H. J. Thim Trust's

THEEM COLLEGE OF ENGINEERING

Assignment - II

Date : ________

Q1. Find the z-transform of $f(k) = \left\{ \left(\frac{1}{3}\right)^{|k|} \right\}$

$\Rightarrow$ Since $f(k) = \left(\frac{1}{3}\right)^{|k|}$

$$\therefore |k| = k \qquad k \geq 0$$
$$= -k \qquad k \leq 0$$

$$f(k) = \left(\frac{1}{3}\right)^{k} \qquad k \geq 0$$

$$= \left(\frac{1}{3}\right)^{-k} \qquad k \leq 0$$

Since $z\left\{f(k)\right\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k}$

$$= \sum_{k=-\infty}^{-1} \left(\frac{1}{3}\right)^{-k} z^{-k} + \sum_{k=\infty}^{\infty} \left(\frac{1}{3}\right)^{k} z^{-k}$$

$$= \left[\dots + \left(\frac{1}{3}\right)^{3} z^{3} + \left(\frac{1}{3}\right)^{2} z^{2} + \left(\frac{1}{3}\right) z \right] + \left[1 + \frac{1}{3} z^{-1} + \right.$$

$$\left. \left(\frac{1}{3}\right)^{2} z^{-2} + \left(\frac{1}{3}\right)^{2} z^{-3} + \dots \right)$$

$$= \left[\frac{z}{3} + \frac{z^2}{3^2} + \frac{z^3}{3^3} + \cdots\right] + \left[1 + \frac{1}{3z} + \frac{1}{(3z)^2} + \frac{1}{(3z)^3} + \cdots\right]$$

$$= \frac{z}{3}\left[1 + \frac{z}{3} + \left(\frac{z}{3}\right)^2 + \cdots\right]\left[1 + \left(\frac{1}{3z}\right) + \left(\frac{1}{3z}\right)^2 + \cdots\right]$$

$$= \frac{z}{3} \cdot \frac{1}{1-(z/3)} + 1 \cdot \frac{1}{1-[1/(3z)]}, \quad \left|\frac{z}{3}\right| < 1, \quad \left|\frac{1}{3z}\right| < 1$$

$$= \frac{z}{3} \cdot \frac{3}{3-z} + \frac{3z}{3z-1} = \frac{z}{3-z} + \frac{3z}{3z-1}, \quad |z| < 3, \quad \frac{1}{3} < |z|$$

$$= \frac{3z^2 - z + 9z - 3z^2}{(3-z)(3z-1)} = \frac{8z}{(3-z)(3z-1)}, \quad \frac{1}{3} < |z| < 3$$

2) Find the z-transform of $f(k) = \begin{cases} b^k, & k < 0 \\ a^k & k \geq 0 \end{cases}$

$\Rightarrow$ Since $Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k}$

$$= \sum_{k=-\infty}^{0} f(k) z^{-k} + \sum_{k=0}^{\infty} f(k) z^{-k}$$

$$= \sum_{k=-\infty}^{0} b^k z^{-k} + \sum_{k=0}^{\infty} a^k z^{-k}$$

$$= \left(\ldots + b^{-3} z^{3} + b^{-2} z^{2} + b^{-1} z^{1} + b^{0} z^{0}\right) + \left(a^{0} z^{0} + b^{1} z^{-1} + b^{2} z^{-2} + b^{3} z^{-3} + \ldots\right)$$

$$= \left(1 + \frac{z}{b} + \frac{z^2}{b^2} + \frac{z^3}{b^3} \ldots\right) + \left(1 + \frac{a}{z} + \frac{a^2}{z^2} + \frac{a^3}{z^3} \ldots\right)$$

$$= \left(1 - \frac{z}{b}\right)^{-1} + \left(1 - \frac{a}{z}\right)^{-1}$$

$$= \left(\frac{b-z}{b}\right)^{-1} + \left(\frac{z-a}{z}\right)^{-1}$$

$$= \frac{b}{b-z} + \frac{z}{z-a}$$

$$= \frac{bz - ab + bz - z^2}{(b-z)(z-a)} = \frac{2bz - z^2 - ab}{bz - ab - z^2 + az}$$

$$= \frac{2bz - z^2 - ab}{bz - ab - z^2 + az}$$

Region of convergence

$$\left|\frac{z}{b}\right| < 1 \ \& \ \left|\frac{a}{z}\right| < 1$$

$$|z| < b \quad \& \quad a < |z|$$

Q3) Find the z-transform of $f(k) = \frac{1}{3^k}$ & $\frac{1}{5^k}$

$\Rightarrow$ Let $f(k) = \frac{1}{3^k}$ & $g(k) = \frac{1}{5^k}$ $k \geq 0$

Since $z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k} = \sum_{k=0}^{\infty} \frac{1}{(3z)^k}$

$$z\{f(k)\} = \sum_{k=-\infty}^{-1} f(k) z^{-k} + \sum_{k=0}^{\infty} f(k) z^{-k}$$

$$= 0 + \sum_{k=0}^{\infty} \frac{1}{3^k} z^{-k}$$

$$= \frac{1}{0\ (3z)^k} = \sum_{k=0}^{\infty} \frac{1}{(3z)^k}$$

$$= \frac{1}{1} + \frac{1}{3z} + \frac{1}{(3z)^2} + \frac{1}{(3z)^3} + \ldots$$

$$= \frac{1}{1-(1/3z)} \quad \left[\because S_\infty = \frac{a}{1-r} \text{ and } |3z| > 1 \right]$$

$$\therefore z\left\{\frac{1}{3^k}\right\} = \frac{3z}{3z-1} \quad \boxed{|z| > \frac{1}{3}}$$

H. J. Thim Trust's

THEEM COLLEGE OF ENGINEERING

Date :

$$g(k) = \frac{1}{5^k} \quad k \geq 0$$

Since $Z\{g(k)\} = \sum_{k=0}^{\infty} g(k) z^{-k} = \sum_{k=0}^{\infty} \frac{1}{(5z)^k}$

$$\therefore Z\{g(k)\} = \sum_{k=-\infty}^{-1} g(k) z^{-k} + \sum_{k=0}^{\infty} g(k) z^{-k}$$

$$= 0 + \sum_{k=0}^{\infty} \frac{1}{5^k} z^{-k}$$

$$= \frac{1}{(5z)^k} = \sum_{k=0}^{\infty} \frac{1}{(5z)^k}$$

$$= 1 + \frac{1}{5z} + \frac{1}{(5z)^2} + \frac{1}{(5z)^2} + \dots$$

$$= \frac{1}{1-(1/5z)} \quad \left[\because S_\infty = \frac{a}{1-r} \text{ and } |5z| > 1\right]$$

$$\therefore Z\left\{\frac{1}{5^k}\right\} = \frac{5z}{5z-1} \quad \boxed{|z| > \frac{1}{5}}$$

= 352

∴ By convolution theorem

$$Z\{f(k) * g(k)\} = F(z) \cdot G(z) = \left(\frac{3z}{3z-1}\right)\left(\frac{5z}{5z-1}\right),$$

$$|z| > \frac{1}{3}.$$

Q4) Find z-transform of $f(k) = a^k \cos bk$, $k \geq 0$

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Date : ____________

Q8) The number of accidents in a year attributed to taxi drivers in a city follows ~~poision~~ poison distribution with mean 3 out of 1,000 taxi drivers find approximately the number of drivers with

i) No accident in a year

ii) more than 3 accident in a year

iii) Atmost two accident in a year

$\Rightarrow$ Since $m=3$ $N=1{,}000$

(i) $P(X=0) = \dfrac{e^{-m} \times m^{x}}{x!}$, $x = 0,1,2 \ldots$

$= \dfrac{e^{-3} \times (3)^{0}}{0!} = 0.049$

(ii) $P(X>3) = P(X = 4,5,6 \ldots)$

$= 1 - P(X = 0,1,2)$

$= 1 - \sum_{x=0}^{3} \dfrac{e^{-3}(3)^{x}}{x!}$

$= 1 - 0.6472$

$= 0.3527$

Expected accident in the year $= Np = 1000 \times 0.3527$

$= 352$

Q12) Samples of two types of electric bulbs were tested for length of life and the following data were obtained

	Type I	Type II
No of Samples	8	7
Mean of the Samples	1134	1024
Standard deviation	35	40

⟹ We have $\overline{X_1} = 1134$ $\overline{X_2} = 1024$ $s_1 = 35$, $s_2 = 40$, $n_1 = 8$ $n_2 = 7$

(i) Null Hypothesis $H_0 : \mu_1 = \mu_2$
Alternative Hypothesis $H_a = \mu \neq \mu_2$

(ii) Calculation of test statistics :- Since the sizes of the samples are small we use t-distribution
The unbiased estimate of the common population is given by

$$S_p = \sqrt{\frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}} = \sqrt{\frac{8 \times 35^2 + 7 \times 40^2}{8 + 7 - 2}} = \sqrt{\frac{21000}{13}}$$

$$= \sqrt{1615.38} = 40.19$$

The standard errors of the difference between the two mean is given

$$S.E = S_p \times \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} = 40.19 \sqrt{\frac{1}{8} + \frac{1}{7}} = 20.8$$

$$\therefore t = \frac{\bar{X}_1 - \bar{X}_2}{S.E} = \frac{1134 - 1024}{20.8} = \frac{110}{20.8} = 5.288$$

(iii) Level of significance : $\alpha = 0.05$

(iv) Critical value :- The table value of t at $\alpha = 0.05$ for $v = 8+7-2$ $= 13$ degrees of freedom is $t_\alpha = 2.16$

(v) Decision :- Since the computed value $|t| = 5.68$ is greater than the table value $t_\alpha = 2.16$, the hypothesis is rejected

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Date :

Q9) The incomes of a group of 10,000 persons were found to be normally distributed with mean ₹520 & standard deviation ₹60

(i) Find the number of persons having incomes ₹400 and 550

(ii) the lowest of the richest 500

⟹ Since $N = 10{,}000$, $m = 520$

$\delta = 60$

$$Z = \frac{X - m}{\delta}$$

(i) $P(400 < X < 520)$

$X = 400 \quad \therefore Z = \frac{400 - 520}{60} = -2$

$X = 520 \quad \therefore Z = \frac{520 - 520}{60} = 0$

$P(400 < X < 520) = P(-2 < Z < 0)$

$=$ area between $(z = -2$ to $0)$

Q 7) Using convolution find inverse z-transform

(i) $\dfrac{z^2}{(z-1)(z-3)}$ (ii) $\dfrac{z}{(z-1)(z-2)}$

$\Rightarrow$ (i) We have $\dfrac{z^2}{(z-1)(z-3)} = \dfrac{z}{z-1} \cdot \dfrac{z}{z-3}$

Let $F(z) = \dfrac{z}{z-1}$ and $G(z) = \dfrac{z}{z-3}$ $\therefore H(z) = \dfrac{z^2}{(z-1)(z-3)}$

Hence $H(z) = F(z)\,G(z)$

Taking inverse z-transform of both sides

$$z^{-1}[H(z)] = z^{-1}[F(z)\cdot G(z)]$$

$$\therefore z^{-1}[F(z)\,G(z)] = z^{-1}[H(z)] = z^{-1}\{z\{h(k)\}\}$$

$$= \{h(k)\} = \{f(k)\} * \{g(k)\}$$

Now, if $F(z) = \dfrac{z}{z-1}$ and $G(z) = \dfrac{z}{z-3}$

$$\{f(k)\} = z^{-1}\{F(z)\} = z^{-1}\left[\frac{z}{z-1}\right] = \{1^k\},\ k \geq 0$$

H. J. Thim Trust's
THEEM COLLEGE OF ENGINEERING

Date : ____________

$$\{g(k)\} = Z^{-1}\{G(z)\} = Z^{-1}\left[\frac{z}{z-3}\right] = \{3^k\},\ k \geq 0$$

Now, $Z^{-1}[F(z)G(z)] = \{f(k)\} * \{g(k)\}$

$$\therefore Z^{-1}[F(z)G(z)] = \sum_{0}^{k} f(m)\, g(k-m) =$$

$$= \sum_{m=0}^{k} (1)^m (3)^{k-m}$$

$$= 3^k \sum_{m=0}^{k} (1)^m (3)^{-m}$$

$$= 3^k \sum_{m=0}^{k} 1 \cdot 3^{-m}$$

$$= 3^k \sum_{m=0}^{k} \left(\frac{1}{3}\right)^m$$

$$\therefore Z^{-1}[F(z)G(z)] = 3^k \left[\frac{(1/3)^{k+1} - 1}{(1/3) - 1}\right] \quad (G.P)$$

$$= 3^k \left[\frac{(1^{k+1} - 3^{k+1}) / 3^{k+1}}{(1-3)/3}\right]$$

$$= 3^k \cdot \frac{1 - 3^{k+1}}{3^{k+1}} \cdot \frac{3}{-2} = \frac{1}{2}(3^{k+1} - 1)$$

(ii) $Z^{-1}\left\{\dfrac{z}{(z-1)(z-2)}\right\}$

We have, $\dfrac{z}{(z-1)(z-2)} = \dfrac{z}{z-1} \cdot \dfrac{1}{z-2}$

Let $F(z) = \dfrac{z}{z-1}$ and $G(z) = \dfrac{1}{z-2}$ $\therefore H(z) = \dfrac{z}{z-1} \cdot \dfrac{1}{z-2}$

$\therefore H(z) = F(z)G(z)$

Taking inverse z-transform of both sides

$\therefore Z^{-1}[H(z)] = Z^{-1}[F(z)G(z)]$

$\therefore Z^{-1}[F(z)G(z)] = Z^{-1}[H(z)] = Z^{-1}[Z\{h(k)\}]$

$= \{h(k)\} = \{f(k)\} * \{g(k)\}$

if $F(z) = \dfrac{z}{z-1}$ and $G(z) = \dfrac{1}{z-2}$

$\{f(k)\} = Z^{-1}\{F(z)\} = Z^{-1}\left[\dfrac{z}{z-1}\right] = \{1^k\}$

Similary $\{g(k)\} = Z^{-1}\{G(z)\} = Z^{-1}\left[\dfrac{1}{z-2}\right] = \{2^{k-1}\}, K \geq 1$

H. J. Thim Trust's

THEEM COLLEGE OF ENGINEERING

Date :

$\therefore k'-1>0$ putting $k'-1=k$, $k>0$

$\therefore \{g(k)\} = \{2^k\}$, $k>0$

$\therefore z^{-1}[F(z)G(z)] = \sum_{m=0}^{k} f(m)g(k-m)$

$= \sum_{m=0}^{k} \{1^m\}\{2^{k-m}\}$

$= \sum_{m=0}^{k} \{2^{k-m}\} = 2^k \sum_{m=0}^{k} 2^{-m} = 2^k \sum_{m=0}^{k} \left(\frac{1}{2}\right)^m$

$= 2^k \left[\frac{(1/2)^{k+1} - 1}{(1/2) - 1}\right]$ (G. P)

$= 2^k \left[\frac{(1-2^{k+1})}{2^{k+1}} \cdot \frac{2}{-1}\right]$

$\therefore z^{-1}[F(z)G(z)] = \frac{2^{k+1}}{2^{k+1}} \cdot \frac{1-2^{k+1}}{-1} = 2^{k+1}-1$

But as we know $k = k'-1$ $= k+1 = k'$

$\therefore z^{-1}[F(z)G(z)] = 2^k+1$, $k>1$

H. J. Thim Trust's

THEEM COLLEGE OF ENGINEERING

Date : ________

Q5) Find inverse z-transform of $f(z) = \dfrac{z}{(z-1)(z-2)}$ for $|z|>2$

$\Rightarrow$ We have $F(z) = \dfrac{1}{(z-1)(z-2)} = \dfrac{2}{z-2} - \dfrac{1}{z-1}$

Since $|z|>2$ clearly $|z|>1$

$\therefore |z/2| > 1$ and $|z|>1$

$\therefore |2/z| < 1$ and $|1/z| < 1$

$$\therefore F(z) = \frac{2}{z[1-(2/z)]} - \frac{1}{z[1-(1/z)]}$$

$$= \frac{2}{z}\left(1-\frac{2}{z}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= \frac{2}{z}\left(1+\frac{2}{z}+\frac{2^2}{z^2}+\dots+\frac{2^{k-1}}{z^{k-1}}+\dots\right) - \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\dots \dots \frac{1}{z^{k+1}}+\dots\right)$$

$$= \left(\frac{2}{z}+\frac{2^2}{z^2}+\frac{2^3}{z^3}+\dots \frac{2^k}{z^k}+\dots\right) - \left(\frac{1}{z}+\frac{1}{z^2}+\dots \frac{1}{z^k}+\dots\right)$$

Coefficient of $z^{-k} = 2^k - 1,\ k \geq 1$

$\therefore z^{-1}[F(z)] = \{2^k - 1\},\ k \geq 1$

Q6) Find the inverse z-transform of $\dfrac{2z^2 - 10z + 13}{(z-3)^2(z-2)}$ $\quad 2 < |z| < 3$

$\Rightarrow$ We have $F(z) = \dfrac{2z^2 - 10z + 13}{(z-3)^2(z-2)} = \dfrac{1}{z-2} + \dfrac{1}{z-3} + \dfrac{1}{(z-3)^2}$

Since, $2 < |z|$, $\left|\dfrac{2}{z}\right| < 1$ and since $|z| < 3$, $\left|\dfrac{z}{3}\right| < 1$

$\therefore$ We take out z from the first bracket and 3 out from the last two bracket

$$\therefore F(z) = \frac{1}{z} \cdot \frac{1}{[1-(2/z)]} + \frac{1}{3} \cdot \frac{1}{[(z/3)-1]} + \frac{1}{9} \cdot \frac{1}{[(z/3)-1]^2}$$

$$F(z) = \frac{1}{z} \cdot \left(1 - \frac{2}{z}\right)^{-1} - \frac{1}{3}\left(1 - \frac{z}{3}\right)^{-1} + \frac{1}{9}\left(1 - \frac{z}{3}\right)^{-2}$$

$$= \frac{1}{z} \cdot \left(1 + \frac{2}{z} + \frac{2^2}{z^2} + \frac{2^3}{z^3} + \dots + \frac{2^{k-1}}{z^{k-1}} + \dots\right) - \frac{1}{3}\left(1 + \frac{z}{3}\right.$$

$$\left. + \frac{z^2}{3^2} + \dots + \frac{z^k}{3^k} + \dots\right) + \frac{1}{9}\left(1 + 2 \cdot \frac{z}{3} + 3 \cdot \frac{z^2}{3^2} + \dots + \frac{(k+1)z^k}{3^k} + \dots\right)$$