Near-Optimal Embeddings of Trees into Fibonacci Cubes

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Abstract

The Fibonacci cube network was proposed recently as an alternative to the hypercube network. In this paper, we consider the problems of simulating tree and X-tree structures by the Fibonacci cube. Such problems can be characterized as network embeddings. There is a big gap between the hypercube and the Fibonacci cube with respect to the availability of routing and embedding results. We present near-optimal embedding results, which show that the Fibonacci cube can simulate trees and X-trees almost as efficiently as the hypercube, even though the Fibonacci cube is a network much sparser than the hypercube.

1 Introduction

The Fibonacci cubes are a class of new interconnection networks that are inspired by the Fibonacci numbers [10-13]. Because of the unique features of the Fibonacci numbers [8], these new networks have interesting properties. For a network with N nodes, it has been shown [11] that the diameter, the edge connectivity and the node connectivity of the Fibonacci cube are in the order of $O(\log N)$, which are comparable to the hypercube network, even it is sparser than the hypercube (it contains about $1/5$ fewer edges than the hypercube for the same number of nodes). Also, the size of the Fibonacci cube increases not as fast as the hypercube, namely, the size of the hypercube increases by factor 2 from dimension n to $n+1$, whereas the size of the Fibonacci cube increases by factor less than 1.63 from dimension n to $n+1$. Since a Fibonacci cube is a subgraph of some hypercubes, it could be considered as a hypercube with many faulty nodes and links. Like hypercubes, the Fibonacci cube has very attractive recurrent structures, i.e. a Fibonacci cube can be decomposed into subgraphs which are also Fibonacci cubes. The objective of this paper is to address the relations (in terms of embedding) of Fibonacci cubes with binary trees and X-trees.

As trees capture the essence of divide-and-conquer strategies, tree machines support several useful logarithmic-time functions. Designing efficient parallel algorithms for tree machines has received extensive attention (see[1-3,5,9,14] for examples). Since a tree has only one path between pairs of nodes, if any single node or link fails the network is broken and the whole system fails. In [7,15], it is proposed to add "lateral" connections between offsprings at the same depth, in terms of distance from the root of the tree. The resulting structures are called X-trees. The additional links not only improve the reliability of the system, but also admit more efficient data communication among processors.

An embedding is defined as a function f, which maps the vertices of a guest graph G to the vertices of the host graph H. The important characteristics of an embedding are dilation, and expansion. The dilation of an embedding is defined as the maximum distance between images of the adjacent guest nodes. The expansion of an embedding f is the ratio of the number of nodes in the host to the number of the nodes in the guest. The dilation gives a lower bound for communication delay when H simulates G. The expansion is a measurement for processor utilization.

Much research has been devoted to embeddings to hypercubes. So far, there is a big gap between hypercubes and Fibonacci Cubes in relation to the availability of embedding results. The following embedding results are presented in [4] and [5]:

(1) a linear arrays can be embedded into a Fibonacci cube with dilation 1 and expansion 1;

(2) a cycle with even number of nodes can be embedded into a Fibonacci cube with dilation 1 and expansion 1;

(3) a cycle with odd number of nodes can be embed-

ded into a Fibonacci cube with near dilation 1 (only one edge with dilation 2) and expansion 1;

(4) a 2-D mesh can be embedded in a Fibonacci cube with dilation 2 and expansion 1.

(5) a complete binary tree of height n and a X-tree of height n can be embedded into $FC(2n+2)$ with dilation 3.

In this paper, we consider the problem of embedding trees and X-trees into Fibonacci cubes. Our results considerablly improve the previous results and are near optimum.

2 Fibonacci Cube and Its Properties

A Fibonacci cube of dimension n, denoted by $FC(n)$, is an undirected graph of f_n (which is the n-th Fibonacci number) nodes, each is labeled by a distinct $(n-2)$-bit binary number such that no two 1's occur consecutively. Two nodes in $FC(n)$ are connected by an edge if and only if their labels differ in exactly one bit position. Figure 1 shows $FC(3)$,$FC(4)$,$FC(5)$ and $FC(6)$.

The following are some basic properties of $FC(n)$ [11]:

1. The number of nodes in a Fibonacci cube $FC(n)$ is $O(1.6)^n$.

2. Assume $FC(n) = (V_n, E_n)$, $FC(n-1) = (V_{n-1}, E_{n-1})$, $FC(n-2) = (V_{n-2}, E_{n-2})$, then $V_n = 0V_{n-1} \cup 10V_{n-2}$. For example, $V_5 = 0V_4 \cup 10V_3 = 0\{00, 01, 10\} \cup 10\{0, 1\} = \{000, 001, 010, 100, 101\}$.

3. $FC(n+2)$ can be decomposed into subgraphs that are isomorphic to, respectively, $FC(n+1)$ and $FC(n)$.

4. $FC(n)$ is a subgraph of $H(n-2)$, the hypercube of dimension $n-2$.

5. $H(n)$, the hypercube of dimension n, is a subgraph of $FC(2n+1)$.

3 Embedding Results

Several one-to-one embeddings (simulation) of binary trees and X-trees into Fibonacci cubes are described in this section.

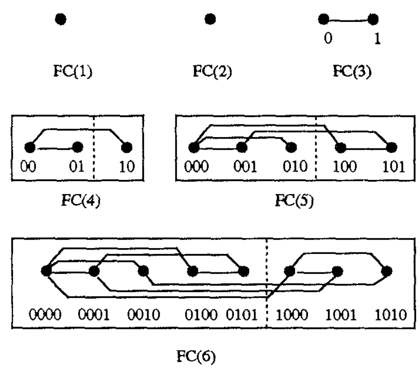

Figure 1: Fibonacci cubes

A complete binary tree of height n, denoted by $B(n)$, has $(2^{n+1}-1)$ nodes and $(2^{n+1}-2)$ edges. One can easily verify the following observations:

Lemma 1 *The smallest Fibonacci cube with at least $2^{n+1}-1$ nodes is $FC(t)$, where $t \approx 1.47n$.*

Lemma 2 *$B(n)$ can be embedded into $FC(2n+3)$ with dilation 2.*

Proof: As $B(n)$ can be embedded into $H(n+1)$ with dilation 2, and $H(n+1)$ is a subgraph of $FC(2(n+1)+1)$, thus $B(n)$ can also be embedded into $FC(2n+3)$ with dilation 2. □

Furthermore, we are able to show that $B(n)$ can be embedded into $FC(2n+2)$ with dilation 2, and $B(n)$ can also be embedded into $FC(1.5n+3)$ with dilation 3. The second result is close to optimum in terms of expansion. It is easy to verify that the expansion factor is optimum when n is less than 15. The inductive technique is used in these embeddings. Before describing the embeddings, we need to define the following notations.

Consider $FC(n+2)$ as $G = (V, E)$, where $V = \{X_nX_{n-1}\cdots X_2X_1 \mid X_i \in \{0,1\} \text{ and } X_i * X_{i+1} = 0\}$.

(1) Let b be any binary string, $bG = (bV, E)$, where $bV = \{bX_nX_{n-1}\cdots X_2X_1 \mid X_i \in \{0,1\} \text{ and } X_i * X_{i+1} = 0\}$.

(2) $(a+b)V = aV \cup bV$, where a and b are binary strings.

(3) $G^f = (E^f, V^f)$, where $V^f = \{X_n010X_{n-1}\cdots X_2X_1 \mid X_nX_{n-1}\cdots X_2X_1n \in V\}$, $E^f = \{(X_n010X_{n-1}\cdots X_2X_1,\ Y_n010Y_{n-1}\cdots Y_2Y_1) \mid (X_nX_{n-1}\cdots X_2X_1, Y_nY_{n-1}\cdots Y_2Y_1 \in E\}$.

Lemma 3 *Let r be the root node of $B(k)$ and f be an embedding of $B(k)$ to $FC(n)$ $(n \geq 4)$ with dilation*

d_0. If the leftmost bit of r is 0, then $B(k+j)$ can also be embedded into $FC(n+2j)$ with dilation $d = \max\{d_0, 2\}$.

Proof: Assume that $B(k)$ can be embedded into $FC(n)$ with dilation d_0 by embedding f, and the root r is mapped into $0a$, where $0a$ belongs to V_n. Note by definition the following is true:

$$V_{n+2} = 0V_{n+1} + 10V_n = 00V_n + 10V_n + 010V_n.$$

Our proof technique is illustrated for the embedding of $B(k+1)$ into $FC(n+2)$.

The root of $B(k+1)$ is mapped to $010a$, and its two children are mapped to $00a$ and $100a$, respectively (note: they are distance 1 and 2 apart in $FC(n+2)$). Using embedding f to map two subtrees of $B(k+1)$ into $00V_n$ and $10V_n$ with dilation d_0, one can obtain an embedding of $B(k+1)$ into $FC(n+2)$ with dilation $d = \max\{d_0, 2\}$. Figure 2 illustrates this embedding. Note that the leftmost bit of the new root $010a$ is still 0. Therefore, one can use this embedding repeatedly without increasing the dilation. □

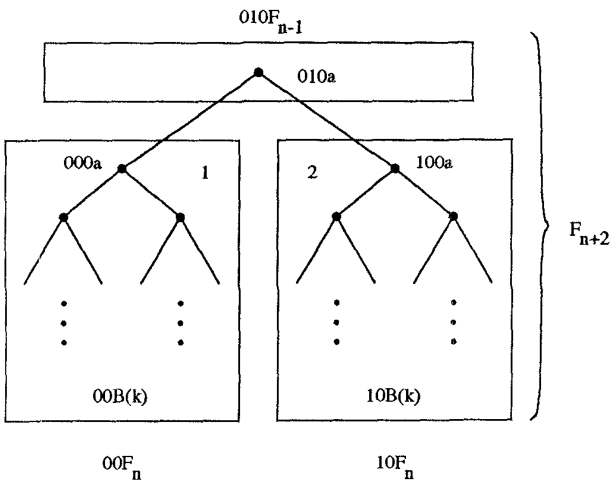

Figure 2: Embedding of $B(k+1)$ into $FC(n+2)$.

Theorem 1 The complete binary tree $B(k)$, $k \geq 2$, can be embedded into Fibonacci cube $FC(2k+2)$ with dilation 2.

Proof: Inductive embedding technique is used.

Basis: Figure 3 describes an embedding of $B(2)$ into $FC(6)$ with dilation 2, and the root is mapped to 0100 whose leftmost bit is 0.

Inductive step: Using Lemma 3, one can embed $B(2+j)$ into $FC(6+2j)$ with dilation $d = \max\{2, 2\} = 2$. This implies $B(k)$ can be embedded into $FC(2k+2)$ with dilation 2 for $k \geq 2$. □

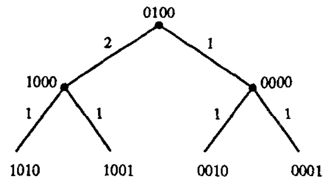

Figure 3: Embedding of $B(k+1)$ into $FC(n+2)$.

It is easy to verify that the expansion of embedding $B(k)$ into $FC(2k+2)$ is $O(1.3)^k$. Can we reduce the expansion factor without increasing the dilation too much? The following embedding gives a positive answer.

Consider a complete double rooted tree of height k, denoted by $DRB(k)$, which consists of 2^{k+1} nodes. Figure 4 illustrates $DRB(2)$.

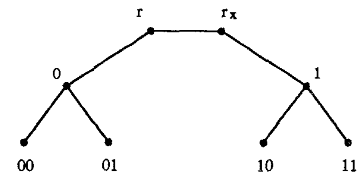

Figure 4: Embedding of $B(k+1)$ into $FC(n+2)$.

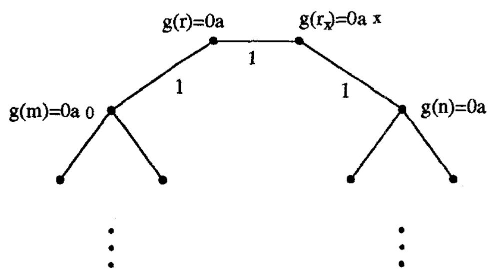

Figure 5: Embedding of $B(k+1)$ into $FC(n+2)$.

Lemma 4 Assume that $DRB(k)$ can be embedded into $FC(n)$ for $n \geq 4$ by an embedding g with dilation d_0. $DRB(k+2j)$ or $B(k+2j)$ can be embedded into $FC(n+3j)$ with dilation $d = \max\{d_0, 3\}$ if the following two conditions are true:

(1) $g(r)$, $g(r_x)$, $g(m)$, $g(n)$ all begin with 0 (r, r_x, m, n are defined in Figure 5);

(2) the distance from $g(m)$ to $g(r)$, $g(r)$ to $g(r_x)$, and $g(r_x)$ to $g(n)$ are all 1.

Proof: By definition, we know:

$$\begin{aligned} V_{n+3} &= 0V_{n+2} + 10V_{n+1} \\ &= (00+10)V_{n+1} + 010V_n \\ &= (000+010+100)V_n + 0010V_{n-1} \\ &\quad +10100V_{n-2} + 101010V_{n-3} \\ &= (000+010+100)V_n \\ &\quad +V_n^f + 101010V_{n-3} \end{aligned}$$

where $V_n^f = \{b_{n-1}010b_{n-2}\cdots b_3b_2 \mid b_{n-1}b_{n-2}\cdots b_3b_2 \in V_n\}$, as defined before.

Let g be a dilation d_0 embedding of $DRB(k)$ into $FC(n)$, and g satisfies (1) and (2) described in Lemma 4 (see Figure 5., note a_0 and a differ in exactly one position, so do a and a_x, a_x and a_1).

Using above formula and embedding g, one can map four $DRB(k)$s into four different $FC(n)$'s in $FC(n+3)$ with dilation d_0. This is illustrated in Figure 6.

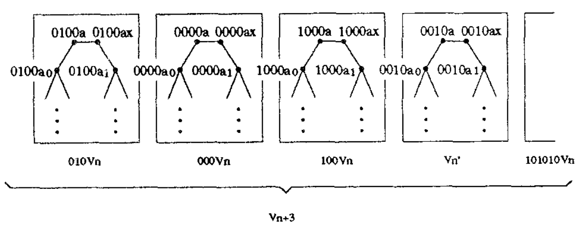

Figure 6: Embedding of $B(k+1)$ into $FC(n+2)$.

By rearranging the nodes in the top two levels of the four $DRB(k)$'s, we can obtain an embedding of $DRB(k+2)$ into $FC(n+3)$ with minimized distances between images of these nodes. Figure 7 illustrates this embedding. One can easily verify that the dilation of this embedding is $d = \max\{d_0, 3\}$. This embedding can be better seen in Figure 8 which is the same as Figure 7.

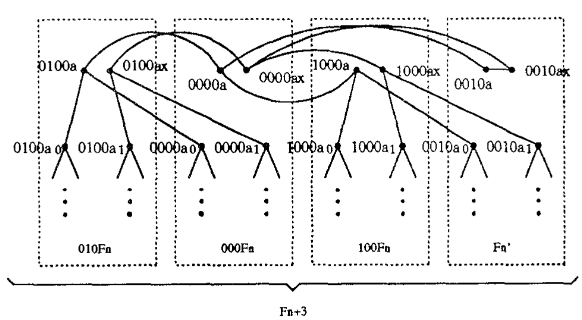

Figure 7: Embedding of $B(k+1)$ into $FC(n+2)$.

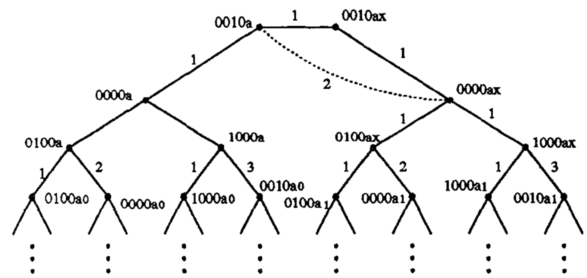

Figure 8: Embedding of $B(k+1)$ into $FC(n+2)$.

Note that the new embedding of $DRB(k+2)$ also satisfies the conditions given in Lemma 4. Thus, this embedding can be used repeatedly without increasing the dilation. Moreover, if one wants to embed complete binary tree $B(k+2)$ into $FC(n+3)$, it can be done by linking the new root (left root) with the son of the right root as described in Figure 8. The number on each edge is the dilation over that edge. □

Theorem 2 Double rooted tree $DRB(2k)$ or complete binary tree $B(2k)$ can be embedded into $FC(3k+3)$ with dilation 3.

Proof: Induction is used.

Basis: Figure 9 describes a dilation 2 embedding of $DRB(2)$ (or $B(2)$) into $FC(6)$. Note that this embedding satisfies the conditions given in Lemma 4.

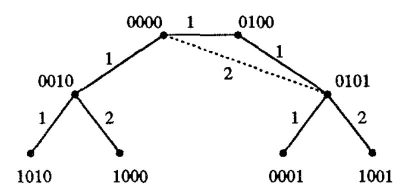

Figure 9: Embedding of $B(k+1)$ into $FC(n+2)$.

Inductive step: By Lemma 4, one can embed $DRB(2+2j)$ (or $B(2+2j)$) into $FC(6+3j)$ with dilation $d = \max\{2,3\} = 3$. That is, $DRB(2k)$ (or $B(2k)$) can be embedded into $FC(3k+3)$ with dilation 3. □

Theorem 3 $B(2k+1)$ can be embedded into $FC(3k+5)$ with dilation 3.

Proof: From Theorem 2, one can embed $B(2k)$ to $FC(3k+3)$ with dilation 3. Using Lemma 4 again, we can map $B(2k+1)$ to $FC((3k+3)+2) = FC(3k+5)$ with dilation 3. □

Theorem 4 $B(k)$ can be embedded into $FC(1.5k+3)$ with dilation 3.

Proof: Directly from Theorem 3 and Theorem 4. □

Embeddings of X-tree to Fibonacci cubes can be obtained using the similar technique. The X-tree of height n, denoted by $X(n)$, is a graph obtained from $B(n)$ by adding edges to connect the nodes in the same level of $B(n)$ to form a path from left to right.

Theorem 5 $X(n)$ can be embedded into $FC(2n+2)$ with dilation 2. $X(n)$ can also be embedded into $FC(1.5n+3)$ with dilation 3.
Proof: Similar inductive embedding techniques can be used. In our inductive step, the embedding of right side sub-tree is always turned 180 degrees. Thus the distance of cross edges in the X- tree will always be kept 1, as the nodes involved are mirror image of each other. For brevity, the details of the proof are omitted. □

4 Final Remarks

We presented several near optimal tree embedding results. We pose a couple of open problems. Can $B(n)$ and $X(n)$ be embedded into $FC(1.5n+3)$ with dilation 2? Can embeddings of d-dimensional meshes into Fibonacci cubes be obtained with dilations and expansions comparable to those of the embeddings of d-dimensional meshes into hypercubes? Can a Fibonacci cube be embedded into a hypercube with constant dilation and expansion less than 2?

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