7. TURBULENCE SPRING 2015

7.1 What is Turbulence?

- A "random", 3-d, time-dependent eddying motion with many scales, superposed on an often drastically simpler *mean* flow.

- A solution of the Navier-Stokes equations.

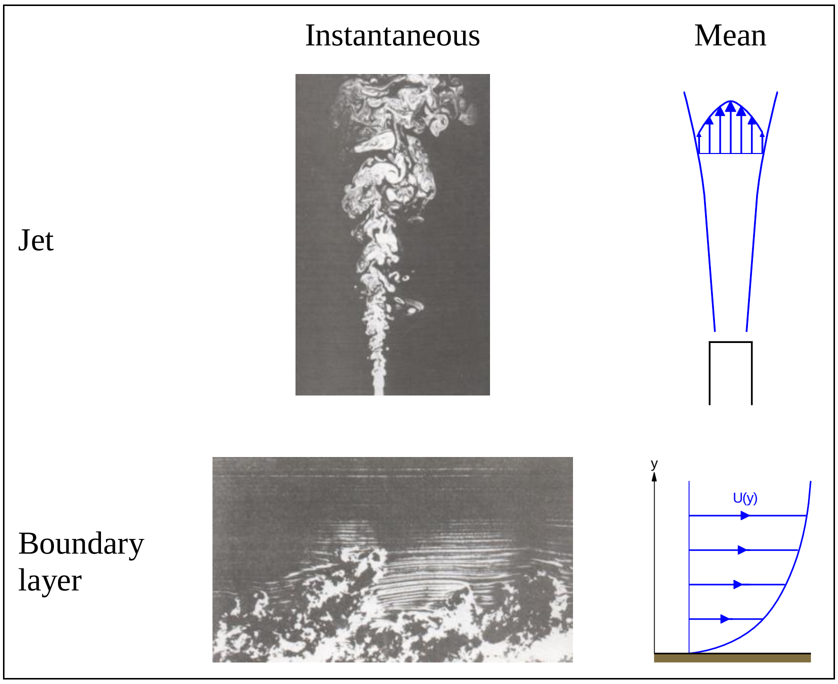
- The natural state at high Reynolds numbers.

- An efficient transporter and mixer ... of momentum, energy, constituents.

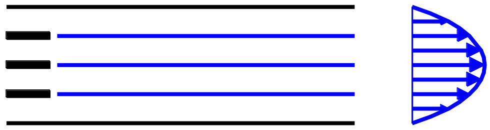
- A major source of energy loss.

- A significant influence on drag and boundary-layer separation.

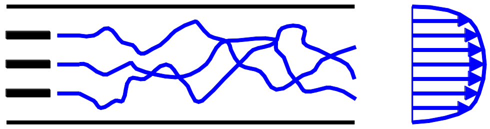
- "The last great unsolved problem of classical physics"; (*variously attributed to Sommerfeld, Einstein and Feynman*).



7.2 Momentum Transfer in Laminar and Turbulent Flow



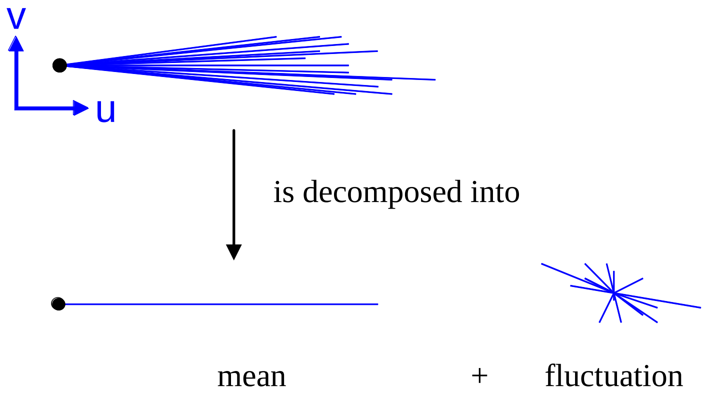
In *laminar* flow adjacent layers of fluid slide past each other **without mixing**. Transfer of momentum occurs between layers moving at different speeds because of **viscous stresses**.



In *turbulent* flow adjacent layers continually intermingle. A **net** transfer of momentum occurs because of the **mixing** of fluid elements from layers with different mean velocity. This mixing is a far more effective means of transferring momentum than viscous stresses. Consequently, the mean-velocity profile tends to be more uniform in turbulent flow.

7.3 Turbulence Notation

The instantaneous value of any flow variable can be decomposed into *mean* + *fluctuation*.



Mean and fluctuating parts are denoted by either:

- an overbar and prime: $u = \bar{u} + u'$

  or
- upper case and lower case: $U + u$

The first is useful in deriving theoretical results, but cumbersome in general use. The notation being used is, hopefully, obvious from the context.

By definition, the average fluctuation is zero:

$$\overline{u'} = 0$$

In experimental work and in steady flow the "mean" is usually a time mean, whilst in theoretical work it is the probabilistic (or "ensemble") mean. The process of taking the mean of a turbulent quantity or a product of turbulent quantities is called *Reynolds averaging*.

The normal rules for averages of products apply:

$$\overline{u^2} = \bar{u}^2 + \overline{u'^2} \qquad (\textit{variance})$$

$$\overline{uv} = \bar{u}\,\bar{v} + \overline{u'v'} \qquad (\textit{covariance})$$

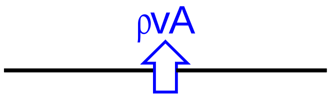
Thus, in turbulent flow the "mean of a product" is not equal to the "product of the means" but includes an (often significant) contribution from the net effect of turbulent fluctuations.

7.4 Effect of Turbulence on the Mean Flow

Engineers are usually only interested in the **mean** flow. However, turbulence must still be considered because, although the averages of individual fluctuations (e.g. u' or v') are zero, the average of a **product** (e.g. $\overline{u'v'}$) is not and may lead to a significant net flux.

Consider mass and x-momentum fluxes in the y direction across surface area A. For simplicity, assume constant density.

7.4.1 Continuity



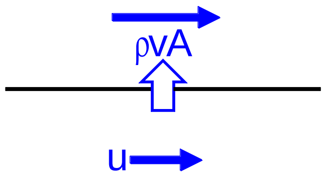
Mass flux: $\rho v A$

Average mass flux: $\rho \bar{v} A$

The only change is that the **instantaneous** velocity is replaced by the **mean** velocity.

> *The mean velocity satisfies the same continuity equation as the instantaneous velocity.*

7.4.2 Momentum



x-momentum flux: $(\rho v A)u \quad = \rho(uv)A$

Average x-momentum flux: $\overline{(\rho v A)u} \quad = \rho(\bar{u}\,\bar{v} + \overline{u'v'})A$

The **average** momentum flux has the same form as the **instantaneous** momentum flux … except for additional fluxes $\rho\overline{u'v'}A$ due to the net effect of turbulent fluctuations. These additional terms arise because of the averaging of a **product** of fluctuating quantities.

A net rate of transport of momentum $\rho\overline{u'v'}A$ from **lower** to **upper** side of an interface ...

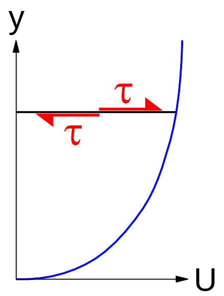
- is equivalent to a net rate of transport of momentum $-\rho\overline{u'v'}A$ from **upper** to **lower**;
- has the same **dynamic effect** (rate of transfer of momentum) as a *stress* (force per unit area) equal to $-\rho\overline{u'v'}$.

This apparent stress is called a *Reynolds stress*.

Other Reynolds stresses ($-\rho\overline{u'u'}$, $-\rho\overline{v'v'}$, etc.) emerge when considering the average flux of the different momentum components in different directions.

> *The mean velocity satisfies the same momentum equation as the instantaneous velocity, except for additional apparent stresses: the Reynolds stresses* $-\rho\overline{u_i'u_j'}$

In a *simple shear flow* the *total stress* really means the net rate of transfer (per unit area) and is given by



$$\tau = \underbrace{\mu \frac{\partial \bar{u}}{\partial y}}_{\substack{viscous\\stress}} \quad \underbrace{- \rho \overline{u'v'}}_{\substack{turbulent\\stress}} \qquad (1)$$

In fully-turbulent flow the turbulent contribution (Reynolds stress) is usually substantially larger than the viscous stress.

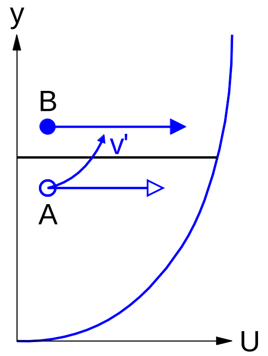
τ can be interpreted as either:

- the *apparent force* (per unit area) exerted by the upper fluid on the lower,

  or
- the *rate of transport of momentum* (per unit area) from upper fluid to lower.

The dynamic effect – a net transfer of momentum – is the same.

The nature of the turbulent stress can be illustrated by considering the motion of particles whose fluctuating velocities allow them to cross an interface.



- If particle A migrates upward ($v' > 0$) then it tends to retain its original momentum, which is now **lower** than its surrounds ($u' < 0$).

- If particle B migrates downward ($v' < 0$) it tends to retain its original momentum which is now **higher** than its surrounds ($u' > 0$).

In both cases, $-\rho u'v'$ is positive and, **on average**, tends to reduce the momentum in the upper fluid and increase the momentum in the lower fluid. Hence there is a net transfer of momentum from upper to lower fluid, equivalent to the effect of an additional mean stress.

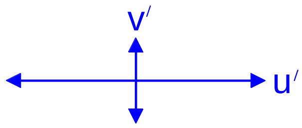
<u>Velocity Fluctuations</u>

Normal stresses: $\overline{u'^2}, \ \overline{v'^2}, \ \overline{w'^2}$

Shear stresses: $\overline{v'w'}, \ \overline{w'u'}, \ \overline{u'v'}$

(In slightly careless usage both $-\rho\overline{u'v'}$ and $\overline{u'v'}$ are referred to as "stresses".)

Most turbulent flows are *anisotropic*; i.e. $\overline{u'^2}$, $\overline{v'^2}$, $\overline{w'^2}$ are different.



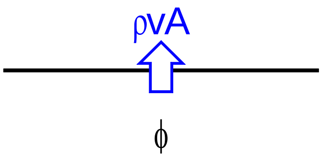
The level of turbulence may be quantified by either

turbulent kinetic energy: $k = \tfrac{1}{2}(\overline{u'^2} + \overline{v'^2} + \overline{w'^2})$

or

turbulence intensity: $i = \dfrac{\textit{root-mean-square fluctuation}}{\textit{mean velocity}} = \dfrac{u'_{rms}}{U} = \dfrac{\sqrt{\frac{2}{3}k}}{U}$

7.4.3 General Scalar



In general, the advection of any scalar quantity ϕ gives rise to an additional scalar flux in the mean-flow equations; e.g.

$$\rho\overline{v\phi} = \rho\bar{v}\,\bar{\phi} + \underbrace{\rho\overline{v'\phi'}}_{additional flux} \qquad (2)$$

Again, the extra term is the result of averaging a *product* of fluctuating quantities.

7.4.4 Turbulence Modelling

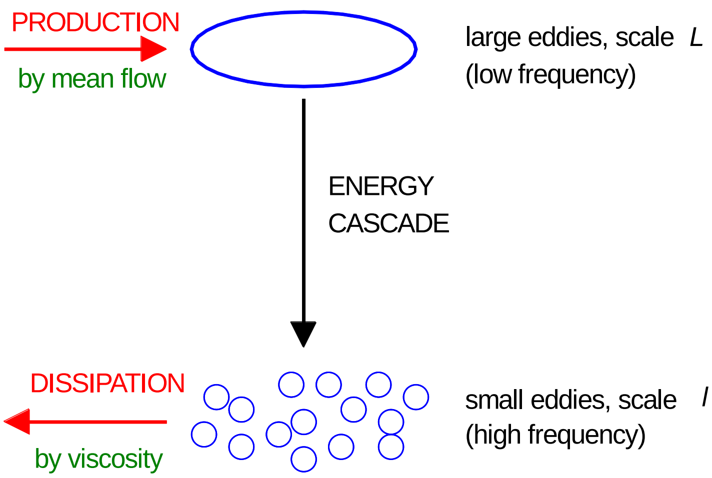
At high Reynolds numbers, turbulent fluctuations cause a much greater net momentum transfer than viscous forces throughout most of the flow. Thus, accurate modelling of the Reynolds stresses is vital.

A *turbulence model* or *turbulence closure* is a means of approximating the Reynolds stresses (and other turbulent fluxes) in order to close the mean-flow equations. Section 8 will describe some of the commoner turbulence models used in engineering.

7.5 Turbulence Generation and Transport

7.5.1 Production and Dissipation

Turbulence is initially generated by instabilities in the flow caused by mean velocity gradients. These eddies in their turn breed new instabilities and hence smaller eddies. The process continues until the eddies become sufficiently small (and fluctuating velocity gradients sufficiently large) that viscous effects eventually become significant and dissipate turbulence energy as heat.



This process – the continual creation of turbulence energy at large scales, transfer of energy to smaller and smaller eddies, and the ultimate dissipation of turbulence energy by viscosity – is called the *turbulent energy cascade*.

7.5.2 Turbulent Transport Equations

It is common experience that turbulence can be transported with the flow. (Think of the turbulent wake behind a vehicle or downwind of a large building.) The following can be proved mathematically.

- Each Reynolds stress $\overline{u_i u_j}$ satisfies its own scalar-transport equation. (This can be found in the optional Section 10; however, a transport equation for turbulent kinetic energy k, which is the sum of the normal stresses, is derived in an Appendix to this section.) These equations contain the usual time-dependence, advection and diffusion terms.

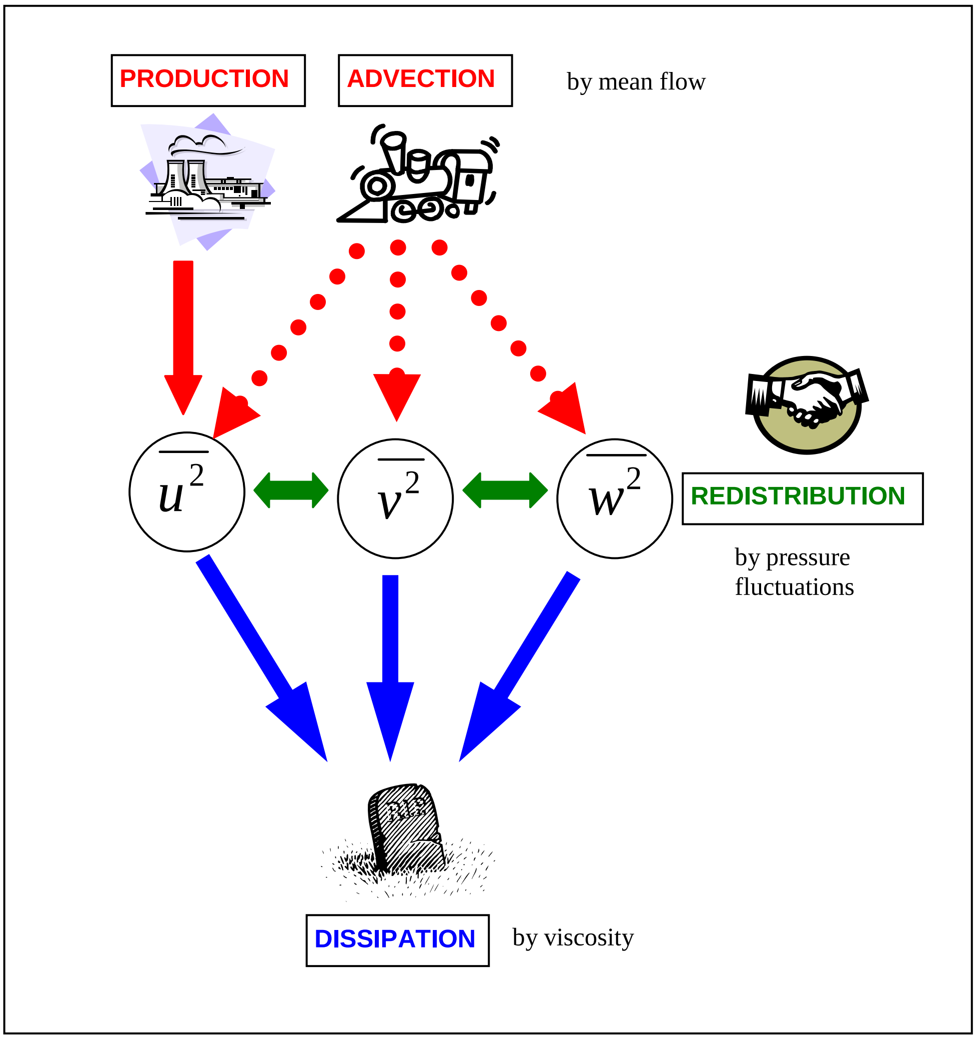
- The source term for an individual Reynolds stress $\overline{u_i u_j}$ has the form:

  $$production + redistribution - dissipation$$

  where:

  production P_{ij} is determined by **mean velocity gradients**;
  redistribution Φ_{ij} transfers energy between stresses via **pressure fluctuations**;
  dissipation ε_{ij} involves **viscosity** acting on fluctuating velocity gradients.

  The k equation just contains production ($P^{(k)}$) and dissipation (ε) terms.



- The production terms for different Reynolds stresses involve different mean velocity gradients; for example, it may be shown that the rates of production (per unit mass) of $\overline{u_1 u_1} \equiv \overline{u^2}$ and $\overline{u_1 u_2} \equiv \overline{uv}$ are, respectively,

$$\begin{aligned} P_{11} &= -2(\overline{uu}\frac{\partial U}{\partial x} + \overline{uv}\frac{\partial U}{\partial y} + \overline{uw}\frac{\partial U}{\partial z}) \\ P_{12} &= -(\overline{uu}\frac{\partial V}{\partial x} + \overline{uv}\frac{\partial V}{\partial y} + \overline{uw}\frac{\partial V}{\partial z}) - (\overline{vu}\frac{\partial U}{\partial x} + \overline{vv}\frac{\partial U}{\partial y} + \overline{vw}\frac{\partial U}{\partial z}) \end{aligned} \qquad (3)$$

(*Exercise*: by "pattern-matching" write production terms for the other stresses).

- Turbulence is usually *anisotropic*; i.e. $\overline{u^2}$, $\overline{v^2}$, $\overline{w^2}$ are all different. This is because:
  (i) mean velocity gradients are greater in some directions than others,
  (ii) motions in certain directions are selectively damped (especially by impermeable boundaries).

- In practice, most turbulence models in regular use do not solve transport equations for all turbulent stresses, but only for the turbulent kinetic energy $k = \frac{1}{2}(\overline{u^2} + \overline{v^2} + \overline{w^2})$, relating the other stresses to this by an *eddy-viscosity* formula (see Section 8).

7.6 Simple Shear Flows

Flows for which there is only one non-zero mean velocity gradient, $\partial U/\partial y$, are called *simple shear flows*. Because they form a good approximation to many real flows, have been extensively researched in the laboratory and are amenable to basic theory they are an important starting point for many turbulence models.

For such a flow, the first of (3) and similar expressions show that $P_{11} > 0$ but that $P_{22} = P_{33} = 0$, and hence $\overline{u^2}$ tends to be the largest of the normal stresses. If there is a rigid boundary on $y = 0$ then it will selectively damp wall-normal fluctuations; hence $\overline{v^2}$ is the smallest of the normal stresses.

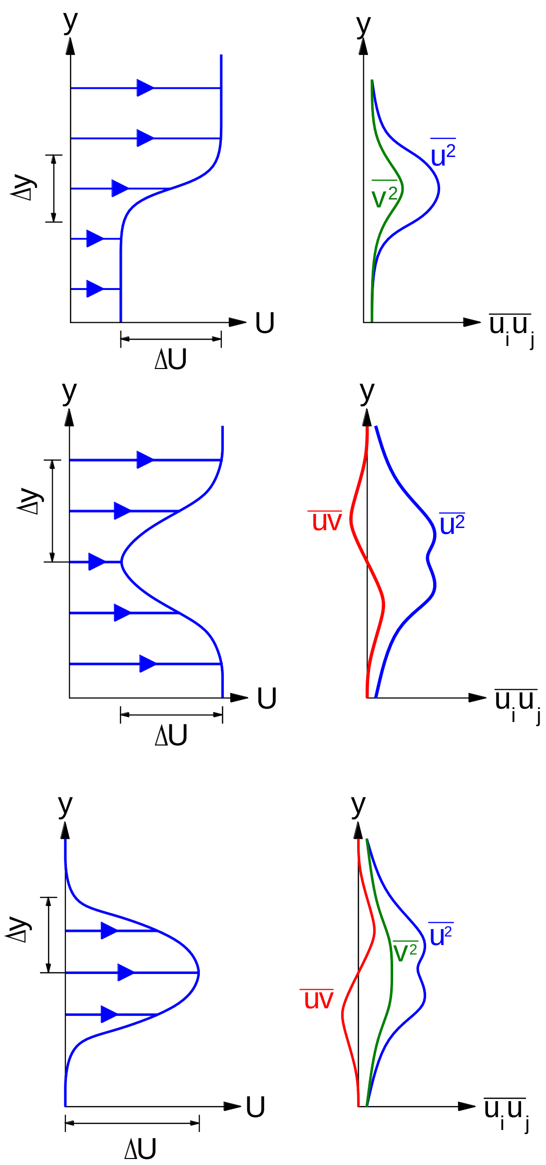
If there are density gradients (for example in atmospheric or oceanic flows, in fires or near heated surfaces) then buoyancy forces will either damp (stable density gradient) or enhance (unstable density gradient) vertical fluctuations.

7.6.1 Free Flows

Mixing layer

Wake
(plane or axisymmetric)

Jet
(plane or axisymmetric)

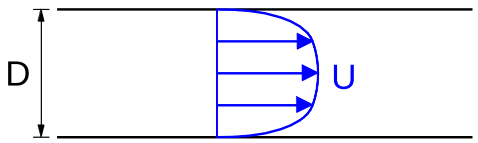


For these simple flows:

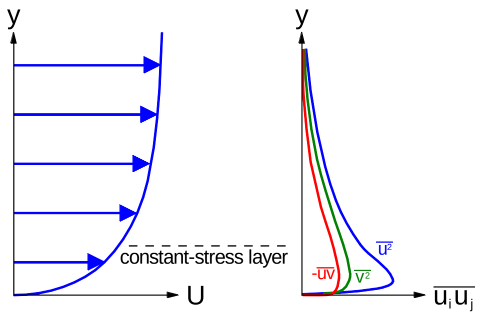
- maximum turbulence occurs where $|\partial U/\partial y|$ (and hence turbulence **production**) is largest;
- $\overline{uv}$ has the opposite sign to $\partial U/\partial y$ and vanishes when this derivative vanishes;
- these turbulent flows are anisotropic: $\overline{u^2} > \overline{v^2}$.

7.6.2 Wall-Bounded Flows

Pipe or channel flow



Flat-plate boundary layer



Even though the **overall** Reynolds number $\mathrm{Re} = U_0 L/\nu$ may be large, and hence viscous transport much smaller than turbulent transport in the majority of the flow, there must be a thin layer very close to the wall where the **local** Reynolds number based on distance from the wall, $\mathrm{Re}_y = \bar{u}y/\nu$, is small and hence molecular viscosity is important.

<u>Wall Units</u>

An important parameter is the *wall shear stress* τ_w. Like any other stress this has dimensions of [density] × [velocity]^2 and hence it is possible to define an important velocity scale called the *friction velocity* u_τ (sometimes written u_*):

$$u_\tau = \sqrt{\tau_w/\rho} \tag{4}$$

From u_τ and ν it is possible to form a viscous length scale $l_\nu = \nu/u_\tau$. Hence, we may define non-dimensional velocity and distance from the wall in so-called *wall units*:

$$U^+ = \frac{U}{u_\tau} \quad , \qquad\qquad y^+ = \frac{yu_\tau}{\nu} \tag{5}$$

y^+ is a measure of proximity to a boundary and how important molecular viscosity is compared with turbulent transport. When y^+ is small ($y^+ < 5$) viscous stresses dominate; when y^+ is large ($y^+ > 30$) turbulent stresses dominate. In between there is a *buffer layer*.

The total mean shear stress is made up of viscous and turbulent parts:

$$\tau = \underbrace{\mu \frac{\partial U}{\partial y}}_{viscous} \quad \underbrace{-\rho \overline{uv}}_{turbulent}$$

When there is no streamwise pressure gradient, τ is approximately constant over a significant depth and is equal to the wall stress τ_w.

Viscous Sublayer (typically, $y^+ < 5$)

Very close to a smooth wall, turbulence is damped by the presence of the boundary. In this region the shear stress is predominantly viscous. Assuming constant shear stress,

$$\tau_w = \mu \frac{\partial U}{\partial y}$$

$$\Rightarrow \qquad U = \frac{\tau_w y}{\mu} \qquad (6)$$

i.e. *the mean velocity profile in the viscous sublayer is linear*.

Log-Law Region (typically, $y^+ > 30$)

At large Reynolds numbers, the turbulent part of the shear stress dominates throughout most of the boundary layer so that on dimensional grounds, since u_τ and y are the only possible velocity and length scales,

$$\frac{\partial U}{\partial y} \propto \frac{u_\tau}{y}$$

or

$$\frac{\partial U}{\partial y} = \frac{u_\tau}{\kappa y}$$

where κ is a constant. Integrating, and putting part of the constant of integration inside the logarithm:

$$U = u_\tau \left(\frac{1}{\kappa} \ln \frac{u_\tau y}{\nu} + B\right) \qquad (7)$$

κ (*von Kármán's constant*) and B are universal constants with experimentally-determined values of about 0.41 and 5 respectively.

Using the definition of wall units (equation (5)) these velocity profiles are often written in non-dimensional form:

Viscous sublayer:	$U^+ = y^+$	
Log layer (smooth wall):	$U^+ = \frac{1}{\kappa} \ln y^+ + B$	(8)

Experimental measurements indicate that the log law actually holds to a good approximation over a substantial proportion of the boundary layer. (This is where the logarithm originates in common friction-factor formulae such as the Colebrook-White formula for pipe flow). *Consistency with the log law is probably the single most important consideration in the construction of turbulence models.*

Summary

- Turbulence is a 3-d, time-dependent, eddying motion with many scales, causing continuous mixing of fluid.

- Each flow variable may be decomposed as *mean* + *fluctuation*.

- The process of averaging turbulent variables or their products is called *Reynolds averaging* and leads to the *Reynolds-averaged Navier-Stokes* (RANS) equations.

- Turbulent fluctuations make a net contribution to the transport of momentum and other quantities. Turbulence enters the mean momentum equations via the *Reynolds stresses*; e.g.
$$\tau_{turb} = -\rho\overline{u'v'}$$

- A means of specifying the Reynolds stresses (and hence solving the mean flow equations) is called a *turbulence model* or *turbulence closure*.

- Turbulence energy is *generated* at large scales by mean-velocity gradients (and, sometimes, body forces such as buoyancy). Turbulence is *dissipated* (as heat) at small scales by viscosity.

- Due to the directional nature of the generating process turbulence is initially anisotropic. Energy is subsequently *redistributed* amongst the different components by the action of pressure fluctuations.

- Turbulence modelling is guided by experimental and theoretical observations for simple free shear flows (mixing layer, jet, wake) and wall-bounded flows (pipe, flat plate).

- Theory and experiment suggest that, throughout much of a turbulent boundary layer the shear stress is constant:
$$\tau = \tau_w = \rho u_\tau^2$$
and the velocity profile is logarithmic:
$$\frac{\partial U}{\partial y} = \frac{u_\tau}{\kappa y}$$

Appendix: Derivation of the Transport Equation for Turbulent Kinetic Energy

For simplicity, restrict to constant-density, constant-viscosity fluids, with no body forces. A more advanced derivation for the individual Reynolds stresses, including body forces, is given in Section 10.

The summation convention is used throughout.

Continuity

Instantaneous: $\frac{\partial u_j}{\partial x_j} = 0$

Average: $\frac{\partial \bar{u}_j}{\partial x_j} = 0$

Subtract: $\frac{\partial u'_j}{\partial x_j} = 0$

Consequently:

- both mean and fluctuating velocities satisfy the incompressibility condition;
- u'_j and $\partial/\partial x_j$ commute when there is an implied summation; i.e.

$$\frac{\partial (u'_j \phi)}{\partial x_j} = u'_j \frac{\partial \phi}{\partial x_j} \qquad \text{for all } \phi$$

This last result will be used several times in what follows.

Momentum

Instantaneous: $\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}(\nu \frac{\partial u_i}{\partial x_j})$

Average: $\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} + \overline{u'_j \frac{\partial u'_i}{\partial x_j}} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j}(\nu \frac{\partial \bar{u}_i}{\partial x_j})$

Subtract: $\frac{\partial u'_i}{\partial t} + \bar{u}_j \frac{\partial u'_i}{\partial x_j} + u'_j \frac{\partial \bar{u}_i}{\partial x_j} + u'_j \frac{\partial u'_i}{\partial x_j} - \overline{u'_j \frac{\partial u'_i}{\partial x_j}} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_i} + \frac{\partial}{\partial x_j}(\nu \frac{\partial u'_i}{\partial x_j})$

Multiply by u'_i, sum over i and average, noting that, e.g., $\overline{u'_i \frac{\partial u'_i}{\partial t}} = \frac{\partial (\frac{1}{2}\overline{u'_i u'_i})}{\partial t} = \frac{\partial k}{\partial t}$:

$$\frac{\partial k}{\partial t} + \bar{u}_j \frac{\partial k}{\partial x_j} + \overline{u'_i u'_j}\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial}{\partial x_j}(\overline{\tfrac{1}{2}u'_i u'_i u'_j}) - 0 = -\frac{\partial}{\partial x_i}(\frac{\overline{p' u'_i}}{\rho}) + \frac{\partial}{\partial x_j}(\nu \overline{u'_i \frac{\partial u'_i}{\partial x_j}}) - \nu \overline{\frac{\partial u'_i}{\partial x_j}\frac{\partial u'_i}{\partial x_j}}$$

where we have used the commuting of u'_j and $\partial/\partial x_j$ from above and rearranged the viscous term into one involving a derivative of a product. Changing the dummy summation index for the pressure term and rearranging gives:

$$\frac{\partial k}{\partial t} + \bar{u}_j \frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_j}\left\{ \nu \frac{\partial k}{x_j} - \overline{(\frac{p}{\rho} + \tfrac{1}{2}u'_i u'_i) u'_j} \right\} - \overline{u'_i u'_j}\frac{\partial \bar{u}_i}{\partial x_j} - \nu \overline{\frac{\partial u'_i}{\partial x_j}\frac{\partial u'_i}{\partial x_j}}$$

or

$$\boxed{\frac{\mathrm{D}k}{\mathrm{D}t} = \frac{\partial d_i^{(k)}}{\partial x_i} + P^{(k)} - \varepsilon}$$

where:

$$d_j^{(k)} = \nu \frac{\partial k}{\partial x_j} - \frac{1}{\rho}\overline{(p' + \tfrac{1}{2} u_i' u_i') u_j'}$$ *diffusion*

$$P^{(k)} = -\overline{u_i' u_j'} \frac{\partial \overline{u}_i}{\partial x_j}$$ *production* (by mean velocity gradients)

$$\varepsilon = \nu \overline{(\frac{\partial u_i'}{\partial x_j})^2}$$ *dissipation* (by viscosity)

Examples

Q1. Which is more viscous, air or water?

Air:	$\rho = 1.20$ kg m^{-3}	$\mu = 1.80\times10^{-5}$ kg m^{-1} s^{-1}
Water:	$\rho = 1000$ kg m^{-3}	$\mu = 1.00\times10^{-3}$ kg m^{-1} s^{-1}

Q2. The accepted critical Reynolds number in a round pipe (based on bulk velocity and diameter) is 2300. At what speed is this attained in a 5 cm diameter pipe for (a) air; (b) water?

Q3. Sketch the mean velocity profile in a pipe at Reynolds numbers of (a) 500; (b) 50 000. What is the shear stress along the pipe axis in either case?

Q4. Explain the process of flow separation. How does deliberately "tripping" a developing boundary layer help to prevent or delay separation on a convex curved surface?

Q5. The following couplets are measured values of (u,v) in an idealised 2-d turbulent flow. Calculate $\bar{u}$, $\bar{v}$, $\overline{u'^2}$, $\overline{v'^2}$, $\overline{u'v'}$ from this set of numbers.

(3.6,0.2)	(4.1,–0.4)	(5.2,–0.2)	(4.6,–0.4)	(3.4,0.0)
(3.8,–0.4)	(4.4,0.2)	(3.9,0.4)	(3.0,0.4)	(4.4,–0.3)
(4.0,–0.1)	(3.4,0.1)	(4.6,-0.2)	(3.6,0.4)	(4.0,0.3)

Q6. The rate of production (per unit mass of fluid) of $\overline{u^2}$ and $\overline{uv}$ are, respectively,

$$P_{11} = -2(\overline{uu}\frac{\partial U}{\partial x} + \overline{uv}\frac{\partial U}{\partial y} + \overline{uw}\frac{\partial U}{\partial z})$$

$$P_{12} = -(\overline{uu}\frac{\partial V}{\partial x} + \overline{uv}\frac{\partial V}{\partial y} + \overline{uw}\frac{\partial V}{\partial z}) - (\overline{vu}\frac{\partial U}{\partial x} + \overline{vv}\frac{\partial U}{\partial y} + \overline{vw}\frac{\partial U}{\partial z})$$

(a) By inspection, write down similar expressions for P_{22}, P_{33}, P_{23}, P_{31}, the rates of production of $\overline{v^2}$, $\overline{w^2}$, $\overline{vw}$ and $\overline{wu}$ respectively.

(b) Write down expressions for P_{11}, P_{22}, P_{33} and P_{12}, P_{23}, P_{31} in simple shear flow (where $\partial U/\partial y$ is the only non-zero mean velocity gradient). What does this indicate about the relative distribution of turbulence energy amongst the various Reynolds-stress components? Write down also an expression for $P^{(k)}$, the rate of production of turbulence kinetic energy.

(c) A mathematician would summarise the different production terms by a single compact formula

$$P_{ij} = -(\overline{u_i u_k}\frac{\partial U_j}{\partial x_k} + \overline{u_j u_k}\frac{\partial U_i}{\partial x_k})$$

using the *Einstein summation convention* – implied summation over a repeated index (in this case, k). See if you can relate this to the above expressions for the P_{ij}.