Classification of Second-Order PDEs

Given a generalised PDE on $\phi(x,y)$:

$$a\phi_{xx} + b\phi_{xy} + c\phi_{yy} + d\phi_x + e\phi_y + f = 0 \quad (1)$$

where "a, b, c, d, e" and "f" are functions that <u>are not dependent</u> on ϕ.

If we change the coordinates from x and y to ν and η (independent variables):

$$\begin{cases} x = x(\nu,\eta) \\ y = y(\nu,\eta) \end{cases} \quad (2)$$

This is similar to change coordinates of the system; now using the <u>chain rule of differentiation</u>:

$$\phi_x = \phi_\nu \nu_x + \phi_\eta \eta_x \quad (3.1)$$

$$\phi_y = \phi_\nu \nu_y + \phi_\eta \eta_y \quad (3.2)$$

$$\phi_{xx} = \phi_{\nu\nu}\nu_x^2 + 2\phi_{\nu\eta}\nu_x\eta_x + \phi_{\eta\eta}\eta_x^2 + \phi_\nu \nu_{xx} + \phi_\eta \eta_{xx} \quad (3.3)$$

$$\phi_{yy} = \phi_{\nu\nu}\nu_y^2 + 2\phi_{\nu\eta}\nu_y\eta_y + \phi_{\eta\eta}\eta_y^2 + \phi_\nu \nu_{yy} + \phi_\eta \eta_{yy} \quad (3.4)$$

$$\phi_{xy} = \phi_{\nu\nu}\nu_x\nu_y + \phi_{\nu\eta}(\nu_x\eta_y + \nu_y\eta_x) + \phi_{\eta\eta}\eta_x\eta_y + \phi_\nu\nu_{xy} + \phi_\eta\eta_{xy} \tag{3.5}$$

Now, replacing Eqns (3.1-5) in (1):

$$A\phi_{\nu\nu} + B\phi_{\nu\eta} + C\phi_{\eta\eta} + D\phi_\nu + E\phi_\eta + F = 0 \tag{4}$$

where:

$$A = a\nu_x^2 + b\nu_x\nu_y + c\nu_y^2 \tag{5.1}$$

$$B = 2a\nu_x\eta_x + b(\nu_x\eta_y + \nu_y\eta_x) + 2c\nu_y\eta_y \tag{5.2}$$

$$C = a\eta_x^2 + b\eta_x\eta_y + c\eta_y^2 \tag{5.3}$$

$$D = d\nu_x + e\nu_y \tag{5.4}$$

$$E = d\eta_x + e\eta_y \tag{5.5}$$

$$F = f \tag{5.6}$$

Equation (4) can be simplified if we choose $\underline{\nu}$ and η in a way that $\underline{A = C = 0}$ – thus eqns. (5.1,3):

$$\begin{cases} a\nu_x^2 + b\nu_x\nu_y + c\nu_y^2 = 0 \\ a\eta_x^2 + b\eta_x\eta_y + c\eta_y^2 = 0 \end{cases} \tag{6.}$$

We can assume that neither $\underline{\nu_y}$ nor η_y are $\underline{\text{zero}}$, and rearrange Eqn. 6 as,

$$\begin{cases} a\dfrac{\nu_x^2}{\nu_y^2} + b\dfrac{\nu_x}{\nu_y} + c = 0 \\ a\dfrac{\eta_x^2}{\eta_y^2} + b\dfrac{\eta_x}{\eta_y} + c = 0 \end{cases} \Rightarrow a f^2 + b f + c = 0 \quad (7)$$

$$\text{where } f = \begin{cases} \nu_x/\nu_y \\ \eta_x/\eta_y \end{cases}$$

This resulting quadratic equation (Eqn. 7) has roots that can be either real or imaginary (complex) depending on the ~~determinant~~ determinant

$$D = b^2 - 4ac \quad (8)$$

(i) <u>Hyperbolic Equation: $D > 0$</u>

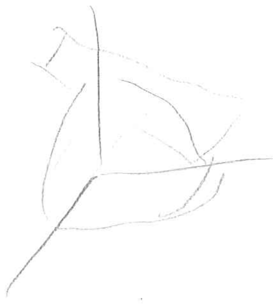
- There are <u>2</u> <u>real</u> <u>distincts</u> <u>roots</u>
- The roots are

$$\nu_x/\nu_y = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad (9.1)$$

$$\eta_x/\eta_y = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad (9.2)$$

These roots can be interpreted as surfaces tangents of a hypersurface where

$$d\nu = \nu_x dx + \nu_y dy = 0$$



Thus

$$\begin{cases} \left.\dfrac{dy}{dx}\right|_\nu = -\dfrac{\nu_x}{\nu_y} = \dfrac{b - \sqrt{b^2-4ac}}{2a} \\[2ex] \left.\dfrac{dy}{dx}\right|_\eta = -\dfrac{\eta_x}{\eta_y} = \dfrac{b + \sqrt{b^2-4ac}}{2a} \end{cases} \quad (10)$$

In other words, the roots of quadratic eqn. are slopes of the ν and η space-curves. These curves are called characteristic curves, which are the preferred direction along which information propagates in a hyperbolic system.

(ii) Parabolic Equation: $\mathcal{D} = 0$

- There is only one double root

- The two characteristic curves are:

$$\left.\frac{dy}{dx}\right|_{\nu} = \left.\frac{dy}{dx}\right|_{\eta} = -\frac{b}{2a} \qquad (11)$$

(iii) <u>Elliptic Equation: $\mathcal{D} < 0$</u>

- There is <u>no real roots</u>;
- There ~~~~ are no transformation that can help eliminate the second derivatives in ν and η.