Essay 4

Numerical Solutions of the Equations of Heat Transfer and Fluid Flow

4.1 Introduction

In Essay 3, it was shown that heat conduction is governed by a partial differential equation. It will also be demonstrated later that fluid flow and convective heat transfer are also described by partial differential equations. The capability of solving partial differential equations in their unapproximated form is confined to analog devices, a prominent example of which is the *differential analyzer*. That device, which was prominent between 1930 and 1945, was not able to provide solutions of sufficient accuracy to meet the exacting needs of critical engineering projects. Digital computers are not capable of solving unapproximated differential equations. To enable differential equations to be solved digitally, it is necessary that they be approximated.

There are three prominent strategies for approximating partial differential equations that are used at present. They are:

    (a) Finite Element Method (FEM)
    (b) Finite Difference Method (FDM)
    (c) Finite Volume Method (FVM)

All of these approaches transform the solution task to that of solving a set of simultaneous algebraic equations in a solution space which is subdivided into a number of small elements. The process of transforming a solution space in which the variables are continuous to a space which there is an assemblage of small elements is called *discretization*. An alternative descriptions of this process is termed meshing. In the discretized space, the variables are not continuous.

It is useful to demonstrate, at least for one of the discretization approaches, how the governing differential equations are discretized. In that regard, it is important to recognize that different degrees of mathematical sophistication are needed for the discretization depending on the specific approach. Among the three listed strategies, the finite element method is based on the most sophisticated mathematics. The degree of mathematical involvement is much less for both the finite difference and finite volume methods. In what follows, the discretization of the heat conduction equation, Eq. (3.18), will be performed by the finite volume method.

4.2 Discretization of the heat conduction equation

The analysis begins by the establishment of a control volume having small, but finite dimensions Δx, Δy, and Δz. That control volume is shown in Fig. 4.1. It may be recognized that this figure is very similar to what has already been displayed in Fig. 3.5, with the main difference being that the differential volume of the latter is now a small finite volume. The analysis of the heat transfers in and out of the finite control volume follows the same path as that used in the previous essay in the derivation of the governing differential equation for heat conduction. However, there are several details which are different and which will be explained by means of Fig. 4.2.
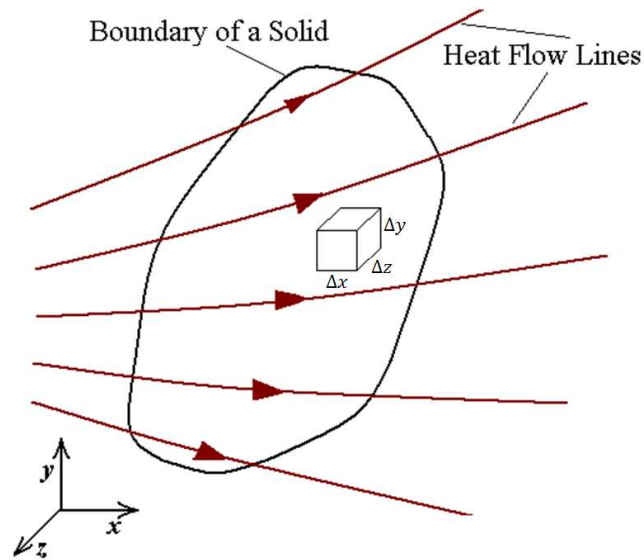

Fig. 4. 1 A solid of arbitrary shape across which heat is passing and a control volume of finite size used to obtain a discretized solution space

In Fig. 4.2, a cluster of seven adjacent, finite-size control volumes is displayed. Every one of these control volumes has dimensions Δx, Δy, and Δz. It is common to identify each control volume by a trio of numbers. The shaded control volume exhibited in the figure is at the center of the cluster. It is identified by the trio of numbers I, J, K. The control volume to the right of the center volume is designated as $I + 1, J, K$. In contrast, the volume to the left of the center element is identified by $I - 1, J, K$. In the same way, the volumes that are respectively situated above and below the center volume are denoted by $I, J + 1, K$ and $I, J - 1, K$. For the volumes that are in front and in back of the center volume, the designations are $I, J, K + 1$ and $I, J, K - 1$.
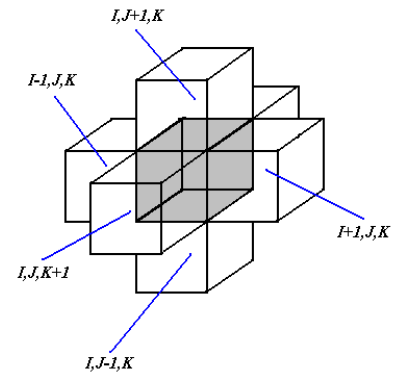

Fig. 4. 2 A cluster of finite control volumes

4.2

To continue, it is convenient for the sake of geometric clarity to look at a two-dimensional version of Fig. 4.2 as shown in Fig. 4.3. In the figure, there is a cluster of five control volumes, each having dimensions Δx by Δy. For the discretized version of the solution space, the temperatures will be known, after a solution has been obtained, at a *finite* number of points. Specifically, these points are at the centers of the respective control volumes. For the two-dimensional array, the known temperatures will be at locations (I,J), $(I-1,J)$, $(I+1,J)$, $(I,J-1)$, and $(I,J+1)$. These locations are called *nodes*.
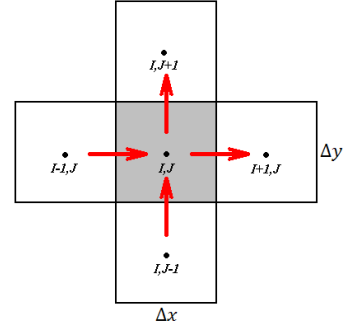

Fig. 4. 3 Two-dimensional cluster of finite-size control volumes

The next step in the derivation of the discretized form of energy conservation is to specify whether the problem in question is in the steady state or in the unsteady state. Here, the steady state situation will be dealt with first. From Eq. (3.10), which is repeated here as:

$$\left(\dot{Q}\right)_{inflow} = \left(\dot{Q}\right)_{outflow} \tag{4.1}$$

The directions of heat flow are taken to be positive in the positive coordinate directions, as illustrated by the red vectors in Fig. 4.3. If attention is focused on the shaded control volume in Fig. 4.3, Eq. (4.1) can be made specific as:

$$\left(\dot{Q}\right)_{inflow} = \dot{Q}_{(I-1,J)\to(I,J)} + \dot{Q}_{(I,J-1)\to(I,J)} \tag{4.2}$$

$$\left(\dot{Q}\right)_{outflow} = \dot{Q}_{(I,J)\to(I+1,J)} + \dot{Q}_{(I,J)\to(I,J+1)} \tag{4.3}$$

To evaluate these equations in terms of the temperatures at the center points of the respective volumes, it is necessary to make an assumption about the manner in which the temperature varies between these points. The usual approach in to use a linear variation. For example, consider the temperature variation between the points $(I-1,J)$ and (I,J). If references made to Fig. 4.4, a linear temperature variation between these points is given by:

$$T(x) = T_{(I-1,J)} + \left(T_{(I,J)} - T_{(I-1,J)}\right)\frac{x}{\Delta x} \qquad 0 \le x \le \Delta x \tag{4.4}$$
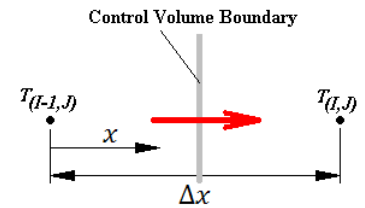

Fig. 4. 4 Diagram for the evaluation of the heat flux entering the left-hand face of the control volume

The rate of heat flow crossing the control volume boundary and shown by the red vector in Fig. 4.4 is obtained by applying Fourier's law, Eq. (3.6), with the result:

$$\dot{Q}_{(I-1,J)\to(I,J)} = -k\Delta y\Delta z \left(\frac{\partial T}{\partial x}\right)_{boundary} = -k\Delta y\Delta z \frac{T_{(I,J)}-T_{(I-1,J)}}{\Delta x} \tag{4.5}$$

The other terms in Eqs. (4.2) and (4.3) are evaluated in a similar manner. When all of the rates of heat transfer are collected according to these equations, there is obtained:

$$\left[-k\Delta y\Delta z\frac{T_{(I,J)}-T_{(I-1,J)}}{\Delta x}\right] + \left[-k\Delta x\Delta z\frac{T_{(I,J)}-T_{(I,J-1)}}{\Delta y}\right] =$$
$$\left[-k\Delta y\Delta z\frac{T_{(I+1,J)}-T_{(I,J)}}{\Delta x}\right] + \left[-k\Delta x\Delta z\frac{T_{(I,J+1)}-T_{(I,J)}}{\Delta y}\right] \tag{4.6}$$

When this equation is solved for $T_{(I,J)}$, the discretized form of energy conservation for the selected control volume (I,J) is:

$$T_{(I,J)} = \frac{\Delta y}{\Delta x}\left[T_{(I-1,J)} + T_{(I+1,J)}\right] + \frac{\Delta x}{\Delta y}\left[T_{(I,J-1)} + T_{(I,J+1)}\right] \tag{4.7}$$

An algebraic equation, such as Eq. (4.7), can be written at all nodes that are contained within the solution space. If there are M such nodes, there would be M algebraic equations for the M unknown temperatures. Since each equation is linked to the neighboring equations, simultaneous solution is necessary.

Equations such as Eq. (4.7) are written at all nodes that are interior to the solution space. Special consideration is needed at nodes that are on the boundaries of the space. The software that will be used for the solution of heat conduction problems permits five types of conditions to be specified at boundary nodes. They are:

    (a) Specified temperature
    (b) Specified heat flux
    (c) Convective interchange between a surface node and a surrounding medium
    (d) Radiative interchange between a surface node and its radiative environment (available in ANSYS 12)
    (e) No heat transfer (adiabatic)

Among these, condition (e) is the default condition; that is, if no specification is made of a boundary condition, the software will apply condition (e).

The classical thermal boundary conditions are conveyed in items (a)-(c). Although the specified temperature case is the most popular for academic problems, there is some difficulty in controlling the surface temperature in practice. The specified heat flux condition can be realized by making use of pre-packaged heating elements which may be adhered to a surface. The most realistic of the listed boundary conditions is item (c). In order to implement this boundary condition, it is necessary to provide information for a quantity called the *heat transfer coefficient*. To illuminate the function of the heat transfer coefficient, it is convenient to start with the thermal Ohm's law to which reference has already been made earlier. That law, which is analogous to the Ohm's law for the electricity, is:

$$Q = \frac{\Delta T}{R} \tag{4.8}$$

It can be reasoned that the thermal resistance diminishes when the surface area through which heat is passing is increased. Therefore,

$$R \sim \frac{1}{A} \qquad (4.9)$$

In addition, at the surface of a solid which is washed by a moving fluid, the capability of the fluid to transfer heat to or from the surface plays a major role in the magnitude of the resistance. This capability is expressed by a quantity called the convective heat transfer coefficient h. The larger the value of h, the smaller is the thermal resistance ($R \sim 1/h$). In light of the foregoing reasoning, it follows that:

$$R \sim \frac{1}{hA} \qquad (4.10)$$

The proportionality conveyed by Eq. (4.10) can be taken as an equality by adjusting the value of h. Therefore, the characterization of heat exchange by convection between a moving fluid and a surface is almost universally represented by:

$$Q = hA(T_{surf} - T_{fluid}) \qquad or \qquad Q = hA(T_{fluid} - T_{surf}) \qquad (4.11)$$

The first of Eqs. (4.11) pertains to heat transfer from a surface to a fluid, and the second is applicable when heat flows from a fluid to a surface.

To implement item (c), it is necessary to convey numerical values of both h and T_{fluid} to the software.