Answers 4

<u>Classroom Example 1</u>

(a)
Conservation:

$$flux_e - flux_w = source$$

$$\Rightarrow \qquad flux_e - flux_w = s\Delta x\,, \qquad \text{where} \qquad flux = -kA\frac{\mathrm{d}T}{\mathrm{d}x}\,, \quad s = -c(T - T_\infty)$$

$$\Rightarrow \qquad \frac{flux_e - flux_w}{\Delta x} = s$$

Taking the limit as $\Delta x \to 0$:

$$\frac{\mathrm{d}}{\mathrm{d}x}(flux) = s$$

Substituting for *flux* and *source*:

$$\frac{\mathrm{d}}{\mathrm{d}x}(-kA\frac{\mathrm{d}T}{\mathrm{d}x}) = -c(T - T_\infty)$$

Substituting constants:

$$-0.1\frac{\mathrm{d}^2T}{\mathrm{d}x^2} = -2.5(T - 20)$$

Multiplying by −10 and rearranging:

$$\frac{\mathrm{d}^2T}{\mathrm{d}x^2} - 25T = -500$$

This is a 2nd-order linear differential equation with constant coefficients (first-year maths course), so use "complementary function plus particular integral" to get general solution:

$$T = A\mathrm{e}^{5x} + B\mathrm{e}^{-5x} + 20$$

where A and B are constants. To satisfy boundary conditions $T = 100$ at $x = 0$ and $\mathrm{d}T/\mathrm{d}x = 0$ at $x = 1$,

$$A = \frac{80}{1 + \mathrm{e}^{10}} = 0.0036\,, \quad B = \frac{80\mathrm{e}^{10}}{1 + \mathrm{e}^{10}} = 79.9964$$

This gives: $T(0.1) = 68.53$, $T(0.3) = 37.87$, $T(0.5) = 26.61$, $T(0.7) = 22.54$, $T(0.9) = 21.22$

(b)
Conservation:

$$flux_e - flux_w = source$$

where

$$flux = -kA\frac{\mathrm{d}T}{\mathrm{d}x}, \qquad source = -c(T - T_\infty)\Delta x$$

<u>Fluxes</u>

Interior faces:

$$flux_e = -D(T_E - T_P)$$
$$flux_w = -D(T_P - T_W)$$

Left boundary:

$$flux_L = -2D(T_P - T_L)$$

Right boundary:

$$flux_R = 0$$

$$D = \frac{\Gamma A}{\Delta x} = 0.5$$

<u>Source</u>

$$source = b_p + s_P T_P$$

where

$$b_P = cT_\infty \Delta x = 10, \quad s_P = -c\Delta x = -0.5$$

Substituting in the conservation equation:

Cells i = 2, 3, 4: $\quad -0.5T_{i-1} + 1.5T_i - 0.5T_{i+1} = 10$

Cell 1: $\quad 2T_1 - 0.5T_2 = 110$

Cell 5: $\quad -0.5T_4 + T_5 = 10$

$$\begin{pmatrix} 2 & -0.5 & 0 & 0 & 0 \\ -0.5 & 1.5 & -0.5 & 0 & 0 \\ 0 & -0.5 & 1.5 & -0.5 & 0 \\ 0 & 0 & -0.5 & 1.5 & -0.5 \\ 0 & 0 & 0 & -0.5 & 1 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{pmatrix} = \begin{pmatrix} 110 \\ 10 \\ 10 \\ 10 \\ 10 \end{pmatrix}$$

Solution (e.g. by Gaussian elimination):

$T_1 = 64.23$, $T_2 = 36.91$, $T_3 = 26.50$, $T_4 = 22.60$, $T_5 = 21.30$

Classroom Example 2

(a) $u = 0$ on upper and lower walls.

(b)

$$net\ \ pressure\ \ force = (p_L - p_R) \times (\Delta y \times 1)$$
$$= -\frac{\mathrm{d}p}{\mathrm{d}x} \Delta x \times \Delta y$$

Hence,

$$net\ \ pressure\ \ force = G \Delta x \Delta y$$

(As expected, the force per unit volume is minus the pressure gradient.)

(c) On upper face of an interior cell,

$$viscous\ \ force = \mu \frac{\mathrm{d}u}{\mathrm{d}y} \times (\Delta x \times 1) \quad \approx \mu \left(\frac{u_{j+1} - u_j}{\Delta y} \right) \Delta x$$

Hence,

$$viscous\ \ force\ on\ upper\ \ face = \mu \frac{\Delta x}{\Delta y} (u_{j+1} - u_j)$$
$$viscous\ \ force\ on\ lower\ \ face = -\mu \frac{\Delta x}{\Delta y} (u_j - u_{j-1})$$

(The minus sign in the latter is because here the force is that exerted by lower fluid on upper).

At the channel boundaries replace Δy by $\frac{1}{2} \Delta y$ and the relevant velocity by 0:

$$viscous\ \ force\ on\ upper\ \ face\ of\ \ top\ cell = -2\mu \frac{\Delta x}{\Delta y} (u_N)$$
$$viscous\ \ force\ on\ lower\ \ face\ of\ \ bottom\ cell = -2\mu \frac{\Delta x}{\Delta y} (u_1)$$

(d) In fully-developed (non-accelerating) flow the net pressure + viscous force is zero.

Interior cell

$$G \Delta x \Delta y + \mu \frac{\Delta x}{\Delta y} (u_{j+1} - u_j) - \mu \frac{\Delta x}{\Delta y} (u_j - u_{j-1}) = 0$$

$$\Rightarrow \qquad \frac{G \Delta y^2}{\mu} = -u_{j-1} + 2u_j - u_{j+1}$$

Top cell

$$G \Delta x \Delta y - 2\mu \frac{\Delta x}{\Delta y} u_N - \mu \frac{\Delta x}{\Delta y} (u_N - u_{N-1}) = 0$$

$$\Rightarrow \qquad \frac{G \Delta y^2}{\mu} = -u_{N-1} + 3u_N$$

<u>Bottom cell</u>

$$G\Delta x\Delta y+\mu\frac{\Delta x}{\Delta y}(u_2-u_1)-2\mu\frac{\Delta x}{\Delta y}(u_1)=0$$

$$\Rightarrow \qquad \frac{G\Delta y^2}{\mu}=3u_1-u_2$$

(e) Symmetry implies $u_1 = u_6$, $u_2 = u_5$, $u_3 = u_4$; it is only necessary to solve for j = 1, 2, 3.

For N = 6,

$$\frac{G\Delta y^2}{\mu}=\left(\frac{\Delta y}{H}\right)^2\times\frac{GH^2}{\mu} \quad =\frac{1}{36}U_0$$

Hence, for the lowest 3 cells:

$j = 1 \quad \Rightarrow \qquad 3u_1-u_2=\frac{1}{36}U_0$

$j = 2 \quad \Rightarrow \qquad -u_1+2u_2-u_3=\frac{1}{36}U_0$

$j = 3 \quad \Rightarrow \qquad -u_2+2u_3-u_4=\frac{1}{36}U_0 \qquad \Rightarrow \qquad -u_2+u_3=\frac{1}{36}U_0 \qquad$ (since $u_3 = u_4$)

From the first and last of these:

$$u_1=\frac{1}{3}(u_2+\frac{1}{36}U_0)$$
$$u_3=u_2+\frac{1}{36}U_0$$

Substituting these in the second,

$$-\frac{1}{3}(u_2+\frac{1}{36}U_0)+2u_2-(u_2+\frac{1}{36}U_0)=\frac{1}{36}U_0$$

$$\Rightarrow \qquad \frac{2}{3}u_2=\frac{7}{3}\times\frac{1}{36}U_0$$

$$\Rightarrow \qquad u_2=\frac{7}{72}U_0$$

Then,

$$u_1=\frac{1}{3}(u_2+\frac{1}{36}U_0) \quad =\frac{3}{72}U_0$$
$$u_3=u_2+\frac{1}{36}U_0 \quad =\frac{9}{72}U_0$$

Answer: $u_1=u_6=\frac{3}{72}U_0, \quad u_2=u_5=\frac{7}{72}U_0, \quad u_3=u_4=\frac{9}{72}U_0$

(f) Adding flow-rate contributions for each cell:

$$Q=\sum_{j=1}^{6}u_j\Delta y$$
$$=2\sum_{j=1}^{3}u_j\frac{H}{6}$$
$$=\frac{1}{3}H\sum_{j=1}^{3}u_j$$

Hence,

$$Q=\frac{1}{3}H(\frac{3}{72}+\frac{7}{72}+\frac{9}{72})U_0 \quad =\frac{19}{216}U_0H$$

(g) The wall stress can be found from the finite-difference approximation for the stress on the lower face of cell 1 (you should check this), but for the fully-developed flow here it is more easily found by balancing pressure force over the whole depth of the channel against the viscous drag on the top and bottom walls:

$$(p_L - p_R)H = 2\times\tau_w\Delta x$$

$$\Rightarrow \qquad \tau_w = \tfrac{1}{2}\left(\frac{p_L - p_R}{\Delta x}\right)H$$

$$\Rightarrow \qquad \tau_w = \tfrac{1}{2}GH$$

(h) The Navier-Stokes equation is

$$\rho\frac{\mathrm{D}u}{\mathrm{D}t} = -\frac{\partial p}{\partial x} + \mu\nabla^2 u\,,$$

with boundary conditions

$$u(0) = u(H) = 0$$

For fully-developed flow,

$$\frac{\mathrm{D}u}{\mathrm{D}t} = 0\,, \qquad \frac{\partial p}{\partial x} = -G\text{, constant}\,, \qquad \nabla^2 u = \frac{\partial^2 u}{\partial y^2}$$

Hence,

$$0 = G + \mu\frac{\partial^2 u}{\partial y^2}$$

Integrating twice,

$$u = -\frac{G}{2\mu}y^2 + Ay + B$$

Applying the boundary conditions gives

$$B = 0, \qquad A = \frac{GH}{2\mu}$$

$$\Rightarrow \qquad u = -\frac{G}{2\mu}y^2 + \frac{GH}{2\mu}y$$

which is conveniently rearranged in non-dimensional form as

$$\frac{u}{U_0} = \frac{1}{2}\frac{y}{H}\left(1-\frac{y}{H}\right), \qquad \text{where} \qquad U_0 = \frac{GH^2}{\mu}$$

<u>Comparing with the velocity solution, part (e)</u>

Comparing solutions at the cell-centre points $\frac{y}{H} = \frac{1}{12}, \frac{3}{12}, \frac{5}{12}, \frac{7}{12}, \frac{9}{12}, \frac{11}{12}$ this gives

$$\frac{U}{U_0} = \frac{11}{288}, \frac{27}{288}, \frac{35}{288}, \frac{35}{288}, \frac{27}{288}, \frac{11}{288} \qquad \text{(exact)}$$

$$\frac{U}{U_0} = \frac{12}{288}, \frac{28}{288}, \frac{36}{288}, \frac{36}{288}, \frac{28}{288}, \frac{12}{288} \qquad \text{(numerical, part(e))}$$

Comparing with the flow rate, part (f)

For the overall flow rate per unit span we find

$$q = \int_0^H u \quad dy$$

which integrates to give

$$q = \frac{1}{12} U_0 H$$

or, multiplying by 18 to get a denominator of 216 for comparison:

$$q = \frac{18}{216} U_0 H \qquad \text{(exact)}$$

$$q = \frac{19}{216} U_0 H \qquad \text{(numerical, part(f))}$$

Comparing with the wall shear stress, part (g)

$$\tau_w = \mu \left. \frac{\partial u}{\partial y} \right|_{y=0}$$

$$= \mu U_0 \times \left(\frac{1}{2H} - \frac{y}{H^2} \right)_{y=0}$$

$$= \frac{\mu U_0}{2H}$$

With $U_0 = GH^2 / \mu$ this gives

$$\tau_w = \tfrac{1}{2} GH$$

(exactly the same as in part (g)).

Classroom Example 3

Conservation:

$$flux_e - flux_w = source$$

where

$$flux = \rho u \phi - \Gamma A \frac{\mathrm{d}\phi}{\mathrm{d}x}$$

Fluxes

Interior faces:

$$flux_e = C\phi_e - D(\phi_E - \phi_P)$$

$$flux_w = C\phi_w - D(\phi_P - \phi_W)$$

Left boundary:

$$flux_L = 0 - 2D\phi_P$$

Right boundary:

$$flux_R = C\phi_P$$

$$C = \rho u A = 1.0\,, \quad D = \frac{\Gamma A}{\Delta x} = 0.007$$

Source

$$source = \underbrace{S_{pt}}_{cell\ 4\ only} - \gamma\phi_P \Delta x = b_p + s_P \phi_P$$

where

$$b_P = 0.01(\text{cell } 4), \quad s_P = -\gamma \Delta x = -0.071$$

(a) **Central differencing**: $\phi_e = \tfrac{1}{2}(\phi_P + \phi_E)\,, \quad \phi_w = \tfrac{1}{2}(\phi_W + \phi_P)$

Interior cells: $flux_e - flux_w = -(\frac{C}{2} + D)\phi_W + 2D\phi_P - (-\frac{C}{2} + D)\phi_E = b_P + s_P\phi_P$

Cell 1: $flux_e - flux_L = (\frac{C}{2} + 3D)\phi_P - (-\frac{C}{2} + D)\phi_E = s_P\phi_P$

Cell 7: $flux_R - flux_w = -(\frac{C}{2} + D)\phi_W + (\frac{C}{2} + D)\phi_P = s_P\phi_P$

Hence:

Interior cells: $-0.507\phi_W + 0.085\phi_P + 0.493\phi_E = \begin{cases} 0.01 & \textit{in cell } 4 \\ 0 & \textit{otherwise} \end{cases}$

Cell 1: $0.592\phi_P + 0.493\phi_E = 0$

Cell 7: $-0.507\phi_W + 0.578\phi_P = 0$

$$\begin{pmatrix} 0.592 & 0.493 & 0 & 0 & 0 & 0 & 0 \\ -0.507 & 0.085 & 0.493 & 0 & 0 & 0 & 0 \\ 0 & -0.507 & 0.085 & 0.493 & 0 & 0 & 0 \\ 0 & 0 & -0.507 & 0.085 & 0.493 & 0 & 0 \\ 0 & 0 & 0 & -0.507 & 0.085 & 0.493 & 0 \\ 0 & 0 & 0 & 0 & -0.507 & 0.085 & 0.493 \\ 0 & 0 & 0 & 0 & 0 & -0.507 & 0.578 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \\ \phi_7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0.01 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

(b) **Upwind differencing** (when $C > 0$): $\phi_e = \phi_P$, $\phi_w = \phi_W$

Interior cells: $flux_e - flux_w = -(C+D)\phi_W + (C+2D)\phi_P - D\phi_E = b_P + s_P\phi_P$

Cell 1: $flux_e - flux_L = (C+3D)\phi_P - D\phi_E = s_P\phi_P$

Cell 7: $flux_R - flux_w = -(C+D)\phi_W + (C+D)\phi_P = s_P\phi_P$

Hence:

Interior cells: $-1.007\phi_W + 1.085\phi_P - 0.007\phi_E = \begin{cases} 0.01 & \textit{in cell } 4 \\ 0 & \textit{otherwise} \end{cases}$

Cell 1: $1.092\phi_P - 0.007\phi_E = 0$

Cell 7: $-1.007\phi_W + 1.078\phi_P = 0$

$$\begin{pmatrix} 1.092 & -0.007 & 0 & 0 & 0 & 0 & 0 \\ -1.007 & 1.085 & -0.007 & 0 & 0 & 0 & 0 \\ 0 & -1.007 & 1.085 & -0.007 & 0 & 0 & 0 \\ 0 & 0 & -1.007 & 1.085 & -0.007 & 0 & 0 \\ 0 & 0 & 0 & -1.007 & 1.085 & -0.007 & 0 \\ 0 & 0 & 0 & 0 & -1.007 & 1.085 & -0.007 \\ 0 & 0 & 0 & 0 & 0 & -1.007 & 1.078 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \\ \phi_7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0.01 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The analytical solution (from solution of a homogeneous equation on either side, plus a jump condition in the flux at the discontinuity) is:

$$\phi = \frac{S_{pt}}{\Gamma A}(\alpha e^{k_1 x} + \beta e^{k_2 x})$$

where

$$k_{1,2} = \frac{1}{2}\left\{\frac{\rho u}{\Gamma} \pm \sqrt{\left(\frac{\rho u}{\Gamma}\right)^2 + \left(\frac{4\gamma}{\Gamma A}\right)}\right\} \qquad \text{(in either order)}$$

and

$$\alpha = -\beta = \frac{1}{k_1 - k_2}\left(\frac{k_1 e^{k_1/2} - k_2 e^{k_2/2}}{k_1 e^{k_1} - k_2 e^{k_2}}\right) \qquad (x \le ½)$$

$$\alpha = -\frac{k_2}{k_1} e^{k_2 - k_1}\beta = \frac{k_2 e^{k_2}}{k_1 - k_2}\left(\frac{e^{-k_1/2} - e^{-k_2/2}}{k_1 e^{k_1} - k_2 e^{k_2}}\right) \qquad (x \ge ½)$$

Classroom Example 4

(a) If the velocity is positive:

$$r_w = \frac{\phi_W - \phi_{WW}}{\phi_P - \phi_W} \quad = \frac{4-2}{6-4} \quad = 1$$

$$\psi_w = \frac{2\times 1}{1+1} \quad = 1$$

$$\phi_w = \phi_W + \tfrac{1}{2}\psi_w(\phi_P - \phi_W) \quad = 4 + \tfrac{1}{2}\times 1\times(6-4) \quad = 5$$

$$r_e = \frac{\phi_P - \phi_W}{\phi_E - \phi_P} \quad = \frac{6-4}{7-6} \quad = 2$$

$$\psi_e = \frac{2\times 2}{1+2} \quad = \frac{4}{3}$$

$$\phi_e = \phi_P + \tfrac{1}{2}\psi_e(\phi_E - \phi_P) \quad = 6 + \tfrac{1}{2}\times\tfrac{4}{3}\times(7-6) \quad = 6\tfrac{2}{3}$$

(b) If the velocity is negative:

$$r_w = \frac{\phi_P - \phi_E}{\phi_W - \phi_P} \quad = \frac{6-7}{4-6} \quad = \frac{1}{2}$$

$$\psi_w = \frac{2\times\frac{1}{2}}{1+\frac{1}{2}} \quad = \frac{2}{3}$$

$$\phi_w = \phi_P + \tfrac{1}{2}\psi_w(\phi_W - \phi_P) \quad = 6 + \tfrac{1}{2}\times\tfrac{2}{3}\times(4-6) \quad = 5\tfrac{1}{3}$$

$$r_e = \frac{\phi_E - \phi_{EE}}{\phi_P - \phi_E} \quad = \frac{7-6}{6-7} \quad = -1$$

$$\psi_e = 0$$

$$\phi_e = \phi_E + \tfrac{1}{2}\psi_e(\phi_P - \phi_E) \quad = 7 + \tfrac{1}{2}\times 0\times(6-7) \quad = 7$$

Classroom Example 5.

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Start
         A           B           C           D
     0.000       0.000       0.000       0.000

Sweep 1:
     0.500       1.125       1.781       3.695

Sweep 2:
     0.781       1.641       2.834       3.958

Sweep 3:
     0.910       1.936       2.974       3.993

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After 7 sweeps:
         A           B           C           D
     1.000       2.000       3.000       4.000
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Q1.
(a)

$$\phi_E = \phi_e + \left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)_e (\frac{\Delta x}{2}) + \frac{1}{2!}\left(\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}\right)_e (\frac{\Delta x}{2})^2 + \frac{1}{3!}\left(\frac{\mathrm{d}^3\phi}{\mathrm{d}x^3}\right)_e (\frac{\Delta x}{2})^3 + \frac{1}{4!}\left(\frac{\mathrm{d}^4\phi}{\mathrm{d}x^4}\right)_e (\frac{\Delta x}{2})^4 + \cdots$$

$$\phi_P = \phi_e - \left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)_e (\frac{\Delta x}{2}) + \frac{1}{2!}\left(\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}\right)_e (\frac{\Delta x}{2})^2 - \frac{1}{3!}\left(\frac{\mathrm{d}^3\phi}{\mathrm{d}x^3}\right)_e (\frac{\Delta x}{2})^3 + \frac{1}{4!}\left(\frac{\mathrm{d}^4\phi}{\mathrm{d}x^4}\right)_e (\frac{\Delta x}{2})^4 + \cdots$$

Adding and dividing by 2:

$$\tfrac{1}{2}(\phi_P + \phi_E) = \phi_e \quad + \quad \frac{1}{2!}\left(\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}\right)_e (\frac{\Delta x}{2})^2 \quad + \quad \frac{1}{4!}\left(\frac{\mathrm{d}^4\phi}{\mathrm{d}x^4}\right)_e (\frac{\Delta x}{2})^4 + \cdots$$

The first term is ϕ_e and the leading-order error term is proportional to Δx^2. Hence, this is a second-order approximation for ϕ_e.

Alternatively, subtracting the two Taylor series:

$$\phi_E - \phi_P = 2\times\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)_e \frac{\Delta x}{2} + 2\times\frac{1}{3!}\left(\frac{\mathrm{d}^3\phi}{\mathrm{d}x^3}\right)_e (\frac{\Delta x}{2})^3 + 2\times\frac{1}{5!}\left(\frac{\mathrm{d}^5\phi}{\mathrm{d}x^5}\right)_e (\frac{\Delta x}{2})^5 + \ldots$$

or

$$\frac{\phi_E - \phi_P}{\Delta x} = \left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)_e + \frac{1}{3!}\left(\frac{\mathrm{d}^3\phi}{\mathrm{d}x^3}\right)_e (\frac{\Delta x}{2})^2 + \frac{1}{5!}\left(\frac{\mathrm{d}^5\phi}{\mathrm{d}x^5}\right)_e (\frac{\Delta x}{2})^4 + \ldots$$

The first term is $(\mathrm{d}\phi/\mathrm{d}x)_e$ and the leading-order error term is proportional to Δx^2. Hence, this is a second-order approximation for $(\mathrm{d}\phi/\mathrm{d}x)_e$.

(b) Advection scheme (i.e. approximation for ϕ_e).
From part (a):

$$\tfrac{1}{2}(\phi_P + \phi_E) = \phi_e \quad + \quad \frac{1}{2!}\left(\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}\right)_e (\frac{\Delta x}{2})^2 \quad + \quad \frac{1}{4!}\left(\frac{\mathrm{d}^4\phi}{\mathrm{d}x^4}\right)_e (\frac{\Delta x}{2})^4 + \cdots$$

A second symmetric approximation can be obtained from W and EE nodes simply by replacing Δx by $3\Delta x$:

$$\tfrac{1}{2}(\phi_W + \phi_{EE}) = \phi_e \quad + \quad \frac{1}{2!}\left(\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}\right)_e (\frac{3\Delta x}{2})^2 \quad + \quad \frac{1}{4!}\left(\frac{\mathrm{d}^4\phi}{\mathrm{d}x^4}\right)_e (\frac{3\Delta x}{2})^4 + \cdots$$

A 4th-order approximation to ϕ_e can be obtained by an appropriately weighted combination of these two in order to eliminate the Δx^2 term:

$$\alpha\times\tfrac{1}{2}(\phi_P + \phi_E) + \beta\times\tfrac{1}{2}(\phi_W + \phi_{EE})$$

$$= (\alpha+\beta)\phi_e \quad + \quad (\alpha+9\beta)\frac{1}{2!}\left(\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}\right)_e (\frac{\Delta x}{2})^2 \quad + \quad (\alpha+81\beta)\frac{1}{4!}\left(\frac{\mathrm{d}^4\phi}{\mathrm{d}x^4}\right)_e (\frac{\Delta x}{2})^4 + \cdots$$

where:

$$\alpha + \beta = 1$$
$$\alpha + 9\beta = 0$$

Subtracting gives

$$-8\beta = 1$$

whence

$$\beta = -\frac{1}{8}, \quad \alpha = \frac{9}{8}$$

With these values of α and β the $O(\Delta x^4)$ term in the expansion does not vanish. Hence a fourth-order approximation to ϕ_e is

$$\phi_e = \frac{9}{16}(\phi_P + \phi_E) - \frac{1}{16}(\phi_W + \phi_{EE}) \quad = \frac{-\phi_W + 9\phi_P + 9\phi_E - \phi_{EE}}{16}$$

<u>Diffusion scheme</u> (i.e. approximation for $(\mathrm{d}\phi/\mathrm{d}x)_e$).
From part (a):

$$\frac{\phi_E - \phi_P}{\Delta x} = \left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)_e + \frac{1}{3!}\left(\frac{\mathrm{d}^3\phi}{\mathrm{d}x^3}\right)_e \left(\frac{\Delta x}{2}\right)^2 + \frac{1}{5!}\left(\frac{\mathrm{d}^5\phi}{\mathrm{d}x^5}\right)_e \left(\frac{\Delta x}{2}\right)^4 + \ldots$$

A second symmetric approximation can be obtained from W and EE nodes simply by replacing Δx by $3\Delta x$:

$$\frac{\phi_{EE} - \phi_W}{3\Delta x} = \left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)_e + \frac{1}{3!}\left(\frac{\mathrm{d}^3\phi}{\mathrm{d}x^3}\right)_e \left(\frac{3\Delta x}{2}\right)^2 + \frac{1}{5!}\left(\frac{\mathrm{d}^5\phi}{\mathrm{d}x^5}\right)_e \left(\frac{3\Delta x}{2}\right)^4 + \ldots$$

A 4th-order approximation to ϕ_e can be obtained by an appropriately weighted combination of these two in order to eliminate the Δx^2 term:

$$\alpha\frac{\phi_E - \phi_P}{\Delta x} + \beta\frac{\phi_{EE} - \phi_W}{3\Delta x} = (\alpha + \beta)\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)_e + \frac{\alpha + 9\beta}{3!}\left(\frac{\mathrm{d}^3\phi}{\mathrm{d}x^3}\right)_e \left(\frac{\Delta x}{2}\right)^2 + \frac{\alpha + 81\beta}{5!}\left(\frac{\mathrm{d}^5\phi}{\mathrm{d}x^5}\right)_e \left(\frac{\Delta x}{2}\right)^4 + \ldots$$

For a 4th-order approximation for $(\mathrm{d}\phi/\mathrm{d}x)_e$ we require that

$$\alpha + \beta = 1$$

$$\alpha + 9\beta = 0$$

whence

$$\alpha = \frac{9}{8}, \qquad \beta = -\frac{1}{8}$$

With these values of α and β the $O(\Delta x^4)$ term in the expansion does not vanish. Hence a fourth-order approximation to $(\mathrm{d}\phi/\mathrm{d}x)_e$ is

$$\frac{9}{8}\left(\frac{\phi_E - \phi_P}{\Delta x}\right) - \frac{1}{8}\left(\frac{\phi_{EE} - \phi_W}{3\Delta x}\right) = \frac{-\phi_{EE} + 27\phi_E - 27\phi_P + \phi_W}{24\Delta x}$$

Q2.
(a) For convenience write

$$X = \frac{x}{\Delta x}$$

as the number of mesh spacings. Then:

$$\phi = \begin{cases} \phi_W & \text{at} \quad X = -\frac{3}{2} \\ \phi_P & \text{at} \quad X = -\frac{1}{2} \\ \phi_E & \text{at} \quad X = +\frac{1}{2} \end{cases}$$

A long way is to assume a quadratic:

$$\phi = \alpha X^2 + \beta X + \gamma$$

and find α, β, γ by substituting at X = –3/2, –1/2 and 1/2 and solving simultaneous equations.

A much faster method, producing the same result directly, uses Lagrange interpolation:

$$\phi = \frac{(X+\frac{1}{2})(X-\frac{1}{2})}{(-\frac{3}{2}+\frac{1}{2})(-\frac{3}{2}-\frac{1}{2})}\phi_W + \frac{(X+\frac{3}{2})(X-\frac{1}{2})}{(-\frac{1}{2}+\frac{3}{2})(-\frac{1}{2}-\frac{1}{2})}\phi_P + \frac{(X+\frac{3}{2})(X+\frac{1}{2})}{(\frac{1}{2}+\frac{3}{2})(\frac{1}{2}+\frac{1}{2})}\phi_E$$

To evaluate at a single point it is not necessary to simplify this. Putting X = 0:

$$\begin{aligned} \phi_e &= \frac{(\frac{1}{2})(-\frac{1}{2})}{(-1)(-2)}\phi_W + \frac{(\frac{3}{2})(-\frac{1}{2})}{(1)(-1)}\phi_P + \frac{(\frac{3}{2})(\frac{1}{2})}{(2)(1)}\phi_E \\ &= -\tfrac{1}{8}\phi_W + \tfrac{3}{4}\phi_P + \tfrac{3}{8}\phi_E \end{aligned}$$

(b) Taylor series expansions about the cell face e:

$$\phi_W = \phi_e - (\frac{3\Delta x}{2})\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)_e + \frac{1}{2!}(\frac{3\Delta x}{2})^2\left(\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}\right)_e - \frac{1}{3!}(\frac{3\Delta x}{2})^3\left(\frac{\mathrm{d}^3\phi}{\mathrm{d}x^3}\right)_e + \cdots$$

$$\phi_P = \phi_e - (\frac{\Delta x}{2})\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)_e + \frac{1}{2!}(\frac{\Delta x}{2})^2\left(\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}\right)_e - \frac{1}{3!}(\frac{\Delta x}{2})^3\left(\frac{\mathrm{d}^3\phi}{\mathrm{d}x^3}\right)_e + \cdots$$

$$\phi_E = \phi_e + (\frac{\Delta x}{2})\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)_e + \frac{1}{2!}(\frac{\Delta x}{2})^2\left(\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}\right)_e + \frac{1}{3!}(\frac{\Delta x}{2})^3\left(\frac{\mathrm{d}^3\phi}{\mathrm{d}x^3}\right)_e + \cdots$$

Choose a linear combination of these:

$$\begin{aligned} a\phi_W + b\phi_P + c\phi_E = {} & (a+b+c)\phi_e \\ & + (-3a-b+c)(\frac{\Delta x}{2})\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)_e \\ & + (9a+b+c)\frac{1}{2!}(\frac{\Delta x}{2})^2\left(\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}\right)_e \\ & + (-27a-b+c)\frac{1}{3!}(\frac{\Delta x}{2})^3\left(\frac{\mathrm{d}^3\phi}{\mathrm{d}x^3}\right)_e \\ & + \cdots \end{aligned}$$

To get a 3rd-order approximation to ϕ_e requires:

$$
\begin{aligned}
a \quad &+b \quad +c \quad =1 \\
-3a \quad &-b \quad +c \quad =0 \\
9a \quad &+b \quad +c \quad =0
\end{aligned}
$$

Eliminating c by subtracting the second and third equations from the first:

$$
\begin{aligned}
4a \quad &+2b \quad =1 \\
-8a \quad & \qquad =1
\end{aligned}
$$

Hence

$$a = -\tfrac{1}{8}, \; b = \tfrac{3}{4}$$

$$c = 1 - a - b \quad = \tfrac{3}{8}$$

Thus,

$$\phi_e = -\tfrac{1}{8}\phi_W + \tfrac{3}{4}\phi_P + \tfrac{3}{8}\phi_E$$

Q3.
$\phi = 1$
$\Gamma = 0$
$S = 0$

Q4.
(a) Might work, but isn't very efficient (and could fail for positive-definite quantities).

(b) Wrong source if $\phi_P < 0$.

(c) Unstable, because $s_P > 0$.

(d) This is the best method.

Q5.

From the given equation,

$$\frac{\mathrm{d}}{\mathrm{d}x}(\rho u\phi - \Gamma\frac{\mathrm{d}\phi}{\mathrm{d}x}) = 0$$

$$\Rightarrow \qquad \rho u\phi - \Gamma\frac{\mathrm{d}\phi}{\mathrm{d}x} = \mathit{constant}, F \text{ say}$$

$$\Rightarrow \qquad \frac{\mathrm{d}\phi}{\mathrm{d}x} - \frac{\rho u}{\Gamma}\phi = -\frac{F}{\Gamma}$$

The integrating factor which will make the LHS a total derivative is (by inspection, or by formula: refer to your first-year maths notes) $\mathrm{e}^{-\frac{\rho u}{\Gamma}x}$, whence:

$$\mathrm{e}^{-\frac{\rho u}{\Gamma}x}\frac{\mathrm{d}\phi}{\mathrm{d}x} - \frac{\rho u}{\Gamma}\mathrm{e}^{-\frac{\rho u}{\Gamma}x}\phi = -\frac{F}{\Gamma}\mathrm{e}^{-\frac{\rho u}{\Gamma}x}$$

$$\Rightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{-\frac{\rho u}{\Gamma}x}\phi) = -\frac{F}{\Gamma}\mathrm{e}^{-\frac{\rho u}{\Gamma}x}$$

Integrating between $x = 0$ (where $\phi = \phi_P$) and $x = \Delta x$ (where $\phi = \phi_E$):

$$\left[\mathrm{e}^{-\frac{\rho u}{\Gamma}x}\phi\right]_{x=0}^{\Delta x} = \frac{F}{\rho U}\left[\mathrm{e}^{-\frac{\rho u}{\Gamma}x}\right]_{x=0}^{\Delta x}$$

$$\Rightarrow \qquad \mathrm{e}^{-\frac{\rho u\Delta x}{\Gamma}}\phi_E - \phi_P = \frac{F}{\rho u}(\mathrm{e}^{-\frac{\rho u\Delta x}{\Gamma}} - 1)$$

Hence,

$$F = \rho u\left(\frac{\mathrm{e}^{-\mathrm{Pe}}\phi_E - \phi_P}{\mathrm{e}^{-\mathrm{Pe}} - 1}\right) \qquad \text{where} \qquad \mathrm{Pe} = \frac{\rho u\Delta x}{\Gamma}$$

or, multiplying numerator and denominator by – e^{Pe} (for convenience) and then rearranging:

$$F = \rho u\left(\frac{\phi_P\mathrm{e}^{\mathrm{Pe}} - \phi_E}{\mathrm{e}^{\mathrm{Pe}} - 1}\right)$$

Q6.
(a)

$$\begin{aligned}
\phi_e &= -\tfrac{1}{8}\phi_W + \tfrac{3}{4}\phi_P + \tfrac{3}{8}\phi_E \\
\phi_w &= -\tfrac{1}{8}\phi_{WW} + \tfrac{3}{4}\phi_W + \tfrac{3}{8}\phi_P \\
\phi_n &= -\tfrac{1}{8}\phi_S + \tfrac{3}{4}\phi_P + \tfrac{3}{8}\phi_N \\
\phi_s &= -\tfrac{1}{8}\phi_{SS} + \tfrac{3}{4}\phi_S + \tfrac{3}{8}\phi_P
\end{aligned}$$

(b)

$$a_P\phi_P - \sum a_F\phi_F = b_P$$

where:

$$\begin{aligned}
a_E &= -\tfrac{3}{8}C_e & \qquad a_N &= -\tfrac{3}{8}C_n \\
a_W &= \tfrac{3}{4}C_w + \tfrac{1}{8}C_e & \qquad a_S &= \tfrac{3}{4}C_s + \tfrac{1}{8}C_n \\
a_{WW} &= -\tfrac{1}{8}C_w & \qquad a_{SS} &= -\tfrac{1}{8}C_s
\end{aligned}$$

$$a_P = a_{WW} + a_W + a_E + a_{SS} + a_S + a_N + \underbrace{(C_e - C_w + C_n - C_s)}_{=0 \text{ by mass conservation}}$$

$$b_P = sV$$

and

$$C_e = \rho u A_e, \quad C_n = \rho v A_n, \quad etc.$$

(c) If $u < 0$ then we use a different set of nodes:

$$\begin{aligned}
\phi_e &= -\tfrac{1}{8}\phi_{EE} + \tfrac{3}{4}\phi_E + \tfrac{3}{8}\phi_P \\
\phi_w &= -\tfrac{1}{8}\phi_E + \tfrac{3}{4}\phi_P + \tfrac{3}{8}\phi_W
\end{aligned}$$

(d)
QUICK is transportive; (choice and weighting of nodes is upwind biased).
QUICK is not bounded (check the sign of a_E or a_N in part (b) above).

(e)

$$\begin{aligned}
\phi_{face} &= -\tfrac{1}{8}\phi_{UU} + \tfrac{3}{4}\phi_U + \tfrac{3}{8}\phi_D \\
&= \phi_U + \underbrace{\{-\tfrac{1}{8}\phi_{UU} - \tfrac{1}{4}\phi_U + \tfrac{3}{8}\phi_D\}}_{\text{deferred correction}}
\end{aligned}$$

The deferred correction is transferred to the RHS (source) of the equation at each iteration, to make the matrix coefficients diagonally dominant ($a_P \ge \sum |a_F|$), as in the upwind scheme. This is required by many iterative matrix solution algorithms in order to obtain a converged solution.

Q7.
(a)

$$\left(\rho u\phi - \Gamma\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)_e - \left(\rho u\phi - \Gamma\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)_w = s_i\Delta x$$

(b)
East face (i + 1/2):

$$\phi_e = \phi_i$$

$$\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)_e = \frac{\phi_{i+1}-\phi_i}{\Delta x}$$

West face (i –1/2):

$$\phi_w = \phi_{i-1}$$

$$\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)_w = \frac{\phi_i-\phi_{i-1}}{\Delta x}$$

(c) A numerical scheme is "order n" if

error $\propto$ (grid spacing)^n as grid spacing $\to 0$

Upwind differencing for advection: order 1
Central differencing for diffusion: order 2

(d) With constant coefficients $\rho u = 1$, $\Gamma = 0.5$, $s = 2$, $\Delta x = 0.5$:

$$\left(\rho u\phi - \Gamma\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)_e - \left(\rho u\phi - \Gamma\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)_w = s\Delta x$$

Interior cell (i = 2, 3):

$$\Rightarrow \quad \left(\rho u\phi_i - \Gamma\frac{\phi_{i+1}-\phi_i}{\Delta x}\right) - \left(\rho u\phi_{i-1} - \Gamma\frac{\phi_i-\phi_{i-1}}{\Delta x}\right) = s\Delta x$$

$$\Rightarrow \quad -\left(\rho u + \frac{\Gamma}{\Delta x}\right)\phi_{i-1} + \left(\rho u + \frac{2\Gamma}{\Delta x}\right)\phi_i - \left(\frac{\Gamma}{\Delta x}\right)\phi_{i+1} = s\Delta x$$

$$-2\phi_{i-1} + 3\phi_i - \phi_{i+1} = 1$$

Leftmost cell (i = 1):

$$\Rightarrow \quad \left(\rho u\phi_1 - \Gamma\frac{\phi_2-\phi_1}{\Delta x}\right) - \left(0 - \Gamma\frac{\phi_1 - 0}{\frac{1}{2}\Delta x}\right)_e = s\Delta x$$

$$\Rightarrow \quad \left(\rho u + \frac{3\Gamma}{\Delta x}\right)\phi_1 - \left(\frac{\Gamma}{\Delta x}\right)\phi_2 = s\Delta x$$

$$4\phi_1 - \phi_2 = 1$$

Rightmost cell (i = 4):

$$\Rightarrow \qquad (\rho u\phi_4 - 0) - \left(\rho u\phi_3 - \Gamma\frac{\phi_4 - \phi_3}{\Delta x}\right) = s\Delta x$$

$$\Rightarrow \qquad -\left(\rho u + \frac{\Gamma}{\Delta x}\right)\phi_3 + \left(\rho u + \frac{\Gamma}{\Delta x}\right)\phi_4 = s\Delta x$$

$$-2\phi_3 + 2\phi_4 = 1$$

Assembling equations:

$$\begin{pmatrix} 4 & -1 & 0 & 0 \\ -2 & 3 & -1 & 0 \\ 0 & -2 & 3 & -1 \\ 0 & 0 & -2 & 2 \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Solve by Gaussian elimination:

$$R2 \to 2R2 + R1 \qquad \begin{pmatrix} 4 & -1 & 0 & 0 \\ 0 & 5 & -2 & 0 \\ 0 & -2 & 3 & -1 \\ 0 & 0 & -2 & 2 \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 1 \\ 1 \end{pmatrix}$$

$$R3 \to 5R3 + 2R2 \qquad \begin{pmatrix} 4 & -1 & 0 & 0 \\ 0 & 5 & -2 & 0 \\ 0 & 0 & 11 & -5 \\ 0 & 0 & -2 & 2 \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 11 \\ 1 \end{pmatrix}$$

$$R4 \to 11R4 + 2R3 \qquad \begin{pmatrix} 4 & -1 & 0 & 0 \\ 0 & 5 & -2 & 0 \\ 0 & 0 & 11 & -5 \\ 0 & 0 & 0 & 12 \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 11 \\ 33 \end{pmatrix}$$

Back-substituting:

$$12\phi_4 = 33 \qquad \Rightarrow \qquad \phi_4 = \frac{33}{12} \quad = 2.75$$

$$11\phi_3 - 5\phi_4 = 11 \qquad \Rightarrow \qquad \phi_3 = \frac{5\phi_4 + 11}{11} \quad = 2.25$$

$$5\phi_2 - 2\phi_3 = 3 \qquad \Rightarrow \qquad \phi_2 = \frac{2\phi_3 + 3}{5} \quad = 1.5$$

$$4\phi_1 - \phi_2 = 1 \qquad \Rightarrow \qquad \phi_1 = \frac{\phi_2 + 1}{4} \quad = 0.625$$

Answer: $(\phi_1, \phi_2, \phi_3, \phi_4) = (0.625, 1.5, 2.25, 2.75)$

(e) This is a second-order order linear differential equation with constant coefficients:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\phi - \frac{1}{2}\frac{\mathrm{d}\phi}{\mathrm{d}x}) = 2$$

Rearranging in standard form:

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} - 2\frac{\mathrm{d}\phi}{\mathrm{d}x} = -4$$

Auxiliary equation:

$$k^2 - 2k = 0$$

$\Rightarrow \quad k(k-2) = 0$

$\Rightarrow \quad k = 0$ or $k = 2$

Complementary function (solution of LHS = 0):

$$\phi = A + B\mathrm{e}^{2x}$$

Particular integral (by inspection):

$$\phi = 2x$$

General solution:

$$\phi = A + B\mathrm{e}^{2x} + 2x$$

Boundary condition $\phi = 0$ when $x = 0$:

$\Rightarrow \quad 0 = A + B$

Boundary condition $\mathrm{d}\phi/\mathrm{d}x = 0$ when $x = 2$:

$\Rightarrow \quad 0 = 2B\mathrm{e}^4 + 2$

These give

$$B = -\mathrm{e}^{-4}, \qquad A = \mathrm{e}^{-4}$$

The general solution is then:

$$\phi = \mathrm{e}^{-4}(1 - \mathrm{e}^{2x}) + 2x$$

At the nodal values corresponding to $x = 0.25, 0.75, 1.25, 1.75$ this then gives

$(\phi_1, \phi_2, \phi_3, \phi_4) = (0.49, 1.44, 2.30, 2.91)$ (exact)

compared with

$(\phi_1, \phi_2, \phi_3, \phi_4) = (0.63, 1.50, 2.25, 2.75)$ (numerical, part (d))

Answer: $\phi = \mathrm{e}^{-4}(1 - \mathrm{e}^{2x}) + 2x$; $(\phi_1, \phi_2, \phi_3, \phi_4) = (0.49, 1.44, 2.30, 2.91)$

Alternative Method

The original equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\phi - \frac{1}{2}\frac{\mathrm{d}\phi}{\mathrm{d}x}\right) = 2$$

can be integrated directly to give, with one constant of integration:

$$\phi - \frac{1}{2}\frac{\mathrm{d}\phi}{\mathrm{d}x} = 2x + C$$

This can then be solved as a *first-order* equation (using an integrating factor) if required. The constant C and the subsequent constant that arises at the next integration are found by applying the boundary conditions.

Q8.

(a)

Upwind: $\psi = 0$

Central: $\psi = 1$

(b)

(i) If the velocity is positive:

$$r = \frac{\phi_P - \phi_W}{\phi_E - \phi_P} \quad = \frac{4-3}{7-4} \quad = \frac{1}{3}$$

$$\psi = \frac{1}{3}$$

$$\phi_e = \phi_P + \tfrac{1}{2}\psi_e(\phi_E - \phi_P) \quad = 4 + \tfrac{1}{2} \times \tfrac{1}{3} \times (7-4) \quad = 4\tfrac{1}{2}$$

(ii) If the velocity is negative:

$$r_e = \frac{\phi_E - \phi_{EE}}{\phi_P - \phi_E} \quad = \frac{7-5}{4-7} \quad = -\frac{2}{3}$$

$$\psi_e = 0 \text{ (non-monotonic)}$$

$$\phi_e = \phi_E + \tfrac{1}{2}\psi_e(\phi_P - \phi_E) \quad = 7$$

Q9.

(a)

LUD

$$\phi_{face} = \phi_U + \tfrac{1}{2}(\phi_U - \phi_{UU})$$
$$= \phi_U + \tfrac{1}{2}\left(\frac{\phi_U - \phi_{UU}}{\phi_D - \phi_U}\right)(\phi_D - \phi_U)$$
$$= \phi_U + \tfrac{1}{2} r(\phi_D - \phi_U)$$

Hence, by comparison with $\phi_{face} = \phi_U + \tfrac{1}{2}\psi(r)(\phi_D - \phi_U)$,

$$\psi(r) = r$$

QUICK

$$\phi_{face} = \phi_U + (-\tfrac{1}{8}\phi_{UU} - \tfrac{1}{4}\phi_U + \tfrac{3}{8}\phi_D)$$
$$= \phi_U + \tfrac{1}{8}(\phi_U - \phi_{UU}) + \tfrac{3}{8}(\phi_D - \phi_U)$$
$$= \phi_U + \tfrac{1}{8}\left(\frac{\phi_U - \phi_{UU}}{\phi_D - \phi_U} + 3\right)(\phi_D - \phi_U)$$
$$= \phi_U + \tfrac{1}{8}(r+3)(\phi_D - \phi_U)$$

Hence, by comparison with $\phi_{face} = \phi_U + \tfrac{1}{2}\psi(r)(\phi_D - \phi_U)$,

$$\psi(r) = \tfrac{1}{4}(r+3)$$

(b)

LUD

$\psi(r) = r$, so $\psi > 2$ if $r > 2$.

QUICK

For $0 < r \le 1$ then Sweby's upper limit on ψ is $2r$. In this range of r, Sweby's condition is contravened if

$$\tfrac{1}{4}r + \tfrac{3}{4} > 2\text{r}$$
$$\Leftrightarrow \quad \tfrac{3}{4} > \tfrac{7}{4}\text{r}$$
$$\Leftrightarrow \quad r < \tfrac{3}{7}$$

For $r > 1$, Sweby's upper limit on ψ is 2. In this range of r, Sweby's condition is contravened if

$$\tfrac{1}{4}r + \tfrac{3}{4} > 2$$
$$\Leftrightarrow \quad \tfrac{1}{4}r > \tfrac{5}{4}$$
$$\Leftrightarrow \quad r > 5$$

Hence, Sweby's condition is contravened if

$$0 < r < \tfrac{3}{7} \qquad \text{or} \qquad r > 5$$

(Either will do in answer to the question.)

(c) OK for $r \le 0$.

If $0 < r \le 1$ then the required upper limit on ψ is $2r$. Here,

$$\psi(r) = r\left(\frac{1}{1+r^2}\right) + r\left(\frac{r}{1+r^2}\right)$$
$$< r + r$$

Hence, $\psi(r) < 2r$ as required.

If $r > 1$ then the required upper limit on ψ is 2. Here,

$$\psi(r) = \frac{r}{1+r^2} + \frac{r^2}{1+r^2}$$
$$< \frac{r}{r^2} + \frac{r^2}{1+r^2}$$
$$< \frac{1}{r} + 1$$
$$< 2$$

Hence, $\psi < 2$ as required.

For the symmetry property,

$$\frac{\psi(r)}{r} = \frac{1+r}{1+r^2}$$
$$= \frac{\frac{1}{r^2} + \frac{1}{r}}{\frac{1}{r^2} + 1} \qquad \text{(dividing numerator and denominator by } r^2\text{)}$$
$$= \frac{\frac{1}{r} + \left(\frac{1}{r}\right)^2}{1 + \left(\frac{1}{r}\right)^2} \qquad \text{(reversing summands)}$$
$$= \psi(\frac{1}{r})$$

Q10.

(a)

"Transportive" – upstream-biased

"Bounded" – for pure advection, without sources:

(i) the value at a node lies between maximum and minimum at surrounding nodes;

(ii) ϕ = *constant* is a possible solution.

QUICK is transportive but not bounded.

(b) Take the x direction in the direction of the flow for this face. Let the mesh spacing be Δx. Using Taylor-series expansions and using subscript f for 'face':

$$\phi_D = \phi_f \quad + \quad \phi'_f(\frac{\Delta x}{2}) \quad + \quad \frac{1}{2!}\phi''_f(\frac{\Delta x}{2})^2 \quad + \quad \frac{1}{3!}\phi'''_f(\frac{\Delta x}{2})^3 \quad + \quad \cdots$$

$$\phi_U = \phi_f \quad - \quad \phi'_f(\frac{\Delta x}{2}) \quad + \quad \frac{1}{2!}\phi''_f(\frac{\Delta x}{2})^2 \quad - \quad \frac{1}{3!}\phi'''_f(\frac{\Delta x}{2})^3 \quad + \quad \cdots$$

$$\phi_{UU} = \phi_f \quad - \quad \phi'_f(\frac{3\Delta x}{2}) \quad + \quad \frac{1}{2!}\phi''_f(\frac{3\Delta x}{2})^2 \quad - \quad \frac{1}{3!}\phi'''_f(\frac{3\Delta x}{2})^3 \quad + \quad \cdots$$

Hence,

$$\begin{aligned}
-\tfrac{1}{8}\phi_{UU} + \tfrac{3}{4}\phi_U + \tfrac{3}{8}\phi_D = {} & (-\tfrac{1}{8}+\tfrac{3}{4}+\tfrac{3}{8})\phi_f \\
& +(\tfrac{3}{8}-\tfrac{3}{4}+\tfrac{3}{8})\phi'_f(\frac{\Delta x}{2}) \\
& +(-\tfrac{9}{8}+\tfrac{3}{4}+\tfrac{3}{8})\frac{1}{2!}\phi''_f(\frac{\Delta x}{2})^2 \\
& +(\tfrac{27}{8}-\tfrac{3}{4}+\tfrac{3}{8})\frac{1}{3!}\phi'''_f(\frac{\Delta x}{2})^3 \\
& + \quad \cdots \\
= {} & \phi_f \quad + \quad \phi'''_f\frac{\Delta x^3}{16} \quad + \quad \cdots
\end{aligned}$$

The leading-order error term in approximating ϕ_f is proportional to Δx^3; hence, the scheme is 3^{rd}-order accurate.

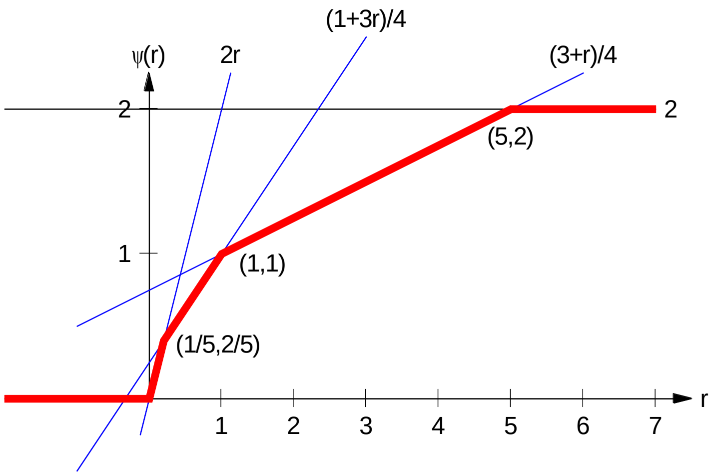
(c) Since the flow is from left to right:

$$\begin{aligned}
\phi_w &= -\frac{1}{8}\phi_{WW} + \frac{3}{4}\phi_W + \frac{3}{8}\phi_P \\
&= -\frac{1}{8}\times 1 + \frac{3}{4}\times 2 + \frac{3}{8}\times 5 \quad = \frac{13}{4} \quad = 3\tfrac{1}{4}
\end{aligned}$$

$$\begin{aligned}
\phi_e &= -\frac{1}{8}\phi_W + \frac{3}{4}\phi_P + \frac{3}{8}\phi_E \\
&= -\frac{1}{8}\times 2 + \frac{3}{4}\times 5 + \frac{3}{8}\times 3 \quad = \frac{37}{8} \quad = 4\tfrac{5}{8}
\end{aligned}$$

Answer: $\phi_w = 3.25$, $\phi_e = 4.625$.

(d)



The key intersection points are:

$$2r = \tfrac{1}{4}(1+3r)$$
$$\Rightarrow \quad 8r = 1+3r$$
$$\Rightarrow \quad 5r = 1$$
$$\Rightarrow \quad r = \tfrac{1}{5}, \quad \text{whence} \quad \psi = \tfrac{2}{5}$$

$$\tfrac{1}{4}(1+3r) = \tfrac{1}{4}(3+r)$$
$$\Rightarrow \quad 1+3r = 3+r$$
$$\Rightarrow \quad 2r = 2$$
$$\Rightarrow \quad r = 1, \quad \text{whence} \quad \psi = 1$$

$$\tfrac{1}{4}(3+r) = 2$$
$$\Rightarrow \quad 3+r = 8$$
$$\Rightarrow \quad r = 5, \quad \text{and} \quad \psi = 2$$

Method 1: find $\psi(r)$ for the QUICK scheme and see where it corresponds to that for UMIST.

The QUICK scheme is

$$\phi_{face} = -\frac{1}{8}\phi_{UU} + \frac{3}{4}\phi_U + \frac{3}{8}\phi_D$$
$$= \phi_U + \frac{1}{8}(-\phi_{UU} - 2\phi_U + 3\phi_D)$$
$$= \phi_U + \frac{1}{8}(3(\phi_D - \phi_U) + (\phi_U - \phi_{UU}))$$
$$= \phi_U + \frac{1}{2}\times\frac{1}{4}(3 + (\frac{\phi_U - \phi_{UU}}{\phi_D - \phi_U}))(\phi_D - \phi_U)$$
$$= \phi_U + \frac{1}{2}\times\frac{1}{4}(3+r)(\phi_D - \phi_U)$$

This corresponds to $\psi = \tfrac{1}{4}(3+r)$ which, from the graph and key points, occurs for $1 \le r \le 5$.

Method 2: Start with $\psi(r)$ in the given range and show that it corresponds to QUICK.

In the range $1 \le r \le 5$ the ψ function is given by

$$\psi = \tfrac{1}{4}(3+r)$$

Hence,

$$\begin{aligned}\phi_{face} &= \phi_U + \tfrac{1}{2}\psi(r)(\phi_D - \phi_U) \\ &= \phi_U + \tfrac{1}{2}\times\tfrac{1}{4}(3+\frac{\phi_U - \phi_{UU}}{\phi_D - \phi_U})(\phi_D - \phi_U) \\ &= \phi_U + \tfrac{1}{8}[3(\phi_D - \phi_U) + \phi_U - \phi_{UU}] \\ &= -\tfrac{1}{8}\phi_{UU} + \tfrac{3}{4}\phi_U + \tfrac{3}{8}\phi_D\end{aligned}$$

which is the QUICK scheme.

(e)

West face:

$$r = \frac{\phi_W - \phi_{WW}}{\phi_P - \phi_W} \quad = \frac{2-1}{5-2} \quad = \frac{1}{3}$$

$$\psi(r) = \max[0, \min\{2, \frac{2}{3}, \frac{1}{2}, \frac{5}{6}\}] \quad = \frac{1}{2}$$

$$\phi_{face} = \phi_W + \tfrac{1}{2}\psi(r)(\phi_P - \phi_W) \quad = 2 + \tfrac{1}{2}\times\tfrac{1}{2}\times(5-2) \quad = 2\tfrac{3}{4}$$

East face:

$$r = \frac{\phi_P - \phi_W}{\phi_E - \phi_P} \quad = \frac{5-2}{3-5} \quad = -\frac{3}{2}$$

$$\psi(r) = 0$$

$$\phi_{face} = \phi_P + \tfrac{1}{2}\psi(r)(\phi_E - \phi_P) \quad = 5 + 0\times(3-5) \quad = 5$$

Answer: $\phi_w = 2.75$, $\phi_e = 5$.

Q11.

(a)

$$(\rho u\phi - \Gamma \frac{d\phi}{dx})_e - (\rho u\phi - \Gamma \frac{d\phi}{dx})_w = \int_w^e s \; dx$$

(b)

$$\left(\frac{d\phi}{dx}\right)_e \quad \to \quad \frac{\phi_E - \phi_P}{\Delta x}$$

$$\left(\frac{d\phi}{dx}\right)_w \quad \to \quad \frac{\phi_P - \phi_W}{\Delta x}$$

(c)

(i) First-order upwind:

$$\phi_e = \phi_P$$

$$\phi_w = \phi_W$$

(ii) Central:

$$\phi_e = \tfrac{1}{2}(\phi_P + \phi_E)$$

$$\phi_w = \tfrac{1}{2}(\phi_W + \phi_P)$$

(d) A flux-differencing scheme is bounded if, for advection and diffusion, with no sources,

(i) values at one node lie within the maximum and minimum values at surrounding nodes;

(ii) ϕ = constant is a possible solution.

With the central scheme for advection the discretised equation (with no source) is

$$\left[\rho u \frac{\phi_P + \phi_E}{2} - \Gamma \frac{\phi_E - \phi_P}{\Delta x}\right] - \left[\rho u \frac{\phi_W + \phi_P}{2} - \Gamma \frac{\phi_P - \phi_W}{\Delta x}\right] = 0$$

or

$$-a_W \phi_W + a_P \phi_P - a_E \phi_E = 0$$

where

$$a_w = \frac{\rho u}{2} + \frac{\Gamma}{\Delta x}$$

$$a_P = \frac{2\Gamma}{\Delta x}$$

$$a_E = -\frac{\rho u}{2} + \frac{\Gamma}{\Delta x}$$

For boundedness we require all the a's to be non-negative, and hence, from a_E:

$$\frac{\rho u}{2} \le \frac{\Gamma}{\Delta x}$$

or

$$\frac{\rho u \Delta x}{\Gamma} \le 2$$

(This quantity is called the cell Peclet number.)

(e)

(i) first-order upwind: $\psi = 0$;

(ii) central: $\psi = 1$.

(f) For the given values of ϕ:

– on the east face ϕ is not monotonic for the W, P, E nodes; hence $\psi = 0$;

– on the west face,

$$r = \frac{\phi_W - \phi_{WW}}{\phi_P - \phi_W} = \frac{1}{2}, \qquad \psi = \frac{2 \times \frac{1}{2}}{1 + \frac{1}{2}} = \frac{2}{3}$$

Hence,

$$\phi_e = \phi_P \quad = 4$$

$$\phi_w = \phi_W + \tfrac{1}{3}(\phi_P - \phi_W) \quad = 2 + \tfrac{1}{3}(4 - 2) \quad = \frac{8}{3}$$

$$\left(\frac{d\phi}{dx}\right)_e = \frac{\phi_E - \phi_P}{\Delta x} \quad = \frac{-2}{0.1} \quad = -20$$

$$\left(\frac{d\phi}{dx}\right)_w = \frac{\phi_P - \phi_W}{\Delta x} \quad = \frac{2}{0.1} \quad = 20$$

Substituting in the integral equation from part (a):

$$(\rho u \phi - \Gamma \frac{d\phi}{dx})_e - (\rho u \phi - \Gamma \frac{d\phi}{dx})_w = s \Delta x$$

$$\rho u (\phi_e - \phi_w) - \Gamma \left\{ (\frac{d\phi}{dx})_e - (\frac{d\phi}{dx})_w \right\} = s \Delta x$$

$$\Rightarrow \qquad 1 \times 1 \times \frac{4}{3} - 0.02 \times (-40) = 0.1 s$$

Hence,

$$s = 10 \times (\frac{4}{3} + 0.8) \quad = \frac{64}{3}$$

Q12.

(a)

Transportive: upstream-biased

Bounded: for an advection-diffusion process without sources:

(i) the value at one node should not lie outside the range of values at surrounding nodes;

(ii) ϕ = constant is a possible solution.

(b) An advection scheme is TVD if the *total variation*

$$\sum |\phi_{r+1} - \phi_r|$$

between a sequence of ϕ values is not increased by the insertion of a cell-face value between the upwind and downstream nodes.

(c)

Upwind

- TVD? yes;
- order = 1 ($\psi \neq 1$ when $r = 1$)

Central

- TVD? no (fails whenever $r < ½$)
- order = 2 ($\psi = 1$ when $r = 1$, but the slope here is zero)

QUICK

- TVD? no (fails when, e.g., $r = 0$; actually it fails whenever $r < \frac{3}{7}$ or $r > 5$)
- order = 3 ($\psi = 1$ when $r = 1$ and the slope here is ¼)

Min-mod

- TVD? yes;
- order = 2 ($\psi = 1$ when $r = 1$, but the slope here is 1 from the left, 0 from the right)

Van Leer

- TVD? yes (immediate for $r \leq 0$; if $r > 0$ then $r < 2r/1 = 2r$ and $r < 2r/r = 2$)
- order = 2 ($\psi = 1$ when $r = 1$, but the slope here is ½)

Q13.
(a)
In 2-d (and with $\rho = 1$) considering the projected areas the mass flux through one face is

$$u\Delta y - v\Delta x$$

where the cell is traversed anticlockwise.

Hence, the outward mass fluxes are:

$$(mass\ \ flux)_n = 6\times(-1) - 3\times(-3) \quad = 3$$

$$(mass\ \ flux)_w = 5\times(-3) - 2\times 0 \quad = -15$$

$$(mass\ \ flux)_s = 5\times 0 - 2\times 2 \quad = -4$$

(b) The mass flux through the *east* face is, in terms of u_e:

$$(mass\ \ flux)_e = u_e \times 4 - 0\times 1 \quad = 4u_e$$

Since the total mass flux out of the cell is zero (by continuity) then

$$4u_e + 3 - 15 - 4 = 0$$

$$\Rightarrow \qquad 4u_e - 16 = 0$$

$$\Rightarrow \qquad u_e = 4$$

(c) If there is no source term or diffusion then the net advective flux of the scalar out of the cell is zero. Hence,

$$\sum_{face\ f} (mass\ \ flux \times \phi_f) = 0$$

Hence:

$$16\times\phi_e + 3\times 4 - 15\times 2 - 4\times 3.5 = 0$$

$$\Rightarrow \qquad 16\phi_e - 32 = 0$$

$$\Rightarrow \qquad \phi_e = 2$$

Q14.

(a)

The *outward* volume flux through each face is

$$u\Delta y - v\Delta x$$

when the cell is traversed in the positive sense (i.e. anticlockwise).

Hence,

$$Q_n = 9\times 0 - (-3)\times(-3) \quad = -9$$

$$Q_w = 4\times(-3) - (-2)\times 1 \quad = -10$$

$$Q_s = 3\times 0 - 3\times 1 \quad = -3$$

(b) On the east face,

$$Q_e = u_e \times 3 - v_e \times 1$$

From the incompressibility condition (net outward volume flux = 0):

$$Q_e + Q_n + Q_w + Q_s = 0$$

$\Rightarrow \qquad Q_e - 9 - 10 - 3 = 0$

$\Rightarrow \qquad 3u_e - v_e = 22 \qquad (*)$

By the circulation condition for irrotational flows ($\oint \mathbf{u}\bullet \mathrm{d}\mathbf{x} = 0$):

$$[u_e \times 1 + v_e \times 3] + [9\times(-3) + (-3)\times 0] + [4\times 1 + (-2)\times(-3)] + [3\times 1 + 3\times 0] = 0$$

$\Rightarrow \qquad u_e + 3v_e - 27 + 10 + 3 = 0$

$\Rightarrow \qquad u_e + 3v_e = 14 \qquad (**)$

Eliminating v_e from (*) and (**) gives

$$10u_e = 80$$

whence,

$$u_e = 8, \quad v_e = 2$$

(c) In the absence of source or diffusion terms, the net advective flux of ϕ out of the cell is zero; i.e.

$$Q_e\phi_e + Q_n\phi_n + Q_w\phi_w + Q_s\phi_s = 0$$

$\Rightarrow \qquad 22\times\phi_e + (-9)\times 0 + (-10)\times 6 + (-3)\times 2 = 0$

$\Rightarrow \qquad 22\phi_e = 66$

$\Rightarrow \qquad \phi_e = 3$

Q15.

(a) Non-slip conditions:

on the lower wall: $u = 0$

on the upper wall: $u = U_0$

(b) Net pressure force (per unit span) is

$$(p_w - p_e)\Delta y$$

But, from the given pressure gradient:

$$-G = \frac{p_e - p_w}{\Delta x}$$

i.e. $p_w - p_e = G\Delta x$

Hence, the net pressure force is

$$G\Delta x\Delta y$$

(c)

$$\tau^{(north)} = \mu \frac{\partial u}{\partial y}\bigg|_{north} \approx \mu \frac{u_{j+1} - u_j}{\Delta y}$$

(d)

On the upper wall:

$$\tau = \mu \frac{\partial u}{\partial y} \approx \mu \frac{U_0 - u_N}{\frac{1}{2}\Delta y} = 2\mu \frac{U_0 - u_N}{\Delta y}$$

On the lower wall:

$$\tau = \mu \frac{\partial u}{\partial y} \approx \mu \frac{u_1 - 0}{\frac{1}{2}\Delta y} = 2\mu \frac{u_1}{\Delta y}$$

(e)

Force balance in fully-developed (non-accelerating) flow:

$$G\Delta x\Delta y + \tau^{(north)}\Delta x - \tau^{(south)}\Delta x = 0$$

<u>Internal Cells</u>

$$G\Delta x\Delta y + \mu\left(\frac{u_{j+1} - u_j}{\Delta y}\right)\Delta x - \mu\left(\frac{u_j - u_{j-1}}{\Delta y}\right)\Delta x = 0$$

Multiplying by $\dfrac{\Delta y}{\mu\Delta x}$,

$$\frac{G\Delta y^2}{\mu} + u_{j+1} - 2u_j + u_{j-1} = 0$$

or

$$-u_{j-1}+2u_j-u_{j+1}=\frac{G\Delta y^2}{\mu}$$

Top Cell

$$G\Delta x\Delta y+2\mu\left(\frac{U_0-u_N}{\Delta y}\right)\Delta x-\mu\left(\frac{u_N-u_{N-1}}{\Delta y}\right)\Delta x=0$$

Multiplying by $\frac{\Delta y}{\mu\Delta x}$,

$$\frac{G\Delta y^2}{\mu}+2U_0-3u_N+u_{N-1}=0$$

or

$$-u_{N-1}+3u_N=\frac{G\Delta y^2}{\mu}+2U_0$$

Bottom Cell

$$G\Delta x\Delta y+\mu\left(\frac{u_2-u_1}{\Delta y}\right)\Delta x-2\mu\left(\frac{u_1}{\Delta y}\right)\Delta x=0$$

Multiplying by $\frac{\Delta y}{\mu\Delta x}$,

$$\frac{G\Delta y^2}{\mu}+u_2-3u_1=0$$

or

$$3u_1-u_2=\frac{G\Delta y^2}{\mu}$$

(f) In the given case, with N = 4 and $\Delta y = H/4$:

$$\frac{G\Delta y^2}{\mu}=\frac{GH^2}{\mu}\times\left(\frac{\Delta y}{H}\right)^2 \quad =\frac{1}{16}\frac{GH^2}{\mu} \quad =\frac{1}{8}U_0$$

The equations are:

$$3u_1-u_2=\frac{1}{8}U_0$$

$$-u_1+2u_2-u_3=\frac{1}{8}U_0$$

$$-u_2+2u_3-u_4=\frac{1}{8}U_0$$

$$-u_3+3u_4=(\frac{1}{8}+2)U_0$$

In matrix form:

$$\begin{pmatrix} 3 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 3 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \frac{1}{8}U_0\begin{pmatrix} 1 \\ 1 \\ 1 \\ 17 \end{pmatrix}$$

Solve by Gaussian elimination:

$$R2 \to 3R2 + R1 \qquad \begin{pmatrix} 3 & -1 & 0 & 0 \\ 0 & 5 & -3 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 3 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \frac{1}{8}U_0\begin{pmatrix} 1 \\ 4 \\ 1 \\ 17 \end{pmatrix}$$

$$R3 \to 5R3 + R2 \qquad \begin{pmatrix} 3 & -1 & 0 & 0 \\ 0 & 5 & -3 & 0 \\ 0 & 0 & 7 & -5 \\ 0 & 0 & -1 & 3 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \frac{1}{8}U_0\begin{pmatrix} 1 \\ 4 \\ 9 \\ 17 \end{pmatrix}$$

$$R4 \to 7R4 + R3 \qquad \begin{pmatrix} 3 & -1 & 0 & 0 \\ 0 & 5 & -3 & 0 \\ 0 & 0 & 7 & -5 \\ 0 & 0 & 0 & 16 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \frac{1}{8}U_0\begin{pmatrix} 1 \\ 4 \\ 9 \\ 128 \end{pmatrix}$$

Back-substituting:

$$16u_4 = 16U_0 \qquad \Rightarrow \qquad u_4 = U_0$$

$$7u_3 - 5u_4 = \tfrac{9}{8}U_0 \qquad \Rightarrow \qquad u_3 = \frac{\tfrac{9}{8}U_0 + 5u_4}{7} \quad = \frac{7}{8}U_0$$

$$5u_2 - 3u_3 = \tfrac{4}{8} \qquad \Rightarrow \qquad u_2 = \frac{\tfrac{4}{8}U_0 + 3u_3}{5} \quad = \frac{5}{8}U_0$$

$$3u_1 - u_2 = \tfrac{1}{8} \qquad \Rightarrow \qquad u_1 = \frac{\tfrac{1}{8}U_0 + u_2}{3} \quad = \frac{2}{8}U_0$$

Answer: $u_1 = \frac{1}{4}U_0, \quad u_2 = \frac{5}{8}U_0, \quad u_3 = \frac{7}{8}U_0, \quad u_4 = U_0$

(g)

The flow rate per unit width is

$$\begin{aligned} q &= \sum u_j \Delta y \\ &= (\sum u_j)\frac{H}{4} \\ &= \frac{22}{8}U_0 \times \frac{H}{4} \\ &= \frac{22}{32}U_0 H \end{aligned}$$

Answer: $q = \frac{11}{16}U_0 H$

(h) The Navier-Stokes equation is

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \mu \nabla^2 u$$

with boundary conditions

$$u = 0 \quad \text{on } y = 0$$

$$u = U_0 \quad \text{on } y = H$$

For fully-developed flow,

$$\frac{\mathrm{D}u}{\mathrm{D}t} = 0, \qquad \frac{\partial p}{\partial x} = -G, \text{ constant}, \qquad \nabla^2 u = \frac{\partial^2 u}{\partial y^2}$$

Hence,

$$0 = G + \mu \frac{\partial^2 u}{\partial y^2}$$

Integrating twice,

$$u = -\frac{G}{2\mu} y^2 + Ay + B$$

Applying the boundary conditions gives

$$B = 0$$

$$A = \frac{U_0}{H} + \frac{GH}{2\mu}$$

Hence,

$$u = -\frac{G}{2\mu} y^2 + \left(\frac{U_0}{H} + \frac{GH}{2\mu} \right) y$$

which is conveniently rearranged in non-dimensional form as

$$\frac{u}{U_0} = \frac{1}{2} \left(\frac{GH^2}{\mu U_0} \right) \frac{y}{H} (1 - \frac{y}{H}) + \frac{y}{H}$$

(i.e. the sum of plane-Poiseuille and Couette-flow solutions).

For the case $\frac{GH^2}{\mu U_0} = 2$ this gives values at the cell-centre points $\frac{y}{H} = \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}$ of

$$\frac{U}{U_0} = \frac{15}{64}, \frac{39}{64}, \frac{55}{64}, \frac{63}{64}$$

respectively – these can be compared with part (f).

For the overall flow rate we find

$$q = \int_0^H u \ \ dy$$

which integrates to give (compare part (g)):

$$q = \frac{2}{3} U_0 H$$

Q16.
Write the original equations as:

$$\begin{array}{llll} 5A & -B & -D & =9.5 \\ -3A & +8B & -C & =-3.5 \\ -4B & +10C & -D & =30.5 \\ -2A & -3C & +10D & =-37 \end{array}$$

Rewrite with the dominant term as the subject:

$$A = \frac{9.5+B+D}{5}$$
$$B = \frac{-3.5+3A+C}{8}$$
$$C = \frac{30.5+4B+D}{10}$$
$$D = \frac{-37+2A+3C}{10}$$

Iterate by the method of Gauss-Seidel:

A	B	C	D
0	0	0	0
1.900	0.275	3.160	–2.372
1.481	0.513	3.018	–2.498
1.503	0.503	3.001	–2.499
1.501	0.501	3.001	–2.500
1.500	0.500	3.000	–2.500
1.500	0.500	3.000	–2.500

Answer: A = 1.50, B = 0.50, C = 3.00, D = –2.50; (these are, in fact, exact).

Q17.
If the equations are rearranged for iteration as, e.g.,

$$A = \frac{1 + B - 9|A|A}{5}$$

then they will diverge because of the strong dependence on A on the RHS. Instead, rearrange the first equation as

$$(5 + 9|A|)A = 1 + B$$

$$\Rightarrow \qquad A = \frac{1 + B}{(5 + 9|A|)}$$

The complete iterative set is

$$A = \frac{1 + B}{5 + 9|A|}$$

$$B = \frac{2 + 2A + C}{5 + 9|B|}$$

$$C = \frac{3 + 2B}{5 + 9|C|}$$

Starting from $A = B = C = 0$ this gives the following iterative sequence (stopping when successive rows are equal to 2 sig figs).

A	***B***	***C***
0	0	0
0.2	0.48	0.792
0.2176	0.3463	0.3045
0.1935	0.3316	0.4733
0.1975	0.3592	0.4016
0.2005	0.3404	0.4273
0.197	0.3499	0.4183
0.1993	0.3457	0.4212

Answer: A = 0.20, B = 0.35, C = 0.42 (2 sig figs)

Q18.

Transferring the solution-dependent part to the LHS (this isn't vital, but it would be expected to improve convergence):

$$-2\phi_{i-1} + (6 + 2|\phi_i|)\phi_i - \phi_{i+1} = 2$$

Rearrange for ϕ_i as the subject:

$$\phi_i = \frac{2 + 2\phi_{i-1} + \phi_{i+1}}{6 + 2|\phi_i|}$$

The case $N = 3$ with $\phi_0 = \phi_4 = 0$ gives the following sequence of equations for iteration:

$$\phi_1 = \frac{2 + \phi_2}{6 + 2|\phi_1|}$$

$$\phi_2 = \frac{2 + 2\phi_1 + \phi_3}{6 + 2|\phi_2|}$$

$$\phi_3 = \frac{2 + 2\phi_2}{6 + 2|\phi_3|}$$

Successive values, by Gauss-Seidel iteration, starting from all $\phi_i = 0$ are as follows.

ϕ_1	ϕ_2	ϕ_3
0	0	0
0.3333	0.4444	0.4815
0.3667	0.4667	0.4213
0.3663	0.4549	0.4252
0.3646	0.4565	0.4252
0.3651	0.4564	0.4252
0.3650	0.4564	0.4252
0.3650	0.4564	0.4252

Answer: $\phi_1 = 0.365$, $\phi_2 = 0.456$, $\phi_3 = 0.425$.

Q19.
The equation set is

$$-a_i\phi_{i-1} + b_i\phi_i - c_i\phi_{i+1} = d_i$$

The proof is by induction.

For $i = 1$ the original equation set rearranges to

$$\phi_1 = \frac{c_1}{b_1}\phi_2 + \frac{d_1 + a_1\phi_0}{b_1}$$

which implies that the formula is OK for $i = 1$ provided that we take

$$P_0 = 0, \qquad Q_0 = \phi_0$$

For $i > 1$, assume

$$\phi_{i-1} = P_{i-1}\phi_i + Q_{i-1}$$

Then, by substitution:

$$-a_i(P_{i-1}\phi_i + Q_{i-1}) + b_i\phi_i - c_i\phi_{i+1} = d_i$$

$$\Rightarrow \qquad (b_i - a_iP_{i-1})\phi_i = c_i\phi_{i+1} + d_i + a_iQ_{i-1}$$

$$\Rightarrow \qquad \phi_i = \frac{c_i}{b_i - a_iP_{i-1}}\phi_{i+1} + \frac{d_i + a_iQ_{i-1}}{b_i - a_iP_{i-1}}$$

This is of the form

$$\phi_i = P_i\phi_{i+1} + Q_i$$

where

$$P_i = \frac{c_i}{b_i - a_iP_{i-1}}$$

$$Q_i = \frac{d_i + a_iQ_{i-1}}{b_i - a_iP_{i-1}}$$

Hence, if the formula holds for $i-1$ then it holds for i. But it is true for 1, and hence, by induction, for 2, 3, 4, …