3-CNF-SAT $\leq_p$ Dir-Ham-Cyc

Transformation illustration

Instance of 3-CNF-SAT

- N variables/literals:
  - $X_1, X_2, X_3, \ldots, X_N$
- K clauses (3-CNF)
  - $\phi = C_1 \wedge C_2 \wedge C_3 \wedge \ldots \wedge C_k$
  - $C_k = X_u \vee X_v \vee X_t$
- Example:

$$\phi = (X_1 \vee X_2 \vee X_4) \wedge (\neg X_2 \vee X_3 \vee \neg X_5) \wedge (\neg X_3 \vee X_4 \vee \neg X_5) \wedge (\neg X_3 \vee X_5 \vee X_6)$$

Instance of Dir-Ham-Cyc

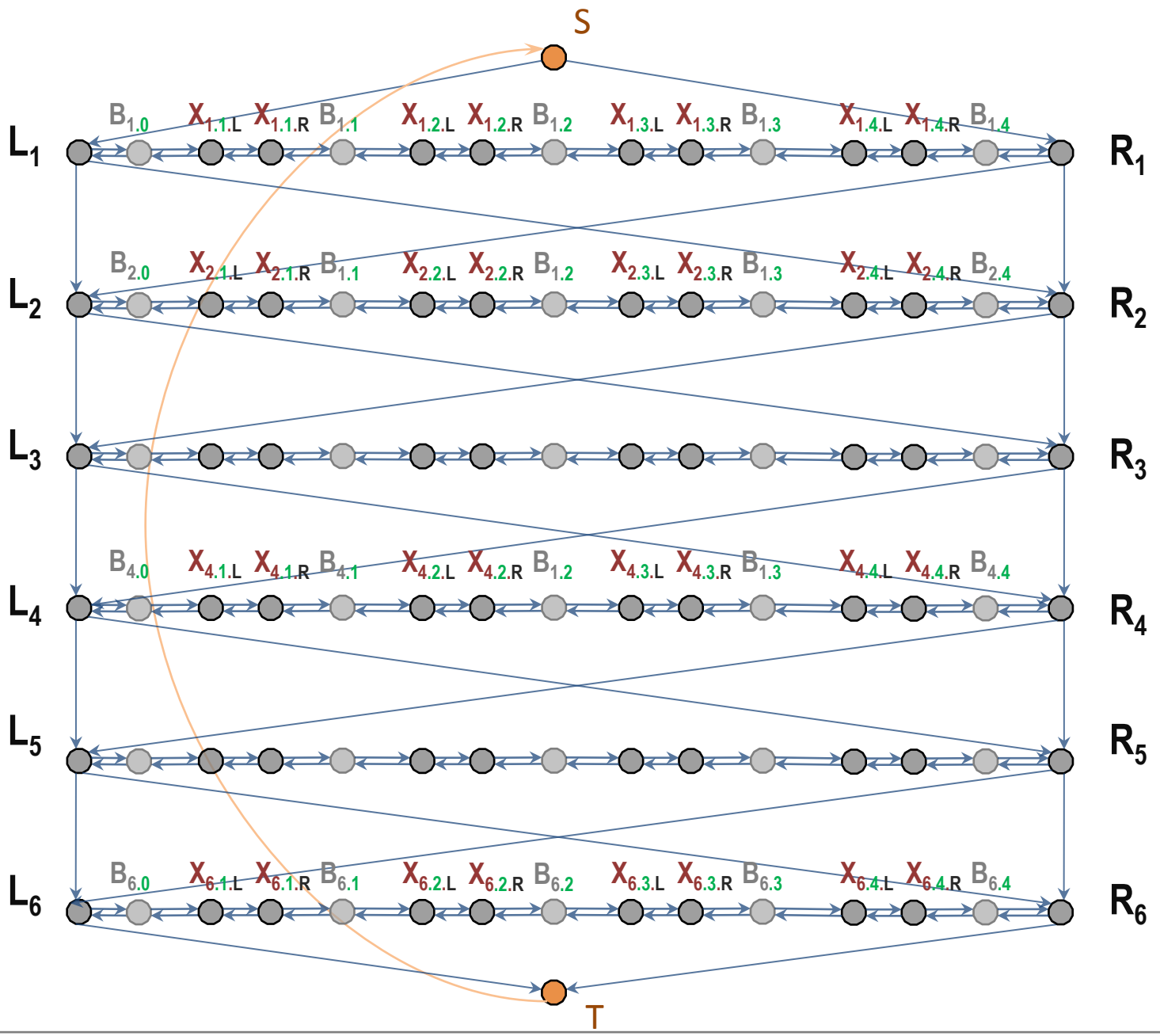
Graph G: *N(3K+3) + K + 2* vertices

- N(3K+3) literal vertices:
  - $L_1, B_{1.0}, X_{1.1.L}, X_{1.1.R}, B_{1.1}, X_{1.2.L}, X_{1.2.R}, \ldots, X_{1.K.L}, X_{1.K.R}, B_{1.K}, R_1$
  - $L_2, B_{2.0}, X_{2.1.L}, X_{2.1.R}, B_{2.1}, X_{2.2.L}, X_{2.2.R}, \ldots, X_{2.K.L}, X_{2.K.R}, B_{2.K}, R_2$
  - $L_N, B_{N.0}, X_{N.1.L}, X_{N.1.R}, B_{N.1}, X_{N.2.L}, X_{N.2.R}, \ldots, X_{N.K.L}, X_{N.K.R}, B_{N.K}, R_N$
- K clausal vertices
  - $C_1, C_2, C_3, \ldots, C_K$
- 2 extra vertices: source S and sink T

Convention:

L: left, R: right, B: buffer, X: literal, C: clause, n: literal index, k: clause index

True/positive: direction from Left to Right; False/negative: direction from Right to Left


S
T
L1
L2
L3
L4
L5
L6
R1
R2
R3
R4
R5
R6
B1.0 X1.1.L X1.1.R B1.1 X1.2.L X1.2.R B1.2 X1.3.L X1.3.R B1.3 X1.4.L X1.4.R B1.4
B2.0 X2.1.L X2.1.R B1.1 X2.2.L X2.2.R B1.2 X2.3.L X2.3.R B1.3 X2.4.L X2.4.R B2.4
B4.0 X4.1.L X4.1.R B4.1 X4.2.L X4.2.R B1.2 X4.3.L X4.3.R B1.3 X4.4.L X4.4.R B4.4
B6.0 X6.1.L X6.1.R B6.1 X6.2.L X6.2.R B6.2 X6.3.L X6.3.R B6.3 X6.4.L X6.4.R B6.4

Edges in G

Edges between literal vertices

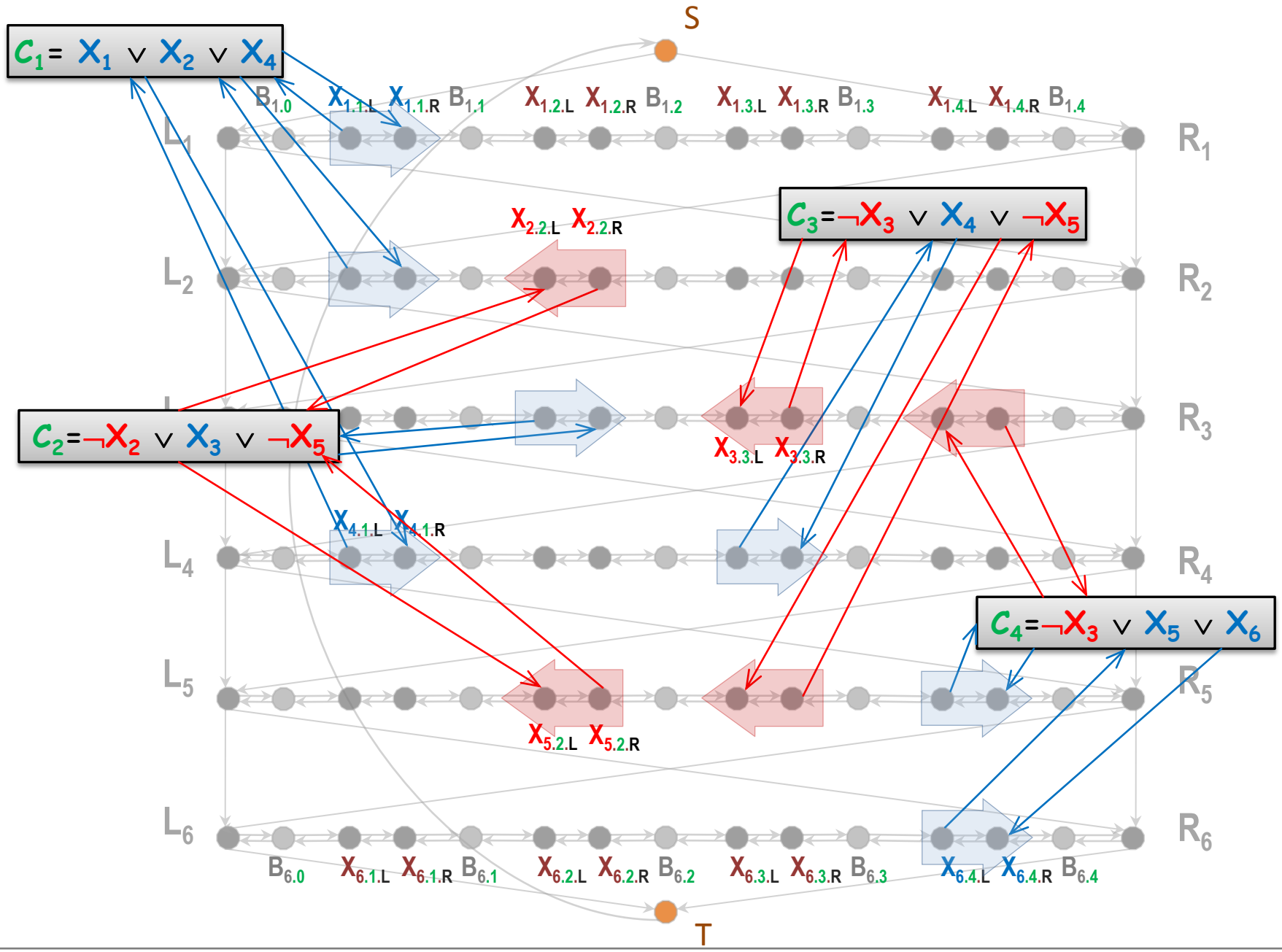
- Horizontal edges: $n = 1..N$
  - $L_n \leftrightarrow B_{n.0} \leftrightarrow X_{n.1.L} \leftrightarrow X_{n.1.R} \leftrightarrow B_{n.1} \leftrightarrow \ldots \leftrightarrow X_{n.K.L} \leftrightarrow X_{n.K.R} \leftrightarrow B_{n.K} \leftrightarrow R_n$
- Vertical edges: $i = 1..(N-1)$
  - $L_i \rightarrow L_{i+1}$ and $L_i \rightarrow R_{i+1}$
  - $R_i \rightarrow L_{i+1}$ and $R_i \rightarrow L_{i+1}$

Edges with S and T

- $S \rightarrow L_1$ and $S \rightarrow R_1$
- $L_N \rightarrow T$ and $R_N \rightarrow T$
- $T \rightarrow S$

Edges in G (2)

- Edges formed by clausal vertices
  - For each clause $C_k = X_n \vee X_m \vee X_p$
  - Connect C_k with $X_{n.k.L}$ and $X_{n.k.R}$
    - If X_n is a positive literal: (dir. Left to Right)
      - Connect $X_{n.k.L} \rightarrow C_k$ and $C_k \rightarrow X_{n.k.R}$
    - If X_n is a negative literal: (dir. Right to Left)
      - Connect $X_{n.k.R} \rightarrow C_k$ and $C_k \rightarrow X_{n.k.L}$
  - Same for X_m and X_p


S
$C_1 = X_1 \vee X_2 \vee X_4$
$C_2 = \neg X_2 \vee X_3 \vee \neg X_5$
$C_3 = \neg X_3 \vee X_4 \vee \neg X_5$
$C_4 = \neg X_3 \vee X_5 \vee X_6$
L_1 L_2 L_3 L_4 L_5 L_6
R_1 R_2 R_3 R_4 R_5 R_6
$B_{1.0}$ $X_{1.1.L}$ $X_{1.1.R}$ $B_{1.1}$ $X_{1.2.L}$ $X_{1.2.R}$ $B_{1.2}$ $X_{1.3.L}$ $X_{1.3.R}$ $B_{1.3}$ $X_{1.4.L}$ $X_{1.4.R}$ $B_{1.4}$
$X_{2.2.L}$ $X_{2.2.R}$
$X_{3.3.L}$ $X_{3.3.R}$
$X_{4.1.L}$ $X_{4.1.R}$
$X_{5.2.L}$ $X_{5.2.R}$
$B_{6.0}$ $X_{6.1.L}$ $X_{6.1.R}$ $B_{6.1}$ $X_{6.2.L}$ $X_{6.2.R}$ $B_{6.2}$ $X_{6.3.L}$ $X_{6.3.R}$ $B_{6.3}$ $X_{6.4.L}$ $X_{6.4.R}$ $B_{6.4}$
T

Example of 3-CNF-SAT

- 3-CNF-SAT:

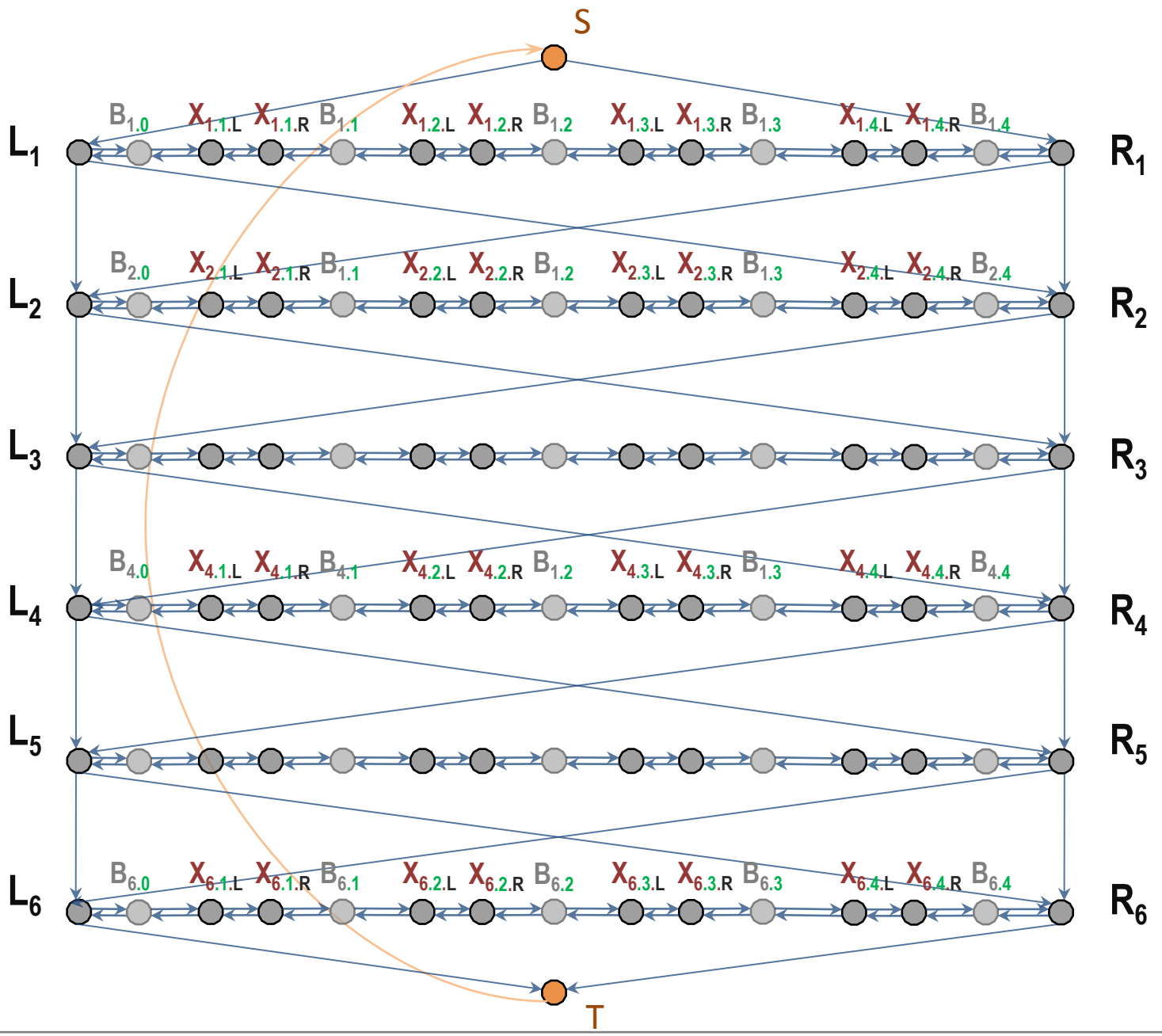
$$\phi = (X_1 \vee X_2 \vee X_4) \wedge (\neg X_2 \vee X_3 \vee \neg X_5) \wedge (\neg X_3 \vee X_4 \vee \neg X_5) \wedge (\neg X_3 \vee X_5 \vee X_6)$$

$C_1 = X_1 \vee X_2 \vee X_4$

$C_2 = \neg X_2 \vee X_3 \vee \neg X_5$

$C_3 = \neg X_3 \vee X_4 \vee \neg X_5$

$C_4 = \neg X_3 \vee X_5 \vee X_6$


S
T
L1
L2
L3
L4
L5
L6
R1
R2
R3
R4
R5
R6
B1.0 X1.1.L X1.1.R B1.1 X1.2.L X1.2.R B1.2 X1.3.L X1.3.R B1.3 X1.4.L X1.4.R B1.4
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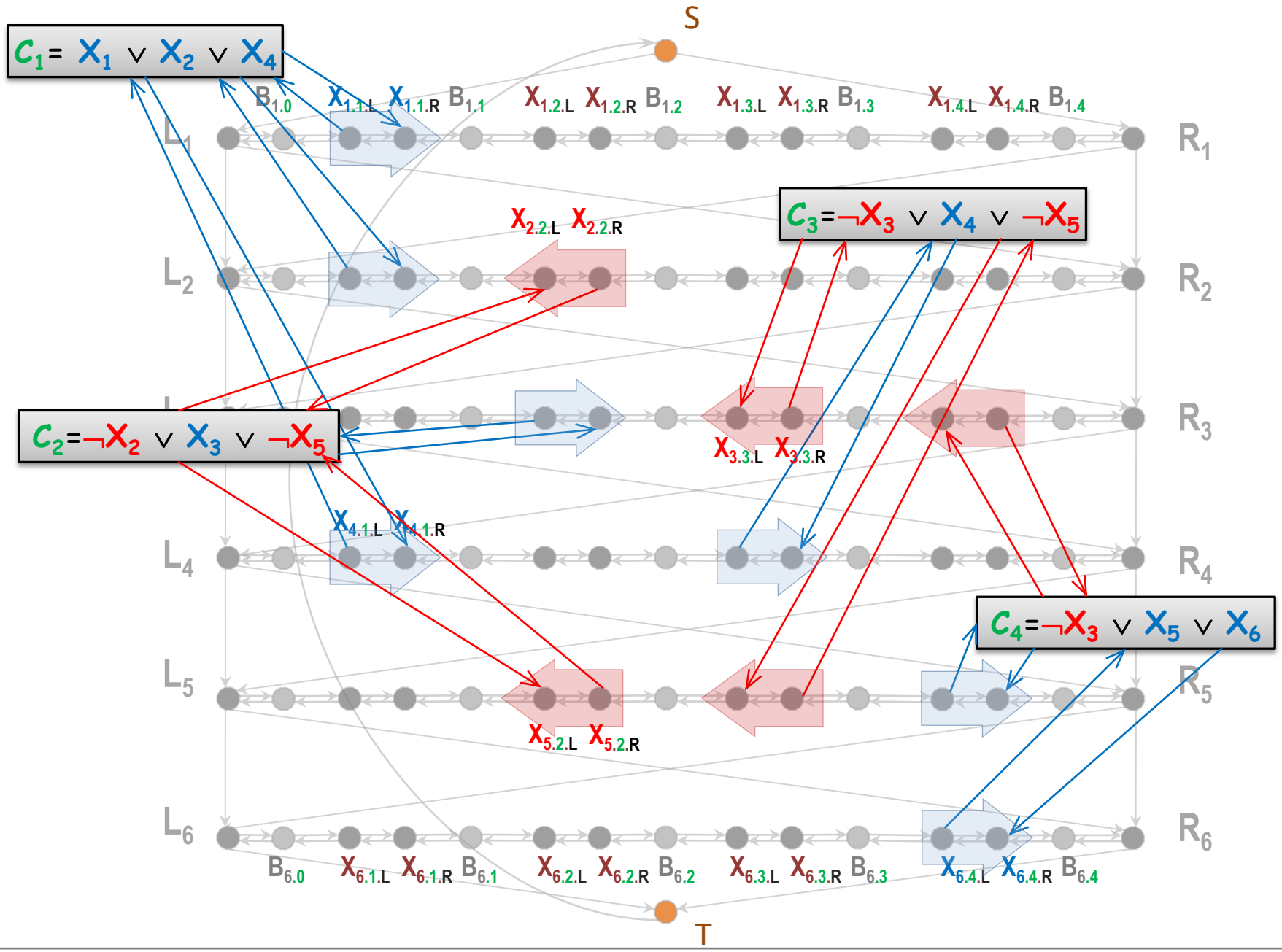
B6.0 X6.1.L X6.1.R B6.1 X6.2.L X6.2.R B6.2 X6.3.L X6.3.R B6.3 X6.4.L X6.4.R B6.4


S
$C_1 = X_1 \vee X_2 \vee X_4$
$C_2 = \neg X_2 \vee X_3 \vee \neg X_5$
$C_3 = \neg X_3 \vee X_4 \vee \neg X_5$
$C_4 = \neg X_3 \vee X_5 \vee X_6$
L_1
L_2
L_3
L_4
L_5
L_6
R_1
R_2
R_3
R_4
R_5
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$B_{1.0}$ $X_{1.1.L}$ $X_{1.1.R}$ $B_{1.1}$ $X_{1.2.L}$ $X_{1.2.R}$ $B_{1.2}$ $X_{1.3.L}$ $X_{1.3.R}$ $B_{1.3}$ $X_{1.4.L}$ $X_{1.4.R}$ $B_{1.4}$
$X_{2.2.L}$ $X_{2.2.R}$
$X_{3.3.L}$ $X_{3.3.R}$
$X_{4.1.L}$ $X_{4.1.R}$
$X_{5.2.L}$ $X_{5.2.R}$
$B_{6.0}$ $X_{6.1.L}$ $X_{6.1.R}$ $B_{6.1}$ $X_{6.2.L}$ $X_{6.2.R}$ $B_{6.2}$ $X_{6.3.L}$ $X_{6.3.R}$ $B_{6.3}$ $X_{6.4.L}$ $X_{6.4.R}$ $B_{6.4}$
T