Albert Wandui (awandui@stanford.edu)
Lawrence Lin Murata (lmurata@stanford.edu)

1. (a) Let E be the event that there is a student enrolled in CS103 and F the event that there is a student enrolled in CS107.
   $P(E) = 0.3$ and $P(F) = 0.25$ Then: $P(EF) = P(F|E)P(E)$ which gives $0.05 * 0.30 = 0.015$
   There is a 1.5% chance that a randomly selected student is enrolled in both CS107 and CS103
   (b) Using the definition of conditional probability gives $P(F|E^c) = \frac{P(FE^c)}{P(E^c)}$
   We can use the fact that:
   $P(E^c) = 1 - P(E)$ and that $P(F) = P(EF) + P(FE^c)$.
   The final answer is $\frac{P(FE^c)}{1-P(E)} = \frac{P(F)-P(FE)}{1-P(E)}$

2. Let E be the event that a player stays up and F the event that a player makes an error
   Using Bayes theorem:
   $P(E|F) = \frac{P(F|E)P(E)}{P(F|E)P(E)+P(F|E^c)P(E^c)}$
   We know $P(F|E) = 3P(F|E^c)$ and thus we can substitute for this in the expression to get:
   $P(E|F) = \frac{3P(E)}{2P(E)+1}$
   Now, since $P(E) = 0.4$ we can substitute this to get $P(E|F) = 2/3$ We conclude that two-thirds of the players who made critical errors stayed up late.

3. (a) Using Bayes Theorem: $P(E|F) = \frac{P(F|E)P(E)}{P(F)}$
   $P(F|E)$ is the probability that one of the cards is an ace of spades given that both cards are aces. We have $\binom{4}{2} = 6$ such possibilities and $\binom{3}{1} = 3$ combinations with at least 1 ace of spades. Therefore $P(F|E) = 1/2$
   $P(E) = \frac{\binom{4}{2}}{\binom{52}{2}} = 1/221$ where out of the 4 aces we pick any two and the size of the sample space is the number of ways we can pick any two cards out of the 52 original cards.
   $P(F) = \frac{\binom{51}{1}}{\binom{52}{2}} = 1/26$ since we have 51 possible groups of two cards with one ace of spades and the denominator is the size of the sample space.
   This gives: $P(E|F) = 1/17$
   (b) Using Bayes Theorem: $P(E|G) = \frac{P(G|E)P(E)}{P(G)}$
   $P(G|E) = 1$ since if we know that both cards are aces, G is automatically satisfied. We simply need $P(G) = \frac{\binom{4}{1}\binom{51}{1}}{\binom{52}{2}} = 2/13$ since the size of the event space is the number of ways we can pick at least 1 ace from the 4 possible aces and pick the second card from the remaining 51 cards.
   This gives: $P(E|G) = 1/34$

4. (a) Let F_i be the event that at least 1 string is hashed into the ith bucket. and p_i the probability that a string is hashed to the ith bucket. There are $m = 6$ strings in total and $n = 4$ buckets
   $P(F_1F_2F_3) = 1 - P((F_1F_2F_3)^c) = 1 - P(F_1^c \cup F_2^c \cup F_3^c)$ using De Morgan's Laws
   We can use the inclusion exclusion principle to expand the union:
   $P(F_1^c \cup F_2^c \cup F_3^c) = P(F_1^c) + P(F_2^c) + P(F_3^c) - P(F_1^cF_2^c) - P(F_1^cF_3^c) - P(F_2^cF_3^c) + P(F_1^cF_2^cF_3^c)$
   We also know that $P(F_1^c ... F_k^c) = (1 - \sum_{i=1}^{k} p_i)^m$ since if no string is not hashed to the first k buckets, it must be hashed to the remaining $n - k$ buckets.
   This gives:
   $P(F_1F_2F_3) = 1 - (1-p_1)^m - (1-p_2)^m - (1-p_1)^m + (1-p_1-p_2)^m +$
   $(1-p_1-p_3)^m + (1-p_2-p_3)^m - (1-p_1-p_2-p_3)^m$
   (b) Substituting $p_1 = 0.25, p_2 = 0.2, p_3 = 0.1, p_4 = 0.45$ gives
   $P(F_1F_2F_3) = 0.240881$

Albert Wandui (awandui@stanford.edu)
Lawrence Lin Murata (lmurata@stanford.edu)

5. (a) Let E be the event that at least one of the emails is spam and F the event that both emails are spam. $p = 0.93$ is the probability that a given email is spam.
$P(F|E) = \frac{P(E|F)P(F)}{P(E)}$ from Bayes' Theorem
$P(E|F) = 1$ since E is satisfied if F is given.
$P(F) = p^2$ and $P(E) = P(E|F)P(F) + P(EF^c)$
$P(EF^c) = 2p(1-p)$ which is the probability of having just 1 spam email at the server.
Putting it all together gives: $P(F|E) = \frac{p^2}{p^2+2p(1-p)}$ where $p = 0.93$
(b) Let G be the event that the email forwarded is spam
$P(F|G) = \frac{P(G|F)P(F)}{P(G)}$
$P(G|F) = 1$ since if both emails received are spam I will undoubtedly get spam in my inbox.
$P(F) = p^2$ as before.
$P(G) = p^2 + 2 \times p(1-p) \times 1/2$ the first term is for the case that both emails received are spam and the second for one spam and one non-spam email received while accounting for the probability of choosing the spam email. We finally arrive at: $P(F|G) = \frac{p^2}{p^2+p(1-p)}$

6. Let C_i be the event that the applicant is contacted on day i.
(a) $P(C_1) = P(C_1|H)P(H) + P(C_1|H^c)(1-P(H))$
This gives $P(C_1) = 0.34$
(b) $P(C_2|C_1^c) = \frac{P(C_1^c|C_2)P(C_2)}{1-P(C_1)}$
$P(C_1^c|C_2) = 1$ since an applicant is contacted only once. If she was contacted on Tuesday, the applicant must not have been contacted on Monday.
Similarly to part a)$P(C_2) = P(C_2|H)P(H) + P(C_2|H^c)(1-P(H)) = 0.16$
This gives: $P(C_2|C_1^c) = \frac{P(C_2)}{1-P(C_1)} = \frac{0.16}{1-0.34} = 0.4706$
Given that the applicant was not contacted on Monday,there is a close to 50% chance that the applicant will be contacted on Tuesday.
(c) $P(H|C_1^cC_2^cC_3^c) = \frac{P(HC_1^cC_2^cC_3^c)}{P(C_1^cC_2^cC_3^c)}$ De Morgan's Laws come in handy allowing us to expand $C_1^cC_2^cC_3^c$ as $(C_1 \cup C_2 \cup C_3)^c$.
Thus:
$P(HC_1^cC_2^cC_3^c) = P((C_1 \cup C_2 \cup C_3)^c|H)P(H) = (1 - P((C_1 \cup C_2 \cup C_3)|H))P(H)$
The conditional can distribute over the unions since the events are mutually exclusive and we can write the expression as
$P(HC_1^cC_2^cC_3^c) = P(H)(1 - P(C_1|H) - P(C_2|H) - P(C_3|H))$
We also note that: $P(C_3^cC_2^cC_1^c) = P((C_1 \cup C_2 \cup C_3)^c) = 1 - P(C_1 \cup C_2 \cup C_3) = 1 - P(C_1) - P(C_2) - P(C_3)$
This gives the final expression as: $P(H|C_1^cC_2^cC_3^c) = \frac{P(H)(1-P(C_1|H)-P(C_2|H)-P(C_3|H))}{1-P(C_1)-P(C_2)-P(C_3)}$
(d) Let $C_{>5}$ be the event that you are contacted after Friday
$P(H|C_{>5}) = \frac{P(C_{>5}|H)P(H)}{P(C_{>5})}$
We note that: $P(C_{>5}|H) = 1 - P((C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5)|H)$
Expanding this gives: $P(C_{>5}|H) = 1 - P(C_1|H) - P(C_2|H) - P(C_3|H) - P(C_4|H) - P(C_5|H)$
$P(C_{>5}) = 1 - P(C_1) - P(C_2) - P(C_3) - P(C_4) - P(C_5)$
Putting all the pieces together gives:
$P(C_{>5}|H) = \frac{1-P(C_1|H)-P(C_2|H)-P(C_3|H)-P(C_4|H)-P(C_5|H)}{1-P(C_1)-P(C_2)-P(C_3)-P(C_4)-P(C_5)}$

7. When the 16 bit string passes through the channel X, either both bits that were corrupted were 0's or one 0 and a 1 were corrupted or both bits that were corrupted were 1's.

Let X be the random variable whose value is number of zeros flipped. X Bin(p, 7) where p is the probability of flipping a zero.

P(the two bits that are corrupted by channel Y are 0's) is given by:
$(\binom{7}{2}p^2(1-p)^5)(\binom{5}{2}p^2(1-p)^3) + (\binom{7}{1}p^2(1-p)^6)(\binom{6}{2}p^2(1-p)^4) + (\binom{7}{0}p^2(1-p)^7)(\binom{7}{2}p^2(1-p)^5)$

Albert Wandui (awandui@stanford.edu)
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8. Die 1: 1, 6, 1, 4, 5, 6

   Die 2: 2, 2, 2, 4, 4, 4

   Let E be the event that you roll a 2 and F the event that you roll a 4

   $P(E|F) = \frac{P(F|E)P(E)}{P(F)}$

   $P(F|E) = 1/2$ since if you rolled a two, you are limited solely to the second die

   $P(E) = (1/2) \cdot (1/2) = 1/4$ which is the product of the probability of picking the second die and the probability of rolling a 2 with the second die.

   $P(F) = (1/2) \cdot (1/6) + (1/2) \cdot (1/2) = 1/3$ which is the probability of rolling a 4 with the first die or rolling a 4 with the second die.

   This gives:

   $P(E|F) = \frac{(1/2)(1/4)}{(1/3)} = 3/8$

9. Let B be the event that a person inherits a blue eyes gene.

   Let Br be the event that a person inherits a brown eyes gene.

   (a) If William's sister has blue eyes, then each of his parents must carry the B recessive gene.
   P(William has a blue gene |Has brown eyes) = $\frac{P(BrB)+P(BBr)}{P(BrBr)+P(BBr)+P(BrB)} = \frac{(1/4)+(1/4)}{(3/4)} = 2/3$

   (b) If the child has blue eyes, William must necessarily have inherited a blue gene.
   P(child has blue eyes) = P(William inherited a blue gene) $\cdot(1/2)$ which is the probability of William and his wife getting a blue eyed child.
   Possible combinations: BB BrB BB BrB
   P(child has blue eyes) = $(2/3) \cdot (1/2) = 1/3$

   (c) The probability any child getting any eye color is not dependent on the eye color of the previous child.
   P(child has brown eyes) = P(William has B Br genes) $\cdot$ P(getting a child with brown eyes in this case) + P(William has Br Br genes) $\cdot$ P(getting a child with brown eyes in this case)
   Possible combinations when William has BrBr: BrB BrB BrB BrB
   Possible combinations when William has BBr: BB BrB BB BrB
   P(child has brown eyes) = $(1/3) \cdot 1 + (2/3) \cdot (1/2) = 2/3$

10. Let X be the random variable that equals the number of 1s in a bit string of length n.

    X $\sim$ Bin$(1/4, n)$

    $P(X \geq 1) = 1 - P(X = 0) = \binom{n}{0}p^0(1-p)^n = 1 - (1-p)^n \geq 0.8$

    Rearranging the equation gives:

    $\frac{log_{10}0.2}{log_{10}(1-p)} \leq n \implies n \geq \lceil \frac{log_{10}0.2}{log_{10}(1-p)} \rceil$

11. Let E be the event that a person tests positive.

    X is a random variable that is the number of tests a group takes.

    P(X = 1) is the probability that no one in the group has measles.

    $P(X = 1) = (1-p)^5$ where p is the probability that a person has measles.

    P(X = 6) is the probability that at least 1 person in the group has measles which is 1 less the probability that no one has measles.

    $P(X = 6) = 1 - (1-p)^5$

    $E[X] = \sum_x x \cdot p(x) = 1(1-p)^5 + 6(1 - (1-p)^5) = 6 - 5(1-p)^5.$

    Substituting p = 0.1 gives E[X] = 3.0476

Albert Wandui (awandui@stanford.edu)
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12. Y is a random variable whose value is your winnings in the game. $E[Y] = \sum_x x \cdot p(x)$

    $p(x) = P(Y = 2^x) = (\frac{1}{2})^x$

    (a) For 2^x greater than X = 250, Y is capped at the value 250. This happens for $x > 7$.
    This gives:

    $$E[Y] = \underbrace{\sum_{k=0}^{7}(\frac{1}{2})^{k+1} \cdot 2^k}_{\text{contribution from } < 7 \text{ coin flips}} + \underbrace{\sum_{k=8}^{\infty}(\frac{1}{2})^{k+1} \cdot (250)}_{\text{contribution from earnings above the cap of \$250}}$$

    $E[Y] = 4 + \frac{125}{128} = 4.9766$

    (b) For 2^x greater than X = 25000, Y is capped at the value 250. This happens for $x > 14$.
    This gives:

    $$E[Y] = \underbrace{\sum_{k=0}^{1}4(\frac{1}{2})^{k+1} \cdot 2^k}_{\text{contribution from } < 14 \text{ coin flips}} + \underbrace{\sum_{k=15}^{\infty}(\frac{1}{2})^{k+1} \cdot (25000)}_{\text{contribution from earnings above the cap of \$25000}}$$

    $E[Y] = \frac{15}{2} + \frac{15625}{2048} = 15.1294$

13. (a) X is the random variable that is the value that is returned when arr[i] == key.
    $P(X = 1) = \frac{1}{14}$ since for every i there is only 1 key such that arr[i] == key.
    $E[X] = \sum_{i=0}^{1} 3\frac{i}{14} = \frac{91}{14} = 6.5$

    (b) X is the random variable that is the value that is returned when arr[mid] == key.
    We can enumerate P(X = i) ∀ mid = i in [0,13]
    P(X = 13) = (7/14)(3/7)(1/3) = 1/14
    P(X = 12) = (7/14)(3/7)(1/3) = 1/14
    P(X = 11) = (7/14)(3/7)(1/3) = 1/14
    P(X = 10) = (7/14)(1/7)= 1/14
    P(X = 9) = (7/14)(3/7)(1/3) = 1/14
    P(X = 8) = (7/14)(3/7)(1/3) = 1/14
    P(X = 7) = (7/14)(3/7)(1/3) = 1/14
    P(X = 6) = 1/14
    P(X = 5) = (6/14)(3/6)(1/3) = 1/14
    P(X = 4) = (6/14)(3/6)(1/3) = 1/14
    P(X = 3) = (6/14)(3/6)(1/3) = 1/14
    P(X = 2) = (6/14)(1/6) = 1/14
    P(X = 1) = (6/14)(2/6)(1/2) = 1/14
    P(X = 0) = (6/14)(2/6)(1/2) = 1/14
    The subdivision of the array in each successive iteration ensures that all the probabilities match.
    The expected value is exactly as in the previous case: 6.5

14. X is a random variable that is the number of flowers that survive after 1 year $X \sim$ Bin(p, 10)

    (a) $P(X \geq 1) = 1 - P(X = 0) = 1 - \binom{10}{0}p^0(1-p)^10 = 1 - (1-p)^10$

    (b) $P(X = 2) = \binom{10}{2}p^2(1-p)^8$

    (c) $P(X \geq 2) = 1 - P(X = 0) - P(X = 1)$
    $= 1 - \binom{10}{0}p^0(1-p)^10 - \binom{10}{1}p^2(1-p)^9$

Albert Wandui (awandui@stanford.edu)
Lawrence Lin Murata (lmurata@stanford.edu)

15. X is the random variable whose value is the number of requests received. $X \sim Poi(\lambda)$ where $\lambda = 2$

    $P(X = k) = e^{-\lambda}\frac{\lambda^k}{k!}$

    P(No server has crashed) $= P(\bigcup_{k=0}^{6} X = k) = e^{-\lambda}\sum_{k=0}^{6}\frac{\lambda^k}{k!}$

16. E is the event that your computer crashes 4 times.

    F is the event that the patch has an effect.

    $P(F|E) = \frac{P(E|F)P(F)}{P(E|F)P(F)+P(E|F^c)(1-P(F))}$ using Bayes Theorem.

    $P(F) = 0.7$ $P(E|F)$ has a random variable $X \sim Poi(3)$ while $P(E|F^c)$ has a random variable $Y \sim Poi(6)$

    $P(E|F) = e^{-3}\frac{3^4}{4!} = 0.168031$

    $P(E|F^c) = e^{-6}\frac{6^4}{4!} = 0.133853$

    Therefore $P(F|E) = \frac{(0.168031)(0.7)}{(0.168031)(0.7)+(0.133853)(0.3)} = 0.74549$

17. (a) 1

    (b) 10 hours

    (c) 7,6,11