Transient Heat Conduction

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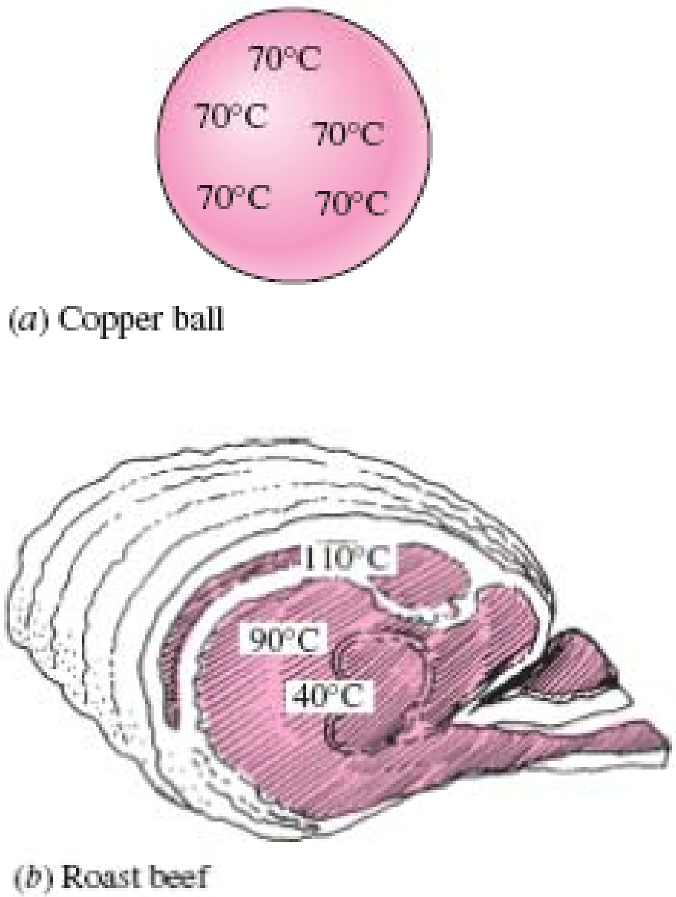
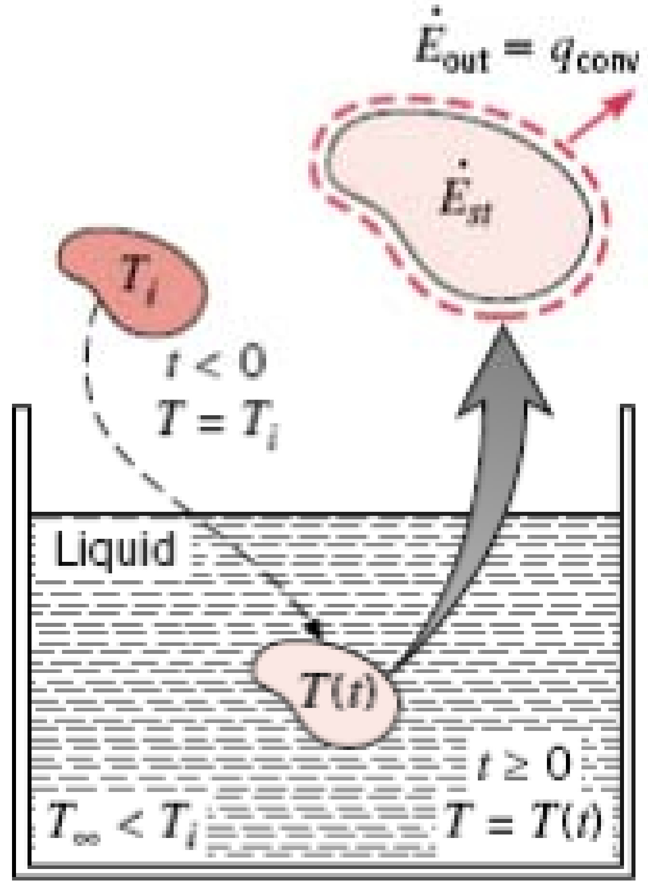
1. Lumped System Analysis
2. Transient heat conduction in large plane walls, long cylinders, and spheres with spatial effects
3. Transient heat conduction in semi-infinite solids

Lumped System





Cooling of a hot metal forging

Energy balance

Heat transfer into the body during dt = The increase in the energy of the body during dt

$$h.A_s(T_\infty - T)dt = mC_p dT$$

$$m = \rho V$$

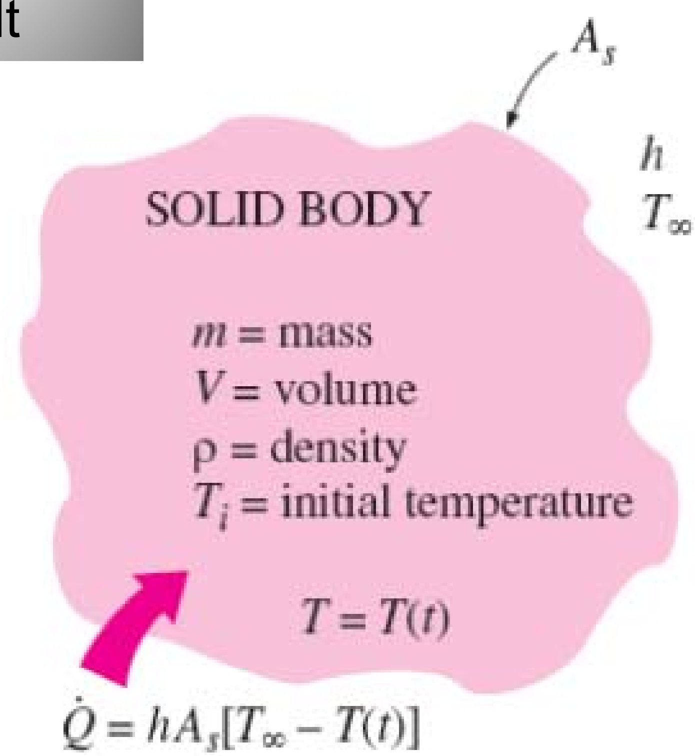
$$dT = d(T - T_\infty)$$

$$T_\infty = \text{constant}$$

$$\frac{d(T - T_\infty)}{T - T_\infty} = -\frac{h.A_s}{\rho V C_p} dt$$

$$\ln \frac{T(t) - T_\infty}{T_i - T_\infty} = -\frac{h.A_s}{\rho V C_p} t$$

$$\frac{T(t) - T_\infty}{T_i - T_\infty} = e^{-bt} \qquad b = \frac{h.A_s}{\rho V C_p} (s^{-1})$$



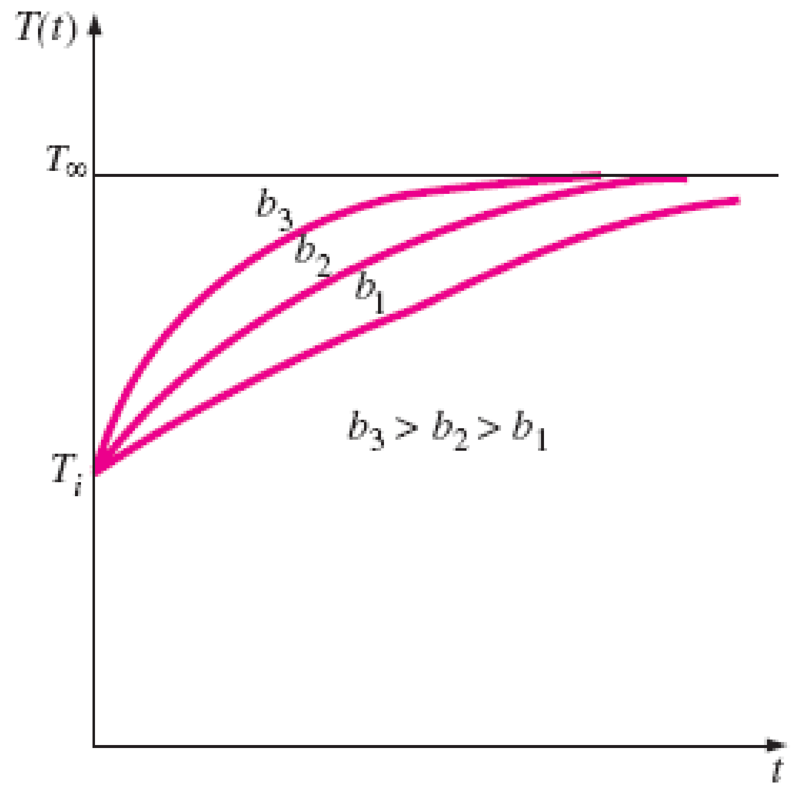
The geometry and parameters involved in the lumped system analysis.

Effect of b

$$b = \frac{h.A_s}{\rho V C_p}(s^{-1})$$

b has the unit of 1/s.
Reciprocal of b is the time constant

$$\frac{T(t) - T_\infty}{T_i - T_\infty} = e^{-bt}$$

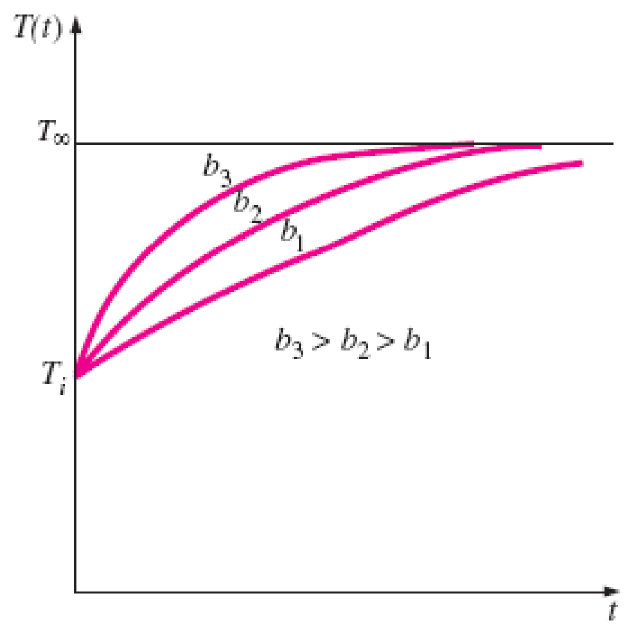


This equation enables us to determine the temperature T(t) of a body at time t, or alternatively, the time t required for the temperature to reach a specified value T(t).

Effect of b

$$b = \frac{h.A_s}{\rho V C_p} (s^{-1})$$

- The temperature of a body approaches the ambient temperature *T* exponentially.
- The temperature of the body changes rapidly at the beginning, but rather slowly later on.
- A large value of *b* indicates that the body will approach the environment temperature in a short time.
- The larger the value of the exponent *b*, the higher the rate of decay in temperature.
- Note that *b* is proportional to the surface area, but inversely proportional to the mass and the specific heat of the body. This is not surprising since it takes longer to heat or cool a larger mass, especially when it has a large specific heat.



Heat Transfer

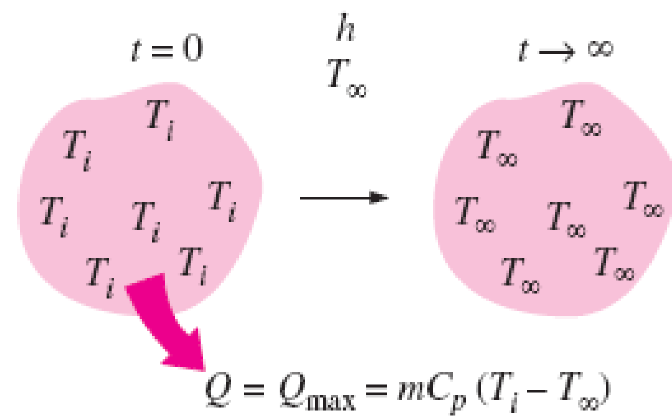
Rate of convection heat transfer between the body and its environment at that time

$$\dot{Q}(t) = hA_s\left[T(t) - T_\infty\right] \quad \text{Watt}$$

The total amount of heat transfer between the body and the surrounding medium over the time interval t = 0 to t is simply the change in the energy content of the body:

$$Q = mC_p\left[T(t) - T_i\right] \qquad \text{kJ}$$

$$Q_{max} = mC_p\left[T_\infty - T_i\right] \qquad \text{kJ}$$



Heat transfer to or from a body reaches its maximum value when the body reaches the environment temperature.

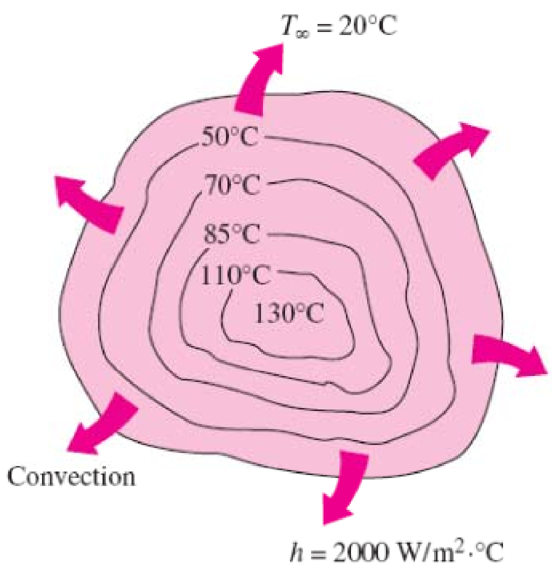
Criterion for lumped system

characteristic length $L_c = \dfrac{V}{A_s}$

Biot number $Bi = \dfrac{hL_c}{k}$

$$Bi = \frac{h}{k/L_c}\frac{\Delta T}{\Delta T} = \frac{\text{Convection at the surface of the body}}{\text{Conduction within the body}}$$

$$Bi = \frac{L_c/k}{1/h} = \frac{\text{Conduction resistance within the body}}{\text{Convection resistance at the surface of the body}}$$

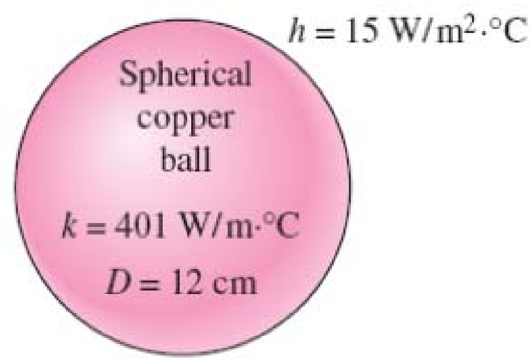


When the convection coefficient h is high and k is low, large temperature differences occur between the inner and outer regions of a large solid.

$$Plane\ wall\ (thickness\ 2L)\quad L_c = \frac{2L\ A}{2\ A} = L$$

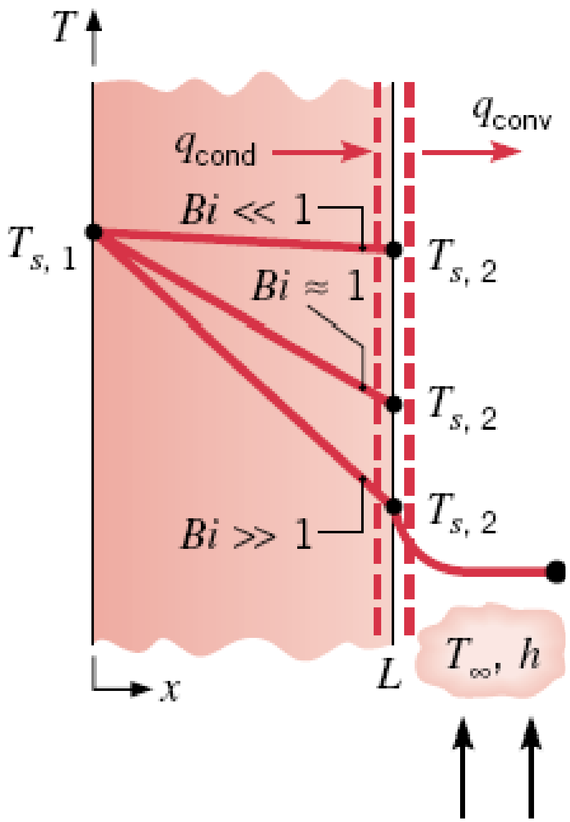
$$Bi = \frac{hL_c}{k} < 0.1$$

Small bodies with high thermal conductivities and low convection coefficients are most likely to satisfy the criterion for lumped system analysis.



$$L_c = \frac{V}{A_s} = \frac{\frac{1}{6}\pi D^3}{\pi D^2} = \frac{1}{6}D = 0.02\ m$$

$$Bi = \frac{hL_c}{k} = \frac{15 \times 0.02}{401} = 0.00075 < 0.1$$

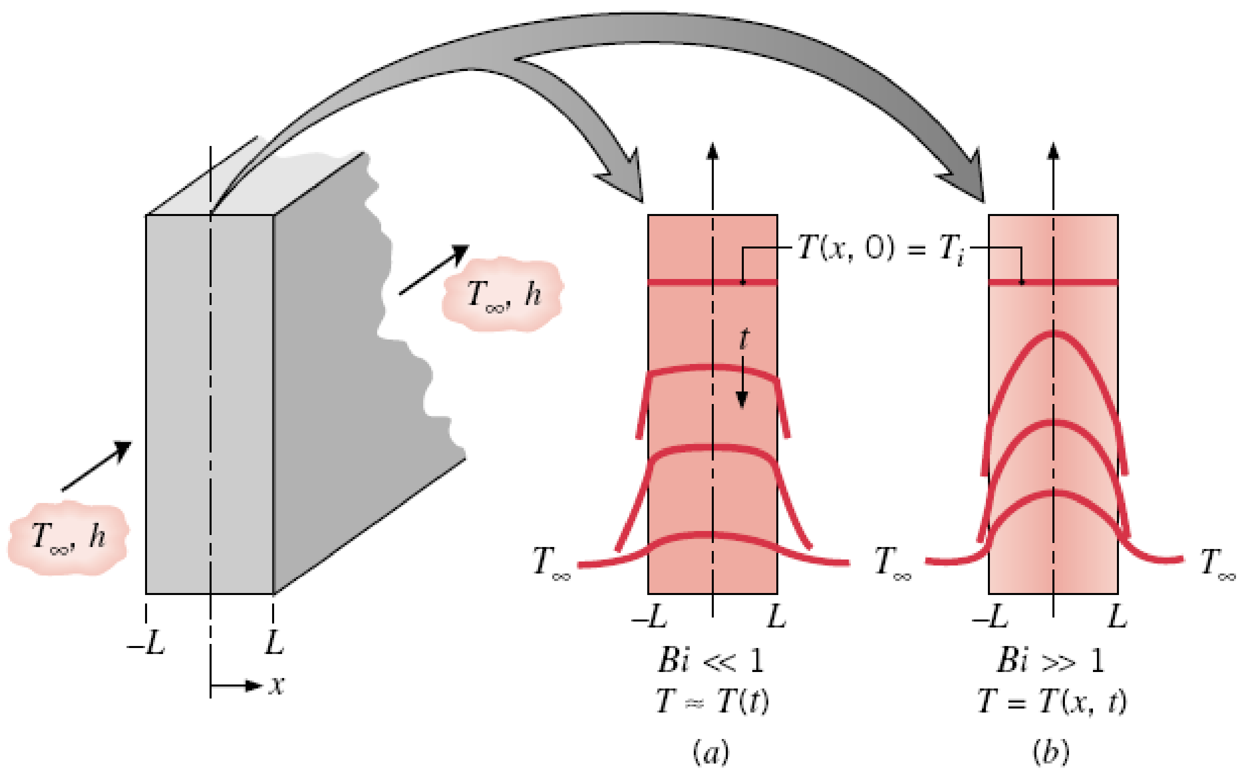


under steady-state conditions

$$\frac{kA}{L}(T_{s,1} - T_{s,2}) = hA(T_{s,2} - T_\infty)$$

$$\frac{T_{s,1} - T_{s,2}}{T_{s,2} - T_\infty} = \frac{(L/kA)}{(1/hA)} = \frac{R_{cond}}{R_{conv}} = \frac{hL}{k} \equiv Bi$$

Transient temperature distribution →



Fourier Number

$$\ln \frac{T(t)-T_\infty}{T_i - T_\infty} = -\frac{h.A_s}{\rho V C_p} t$$

$$\frac{T(t)-T_\infty}{T_i - T_\infty} = \exp\left(-\frac{h.A_s}{\rho V C_p} t\right)$$

(Fourier number, a dimensionless time) $\tau = \frac{\alpha t}{L_c^2}$

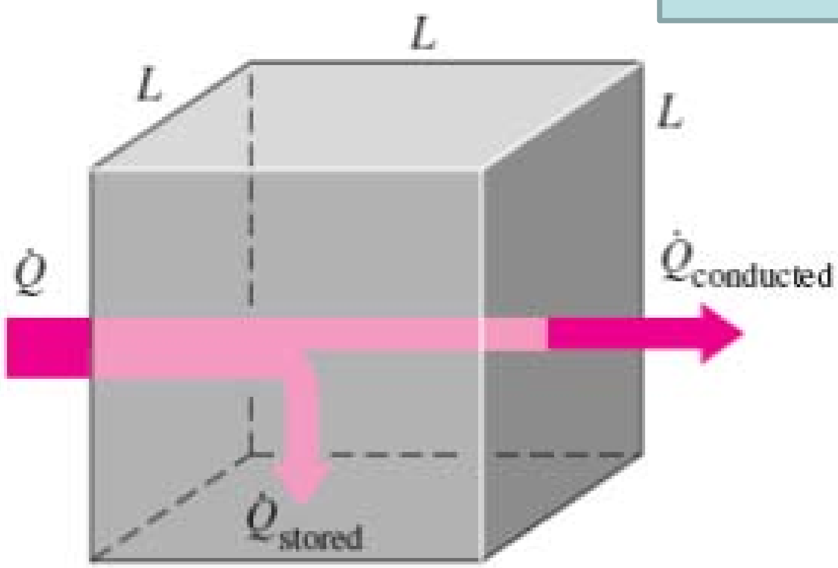
$$\frac{h.A_s t}{\rho V C_p} = \frac{ht}{\rho C_p L_c} = \frac{hL_c}{k}\frac{k}{\rho C_p}\frac{t}{L_c^2} = \frac{hL_c}{k}\frac{\alpha t}{L_c^2} = Bi\,\tau$$

$$\frac{h.A_s t}{\rho V C_p} = Bi\,\tau$$

$$\boxed{\frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = \exp(-Bi.\tau)}$$

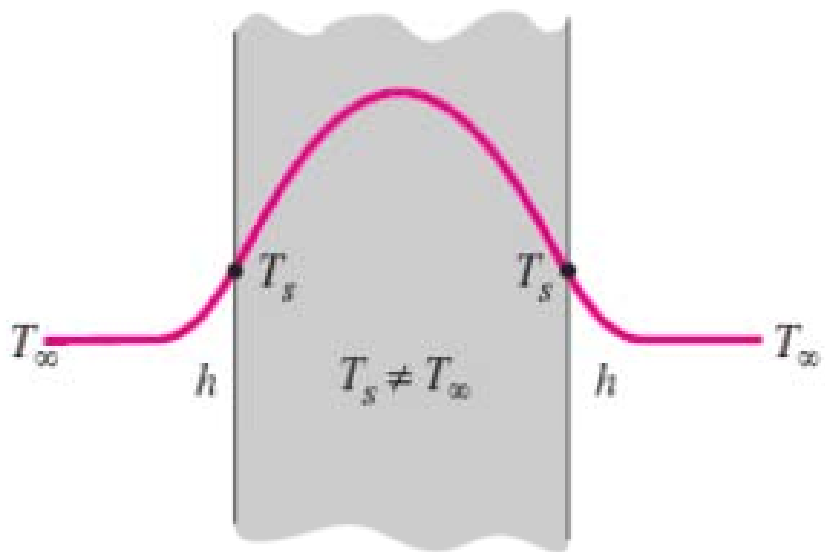
Fourier Number (τ)

$$\tau = \frac{\alpha t}{L^2} = \frac{kL^2\,(1/L)}{\rho C_p L^3/t}\,\frac{\Delta T}{\Delta T} =$$

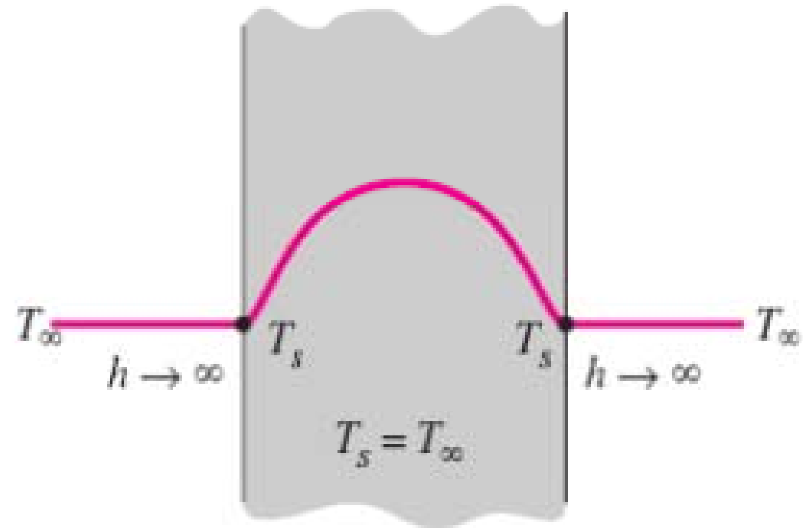


Fourier number at time t can be viewed as the ratio of the rate of heat conducted to the rate of heat stored at that time.

Fourier number: $\tau = \dfrac{\alpha t}{L^2} = \dfrac{\dot{Q}_{conducted}}{\dot{Q}_{stored}}$



(*a*) Finite convection coefficient

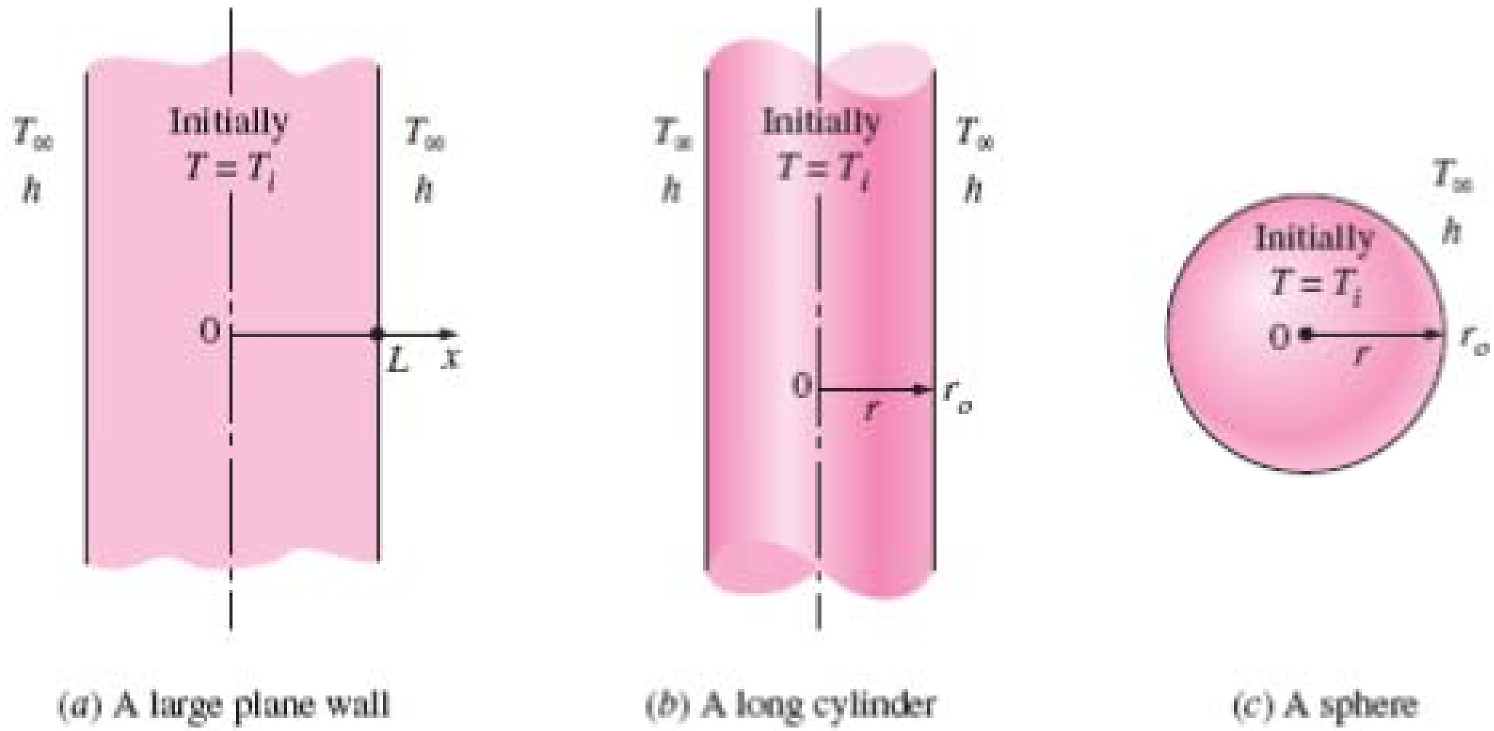


(*b*) Infinite convection coefficient

The specified surface temperature corresponds to the case of convection to an environment at T with a convection coefficient h that is infinite

- Transient heat conduction in large plane walls, long cylinders, and spheres with spatial effects

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad ?$$

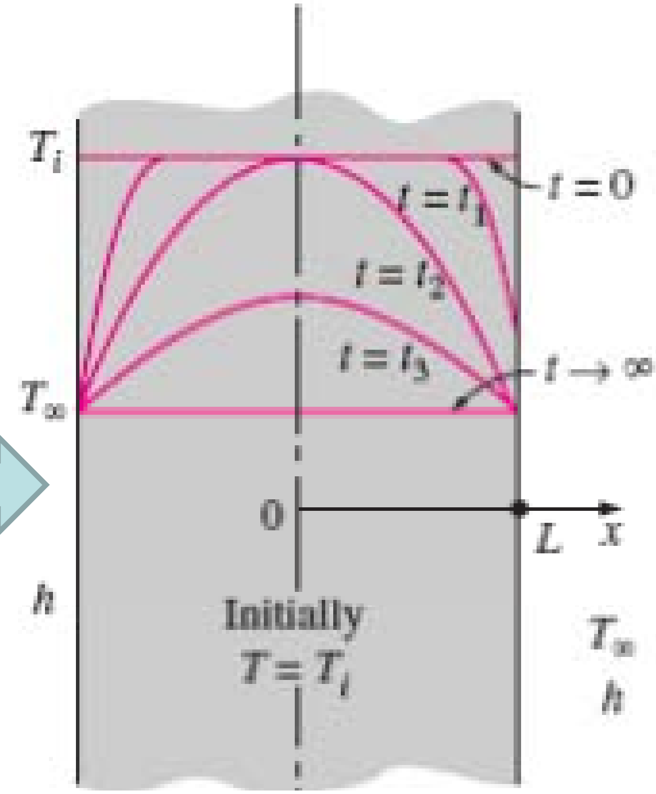


Schematic of the simple geometries in which heat transfer is 1-D

Plane Wall

Governing equation $$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Transient temperature profiles in a plane wall exposed to convection from its surfaces for $T_i > T_\infty$



Governing equation along with boundary conditions shows that T is a function of x, t, k, α, h, T_i and T_∞

Dimensionless temperature: $$\theta(x, t) = \frac{T(x, t) - T_\infty}{T_i - T_\infty}$$

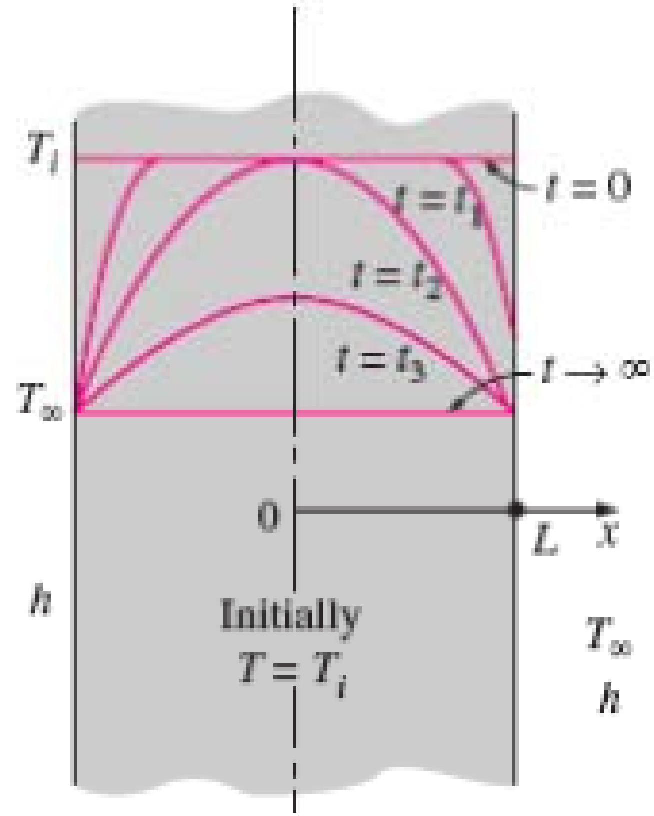
Dimensionless distance from the center: $$X = \frac{x}{L}$$

Dimensionless heat transfer coefficient: $$\text{Bi} = \frac{hL}{k} \quad \textbf{(Biot number)}$$

Dimensionless time: $$\tau = \frac{\alpha t}{L^2} \quad \textbf{(Fourier number)}$$

Non-dimensional Eq

$$\theta = \frac{T - T_\infty}{T_i - T_\infty}$$



Governing equation (Non-dimensional form)

$$\frac{\partial^2 \theta}{\partial X^2} = \frac{\partial \theta}{\partial \tau}$$

Initial condition

$$\theta(X,0) = 1$$

Boundary conditions

At X=0 and τ >0

$$\frac{\partial \theta}{\partial X} = 0$$

At X=L and τ >0

$$\frac{\partial \theta}{\partial X} = -\mathrm{Bi}.\theta$$

The non-dimensionalization enables us to present the temperature in terms of three parameters only: *X*, Bi, and τ

Solution

$$\frac{\partial^2\theta}{\partial X^2} = \frac{\partial\theta}{\partial\tau}$$

Solution takes the form of an

infinite series

$$\theta = \sum_{n=1}^{\infty} A_n \exp(-\lambda_n^2\tau)\cos(\lambda_n X)$$

$$A_n = \frac{4\sin\lambda_n}{2\lambda_n + \sin 2\lambda_n}$$

$$\lambda_n \tan\lambda_n = Bi$$

The solution of 1D transient heat conduction involves infinite series, which are difficult to deal with. However, the terms in the solutions converge rapidly with increasing time, and for τ > 0.2, keeping the first term and neglecting all the remaining terms in the series results in an error under 2 percent. We are usually interested in the solution for times with τ > 0.2, and thus it is very convenient to express the solution using an **one term approximation.**

See next slide for this expressions

Plane wall $$\theta(x,t)_{wall} = \frac{T(x,t)-T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau} \cos\left(\frac{\lambda_1 x}{L}\right) \qquad \tau > 0.2$$

Cylinder $$\theta(r,t)_{cyl} = \frac{T(r,t)-T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau} J_0\left(\frac{\lambda_1 r}{r_0}\right) \qquad \tau > 0.2$$

Sphere $$\theta(r,t)_{sph} = \frac{T(r,t)-T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau} \frac{\sin\left(\frac{\lambda_1 r}{r_0}\right)}{\frac{\lambda_1 r}{r_0}} \qquad \tau > 0.2$$

Center of Plane wall (x=0) $$\theta_{0,wall} = \frac{T_0 - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau}$$

Center of cylinder (r=0) $$\theta_{0,cyl} = \frac{T_0 - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau}$$

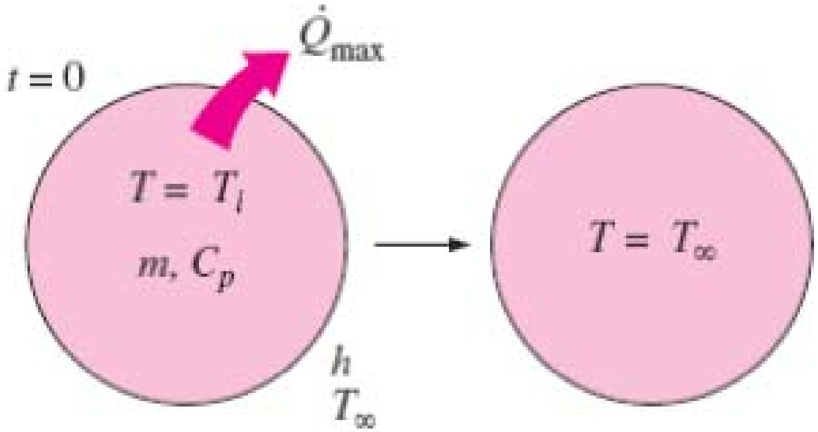
Center of sphere (r=0) $$\theta_{0,sph} = \frac{T_0 - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau}$$

Coefficients used in the one-term approximate solution of transient one-dimensional heat conduction in plane walls, cylinders, and spheres (Bi = hL/k for a plane wall of thickness $2L$, and Bi = hr_o/k for a cylinder or sphere of radius r_o)

	Plane Wall		*Cylinder*		*Sphere*	
Bi	λ_1	A_1	λ_1	A_1	λ_1	A_1
0.01	0.0998	1.0017	0.1412	1.0025	0.1730	1.0030
0.02	0.1410	1.0033	0.1995	1.0050	0.2445	1.0060
0.04	0.1987	1.0066	0.2814	1.0099	0.3450	1.0120
0.06	0.2425	1.0098	0.3438	1.0148	0.4217	1.0179
0.08	0.2791	1.0130	0.3960	1.0197	0.4860	1.0239
0.1	0.3111	1.0161	0.4417	1.0246	0.5423	1.0298
0.2	0.4328	1.0311	0.6170	1.0483	0.7593	1.0592
0.3	0.5218	1.0450	0.7465	1.0712	0.9208	1.0880
0.4	0.5932	1.0580	0.8516	1.0931	1.0528	1.1164
0.5	0.6533	1.0701	0.9408	1.1143	1.1656	1.1441
0.6	0.7051	1.0814	1.0184	1.1345	1.2644	1.1713
0.7	0.7506	1.0918	1.0873	1.1539	1.3525	1.1978
0.8	0.7910	1.1016	1.1490	1.1724	1.4320	1.2236
0.9	0.8274	1.1107	1.2048	1.1902	1.5044	1.2488
1.0	0.8603	1.1191	1.2558	1.2071	1.5708	1.2732
2.0	1.0769	1.1785	1.5995	1.3384	2.0288	1.4793
3.0	1.1925	1.2102	1.7887	1.4191	2.2889	1.6227
4.0	1.2646	1.2287	1.9081	1.4698	2.4556	1.7202
5.0	1.3138	1.2403	1.9898	1.5029	2.5704	1.7870
6.0	1.3496	1.2479	2.0490	1.5253	2.6537	1.8338
7.0	1.3766	1.2532	2.0937	1.5411	2.7165	1.8673
8.0	1.3978	1.2570	2.1286	1.5526	2.7654	1.8920
9.0	1.4149	1.2598	2.1566	1.5611	2.8044	1.9106
10.0	1.4289	1.2620	2.1795	1.5677	2.8363	1.9249
20.0	1.4961	1.2699	2.2880	1.5919	2.9857	1.9781
30.0	1.5202	1.2717	2.3261	1.5973	3.0372	1.9898
40.0	1.5325	1.2723	2.3455	1.5993	3.0632	1.9942
50.0	1.5400	1.2727	2.3572	1.6002	3.0788	1.9962
100.0	1.5552	1.2731	2.3809	1.6015	3.1102	1.9990
∞	1.5708	1.2732	2.4048	1.6021	3.1416	2.0000

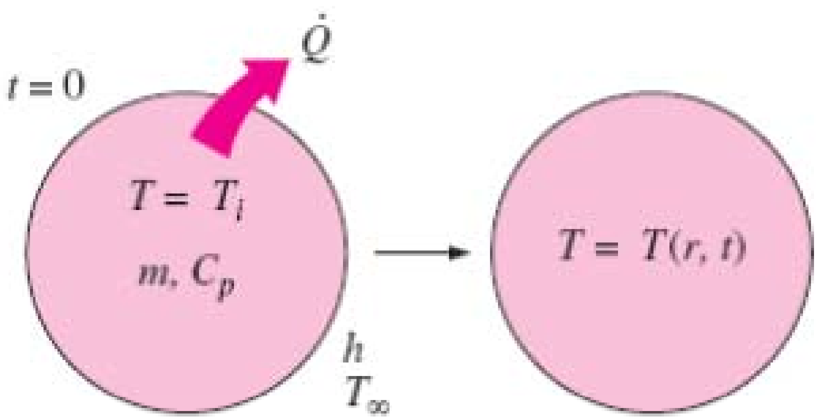
The zeroth- and first-order Bessel functions of the first kind

ξ	$J_0(\xi)$	$J_1(\xi)$
0.0	1.0000	0.0000
0.1	0.9975	0.0499
0.2	0.9900	0.0995
0.3	0.9776	0.1483
0.4	0.9604	0.1960
0.5	0.9385	0.2423
0.6	0.9120	0.2867
0.7	0.8812	0.3290
0.8	0.8463	0.3688
0.9	0.8075	0.4059
1.0	0.7652	0.4400
1.1	0.7196	0.4709
1.2	0.6711	0.4983
1.3	0.6201	0.5220
1.4	0.5669	0.5419



(*a*) Maximum heat transfer ($t \to \infty$)

$$Q_{max} = mC_p(T_\infty - T_i) = \rho V C_p(T_\infty - T_i) \qquad \text{(kJ)}$$



$$\left.\begin{aligned} &\text{Bi} = \ldots \\ &\frac{h^2 \alpha t}{k^2} = \text{Bi}^2 \tau = \ldots \end{aligned}\right\} \quad \frac{Q}{Q_{max}} = \ldots$$

(Gröber chart)

(*b*) Actual heat transfer for time *t*

Plane wall $$\left.\frac{Q}{Q_{max}}\right|_{wall} = 1 - \theta_{0,wall} \frac{\sin \lambda_1}{\lambda_1}$$

Cylinder $$\left.\frac{Q}{Q_{max}}\right|_{cyl} = 1 - 2\theta_{0,cyl} \frac{J_1(\lambda_1)}{\lambda_1}$$

Sphere $$\left.\frac{Q}{Q_{max}}\right|_{sph} = 1 - 3\theta_{0,sph} \frac{\sin \lambda_1 - \lambda_1 \cos \lambda_1}{\lambda_1^3}$$

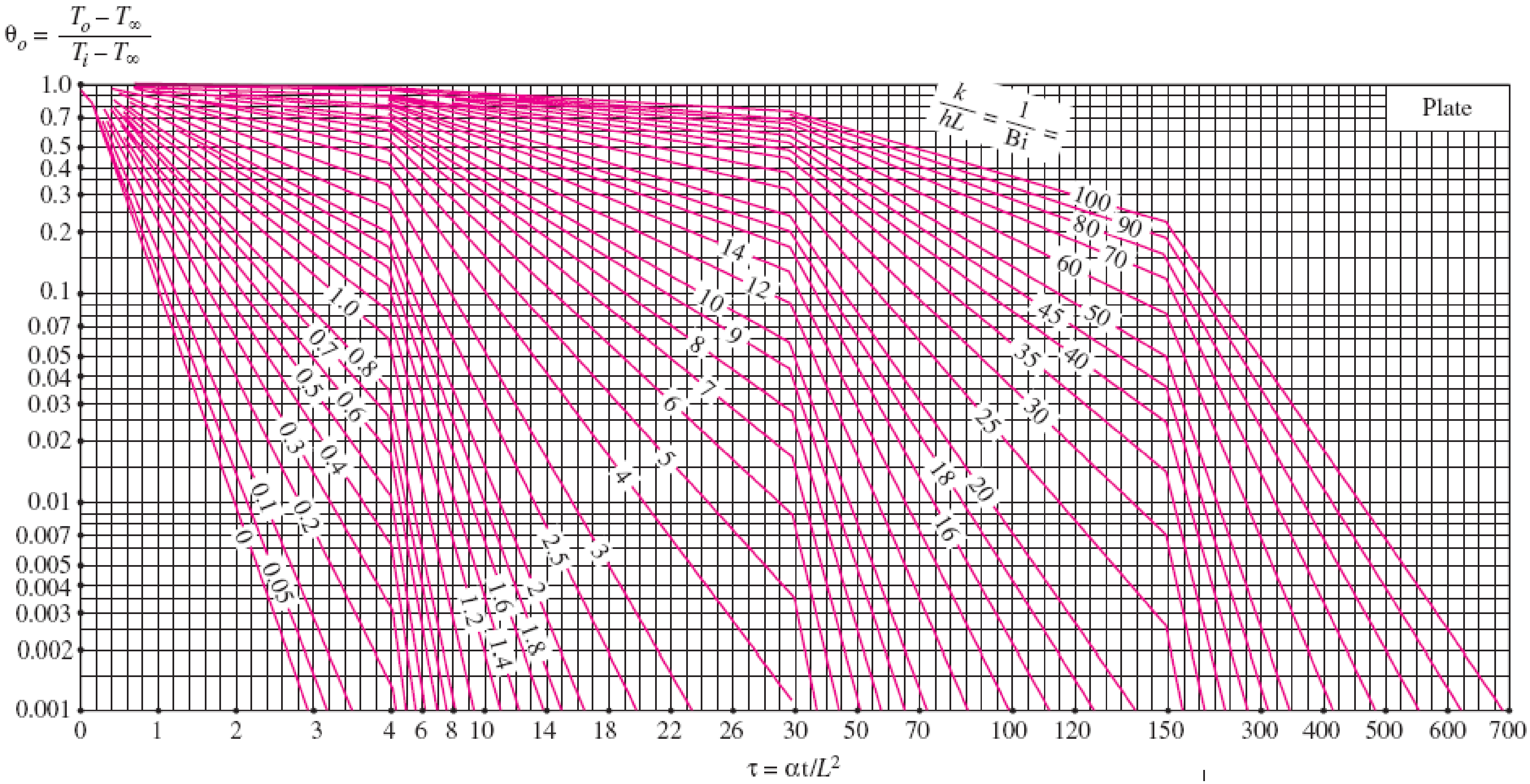
There are three charts associated with each geometry:

The first chart is to determine the temperature T_o at the center of the geometry at a given time t.

The second chart is to determine the temperature at other locations at the same time in terms of T_o.

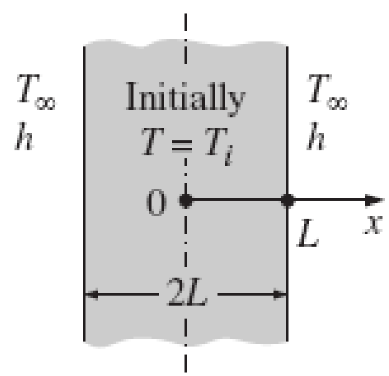
The third chart is to determine the total amount of heat transfer up to the time t.

These plots are valid for $\tau > 0.2$.

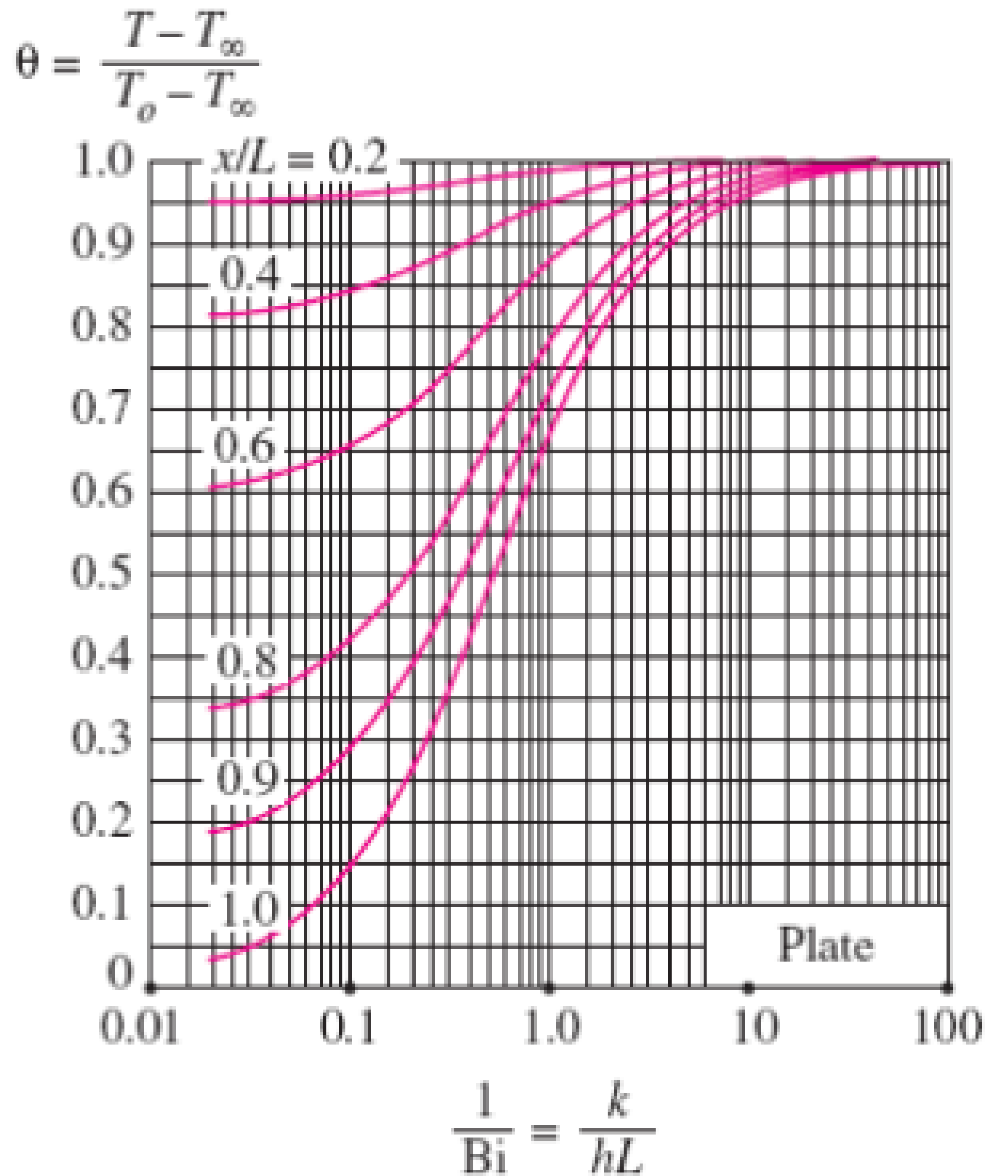


(*a*) Midplane temperature (from M. P. Heisler)

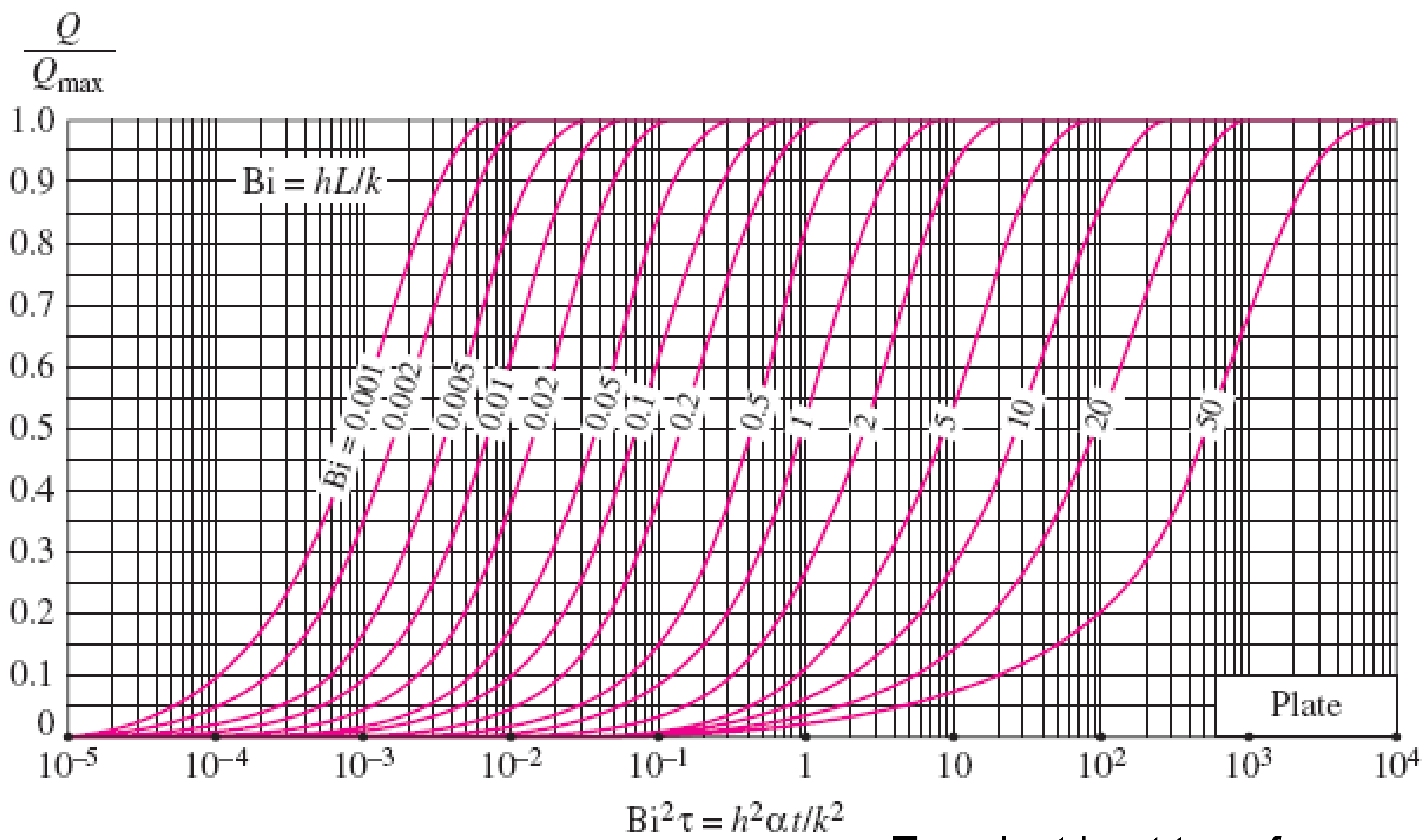
Transient mid-plane temperature chart for a plane wall of thickness 2L initially at a uniform temperature T_i subjected to convection from both sides to an environment at temperature T_∞ with a convection coefficient of h.



Transient temperature chart for a plane wall of thickness 2L initially at a uniform temperature T_i subjected to convection from both sides to an environment at temperature T_∞ with a convection coefficient of h.



(b) Temperature distribution (from M. P. Heisler)



(*c*) Heat transfer (from H. Gröber et al.)

Transient heat transfer chart for a plane wall of thickness 2L initially at a uniform temperature T_i subjected to convection from both sides to an environment at temperature T_∞ with a convection coefficient of h.

$$Q_{max} = m.C_p[T_\infty - T_i] = \rho V C_p[T_\infty - T_i] \quad kJ$$

Transient Conduction in Semi-Infinite Solid

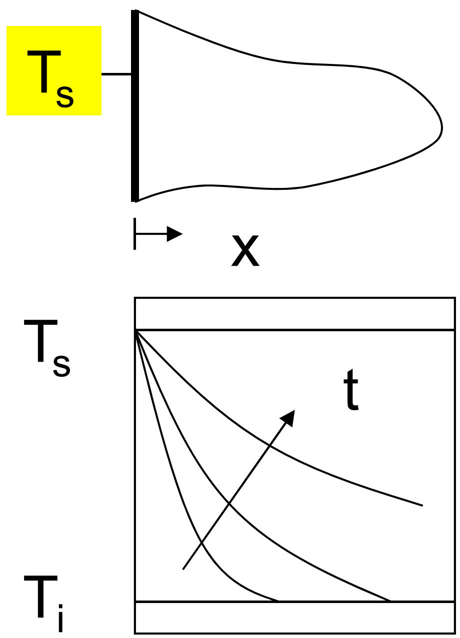
Consider a semi-infinite solid (extends to infinity in all but one direction with a single identifiable surface)

$$T(x \to \infty) = T_i$$

Case 1

$T(x,0) = T_i$

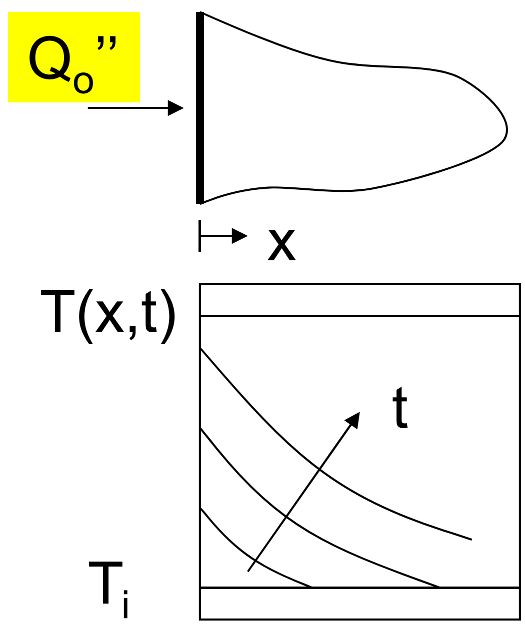
$T(0,t) = T_s$



Case 2

$T(x,0) = T_i$

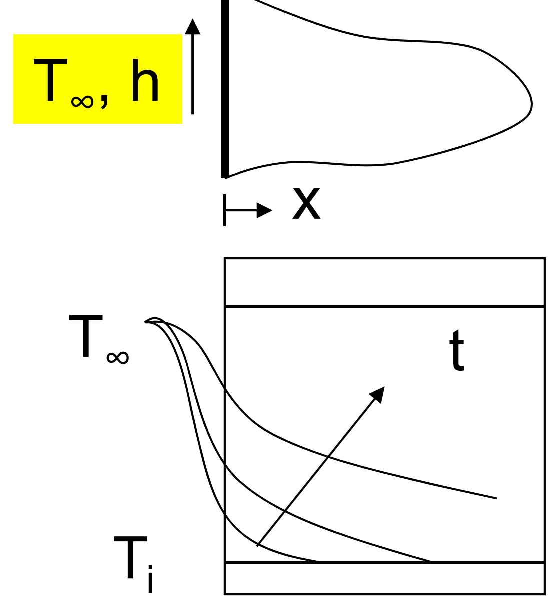
$$-k\left.\frac{\partial T}{\partial x}\right|_{x=0} = \dot{Q}''_{x=0}$$



Case 3

$T(x,0) = T_i$

$$-k\left.\frac{\partial T}{\partial x}\right|_{x=0} = h[T_\infty - T(0,t)]$$



Case 1 Constant Surface Temperature: T(0,t) = T_s

$$\frac{T(x,t)-T_s}{T_i-T_s}=\operatorname{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right) \quad \text{and} \quad \dot{Q}''_s(t)=\frac{k(T_s-T_i)}{\sqrt{\pi\alpha t}}$$

Case 2 Constant Surface Heat Flux: $Q_s'' = Q_o''$

$$T(x,t)-T_i=\frac{2q''_o\left(\alpha t/\pi\right)^{1/2}}{k}\exp\left(\frac{-x^2}{4\alpha t}\right)-\frac{\dot{Q}''_o x}{k}\operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

Case 3 Surface Convection: $-k\left.\frac{\partial T}{\partial x}\right|_{x=0}=h\left[T_\infty-T(0,t)\right]$

$$\frac{T(x,t)-T_i}{T_\infty-T_i}=\operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right)-\left[\exp\left(\frac{hx}{k}+\frac{h^2\alpha t}{k^2}\right)\right]\left[\operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}+\frac{h\sqrt{\alpha t}}{k}\right)\right]$$

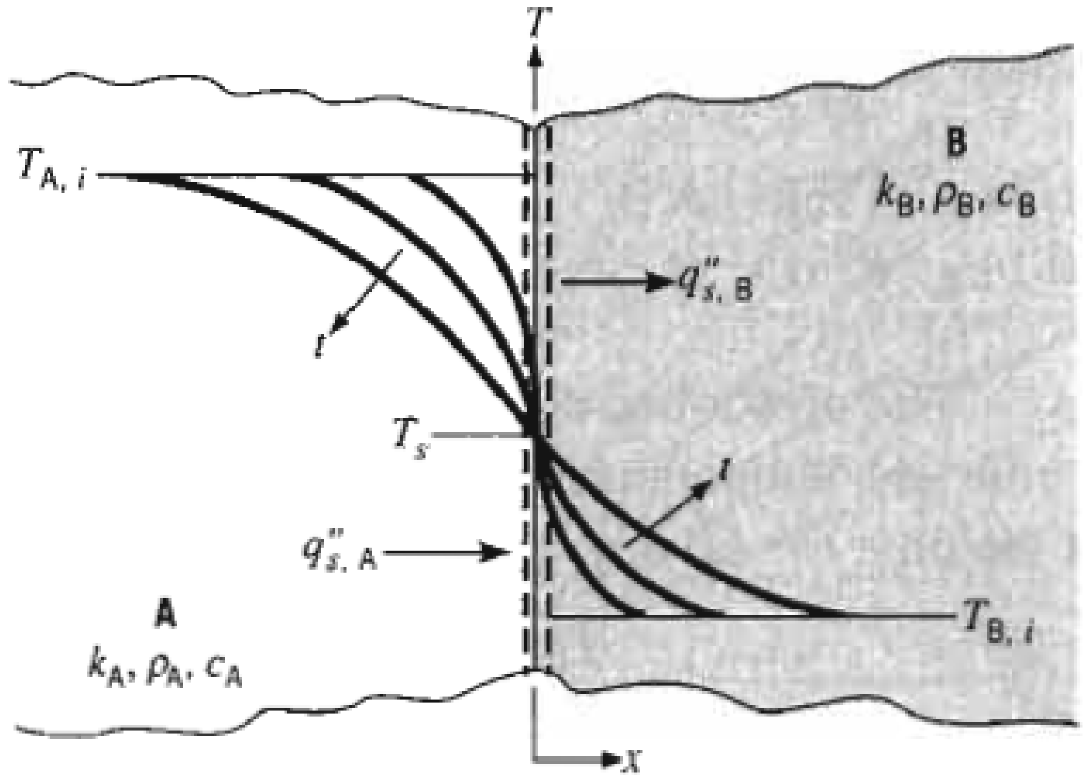
The Gaussian error function, erf (w), is a standard mathematical function (see Table B.2 for values)
Complementary Error Function, erfc(w) = 1 – erf(w)

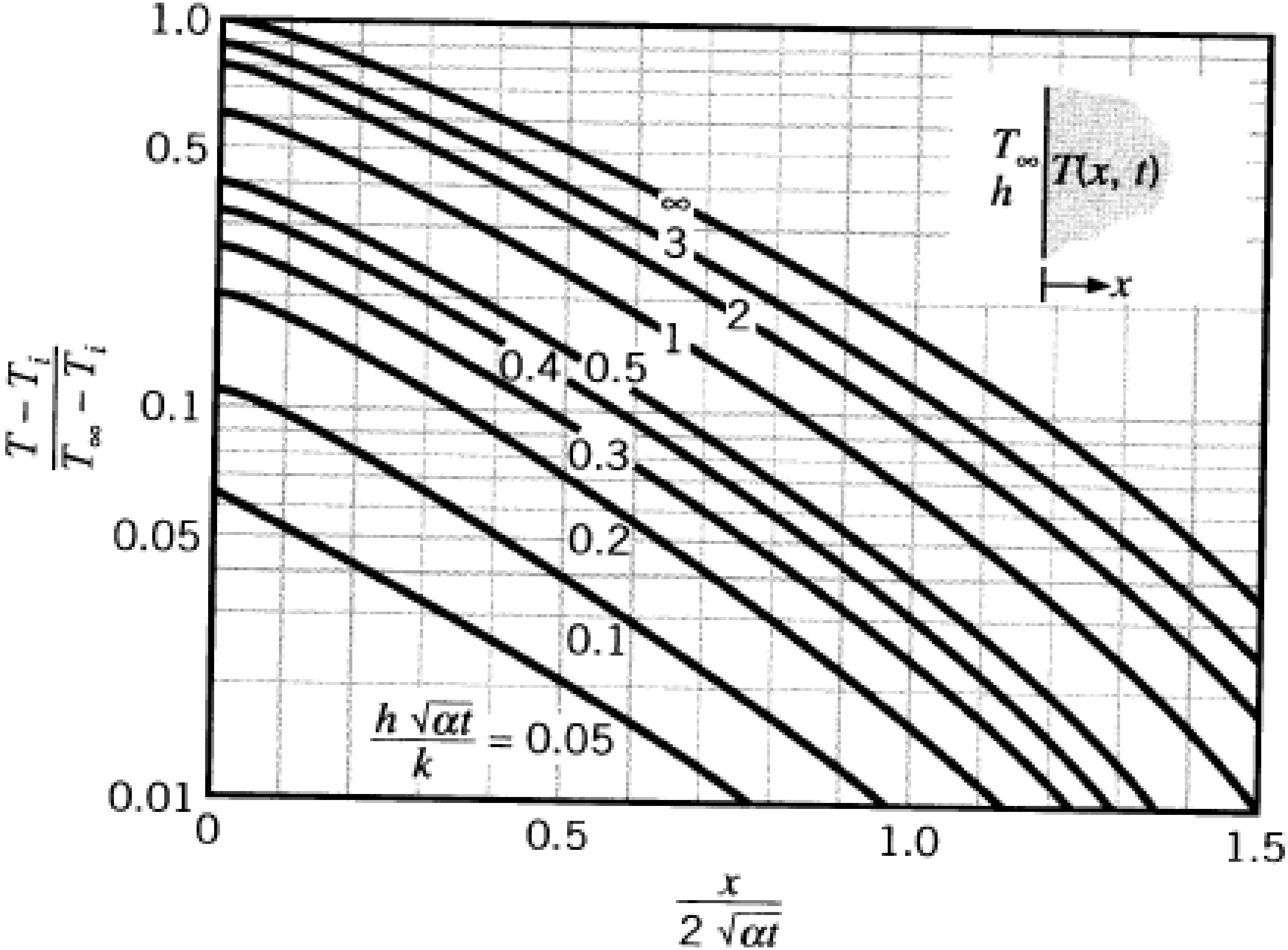
$$\operatorname{erf}(w)=\frac{2}{\sqrt{\pi}}\int_0^w e^{-v^2}dv$$

$$\frac{-k_A(T_s - T_{A,i})}{(\pi\alpha_A t)^{1/2}} = \frac{k_B(T_s - T_{B,i})}{(\pi\alpha_B t)^{1/2}}$$

T_s between 2 solids $$T_s = \frac{m_A T_{A,i} + m_B T_{B,i}}{m_A + m_B}$$

$$m = \sqrt{k\rho Cp}$$





1.0
0.5
0.1
0.05
0.01
$\frac{T - T_i}{T_\infty - T_i}$
0
0.5
1.0
1.5
$\frac{x}{2\sqrt{\alpha t}}$
T_∞
h
$T(x, t)$
x
∞
3
2
1
0.4
0.5
0.3
0.2
0.1
$\frac{h\sqrt{\alpha t}}{k} = 0.05$