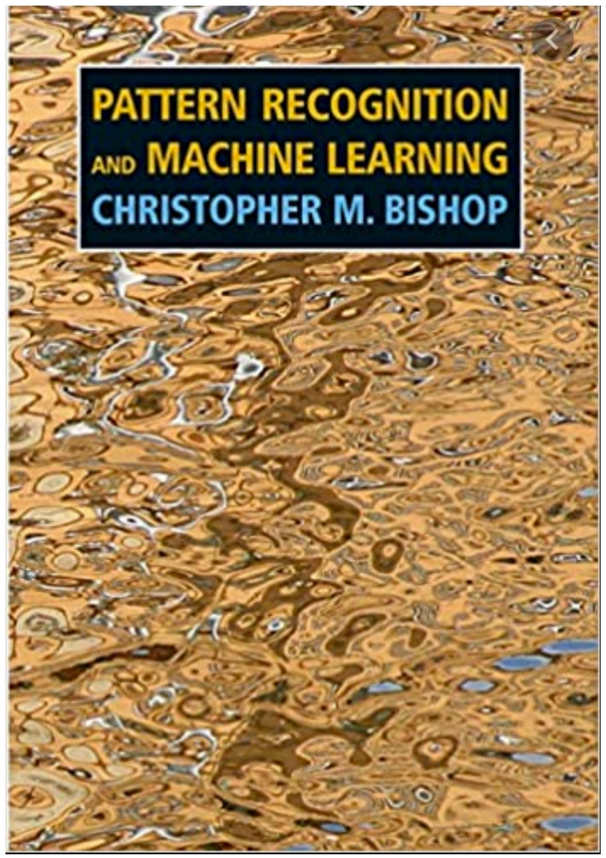
Limitations (among many others)

1. in general, not representative of bayesian methods : $\hat{\theta}_{MAP}$ is a point estimate like $\hat{\theta}_{MLE}$
   - can't compute credible intervals because it doesn't return a pdf/pmf (not a bayesian inference)
2. can't use online learning : the prior is not well updated

3. Analytical Inference

Maximum a posteriori (MAP) : book

For more theoretical details (and example on analytical inference) :



4 Conjugate distributions

4. Conjugate distributions

Conjugate distributions : avoid computing evidence

Posterior distribution

Fixed by **model**

Fixed by **us**

Likelihood **Prior**

$$P(\theta \mid X) = \frac{P(X, \theta)}{P(X)} = \frac{P(X \mid \theta) \times P(\theta)}{P(X)}$$

Posterior

Evidence

Fixed by **data**

4. Conjugate distributions

Conjugate distributions : avoid computing evidence

Posterior distribution

Fixed by **model**

Fixed by **us**

Likelihood

Prior

$$P(\theta|X) = \frac{P(X,\theta)}{P(X)} = \frac{P(X|\theta) \times P(\theta)}{P(X)}$$

Posterior

Evidence

Fixed by **data**

Remarks

- We have to avoid computing the evidence
- We can choose a convenient prior which enable us to compute the **posterior :**

  Conjugate prior

4. Conjugate distributions

Conjugate distributions : avoid computing evidence

Posterior distribution

Fixed by **model**

Fixed by **us**

Likelihood

Prior

$$P(\theta|X) = \frac{P(X,\theta)}{P(X)} = \frac{P(X|\theta) \times P(\theta)}{P(X)}$$

Posterior

Evidence

Fixed by **data**

Remarks

- We have to avoid computing the evidence
- We can choose a convenient prior which enable us to compute the **posterior :**

  Conjugate prior

Conjugate prior

$P(\theta)$ is **conjugate** to $P(X|\theta)$ if the $P(\theta)$ and $P(X|\theta)$ lie in the same family of distributions (gaussian for example)

4. Conjugate distributions

Conjugate distributions : avoid computing evidence

Posterior distribution

Fixed by model

Fixed by us

Likelihood

Prior

$$P(\theta|X) = \frac{P(X,\theta)}{P(X)} = \frac{P(X|\theta) \times P(\theta)}{P(X)}$$

Posterior

Evidence

Fixed by data

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Conjugate prior

$P(\theta)$ is **conjugate** to $P(X|\theta)$ if the $P(\theta)$ and $P(X|\theta)$ lie in the same family of distributions (gaussian for example)

Example

$$P(\theta|X) = \frac{\mathcal{N}(X|\theta,\sigma^2) \times P(\theta)}{P(X)}$$

$\mathcal{N}(\theta|\mu_{posterior},\sigma^2_{posterior})$

$\mathcal{N}(\theta|\mu_{prior},\sigma^2_{prior})$

In the context of a gaussian, the prior for the **mean** is a gaussian !

4. Conjugate distributions

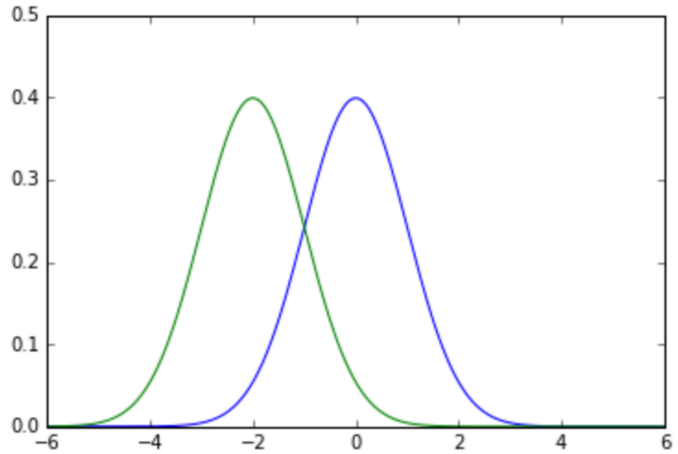
Conjugate distributions : example

Example

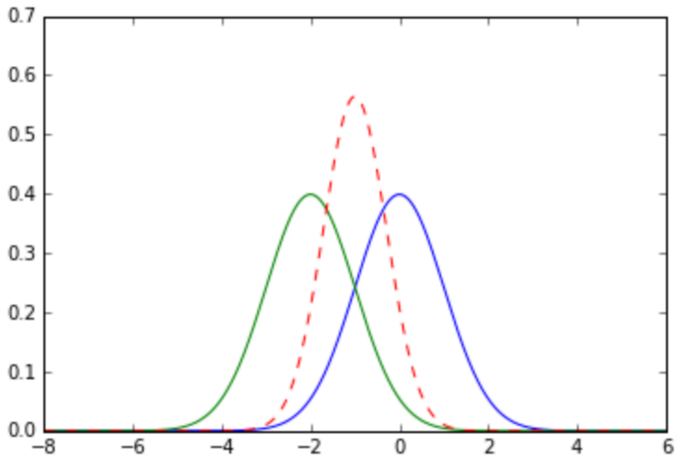
$$P(\theta|X) = \frac{\mathcal{N}(X|\theta,\sigma^2) \times P(\theta)}{P(X)}$$

$\mathcal{N}(\theta|\mu_{prior},\sigma^2_{prior})$

$\mathcal{N}(\theta|\mu_{posterior},\sigma^2_{posterior})$



pointwise product



Exercice (left as an exercice, correction in the next lecture)

Show that $\mathcal{N}(\theta|x/2,1/2) = \frac{\mathcal{N}(x|\theta,1) \times \mathcal{N}(\theta|0,1)}{P(x)}$

4. Conjugate distributions

Usual distributions : Gamma distribution

Gamma distribution

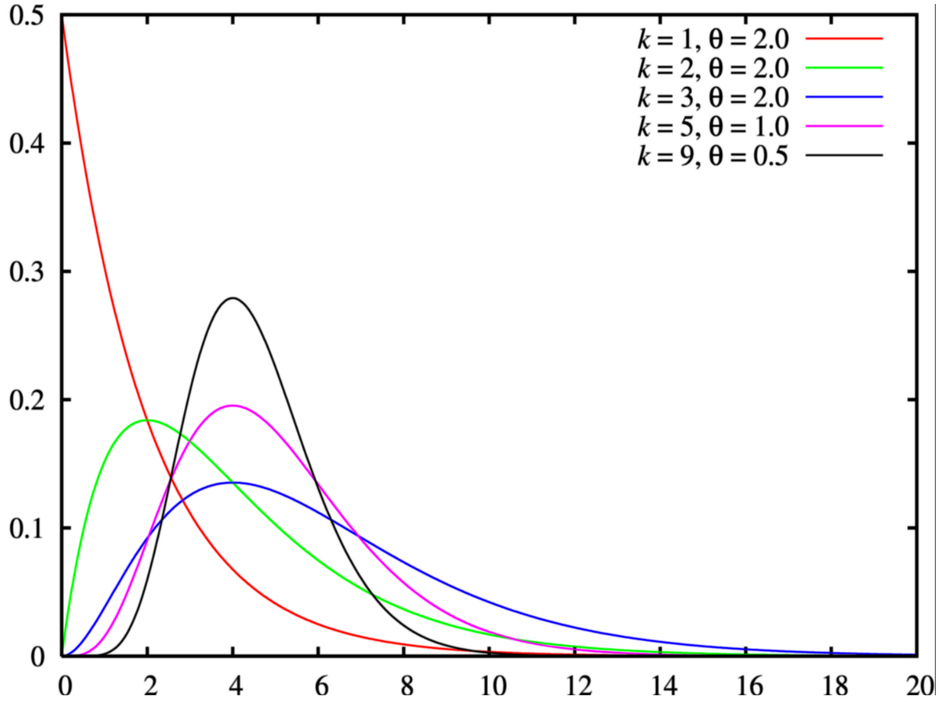
PDF : $\Gamma(x \mid k, \theta) = \dfrac{\theta^k}{\Gamma(k)} x^{k-1} e^{-\theta x}$ with $x, k, \theta > 0$

$\Gamma(k) = (k-1)!$

mean : $\mathbb{E}[x] = \dfrac{k}{\theta}$

variance : $V(x) = \dfrac{k}{\theta^2}$

mode : $Mode\,[x] = \dfrac{k-1}{\theta}$



Example

$$\Gamma(\gamma \mid k_{posterior}, \theta_{posterior}) \longleftarrow P(\gamma \mid x) = \frac{\mathcal{N}(x \mid \mu, \gamma^{-1}) \times P(\gamma)}{P(x)} \longleftarrow \Gamma(\gamma \mid k_{prior}, \theta_{prior})$$

In the context of a gaussian, the prior for the **precision** is a gamma !

4. Conjugate distributions

Usual distributions : Gamma distribution

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PDF : $\Gamma(x \mid k, \theta) = \dfrac{\theta^k}{\Gamma(k)} x^{k-1} e^{-\theta x}$ with $x, k, \theta > 0$

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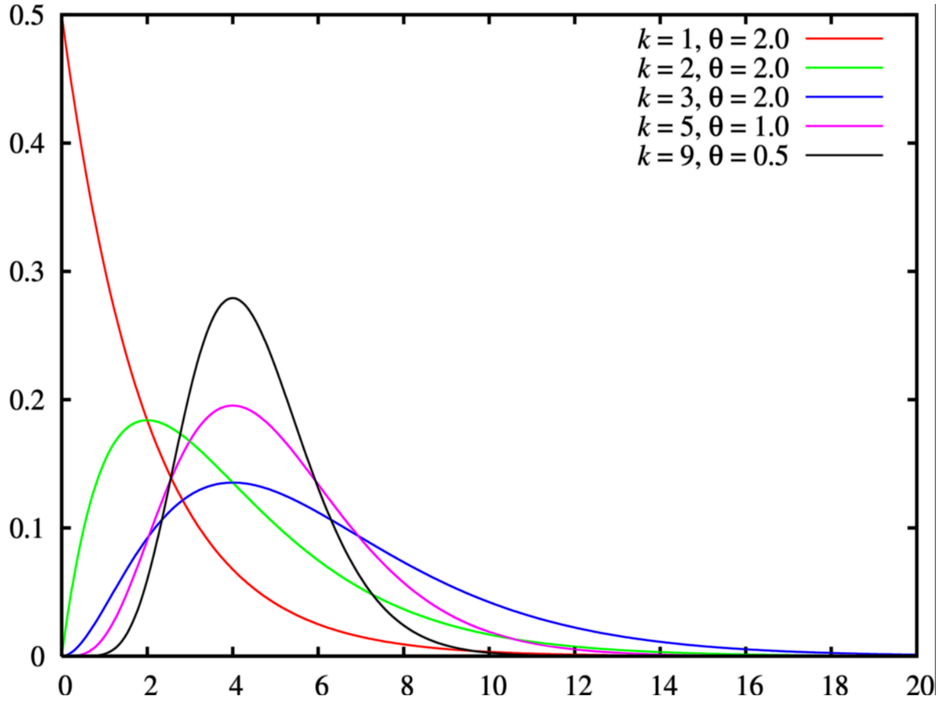


Exercice (left as an exercice, correction in the next lecture)

$$P(\gamma \mid x) = \frac{\mathcal{N}(x \mid \mu, \gamma^{-1}) \times P(\gamma)}{P(x)}$$

$\Gamma(\gamma \mid k_{prior}, \theta_{prior})$

$\Gamma(\gamma \mid k_{posterior}, \theta_{posterior})$

$\Gamma(\gamma \mid k_{prior} + 1/2,\ \theta_{prior} + (x-\mu)^2/2)$

In the context of a gaussian, the prior for the **precision** is a gamma !

4. Conjugate distributions

Usual distributions : Beta distribution

Gamma distribution

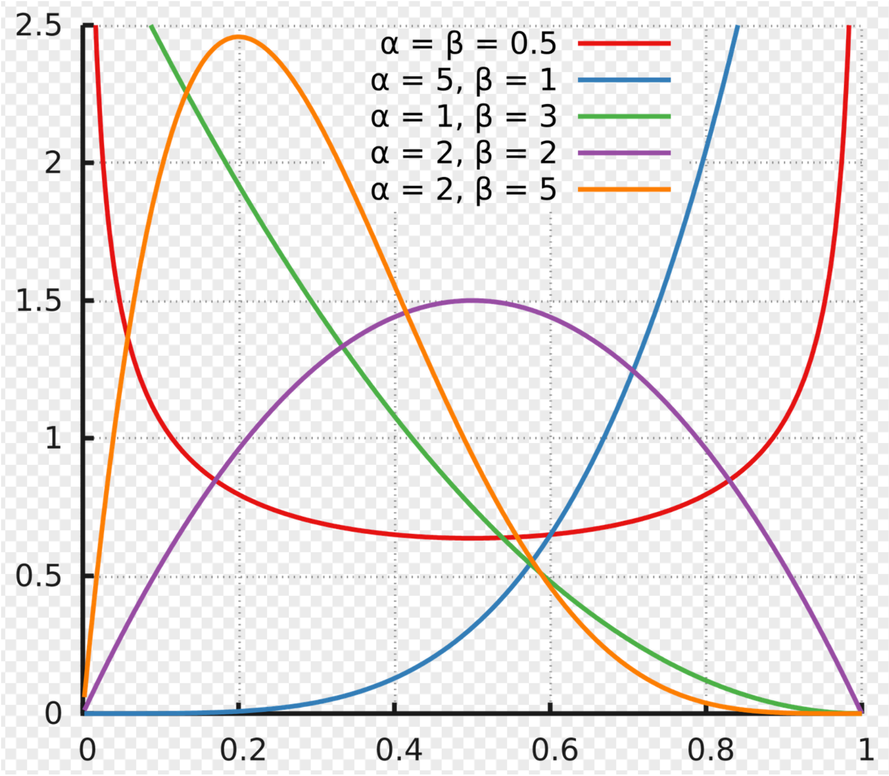
PDF : $B(x \mid \alpha, \beta) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1}(1-x)^{\beta-1}$ with $\alpha, \beta > 0$ and $x \in [0,1]$

$$B(\alpha, \beta) = \frac{\Gamma(\alpha) \cdot \Gamma(\beta)}{\Gamma(\alpha + \beta)}$$

mean : $\mathbb{E}[x] = \frac{\alpha}{\alpha + \beta}$

variance : $V(x) = \frac{\alpha\beta}{(\alpha + \beta)^2 \cdot (\alpha + \beta - 1)}$

mode : $Mode\,[x] = \frac{\alpha - 1}{\alpha + \beta - 2}$



Example

$$B(\theta \mid \alpha_{posterior}, \beta_{posterior}) \longleftarrow P(\theta \mid x) = \frac{Ber(x \mid \theta) \times P(\theta)}{P(x)} \longleftarrow B(\theta \mid \alpha_{prior}, \beta_{prior})$$

In the context of a Bernoulli distribution, the prior is a beta !

4. Conjugate distributions

Usual distributions : Beta distribution

Gamma distribution

PDF : $B(x\,|\,\alpha,\beta) = \dfrac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}$ with $\alpha,\beta > 0$ and $x \in [0,1]$

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\cdot\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

mean : $\mathbb{E}[x] = \dfrac{\alpha}{\alpha+\beta}$

variance : $V(x) = \dfrac{\alpha\beta}{(\alpha+\beta)^2\cdot(\alpha+\beta-1)}$

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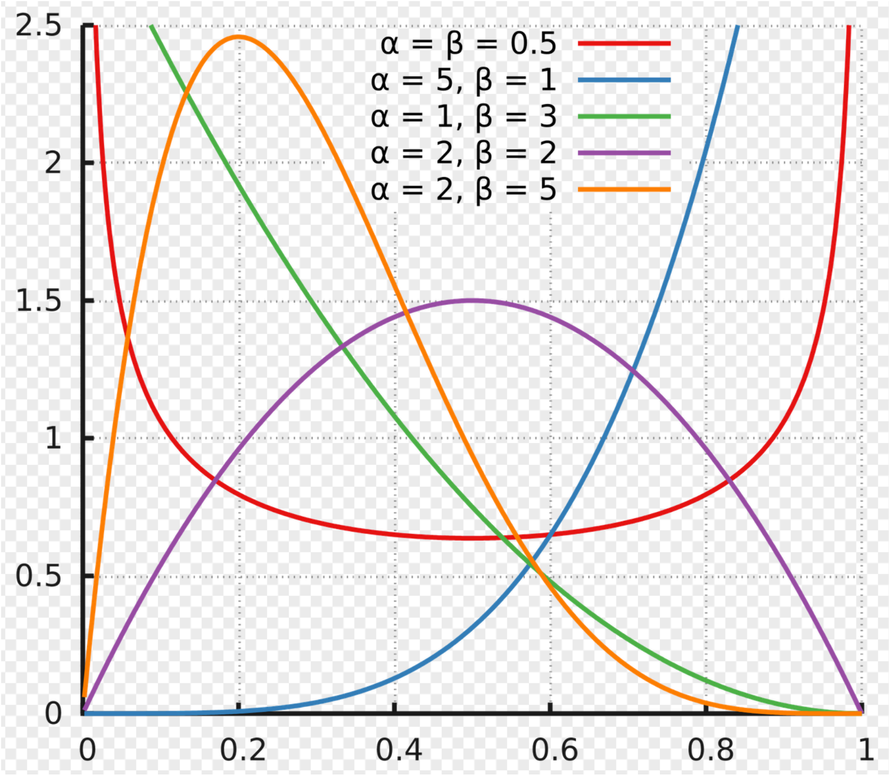

Exercice (left as an exercice, correction in the next lecture)

$$P(\theta\,|\,x) = \frac{Ber(x\,|\,\theta)\times P(\theta)}{P(x)}$$

$Ber(x\,|\,\theta)$: $\theta^{n_1}\cdot(1-\theta)^{n_0}$

$P(\theta)$: $B(\theta\,|\,\alpha_{prior},\beta_{prior})$

$P(\theta\,|\,x)$: $B(\theta\,|\,\alpha_{posterior},\beta_{posterior})$

$B(\theta\,|\,n_1+\alpha_{posterior},n_0+\beta_{posterior})$

In the context of a Bernoulli distribution, the prior is a beta !

4. Conjugate distributions

Limitations

Posterior distribution

Fixed by **model**

Fixed by **us**

Likelihood Prior

$$P(\theta|X) = \frac{P(X,\theta)}{P(X)} = \frac{P(X|\theta) \times P(\theta)}{P(X)}$$

Posterior

Evidence

Fixed by **data**

Remarks

- We have to avoid computing the evidence
- We can choose a convenient prior which enable us to compute the **posterior :**

  Conjugate prior

- It computes the exact posterior
- Easy for online learning

- For some (complex) models, the conjugate prior can be inadequate (improper prior)
- Can be unrealistic (non-informative prior)

Road map

Bayesian statistics (02/05/21)

1

Latent variable models (17/05/21)

2

Variational Inference (31/05/21)

3

Bayesian perspective :

Likelihood — Prior distribution

$$P(\theta \mid X) = \frac{P(X,\theta)}{P(X)} = \frac{P(X \mid \theta) \cdot P(\theta)}{P(X)}$$

Posterior distribution

Evidence

Hard to compute !

θ parameters

X observations

Exemple :
Naive Bayes classifier,
Linear regression, ….

MAP : $\arg\max_{\theta} P(X \mid \theta) \cdot P(\theta)$

Conjugate distribution

Pros :
- exact posterior

Cons :
- conjugate prior maybe inadequate

Markov Chain Monte Carlo (07/06/21)

4

Extensions (14/06/21)

5