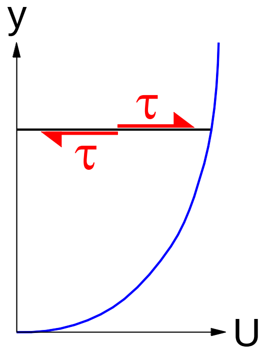
8.1 Eddy-viscosity models
8.2 Advanced turbulence models
8.3 Wall boundary conditions
Summary
References
Examples

The *Reynolds-averaged Navier-Stokes* (RANS) equations are transport equations for the *mean* variables in a turbulent flow. These equations contain net fluxes due to turbulent fluctuations. Turbulence models are needed to specify these fluxes.

8.1 Eddy-Viscosity Models



8.1.1 The Eddy-Viscosity Hypothesis

The mean shear stress has both viscous and turbulent parts. In simple shear:

$$\tau = \underbrace{\mu \frac{\partial U}{\partial y}}_{viscous} \quad \underbrace{-\rho \overline{uv}}_{turbulent} \tag{1}$$

The most popular type of turbulence model is an *eddy-viscosity model* (EVM) which assumes that turbulent stress is proportional to mean velocity gradient in a manner similar to viscous stress. In simple shear,

$$-\rho \overline{uv} = \mu_t \frac{\partial U}{\partial y} \tag{2}$$

μ_t is called an *eddy viscosity* or *turbulent viscosity*. The mean shear stress is then

$$\tau = \mu_{eff} \frac{\partial U}{\partial y} \tag{3}$$

where

$$\mu_{eff} = \mu + \mu_t \tag{4}$$

Notes.
(1) This is a model!
(2) μ is a physical property of the *fluid* and can be measured;
μ_t is a hypothetical property of the *flow* and must be modelled.
(3) μ_t varies with position.
(4) At high Reynolds numbers, $\mu_t \gg \mu$ throughout much of the flow.

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[1] Better, but very mathematical, descriptions of turbulence and its modelling can be found in:
Pope, S.B., 2000, "*Turbulent flows*", Cambridge University Press.
Schlichting, H. and Gersten, K., 1999, *Boundary layer theory*, 8th English Edition, Springer-Verlag.
Wilcox, D.C., 2006, "*Turbulence Modelling for CFD*", 3rd Edition, DCW Industries.

Eddy-viscosity models are widely used and popular because:

- they are easy to implement in existing viscous solvers;
- extra viscosity aids stability;
- they have some theoretical foundation in simple shear flows.

However, one should exercise caution because:

- there is little theoretical foundation in complex flows;
- modelling turbulent transport is reduced to a single scalar μ_t and hence at most one Reynolds stress can be represented accurately.

8.1.2 The Eddy Viscosity in the Log-Law Region

In the log-law region of a turbulent boundary layer it is assumed that:

(a) (i) total stress is constant (and equal to that at the wall);
(ii) viscous stress is negligible compared to turbulent stress;

$$\tau^{(turb)} = \tau_w \equiv \rho u_\tau^2$$

(b) the mean velocity profile is logarithmic;

$$\frac{\partial U}{\partial y} = \frac{u_\tau}{\kappa y}$$

The eddy viscosity is then

$$\mu_t \equiv \frac{\tau^{(turb)}}{\partial U/\partial y} = \frac{\rho u_\tau^2}{u_\tau / \kappa y} = \rho(\kappa u_\tau y)$$

Hence, in the log-law region, with $\nu_t = \mu_t/\rho$ as the *kinematic eddy viscosity*,

$$\boxed{\nu_t = \kappa u_\tau y} \qquad (5)$$

8.1.3 General Stress-Strain Relationship

The stress-strain relationship (2) applies only in simple shear and cannot hold in general because the LHS is symmetric in x and y components but the RHS is not. The appropriate generalisation gives representative shear and normal stresses (from which others can be obtained by "pattern-matching"):

$$-\rho\overline{uv} = \mu_t\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right)$$

$$-\rho\overline{u^2} = 2\mu_t \frac{\partial U}{\partial x} - \frac{2}{3}\rho k$$

where k is turbulent kinetic energy. The $-\frac{2}{3}\rho k$ part ensures the correct sum of normal stresses, $-\rho(\overline{u^2} + \overline{v^2} + \overline{w^2}) = -2\rho k$ because, in incompressible flow, $\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = 0$.

Using suffix notation the shear and normal stresses can be summarised by

$$-\rho\overline{u_i u_j} = \mu_t\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) - \frac{2}{3}\rho k \delta_{ij}$$

8.1.4 Other Turbulent Fluxes

According to *Reynolds' analogy* it is common to assume a gradient-diffusion relationship between any turbulent flux and the gradient of the corresponding mean quantity; e.g.

$$-\rho\overline{v\phi} = \Gamma_t \frac{\partial \Phi}{\partial y} \qquad (6)$$

The *turbulent diffusivity* Γ_t is proportional to the eddy viscosity:

$$\Gamma_t = \frac{\mu_t}{\sigma_t} \qquad (7)$$

σ_t is called a *turbulent Prandtl number*. Since the same turbulent eddies are responsible for transporting momentum and other scalars its value is approximately 1.

8.1.5 Specifying the Eddy Viscosity

With the eddy-viscosity hypothesis, closure of the mean-flow equations now rests solely on the specification of μ_t, a property of the turbulent flow.

The *kinematic* eddy viscosity $\nu_t = \mu_t/\rho$ has dimensions of [velocity] × [length], which suggests that it be modelled as

$$\nu_t = u_0 l_0 \qquad (8)$$

Physically, u_0 should reflect the magnitude of velocity fluctuations and l_0 be related to the size of turbulent eddies. For example, in the log-law region, $\nu_t = \kappa u_\tau y$, or

$$\nu_t = \textit{velocity}\ (u_\tau) \times \textit{length}\ (\kappa y)$$

For wall-bounded flows a candidate for u_0 is the friction velocity $u_\tau = \sqrt{\tau_w/\rho}$. However, this is not a *local* scale (because you need to know where the nearest wall is); a more appropriate velocity scale in general is $k^{1/2}$, where k is the turbulent kinetic energy.

For simple wall-bounded flows, l_0 is proportional to distance from the boundary (e.g. $l_0 = \kappa y$). For free shear flows (e.g. jet, wake, mixing layer) l_0 is proportional to the width of the shear layer. However, both of these are geometry-dependent. For greater generality, we need to relate l_0 to local turbulence properties.

Common practice is to solve transport equations for one or more turbulent quantities (usually *k + one other*) from which μ_t can be derived on dimensional grounds. The following classification of eddy-viscosity models is based on the number of transport equations.

zero-equation models:
- – constant-eddy-viscosity models;
- – mixing-length models: l_0 specified algebraically; u_0 from mean flow gradients.

one-equation models:
- – l_0 specified algebraically; transport equation to derive u_0;

two-equation models:
- – transport equations for quantities from which u_0 and l_0 can be derived.

Of these, the most popular in general-purpose CFD are two-equation models: in particular, the k-ε and k-ω models.

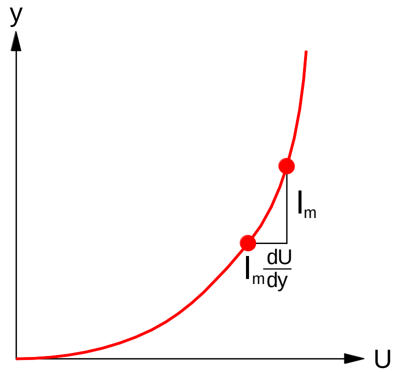
8.1.6 Mixing-Length Models (Prandtl, 1925).

Eddy viscosity:

$$\mu_t = \rho \nu_t \qquad \text{where} \qquad \nu_t = u_0 l_m \tag{9}$$

The *mixing length*, l_m, is specified algebraically and the velocity scale u_0 is then determined from the mean-velocity gradient. In simple shear:

$$u_0 = l_m \left| \frac{\partial U}{\partial y} \right| \tag{10}$$



The model is based on the premise that if a turbulent eddy displaces a fluid particle by distance l_m its velocity will differ from its surrounds by an amount $l_m |\partial U/\partial y|$. (Any constant of proportionality can be absorbed into the definition of l_m.)

The resulting turbulent shear stress is (assuming positive velocity gradient):

$$\tau^{(turb)} = \mu_t \frac{\partial U}{\partial y} \quad = \rho l_m^2 \left(\frac{\partial U}{\partial y} \right)^2 \tag{11}$$

The mixing length l_m depends on the type of flow.

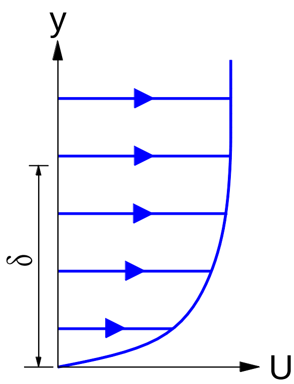
Log Layer

In the log layer,

$$\tau^{(turb)} = \rho u_\tau^2 \qquad \text{and} \qquad \frac{\partial U}{\partial y} = \frac{u_\tau}{\kappa y}$$

Equation (11) then implies that

$$l_m = \kappa y$$



General Wall-bounded flows

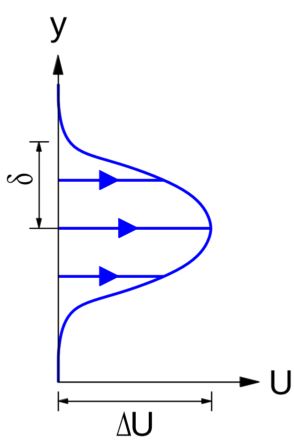
In general, l_m is limited to a certain fraction of the boundary-layer depth δ. Cebeci and Smith (1974) suggest:

$$l_m = \min(\kappa y, 0.09\delta) \tag{12}$$

Free shear flows

l_m is assumed constant and proportional to the shear-layer half-width δ. Wilcox (2006) suggests:

$$\frac{l_m}{\delta} = \begin{cases} 0.071 & \text{(mixing layer)} \\ 0.098 & \text{(plane jet)} \\ 0.080 & \text{(round jet)} \\ 0.180 & \text{(plane wake)} \end{cases} \tag{13}$$



Mixing-length models work well in near-equilibrium boundary layers or free-shear flows. However, although generalisations of (10) exist for arbitrary velocity fields, it difficult to specify l_m for complex flows.

8.1.7 The *k*-ε Model

This is probably the most common type of turbulence model in use today. It is a two-equation eddy-viscosity model with the following specification:

$$\mu_t = \rho \nu_t \ , \qquad \nu_t = C_\mu \frac{k^2}{\varepsilon} \tag{14}$$

k is the turbulent kinetic energy and ε is the rate of dissipation of turbulent kinetic energy. In the standard model C_μ is a constant (with a typical value 0.09).

k and ε are determined by solving transport equations. For the record (i.e. you don't have to learn them!) they are given here in conservative differential form:

$$\begin{array}{lllll}
\dfrac{\partial}{\partial t}(\rho k) & +\dfrac{\partial}{\partial x_i}(& \rho U_i k & -\Gamma^{(k)}\dfrac{\partial k}{\partial x_i} &) = \quad \rho(P^{(k)} - \varepsilon) \\
\dfrac{\partial}{\partial t}(\rho \varepsilon) & +\dfrac{\partial}{\partial x_i}(& \rho U_i \varepsilon & -\Gamma^{(\varepsilon)}\dfrac{\partial \varepsilon}{\partial x_i} &) = \quad \rho(C_{\varepsilon 1} P^{(k)} - C_{\varepsilon 2}\varepsilon)\dfrac{\varepsilon}{k} \\
\textit{rate of change} & & \textit{advection} & \textit{diffusion} & \qquad \textit{source}
\end{array} \tag{15}$$

where $P^{(k)}$ is the rate of production of k, which will be given below. (In these equations there is an implied summation over the repeated index i .)

The diffusivities of k and ε are related to the molecular and turbulent viscosities:

$$\Gamma^{(k)} = \mu + \frac{\mu_t}{\sigma_k}\ , \qquad \Gamma^{(\varepsilon)} = \mu + \frac{\mu_t}{\sigma_\varepsilon}$$

and, in the standard model (Launder and Spalding, 1974), model constants are:

$$C_\mu = 0.09, \quad C_{\varepsilon 1} = 1.44, \quad C_{\varepsilon 2} = 1.92, \quad \sigma_k = 1, \quad \sigma_\varepsilon = 1.3 \tag{16}$$

Notes.

(1) The *k*-ε model is not a single model but a class of slightly different schemes. Variants have different coefficients, some including dependence on molecular-viscosity effects near boundaries ("low-Reynolds-number *k*-ε models") and/or strain-dependent C_μ (e.g. "*realisable*" *k*-ε models). Others have a slightly different ε equation.

(2) Apart from the diffusion term, the k transport equation is that derived from the Navier-Stokes equation. The ε equation is, however, heavily modelled.

(3) Although k is a logical choice (because it has a clear physical definition and can be measured), use of ε as a second scale is not universal and other combinations such as *k*-ω (ω is a frequency), *k*-τ (τ is a timescale) or *k*-*l* (*l* is a length) may be encountered. A popular hybrid of *k*-ω and *k*-ε models is the SST model of Menter, 1994.

Rate of Production of Turbulent Kinetic Energy

The source term in the k equation is a balance between *production* $P^{(k)}$ and *dissipation* ε. The rate of production of turbulent kinetic energy (per unit mass), $P^{(k)}$, is given in simple shear by

$$P^{(k)} = -\overline{uv}\frac{\partial U}{\partial y} \quad = \nu_t \left(\frac{\partial U}{\partial y}\right)^2 \tag{17}$$

or, in general, by

$$P^{(k)} = -\overline{u_i u_j}\frac{\partial U_i}{\partial x_j} \qquad (18)$$

with implied summation over the repeated indices. Under the model assumptions, $P^{(k)}$ is invariably positive; (*exercise*: prove this for a general flow).

A flow for which $P^{(k)} = \varepsilon$ (production equals dissipation) is said to be in *local equilibrium*.

Consistency With the Log Law

The constants in the k-ε model may be chosen for consistency with the log law. In this region, the mean-velocity gradient is

$$\frac{\partial U}{\partial y} = \frac{u_\tau}{\kappa y} \qquad (19)$$

The Reynolds stresses are assumed to be constant and viscous stresses to be negligible. In particular, the kinematic shear stress is

$$-\overline{uv} = \tau_w/\rho = u_\tau^2 \qquad (20)$$

In the log-law region, we have already established that the kinematic eddy viscosity is

$$\nu_t = \kappa u_\tau y \qquad (21)$$

Equation (17) then gives the rate of production of turbulent kinetic energy as

$$P^{(k)} = \frac{u_\tau^3}{\kappa y} \qquad (22)$$

With the further assumption of local equilibrium, $P^{(k)} = \varepsilon$, equations (21) and (22) give

$$\nu_t = \frac{u_\tau^4}{\varepsilon}$$

which, by comparison with the k-ε eddy-viscosity formula (14), shows that C_μ may be determined experimentally from

$$C_\mu = \left(\frac{-\overline{uv}}{k}\right)^2 = \frac{u_\tau^4}{k^2} \qquad \text{or} \qquad u_\tau = C_\mu^{1/4} k^{1/2} \qquad (23)$$

A typical experimentally-measured value is $(-\overline{uv})/k = 0.3$, which gives the standard value $C_\mu = 0.09$.

The high-Reynolds-number (molecular viscosity μ negligible) form of the ε equation (15) is consistent with the log law provided the constants satisfy (see the Examples):

$$(C_{\varepsilon 2} - C_{\varepsilon 1})\sigma_\varepsilon \sqrt{C_\mu} = \kappa^2 \qquad (24)$$

In practice, the standard constants do not quite satisfy this, but have values which give better agreement over a wide range of flows.

8.2 Advanced Turbulence Models

Eddy-viscosity models are popular because:

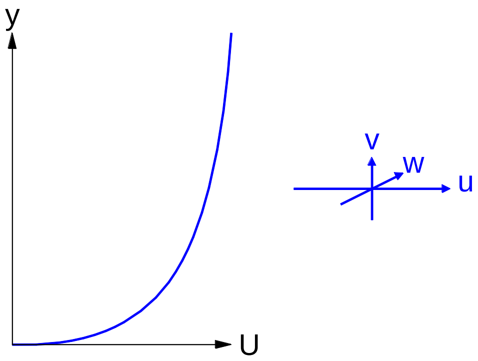
- they are simple to code;
- extra viscosity aids stability;
- they are supported theoretically in some simple but common types of flow;
- they are very effective in many engineering flows.

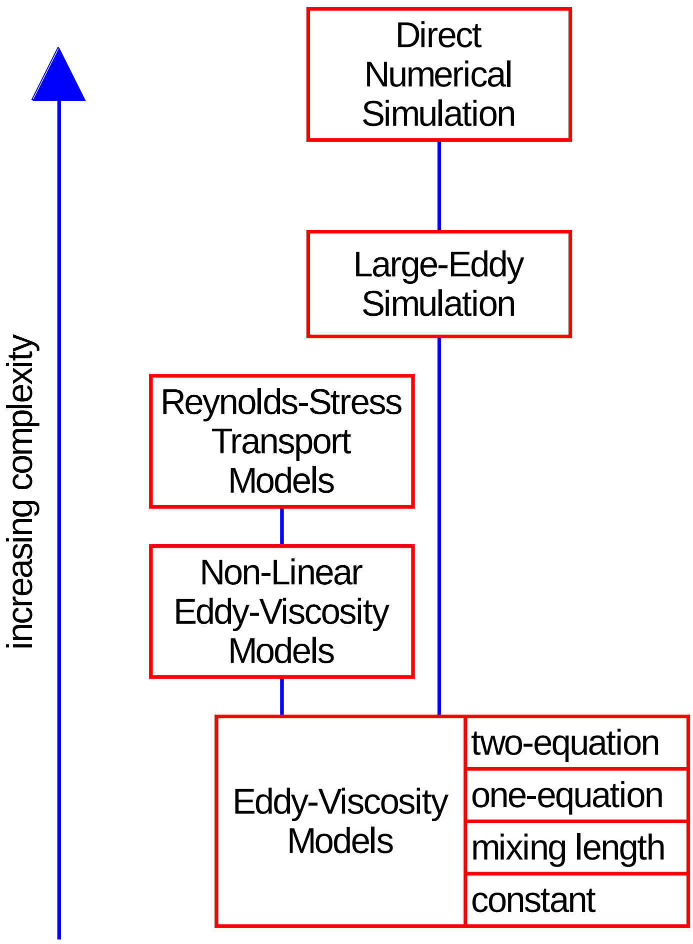
However, the dependence of a turbulence model on a single scalar μ_t is clearly untenable when more than one stress component has an effect on the mean flow. The eddy-viscosity model fails to represent turbulence physics, particularly in respect of the different rates of production of the different Reynolds stresses and the resulting anisotropy.

A classic example occurs in a simple fully-developed boundary-layer where, in the logarithmic region, the various normal stresses are typically in the ratio

$$\overline{u^2} : \overline{v^2} : \overline{w^2} = 1.0 : 0.4 : 0.6 \qquad (25)$$

An eddy-viscosity model would, however, predict all of these to be equal (to $\frac{2}{3}k$).





More advanced types of turbulence model (some of which have a proud history at the University of Manchester) are shown left and described below. A more advanced description of some is given in Section 10.

8.2.1 Reynolds-Stress Transport Models (RSTM)[2]

Also known as *second-order closure* (SOC) or *differential stress models* (DSM) the main idea is to solve individual transport equations for all stresses, $\overline{u^2}$, $\overline{uv}$ etc., rather than just turbulent kinetic energy *k*.

These equations are derived from the Navier-Stokes equations. They can be put in the usual canonical form:

rate of change + advection + diffusion = source

but certain terms have to be modelled. The most important balance is in the "source" term, which, for $\overline{u_i u_j}$, consists of parts that can be identified as:

production of energy from the mean flow, P_{ij};
redistribution of energy amongst different stress components, Φ_{ij};
dissipation of energy by viscosity, ε_{ij}.

The important point is that, at this level of modelling, both the advection term (turbulence carried by the mean flow) and the production term (creation of turbulence by the mean flow) are exact. Thus, everything to "energy in" for a particular Reynolds-stress component is exact and doesn't need modelling. For example, the rate of production of $\overline{u^2}$ per unit mass is:

$$P_{11} = -2(\overline{u^2}\frac{\partial U}{\partial x} + \overline{uv}\frac{\partial U}{\partial y} + \overline{uw}\frac{\partial U}{\partial z})$$

Assessment.

For:

- Advection and turbulence production terms are exact, not modelled; thus, RSTMs should take better account of turbulence physics than eddy-viscosity models.

Against:

- Models are very complex;
- Many important terms (notably redistribution and dissipation) require modelling;
- Models are computationally expensive (6 turbulent transport equations) and tend to be less stable; (only the small molecular viscosity contributes to any sort of gradient diffusion).

[2] The classic reference for this is:
Launder, B.E., Reece, G.J. and Rodi, W., 1975, Progress in the development of a Reynolds-stress turbulence closure, J. Fluid Mech., 68, 537-566.

8.2.2 Non-Linear Eddy-Viscosity Models (NLEVM)[3]

A "half-way house" between eddy-viscosity and Reynolds-stress transport models, the idea behind this type of model is to extend the simple proportionality between Reynolds-stress and mean-velocity gradients:

$$\textit{stress} \propto \textit{velocity gradient}$$

to a non-linear constitutive relation:

$$\textit{stress} = C_1(\textit{velocity gradient}) + C_2(\textit{velocity gradient})^2 + C_3(\textit{velocity gradient})^3 + \ldots$$

(The actual relationship is tensorial and highly mathematical – see the optional Section 10 – so it has been simplified to words here.)

Models can be constructed so as to reproduce the correct anisotropy (25) in simple shear flow, as well as a qualitatively-correct response of turbulence to certain other types of flow: e.g. curved flows. Experience at this university suggests that a cubic stress-strain relationship is desirable.

Assessment.
For:
- produce qualitatively-correct turbulent behaviour in certain important flows;
- only slightly more computationally expensive than linear eddy-viscosity models.

Against:
- doesn't accurately represent the real production and advection processes;
- little theoretical foundation in complex flows.

8.2.3 Large-Eddy Simulation (LES)

Resolving a full, time-dependent turbulent flow at large Reynolds number is impractical as it would require huge numbers of control volumes, all smaller than the tiniest scales of motion. *Large-eddy simulation* solves the time-dependent Navier-Stokes equations for the instantaneous (mean + turbulent) velocity that it can resolve on a moderate size of grid and models the subgrid-scale motions that it cannot resolve. The model for the latter is usually very simple, typically a mixing-length-type model with l_m proportional to the mesh size.

8.2.4 Direct Numerical Simulation (DNS)

This is not a turbulence model! It is a spatially-accurate solution of the complete time-dependent, Navier-Stokes equations without any modelled terms.

This is prohibitively expensive at large Reynolds numbers as huge numbers of grid nodes would be needed to resolve all scales of motion. Nevertheless, supercomputers have extended the Reynolds-number range to a few thousand for simple flows and these results have assisted greatly in the understanding of turbulence physics and development of simpler models.

[3] For some of the mathematical theory see:
Apsley, D.D. and Leschziner, M.A., 1998, A new low-Reynolds-number nonlinear two-equation turbulence model for complex flows, Int. J. Heat Fluid Flow, 19, 209-222

8.3 Wall Boundary Conditions

At walls the no-slip boundary condition applies, so that both mean and fluctuating velocities vanish. At high Reynolds numbers this presents three problems:

- there are very large flow gradients;
- wall-normal fluctuations are selectively damped;
- viscous and turbulent stresses are of comparable magnitude.

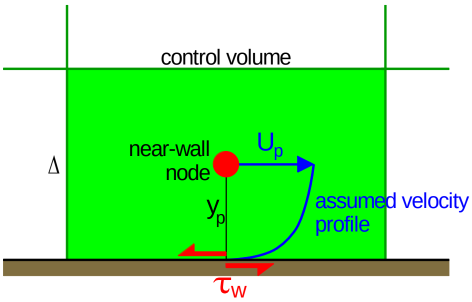
There are two main ways of handling this in turbulent flow.

(1) *Low-Reynolds-number turbulence models*
Resolve the flow right up to solid boundaries. This requires a very large number of cells in the direction perpendicular to the boundary and special viscosity-dependent modifications to the turbulence model.

(2) *Wall functions*
Don't resolve the near-wall flow completely, but assume theoretical profiles between the near-wall node and the surface. This doesn't require as many cells, but the theoretical profiles used are really only justified in near-equilibrium boundary layers.

8.3.1 Wall Functions[4]

The momentum balance for the near-wall cell requires the wall shear stress τ_w $(=\rho u_\tau^2)$. Because the near-wall region isn't resolved, this requires some assumption about what goes on between near-wall node and the surface.



If the near-wall node lies in the logarithmic region then

$$\frac{U_P}{u_\tau} = \frac{1}{\kappa}\ln(Ey_P^+)\,, \qquad y_P^+ = \frac{y_P u_\tau}{\nu} \qquad (26)$$

Subscript P denotes the near-wall node. Given U_P and y_P this is solved (iteratively) for u_τ and hence the wall stress τ_w.

If a transport equation is being solved for k, a better approach when the turbulence can not be assumed in equilibrium (e.g. near separation or reattachment points) is to estimate an "effective" friction velocity from the relationship that holds in the log-layer:

$$u_0 = C_\mu^{1/4} k_P^{1/2}$$

and derive the relationship between U_p and τ_w by assuming a kinematic eddy viscosity

$$\nu_t = \kappa u_0 y$$

(Compare equations (23) and (21) if the boundary layer were actually in equilibrium.) Then

[4] At the risk of self-promotion, an advanced discussion of wall functions (including rough- rather than smooth-walled boundaries) can be found in:
Apsley, D.D., 2007, *CFD calculation of turbulent flow with arbitrary wall roughness*, Flow, Turbulence and Combustion, 78, 153-175.

$$\tau_w = \rho(\kappa u_0 y)\frac{\partial U}{\partial y}$$

which can be integrated for *U* and then applied at the centre of the near-wall cell:

$$\tau_w = \rho\frac{\kappa u_0 U_P}{\ln(E\dfrac{y_P u_0}{\nu})} \qquad (27)$$

Since the code will discretise the velocity gradient at the boundary as U_P/y_P this is conveniently implemented via an *effective wall viscosity* μ_w, such that

$$\tau_w = \mu_w\frac{U_P}{y_P}$$

where

$$\mu_w = \frac{\rho(\kappa u_0 y_P)}{\ln(E\dfrac{y_P u_0}{\nu})}, \qquad u_0 = C_\mu^{1/4}k_P^{1/2}$$

(If the turbulence were genuinely in equilibrium, then u_0 would equal u_τ and (26) and (27) would give equivalent expressions for U_p.)

Amendments also have to be made to the turbulence equations, based on assumed profiles for k and ε. In particular, the production of turbulence energy is a *cell-averaged* quantity, determined by integrating across the cell and the value of ε is specified at the centre of the near-wall cell, not at the boundary.

To use these equilibrium profiles effectively, it is desirable that the grid spacing be such that the near-wall node lies within the logarithmic layer; ideally,

$$30 < y_P^+ < 150$$

This has to be relaxed somewhat in practice, typically to $y_p^+ > 15$, but it means that with wall-function calculations the grid cannot be made arbitrarily fine close to solid boundaries.

Summary

- A turbulence model is a means of specifying the Reynolds stresses (and any other turbulent fluxes), so closing the mean flow equations.

- The most popular types are eddy-viscosity models, which assume that the Reynolds stress is proportional to the mean strain; e.g. in simple shear:

$$\tau^{(turb)} \equiv -\rho\overline{uv} \quad = \mu_t \frac{\partial U}{\partial y}$$

- The eddy viscosity μ_t may be specified geometrically (e.g. mixing-length models) or by solving additional transport equations. The most popular combination is the k-ε model (requiring transport equations for turbulent kinetic energy k and its dissipation rate ε).

- More advanced turbulence models include:
  – Reynolds-stress transport models (solve transport equations for all stresses)
  – non-linear eddy-viscosity models (non-linear stress-strain relationship)
  – large-eddy simulation (time-dependent calculation; model sub-grid scales)

- Wall boundary conditions require special treatment because of large flow gradients and selective damping of wall-normal velocity fluctuations. The main options are low-Reynolds-number models (fine grids) or wall functions (coarse grids).

For the interested, a more advanced treatment of turbulence modelling can be found in the optional Section 10 and in the references.

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Examples

Q1.
In high-Reynolds-number turbulent boundary-layer flow over a flat surface the mean shear stress is made up of viscous and turbulent parts:

$$\tau = \mu \frac{\partial U}{\partial y} - \rho \overline{uv}$$

where μ is the molecular viscosity. In the lower part of the boundary layer the shear stress is effectively constant and equal to the wall shear stress τ_w.

(a) Define the friction velocity u_τ.

(b) Show that, sufficiently close to a smooth wall, the mean velocity profile is linear, and write down an expression for U in terms of τ_w, μ and the distance from the wall, y.

(c) At larger distances from the wall the viscous stress can be neglected, whilst the turbulent stress can be represented by a mixing-length eddy-viscosity model:

$$-\rho \overline{uv} = \mu_t \frac{\partial U}{\partial y}$$

where

$$\mu_t = \rho u_0 l_m , \qquad l_m = \kappa y, \qquad u_0 = l_m \frac{\partial U}{\partial y}$$

and κ (≈ 0.41) is a constant. Again, assuming that $\tau = \tau_w$, show that this leads to a logarithmic velocity profile of the form

$$\frac{U}{u_\tau} = \frac{1}{\kappa} \ln(E \frac{y u_\tau}{\nu})$$

where E is a constant of integration.

(d) Write the velocity profiles in parts (b) and (c) in *wall units*.

(e) In a simple shear flow the rate of production of turbulent kinetic energy per unit mass is

$$P^{(k)} = -\overline{uv} \frac{\partial U}{\partial y}$$

Using the results of (c), prove that, in the logarithmic velocity region,

$$P^{(k)} = \frac{u_\tau^3}{\kappa y}$$

and explain what is meant by the statement that the turbulence is in local equilibrium.

Q2.
On dimensional grounds, an eddy viscosity μ_t can be written as

$$\mu_t = \rho u_0 l_0$$

where u_0 is some representative magnitude of turbulent velocity fluctuations and l_0 is a turbulent length scale. The eddy-viscosity formula for the k-ε turbulence model is

$$\mu_t = C_\mu \rho \frac{k^2}{\varepsilon}$$

where $C_\mu = 0.09$. Identify suitable velocity and length scales u_0 and l_0.

Q3.
(a) The k-ε turbulence model forms an eddy viscosity μ_t from fluid density ρ, the turbulent kinetic energy (per unit mass) k and its dissipation rate ε. Write down the basic physical dimensions of μ_t, ρ, k and ε in terms of the fundamental dimensions of mass M, length L and time T and hence show, on purely dimensional grounds, that any expression for μ_t in terms of the other variables must be of the form

$$\mu_t = constant \times \rho \frac{k^2}{\varepsilon}$$

(b) The k-ω turbulence model forms an eddy viscosity from ρ, k and a quantity ω which has dimensions of frequency (i.e. T^{-1}). Show, on dimensional grounds, that any expression for μ_t in terms of the other variables must be of the form

$$\mu_t = constant \times \rho \frac{k}{\omega}$$

Q4.
In the k-ε turbulence model, k is turbulent kinetic energy and ε is its dissipation rate. A (kinematic) eddy viscosity is defined by

$$\nu_t = C_\mu \frac{k^2}{\varepsilon}$$

where C_μ is a constant. A modeled scalar-transport equation for ε is

$$\frac{\mathrm{D}\varepsilon}{\mathrm{D}t} = \frac{\partial}{\partial x_i}\left(\frac{\nu_t}{\sigma_\varepsilon} \frac{\partial \varepsilon}{\partial x_i} \right) + (C_{\varepsilon 1} P^{(k)} - C_{\varepsilon 2}\varepsilon)\frac{\varepsilon}{k}$$

where D/Dt is a derivative following the flow, $P^{(k)}$ is the rate of production of k and the summation convention is implied by the repeated index i. σ_ε, $C_{\varepsilon 1}$ and $C_{\varepsilon 2}$ are constants.

In the log-law region of a turbulent boundary layer,

$$P^{(k)} = \varepsilon = \frac{u_\tau^3}{\kappa y} \qquad \text{and} \qquad k = C_\mu^{-1/2} u_\tau^2$$

where κ is von Karman's constant, u_τ is the friction velocity and y is the distance from the boundary. Using the scalar-transport equation for ε and the eddy-viscosity formulation, show that this implies the following relationship between coefficients:

$$(C_{\varepsilon 2} - C_{\varepsilon 1})\sigma_\varepsilon \sqrt{C_\mu} = \kappa^2$$

Q5. (Exam 2009 – part)
In the k-ω model of turbulence the kinematic eddy viscosity is given by

$$\nu_t = \frac{k}{\omega}$$

and transport equations are solved for turbulent kinetic energy k and specific dissipation rate ω. A modeled scalar-transport equation for ω is

$$\frac{\mathrm{D}\omega}{\mathrm{D}t} = \frac{\partial}{\partial x_i}\left(\frac{\nu_t}{\sigma_\omega}\frac{\partial\omega}{\partial x_i}\right) + \frac{\alpha}{\nu_t}P^{(k)} - \beta\omega^2$$

where D/Dt is a derivative following the flow, and summation is implied by the repeated index i. Here, $P^{(k)}$ is the rate of production of k, whilst σ_ω, α and β are constants.

In the log-law region of a turbulent boundary layer,

$$P^{(k)} = C_\mu k\omega = \frac{u_\tau^3}{\kappa y} \qquad \text{and} \qquad k = C_\mu^{-1/2}u_\tau^2$$

where κ is von Karman's constant, C_μ is a model constant, u_τ is the friction velocity and y is the distance from the boundary. Show that this implies the following relationship between coefficients in the modeled scalar-transport equation for ω:

$$(\frac{\beta}{C_\mu} - \alpha)\sigma_\omega\sqrt{C_\mu} = \kappa^2$$

Q6.
In the analysis of turbulent flows it is common to decompose the velocity field into mean (U,V,W) and fluctuating (u,v,w) parts as part of the Reynolds-averaging process.

(a) The rate of production of the $\overline{uu}$ stress component per unit mass is given by

$$P_{11} = -2(\overline{uu}\frac{\partial U}{\partial x} + \overline{uv}\frac{\partial U}{\partial y} + \overline{uw}\frac{\partial U}{\partial z})$$

By inspection/pattern-matching, write down an analogous expression for P_{22}.

(b) Define the term *anisotropy* when applied to fluctuating quantities in turbulent flow and give *two* reasons why, for turbulent boundary layers along a plane wall $y = 0$, the wall-normal velocity variance is smaller than the streamwise variance.

(c) Describe the main principles of, and the main differences between
(i) eddy-viscosity
(ii) Reynolds-stress transport
models of turbulence, and give advantages and disadvantages of each type of closure.

Q7. (Exam 2008)

The logarithmic mean-velocity profile for a smooth-wall turbulent boundary layer is given by

$$\frac{U}{u_\tau} = \frac{1}{\kappa}\ln\left(\frac{yu_\tau}{\nu}\right) + B \qquad (*)$$

where U is mean velocity, u_τ is friction velocity, ν is kinematic viscosity, y is the distance from the nearest wall, κ is von Kármán's constant and B is a constant.

(a) Define the friction velocity u_τ in terms of wall shear stress τ_w and fluid density ρ.

(b) Find the mean-velocity gradient $\partial U/\partial y$ from Equation (*), and hence deduce the kinematic eddy viscosity ν_t for this flow.

(c) Write an expression for the eddy viscosity in a *mixing-length* model for a simple shear flow. Hence, using your answer from part (b), deduce the mixing length l_m here.

(d) Define the *turbulent kinetic energy* for a turbulent velocity field.

(e) In a general incompressible velocity field (U, V, W) the turbulent shear stress component τ_{12} is given, for an eddy-viscosity turbulence model, by

$$\tau_{12} = \mu_t\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right)$$

where μ_t ($= \rho\nu_t$) is the dynamic eddy viscosity and ρ is density. Use pattern-matching/index-permutation and the incompressibility condition to write expressions for the other independent stress components: τ_{23}, τ_{31}, τ_{11}, τ_{22}, τ_{33}.

(f) In the k-T turbulence model, T is a turbulent time scale and the eddy-viscosity formulation takes the form

$$\mu_t = C\rho^\alpha k^\beta T^\gamma$$

where C is a dimensionless constant. Use dimensional analysis to find α, β and γ.

Q8. (Exam 2012)

(a) Define the Reynolds stresses in a turbulent flow.

(b) State, mathematically, how the Reynolds stresses are related to the mean-velocity field in an eddy-viscosity model. (You should give either a general form in index notation, or typical normal and shear stresses.)

(c) In the standard k-ε model the dynamic eddy viscosity μ_t is modelled by

$$\mu_t = \rho C_\mu \frac{k^2}{\varepsilon},$$

where C_μ is a constant. What are the quantities denoted by k and ε in this model and how are they calculated?

In an equilibrium turbulent boundary layer, with velocity profile $(U(y),0,0)$ k and ε are related to the friction velocity u_τ by

$$k = C_\mu^{-1/2} u_\tau^2,$$

$$\varepsilon = \frac{u_\tau^3}{\kappa y},$$

where κ is Von Kármán's constant and y is the distance from the boundary.

(d) Define the friction velocity in terms of the wall shear stress.

(e) Deduce how the dynamic eddy viscosity in the k-ε model varies with y, and show that, under the assumption of a constant shear stress, the mean-velocity profile in the fully-turbulent region is logarithmic.

(f) Explain (but without mathematical detail) how Reynolds-stress-transport models differ from eddy-viscosity models of turbulence. State their advantages and disadvantages over the eddy-viscosity approach.

Q9. (Exam 2013)

(a) By consideration of the net momentum transport by turbulent fluctuations show that the quantity $-\rho\overline{u_i'u_j'}$ can be interpreted as an additional effective stress in the mean momentum equation.

(b) For a linear eddy-viscosity turbulence model with strain-independent eddy viscosity μ_t, write down expressions for the typical shear stress $-\rho\overline{u'v'}$ and normal stress $-\rho\overline{u'^2}$ in an arbitrary velocity field. Show that, for certain mean-velocity gradients, this type of model may predict physically unrealisable stresses.

(c) Explain why, for a zero-pressure-gradient, fully-developed boundary-layer flow of the form $(\overline{u}(y),0,0)$, the mean shear stress τ is independent of distance y from the boundary. Hence, find the form of the mean-velocity profile if the total effective viscosity μ_{eff}:

(i) is constant;

(ii) varies linearly with wall distance: $\mu_{eff} = Cy$, where C is a constant.

(d) In the widely-used k-ε model of turbulence:

(i) state the physical quantities represented by k and ε;

(ii) write down the expression for eddy viscosity μ_t in terms of ρ, k and ε in the standard k-ε model;

(iii) explain briefly (and without detailed mathematics) how k and ε are calculated.

(e) What special issues arise in the modeling and computation of near-wall turbulent flow? State the two main methods for dealing with the solid-wall boundary condition and give a brief summary of the major elements of each.