5. Pressure and Velocity

Part 1. Pressure-Velocity Coupling

Part 2. Pressure-Correction Methods

Scalar-Transport Equation For Momentum

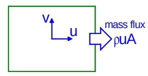
Each momentum component satisfies its own scalar-transport equation:

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}(mass \times \phi)}_{rate\ of\ change} + \sum_{faces}(\underbrace{mass\ flux \times \phi}_{advection} \underbrace{- \Gamma \frac{\partial \phi}{\partial n} A}_{diffusion}) = \underbrace{S}_{source}$$

- "concentration", ϕ ← velocity (u,v,w)
- "diffusivity", Γ ← viscosity, μ
- "source", S ← non-viscous forces

Special Features of the Momentum Equations

- The momentum equations are:
  - **non-linear** (*mass flux*) × *velocity* $\rightarrow (\rho u A)u$
  - **coupled** $(\rho u A)v$
  - required also to be **mass-consistent**



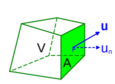
- As a result they must be solved:
  - **iteratively**
  - **together**
  - in conjunction with the **continuity equation**
- And we also need to specify **pressure** …

Solving Mass and Momentum Equations

```
DO WHILE (not_converged)

    CALL SCALAR_TRANSPORT( u )
    CALL SCALAR_TRANSPORT( v )
    CALL SCALAR_TRANSPORT( w )

    CALL MASS_CONSERVATION

END DO
```

Scalar-Transport Equation

$$\frac{d}{dt}(mass \times \phi) + \sum_{faces}(mass\ flux \times \phi - \Gamma \frac{\partial \phi}{\partial n} A) = S$$



Mass Equation

$$\frac{d}{dt}(mass) + \sum_{faces}(mass\ flux) = 0$$

For **compressible** flow, the mass equation is a transport equation for density

For **incompressible** flow, the mass equation *could* be regarded as a special case of the **scalar-transport** equation ... but only if:

$\phi \equiv 1$ **!!!**
$\Gamma \equiv 0$
$S \equiv 0$

How is Pressure Determined?

- **Compressible flow:**
  - mass conservation → transport equation for **density**, ρ
  - transport equation for energy → **temperature**, T
  - equation of state (e.g. ideal gas law $p = \rho RT$) → **pressure**, p
- **Incompressible flow:**
  - the momentum equations link velocity and pressure
  - substitution in the mass equation yields an equation for pressure

A **pressure equation** arises from the requirement that **solutions of the momentum equation be mass-consistent**.

Solving Mass and Momentum Equations

```
DO WHILE (not_converged)

     CALL SCALAR_TRANSPORT( u )
     CALL SCALAR_TRANSPORT( v )
     CALL SCALAR_TRANSPORT( w )

     CALL MASS_CONSERVATION( p )

END DO
```

Pressure-Velocity Coupling

Q1. How are velocity and pressure linked?

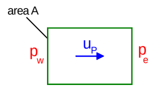
Q2. How does a pressure equation arise?

Q3. Should velocity and pressure be stored at the same positions?

Q1. How are Pressure and Velocity Linked?

Forces:

$$net\ pressure\ force = p_w A - p_e A$$



Momentum equation:

$$\underbrace{a_P u_P - \sum a_F u_F}_{net\ flux} = \underbrace{A(p_w - p_e)}_{pressure\,force} + other\ \ forces$$

Velocity-pressure linkage:

$$u_P = d_P(p_w - p_e) + \ldots$$

$$u = -d\Delta p + \ldots \qquad d_p = \frac{A}{a_P}$$

A1. (a) The **force terms** in the **momentum equation** provide a link between velocity and pressure.

(b) Velocity depends on the **pressure gradient** or, when discretised, on the **difference** between pressure values ½ cell either side.

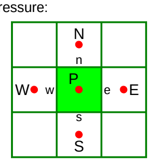
Q2. How Does a Pressure Equation Arise?

The **momentum equation** links velocity and pressure:

$$u = -d\Delta p + \cdots$$

Substituting in the **mass equation** gives an equation for pressure:

$$\begin{aligned} 0 &= (\rho uA)_e - (\rho uA)_w + \cdots \\ &= (\rho Ad)_e(p_P - p_E) - (\rho Ad)_w(p_W - p_P) + \cdots \\ &= -a_W p_W + a_p p_p - a_E p_E + \cdots \end{aligned}$$

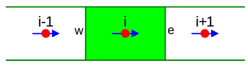


A2. A pressure equation arises from the requirement that **solutions of the momentum equation be mass-consistent**.

Q3. Should velocity and pressure be stored at the same positions?

Co-located Pressure and Velocity

(i) Effect in the Momentum Equation

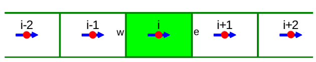


$p_w = \tfrac{1}{2}(p_{i-1} + p_i) \qquad p_e = \tfrac{1}{2}(p_i + p_{i+1})$

Net pressure force: $A(p_w - p_e) = \dfrac{A}{2}(p_{i-1} - p_{i+1})$ No p_i **!!!**

Co-located Pressure and Velocity

(ii) Effect in the Continuity Equation



$u_w = \tfrac{1}{2}(u_{i-1} + u_i) \qquad u_e = \tfrac{1}{2}(u_i + u_{i+1})$

Continuity:

$$
\begin{aligned}
0 = (\rho A u)_e - (\rho A u)_w + \cdots &= \rho A\{\tfrac{1}{2}(u_i + u_{i+1}) - \tfrac{1}{2}(u_{i-1} + u_i)\} + \cdots \\
&= \tfrac{1}{2}\rho A\{u_{i+1} - u_{i-1}\} + \cdots \\
&= \tfrac{1}{2}\rho A\{\tfrac{1}{2}d_{i+1}(p_i - p_{i+2}) - \tfrac{1}{2}d_{i-1}(p_{i-2} - p_i)\} + \cdots
\end{aligned}
$$

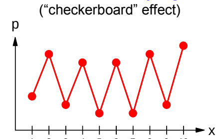
Involves alternate p values only

Pressure/Velocity Coupling

momentum & continuity → pressure equation

but the combination

$\left.\begin{array}{l}\textbf{co-located } u, p \text{ and} \\ \textbf{linear interpolation} \text{ for advective velocities}\end{array}\right\} \longrightarrow$ **odd-even decoupling** ("checkerboard" effect)



Remedies:

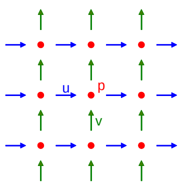
(1) staggered grid

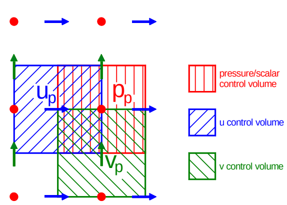
or

(2) Rhie-Chow interpolation for advective velocities

Staggered Grid

(Harlow and Welch, 1965)

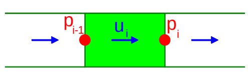




Staggered Grid: Advantages

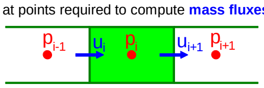
On a **Cartesian** mesh:

- Pressure is stored at points required to compute **pressure forces**



$$u_i = d_i(p_{i-1} - p_i) + \cdots$$

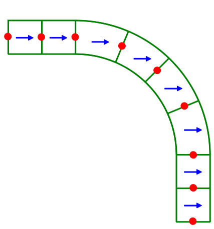
- Velocity is stored at points required to compute **mass fluxes**



$$0 = u_{i+1} - u_i + \cdots \quad = d_{i+1}(p_i - p_{i+1}) - d_i(p_{i-1} - p_i) + \cdots$$
$$= -d_i p_{i-1} + (d_i + d_{i+1}) p_i - d_{i+1} p_{i+1} + \cdots$$

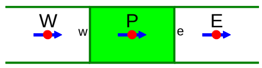
Staggered Grid: Disadvantages

- Added geometric complexity
- Rationale fails on non-Cartesian grids



Non-Staggered/Collocated Grid

(Rhie and Chow, 1983)



Momentum equation: $u_P = \dfrac{\sum a_F u_F}{a_P} - \dfrac{A}{a_P}(p_e - p_w) + \ldots$

$$u_P = \dfrac{\sum a_F u_F}{a_P} - d_P(p_e - p_w) + \ldots \qquad u = \hat{u} - d\Delta p$$

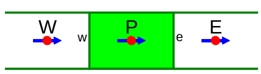
Rhie-Chow interpolation:

$\hat{u} = u + d\Delta p$ — pseudovelocity worked out at **nodes** …

$u_{face} = \hat{u}_{face} - d_{face}\Delta p$ — … then interpolated to **faces**

$$u_e = \overline{(u + d\Delta p)}_e - \overline{d}_e(p_E - p_P)$$

Mass Conservation → Pressure Equation



$u = \hat{u} - d\Delta p$

$$\begin{aligned} 0 &= (\rho A u)_e - (\rho A u)_w + \cdots \\ &= (\rho A \hat{u})_e - (\rho A \hat{u})_w \quad + \quad (\rho A d)_e(p_P - p_E) - (\rho A d)_w(p_W - p_P) + \cdots \\ &= (\rho A \hat{u})_e - (\rho A \hat{u})_w \quad - \quad a_W p_W + a_P p_P - a_E p_E + \cdots \end{aligned}$$

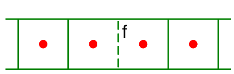
In practice, one solves for a pressure **correction**:

$$0 = (\rho A u^*)_e - (\rho A u^*)_w \quad - \quad a_W p'_W + a_P p'_P - a_E p'_E + \cdots$$

$$- \quad a_W p'_W + a_P p'_P - a_E p'_E + \cdots = -\,\textit{current mass outflow}$$

Example

For the uniform Cartesian mesh shown below the momentum equation gives a velocity/pressure relationship $u = -4\Delta p + \cdots$



u =	1	2	3	3
p =	p_A	0.8	0.7	0.6

For the u and p values given, calculate the advective velocity on the cell face *f*:

(a) by linear interpolation;

(b) by Rhie-Chow interpolation if $p_A = 0.6$;

(c) by Rhie-Chow interpolation if $p_A = 0.9$.

Looking Ahead ...

- Pressure/velocity coupling is <u>the</u> dominant feature in incompressible flow
- Mass and momentum equations must be satisfied simultaneously
- The most popular type of solution algorithm is called a **pressure-correction method**

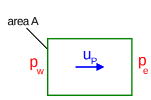
Part 1. Pressure-Velocity Coupling

Part 2. Pressure-Correction Methods

Q1. How are Pressure and Velocity Linked?

Forces:

$$net\ pressure\ force = p_w A - p_e A$$



Momentum equation:

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Velocity-pressure linkage:

$$u_P = d_P (p_w - p_e) + \ldots$$

$$u = -d \Delta p + \ldots \qquad d_p = \frac{A}{a_P}$$

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Q2. How Does a Pressure Equation Arise?

The **momentum equation** links velocity and pressure:

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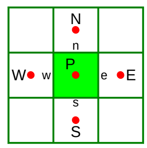
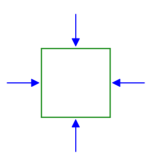
Substituting in the **mass equation** gives an equation for pressure:

$$\begin{aligned} 0 &= (\rho u A)_e - (\rho u A)_w + \cdots \\ &= (\rho A d)_e (p_P - p_E) - (\rho A d)_w (p_W - p_P) + \cdots \\ &= -a_W p_W + a_P p_P - a_E p_E + \cdots \end{aligned}$$

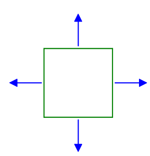


A2. A pressure equation arises from the requirement that **solutions of the momentum equation be mass-consistent**.

Correcting Pressure to Enforce Mass Conservation



Net mass flux **in**;
increase cell pressure



Net mass flux **out**;
decrease cell pressure

Pressure-Correction Methods

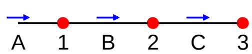
- Iterative numerical schemes for pressure-linked equations
- Used to derive velocity and pressure fields satisfying both mass and momentum equations
- Consist of alternating updates of velocity and pressure:
  - solve momentum equation with current pressure
  - solve equation for a pressure correction field p' to “nudge” velocity and pressure toward mass-conservation
- Popular methods:
  - **SIMPLE**
  - **PISO**

Velocity and Pressure Corrections

- Momentum equation links velocity and pressure: $u = d(p_{-1/2} - p_{+1/2}) + \ldots$
- Must correct **velocity** to satisfy **continuity**: $u \;\rightarrow\; u^* + u'$
- Must simultaneously correct **pressure** to retain a solution of the **momentum** equation: $u' = d(p'_{-1/2} - p'_{+1/2}) + \ldots$

Velocity correction formula: $u \;\rightarrow\; u^* + d(p'_{-1/2} - p'_{+1/2})$

Example



In the steady, one-dimensional, constant-density situation shown, the velocity u is calculated for locations *A*, *B* and *C*, whilst the pressure p is calculated for locations 1, 2 and 3. The velocity-correction formula is:

$$u = u^* + u' \qquad \text{where} \qquad u' = d(p'_{i-1} - p'_i)$$

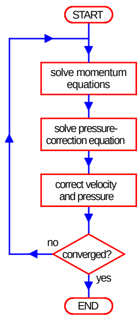
where the locations $i-1$ and i lie on either side of the location for u. The value of d is 2 everywhere. The boundary condition is u_A=10. If, at a given stage in the iteration process, the momentum equations give $u^*_B = 8$ and $u^*_C = 11$, calculate the values of p'_1, p'_2, p'_3 and the resulting velocity corrections.

Comments

- The continuity equation doesn't explicitly contain pressure ... but constraints imposed by the momentum equation lead to a pressure equation
- Matrix equations for pressure are similar to those for the scalar-transport equation
- The source term is minus the current mass outflow
- Pressure is fixed only up to a constant

SIMPLE

Semi-Implicit Method for Pressure-Linked Equations



SIMPLE

1. Solve **momentum** equation with the current pressure. $$u_p = \frac{\Sigma a_F u_F}{a_P} + d_P (p_w^* - p_e^*) + \cdots$$

2. Formulate **pressure-correction equation**:

   (i) relate changes in u and p; $$u'_P = \cancel{\frac{\sum a_F u'_F}{a_P}} + d_P (p'_w - p'_e)$$

   (ii) make SIMPLE approximation;

   (iii) apply mass conservation; $$[\rho A(u^* + u')]_e - [\rho A(u^* + u')]_w + \cdots = 0$$ $$(\rho A u')_e - (\rho A u')_w + \cdots = -\dot{m}^*$$

   (iv) rewrite in terms of p'. $$(\rho A d)_e (p'_P - p'_E) - (\rho A d)_w (p'_W - p'_P)] + \cdots \quad = -\dot{m}^*$$

3. **Solve** pressure-correction equation: $a_P p'_P - \sum a_F p'_F = -\dot{m}^*$

4. **Correct** velocity and pressure: $$p_P \rightarrow p_P^* + p'_P$$ $$u_P \rightarrow u_P^* + d_P (p'_w - p'_e)$$

SIMPLE (continued)

- Minor differences of detail on **staggered** and **unstaggered** grids
- The **source** of the pressure-correction equation is **minus the current mass imbalance**
- (Substantial) **under-relaxation** is usually required
- **Iterative** process – no need to solve equations exactly at each stage

Example

In the 2-d staggered-grid arrangement shown below, u and v (the x and y components of velocity), are stored at nodes indicated by arrows, whilst pressure p is stored at the intermediate nodes A-D. The grid spacing is uniform and the same in both directions. Velocity is fixed on boundaries as shown. The velocity components at the interior nodes (u_B, u_D, v_C and v_D) are to be found.

At an intermediate stage of calculation the internal velocity values are found to be

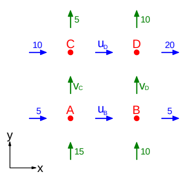
$$u_B = 11, \quad u_D = 14, \quad v_C = 8, \quad v_D = 5$$

whilst correction formulae derived from the momentum equation are

$$u' = 2(p'_w - p'_e), \qquad v' = 3(p'_s - p'_n)$$

with geographical (w,e,s,n) notation indicating the **relative** location of pressure nodes.

Show that applying mass conservation to control volumes centred on pressure nodes leads to simultaneous equations for the pressure corrections. Solve for the pressure corrections and use them to generate a mass-consistent flow field.



Variants of SIMPLE

SIMPLE $\quad u'_P = \cancel{\frac{\sum a_F u'_F}{a_P}} + d_P(p'_w - p'_e)$

SIMPLER: precede momentum update with exact pressure equation:

$$a_P p_P - \sum a_F p_F = -Div(\hat{u})$$

SIMPLEC: alternative correction formula:

$$u'_P \approx \frac{d_P}{1 - \sum a_F / a_P}(p'_w - p'_e)$$

SIMPLEX: solve equations for correction coefficients d_p:

$$u'_P \approx \delta_P (p'_w - p'_e)$$

PISO

(Pressure-Implicit with Splitting of Operators)

- **Time-dependent** pressure-correction method
- Each timestep $t^{old} \rightarrow t^{new}$ is a **non-iterative** sequence:
  1. solve time-dependent momentum eqns with t^{old} pressure
  2. formulate and solve a pressure-correction equation and update pressure and velocity
  3. second mass-corrector step with time-advanced pressure
- More efficient than SIMPLE for **time-dependent** problems

Summary (1)

- Each momentum component satisfies its own scalar-transport equation
- The momentum equations require special treatment because they are:
  - non-linear
  - coupled
  - required also to be mass-consistent
- In incompressible flow, continuity (mass conservation) leads to a pressure equation
- Odd-even decoupling of pressure can be addressed by either:
  - staggered velocity grid
  - non-staggered grid, but Rhie-Chow interpolation for advective velocities

Summary (2)

- Pressure-correction methods are iterative schemes for solving mass and momentum equations simultaneously
- They consist of alternating solutions of:
  - the momentum equation (with pressure fixed)
  - a pressure-correction equation to nudge the velocity field towards mass conservation
- Widely used pressure-correction methods are:
  - SIMPLE
  - PISO