
# Bayesian Machine Learning

**May 2022 - François HU**
**https://curiousml.github.io/**

# Outline

1 **Bayesian statistics**

2 **Latent variable models**
- Latent variable models and EM algorithm
- Probabilistic clustering
- Probabilistic dimensionality reduction

3 **Variational Inference**

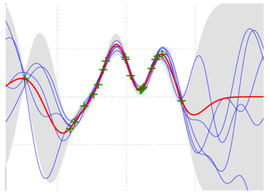
4 **Markov Chain Monte Carlo**

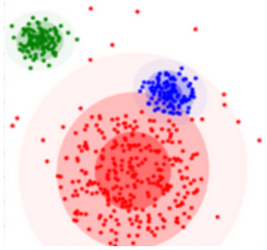
5 **Extensions and oral presentations**

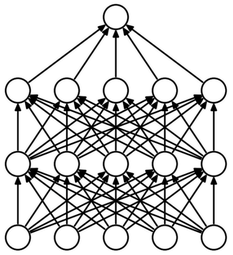
# 0 Evaluation & conjugate prior

# Evaluation

## Group project







- The evaluation will consist of a **group project (4 students max)** based on a research article

- For the last lecture, each student will send me the **codes** and give an **oral presentation** in front of the class. Even if the article is mostly theoretical, each presentation should be understandable by other students (The clarity of the speech will be analysed).

- Initiatives like **more experimentations** or identifying the limits of the article will be greatly appreciated. You are welcome to consult other research articles (it should be cited at the end of your presentation) to boost your knowledge (but don't forget that the proposed paper is the core of your presentation)

- The **evaluation** is as follows :

  - **40% on the clarity of the code** (example : many comments, along with understandable variables/functions names. You can use Jupyter Notebook which might have the advantage to be easy to read for the users). When I run your code, it should be easy to run and easy to understand :)

  - **60% on the clarity of the oral presentation**. Less maths but more experimentations and intuitions. At the beginning a big introduction is expected in order to be understandable by other groups.

# Conjugate priors: Exercices

## Gamma case

**Exercice** (left as an exercice, correction in the next lecture)

$$P(\gamma \mid x) = \frac{\mathcal{N}(x \mid \mu, \gamma^{-1}) \times P(\gamma)}{P(x)}$$

$\Gamma(\gamma \mid \alpha_{prior}, \beta_{prior})$

$\Gamma(\gamma \mid \alpha_{posterior}, \beta_{posterior})$

$\Gamma(\gamma \mid \alpha_{prior} + 1/2,\ \beta_{prior} + (x-\mu)^2/2)$

**Gamma distribution**

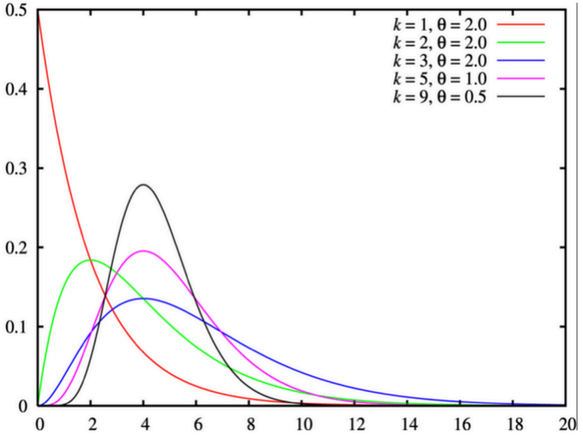
**PDF :** $\Gamma(x \mid \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$ with $x, \alpha, \beta > 0$

$\Gamma(\alpha) = (\alpha - 1)!$

**mean :** $\mathbb{E}[x] = \frac{\alpha}{\beta}$

**variance :** $V(x) = \frac{\alpha}{\beta^2}$

**mode :** $Mode\,[x] = \frac{\alpha - 1}{\beta}$




What we want to compute : $p(\text{parameters} \mid \text{data}) \propto p(\text{data} \mid \text{parameters}) \times p(\text{parameters})$

- $p(\text{data} \mid \text{parameters}) = \mathcal{N}(x \mid \mu, \gamma^{-1}) = \frac{\sqrt{\gamma}}{\sqrt{2\pi}} e^{-\gamma \frac{(x-\mu)^2}{2}} \propto \sqrt{\gamma}\, e^{-\gamma \frac{(x-\mu)^2}{2}}$ (deterministic, deterministic, random)

- $p(\text{parameters}) = \Gamma(\gamma \mid \alpha_{prior}, \beta_{prior}) = \frac{\beta_{prior}^{\alpha_{prior}}}{\Gamma(\alpha_{prior})} \times \gamma^{\alpha_{prior}-1} e^{-\gamma \beta_{prior}} \propto \gamma^{\alpha_{prior}-1} e^{-\gamma \beta_{prior}}$ (random, deterministic, deterministic)

So : $p(\text{parameters} \mid \text{data}) \propto \gamma^{1/2} e^{-\gamma \frac{(x-\mu)^2}{2}} \times \gamma^{\alpha_{prior}-1} e^{-\gamma \beta_{prior}}$

$\propto \gamma^{\frac{1}{2} + \alpha_{prior} - 1} e^{-\gamma (\beta_{prior} + \frac{(x-\mu)^2}{2})}$

$$p(\text{parameters} \mid \text{data}) = \Gamma\left(\gamma \mid \underbrace{\alpha_{prior} + \tfrac{1}{2}}_{\alpha_{posterior}}, \underbrace{\beta_{prior} + \frac{(x-\mu)^2}{2}}_{\beta_{posterior}}\right)$$

# Conjugate priors: Exercices

## Beta case

**Exercice** (left as an exercice, correction in the next lecture)

$$P(\theta \mid x) = \frac{Ber(x \mid \theta) \times P(\theta)}{P(x)}$$

$\theta^{n_1} \cdot (1-\theta)^{n_0}$ — $B(\theta \mid \alpha_{prior}, \beta_{prior})$ — $B(\theta \mid \alpha_{posterior}, \beta_{posterior})$ — $B(\theta \mid n_1 + \alpha_{prior}, n_0 + \beta_{prior})$

**Beta distribution**

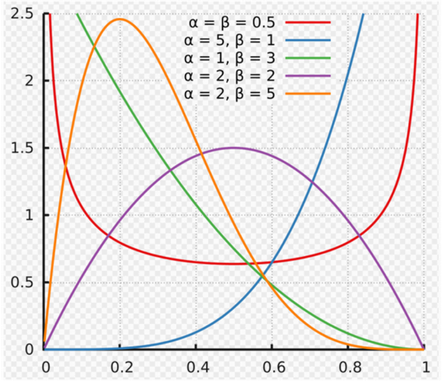
**PDF :** $B(x \mid \alpha, \beta) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}$ with $\alpha, \beta > 0$ and $x \in [0,1]$

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\cdot\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

**mean :** $\mathbb{E}[x] = \frac{\alpha}{\alpha+\beta}$

**variance :** $V(x) = \frac{\alpha\beta}{(\alpha+\beta)^2 \cdot (\alpha+\beta-1)}$

**mode :** $Mode\,[x] = \frac{\alpha-1}{\alpha+\beta-2}$




What we want to compute : $p(\text{parameters} \mid \text{data}) \propto p(\text{data} \mid \text{parameters}) \times p(\text{parameters})$

- $p(\text{data} \mid \text{parameters}) = Ber(x \mid \theta) = \binom{n}{x} \theta^x (1-\theta)^{n-x} \propto \theta^{n_1} (1-\theta)^{n_0}$

- $p(\text{parameters}) = B(\theta \mid \alpha_{prior}, \beta_{prior}) = \frac{\theta^{\alpha_{prior}-1}(1-\theta)^{\beta_{prior}-1}}{B(\alpha_{prior}, \beta_{prior})} \propto \theta^{\alpha_{prior}-1}(1-\theta)^{\beta_{prior}-1}$

So : $p(\text{parameters} \mid \text{data}) \propto \theta^{n_1}(1-\theta)^{n_0} \times \theta^{\alpha_{prior}-1}(1-\theta)^{\beta_{prior}-1}$

$$\propto \theta^{n_1+\alpha_{prior}-1}(1-\theta)^{n_0+\beta_{prior}-1}$$

$$p(\text{parameters} \mid \text{data}) = \boxed{B(\theta \mid \underbrace{n_1+\alpha_{prior}-1}_{\alpha_{posterior}}, \underbrace{n_0+\beta_{prior}-1}_{\beta_{posterior}})}$$

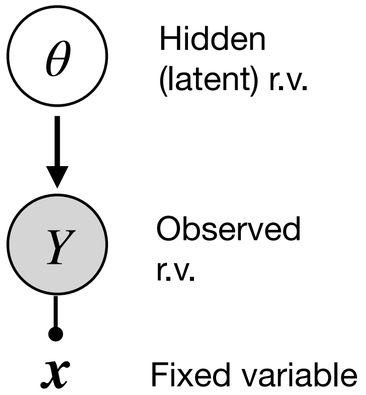
# 1 Latent variable models and mixture models

# 1. Latent Variable Models

## Spoilers

**Latent variable models** : a statistical model that links a set of **observable** variables to a set of **unobservable (latent)** variables

**Example :** Bayesian Linear regression




**Other latent variable models :** unsupervised methods

- **Clustering** models
- **Dimensionality reduction** models

**Questions :**

- **Why do we need latent variable models ?** **simpler models (so fewer parameters) without reducing its flexibility**
- **How to train these models ?** **next section**
- **Any limitations of the proposed training method ?** **last section**

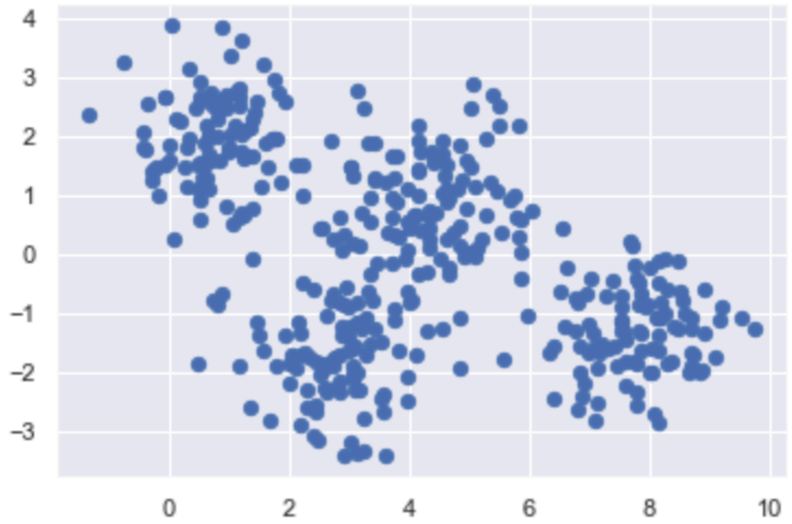
# 1. Latent Variable Models

## Mixture models : Definition

**Mixture models** : a probabilistic model representing a **linear combination** of different distributions

Mixture modeling provides the freedom / flexibility to model the unknown pdf. Downside : more parameters

**Example :** synthetic data



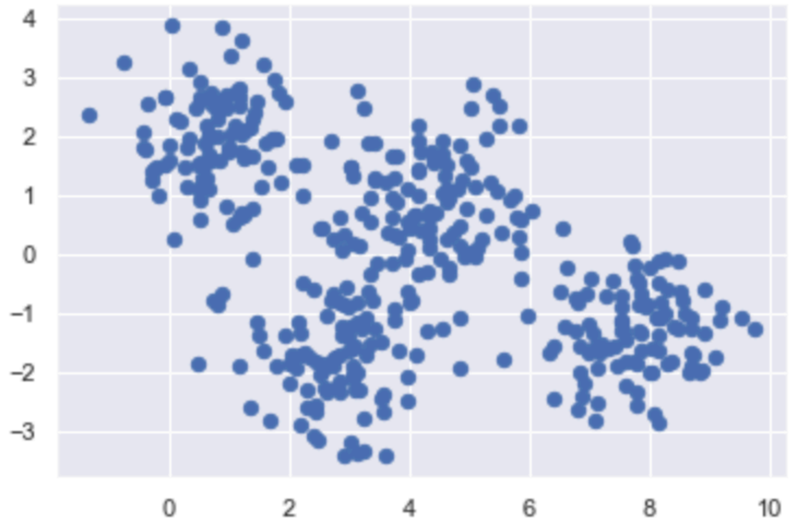
# 1. Latent Variable Models

## Mixture models : Definition

**Mixture models** : a probabilistic model representing a **linear combination** of different distributions

Mixture modeling provides the freedom / flexibility to model the unknown pdf. Downside : more parameters

**Example :** synthetic data




**Let's fit a gaussian !**

$$\mathcal{N}(\mu, \Sigma)$$

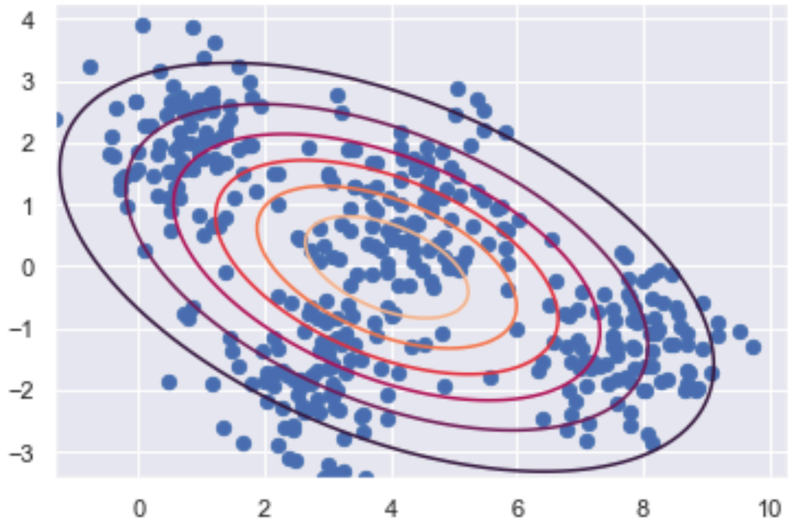
# 1. Latent Variable Models

## Mixture models : Definition

**Mixture models** : a probabilistic model representing a **linear combination** of different distributions

Mixture modeling provides the freedom / flexibility to model the unknown pdf. Downside : more parameters

**Example :** synthetic data




**Let's fit a gaussian !**

$$\mathcal{N}(\mu, \Sigma)$$

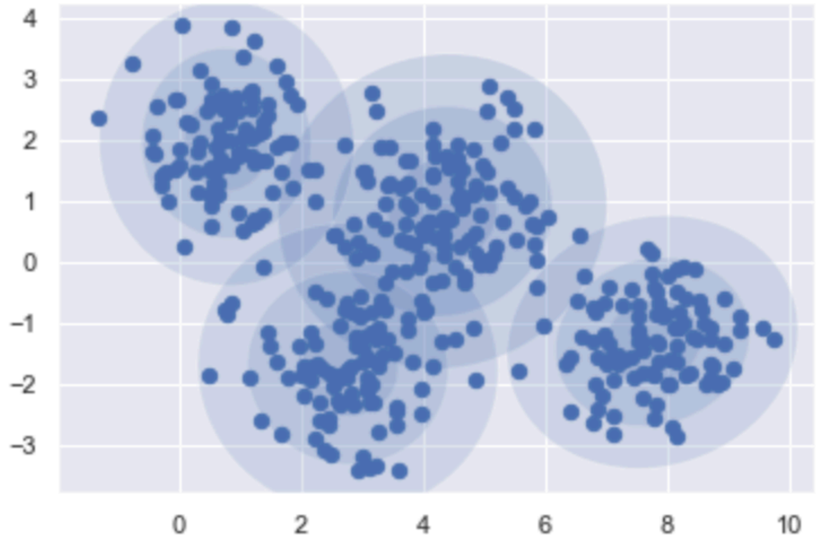
# 1. Latent Variable Models

## Gaussian Mixture Model : Definition

**Mixture models** : a probabilistic model representing a **linear combination** of different distributions

Mixture modeling provides the freedom / flexibility to model the unknown pdf. Downside : more parameters

**Example :** synthetic data




**Let's fit a gaussian !**

$$\mathcal{N}(\mu, \Sigma)$$

**We want to fit a Gaussian Mixture Model (GMM) !**

$$\sum_{k=1}^{4} \pi_k \cdot \mathcal{N}(\mu_k, \Sigma_k)$$

parameters : $\{\pi_k, \mu_k, \Sigma_k\}_{k \in \{1,\dots,4\}} =: \theta$

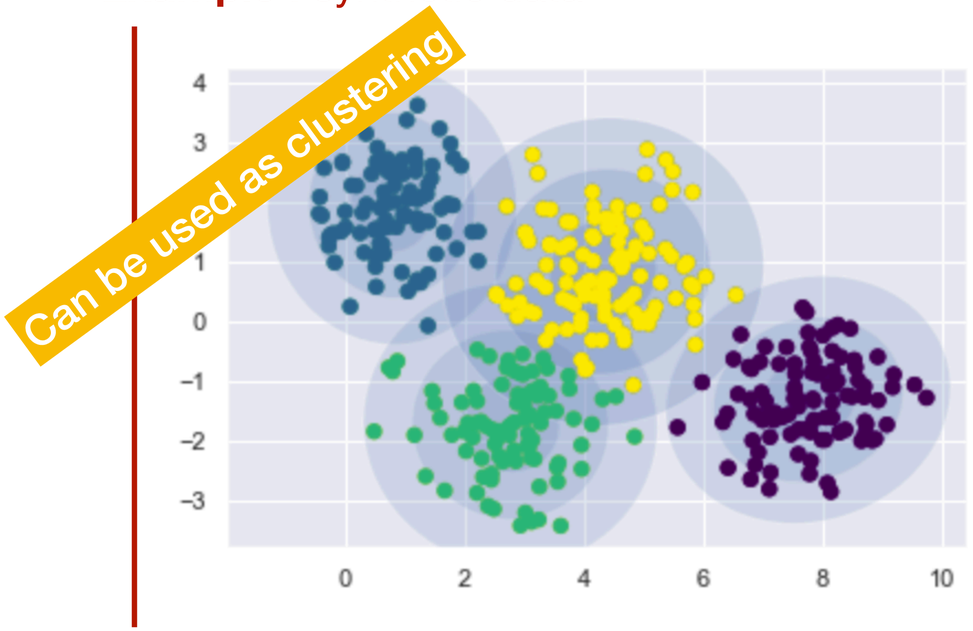
# 1. Latent Variable Models

## Gaussian Mixture Model : Definition

**Mixture models** : a probabilistic model representing a **linear combination** of different distributions

Mixture modeling provides the freedom / flexibility to model the unknown pdf. Downside : more parameters

**Example :** synthetic data




**Let's fit a gaussian !**

$$\mathcal{N}(\mu, \Sigma)$$

**We want to fit a Gaussian Mixture Model (GMM) !**

$$\sum_{k=1}^{4} \pi_k \cdot \mathcal{N}(\mu_k, \Sigma_k)$$

parameters : $\{\pi_k, \mu_k, \Sigma_k\}_{k \in \{1,\dots,4\}} =: \theta$

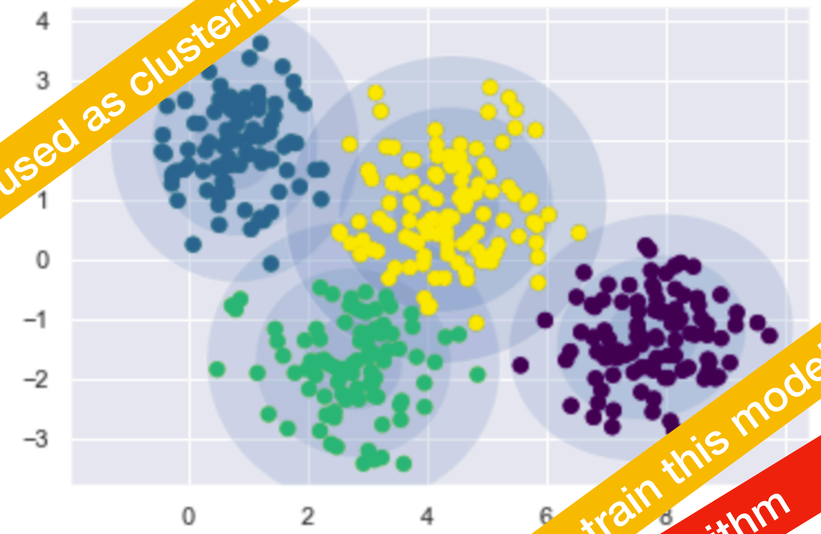
# 1. Latent Variable Models

## Gaussian Mixture Model : Definition

**Mixture models** : a probabilistic model representing a **linear combination** of different distributions

Mixture modeling provides the freedom / flexibility to model the unknown pdf. Downside : more parameters

**Example :** synthetic data




Can be used as clustering

How do we train this model ?

EM algorithm

**Let's fit a gaussian !**

$$\mathcal{N}(\mu, \Sigma)$$

**We want to fit a Gaussian Mixture Model (GMM) !**

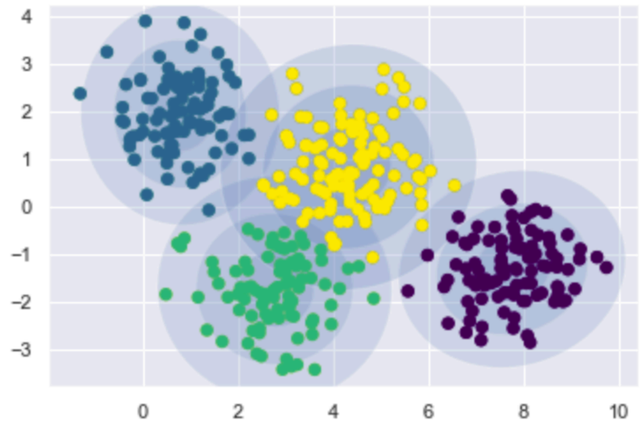
$$\sum_{k=1}^{4} \pi_k \cdot \mathcal{N}(\mu_k, \Sigma_k)$$

parameters : $\{\pi_k, \mu_k, \Sigma_k\}_{k \in \{1,\dots,4\}} =: \theta$

# 2 Probabilistic clustering and EM-algorithm

# 2. Probabilistic clustering

## Gaussian Mixture Model as a Latent variable model



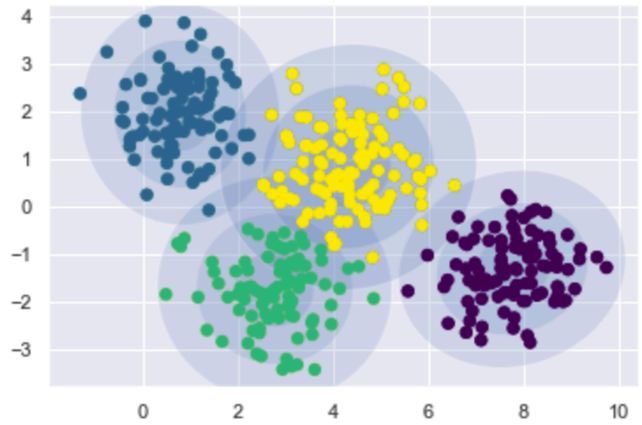

**We want to fit a Gaussian Mixture Model !**

$$\pi_1 \mathcal{N}(\mu_1, \Sigma_1) + \pi_2 \mathcal{N}(\mu_2, \Sigma_2) + \pi_3 \mathcal{N}(\mu_3, \Sigma_3) + \pi_4 \mathcal{N}(\mu_4, \Sigma_4)$$

parameters : $\theta = \{\pi_1, \mu_1, \Sigma_1, \pi_2, \mu_2, \Sigma_2, \pi_3, \mu_3, \Sigma_3, \pi_4, \mu_4, \Sigma_4\}$

# 2. Probabilistic clustering

## Gaussian Mixture Model as a Latent variable model

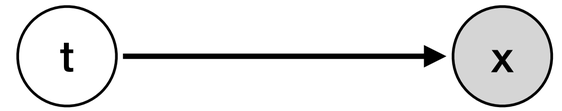



**We want to fit a Gaussian Mixture Model !**

$$\pi_1 \mathcal{N}(\mu_1, \Sigma_1) + \pi_2 \mathcal{N}(\mu_2, \Sigma_2) + \pi_3 \mathcal{N}(\mu_3, \Sigma_3) + \pi_4 \mathcal{N}(\mu_4, \Sigma_4)$$

parameters : $\theta = \{\pi_1, \mu_1, \Sigma_1, \pi_2, \mu_2, \Sigma_2, \pi_3, \mu_3, \Sigma_3, \pi_4, \mu_4, \Sigma_4\}$

**Latent variable model for GMM :**




the source :
from **which** gaussian
{1, 2, 3, 4}
this data came from ?

Gaussian
distribution

$$p(t = k \mid \theta) =$$

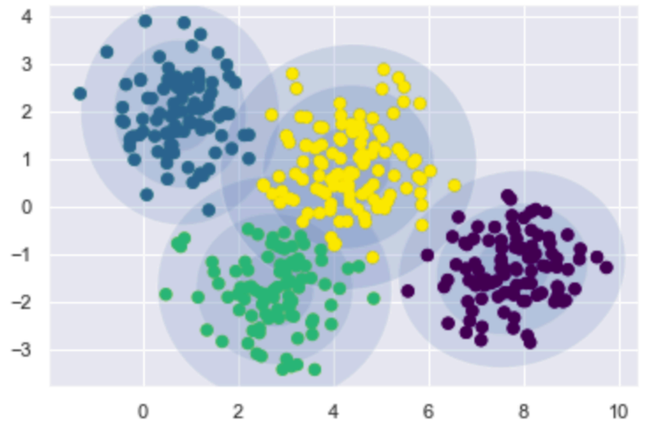
$$p(x \mid t = k, \theta) =$$

$$p(x \mid \theta) =$$

**Reminder :** a PGM models how an observation is generated

# 2. Probabilistic clustering

## Gaussian Mixture Model as a Latent variable model

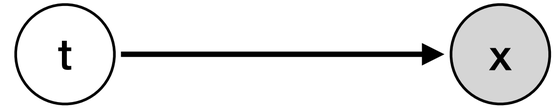



**We want to fit a Gaussian Mixture Model !**

$$\pi_1 \mathcal{N}(\mu_1, \Sigma_1) + \pi_2 \mathcal{N}(\mu_2, \Sigma_2) + \pi_3 \mathcal{N}(\mu_3, \Sigma_3) + \pi_4 \mathcal{N}(\mu_4, \Sigma_4)$$

parameters : $\theta = \{\pi_1, \mu_1, \Sigma_1, \pi_2, \mu_2, \Sigma_2, \pi_3, \mu_3, \Sigma_3, \pi_4, \mu_4, \Sigma_4\}$

**Latent variable model for GMM :**




the source :
from **which** gaussian
{1, 2, 3, 4}
this data came from ?

Gaussian
distribution

$$p(t = k \mid \theta) = \pi_k$$

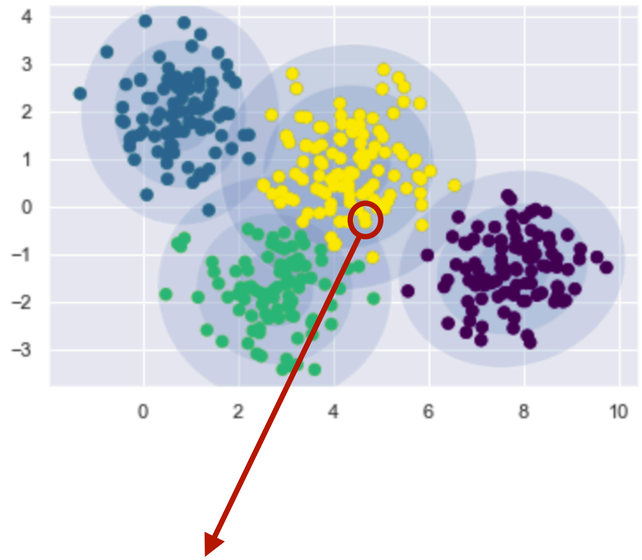
$$p(x \mid t = k, \theta) = \mathcal{N}(x \mid \mu_k, \Sigma_k)$$

$$p(x \mid \theta) = \sum_{k=1}^{4} p(x \mid t = k, \theta) p(t = k \mid \theta)$$

**Reminder :** a PGM models how an observation is generated

# 2. Probabilistic clustering

## Gaussian Mixture Model as a Latent variable model

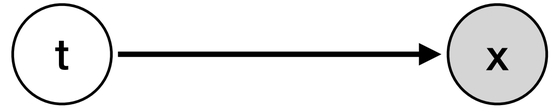



**We want to fit a Gaussian Mixture Model !**

$$\pi_1 \mathcal{N}(\mu_1, \Sigma_1) + \pi_2 \mathcal{N}(\mu_2, \Sigma_2) + \pi_3 \mathcal{N}(\mu_3, \Sigma_3) + \pi_4 \mathcal{N}(\mu_4, \Sigma_4)$$

parameters : $\theta = \{\pi_1, \mu_1, \Sigma_1, \pi_2, \mu_2, \Sigma_2, \pi_3, \mu_3, \Sigma_3, \pi_4, \mu_4, \Sigma_4\}$

**Latent variable model for GMM :**




the source :
from **which** gaussian
{1, 2, 3, 4}
this data came from ?

Gaussian
distribution

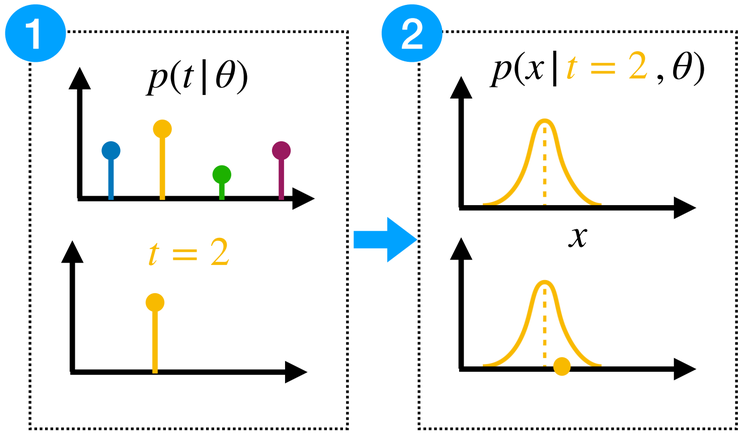
$$p(t = k \mid \theta) = \pi_k$$

$$p(x \mid t = k, \theta) = \mathcal{N}(x \mid \mu_k, \Sigma_k)$$

$$p(x \mid \theta) = \sum_{k=1}^{4} p(x \mid t = k, \theta) p(t = k \mid \theta)$$

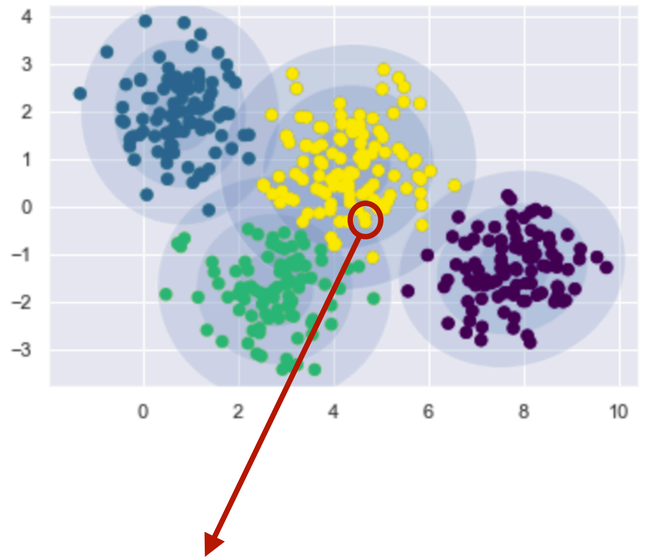
**Reminder :** a PGM models how an observation is generated

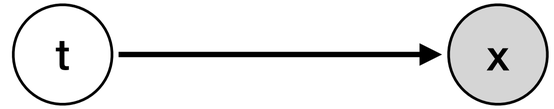
**We assume that this $x$ is generated as follows :**



# 2. Probabilistic clustering

## Gaussian Mixture Model as a Latent variable model




**We want to fit a Gaussian Mixture Model !**

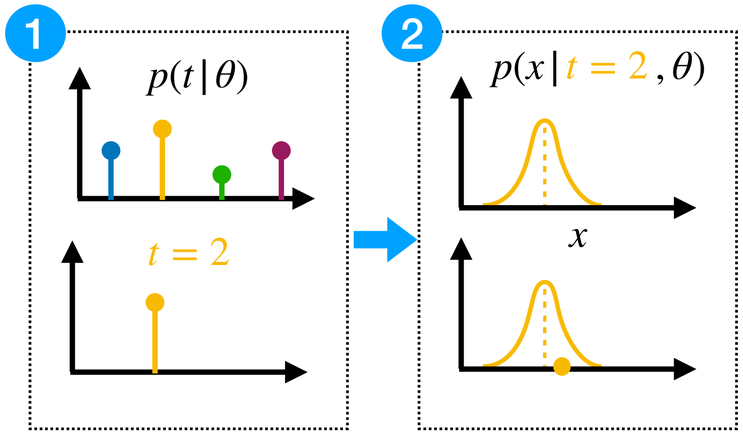
$$\pi_1 \mathcal{N}(\mu_1, \Sigma_1) + \pi_2 \mathcal{N}(\mu_2, \Sigma_2) + \pi_3 \mathcal{N}(\mu_3, \Sigma_3) + \pi_4 \mathcal{N}(\mu_4, \Sigma_4)$$

parameters : $\theta = \{\pi_1, \mu_1, \Sigma_1, \pi_2, \mu_2, \Sigma_2, \pi_3, \mu_3, \Sigma_3, \pi_4, \mu_4, \Sigma_4\}$

**Hard clustering :** if we **know the source** of each instances then,

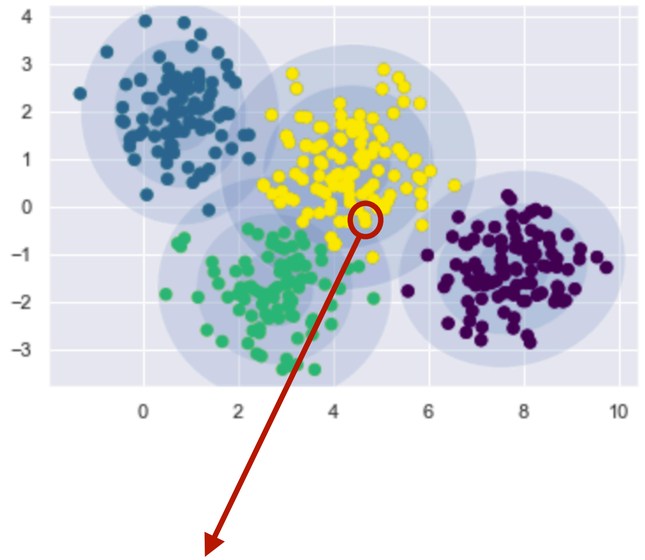
**Soft / probabilistic clustering :** if we **know the source** of each instances then,

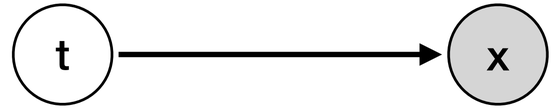
**We assume that this $x$ is generated as follows :**



# 2. Probabilistic clustering

## Gaussian Mixture Model as a Latent variable model




**We want to fit a Gaussian Mixture Model !**

$$\pi_1 \mathcal{N}(\mu_1, \Sigma_1) + \pi_2 \mathcal{N}(\mu_2, \Sigma_2) + \pi_3 \mathcal{N}(\mu_3, \Sigma_3) + \pi_4 \mathcal{N}(\mu_4, \Sigma_4)$$

parameters : $\theta = \{\pi_1, \mu_1, \Sigma_1, \pi_2, \mu_2, \Sigma_2, \pi_3, \mu_3, \Sigma_3, \pi_4, \mu_4, \Sigma_4\}$

**Hard clustering :** if we **know the source** of each instances then,

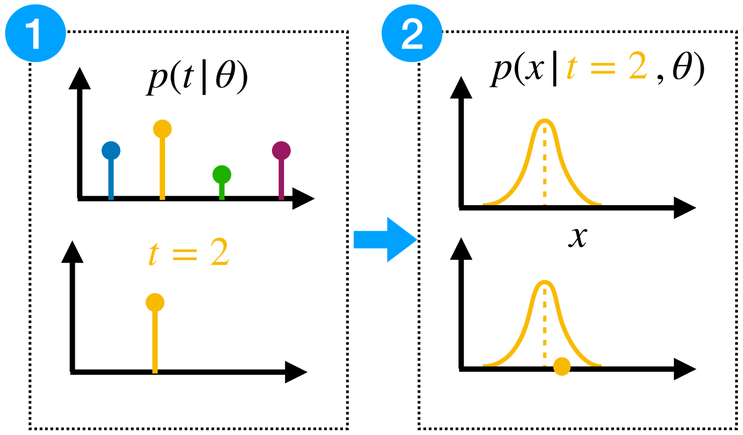
$$p(x \mid t = 2, \theta) = \mathcal{N}(x \mid \mu_{hard}^{MLE}, \Sigma_{hard}^{MLE})$$

$$\mu_{hard}^{MLE} = \frac{\sum_{i \in \text{cluster 2}} x_i}{\text{Number of points in cluster 2}}$$

$$\Sigma_{hard}^{MLE} = \frac{\sum_{i \in \text{cluster 2}} (x_i - \mu_{hard}^{MLE}) \times (x_i - \mu_{hard}^{MLE})^T}{\text{Number of points in cluster 2}}$$

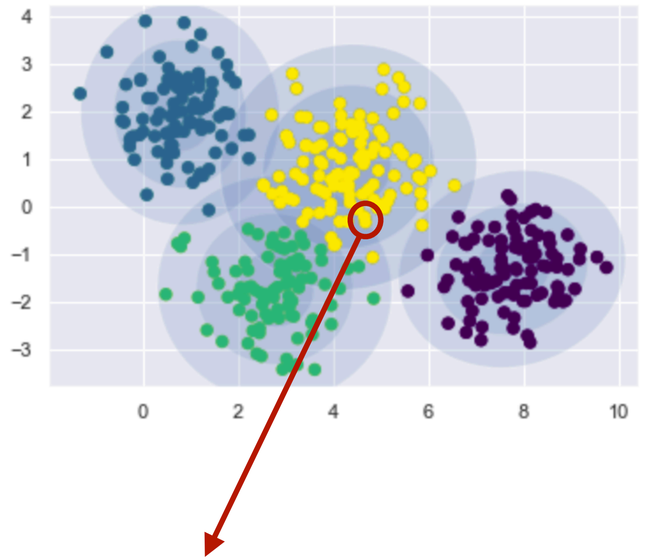
**Soft / probabilistic clustering :** if we **know the source** of each instances then,

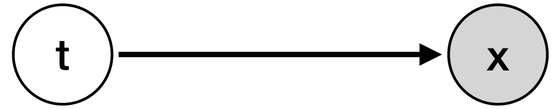
**We assume that this $x$ is generated as follows :**



# 2. Probabilistic clustering

## Gaussian Mixture Model as a Latent variable model



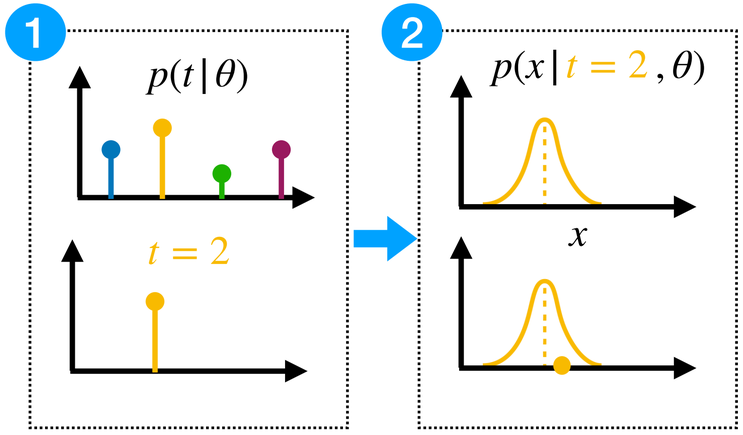




**We want to fit a Gaussian Mixture Model !**

$$\pi_1 \mathcal{N}(\mu_1, \Sigma_1) + \pi_2 \mathcal{N}(\mu_2, \Sigma_2) + \pi_3 \mathcal{N}(\mu_3, \Sigma_3) + \pi_4 \mathcal{N}(\mu_4, \Sigma_4)$$

parameters : $\theta = \{\pi_1, \mu_1, \Sigma_1, \pi_2, \mu_2, \Sigma_2, \pi_3, \mu_3, \Sigma_3, \pi_4, \mu_4, \Sigma_4\}$

**We assume that this $x$ is generated as follows :**




**Hard clustering :** if we **know the source** of each instances then,

$$p(x \mid t = 2, \theta) = \mathcal{N}(x \mid \mu_{hard}^{MLE}, \Sigma_{hard}^{MLE})$$

$$\mu_{hard}^{MLE} = \frac{\sum_{i \in \text{cluster 2}} x_i}{\text{Number of points in cluster 2}}$$

$$\Sigma_{hard}^{MLE} = \frac{\sum_{i \in \text{cluster 2}} (x_i - \mu_{hard}^{MLE}) \times (x_i - \mu_{hard}^{MLE})^T}{\text{Number of points in cluster 2}}$$

**Soft / probabilistic clustering :** if we **know the source** of each instances then,

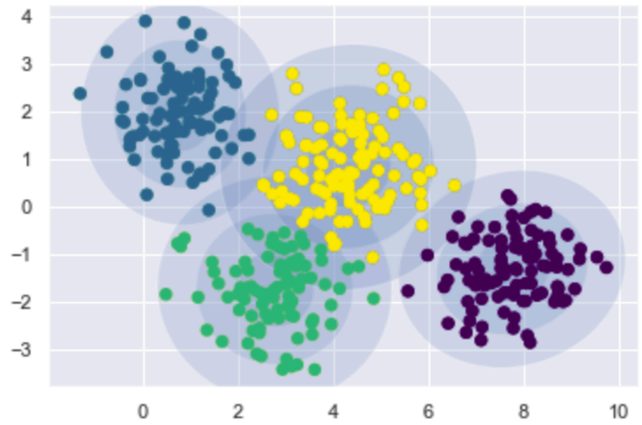
$$p(x \mid t = 2, \theta) = \mathcal{N}(x \mid \mu_{soft}^{MLE}, \Sigma_{soft}^{MLE})$$

$$\mu_{soft}^{MLE} = \frac{\sum_i p(t = 2 \mid x, \theta)\, x_i}{\sum_i p(t = 2 \mid x, \theta)}$$

$$\Sigma_{soft}^{MLE} = \frac{\sum_i p(t = 2 \mid x_i, \theta)\, (x_i - \mu_{soft}^{MLE}) \times (x_i - \mu_{soft}^{MLE})^T}{\sum_i p(t = 2 \mid x_i, \theta)}$$

# 2. Probabilistic clustering

## Gaussian Mixture Model : some intuitions for training this model [0/6]




**Soft / probabilistic clustering** : if we **know the source** of each instances then,

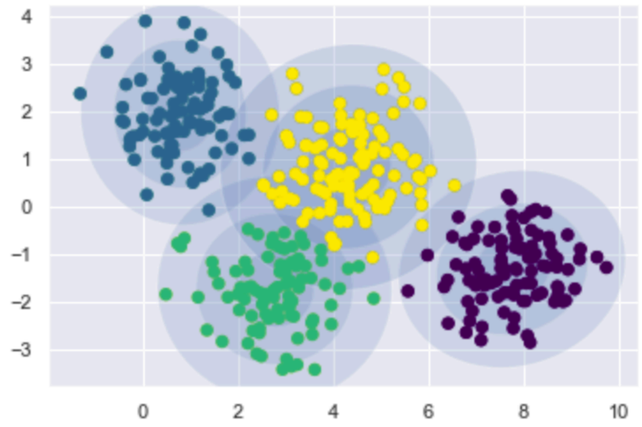
$$p(x \mid t = 2, \theta) = \mathcal{N}(x \mid \mu_{soft}^{MLE}, \Sigma_{soft}^{MLE})$$

$$\mu_{soft}^{MLE} = \frac{\sum_i p(t = 2 \mid x, \theta)\, x_i}{\sum_i p(t = 2 \mid x, \theta)}$$

$$\Sigma_{soft}^{MLE} = \frac{\sum_i p(t = 2 \mid x_i, \theta)\, (x_i - \mu_{soft}^{MLE}) \times (x_i - \mu_{soft}^{MLE})^T}{\sum_i p(t = 2 \mid x_i, \theta)}$$

# 2. Probabilistic clustering

## Gaussian Mixture Model : some intuitions for training this model [0/6]




**Soft / probabilistic clustering :** if we **know the source** of each instances then,

$$p(x \mid t=2, \theta) = \mathcal{N}(x \mid \mu_{soft}^{MLE}, \Sigma_{soft}^{MLE})$$

$$\mu_{soft}^{MLE} = \frac{\sum_i p(t=2 \mid x, \theta)\, x_i}{\sum_i p(t=2 \mid x, \theta)}$$

$$\Sigma_{soft}^{MLE} = \frac{\sum_i p(t=2 \mid x_i, \theta)\,(x_i - \mu_{soft}^{MLE}) \times (x_i - \mu_{soft}^{MLE})^T}{\sum_i p(t=2 \mid x_i, \theta)}$$

**Remarks**: If we **know the parameters** of each instances then,

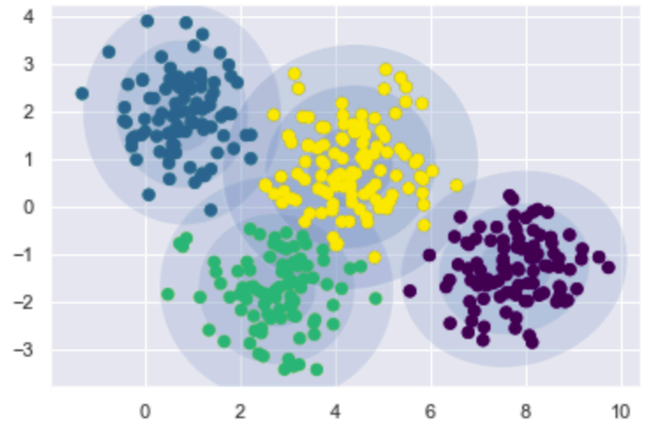
$$p(t=2 \mid x, \theta) = \frac{p(x \mid t=2, \theta) \times p(t=2 \mid \theta)}{\text{Const}}$$

We are now in the following situation :

- **ESTIMATION:**
  If we **knew the parameters**, we could compute the posteriors
- **MAXIMIZATION:**
  If we **knew the posteriors/ sources**, we could easily compute the parameters

# 2. Probabilistic clustering

## Gaussian Mixture Model : some intuitions for training this model [0/6]




**Soft / probabilistic clustering :** if we **know the source** of each instances then,

$$p(x \mid t=2, \theta) = \mathcal{N}(x \mid \mu_{soft}^{MLE}, \Sigma_{soft}^{MLE})$$

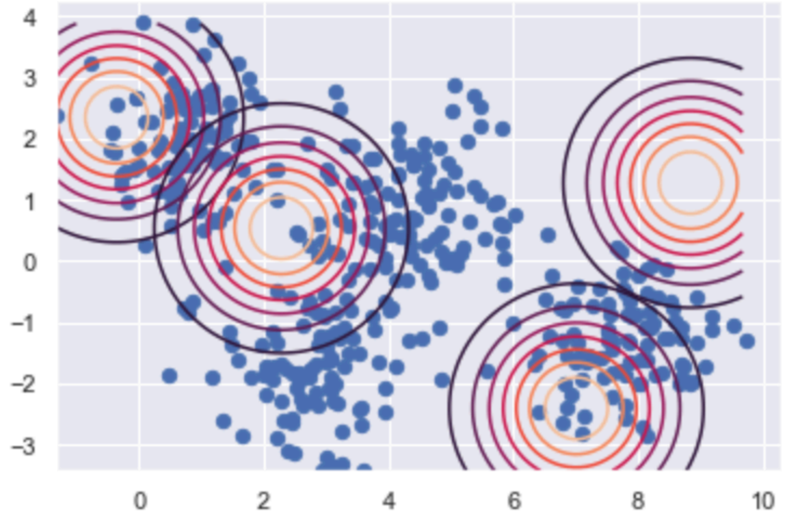
$$\mu_{soft}^{MLE} = \frac{\sum_i p(t=2 \mid x, \theta)\, x_i}{\sum_i p(t=2 \mid x, \theta)}$$

$$\Sigma_{soft}^{MLE} = \frac{\sum_i p(t=2 \mid x_i, \theta)\,(x_i - \mu_{soft}^{MLE}) \times (x_i - \mu_{soft}^{MLE})^T}{\sum_i p(t=2 \mid x_i, \theta)}$$

**Remarks**: If we **know the parameters** of each instances then,

$$p(t=2 \mid x, \theta) = \frac{p(x \mid t=2, \theta) \times p(t=2 \mid \theta)}{\text{Const}}$$

**INITIALISATION : first estimation**

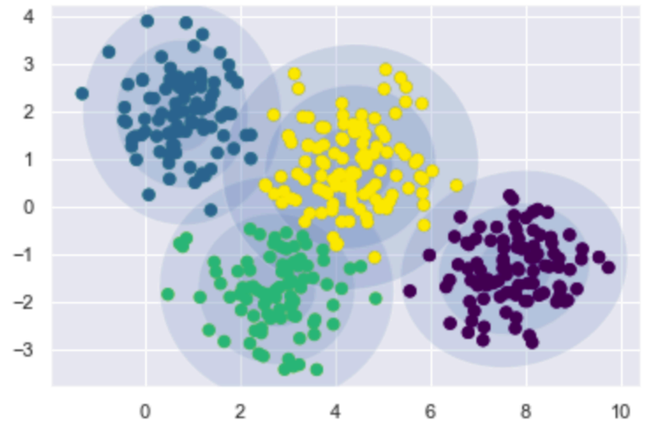



We are now in the following situation :

- **ESTIMATION:**
  If we **knew the parameters**, we could compute the posteriors
- **MAXIMIZATION:**
  If we **knew the posteriors/ sources**, we could easily compute the parameters

# 2. Probabilistic clustering

## Gaussian Mixture Model : some intuitions for training this model [1/6]



**Soft / probabilistic clustering :** if we **know the source** of each instances then,

$$p(x \mid t=2, \theta) = \mathcal{N}(x \mid \mu_{soft}^{MLE}, \Sigma_{soft}^{MLE})$$

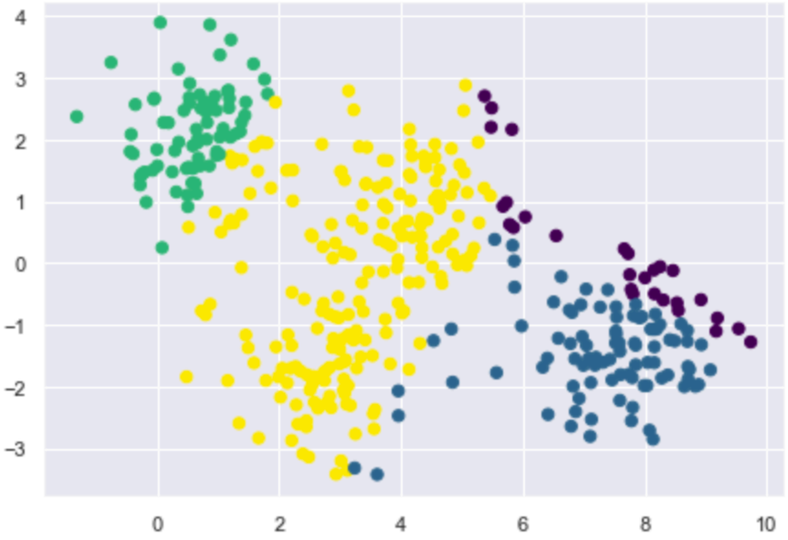
$$\mu_{soft}^{MLE} = \frac{\sum_i p(t=2 \mid x, \theta)\, x_i}{\sum_i p(t=2 \mid x, \theta)}$$

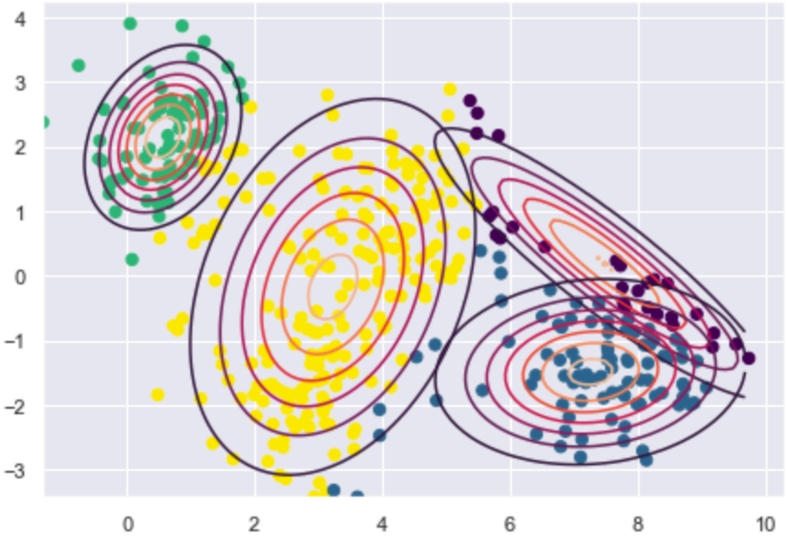
$$\Sigma_{soft}^{MLE} = \frac{\sum_i p(t=2 \mid x_i, \theta)\,(x_i - \mu_{soft}^{MLE}) \times (x_i - \mu_{soft}^{MLE})^T}{\sum_i p(t=2 \mid x_i, \theta)}$$

**Remarks**: If we **know the parameters** of each instances then,

$$p(t=2 \mid x, \theta) = \frac{p(x \mid t=2, \theta) \times p(t=2 \mid \theta)}{\text{Const}}$$

**STEP 1**



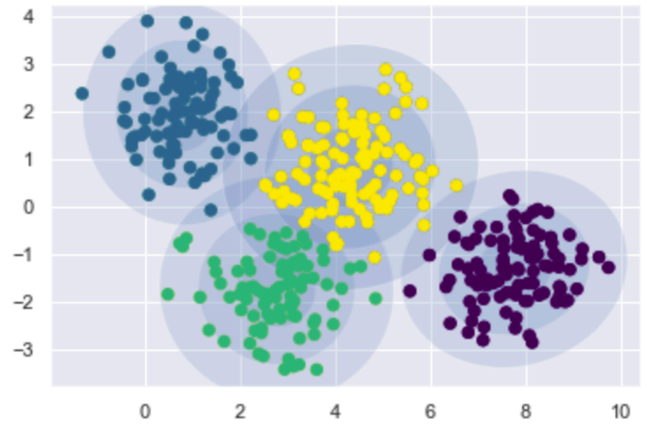


We are now in the following situation :

- **ESTIMATION:**
  If we **knew the parameters**, we could compute the posteriors
- **MAXIMIZATION:**
  If we **knew the posteriors/ sources**, we could easily compute the parameters

# 2. Probabilistic clustering

## Gaussian Mixture Model : some intuitions for training this model [2/6]



**Soft / probabilistic clustering :** if we **know the source** of each instances then,

$$p(x \mid t=2, \theta) = \mathcal{N}(x \mid \mu_{soft}^{MLE}, \Sigma_{soft}^{MLE})$$

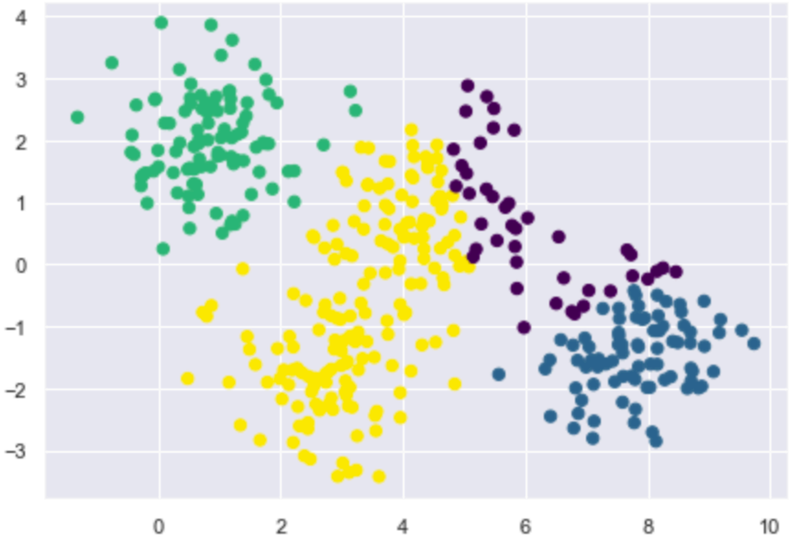
$$\mu_{soft}^{MLE} = \frac{\sum_i p(t=2 \mid x, \theta)\, x_i}{\sum_i p(t=2 \mid x, \theta)}$$

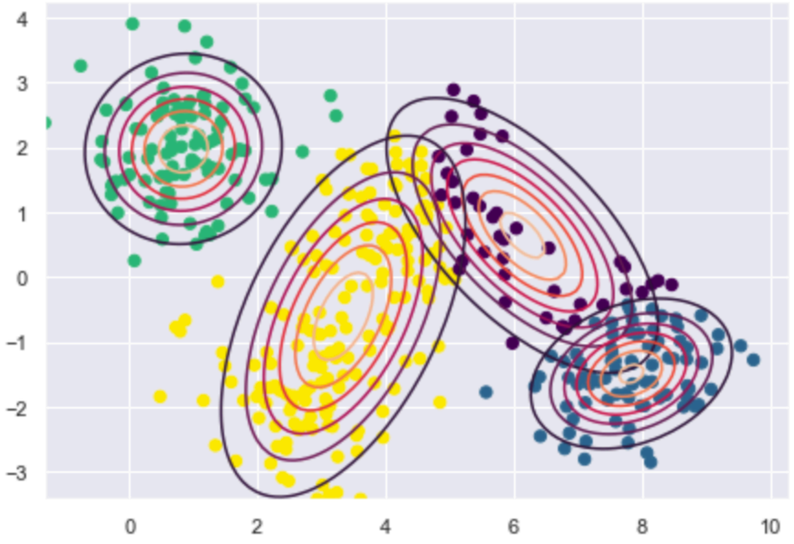
$$\Sigma_{soft}^{MLE} = \frac{\sum_i p(t=2 \mid x_i, \theta)\,(x_i - \mu_{soft}^{MLE}) \times (x_i - \mu_{soft}^{MLE})^T}{\sum_i p(t=2 \mid x_i, \theta)}$$

**Remarks**: If we **know the parameters** of each instances then,

$$p(t=2 \mid x, \theta) = \frac{p(x \mid t=2, \theta) \times p(t=2 \mid \theta)}{\text{Const}}$$

**STEP 2**



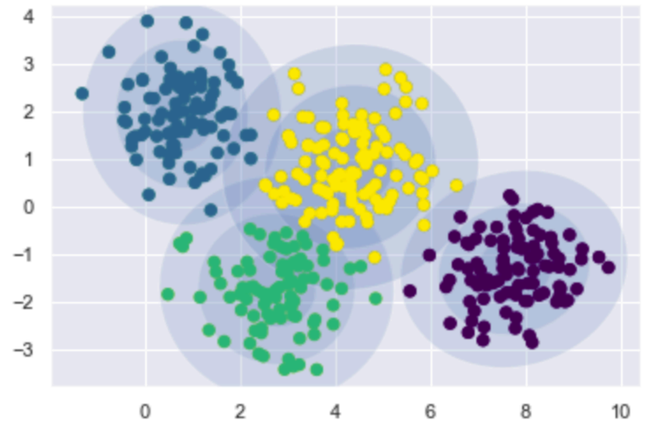


We are now in the following situation :

- **ESTIMATION:**
  If we **knew the parameters**, we could compute the posteriors
- **MAXIMIZATION:**
  If we **knew the posteriors/ sources**, we could easily compute the parameters

# 2. Probabilistic clustering

## Gaussian Mixture Model : some intuitions for training this model [3/6]



**Soft / probabilistic clustering :** if we **know the source** of each instances then,

$$p(x \mid t=2, \theta) = \mathcal{N}(x \mid \mu_{soft}^{MLE}, \Sigma_{soft}^{MLE})$$

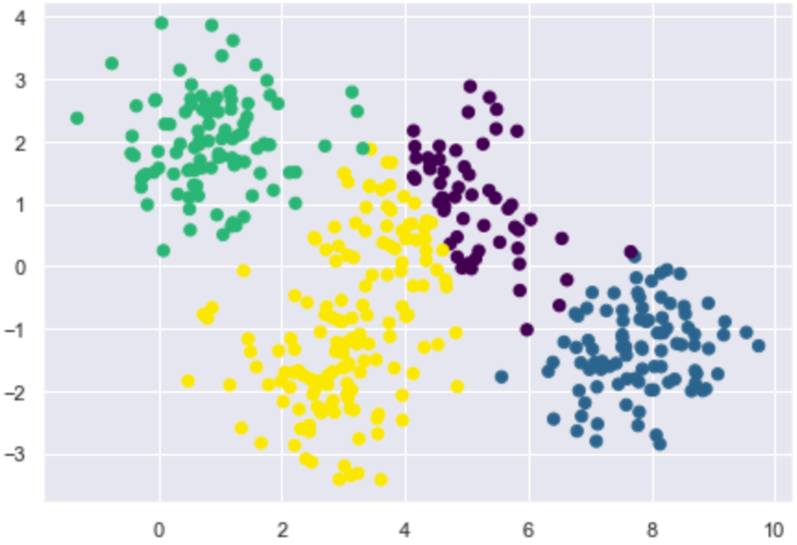
$$\mu_{soft}^{MLE} = \frac{\sum_i p(t=2 \mid x, \theta)\, x_i}{\sum_i p(t=2 \mid x, \theta)}$$

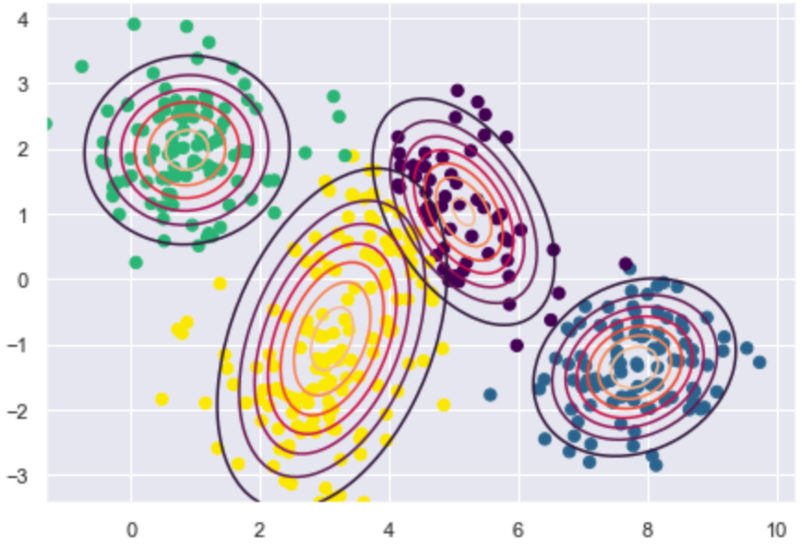
$$\Sigma_{soft}^{MLE} = \frac{\sum_i p(t=2 \mid x_i, \theta)\,(x_i - \mu_{soft}^{MLE}) \times (x_i - \mu_{soft}^{MLE})^T}{\sum_i p(t=2 \mid x_i, \theta)}$$

**Remarks**: If we **know the parameters** of each instances then,

$$p(t=2 \mid x, \theta) = \frac{p(x \mid t=2, \theta) \times p(t=2 \mid \theta)}{\text{Const}}$$

**STEP 3**



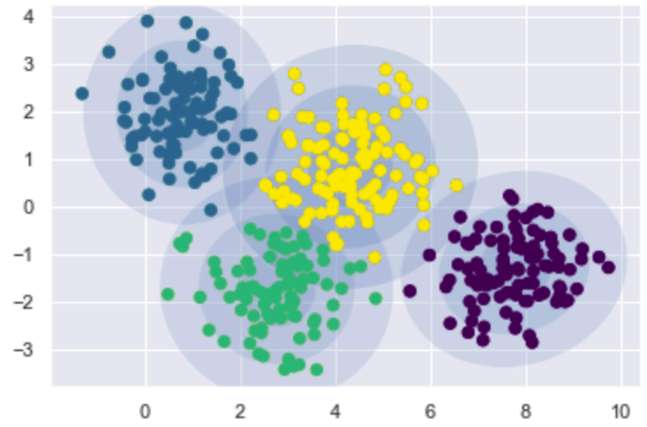


We are now in the following situation :

- **ESTIMATION:**
  If we **knew the parameters**, we could compute the posteriors
- **MAXIMIZATION:**
  If we **knew the posteriors/ sources**, we could easily compute the parameters

# 2. Probabilistic clustering

## Gaussian Mixture Model : some intuitions for training this model [4/6]



**Soft / probabilistic clustering :** if we **know the source** of each instances then,

$$p(x \mid t=2, \theta) = \mathcal{N}(x \mid \mu_{soft}^{MLE}, \Sigma_{soft}^{MLE})$$

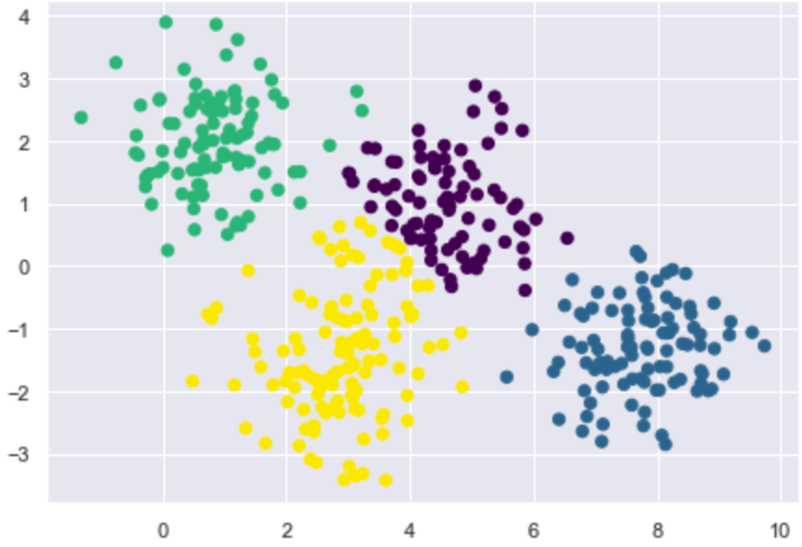
$$\mu_{soft}^{MLE} = \frac{\sum_i p(t=2 \mid x, \theta)\, x_i}{\sum_i p(t=2 \mid x, \theta)}$$

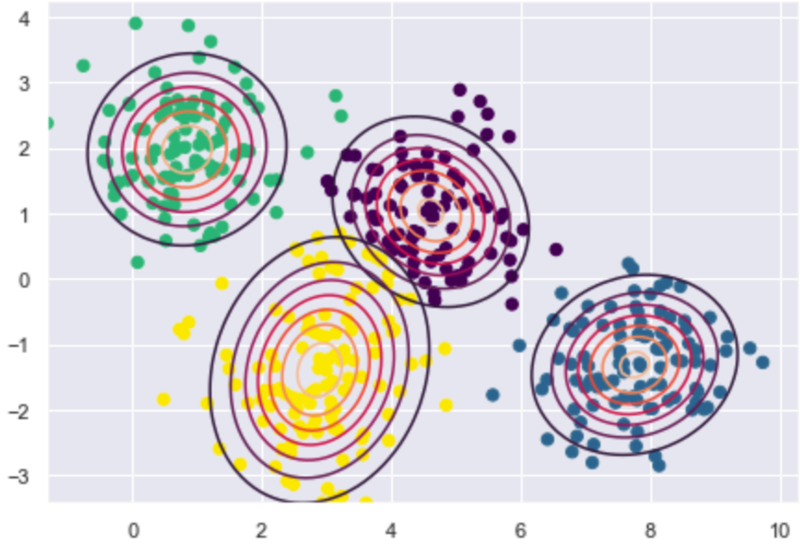
$$\Sigma_{soft}^{MLE} = \frac{\sum_i p(t=2 \mid x_i, \theta)\,(x_i - \mu_{soft}^{MLE}) \times (x_i - \mu_{soft}^{MLE})^T}{\sum_i p(t=2 \mid x_i, \theta)}$$

**Remarks**: If we **know the parameters** of each instances then,

$$p(t=2 \mid x, \theta) = \frac{p(x \mid t=2, \theta) \times p(t=2 \mid \theta)}{\text{Const}}$$

**STEP 4**



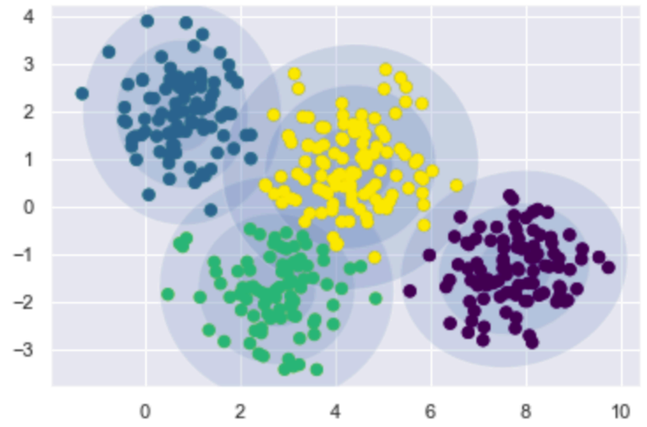


We are now in the following situation :

- **ESTIMATION:**
  If we **knew the parameters**, we could compute the posteriors
- **MAXIMIZATION:**
  If we **knew the posteriors/ sources**, we could easily compute the parameters

# 2. Probabilistic clustering

## Gaussian Mixture Model : some intuitions for training this model [5/6]



**Soft / probabilistic clustering :** if we **know the source** of each instances then,

$$p(x \mid t=2, \theta) = \mathcal{N}(x \mid \mu_{soft}^{MLE}, \Sigma_{soft}^{MLE})$$

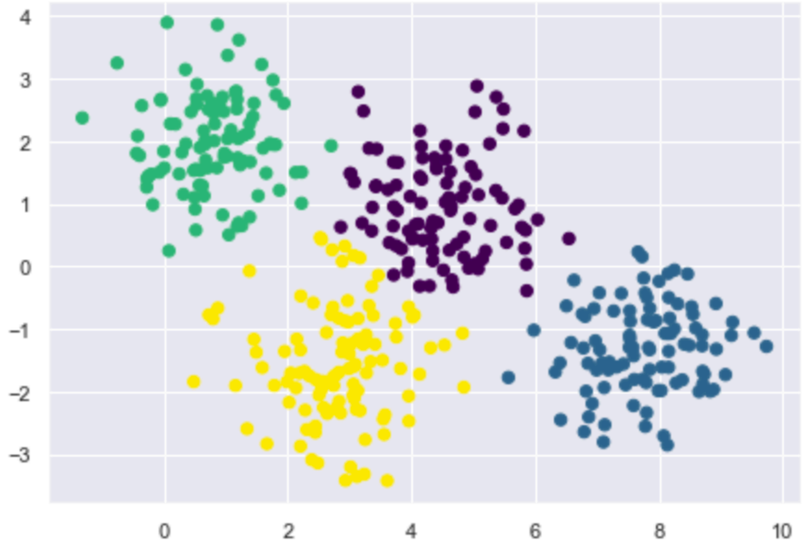
$$\mu_{soft}^{MLE} = \frac{\sum_i p(t=2 \mid x, \theta)\, x_i}{\sum_i p(t=2 \mid x, \theta)}$$

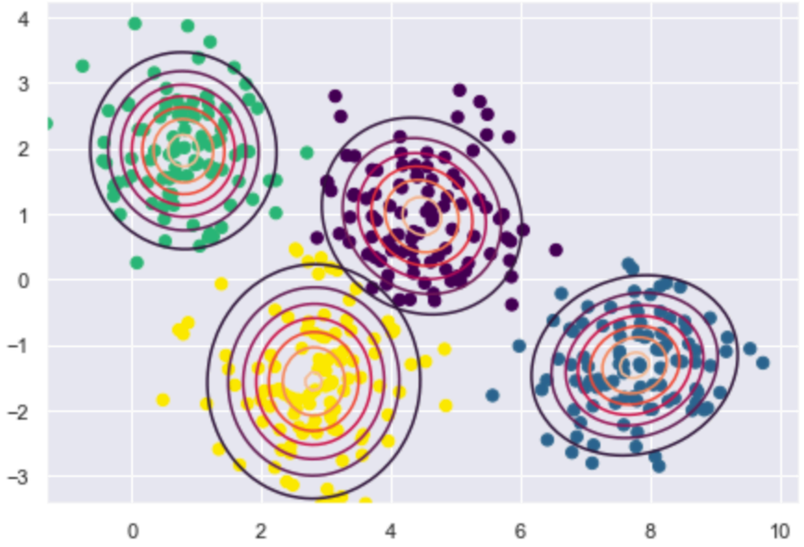
$$\Sigma_{soft}^{MLE} = \frac{\sum_i p(t=2 \mid x_i, \theta)\,(x_i - \mu_{soft}^{MLE}) \times (x_i - \mu_{soft}^{MLE})^T}{\sum_i p(t=2 \mid x_i, \theta)}$$

**Remarks**: If we **know the parameters** of each instances then,

$$p(t=2 \mid x, \theta) = \frac{p(x \mid t=2, \theta) \times p(t=2 \mid \theta)}{\text{Const}}$$

**STEP 5**



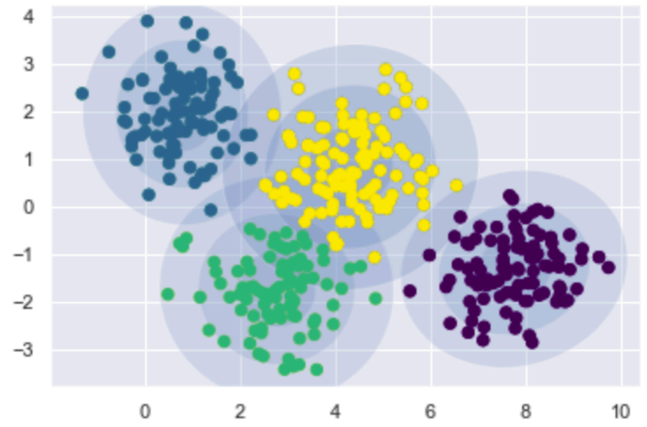


We are now in the following situation :

- **ESTIMATION:**
  If we **knew the parameters**, we could compute the posteriors
- **MAXIMIZATION:**
  If we **knew the posteriors/ sources**, we could easily compute the parameters

# 2. Probabilistic clustering

## Gaussian Mixture Model : some intuitions for training this model [6/6]



**Soft / probabilistic clustering :** if we **know the source** of each instances then,

$$p(x \mid t=2, \theta) = \mathcal{N}(x \mid \mu_{soft}^{MLE}, \Sigma_{soft}^{MLE})$$

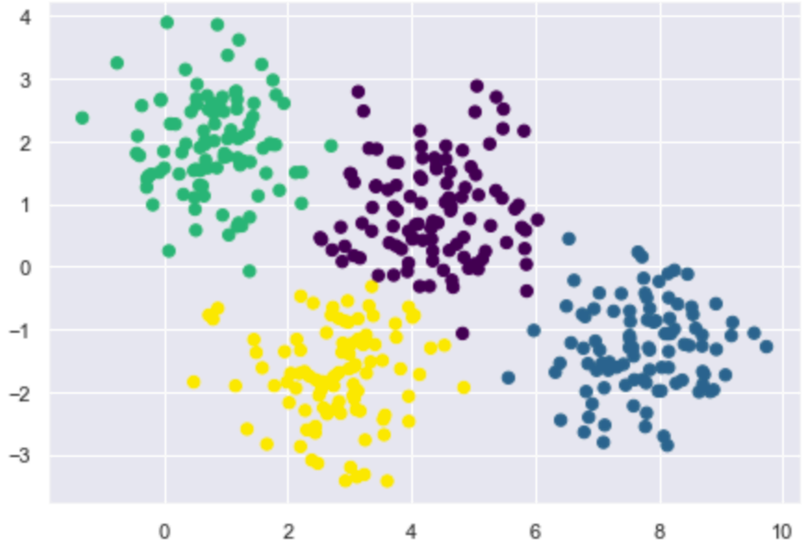
$$\mu_{soft}^{MLE} = \frac{\sum_i p(t=2 \mid x, \theta)\, x_i}{\sum_i p(t=2 \mid x, \theta)}$$

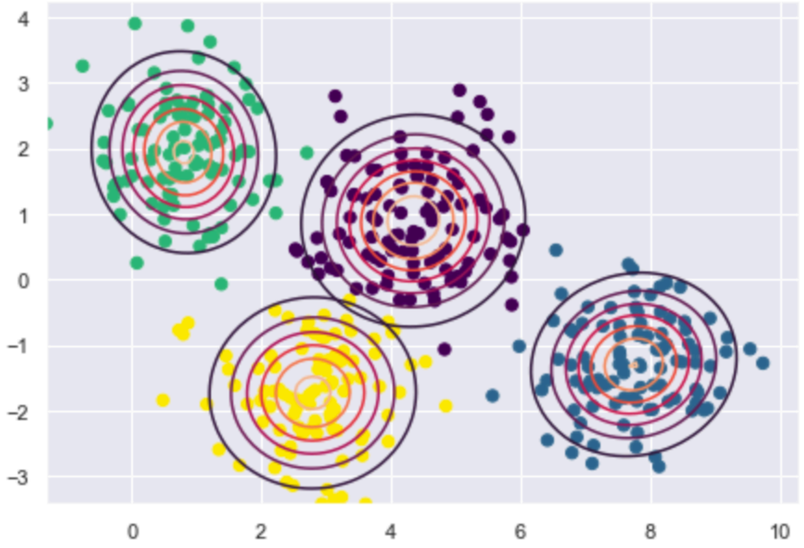
$$\Sigma_{soft}^{MLE} = \frac{\sum_i p(t=2 \mid x_i, \theta)\,(x_i - \mu_{soft}^{MLE}) \times (x_i - \mu_{soft}^{MLE})^T}{\sum_i p(t=2 \mid x_i, \theta)}$$

**Remarks**: If we **know the parameters** of each instances then,

$$p(t=2 \mid x, \theta) = \frac{p(x \mid t=2, \theta) \times p(t=2 \mid \theta)}{\text{Const}}$$

**STEP 6**



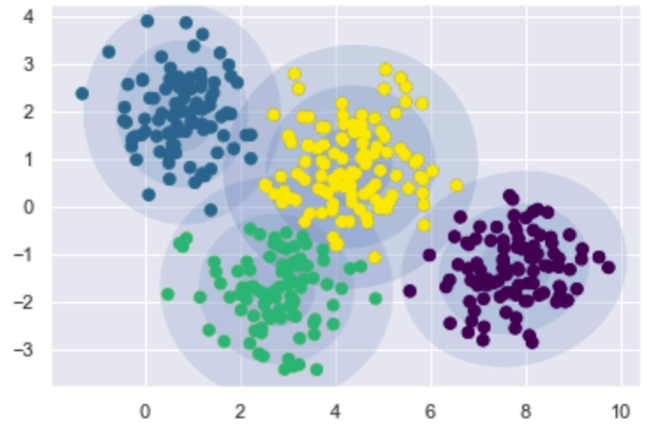


We are now in the following situation :

- **ESTIMATION:**
  If we **knew the parameters**, we could compute the posteriors
- **MAXIMIZATION:**
  If we **knew the posteriors/ sources**, we could easily compute the parameters

# 2. Probabilistic clustering

## Gaussian Mixture Model : some intuitions for training this model [6/6]



**Soft / probabilistic clustering :** if we **know the source** of each instances then,

$$p(x \mid t=2, \theta) = \mathcal{N}(x \mid \mu_{soft}^{MLE}, \Sigma_{soft}^{MLE})$$

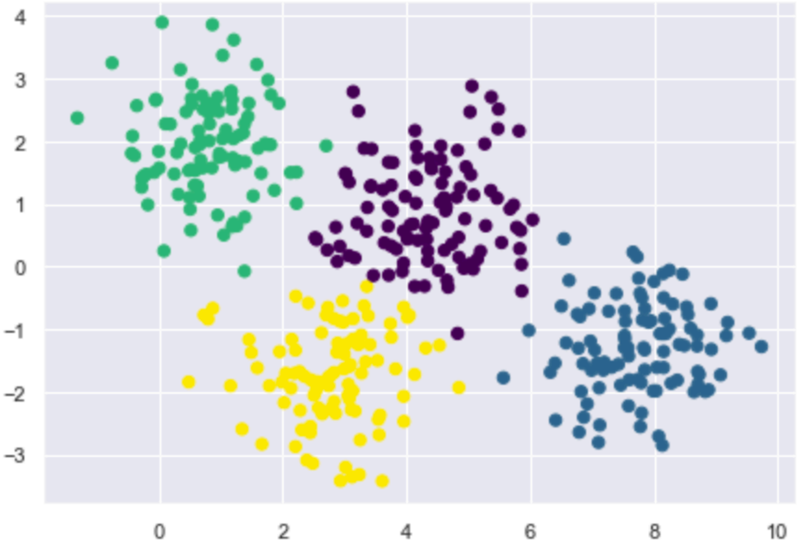
$$\mu_{soft}^{MLE} = \frac{\sum_i p(t=2 \mid x, \theta)\, x_i}{\sum_i p(t=2 \mid x, \theta)}$$

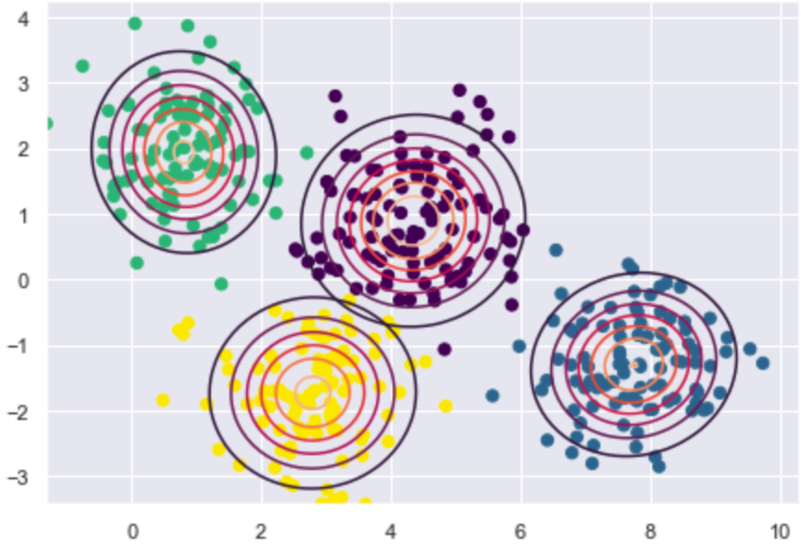
$$\Sigma_{soft}^{MLE} = \frac{\sum_i p(t=2 \mid x_i, \theta)\,(x_i - \mu_{soft}^{MLE}) \times (x_i - \mu_{soft}^{MLE})^T}{\sum_i p(t=2 \mid x_i, \theta)}$$

**Remarks**: If we **know the parameters** of each instances then,

$$p(t=2 \mid x, \theta) = \frac{p(x \mid t=2, \theta) \times p(t=2 \mid \theta)}{\text{Const}}$$

**STEP 6**





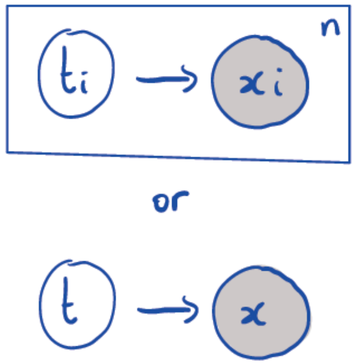
**flexible probabilistic approach to clustering problem**

# 2.b EM-algorithm

# 2.b. Expectation-Maximization algorithm

## Reminder : Maximum Likelihood Estimation (MLE)

Our aim is to find : $\hat{\theta}^{MLE} = \arg\max_{\theta} p(\mathbf{x}\,|\,\theta) = \arg\max_{\theta} \log p(\mathbf{x}\,|\,\theta)$




$$p(x_i|\theta) = \sum_{k=1}^{4} p(x_i, t_i = k|\theta)$$

$$\hat{\theta} = \arg\max_{\theta} \{ \log P(X|\theta) \}$$

$$\log P(X|\theta) = \log \prod_{i=1}^{n} p(x_i|\theta) = \sum_{i=1}^{n} \log p(x_i|\theta)$$

$$= \sum_{i=1}^{n} \log \sum_{k=1}^{4} p(x_i, t_i = k|\theta)$$

$$= \sum_{i=1}^{n} \log \sum_{k=1}^{4} \frac{q(t_i=k)}{q(t_i=k)} p(x_i, t_i = k|\theta) \quad \text{for any distribution } q$$

$$\geq \sum_{i=1}^{n} \sum_{k=1}^{4} q(t_i=k) \log \frac{p(x_i, t_i=k|\theta)}{q(t_i=k)} \quad \text{for any } q$$

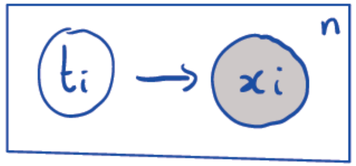
Jensen

$$= \mathcal{L}(\theta, q) \quad \text{for any } \theta \text{ and } q$$

# 2.b. Expectation-Maximization algorithm

## variational lower bound

Our aim is to find : $\hat{\theta}^{MLE} = \arg\max_{\theta} p(\mathbf{x}\,|\,\theta) = \arg\max_{\theta} \log p(\mathbf{x}\,|\,\theta)$




or


$p(x_i|\theta) = \sum_{k=1}^{4} p(x_i, t_i = k|\theta)$

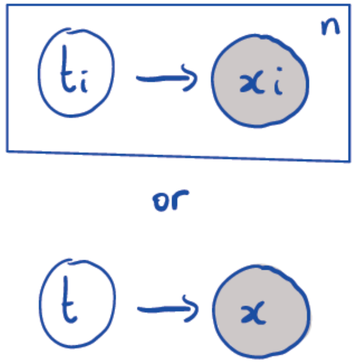
$\log P(X|\theta) \geq \mathcal{L}(\theta, q)$ (Jensen)

$\hat{\theta} = \arg\max_{\theta} \{ \log P(X|\theta) \}$

# 2.b. Expectation-Maximization algorithm

## variational lower bound

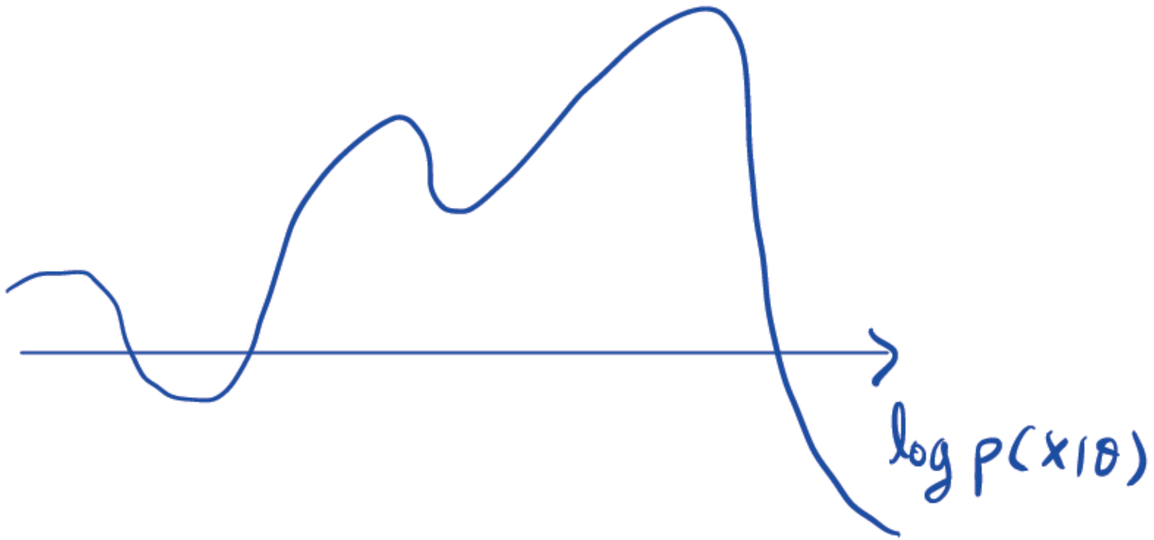
Our aim is to find : $\hat{\theta}^{MLE} = \arg\max_{\theta} p(\mathbf{x}\,|\,\theta) = \arg\max_{\theta} \log p(\mathbf{x}\,|\,\theta)$




$p(x_i|\theta) = \sum_{k=1}^{4} p(x_i, t_i = k|\theta)$

$\log P(X|\theta) \geq \mathcal{L}(\theta, q)$ (Jensen)

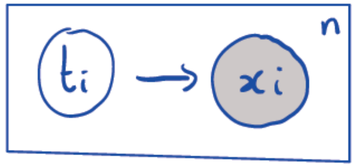
$\hat{\theta} = \arg\max_{\theta} \{ \log P(X|\theta) \}$



# 2.b. Expectation-Maximization algorithm

## EM algorithm : E-step

Our aim is to find : $\hat{\theta}^{MLE} = \arg\max_{\theta} p(\mathbf{x}\,|\,\theta) = \arg\max_{\theta} \log p(\mathbf{x}\,|\,\theta)$

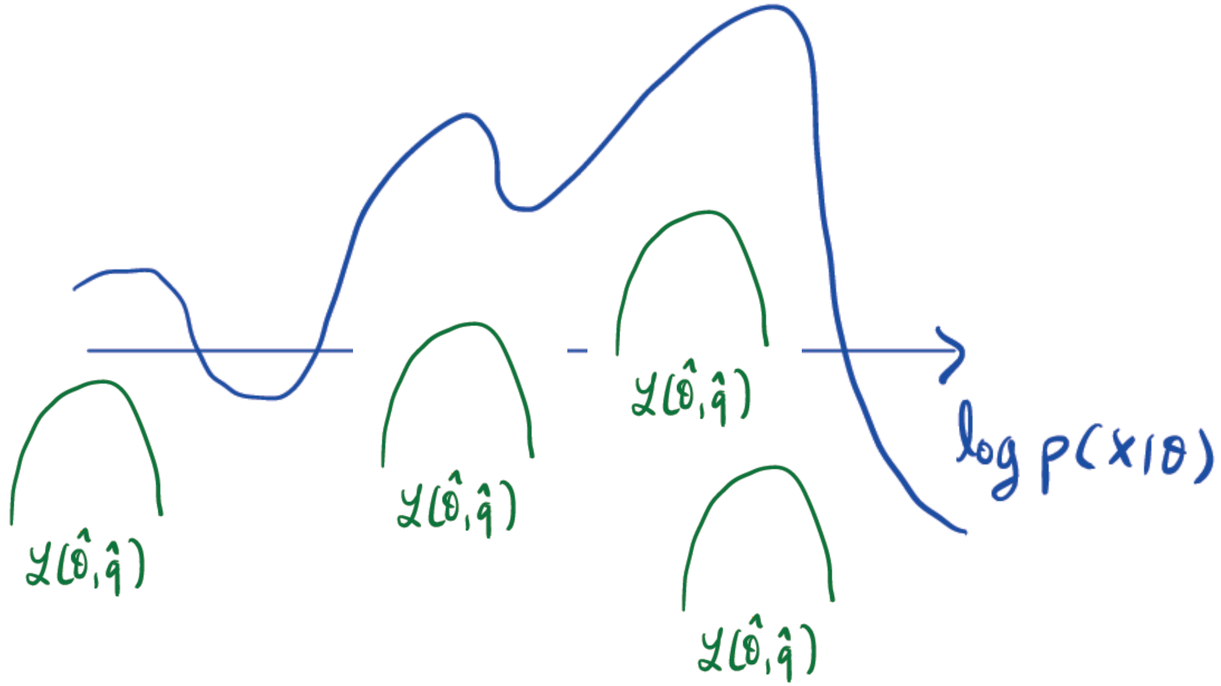



or


$$p(x_i|\theta) = \sum_{k=1}^{4} p(x_i, t_i = k|\theta)$$

$$\hat{\theta} = \arg\max_{\theta} \{ \log P(X|\theta) \}$$

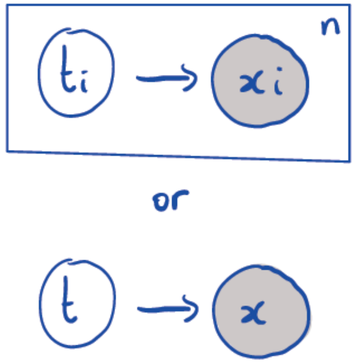
$\log P(X|\theta) \geq \mathcal{L}(\theta, q)$ (Jensen)



# 2.b. Expectation-Maximization algorithm

## EM algorithm : E-step

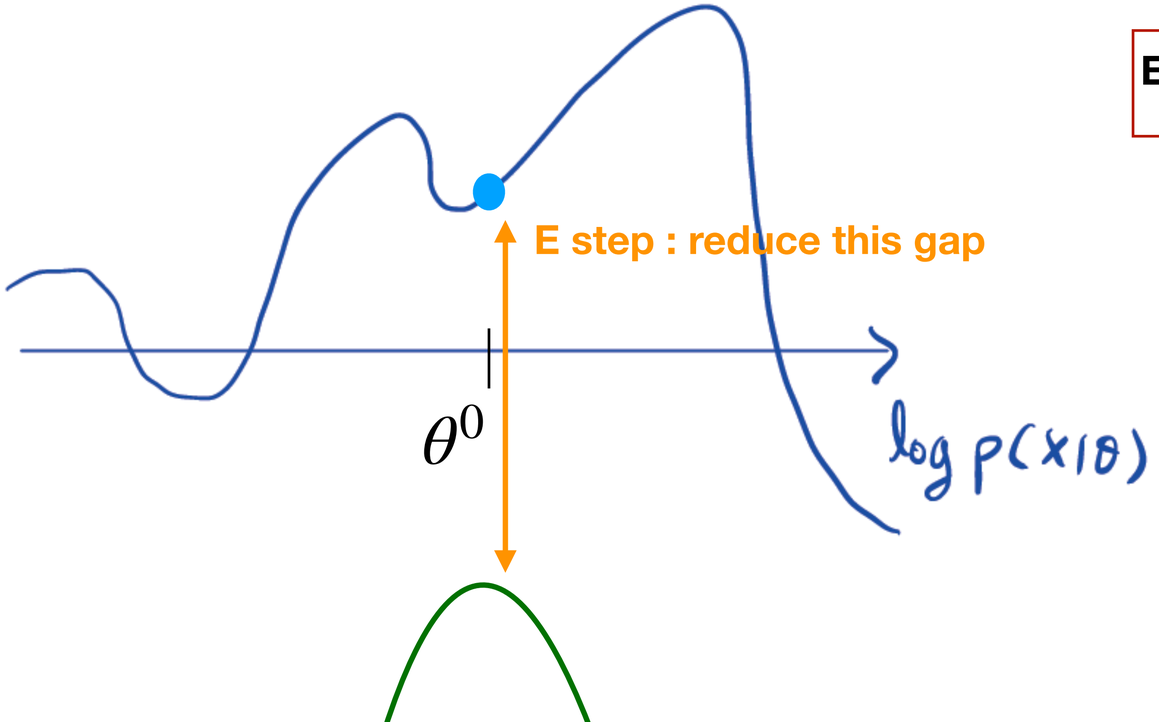
Our aim is to find : $\hat{\theta}^{MLE} = \arg\max_{\theta} p(\mathbf{x}\,|\,\theta) = \arg\max_{\theta} \log p(\mathbf{x}\,|\,\theta)$




$p(x_i|\theta) = \sum_{k=1}^{4} p(x_i, t_i = k|\theta)$

$\log P(X|\theta) \geq \mathcal{L}(\theta, q)$ (Jensen)

$\hat{\theta} = \arg\max_{\theta} \{ \log P(X|\theta) \}$

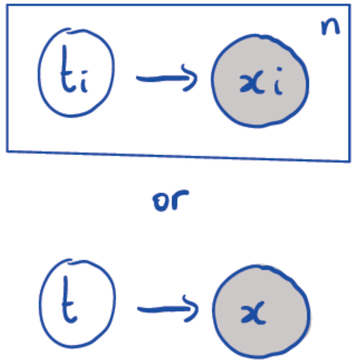



**Expectation step :** $q^{k+1} = \arg\max_{q \in Family} \mathcal{L}(\theta^k, q)$

# 2.b. Expectation-Maximization algorithm

## EM algorithm : E-step

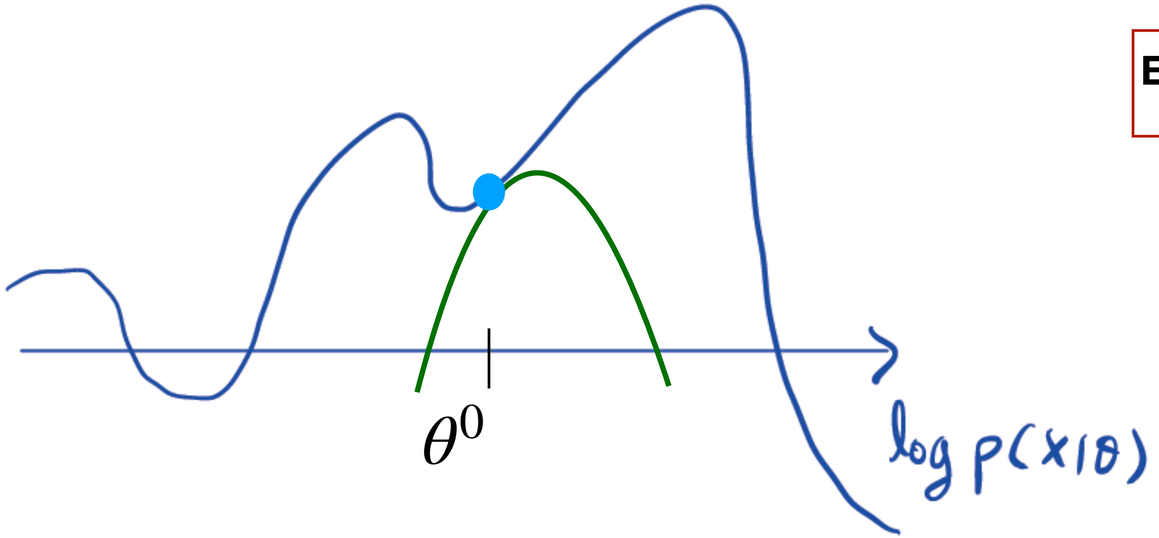
Our aim is to find : $\hat{\theta}^{MLE} = \arg\max_{\theta} p(\mathbf{x}\,|\,\theta) = \arg\max_{\theta} \log p(\mathbf{x}\,|\,\theta)$




$p(x_i|\theta) = \sum_{k=1}^{4} p(x_i, t_i = k|\theta)$

$\hat{\theta} = \arg\max_{\theta} \{ \log P(X|\theta) \}$

$\log P(X|\theta) \geq \mathcal{L}(\theta, q)$ (Jensen)

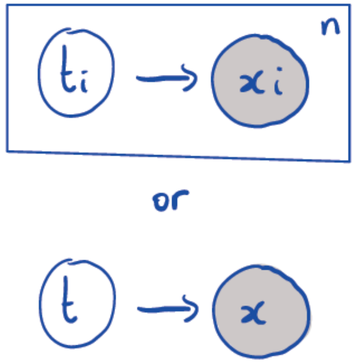



**Expectation step :** $q^{k+1} = \arg\max_{q \in Family} \mathcal{L}(\theta^k, q)$

# 2.b. Expectation-Maximization algorithm

## EM algorithm : M-step

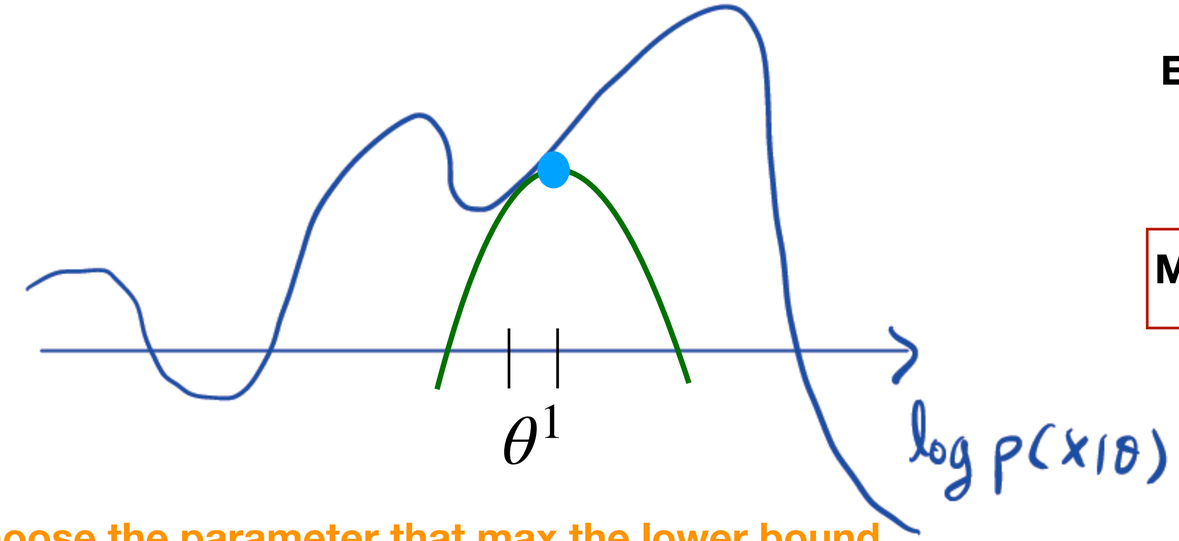
Our aim is to find : $\hat{\theta}^{MLE} = \arg\max_{\theta} p(\mathbf{x}\,|\,\theta) = \arg\max_{\theta} \log p(\mathbf{x}\,|\,\theta)$




$p(x_i|\theta) = \sum_{k=1}^{4} p(x_i, t_i = k|\theta)$

$\hat{\theta} = \arg\max_{\theta} \{ \log P(X|\theta) \}$

$\log P(X|\theta) \geq \mathcal{L}(\theta, q)$ Jensen




**Expectation step :** $q^{k+1} = \arg\max_{q \in Family} \mathcal{L}(\theta^k, q)$

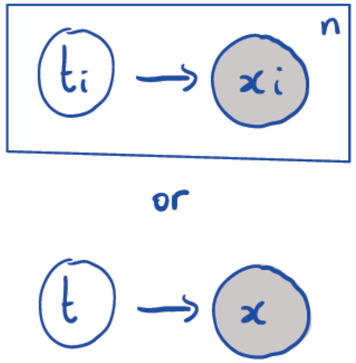
**Maximization step :** $\theta^{k+1} = \arg\max_{\theta} \mathcal{L}(\theta, q^{k+1})$

**M step : choose the parameter that max the lower bound**

# 2.b. Expectation-Maximization algorithm

## EM algorithm

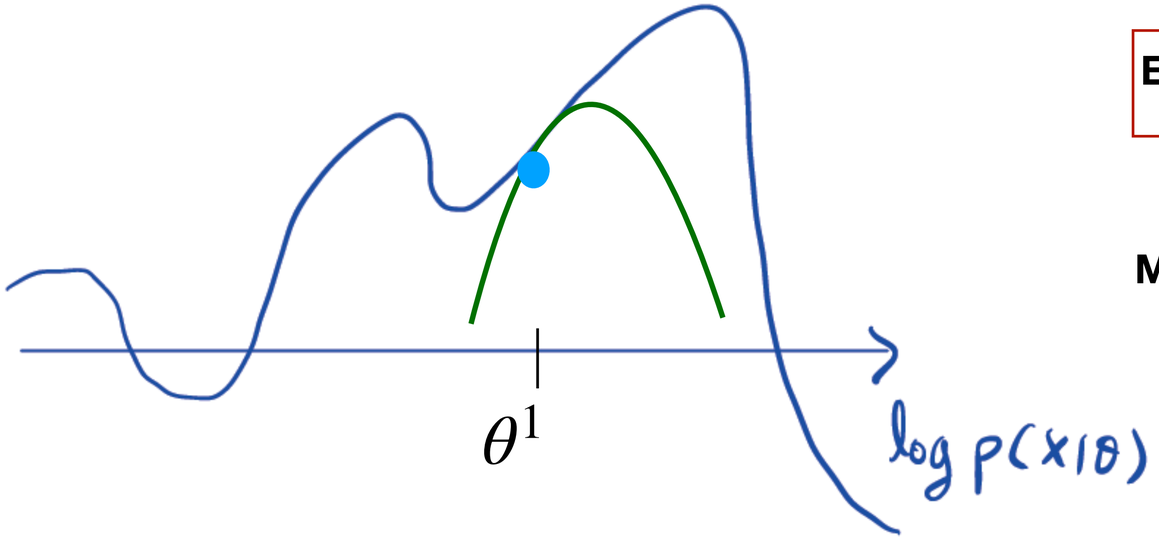
Our aim is to find : $\hat{\theta}^{MLE} = \arg\max_{\theta} p(\mathbf{x}\,|\,\theta) = \arg\max_{\theta} \log p(\mathbf{x}\,|\,\theta)$




$$p(x_i|\theta) = \sum_{k=1}^{4} p(x_i, t_i = k|\theta)$$

$$\hat{\theta} = \arg\max_{\theta} \{ \log P(X|\theta) \}$$

$$\log P(X|\theta) \geq \mathcal{L}(\theta, q)$$ Jensen



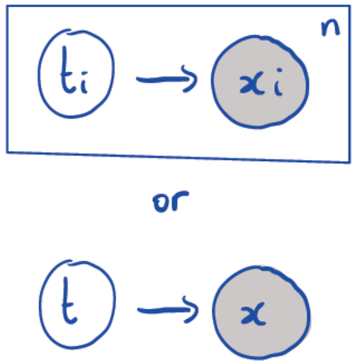

**Expectation step :** $q^{k+1} = \arg\max_{q \in Family} \mathcal{L}(\theta^k, q)$

**Maximization step :** $\theta^{k+1} = \arg\max_{\theta} \mathcal{L}(\theta, q^{k+1})$

# 2.b. Expectation-Maximization algorithm

## EM algorithm

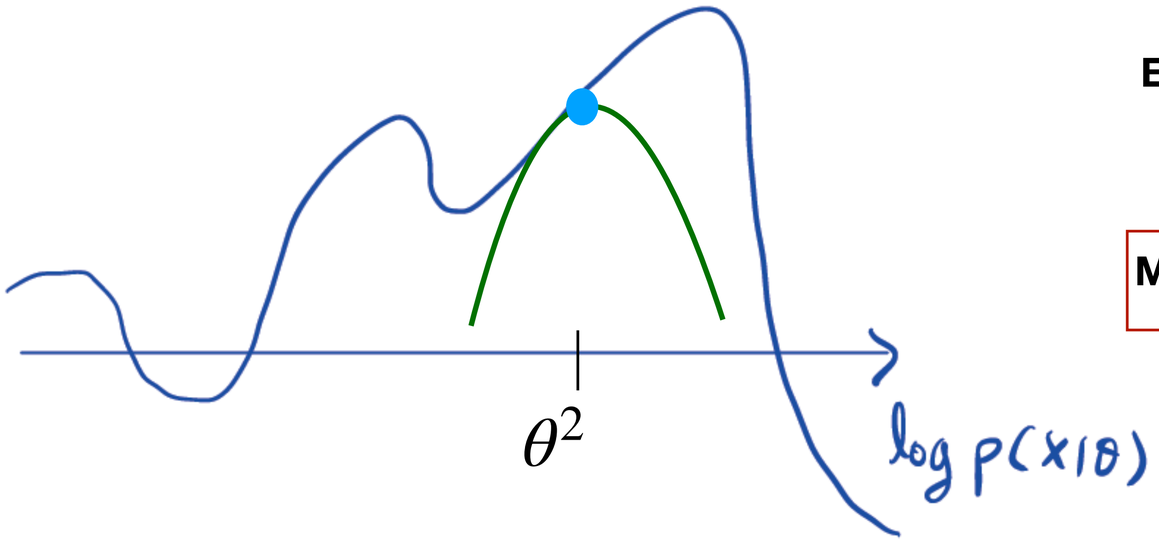
Our aim is to find : $\hat{\theta}^{MLE} = \arg\max_{\theta} p(\mathbf{x}\,|\,\theta) = \arg\max_{\theta} \log p(\mathbf{x}\,|\,\theta)$




$$p(x_i|\theta) = \sum_{k=1}^{4} p(x_i, t_i = k|\theta)$$

$$\hat{\theta} = \arg\max_{\theta} \{ \log P(X|\theta) \}$$

$\log P(X|\theta) \geq \mathcal{L}(\theta, q)$ (Jensen)



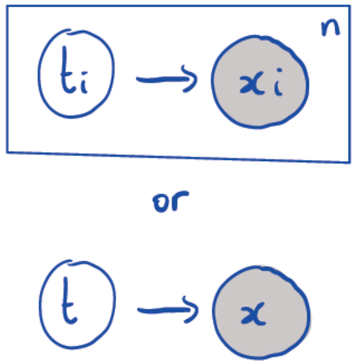

**Expectation step :** $q^{k+1} = \arg\max_{q \in Family} \mathcal{L}(\theta^k, q)$

**Maximization step :** $\theta^{k+1} = \arg\max_{\theta} \mathcal{L}(\theta, q^{k+1})$

# 2.b. Expectation-Maximization algorithm

## EM algorithm

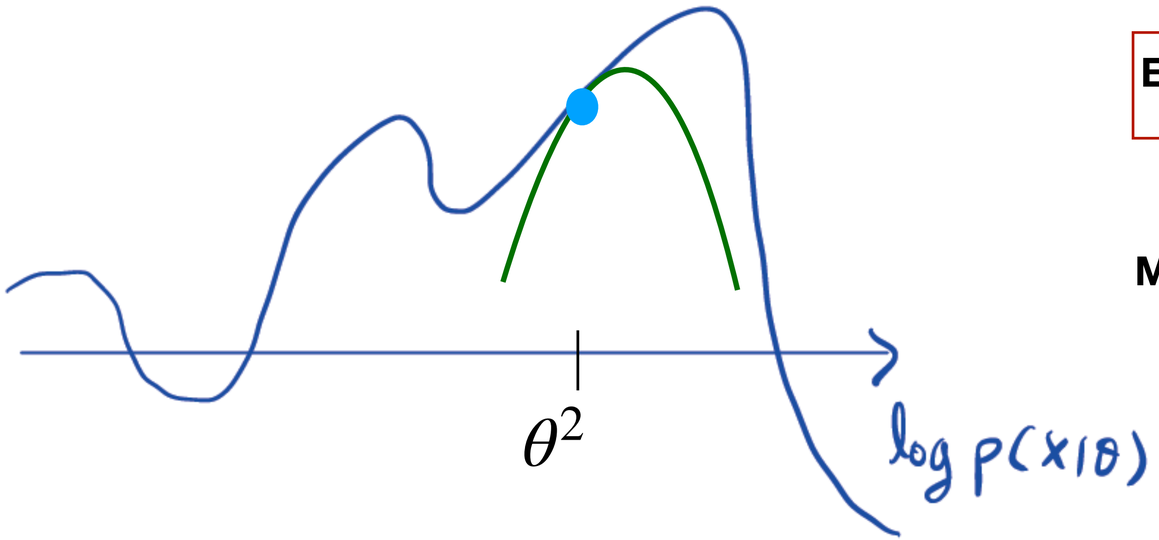
Our aim is to find : $\hat{\theta}^{MLE} = \arg\max_{\theta} p(\mathbf{x}\,|\,\theta) = \arg\max_{\theta} \log p(\mathbf{x}\,|\,\theta)$




$$p(x_i|\theta) = \sum_{k=1}^{4} p(x_i, t_i = k|\theta)$$

$$\hat{\theta} = \arg\max_{\theta} \{ \log P(X|\theta) \}$$

$$\log P(X|\theta) \geq \mathcal{L}(\theta, q)$$ Jensen



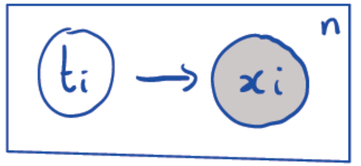

**Expectation step :** $q^{k+1} = \arg\max_{q \in Family} \mathcal{L}(\theta^k, q)$

**Maximization step :** $\theta^{k+1} = \arg\max_{\theta} \mathcal{L}(\theta, q^{k+1})$

# 2.b. Expectation-Maximization algorithm

## EM algorithm

Our aim is to find : $\hat{\theta}^{MLE} = \arg\max_{\theta} p(\mathbf{x}\,|\,\theta) = \arg\max_{\theta} \log p(\mathbf{x}\,|\,\theta)$

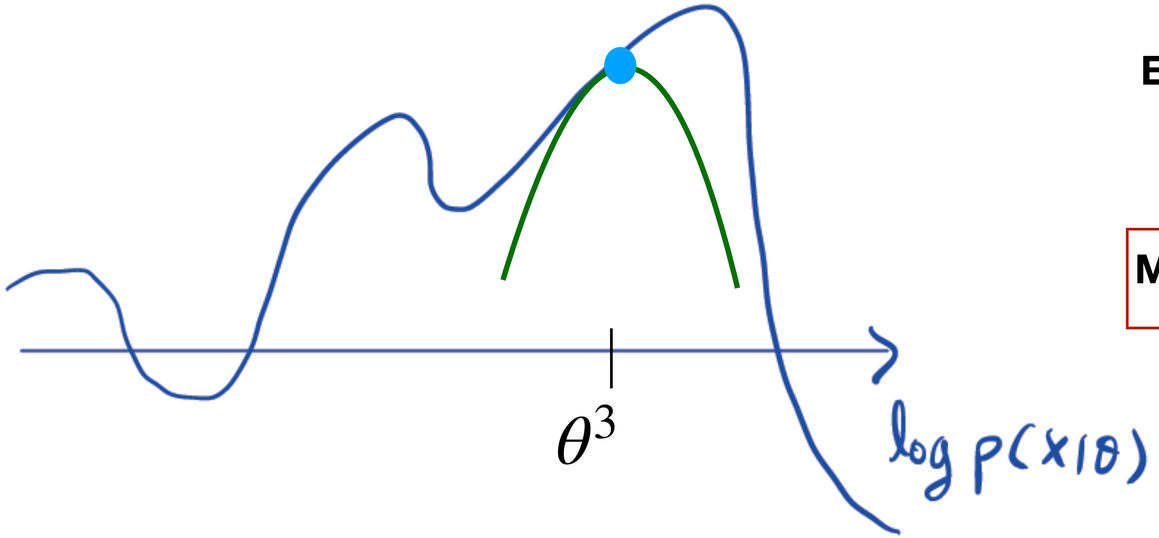



or


$$p(x_i|\theta) = \sum_{k=1}^{4} p(x_i, t_i = k|\theta)$$

$$\hat{\theta} = \arg\max_{\theta} \{ \log P(X|\theta) \}$$

$\log P(X|\theta) \geq \mathcal{L}(\theta, q)$ Jensen




**Expectation step :** $q^{k+1} = \arg\max_{q \in Family} \mathcal{L}(\theta^k, q)$

**Maximization step :** $\theta^{k+1} = \arg\max_{\theta} \mathcal{L}(\theta, q^{k+1})$

**And so on … until we reach a local maximum**

# 2.b. Expectation-Maximization algorithm

## EM algorithm : more details

**E-step :**

$$q^{k+1} = \arg \max_{q \in Family} \mathscr{L}(\theta^k, q) \iff q(t_i) = p(t_i \,|\, x_i, \theta)$$

**M-step :**

$$\theta^{k+1} = \arg\max_{\theta} \mathscr{L}(\theta, q^{k+1}) \iff \theta^{k+1} = \arg\max_{\theta} \mathbb{E}_{q^{k+1}}[\log p(X, T \,|\, \theta)]$$

# 2.b. Expectation-Maximization algorithm

## EM algorithm : back to GMM

**E-step :**

$$q^{k+1} = \arg \max_{q \in Family} \mathscr{L}(\theta^k, q) \iff q(t_i) = p(t_i \,|\, x_i, \theta)$$

**GMM :** for each point we indeed computed $q(t_i) = p(t_i \,|\, x_i, \theta)$

**M-step :**

$$\theta^{k+1} = \arg \max_{\theta} \mathscr{L}(\theta, q^{k+1}) \iff \theta^{k+1} = \arg \max_{\theta} \mathbb{E}_{q^{k+1}}[\log p(X, T \,|\, \theta)]$$

**GMM :** we updated the gaussian parameters with

$$\mu_{soft}^{MLE} = \frac{\sum_i p(t = 2 \,|\, x, \theta)\, x_i}{\sum_i p(t = 2 \,|\, x, \theta)}$$

which indeed is the M-step of the EM algorithm

# 2.b. Expectation-Maximization algorithm

## EM algorithm : back to GMM

**E-step :**

$$q^{k+1} = \arg \max_{q \in Family} \mathscr{L}(\theta^k, q) \iff q(t_i) = p(t_i \,|\, x_i, \theta)$$

**GMM :** for each point we indeed computed $q(t_i) = p(t_i \,|\, x_i, \theta)$

**M-step :**

$$\theta^{k+1} = \arg\max_{\theta} \mathscr{L}(\theta, q^{k+1}) \iff \theta^{k+1} = \arg\max_{\theta} \mathbb{E}_{q^{k+1}}[\log p(X, T \,|\, \theta)]$$

**GMM :** we updated the gaussian parameters with

$$\mu_{soft}^{MLE} = \frac{\sum_i p(t = 2 \,|\, x, \theta)\, x_i}{\sum_i p(t = 2 \,|\, x, \theta)}$$

which indeed is the M-step of the EM algorithm

$$\sum_{i=1}^{n} E_{q(t_i)} \log p(x_i, t_i \,|\, \theta) = \sum_{i=1}^{n} \sum_{k=1}^{4} q(t_i = k) \log \left( \frac{1}{const} e^{-\frac{(x_i - \mu_k)^2}{2\sigma_k^2}} \times \pi_k \right)$$

$$= \sum_{i=1}^{n} \sum_{k=1}^{4} q(t_i = k) \left( \log \left( \frac{\pi_k}{const} \right) - \frac{(x_i - \mu_k)^2}{2\sigma_k^2} \right)$$

$$\frac{\partial}{\partial \mu_2} \left( \sum_{i=1}^{n} \sum_{k=1}^{4} q(t_i = k) \left( \log \left( \frac{\pi_k}{const} \right) - \frac{(x_i - \mu_k)^2}{2\sigma_k^2} \right) \right)$$

$$= \sum_{i=1}^{n} q(t_i = 2) \left( 0 + \frac{(x_i - \mu_2)}{\sigma_2^2} \right) = 0$$

$$\iff \sum_{i=1}^{n} q(t_i = 2) \times x_i - \mu_2 \sum_{i=1}^{n} q(t_i = 2) = 0$$

$$\iff \boxed{\mu_2 = \frac{\sum_{i=1}^{n} q(t_i = 2) \times x_i}{\sum_{i=1}^{n} q(t_i = 2)}}$$

# 3 Probabilistic dimensionality reduction and EM-algorithm

# 3. Probabilistic dimensionality reduction

## Dimensionality reduction : reminder

**Dimensionality reduction** : transformation of data **from a high-dimensional** space **into a low-dimensional** space

# 3. Probabilistic dimensionality reduction

## Dimensionality reduction : reminder

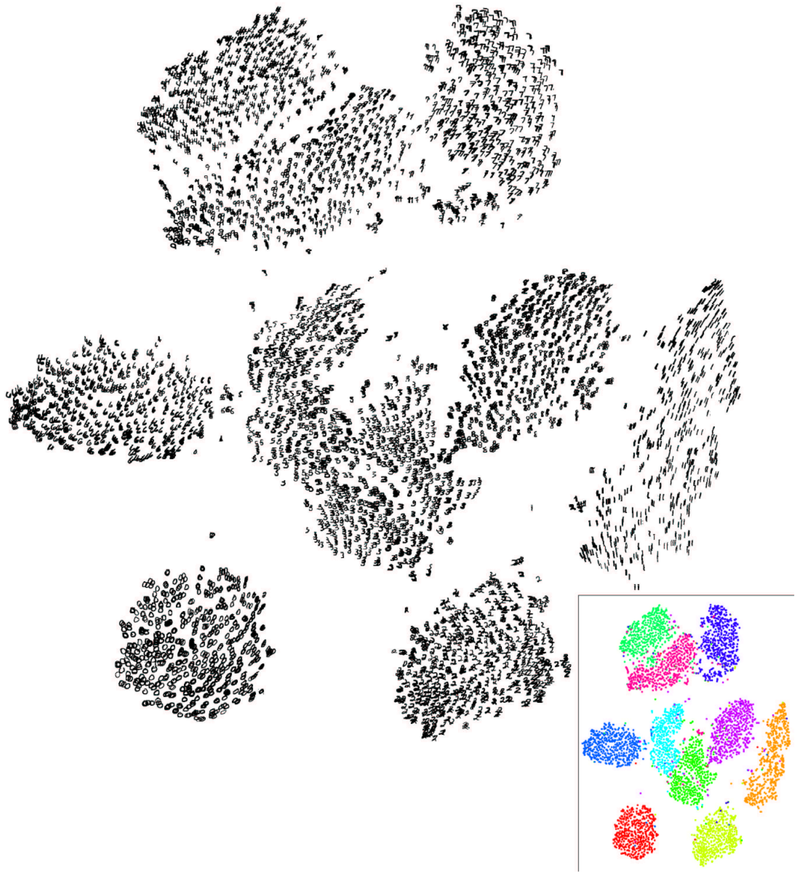
**Dimensionality reduction** : transformation of data **from a high-dimensional** space **into a low-dimensional** space

**Why do we care ?**

- Avoid **curse of dimensionality** :
  a high-dimensional data can be dangerous if the data is too sparse

- **Noise reduction** :
  In a High-dimensional dataset there might be too much noise.

- **Data visualisation (2D or 3D visualisation)** :
  We cannot visualise a high-dimensional data (dimension > 3)



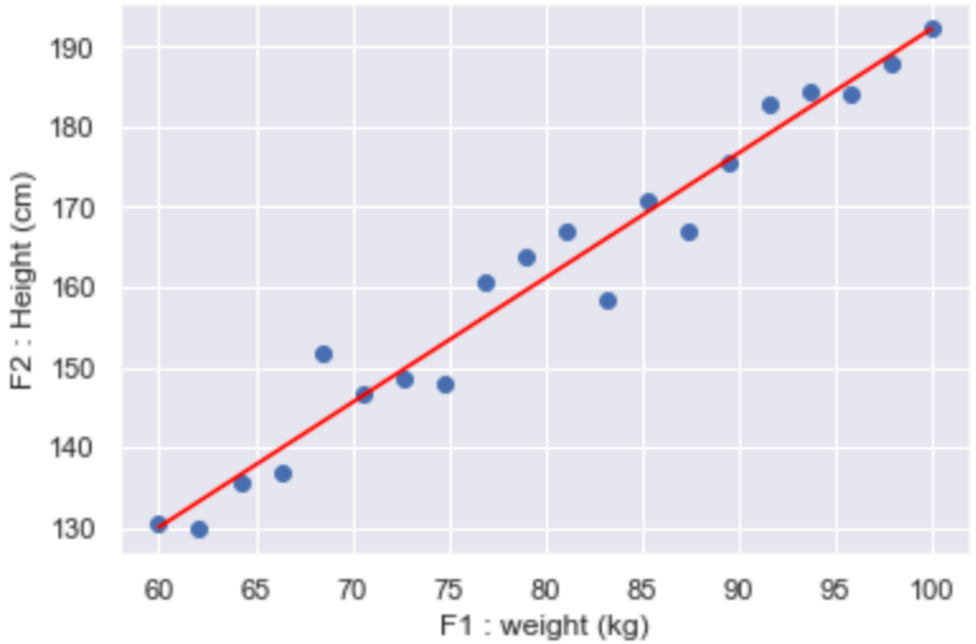
« Visualizing data using t-SNE », JMLR, Laurens et. al, 2008

# 3. Probabilistic dimensionality reduction

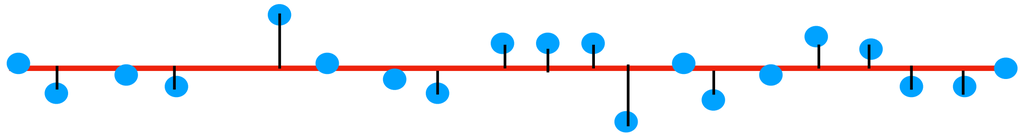
## Dimensionality reduction : PCA

**Dimensionality reduction** : transformation of data **from a high-dimensional** space **into a low-dimensional** space

**Principal Component Analysis (PCA) : Linear approach** to dimensionality reduction : the idea is to linearly project the high-dimensional data into a low-dimensional data




**The two features F1 and F2 have a positive correlation**

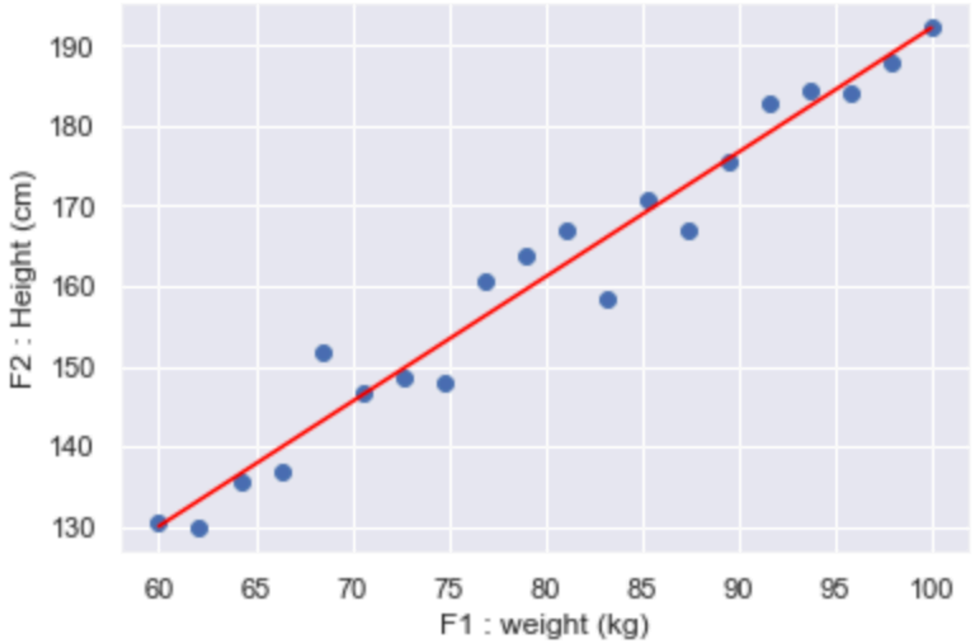


# 3. Probabilistic dimensionality reduction

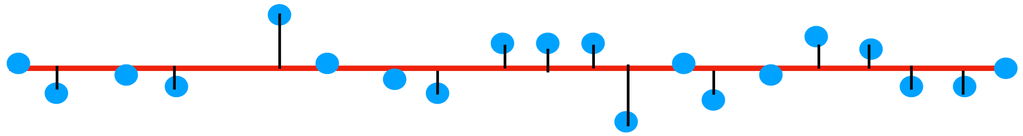
## Dimensionality reduction : PCA

**Dimensionality reduction** : transformation of data **from a high-dimensional** space **into a low-dimensional** space

**Principal Component Analysis (PCA) : Linear approach** to dimensionality reduction : the idea is to linearly project the high-dimensional data into a low-dimensional data




**The two features F1 and F2 have a positive correlation**



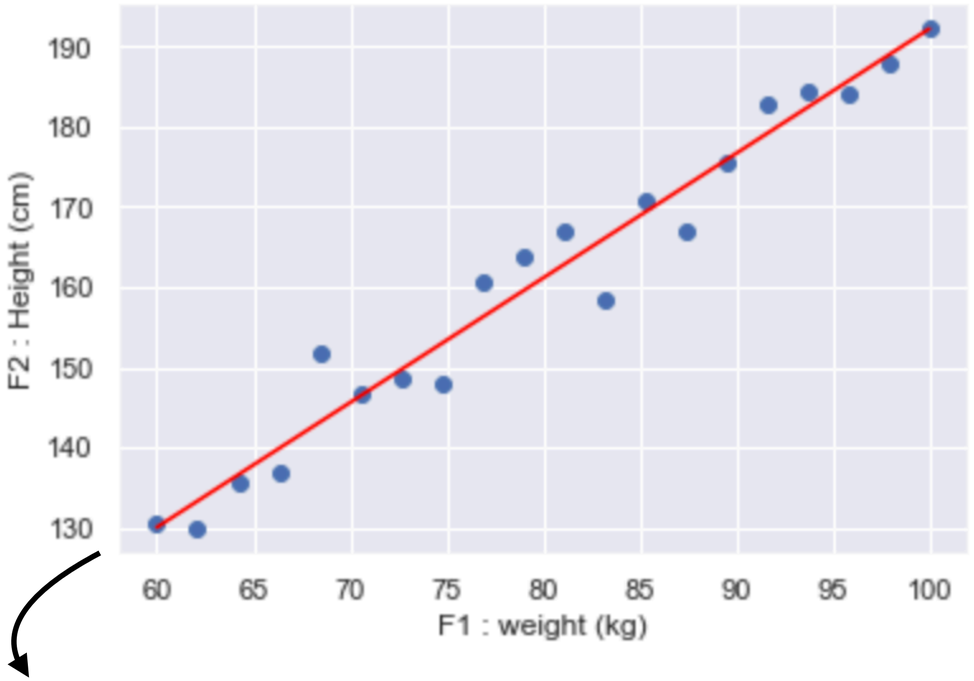
**Combine these two features into one : F**

# 3. Probabilistic dimensionality reduction

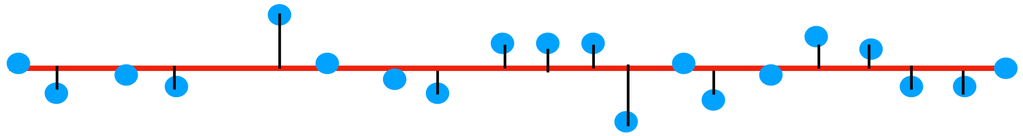
## Dimensionality reduction : PCA

**Dimensionality reduction** : transformation of data **from a high-dimensional** space **into a low-dimensional** space

**Principal Component Analysis (PCA) : Linear approach** to dimensionality reduction : the idea is to linearly project the high-dimensional data into a low-dimensional data




**The two features F1 and F2 have a positive correlation**



**Combine these two features into one : F**

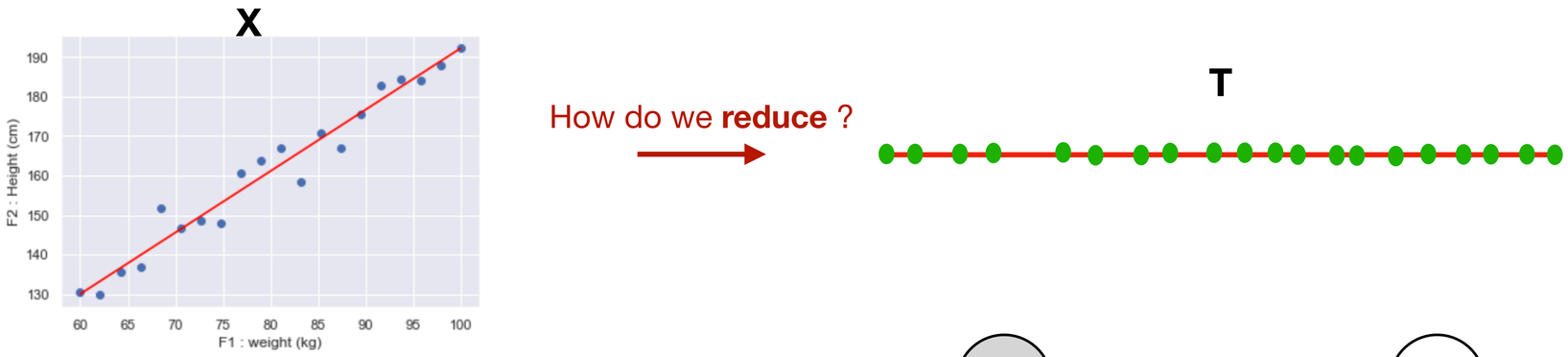
This line corresponds to the eigenvector associated to the greatest eigenvalue of the covariance matrix

# 3. Probabilistic dimensionality reduction

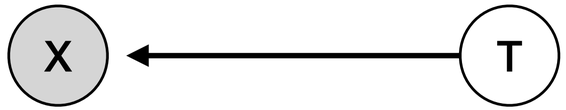
## Dimensionality reduction : probabilistic PCA (PPCA)

**Dimensionality reduction** : transformation of data **from a high-dimensional** space **into a low-dimensional** space

**Principal Component Analysis (PCA) : Linear approach** to dimensionality reduction : the idea is to linearly project the high-dimensional data into a low-dimensional data




**Probabilistic PCA :** a probabilistic point of view of PCA

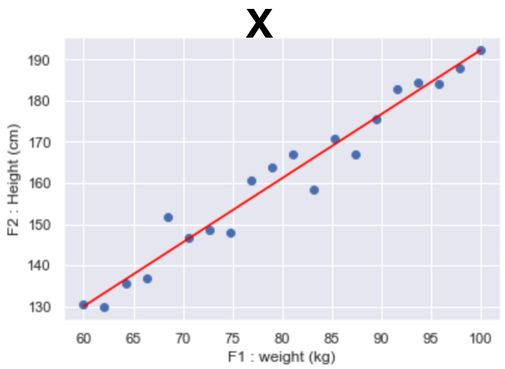


# 3. Probabilistic dimensionality reduction

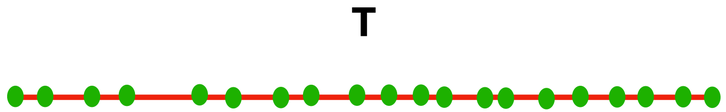
## Dimensionality reduction : probabilistic PCA (PPCA)

**Dimensionality reduction** : transformation of data **from a high-dimensional** space **into a low-dimensional** space

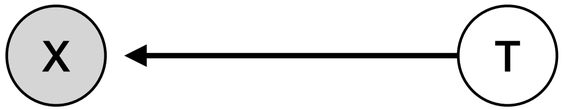
**Principal Component Analysis (PCA) : Linear approach** to dimensionality reduction : the idea is to linearly project the high-dimensional data into a low-dimensional data




How do we **reduce** ?




**Probabilistic PCA :** a probabilistic point of view of PCA

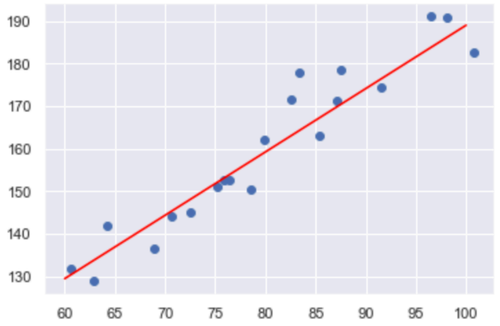



$$p(t_i) = \mathcal{N}(t_i \mid 0,\ I_2)$$

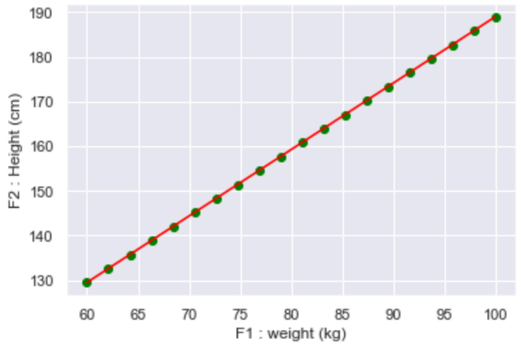
$$x_i = W\, t_i + b$$

$$x_i = W\, t_i + b + \epsilon_i \text{ with } \epsilon_i \sim \mathcal{N}(0, \Sigma)$$

$$p(x_i \mid t_i, \theta) = \ldots$$



How do we **generate** ?

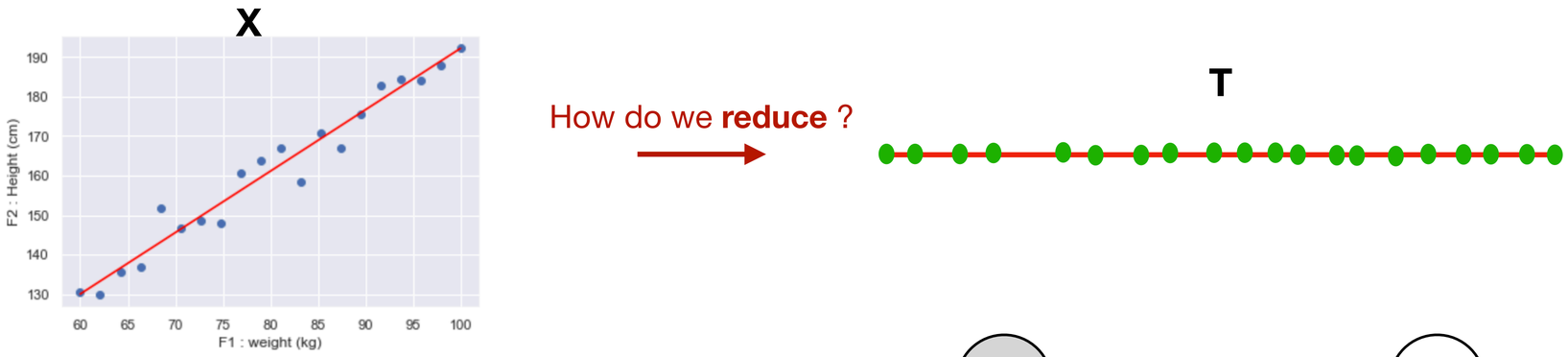


# 3. Probabilistic dimensionality reduction

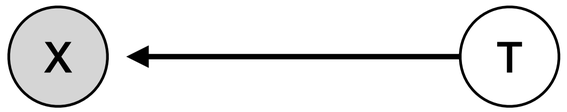
## Dimensionality reduction : probabilistic PCA (PPCA)

**Dimensionality reduction** : transformation of data **from a high-dimensional** space **into a low-dimensional** space

**Principal Component Analysis (PCA) : Linear approach** to dimensionality reduction : the idea is to linearly project the high-dimensional data into a low-dimensional data




**Probabilistic PCA :** a probabilistic point of view of PCA




$$p(t_i) = \mathcal{N}(t_i \mid 0,\ I_2)$$

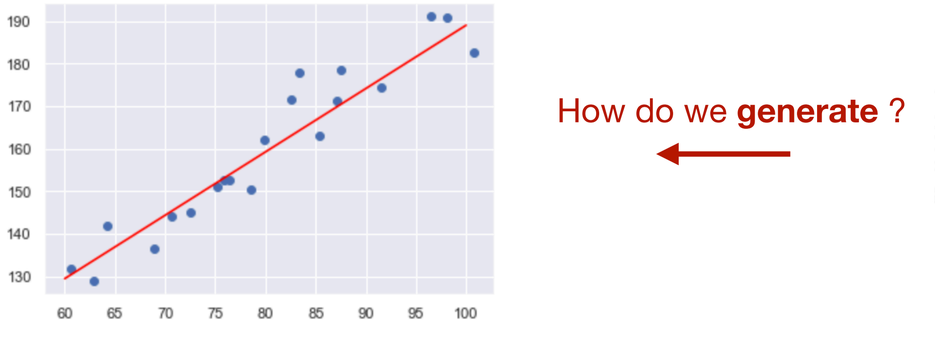
$$x_i = W\,t_i + b$$

$$x_i = W\,t_i + b + \epsilon_i \text{ with } \epsilon_i \sim \mathcal{N}(0,\Sigma)$$

$$p(x_i \mid t_i, \theta) = \mathcal{N}(Wt_i + b, \Sigma)$$

$$p(\mathrm{x} \mid \theta) = \prod_{i=1,\ldots,n} p(x_i \mid \theta)$$

$$= \prod_{i=1,\ldots,n} \int p(x_i \mid t_i, \theta) p(t_i) dt_i$$

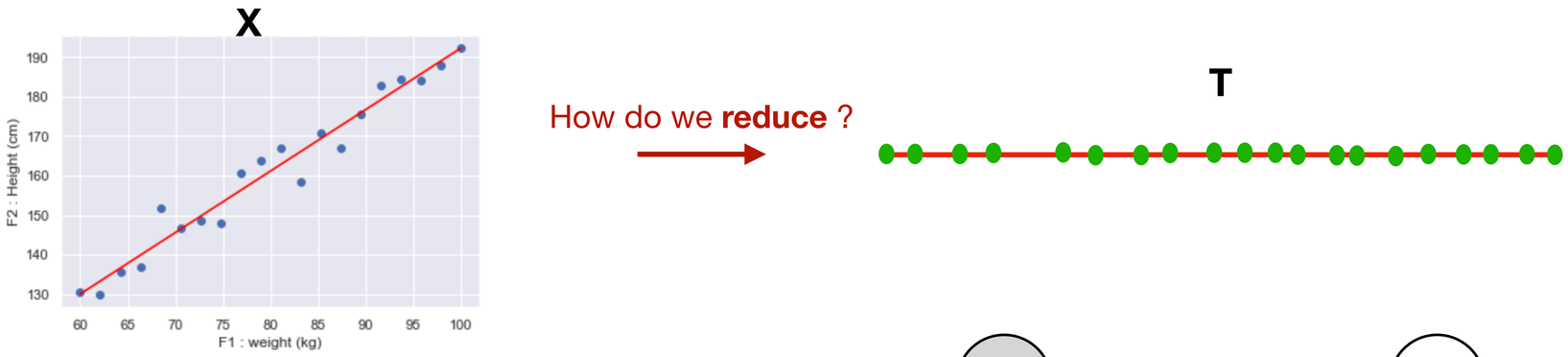


# 3. Probabilistic dimensionality reduction

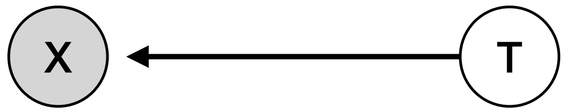
## Dimensionality reduction : probabilistic PCA (PPCA)

**Dimensionality reduction** : transformation of data **from a high-dimensional** space **into a low-dimensional** space

**Principal Component Analysis (PCA) : Linear approach** to dimensionality reduction : the idea is to linearly project the high-dimensional data into a low-dimensional data




**Probabilistic PCA :** a probabilistic point of view of PCA




$$p(t_i) = \mathcal{N}(t_i \mid 0,\ I_2)$$

$$x_i = W\,t_i + b$$

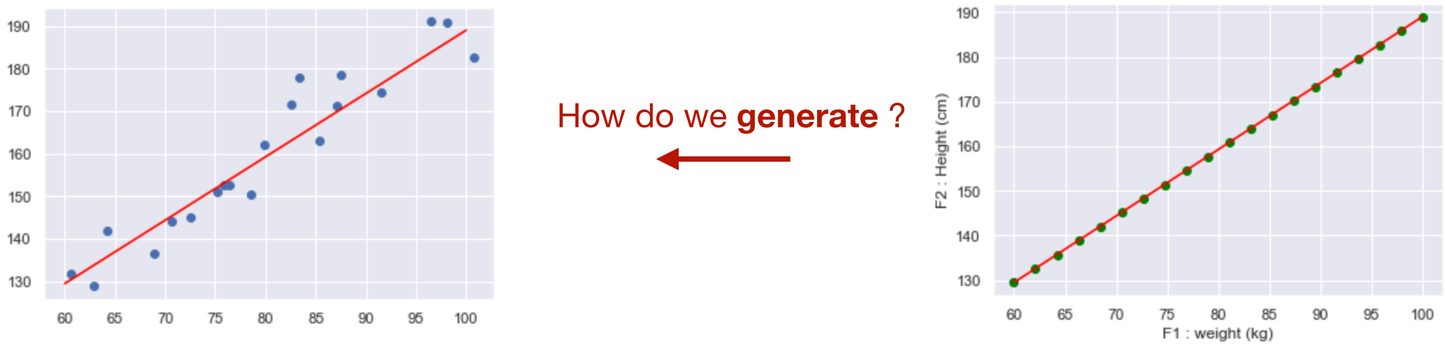
$$x_i = W\,t_i + b + \epsilon_i \text{ with } \epsilon_i \sim \mathcal{N}(0,\Sigma)$$

$$p(x_i \mid t_i, \theta) = \mathcal{N}(Wt_i + b, \Sigma)$$

$$p(\mathrm{x} \mid \theta) = \prod_{i=1,\ldots,n} p(x_i \mid \theta)$$

$$= \prod_{i=1,\ldots,n} \int p(x_i \mid t_i, \theta) p(t_i) dt_i$$

Normal conjugacy !

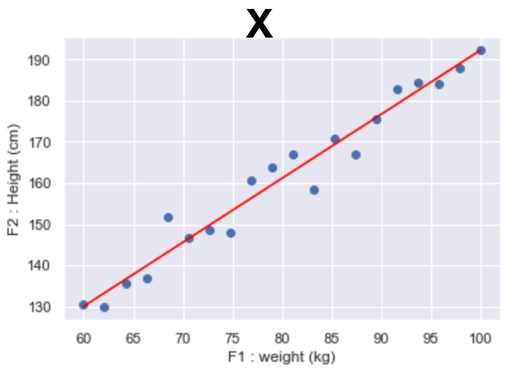


# 3. Probabilistic dimensionality reduction

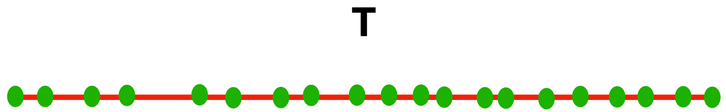
## Dimensionality reduction : probabilistic PCA (PPCA)

**Dimensionality reduction** : transformation of data **from a high-dimensional** space **into a low-dimensional** space

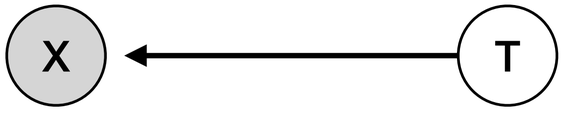
**Principal Component Analysis (PCA) : Linear approach** to dimensionality reduction : the idea is to linearly project the high-dimensional data into a low-dimensional data




How do we **reduce** ?




**Probabilistic PCA :** a probabilistic point of view of PCA




$$p(t_i) = \mathcal{N}(t_i \mid 0,\ I_2)$$

$$x_i = W\,t_i + b$$

$$x_i = W\,t_i + b + \epsilon_i \text{ with } \epsilon_i \sim \mathcal{N}(0, \Sigma)$$

$$p(x_i \mid t_i, \theta) = \mathcal{N}(Wt_i + b, \Sigma)$$

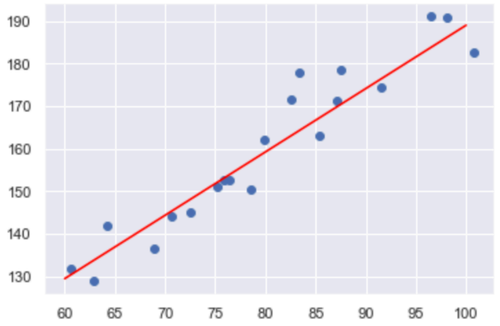
$$p(x \mid \theta) = \prod_{i=1,\ldots,n} p(x_i \mid \theta)$$

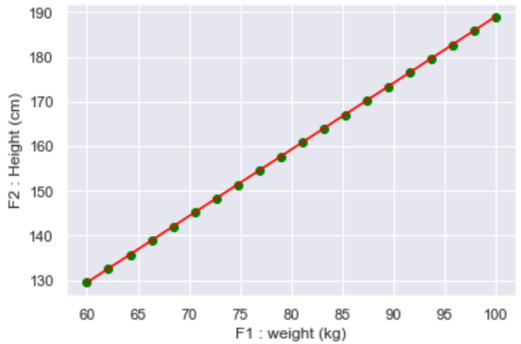
$$= \prod_{i=1,\ldots,n} \int p(x_i \mid t_i, \theta) p(t_i) dt_i$$

Easy to do EM here !

Normal conjugacy !

How do we **generate** ?

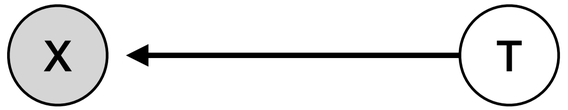




# 3. Probabilistic dimensionality reduction

## Dimensionality reduction : probabilistic PCA (PPCA)

**Probabilistic PCA :** a probabilistic point of view of PCA




EM for PPCA :

E-step : $q(t_i) = p(t_i | x_i, \theta) = \frac{p(x_i | t_i, \theta)\, p(t_i)}{\text{constant}}$ prior conjugacy

M-step : $\max_\theta \leftarrow E_{q(T)} \sum_i \log p(x_i | t_i, \theta)\, p(t_i)$

$= \sum_i E_{q(t_i)} \log \left( \frac{1}{\text{const}} e^{\cdots} e^{\cdots} \right)$

$= \sum_i \log \left( \frac{1}{\text{const}} \right) + \sum_i E_{q(t_i)} \log \left( e^{-\frac{(x - Wt_i - b)^2}{2\sigma^2}} e^{-\frac{t_i^2}{2}} \right)$

quadratic function on $t$, so we can do it analytically

Some cool things with PPCA :

- We can fill **missing values**
- **Hyperparameters** tuning
- We can do **mixture of PPCA**

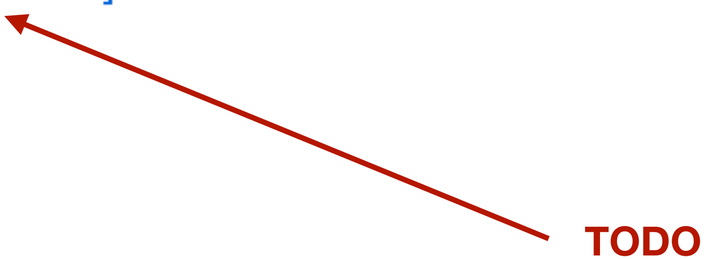
« Probabilistic Principal Component Analysis », JMLR, Michael E. Tipping et. al, 1999

# 4 Applications and examples : notebook

# Application and examples

## website : https://curiousml.github.io/

- Master of Science in Artificial Intelligence Systems : **Bayesian Machine Learning** by François HU
  - **Lecture 1** : Bayesian statistics [Lecture]
  - **Lecture 2** : Latent Variable Models and EM-algorithm [Soon available]
  - **Lecture 3** : Variational Inference and intro to NLP [Soon available]
  - **Lecture 4** : Markov Chain Monte Carlo [Soon available]
  - **Lecture 5** : [Oral presentations]
  - **Training session / prerequisite :** Statistics with python [Notebook], [Data]
  - **Practical work 1** : Conjugate distributions [Notebook] [Correction]
  - **Practical work 2** : Probabilistic K-means and probabilistic PCA [Notebook]
  - **Practical work 3** : Topic Modeling with LDA [Soon available]
  - **Practical work 4** : MCMC samples [Soon available]



# Road map

## Bayesian statistics

1

## Latent variable models

2

## Variational Inference

3

### Bayesian perspective :

$$P(\theta \mid X) = \frac{P(X,\theta)}{P(X)} = \frac{P(X \mid \theta) \cdot P(\theta)}{P(X)}$$

Likelihood — Prior distribution

Posterior distribution

Evidence

Hard to compute !

$\theta$ parameters

$X$ observations

**Exemple** :
Naive Bayes classifier,
Linear regression, ….

MAP : $\arg\max_{\theta} P(X \mid \theta) \cdot P(\theta)$

Conjugate distribution

| Pros : | Cons : |
|---|---|
| - exact posterior | - conjugate prior maybe inadequate |

### Hidden variable models :

$$P(X \mid \theta) = \sum_{t \in T_{indexes}} P(X, T = t \mid \theta)$$

$$P(X, T \mid \theta) = P(X \mid T, \theta) P(T \mid \theta)$$

**Exemple** :
GMM, K-means, PCA/PPCA

| Pros : | Cons : |
|---|---|
| - fewer parameters / simpler models<br>- hidden variable sometimes meaningful<br>- clustering / dimensionality reduction | - harder to work with<br>- requires math<br>- only local maximum or saddle point<br>- EM : the posterior of T could be intractable |

## Markov Chain Monte Carlo

4

## Extensions

5