Q.1 **Question 1**

a) Describe the physical interpretation of the Grashof number for natural convection. Describe each of its terms and write down an equation for the temperature at which temperature-dependent terms in Gr should be evaluated. [5 marks]

Solution:

[1] *The Grashof number is the analogue of the Reynolds number for natural convection*
[1] *and is the ratio of bouyancy and viscous forces in the fluid. It is defined as*

$$Gr = \frac{g\,\rho^2\,\beta\,(T_w - T_\infty)\,L^3}{\mu^2},$$

where g is the gravitational acceleration,
ρ is the density of the fluid,
[1] *β is the thermal expansion coefficient of the fluid,*
T_w is the wall temperature,
T_∞ is the fluid temperature a large distance from the wall (bulk),
[1] *L is a characterstic (and often vertical) length scale,*
and μ is the fluid viscosity.

The properties of the flow for the Grashof number should be evaluated at the so-called film temperature,

$$T_f = (T_w + T_\infty)/2$$

[1]

b) An electric heater of 0.032 m diameter and 0.85 m in length is used to heat a room. Calculate the electrical input (i.e. the sum of heat transferred by convection and radiation) to the heater when the bulk of the air in the room is at 24°C, the walls are at 12°C, and the surface of the heater is at 532°C. For convective heat transfer from the heater, assume the heater is a horizontal cylinder and the Nusselt number is given by

$$\text{Nu} = 0.38(\text{Gr})^{0.25}$$

where all properties are evaluated at the film temperature. You may assume air is an ideal gas, giving $\beta = T^{-1}$. Take the emissivity of the heater surface as $\epsilon = 0.62$ and assume that the surroundings are black. All other properties should be calculated using the steam tables provided. [10 marks]

Solution:

Calculating the film temperature, we have

$$T_f = \frac{532 + 24}{2} = 278^\circ C = 551\ K$$

[1]

[2] *From the tables at 551 K, $\nu = 4.48 \times 10^{-5}$ $m^2\,s^{-1}$ and $k = 0.04375$ $W m^{-1}\,K^{-1}$. The*

[1] *expansion coefficient is* $\beta = 551^{-1}$ *K*$^{-1}$*. Combining these we have:*

$$\textit{Gr} = \frac{9.81\,(532-24)\;0.032^3}{551\,(4.48\times10^{-5})^2} \approx 147\,700$$

[1] *Calculating the Nusselt number, we have*

$$\textit{Nu} = 0.38\,(147\,700)^{1/4} \approx 7.45$$

[1] *Calculating the convective coefficient we have*

$$h = \frac{k\,Nu}{L} = \frac{0.04375\times7.45}{0.032} \approx 10.19\,\textit{W}\,\textit{m}^{-2}\,\textit{K}^{-1}$$

[1] *Heat transfer via convection:*

$$Q_{conv} = h\,A\,\Delta T = 10.19\times\pi\times0.032\times0.85\,(532-24) \approx 442\,\textit{W}$$

[1] *Heat transfer by radiation*

$$\begin{aligned} Q_{rad.} &= \sigma\,\epsilon\,A\,\left(T_w^4 - T_\infty^4\right) \\ &= 5.67\times10^{-8}\times0.62\times\pi\times0.032\times0.85\left(805^4-285^4\right) \\ &\approx 1242 \end{aligned}$$

[1] *Total energy input is*

$$Q_{total} = Q_{rad.} + Q_{conv} = 1242 + 442 = 1684\ \textit{W}$$

[1]

c) Using index notation, prove the following vector calculus identity:

$$\nabla^2 f\,g = f\,\nabla^2 g + 2(\nabla f)\cdot(\nabla g) + g\,\nabla^2 f$$

[5 marks]

Note: You must treat f and g as functions of x, y, z.
Solution:
Converting to index notation in Cartesian coordinates (x,y,z),

$$\nabla^2 f\,g = \frac{\partial}{\partial r_i}\left(\frac{\partial}{\partial r_i}\,f\,g\right)$$

[1] *We can't use* $\partial^2/\partial r_i^2$ *as there is no repeated* i *index. Using the product rule on the term in parenthesis*

$$\frac{\partial}{\partial r_i}\,f\,g = f\frac{\partial g}{\partial r_i} + g\frac{\partial f}{\partial r_i}$$

[1] *Using the product rule again to apply the second derivative to both of these terms* *gives*

$$\frac{\partial}{\partial r_i}\frac{\partial}{\partial r_i}\,f\,g = \frac{\partial f}{\partial r_i}\frac{\partial g}{\partial r_i} + f\frac{\partial}{\partial r_i}\frac{\partial g}{\partial r_i} + \frac{\partial g}{\partial r_i}\frac{\partial f}{\partial r_i} + g\frac{\partial}{\partial r_i}\frac{\partial f}{\partial r_i}$$
$$= f\frac{\partial}{\partial r_i}\frac{\partial g}{\partial r_i} + 2\frac{\partial f}{\partial r_i}\frac{\partial g}{\partial r_i} + g\frac{\partial}{\partial r_i}\frac{\partial f}{\partial r_i}$$

[2] *Converting back to vector notation, yields the identity,*

$$\frac{\partial}{\partial r_i}\frac{\partial}{\partial r_i}\,f\,g = f\,\nabla^2\,g + 2(\nabla\,f)\cdot(\nabla\,g) + g\,\nabla^2\,f$$

[1]

Total Question Marks:20

Q.2 **Question 2**

A wire-coating die consists of a cylindrical wire of radius, $\kappa\,R$, moving horizontally at a constant velocity, v_{wire}, along the axis of a cylindrical die of radius, R. You may assume the pressure is constant within the die (it is not pressure driven flow) but the flow is driven by the motion of the wire (it is "axial annular Couette flow"). Neglect end effects and assume an isothermal system.

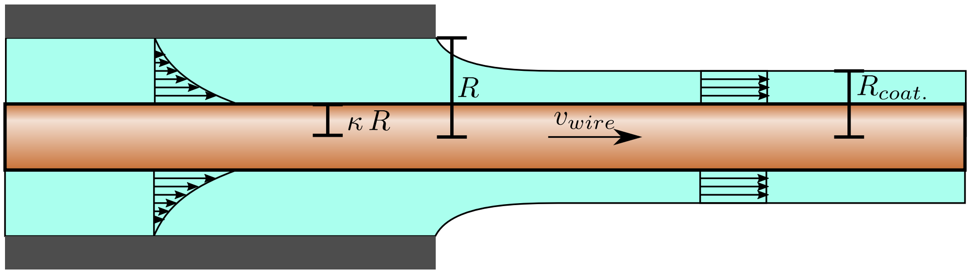


Figure 1: *Diagram of a wire coating die.*

a) State the two relevant boundary conditions for the flow within the die and how they arise. [2 marks]

Solution:

[1] *Both conditions arise from non-slip conditions of the fluid with a solid boundary.*

- $v_z(r = R) = 0$: *At the die wall interface.*
- $v_z(r = \kappa\,R) = v_{wire}$: *At the wire interface.*

[1]

b) The stress profile for an annular system is of the following form

$$\frac{1}{r}\frac{\partial}{\partial r} r\,\tau_{rz} = -\frac{\partial p}{\partial z} + \rho\,g_z.$$

Derive the following expression for the flow profile

$$v_z = \frac{v_{wire}}{\ln\kappa}\ln\left(\frac{r}{R}\right).$$

[9 marks]

Solution:

There is no driving pressure gradient, and as the flow is horizontal, the two terms on the right hand side are zero

$$\frac{1}{r}\frac{\partial}{\partial r} r\,\tau_{rz} = -\overbrace{\cancel{\frac{\partial p}{\partial z}}}^{0} + \rho\,\overbrace{\cancel{g_z}}^{0}.$$

[2]

Performing the integration of the stress profile expression from the previous question,

$$\tau_{rz} = \frac{C_1}{r}.$$

[1]

Assuming the fluid is Newtonian, we have

$$-\mu\frac{\partial v_z}{\partial r} = \frac{C_1}{r}.$$

[1]

Performing the integration

$$v_z = -\mu^{-1}\,C_1 \ln\, r + C_2.$$

[1]

Inserting the two boundary conditions yields the following

$$\begin{aligned} 0 &= -\mu^{-1}\,C_1 \ln R + C_2. \\ v_{wire} &= -\mu^{-1}\,C_1 \ln \kappa\, R + C_2. \end{aligned}$$

[1]

Solving both equations for the constants,

$$\begin{aligned} C_2 &= \mu^{-1}\,C_1 \ln R \\ v_{wire} &= \mu^{-1}\,C_1(\ln R - \ln \kappa\, R) \\ C_1 &= -\frac{\mu\, v_{wire}}{\ln \kappa}. \end{aligned}$$

[2]

Inserting these back in gives the final expression

$$v_z = \frac{v_{wire}}{\ln \kappa} \ln\left(\frac{r}{R}\right)$$

[1]

c) Derive the following expression for the volumetric flow-rate of liquid through the die

$$\dot{V}_z = -\pi\, R^2\, v_{wire}\left(\kappa^2 + \frac{1-\kappa^2}{2\ln \kappa}\right).$$

[5 marks]

Note: You will need the integration identity

$$\int x\, \ln(x)\, \mathrm{d}x = \frac{x^2}{2}\left(\ln(x) - \frac{1}{2}\right).$$

Solution:
To determine the volumetric flow rate, the following integration is performed

$$\dot{V}_z = 2\,\pi \int_{\kappa\,R}^{R} r\,v_z\,\mathrm{d}r$$

[1]

Performing the integration

$$\begin{aligned}\dot{V}_z &= 2\,\pi\,R\frac{v_{wire}}{\ln\kappa}\int_{\kappa\,R}^{R}\frac{r}{R}\ln\left(\frac{r}{R}\right)\mathrm{d}r\\ &= \frac{2\,\pi\,R^2\,v_{wire}}{\ln\kappa}\int_{\kappa}^{1} x\,\ln\left(x\right)\mathrm{d}x\\ &= \frac{2\,\pi\,R^2\,v_{wire}}{\ln\kappa}\left[\frac{x^2}{2}\left(\ln x-\frac{1}{2}\right)\right]_{\kappa}^{1}\\ &= -\frac{2\,\pi\,R^2\,v_{wire}}{\ln\kappa}\left(\frac{\kappa^2}{2}\left(\ln\kappa-\frac{1}{2}\right)+\frac{1}{4}\right)\\ &= -\pi\,R^2\,v_{wire}\left(\kappa^2+\frac{1-\kappa^2}{2\ln\kappa}\right)\end{aligned}$$

[4]

d) Derive an expression for the outer radius of the coating, $R_{coat.}$, far away from the die exit. [4 marks]

Solution:
Solving the stress balance again but for the film coating the wire, the following expression is found again for the stress

$$\tau_{rz} = \frac{C_1}{r}$$

At the exposed surface of the film ($r \neq 0$), the stress is zero (assuming the air exerts close to zero drag). This implies that $C_1 = 0$ as well, as it is the only possible way to set the RHS to zero at finite values of r. As the stress is zero, Newton's law of viscosity then implies the film has a constant velocity which will be the velocity of the wire (note, the diagram gives the student a strong hint that this is true).

[2]

The volumetric flowrate of the wire coating is related to the outer radius of the coating, $R_{coat.}$

$$\dot{V}_{z,coating} = v_{wire}\,\pi\left(R_{coat.}^2-\kappa^2\,R^2\right)$$

[1]

This must be equal to the volumetric flowrate of coating through the die

$$v_{wire}\,\pi\left(R_{coating}^2-\kappa^2\,R^2\right) = -\pi\,R^2\,v_{wire}\left(\kappa^2+\frac{1-\kappa^2}{2\ln\kappa}\right)$$

$$R_{coating} = R\sqrt{\frac{\kappa^2-1}{2\ln\kappa}}$$

[1]

Total Question Marks:20

Q.3 **Question 3**

To explore the effect of using a temperature-dependent thermal conductivity, consider heat flowing through an annular (pipe) wall of inside radius R_0 and an outside radius R_1. It is assumed that thermal conductivity varies linearly with temperature from $k_0(T = T_0)$ to $k_1(T = T_1)$ where T_0 and T_1 are the inner and outer wall temperatures respectively.

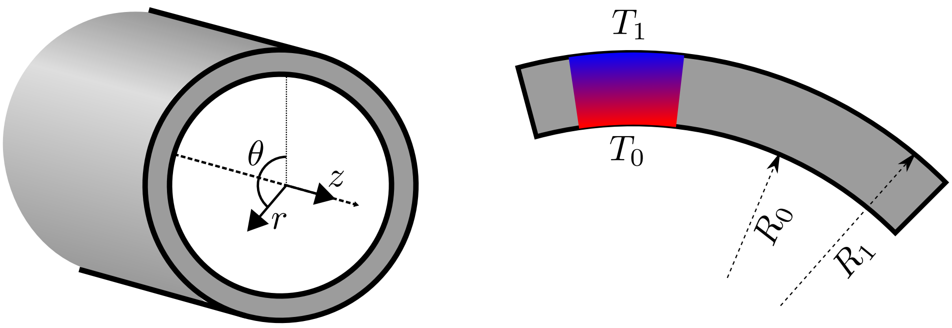


Figure 2: *Conduction through an annular(pipe) wall.*

a) Derive the following energy balance equation

$$\frac{\partial}{\partial r} r\, q_r = 0,$$

and state ALL assumptions required. [7 marks]

Solution:

[1] *Assuming that the pressure dependency of the internal energy of the solid is small, Equation 4 can be used valid.*

As this is heat transfer in solids, we can set the frame of reference to the wall and $\boldsymbol{v} = \boldsymbol{0}$. This greatly simplifies the energy balance equation:

$$\rho\, C_p \frac{\partial T}{\partial t} = -\cancelto{0}{\rho\, C_p\, v_j \nabla_j T} - \nabla_i\, q_i - \cancelto{0}{\tau_{ji} \nabla_j v_i} - \cancelto{0}{p \nabla_i v_i} + \sigma_{energy}$$
$$\rho\, C_p \frac{\partial T}{\partial t} = -\nabla_i\, q_i + \sigma_{energy}$$

[1]

Assuming the wall does not generate heat:

$$\rho\, C_p \frac{\partial T}{\partial t} = -\nabla_i\, q_i + \cancelto{0}{\sigma_{energy}}$$

[1]

And steady state:

$$\cancelto{0}{\rho\, C_p \frac{\partial T}{\partial t}} = -\nabla_i\, q_i$$
$$\nabla_i\, q_i = 0$$

[1]

Finally, inserting the cylindrical coordinate system definition of $\nabla_i\, q_i$:

$$\nabla_i\, q_i = \frac{1}{r}\frac{\partial}{\partial r}\left(r\, q_r\right) + \frac{1}{r}\frac{\partial\, q_\theta}{\partial \theta} + \frac{\partial\, q_z}{\partial z}$$

[2] *Assuming a symmetry of the system ALONG and AROUND the axis, the only remaining derivative is in the r-direction:*

$$\begin{aligned}\nabla_i\, q_i &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\, q_r\right)\\ &= \frac{\partial}{\partial r}\left(r\, q_r\right) = 0\end{aligned}$$

[1] *As required.*

b) Derive the following expression for the temperature profile

$$Q_r = \frac{2\pi\, L}{\ln\left(\frac{R_0}{R_1}\right)}\frac{k_1+k_0}{2}(T_1 - T_0),$$

where L is the length of the pipe/annulus. [10 marks]
Note: You will need the following identity:

$$T_1^2 - T_0^2 = (T_1 + T_0)(T_1 - T_0).$$

Solution:
Performing the integration, we have

$$\begin{aligned}r\, q_r &= C_1\\ q_r &= \frac{C_1}{r}\end{aligned}$$

[1] *Inserting in Fourier's law, we have*

$$-k\frac{\partial T}{\partial r} = \frac{C_1}{r}$$

We need to insert the temperature dependent thermal conductivity, which is given by the following linear relationship

$$k = k_0 + (T - T_0)\,\frac{k_1 - k_0}{T_1 - T_0}$$

[1] *Inserting this,*

$$-\left(k_0 + (T - T_0)\,\frac{k_1 - k_0}{T_1 - T_0}\right)\frac{\partial T}{\partial r} = \frac{C_1}{r}$$

[1] *Integrating between the two limits,*

$$-\int_{R_0}^{R_1}\left(k_0+(T-T_0)\,\frac{k_1-k_0}{T_1-T_0}\right)\frac{\partial T}{\partial r}\mathrm{d}r=\int_{R_0}^{R_1}\frac{C_1}{r}\mathrm{d}r$$
$$-\int_{T_0}^{T_1}\left(k_0+(T-T_0)\,\frac{k_1-k_0}{T_1-T_0}\right)\mathrm{d}T=C_1\ln\left(\frac{R_1}{R_0}\right)$$
$$-\left(k_0\,(T_1-T_0)+\left(\frac{T_1^2-T_0^2}{2}-(T_1-T_0)\,T_0\right)\frac{k_1-k_0}{T_1-T_0}\right)=C_1\ln\left(\frac{R_1}{R_0}\right)$$

[2] *Using the identity* $T_1^2-T_0^2=(T_1+T_0)(T_1-T_0)$*,*

$$-\left(k_0\,(T_1-T_0)+\frac{T_1+T_0}{2}(k_1-k_0)-T_0(k_1-k_0)\right)=C_1\ln\left(\frac{R_1}{R_0}\right)$$
$$-\left(k_0\,T_1+\frac{T_1+T_0}{2}(k_1-k_0)-T_0\,k_1\right)=C_1\ln\left(\frac{R_1}{R_0}\right)$$

[2] *Simple cancellation and factorisation leads to the following*

$$\frac{k_1+k_0}{2\ln\left(\frac{R_0}{R_1}\right)}(T_1-T_0)=C_1$$

[1] *Inserting this back into the expression for the flux, we have*

$$q_r=\frac{C_1}{r}$$
$$=\frac{k_1+k_0}{2\,r\ln\left(\frac{R_0}{R_1}\right)}(T_1-T_0)$$

[1] *The total flux is given by multiplying by the cylindrical area,* $2\,\pi\,r\,L$*,*

$$Q_r=\frac{2\pi\,L}{\ln\left(\frac{R_0}{R_1}\right)}\frac{k_1+k_0}{2}(T_1-T_0)$$

[1]

c) Compare this expression to the standard expression for conduction in pipe walls (with constant thermal conductivity), what can you observe? [3 marks]

Solution:

The expression for pipes is availabe from the tables in the datasheet, and is as follows

$$Q=\frac{2\pi\,L\,k}{\ln\left(\frac{R_1}{R_0}\right)}\Delta T.$$

[1]

On comparision with the derived equation, the only change is to replace the constant thermal conductivity with the average of the thermal conductivity on the inner and outer surfaces. **[1]**

For small temperature differences (where a linear temperature dependence may be assumed) using the average thermal conductivity is a useful strategy. **[1]**

Total Question Marks:20

Q.4 **Question 4**

To maintain a pressure close to 1 atm, an industrial pipeline containing ammonia gas is vented to ambient air. Venting is achieved by tapping the pipe and inserting a 3 mm diameter tube, which extends for 20 m into the atmosphere. With the entire system operating at 25°C and 1 bar, the ideal gas equation of state predicts a total molar concentration of 40.9 mol/m^3. Equimolar counter-diffusion can be assumed, and both the concentration of air in the pipeline and the concentration of ammonia in the atmosphere can be considered negligible. The diffusion coefficient of ammonia through air is approximately 2×10^{-5} m^2/s.

a) Determine the mass rate of ammonia lost in to the atmosphere $\mathbf{N}_A$ in kg/h and the mass rate of contamination of the pipe with air $\mathbf{N}_B$ in the same units. [12 marks]

Solution:

[1] *There is no generation of mass in the flow, and the system is at steady state, thus*

$$\cancelto{0}{\frac{\partial C_A}{\partial t}} = -\nabla \cdot \boldsymbol{N}_A + \cancelto{0}{\sigma_A}$$
$$\nabla \cdot \boldsymbol{N}_A = 0$$

[1] *Using rectangular coordinates, and treating this as a one dimensional flow we find that the fluxes of the ammonia and air are constant.*

$$\frac{\partial N_{A,x}}{\partial x} = 0$$

Thus,

$$N_{A,z} = N_{A,0} \qquad\qquad N_{B,z} = N_{B,0}$$

The boundary conditions are

$$C_A(z = 0\ m) = 40.9\ \mathrm{mol/m^3} \qquad C_A(z = 20\ m) = 0\ \mathrm{mol/m^3}$$
$$C_B(z = 0\ m) = 0\ \mathrm{mol/m^3} \qquad C_B(z = 20\ m) = 40.9\ \mathrm{mol/m^3}$$

[2] *If the system is an ideal gas, and there is no pressure driven flow (assumed by the*
[2] *pipeline being at 1 atm), this is equimolar counterdiffusion, thus $N_{B,0} = -N_{A,0}$.*

For equimolar counterdiffusion we can directly use Fick's law for the fluxes,

$$N_{A,z} = N_{A,0} = -D_{AB}\frac{\partial C_A}{\partial z}$$

Integrating this equation, we find:

$$C_A = C - \frac{N_{A,0}}{D_{AB}} z$$

[1]

From the first boundary condition in the ammonia ($C_A(z = 0\ m) = 40.9\ \mathrm{mol/m^3}$), we find

$$C = 40.9\ \mathrm{mol/m^3})$$

[1] *From the second boundary condition we find*

$$\begin{aligned} 0 &= 40.9 - 20\frac{N_{A,0}}{D_{AB}} \\ N_{A,0} &= \frac{40.9\, D_{AB}}{20} \\ &= \frac{40.9 \times 2 \times 10^{-5}}{20} \approx 4.09 \times 10^{-5}\ \mathrm{mol/m^2 s} \end{aligned}$$

[1]

If we multiply the flux of ammonia by the cross-sectional area of the tube $\pi\, D^2/4$ and its molecular weight (17 g/mol), we will find the mass rate of ammonia lost to the atmosphere:

$$\begin{aligned} \textit{ammonia lost to atmosphere} &= N_{A,0}\frac{\pi}{4}D^2\, M_A \\ &= \left(4.09 \times 10^{-5}\ \frac{\mathrm{mol}}{\mathrm{m^2\ s}}\right)\frac{\pi}{4}(0.003\ \mathrm{m})^2(17\ \mathrm{g/mol}) \\ &\approx 4.91 \times 10^{-9}\ \mathrm{g/s} \\ &\approx 1.77 \times 10^{-8}\ \mathrm{kg/h} \end{aligned}$$

[1] *To determine the mass rate of contamination of the pipe with air, we first note the molar flux of air into the pipe is equal and opposite to the molar flux of ammonia into the atmosphere ($N_{A,0} = -N_{B,0}$ due to the assumption of equimolar counter-diffusion). Multiplying this molar flux by the cross-sectional area of the tube and the molecular weight of air (29 g/mol), we find that the mass flowrate of air into the pipeline is*

$$\begin{aligned} \textit{air entering pipeline} &= -N_{A,0}\frac{\pi}{4}D^2\, M_B \\ &= -\left(4.09 \times 10^{-5}\ \frac{\mathrm{mol}}{\mathrm{m^2\ s}}\right)\frac{\pi}{4}(0.003\ \mathrm{m})^2(29\ \mathrm{g/mol}) \\ &\approx -8.38 \times 10^{-9}\ \mathrm{g/s} \\ &\approx -3.02 \times 10^{-9}\ \mathrm{kg/hr} \end{aligned}$$

[2]

b) A new high-tech membrane, which is impermeable to air, is installed at the bottom of the pipe to prevent air polluting the pipeline. The *air* within the tube is now **stationary** and the mole fraction of ammonia at the surface of the membrane is $x_A(z = 0) = 0.9$. Resolve the problem again to determine the flux of ammonia.
Note: Stefan's law (in mole fractions for ideal gases) is given by the following

$$N_{A,z} = -D_{AB}\frac{C_T}{1 - x_A}\frac{\partial x_A}{\partial z}$$

[8 marks]

Solution:
This problem is similar to diffusion in an Arnold cell. For equimolar counter-diffusion, we have Stefan's law

$$N_{A,z} = -D_{AB}\frac{C_T}{1-x_A}\frac{\partial x_A}{\partial z}$$

The flux of ammonia is still constant along the pipe (the balance equation hasn't changed, only the expression for the flux). So we can try integrating Stefan's law

$$\begin{aligned} N_{A,z} = N_{A,0} &= -D_{AB}\frac{C_T}{1-x_A}\frac{\partial x_A}{\partial z} \\ N_{A,0}\int \mathrm{d}z &= -D_{AB}\,C_T\int\frac{1}{1-x_A}\mathrm{d}x_A \\ N_{A,0}\,z &= D_{AB}\,C_T\ln(1-x_A) + C \end{aligned}$$

[2]

The boundary condition at the bottom of the pipe, in terms of the mole fraction, is $x_A(z=0) = 0.9$ *which gives*

$$\begin{aligned} 0 &= D_{AB}\,C_T\ln(0.1) + C \\ C &= -D_{AB}\,C_T\ln(0.1) \\ &= -2\times10^{-5}\times 40.9\times\ln(0.1) \approx 1.88\times10^{-3} \end{aligned}$$

[1]

The other boundary condition is that the concentration of ammonia is zero at the exit of the tube $x_A(z = 2\text{ m}) = 0$.

$$\begin{aligned} 2\,N_{A,0} &= D_{AB}\,C_T\ln(1) + 1.88\times10^{-3} \\ N_{A,0} &= \frac{1.88\times10^{-3}}{2} = 9.44\times10^{-4}\ \text{mol/m}^2\ \text{s} \end{aligned}$$

[1] *The total mass flowrate of ammonia is*

$$\begin{aligned} \text{ammonia lost to atmosphere} &= N_{A,0}\frac{\pi}{4}D^2\,M_A \\ &= \left(9.44\times10^{-4}\ \frac{\mathrm{mol}}{\mathrm{m}^2\ \mathrm{s}}\right)\frac{\pi}{4}(0.003\ \mathrm{m})^2(17\ \mathrm{g/mol}) \\ &\approx 1.13\times10^{-7}\ \mathrm{g/s} \\ &\approx 4.08\times10^{-7}\ \mathrm{kg/hr} \end{aligned}$$

[2] *The flow rate of ammonia has increased from* $1.77\times10^{-8}\ \mathrm{kg/h}$ *(this is a feature of*
[2] *diffusion through a stationary layer), but it is still small.*

Total Question Marks:20

END OF PAPER

Sum of all question's marks:80

DATASHEET

General balance equations:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \rho\,\boldsymbol{v} \qquad \text{(Mass/Continuity)} \quad (1)$$

$$\frac{\partial C_A}{\partial t} = -\nabla \cdot \boldsymbol{N}_A + \sigma_A \qquad \text{(Species)} \quad (2)$$

$$\rho\frac{\partial \boldsymbol{v}}{\partial t} = -\rho\,\boldsymbol{v} \cdot \nabla \boldsymbol{v} - \nabla \cdot \boldsymbol{\tau} - \nabla\,p + \rho\,\boldsymbol{g} \qquad \text{(Momentum)} \quad (3)$$

$$\rho\,C_p\frac{\partial T}{\partial t} = -\rho\,C_p\,\boldsymbol{v} \cdot \nabla\,T - \nabla \cdot \boldsymbol{q} - \boldsymbol{\tau} : \nabla\,\boldsymbol{v} - p\,\nabla \cdot \boldsymbol{v} + \sigma_{energy} \qquad \text{(Heat/Energy)} \quad (4)$$

In Cartesian coordinate systems, ∇ can be treated as a vector of derivatives. In curvelinear coordinate systems, the directions $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$ depend on the position. For convenience in these systems, look-up tables are provided for common terms involving ∇.

Cartesian coordinates (with index notation examples)
where s is a scalar, $\boldsymbol{v}$ is a vector, and $\boldsymbol{\tau}$ is a tensor.

$$\nabla s = \nabla_i s = \left[\frac{\partial\,s}{\partial x}, \frac{\partial\,s}{\partial y}, \frac{\partial\,s}{\partial z}\right]$$

$$\nabla^2 s = \nabla_i \nabla_i s = \frac{\partial^2\,s}{\partial x^2} + \frac{\partial^2\,s}{\partial y^2} + \frac{\partial^2\,s}{\partial z^2}$$

$$\nabla \cdot \boldsymbol{v} = \nabla_i v_i = \frac{\partial\,v_x}{\partial x} + \frac{\partial\,v_y}{\partial y} + \frac{\partial\,v_z}{\partial z}$$

$$\nabla \cdot \boldsymbol{\tau} = \nabla_i\,\tau_{ij}$$

$$[\nabla \cdot \boldsymbol{\tau}]_x = \frac{\partial\,\tau_{xx}}{\partial x} + \frac{\partial\,\tau_{yx}}{\partial y} + \frac{\partial\,\tau_{zx}}{\partial z}$$

$$[\nabla \cdot \boldsymbol{\tau}]_y = \frac{\partial\,\tau_{xy}}{\partial x} + \frac{\partial\,\tau_{yy}}{\partial y} + \frac{\partial\,\tau_{zy}}{\partial z}$$

$$[\nabla \cdot \boldsymbol{\tau}]_z = \frac{\partial\,\tau_{xz}}{\partial x} + \frac{\partial\,\tau_{yz}}{\partial y} + \frac{\partial\,\tau_{zz}}{\partial z}$$

$$\boldsymbol{v} \cdot \nabla \boldsymbol{v} = v_i\,\nabla_i\,v_j$$

$$[\boldsymbol{v} \cdot \nabla \boldsymbol{v}]_x = v_x\frac{\partial\,v_x}{\partial x} + v_y\frac{\partial\,v_x}{\partial y} + v_z\frac{\partial\,v_x}{\partial z}$$

$$[\boldsymbol{v} \cdot \nabla \boldsymbol{v}]_y = v_x\frac{\partial\,v_y}{\partial x} + v_y\frac{\partial\,v_y}{\partial y} + v_z\frac{\partial\,v_y}{\partial z}$$

$$[\boldsymbol{v} \cdot \nabla \boldsymbol{v}]_z = v_x\frac{\partial\,v_z}{\partial x} + v_y\frac{\partial\,v_z}{\partial y} + v_z\frac{\partial\,v_z}{\partial z}$$

Cylindrical coordinates
where s is a scalar, $\boldsymbol{v}$ is a vector, and $\boldsymbol{\tau}$ is a tensor. All expressions involving $\boldsymbol{\tau}$ are for symmetrical $\boldsymbol{\tau}$ only.

$$\begin{aligned}
\nabla s &= \left[\frac{\partial s}{\partial r},\, \frac{1}{r}\frac{\partial s}{\partial \theta},\, \frac{\partial s}{\partial z}\right] \\
\nabla^2 s &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial s}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 s}{\partial \theta^2} + \frac{\partial^2 s}{\partial z^2} \\
\nabla \cdot \boldsymbol{v} &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\, v_r\right) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} \\
\left[\nabla \cdot \boldsymbol{\tau}\right]_r &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\, \tau_{rr}\right) + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} - \frac{1}{r}\tau_{\theta\theta} + \frac{\partial \tau_{rz}}{\partial z} \\
\left[\nabla \cdot \boldsymbol{\tau}\right]_\theta &= \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{r\theta}}{\partial r} + \frac{2}{r}\tau_{r\theta} + \frac{\partial \tau_{\theta z}}{\partial z} \\
\left[\nabla \cdot \boldsymbol{\tau}\right]_z &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\, \tau_{rz}\right) + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z} \\
\left[\boldsymbol{v} \cdot \nabla \boldsymbol{v}\right]_r &= v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z} \\
\left[\boldsymbol{v} \cdot \nabla \boldsymbol{v}\right]_\theta &= v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r\, v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z} \\
\left[\boldsymbol{v} \cdot \nabla \boldsymbol{v}\right]_z &= v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}
\end{aligned}$$

Spherical coordinates
where s is a scalar, $\boldsymbol{v}$ is a vector, and $\boldsymbol{\tau}$ is a tensor. All expressions involving $\boldsymbol{\tau}$ are for symmetrical $\boldsymbol{\tau}$ only.

$$\begin{aligned}
\nabla s &= \left[\frac{\partial s}{\partial r},\, \frac{1}{r}\frac{\partial s}{\partial \theta},\, \frac{1}{r\,\sin\theta}\frac{\partial s}{\partial \phi}\right] \\
\nabla^2 s &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial s}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial s}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 s}{\partial \phi^2} \\
\nabla \cdot \boldsymbol{v} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\, v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(v_\theta \sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} \\
\left[\nabla \cdot \boldsymbol{\tau}\right]_r &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\, \tau_{rr}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\tau_{r\theta}\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial \tau_{r\phi}}{\partial \phi} - \frac{\tau_{\theta\theta} + \tau_{\phi\phi}}{r} \\
\left[\nabla \cdot \boldsymbol{\tau}\right]_\theta &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\, \tau_{r\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\tau_{\theta\theta}\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial \tau_{\theta\phi}}{\partial \phi} + \frac{\tau_{r\theta}}{r} - \frac{\cot\theta}{r}\tau_{\phi\phi} \\
\left[\nabla \cdot \boldsymbol{\tau}\right]_\phi &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\, \tau_{r\phi}\right) + \frac{1}{r}\frac{\partial \tau_{\theta\phi}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \tau_{\phi\phi}}{\partial \phi} + \frac{\tau_{r\theta}}{r} + \frac{2\cot\theta}{r}\tau_{\theta\phi} \\
\left[\boldsymbol{v} \cdot \nabla \boldsymbol{v}\right]_r &= v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r} \\
\left[\boldsymbol{v} \cdot \nabla \boldsymbol{v}\right]_\theta &= v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_r\, v_\theta - v_\phi^2\cot\theta}{r} \\
\left[\boldsymbol{v} \cdot \nabla \boldsymbol{v}\right]_\phi &= v_r\frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r\, v_\phi + v_\theta\, v_\phi\cot\theta}{r}
\end{aligned}$$

Rectangular		Cylindrical		Spherical	
q_x	$-k\frac{\partial T}{\partial x}$	q_r	$-k\frac{\partial T}{\partial r}$	q_r	$-k\frac{\partial T}{\partial r}$
q_y	$-k\frac{\partial T}{\partial y}$	q_θ	$-k\frac{1}{r}\frac{\partial T}{\partial \theta}$	q_θ	$-k\frac{1}{r}\frac{\partial T}{\partial \theta}$
q_z	$-k\frac{\partial T}{\partial z}$	q_z	$-k\frac{\partial T}{\partial z}$	q_ϕ	$-k\frac{1}{r\,\sin\theta}\frac{\partial T}{\partial \phi}$
τ_{xx}	$-2\,\mu\frac{\partial v_x}{\partial x} + \mu^B\,\nabla\cdot\boldsymbol{v}$	τ_{rr}	$-2\,\mu\frac{\partial v_r}{\partial r} + \mu^B\,\nabla\cdot\boldsymbol{v}$	τ_{rr}	$-2\,\mu\frac{\partial v_r}{\partial r} + \mu^B\,\nabla\cdot\boldsymbol{v}$
τ_{yy}	$-2\,\mu\frac{\partial v_y}{\partial y} + \mu^B\,\nabla\cdot\boldsymbol{v}$	$\tau_{\theta\theta}$	$-2\,\mu\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) + \mu^B\,\nabla\cdot\boldsymbol{v}$	$\tau_{\theta\theta}$	$-2\,\mu\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) + \mu^B\,\nabla\cdot\boldsymbol{v}$
τ_{zz}	$-2\,\mu\frac{\partial v_z}{\partial z} + \mu^B\,\nabla\cdot\boldsymbol{v}$	τ_{zz}	$-2\,\mu\frac{\partial v_z}{\partial z} + \mu^B\,\nabla\cdot\boldsymbol{v}$	$\tau_{\phi\phi}$	$-2\,\mu\left(\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r+v_\theta\cot\theta}{r}\right) + \mu^B\,\nabla\cdot\boldsymbol{v}$
τ_{xy}	$-\mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)$	$\tau_{r\theta}$	$-\mu\left(r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right)$	$\tau_{r\theta}$	$-\mu\left(r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right)$
τ_{yz}	$-\mu\left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right)$	$\tau_{\theta z}$	$-\mu\left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z}\right)$	$\tau_{\theta\phi}$	$-\mu\left(\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{v_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}\right)$
τ_{xz}	$-\mu\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right)$	τ_{zr}	$-\mu\left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right)$	$\tau_{\phi r}$	$-\mu\left(\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} + r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)\right)$

Table 1: *Fourier's law for the heat flux and Newton's law for the stress in several coordinate systems. Please remember that the stress is symmetric, so* $\tau_{ij} = \tau_{ji}$.

Viscous models:
Power-Law Fluid:

$$|\tau_{xy}| = k\left|\frac{\partial v_x}{\partial y}\right|^n \tag{5}$$

Bingham-Plastic Fluid:

$$\frac{\partial v_x}{\partial y} = \begin{cases} -\mu^{-1}\left(\tau_{xy} - \tau_0\right) & \text{if } \tau_{xy} > \tau_0 \\ 0 & \text{if } \tau_{xy} \leq \tau_0 \end{cases}$$

Dimensionless Numbers

$$\mathrm{Re} = \frac{\rho\,\langle v\rangle\,D}{\mu} \qquad \mathrm{Re}_H = \frac{\rho\,\langle v\rangle\,D_H}{\mu} \qquad \mathrm{Re}_{MR} = -\frac{16\,L\,\rho\,\langle v\rangle^2}{R\,\Delta p} \tag{6}$$

The hydraulic diameter is defined as $D_H = 4\,A/P_w$.

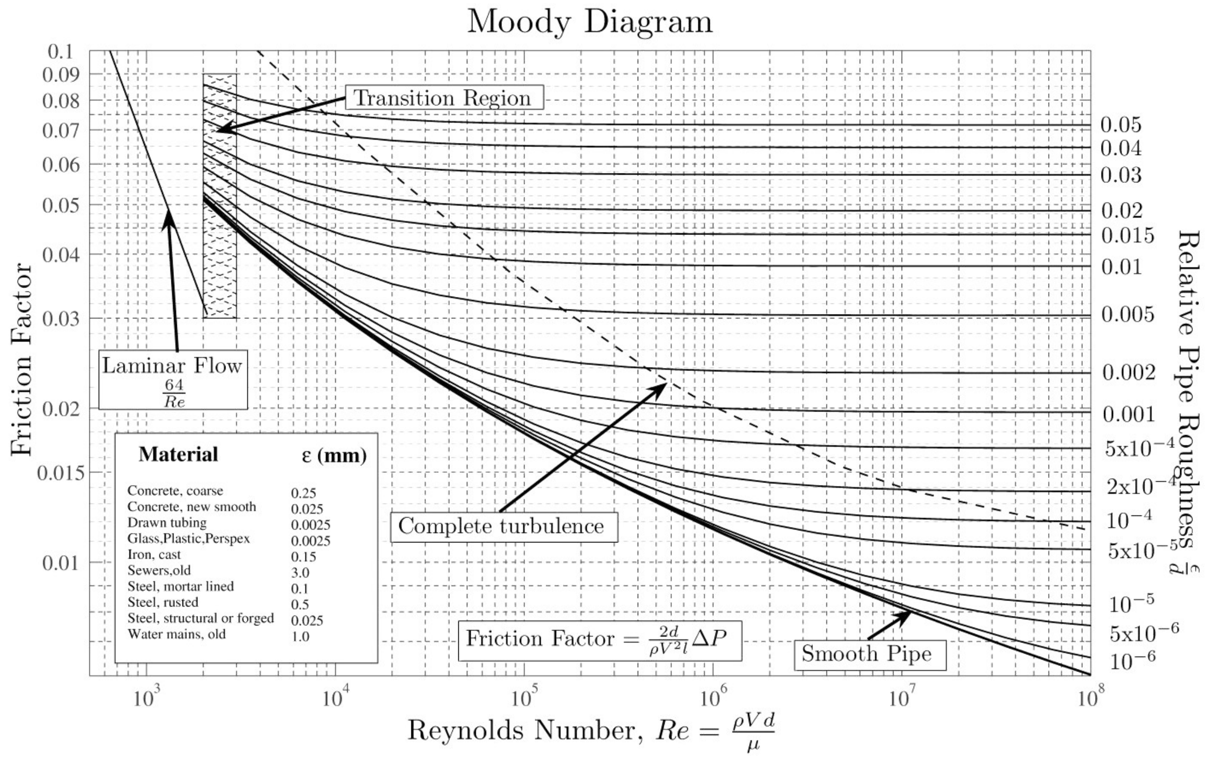
Single phase pressure drop calculations in pipes:
Darcy-Weisbach equation:

$$\frac{\Delta p}{L} = -\frac{C_f\,\rho\,\langle v\rangle^2}{R} \tag{7}$$

where $C_f = 16/Re$ for laminar Newtonian flow. For turbulent flow of Newtonian fluids in smooth pipes, we have the Blasius correlation:

$$C_f = 0.079\,\mathrm{Re}^{-1/4} \qquad \text{for } 2.5\times 10^3 < \mathrm{Re} < 10^5 \text{ and smooth pipes.}$$

Otherwise, you may refer to the Moody diagram.



Laminar Power-Law fluid:

$$\dot{V} = \frac{n\,\pi\,R^3}{3\,n+1}\left(\frac{R}{2\,k}\right)^{\frac{1}{n}}\left(-\frac{\Delta p}{L}\right)^{\frac{1}{n}}$$

Two-Phase Flow:
Lockhart-Martinelli parameter:

$$X^2 = \frac{\Delta p_{liq.-only}}{\Delta p_{gas-only}}$$

Pressure drop calculation:

$$\Delta p_{two-phase} = \Phi^2_{liq.}\,\Delta p_{liq.-only} = \Phi^2_{gas}\,\Delta p_{gas-only}$$

Chisholm's relation:

$$\Phi^2_{gas} = 1 + c\,X + X^2$$

$$\Phi^2_{liq.} = 1 + \frac{c}{X} + \frac{1}{X^2} \qquad c = \begin{cases} 20 & \text{turbulent liquid \& turbulent gas} \\ 12 & \text{laminar liquid \& turbulent gas} \\ 10 & \text{turbulent liquid \& laminar gas} \\ 5 & \text{laminar liquid \& laminar gas} \end{cases}$$

Farooqi and Richardson expression for liquid hold-up in co-current flows of Newtonian fluids and air in horizontal pipes:

$$h = \begin{cases} 0.186 + 0.0191\,X & 1 < X < 5 \\ 0.143\,X^{0.42} & 5 < X < 50 \\ 1/\left(0.97 + 19/X\right) & 50 < X < 500 \end{cases}$$

Heat Transfer:
Stefan-Boltzmann constant $\sigma = 5.6703 \times 10^{-8}$ W/m^2 K^4

Heat Transfer Dimensionless numbers:

$$\text{Nu} = \frac{h\,L}{k} \qquad \text{Pr} = \frac{\mu\,C_p}{k} \qquad \text{Gr} = \frac{g\,\beta\,(T_w - T_\infty)\;L^3}{\nu^2}$$

Resistances

$$Q = U_T\,A_T\,\Delta T = R_T^{-1}\,\Delta T \qquad Q_{rad.} = \sigma\,\varepsilon\,A\left(T_\infty^4 - T_w^4\right) = h_{rad.}\,A\,(T_\infty - T_w)$$

	Conduction Shell Resistances			Radiation
	Rect.	Cyl.	Sph.	
R	$\dfrac{X}{k\,A}$	$\dfrac{\ln\left(R_{outer}/R_{inner}\right)}{2\,\pi\,L\,k}$	$\dfrac{R_{inner}^{-1} - R_{outer}^{-1}}{4\,\pi\,k}$	$\left[A\,\varepsilon\,\sigma\,(T_\infty^2 + T_w^2)\,(T_\infty + T_w)\right]^{-1}$

Natural Convection

Ra $=$ Gr Pr	C	m
$< 10^4$	1.36	1/5
10^4–10^9	0.59	1/4
$> 10^9$	0.13	1/3

Table 2: *Natural convection coefficients for isothermal vertical plates in the empirical relation Nu $\approx C\,(\text{Gr Pr})^m$.*

For isothermal vertical cylinders, the above expressions for isothermal vertical plates may be used but must be scaled by a factor, F:

$$F = \begin{cases} 1 & \text{for } (D/H) < 35\,\text{Gr}_H^{-1/4} \\ 1.3\left[H\,D^{-1}\,\text{Gr}_D^{-1}\right]^{1/4} + 1 & \text{for } (D/H) \geq 35\,\text{Gr}_H^{-1/4} \end{cases}$$

where D is the diameter and H is the height of the cylinder. The subscript on Gr indicates which length is to be used as the critical length to calculate the Grashof number.

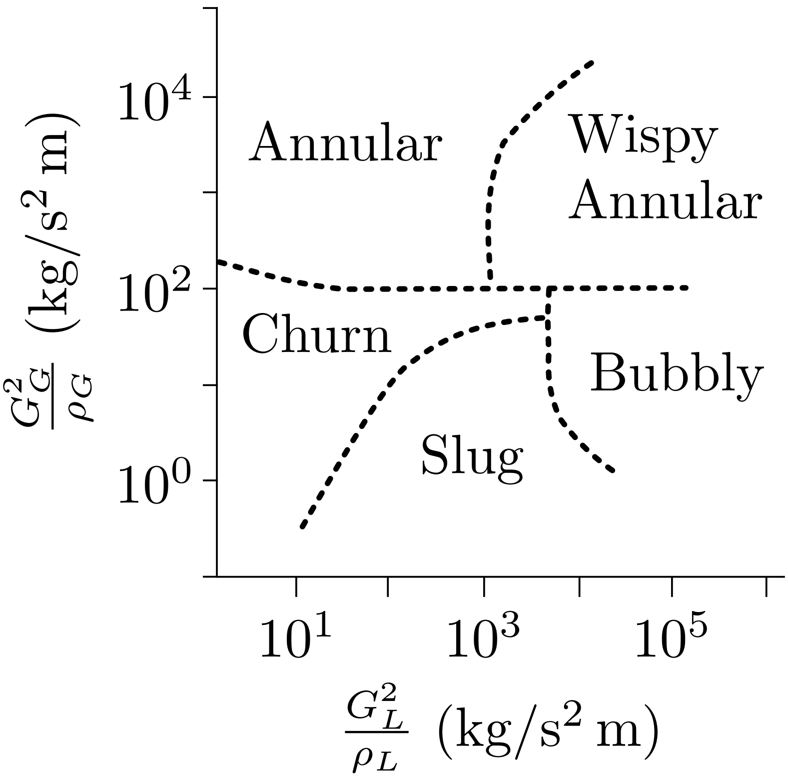


Figure 3: *Hewitt-Taylor flow pattern map for multiphase flows in vertical pipes.*

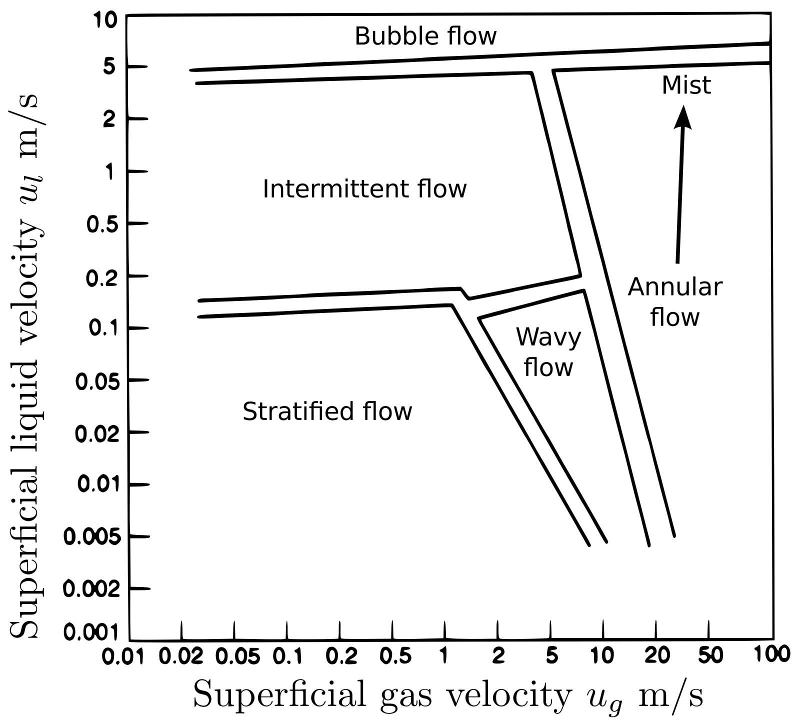


Figure 4: *Chhabra and Richardson flow pattern map for horizontal pipes.*

Churchill and Chu expression for natural convection from a horizontal pipe:

$$\text{Nu}^{1/2} = 0.6 + 0.387 \left\{ \frac{\text{Gr}\,\text{Pr}}{\left[1 + (0.559/\text{Pr})^{9/16}\right]^{16/9}} \right\}^{1/6} \qquad \text{for } 10^{-5} < \text{Gr}\,\text{Pr} < 10^{12}$$

Forced Convection:
Laminar flows:

$$\text{Nu} \approx 0.332\,\text{Re}^{1/2}\,\text{Pr}^{1/3}$$

Well-Developed turbulent flows in smooth pipes:

$$\text{Nu} \approx \frac{(C_f/2)\text{Re}\,\text{Pr}}{1.07 + 12.7(C_f/2)^{1/2}\left(\text{Pr}^{2/3} - 1\right)} \left(\frac{\mu_b}{\mu_w}\right)^{0.14}$$

Boiling:
Forster-Zuber pool-boiling coefficient:

$$h_{nb} = 0.00122 \frac{k_L^{0.79}\, C_{p,L}^{0.45}\, \rho_L^{0.49}}{\gamma^{0.5}\, \mu_L^{0.29}\, h_{fg}^{0.24}\, \rho_G^{0.24}} \left(T_w - T_{sat}\right)^{0.24} \left(p_w - p_{sat}\right)^{0.75}$$

Mostinski correlations:

$$h_{nb} = 0.104\, p_c^{0.69}\, q^{0.7} \left[1.8 \left(\frac{p}{p_c}\right)^{0.17} + 4 \left(\frac{p}{p_c}\right)^{1.2} + 10 \left(\frac{p}{p_c}\right)^{10}\right]$$

$$q_c = 3.67 \times 10^4\, p_c \left(\frac{p}{p_c}\right)^{0.35} \left[1 - \frac{p}{p_c}\right]^{0.9}$$

(**Note**: for the Mostinski correlations, the pressures are in units of bar)
Condensing:
Horizontal pipes

$$h = 0.72 \left(\frac{k^3\, \rho^2\, g_x\, E_{latent}}{D\, \mu\, (T_w - T_\infty)}\right)^{1/4}$$

NTU method:

$$\text{NTU} = \frac{U\,A}{C_{min}} = \frac{t_{C1} - t_{C2}}{\Delta t_{ln}} \qquad\qquad R = \frac{C_{min}}{C_{max}}$$

For counter-current flow:

$$E = \frac{1 - \exp\left[-\text{NTU}(1 - R)\right]}{1 - R \exp\left[-\text{NTU}(1 - R)\right]}$$

For co-current flow:

$$E = \frac{1 - \exp\left[-\mathsf{NTU}(1-R)\right]}{1+R}$$

Lumped capacitance method:

$$\begin{aligned} \mathsf{Bi} &= \frac{h\,L_c}{k} \\ L_c &= V/A \qquad \text{for Bi} < 0.1 \end{aligned}$$

$$\frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = \exp\left[-\frac{h\,A_s}{\rho\,V\,C_p} t\right]$$

Diffusion Dimensionless Numbers

$$\mathsf{Sc} = \frac{\mu}{\rho\,D_{AB}} \qquad \mathsf{Le} = \frac{k}{\rho\,C_p\,D_{AB}}$$

Diffusion

General expression for the flux:

$$\boldsymbol{N}_A = \boldsymbol{J}_A + x_A \sum_B \boldsymbol{N}_B$$

Fick's law:

$$\boldsymbol{J}_A = -D_{AB}\,\nabla C_A$$

Stefan's law:

$$N_{s,r} = -D \frac{c}{1-x} \frac{\partial x}{\partial r}$$

Misc

$$P\,V = n\,R\,T \qquad R \approx 8.314598 \text{ J K}^{-1} \text{ mol}^{-1}$$