# $C^1$-Approximation of Seafloor Surfaces With Large Variations

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In many problems of geophysical interest, when trying to describe surfaces,one has to deal with data that exhibit rapid variations. This occurs for instance when describing the topography of mountain ranges, volcanoes, seafloor surfaces (bathymetry maps), islands, or the shape of geological entities, that can present large and rapid variations due for instance to the presence of faults in the structure. The correct description of such geological surfaces, by a fitting process from a given set of points, is therefore of great importance. Usual methods give good results in the case of curve fitting, but less accurate results in the case of surface fitting. The new method we propose here uses scale transformations (spline under tension), and is applied without any particular a priori knowledge of the data. We first propose a construction of these scale transformations families, and then show the efficiency of this innovative approach by applying it to seafloor surfaces around the Big Island in Hawaii in order to get a regular surface with at least continuity of the first derivatives.

## DESCRIPTION OF THE METHOD

The method we propose uses two scale transformations, namely $\varphi_d$ for the pre-processing and $\psi_d$ for the post-processing. The first one, $\varphi_d$, is used to transform the $z$-values representing the height of the unknown surface $f$ into values $(u_i)$, regularly distributed in an interval chosen by the user. The preprocessing function $\varphi_d$ is such that the transformed data do not exhibit large local variations, and therefore a usual spline operator $T^d$ can subsequently be applied without generating significant oscillations. The second scale transformation $\psi_d$ is then applied to the approximated values to map them back and obtain the approximated values of $z$. It is important to underline that the proposed scale transformations do not create spurious oscillations. Moreover, this method is applied without any particular knowledge of the location of the large variations in the dataset.

Let us consider a dataset $\left(x_i^d, z_i^d\right)_{i=1,..,N(d)}^d$ indexed with a real $d$, such that when $d$ tends to 0, the number of data points $N(d)$ tends to infinity. For the purpose of a theoretical study of the convergence of the approximation, we introduce a function $f : \Omega \longrightarrow [a, b]$, such that the dataset becomes $\left(x_i^d, z_i^d = f(x_i^d)\right)_{i=1,..,N(d)}^d$. The functions introduced above have the following expression, for $m \in \mathbb{N}$:

- $\varphi_d : [a, b] \longrightarrow [\alpha, \beta] \subset \mathbb{R}$,
- $T^d : (\varphi_d \circ f) \in H^m(\Omega, [\alpha, \beta]) \to T^d(\varphi_d \circ f) \in H^m(\Omega, [\alpha, \beta])$,
- $\psi_d \circ \left(T^d(\varphi_d \circ f)\right)$,

where the pre-processing $\varphi_d$ and the post-processing $\psi_d$ are continuous scale transformations families, where $T^d$ is an approximation operator, for instance a spline, and where $H^m(\Omega, .)$ denotes the usual Sobolev space. More precisely, we introduce a bounded non empty connected set $\Omega$ with a Lipschitz-continuous boundary of $\mathbb{R}^2$, and an unknown function $f \in H^{m'}(\Omega, [a, b])$ that we want to approximate. We also introduce the set $Z_1^d$ of $N = N(d)$ real numbers such that

$$\forall x_i^d \in A^d, f(x_i^d) \in Z_1^d,$$

and the sequence $Z_2^d$ of $p(d)$ distinct $z$-values obtained from the ordering of $Z_1^d$, $\forall \tilde{z}_i^d \in Z_2^d, i = 1, ..., p(d)$,

$$a = \tilde{z}_1^d < \tilde{z}_2^d < \tilde{z}_3^d < ... < \tilde{z}_{p(d)-1}^d < \tilde{z}_{p(d)}^d = b,$$

where $[a, b] = Im(f)$.

The sequence $Z_2^d$ will be used for the construction of the scale transformation families in the following section. In what follows, for convenience, we also write $(z_i)$ instead of $\left(\tilde{z}_i^d\right)$.

The scale transformations $\varphi_d$ and $\psi_d$ are one dimensional spline under tension satisfying

$$\varphi_d(z_i) = u_i\,, \qquad \forall i = 1, ..., p(d);$$
$$\psi_d(u_i) = z_i, \qquad \forall i = 1, ..., p(d),$$

where the $\{u_i\}_{i=1,...,p(d)}$ are a regular subdivision of $Imf = [\alpha, \beta]$.

These interval and subdivision are chosen by the user. When dealing with surface approximation from rapidly varying data, we choose the interval to be $[0, 1]$, and an even subdivision of the $\{u_i\}$ that is used to reduce the local variations of the $(z_i)$. After applying $\varphi_d$, we obtain a new dataset $(x_i, u_i)$ related to the initial data by $u_i = \varphi_d(z_i)$. We recall that after the pre-processing, the large local variations in the dataset have been drastically reduced, therefore it is possible to approximate the data using a usual spline operator $T^d$ without generating significant oscillations. To map these values back and obtain the approximated values of $z$, we need to use a post-processing step, and therefore need to introduce a family $(\psi_d)$, which is almost the inverse of $(\varphi_d)$.

*The smoothing spline operator*

Given a Lagrange dataset $(x_i, (\varphi_d \circ f)(x_i) = \varphi_d(z_i))$, we have to solve the classical problem of constructing an approximant $T^d$ of class $C^k$ (with $k = 1$ or 2 in practice). In this work, we use a smoothing $D^m$–spline, as defined in [1], which has many advantages: it is possible to implement a local refinement, the matrix of the linear system is banded, and it is possible to study the convergence of the approximation. We have chosen to use a smoothing $D^m$–spline and not an interpolation spline because we want to be able to work with large datasets of up to several hundreds of thousands of points, and in that case, a smoothing spline is far less expensive than an interpolation spline. In order to compute $\sigma_\varepsilon^d$, we choose to discretize it on a finite element basis, which enables us to obtain a small sparse linear system. In what follows, we use either the BFS of class $C^0$ or of class $C^1$ in order to obtain a $C^0$ or $C^1$ approximant.

## APPLICATION TO THE TOPOGRAPHY AND BATHYMETRY OF COASTAL AREAS IN HAWAII

The topography and bathymetry of the Hawaiian Islands in the Pacific ocean result from the activity of a huge hot spot combined with the effect of erosion. This hot spot has been more or less active since the Late Cretaceous, and as a result the Big Island continues to grow, and to the East a new island is being formed. The hot spot is stationary, but the Pacific plate is moving at a rate of about 3 inches per year, which results in the well-known pattern of older eroded islands to the West, and more recent islands to the East. The next island in Hawaii's volcanic history already has a name: Loihi. With the current rate of volcanic activity, Loihi is expected to surface sometime in the next 10,000 years. Just 3,180 feet (969 m) below the surface, the rising seamount has already built up over 15,000 feet from the ocean floor. The maximum height of the Big Island is 4.7~km, and the depth of the seafloor reaches more than 4~km in several places.

Being able to describe the topography of such regions exhibiting rapid local variations with at least $C^0$ regularity, or even $C^1$ regularity, is important in many fields in geophysics. For example, this description of the topography can be an input to numerical modeling codes that study the propagation of pyroclastic flows or lava flows, and related hazard; other examples are seismic site effects and ground motion amplification due to topographic features. In both cases, to avoid creating numerical artefacts, it is very important not to introduce spurious oscillations in the description of the model itself. Otherwise, [7] and [8] underlined for instance in the context of curvilinear spectral-element modeling of elastic wave propagation that artificial diffraction points appear at the edges between elements, which significantly affects the behavior of surface waves.

To demonstrate the efficiency of our method, we create $C^0$ and $C^1$ approximants from a set of 8000 data points of the Big Island area. The data points in the DEM have been obtained by digitizing a map of the seafloor. In this DEM, the height is given on an evenly spaced grid of $80 \times 100$ points This DEM is shown in Fig. 1 using a top view with isocontours representing the height of the topography.

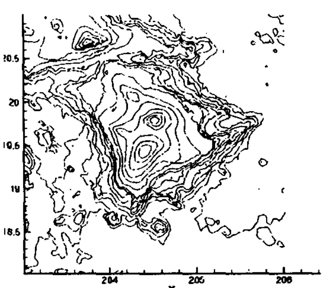


Figure 1 – 2D view of the data

In the pre-processing, we choose a regular distribution of the $\{u_i\}$ in $[\alpha, \beta] = [0, 1]$ in order to reduce the large variations in the dataset. The approximants are subsequently obtained by discretizing the $D_m$-spline in a finite-element space. In the case of the $C^1$ approximant, we use $15 \times 20$ rectangular $C^1$-BFS finite elements, each having sixteen degrees of freedom. In both cases, the smoothing parameter $\varepsilon$ is taken to be $10^{-6}$.

In Fig.2 , we show a 3-D representation of the $C^1$

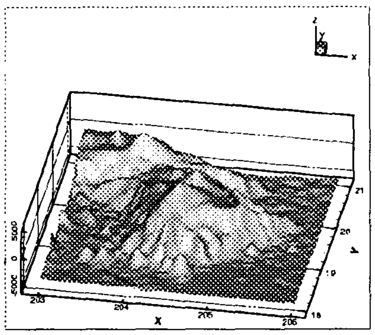

Figure 2 – 3D view of the approximant

approximant after post-processing, evaluated on an evenly spaced grid comprising $200 \times 200$ points. To compare this approximant to the original dataset more precisely, in Fig. 3,

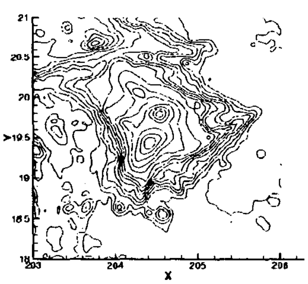

Figure 3 – 2D view of the approximant

we present a top view of the approximated values, with isocontours representing the height every 0.2 km, in addition to the same plot for the original dataset, as in Fig.1 . It is clear from these plots that the approximant is very close to the original data, with local variations smoothed as expected. More detailed studies of the approximation error, and evidence that the rate of convergence is higher in this method than in usual approaches with no pre-processing, such as splines under tension or thin plate splines, can be found in [4].


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