Answers 6

Classroom Example 1

The exact solution is $\phi = t - 1 + 2e^{-t}$

t	ϕ			
	(a) Forward differencing	(b) Backward differencing	(c) Crank-Nicolson	Exact
0	1.00000	1.00000	1.00000	1.00000
0.2	0.80000	0.86667	0.83636	0.83746
0.4	0.68000	0.78889	0.73884	0.74064
0.6	0.62400	0.75741	0.69541	0.69762
0.8	0.61920	0.76451	0.69624	0.69866
1.0	0.65536	0.80376	0.73329	0.73576

Classroom Example 2

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = t - \phi^4$$

Replace by finite-differences:

$$\frac{\Delta\phi}{\Delta t} = (t - \phi^4)^{av}$$

$$\Rightarrow \qquad \frac{\phi^{new} - \phi^{old}}{\Delta t} = (t - \phi^4)^{av}$$

$$\Rightarrow \qquad \phi^{new} = \phi^{old} + \Delta t (t - \phi^4)^{av}$$

(a) Forward differencing

$$\phi^{new} = \phi^{old} + \Delta t (t - \phi^4)^{old}$$

With Δt = 0.1 this gives the following table.

t^{old}	ϕ^{old}	t^{new}	ϕ^{new}
0	2	**0.1**	**0.40000**
0.1	0.4	**0.2**	**0.40744**
0.2	0.40744	**0.3**	**0.42468**
0.3	0.42468	**0.4**	**0.45143**

(b) Backward differencing

$$\phi^{(new)} = \phi^{(old)} + \Delta t (t - \phi^4)^{(new)}$$

If this is iterated in this form from $\phi = 2$ then it will not converge. Instead, rearrange (and drop "*new*" as tacitly understood):

$$(1 + \Delta t \phi^3)\phi = \phi^{(old)} + \Delta t.t$$

$$\Rightarrow \qquad \phi = \frac{\phi^{(old)} + \Delta t.t}{1 + \Delta t \phi^3}$$

With Δt = 0.1 this gives the following table. Iteration is required at every step.

t^{old}	ϕ^{old}	Iteration	t^{new}	ϕ^{new}
0	2	$\phi = \frac{2.0000 + 0.1 \times 0.1}{1 + 0.1\phi^3}$	**0.1**	**1.5016**
0.1	1.5016	$\phi = \frac{1.5016 + 0.1 \times 0.2}{1 + 0.1\phi^3}$	**0.2**	**1.2653**
0.2	1.2653	$\phi = \frac{1.2653 + 0.1 \times 0.3}{1 + 0.1\phi^3}$	**0.3**	**1.1314**
0.3	1.1314	$\phi = \frac{1.1314 + 0.1 \times 0.4}{1 + 0.1\phi^3}$	**0.4**	**1.0499**

(c) Centred differencing

$$\phi^{new} = \phi^{old} + \frac{\Delta t}{2}\left[(t-\phi^4)^{old} + (t-\phi^4)^{new}\right]$$

This does actually work here, but to improve the rate of convergence rearrange it in a similar manner to part (ii) above:

$$(1+\frac{\Delta t}{2}\phi^3)\phi = \phi^{old} + \frac{\Delta t}{2}\left[t^{old} + t - \phi^{old\,4}\right]$$

$$\Rightarrow \qquad \phi = \frac{\phi^{old} + \frac{\Delta t}{2}\left[t^{old} + t - \phi^{old\,4}\right]}{1+\frac{\Delta t}{2}\phi^3}$$

With $\Delta t = 0.1$ this gives the following table. Iteration is required at every step.

t^{old}	ϕ^{old}	Iteration	$\boldsymbol{t^{new}}$	$\boldsymbol{\phi^{new}}$
0	1	$\phi = \frac{2.0000 + 0.05(0.0 + 0.1 - 2.0000^4)}{1 + 0.05\phi^3}$	**0.1**	**1.1249**
0.1	1.1249	$\phi = \frac{1.1249 + 0.05(0.1 + 0.2 - 1.1249^4)}{1 + 0.05\phi^3}$	**0.2**	**1.0082**
0.2	1.0082	$\phi = \frac{1.0082 + 0.05(0.2 + 0.3 - 1.0082^4)}{1 + 0.05\phi^3}$	**0.3**	**0.9421**
0.3	0.9421	$\phi = \frac{0.9421 + 0.05(0.3 + 0.4 - 0.9421^4)}{1 + 0.05\phi^3}$	**0.4**	**0.9043**

Q1.

(a) Forward differencing:

$$\phi^{new} = \phi^{old} + F^{old}\Delta t$$

which, in this case, becomes:

$$\phi^{new} = \phi^{old} + 0.25(t^2 - 2\phi)^{old}$$

The time integration, set out in a table, is as follows.

t^{old}	ϕ^{old}	$\boldsymbol{t^{new}}$	$\boldsymbol{\phi^{new}}$
0.00	0.0000	**0.25**	**0.0000**
0.25	0.0000	**0.50**	**0.0156**
0.50	0.0156	**0.75**	**0.0703**
0.75	0.0703	**1.00**	**0.1758**

(b) Backward differencing:

$$\phi^{new} = \phi^{old} + F^{new}\Delta t$$

which, in this case, becomes

$$\phi^{new} = \phi^{old} + 0.25(t^2 - 2\phi)^{new}$$

An explicit update formula is possible for this particular problem by rearranging for ϕ^{new}:

$$(1 + 0.5)\phi^{new} = \phi^{old} + 0.25t^{new2}$$

Dropping the "*new*" superscript as tacitly understood:

$$\phi^{new} = \frac{\phi^{old} + 0.25t^2}{1.5}$$

The time integration, set out in a table, is as follows.

t^{old}	ϕ^{old}	$\boldsymbol{t^{new}}$	$\boldsymbol{\phi^{new}}$
0.00	0.0000	**0.25**	**0.0104**
0.25	0.0104	**0.50**	**0.0486**
0.50	0.0486	**0.75**	**0.1262**
0.75	0.1262	**1.00**	**0.2508**

Q2.
Expand $\phi^{(n-1)}$ and $\phi^{(n-2)}$ about $\phi^{(n)}$ using Taylor series:

$$\phi^{(n-1)} = \phi^{(n)} - \Delta t\left(\frac{\partial\phi}{\partial t}\right)^{(n)} + \frac{\Delta t^2}{2!}\left(\frac{\partial^2\phi}{\partial t^2}\right)^{(n)} - \frac{\Delta t^3}{3!}\left(\frac{\partial^3\phi}{\partial t^3}\right)^{(n)} + \cdots$$

$$\phi^{(n-2)} = \phi^{(n)} - 2\Delta t\left(\frac{\partial\phi}{\partial t}\right)^{(n)} + \frac{(2\Delta t)^2}{2!}\left(\frac{\partial^2\phi}{\partial t^2}\right)^{(n)} - \frac{(2\Delta t)^3}{3!}\left(\frac{\partial^3\phi}{\partial t^3}\right)^{(n)} + \cdots$$

Form a weighted combination to eliminate the second-derivative term:

$$4\phi^{(n-1)} - \phi^{(n-2)} = 3\phi^{(n)} - 2\Delta t\left(\frac{\partial\phi}{\partial t}\right)^{(n)} + 0 + \frac{4(\Delta t)^3}{3!}\left(\frac{\partial^3\phi}{\partial t^3}\right)^{(n)} + \cdots$$

Hence,

$$-3\phi^{(n)} + 4\phi^{(n-1)} - \phi^{(n-2)} = -2\Delta t\left(\frac{\partial\phi}{\partial t}\right)^{(n)} + O(\Delta t^3)$$

and so, dividing by $-2\Delta t$,

$$\frac{3\phi^{(n)} - 4\phi^{(n-1)} + \phi^{(n-2)}}{2\Delta t} = \left(\frac{\partial\phi}{\partial t}\right)^{(n)} + O(\Delta t^2)$$

The truncation error is proportional to Δt^2; hence, the method is of order 2 in time.

Q3. See notes.

Q4.

(a) The equation to be solved is, in standard form:

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = (1-\phi t)\phi^2$$

with Δt = 0.25.

Forward differencing: $$\frac{\phi^{new}-\phi^{old}}{\Delta t} = \left[(1-\phi t)\phi^2\right]^{old}$$

Backward differencing: $$\frac{\phi^{new}-\phi^{old}}{\Delta t} = \left[(1-\phi t)\phi^2\right]^{new}$$

Rearranging each, and dropping the "*new*" superscript as tacitly understood:

Forward differencing: $\phi = \phi^{old} + \Delta t\left[(1-\phi t)\phi^2\right]^{old}$

Backward differencing: $\phi = \phi^{old} + \Delta t\left[(1-\phi t)\phi^2\right]$

Backward differencing requires iteration.

(i) Forward-differencing:

t^{old}	ϕ^{old}	t^{new}	ϕ^{new}
0.00	1.0000	**0.25**	**1.2500**
0.25	1.2500	**0.50**	**1.5186**
0.50	1.5186	**0.75**	**1.6574**
0.75	1.6574	**1.00**	**1.4905**

(ii) Backward differencing:

t^{old}	ϕ^{old}	Iteration	t^{new}	ϕ^{new}
0.00	1	$\phi = 1.0000 + 0.25(1-\phi\times 0.25)\times\phi^2$	**0.25**	**1.2778**
0.25	1.2778	$\phi = 1.2778 + 0.25(1-\phi\times 0.50)\times\phi^2$	**0.50**	**1.4238**
0.50	1.4238	$\phi = 1.4238 + 0.25(1-\phi\times 0.75)\times\phi^2$	**0.75**	**1.3995**
0.75	1.3995	$\phi = 1.3995 + 0.25(1-\phi\times 1.00)\times\phi^2$	**1.00**	**1.2830**

Note that an alternative and better iteration formula could be used in part (ii). Rearrange as

$$(1+\Delta t.t\phi^2)\phi = \phi^{old} + \Delta t\phi^2$$

whence

$$\phi = \frac{\phi^{old} + \Delta t\phi^2}{1+\Delta t.t\phi^2}$$

(b)

	Advantages	Disadvantages
Forward differencing	Explicit, so easy to implement	Only first-order accurate; Time-step restrictions.
Backward differencing	Unconditionally bounded for all timesteps.	Only first-order accurate; Implicit, so iteration necessary.

Q5.

(a) Integrating from x_w to $x_e = x_w + \Delta x$:

$$\int_{x_w}^{x_e} \frac{\partial}{\partial t}(\rho\phi)\ \ \mathrm{d}x + \int_{x_w}^{x_e} \frac{\partial}{\partial x}(\rho u\phi - \Gamma\frac{\partial\phi}{\partial x})\ \mathrm{d}x = \int_{x_w}^{x_e} s\ \ \mathrm{d}x$$

$$\Rightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}t}\int_{x_w}^{x_e} \rho\phi\ \ \mathrm{d}x + \left[\rho u\phi - \Gamma\frac{\partial\phi}{\partial x}\right]_{x_w}^{x_e} = \int_{x_w}^{x_e} s\ \ \mathrm{d}x$$

A *flux* of a physical quantity is the rate of transport across an area.

In the integral equation the advective and diffusive fluxes (per unit cross-sectional area) are $\rho u\phi$ and $-\Gamma\partial\phi/\partial x$, respectively.

They are conservative in the sense that the flux *out* of one cell is the flux *into* the next.

(b)

<u>(i) Forward differencing</u>

$$\frac{\phi^{new} - \phi^{old}}{\Delta t} = (1 - t - \phi^2)^{old}$$

Rearranging and dropping the superscript '*new*':

$$\phi = \phi^{old} + \Delta t(1 - t - \phi^2)^{old}$$

With $\Delta t = 0.5$ this gives the following table.

t^{old}	ϕ^{old}	t^{new}	ϕ^{new}
0	1.00000	**0.5**	**1.00000**
0.5	1.00000	**1.0**	**0.75000**
1.0	0.75000	**1.5**	**0.46875**
1.5	0.46875	**2.0**	**0.10889**

<u>(ii) Backward differencing</u>

$$\frac{\phi^{new} - \phi^{old}}{\Delta t} = (1 - t - \phi^2)^{new}$$

Dropping "new" as tacitly understood:

$$\phi = \phi^{old} + \Delta t(1 - t - \phi^2)$$

This must be solved iteratively at each timestep. The form above will work, but is slow to converge. An improved rate of convergence may be obtained by first rewriting it as

$$\phi(1 + \Delta t\phi) = \phi^{old} + \Delta t(1 - t)$$

whence a better iterative formula is

$$\phi = \frac{\phi^{old} + \Delta t(1 - t)}{1 + \Delta t\phi}$$

With $\Delta t = 0.5$ this gives the following table. Iteration can start from the previous ϕ value.

t^{old}	ϕ^{old}	Iteration	t^{new}	ϕ^{new}
0	1.00000	$\phi = \dfrac{1.0000 + 0.5 \times 0.5}{1 + 0.5\phi}$	**0.5**	**0.87083**
0.5	0.87083	$\phi = \dfrac{0.87083 + 0.5 \times 0.0}{1 + 0.5\phi}$	**1.0**	**0.65580**
1.0	0.65580	$\phi = \dfrac{0.65580 + 0.5 \times (-0.5)}{1 + 0.5\phi}$	**1.5**	**0.34596**
1.5	0.34596	$\phi = \dfrac{0.34596 + 0.5 \times (-1.0)}{1 + 0.5\phi}$	**2.0**	**–0.16818**

(ii) Centred differencing

$$\frac{\phi^{new} - \phi^{old}}{\Delta t} = \frac{1}{2}\left\{(1 - t - \phi^2)^{old} + (1 - t - \phi^2)^{new}\right\}$$

Dropping the "new" as tacitly understood:

$$\phi = \phi^{old} + \frac{\Delta t}{2}\left\{(1 - t - \phi^2)^{old} + (1 - t - \phi^2)\right\}$$

This can be solved iteratively in this form, but better convergence is obtained by rearranging:

$$\phi(1 + \tfrac{1}{2}\Delta t \phi) = \phi^{old} + \Delta t[1 - \tfrac{1}{2}(t^{old} + t) - \tfrac{1}{2}\phi^{old^2}]$$

$$\Rightarrow \qquad \phi = \frac{\phi^{old} + \Delta t[1 - \tfrac{1}{2}(t^{old} + t) - \tfrac{1}{2}\phi^{old^2}]}{1 + \tfrac{1}{2}\Delta t \phi}$$

With $\Delta t = 0.5$ this gives the following table. Iteration can start from the previous ϕ value.

t^{old}	ϕ^{old}	Iteration	t^{new}	ϕ^{new}
0	1.00000	$\phi = \dfrac{1.0000 + 0.5 \times [0.75 - \frac{1}{2} \times 1.0000^2]}{1 + 0.25\phi}$	**0.5**	**0.91548**
0.5	0.91548	$\phi = \dfrac{0.91548 + 0.5 \times [0.25 - \frac{1}{2} \times 0.91548^2]}{1 + 0.25\phi}$	**1.0**	**0.70626**
1.0	0.70626	$\phi = \dfrac{0.70626 + 0.5 \times [-0.25 - \frac{1}{2} \times 0.70626^2]}{1 + 0.25\phi}$	**1.5**	**0.41376**
1.5	0.41376	$\phi = \dfrac{0.41376 + 0.5 \times [-0.75 - \frac{1}{2} \times 0.41376^2]}{1 + 0.25\phi}$	**2.0**	**-0.00404**

(c)

	Advantages	Disadvantages
Forward differencing	Explicit, so easy to implement	Only first-order accurate; Time-step restrictions.
Backward differencing	Bounded for all timesteps	Only first-order accurate; Implicit, so iteration necessary.
Centred differencing	Second-order accurate	Time-step restrictions. Implicit, so iteration necessary.

Q6.
Both backward differencing and centred differencing require iteration.
On rearrangement, the "new" is dropped as tacitly understood.

(a) Forward differencing

$$\frac{\phi_{new} - \phi_{old}}{\Delta t} = -\left(\frac{\phi^2}{t+\phi}\right)_{old}$$

$$\Rightarrow \qquad \phi = \phi_{old} - \Delta t\left(\frac{\phi_{old}^2}{t_{old} + \phi_{old}}\right)$$

t_{old}	ϕ_{old}	t_{new}	ϕ_{new}
0.00	2.0000	**0.25**	**1.5000**
0.25	1.5000	**0.50**	**1.1786**
0.50	1.1786	**0.75**	**0.9717**
0.75	0.9717	**1.00**	**0.8346**

(b) Backward differencing

$$\frac{\phi_{new} - \phi_{old}}{\Delta t} = -\left(\frac{\phi^2}{t+\phi}\right)_{new}$$

$$\Rightarrow \qquad \phi = \phi_{old} - \Delta t\left(\frac{\phi^2}{t+\phi}\right)$$

t_{old}	ϕ_{old}	Iteration	t_{new}	ϕ_{new}
0.00	2.0000	$\phi = 2.0000 - 0.25\times\left(\frac{\phi^2}{0.25+\phi}\right)$	**0.25**	**1.6434**
0.25	1.6434	$\phi = 1.6434 - 0.25\times\left(\frac{\phi^2}{0.50+\phi}\right)$	**0.50**	**1.3882**
0.50	1.3882	$\phi = 1.3882 - 0.25\times\left(\frac{\phi^2}{0.75+\phi}\right)$	**0.75**	**1.2030**
0.75	1.2030	$\phi = 1.2030 - 0.25\times\left(\frac{\phi^2}{1.00+\phi}\right)$	**1.00**	**1.0656**

(c) Centred differencing

$$\frac{\phi_{new}-\phi_{old}}{\Delta t}=-\frac{1}{2}\left[\left(\frac{\phi^2}{t+\phi}\right)_{old}+\left(\frac{\phi^2}{t+\phi}\right)_{new}\right]$$

$$\Rightarrow \qquad \phi=\phi_{old}-\frac{\Delta t}{2}\left(\frac{\phi_{old}^2}{t_{old}+\phi_{old}}+\frac{\phi^2}{t+\phi}\right)$$

t_{old}	ϕ_{old}	Iteration	t_{new}	ϕ_{new}
0.00	2.0000	$\phi=2.0000-0.125\times\left(\frac{2.0000^2}{0.00+2.0000}+\frac{\phi^2}{0.25+\phi}\right)$	**0.25**	**1.5795**
0.25	1.5795	$\phi=1.5795-0.125\times\left(\frac{1.5795^2}{0.25+1.5795}+\frac{\phi^2}{0.50+\phi}\right)$	**0.50**	**1.2925**
0.50	1.2925	$\phi=1.2925-0.125\times\left(\frac{1.2925^2}{0.50+1.2925}+\frac{\phi^2}{0.75+\phi}\right)$	**0.75**	**1.0948**
0.75	1.0948	$\phi=1.0948-0.125\times\left(\frac{1.0948^2}{0.75+1.0948}+\frac{\phi^2}{1.00+\phi}\right)$	**1.00**	**0.9552**

Q7.

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{-2\phi^3}{t+\phi}$$

Replace by finite-differences:

$$\Rightarrow \qquad \frac{\phi^{new} - \phi^{old}}{\Delta t} = \left(\frac{-2\phi^3}{t+\phi}\right)^{av}$$

$$\Rightarrow \qquad \phi^{new} = \phi^{old} - 2\Delta t\left(\frac{\phi^3}{t+\phi}\right)^{av}$$

(i) Forward differencing

$$\phi^{new} = \phi^{old} - 2\Delta t\left(\frac{\phi^3}{t+\phi}\right)^{old}$$

With Δt = 0.25 this gives the following table.

t^{old}	ϕ^{old}	t^{new}	ϕ^{new}
0	1	**0.25**	**0.50000**
0.25	0.5	**0.50**	**0.41667**
0.50	0.41667	**0.75**	**0.37721**
0.75	0.37721	**1.00**	**0.35340**

(ii) Centred differencing

$$\phi^{new} = \phi^{old} - \Delta t\left[\left(\frac{\phi^3}{t+\phi}\right)^{old} + \left(\frac{\phi^3}{t+\phi}\right)^{new}\right]$$

This will work here, but to improve the rate of convergence one can first rearrange it (dropping the "new" superscript):

$$\left(1 + \Delta t\frac{\phi^2}{t+\phi}\right)\phi = \phi^{old} - \Delta t\left(\frac{\phi^3}{t+\phi}\right)^{old}$$

$$\Rightarrow \qquad \phi = \frac{\phi^{old} - \Delta t\left(\frac{\phi^3}{t+\phi}\right)^{old}}{1 + \Delta t\frac{\phi^2}{t+\phi}}$$

With Δt = 0.25 this gives the following table. Iteration is required at every step.

t^{old}	ϕ^{old}		$\boldsymbol{t^{new}}$	$\boldsymbol{\phi^{new}}$
0	1	$\phi = \dfrac{0.75}{1 + 0.25\dfrac{\phi^2}{0.25 + \phi}}$	**0.25**	**0.66865**
0.25	0.66865	$\phi = \dfrac{0.58729}{1 + 0.25\dfrac{\phi^2}{0.5 + \phi}}$	**0.50**	**0.54803**
0.50	0.54803	$\phi = \dfrac{0.50877}{1 + 0.25\dfrac{\phi^2}{0.75 + \phi}}$	**0.75**	**0.48560**
0.75	0.48560	$\phi = \dfrac{0.46243}{1 + 0.25\dfrac{\phi^2}{1.0 + \phi}}$	**1.0**	**0.44700**

(b) Centred differencing is expected to be more accurate, because it is second-order in time.