Chapter 4:
Transient Heat Conduction

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Objectives

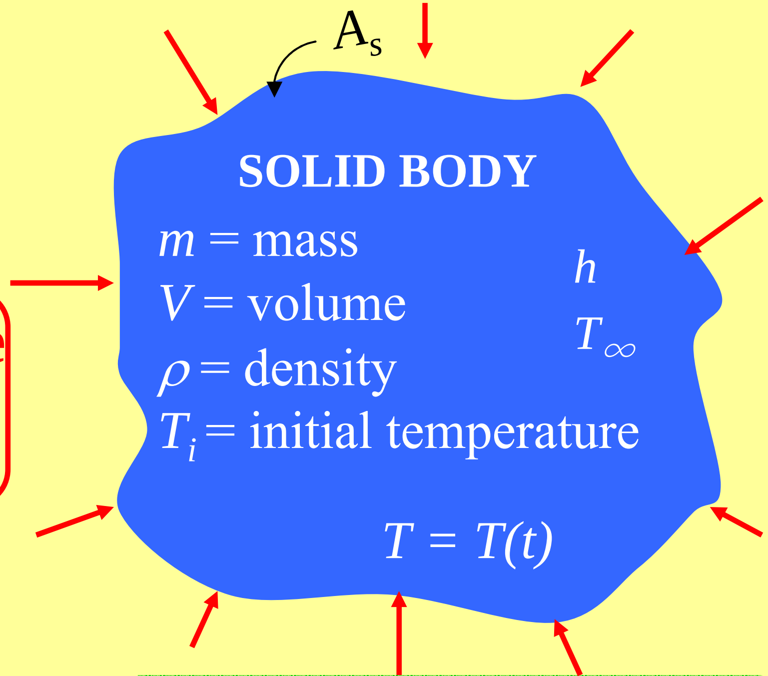
When you finish studying this chapter, you should be able to:

- Assess when the spatial variation of temperature is negligible, and temperature varies nearly uniformly with time, making the simplified lumped system analysis applicable,
- Obtain analytical solutions for transient one-dimensional conduction problems in rectangular, cylindrical, and spherical geometries using the method of separation of variables, and understand why a one-term solution is usually a reasonable approximation,
- Solve the transient conduction problem in large mediums using the similarity variable, and predict the variation of temperature with time and distance from the exposed surface, and
- Construct solutions for multi-dimensional transient conduction problems using the product solution approach.

Lumped System Analysis

- In heat transfer analysis, some bodies are essentially isothermal and can be treated as a “lump” system.

- An energy balance of an isothermal solid for the time interval *dt* can be expressed as



$$\left(\begin{array}{c}\textbf{Heat Transfer}\\ \textbf{into the body}\\ \textbf{during } \boldsymbol{dt}\end{array}\right) = \left(\begin{array}{c}\textbf{The increase in the}\\ \textbf{energy of the body}\\ \textbf{during } \boldsymbol{dt}\end{array}\right)$$

$$hA_s(T_\infty - T)dt = mc_p dT$$

(4–1) $$\dot{Q} = hA_s\left[T_\infty - T(t)\right]$$

- Noting that $m=\rho V$ and $dT=d(T\text{-}T_\infty)$ since T_∞ constant, Eq. 4–1 can be rearranged as

$$\frac{d\left(T-T_\infty\right)}{T-T_\infty}=\frac{hA_s}{\rho V c_p}dt \qquad \textbf{(4–2)}$$

- Integrating from time zero (at which $T=T_i$) to t gives

$$\ln\frac{T(t)-T_\infty}{T_i-T_\infty}=\frac{hA_s}{\rho V c_p}t \qquad \textbf{(4–3)}$$

- Taking the exponential of both sides and rearranging

$$\boxed{\frac{T(t)-T_\infty}{T_i-T_\infty}=e^{-bt} \quad ; \quad b=\frac{hA_s}{\rho V c_p} \quad (1/\text{s})} \qquad \textbf{(4–4)}$$

- b is a positive quantity whose dimension is $(\text{time})^{-1}$, and is called the **time constant**.

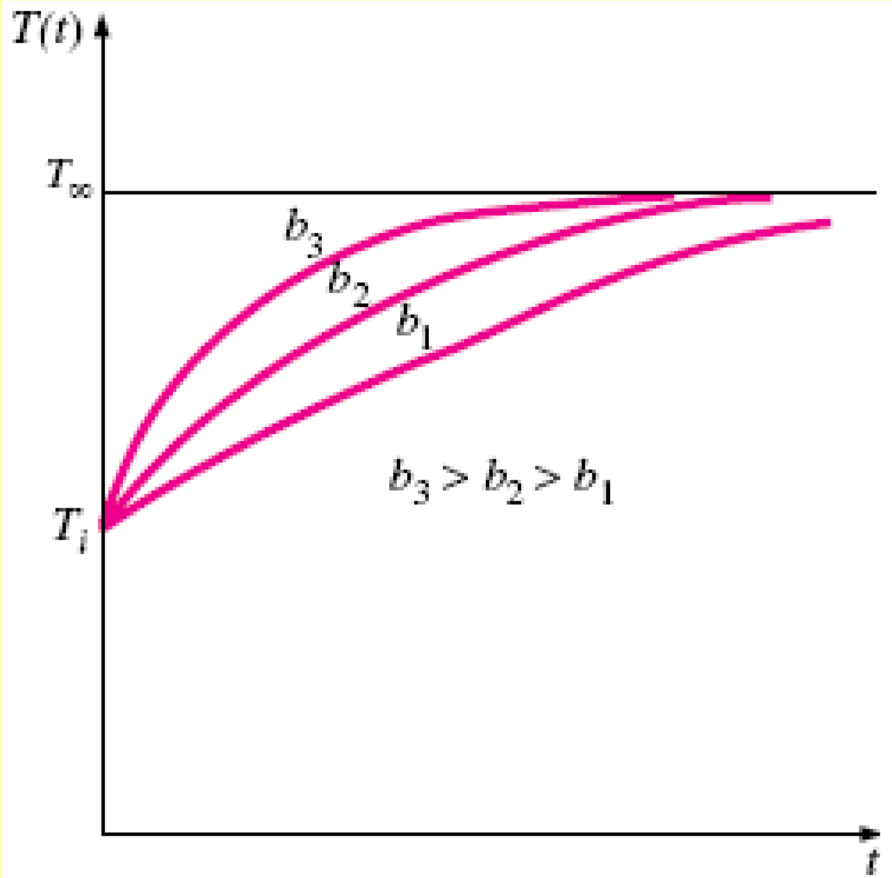
There are several observations that can be made from this figure and the relation above:

1. Equation 4–4 enables us to determine the temperature $T(t)$ of a body at time t, or alternatively, the time t required for the temperature to reach a specified value $T(t)$.

2. The temperature of a body approaches the ambient temperature T exponentially.

3. The temperature of the body changes rapidly at the beginning, but rather slowly later on.

4. A large value of b indicates that the body approaches the ambient temperature in a short time.



Rate of Convection Heat Transfer

- The *rate* of convection heat transfer between the body and the ambient can be determined from Newton's law of cooling

$$\boxed{\dot{Q}(t) = hA_s\left[T(t) - T_\infty\right] \quad \text{(W)}} \qquad \textbf{(4–6)}$$

- The *total* heat transfer between the body and the ambient over the time interval *0* to *t* is simply the change in the energy content of the body:

$$Q = mc_p\left[T(t) - T_\infty\right] \quad \text{(kJ)} \qquad \textbf{(4–7)}$$

- The *maximum* heat transfer between the body and its surroundings (when the body reaches T_∞)

$$Q_{\max} = mc_p\left[T_i - T_\infty\right] \quad \text{(kJ)} \qquad \textbf{(4–8)}$$

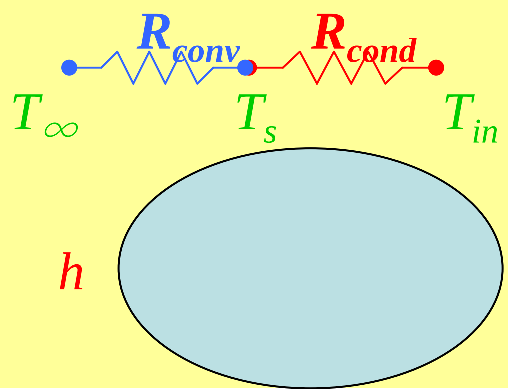
Criteria for Lumped System Analysis

- Lumped system is not always appropriate,
  characteristic length $L_c = V/A_s$

- and a **Biot number** (*Bi*) as

- It can also be expressed as $Bi = \frac{hL_c}{k}$ **(4–9)**

$$Bi = \frac{L_c/k}{1/h} = \frac{R_{cond}}{R_{conv}} = \frac{\text{Conduction resistance within the body}}{\text{Convection resistance at the surface of the body}}$$



- Lumped system analysis assumes a *uniform* temperature distribution throughout the body, which is true only when the thermal resistance of the body to heat conduction is zero.

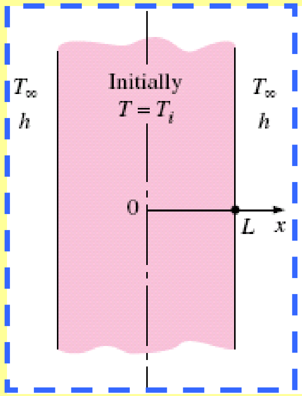
- The smaller the Bi number, the more accurate the lumped system analysis.

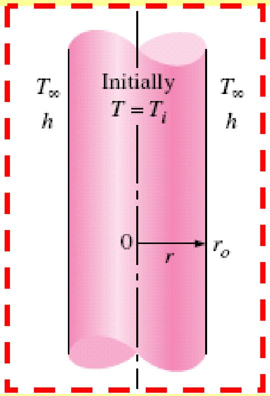
- It is generally accepted that lumped system analysis is *applicable* if

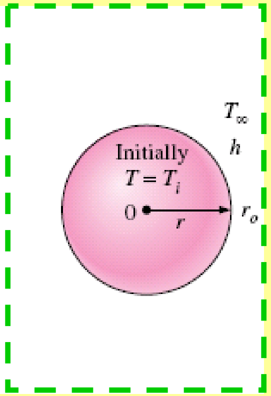
$$\boxed{Bi \le 0.1}$$

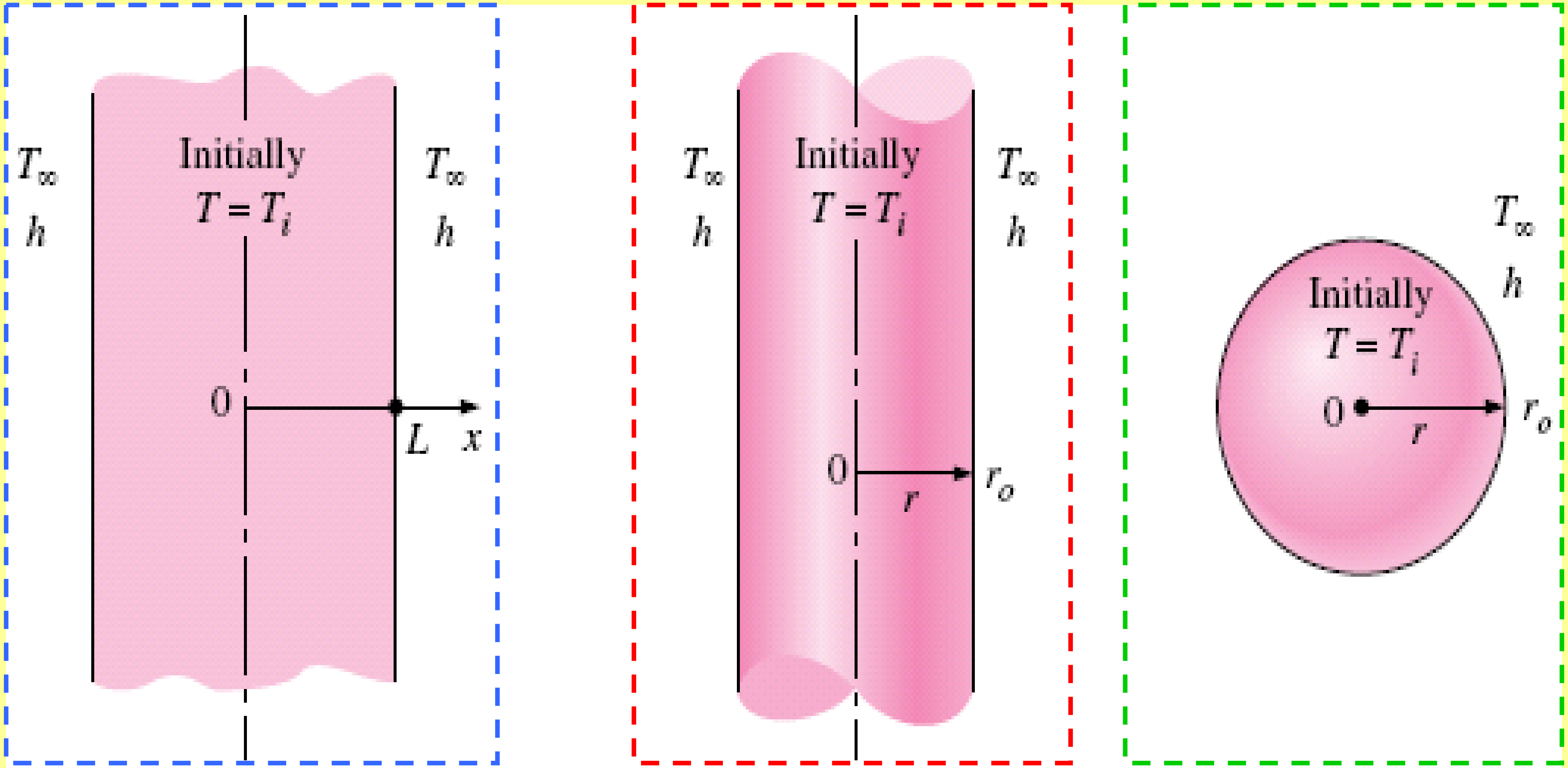
Transient Heat Conduction in Large Plane Walls, Long Cylinders, and Spheres with **Spatial Effects**

- In many transient heat transfer problems the Biot number is larger than 0.1, and lumped system can not be assumed.
- In these cases the temperature within the body changes appreciably from point to point as well as with time.
- It is constructive to first consider the variation of temperature with *time* and *position in* one-dimensional problems of rudimentary configurations such as a **large plane wall**, a **long cylinder**, and a **sphere**.





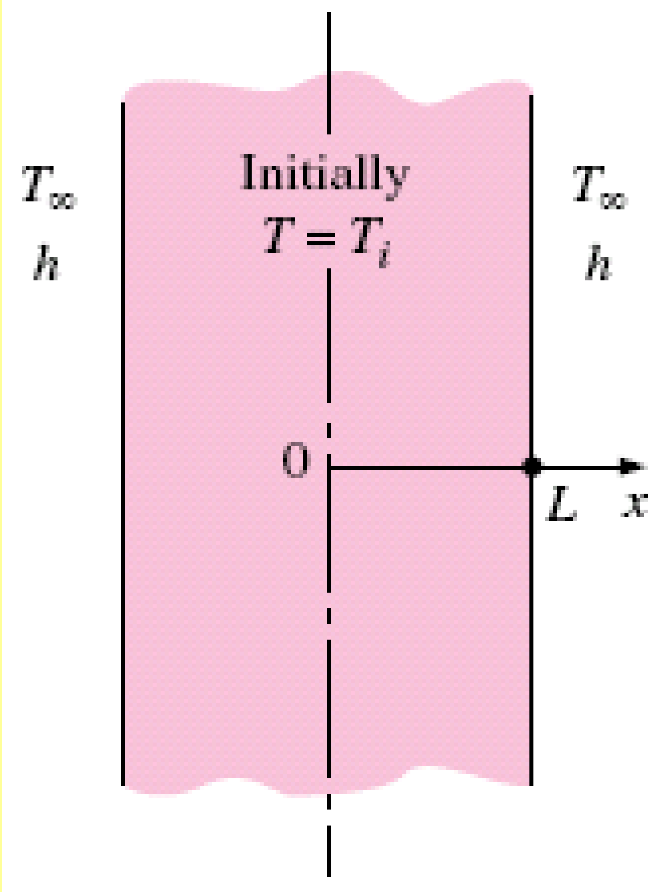




T_∞
h
Initially
$T = T_i$
0
L
x
T_∞
h
Initially
$T = T_i$
0
r
r_o
T_∞
h
Initially
$T = T_i$
0
r
r_o

A large Plane Wall

- A plane wall of thickness **$2L$**.
- Initially at a uniform temperature of $\boldsymbol{T_i}$.
- At time $\boldsymbol{t}$=**0**, the wall is immersed in a fluid at temperature $\boldsymbol{T_\infty}$.
- Constant heat transfer coefficient $\boldsymbol{h}$.
- The height and the width of the wall are large relative to its thickness → one-dimensional approximation is valid.



- Constant thermophysical properties.
- No heat generation.
- There is thermal symmetry about the midplane passing through x=0.

The Heat Conduction Equation

- One-dimensional transient heat conduction equation problem ($0 \le x \le L$):

Differential equation:

$$\boxed{\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}} \qquad \textbf{(4–10a)}$$

Boundary conditions:

$$\begin{cases} \dfrac{\partial T(0,t)}{\partial x} = 0 \\ -k\dfrac{\partial T(L,t)}{\partial x} = h\left[T(L,t) - T_\infty\right] \end{cases} \qquad \textbf{(4–10b)}$$

Initial condition:

$$T(x,0) = T_i \qquad \textbf{(4–10c)}$$

Non-dimensional Equation

- A dimensionless space variable

$$X = x/L$$

- A dimensionless temperature variable

$$\theta(x, t) = [T(x,t) - T_\infty]/[T_i - T_\infty]$$

- The dimensionless time and *h*/*k* ratio will be obtained through the analysis given below
- Introducing the dimensionless variable into Eq. 4-10a

$$\frac{\partial \theta}{\partial X} = \frac{\partial \theta}{\partial (x/L)} = \frac{L}{T_i - T_\infty}\frac{\partial T}{\partial x} \; ; \; \frac{\partial^2 \theta}{\partial X^2} = \frac{L^2}{T_i - T_\infty}\frac{\partial^2 T}{\partial x^2} \; ; \; \frac{\partial \theta}{\partial t} = \frac{1}{T_i - T_\infty}\frac{\partial T}{\partial t}$$

- Substituting into Eqs. 4–10*a* and 4–10*b* and rearranging

$$\frac{\partial^2 \theta}{\partial X^2} = \frac{L^2}{\alpha}\frac{\partial^2 T}{\partial x^2}\frac{\partial \theta}{\partial t} \; ; \; \frac{\partial \theta(1,t)}{\partial X} = \frac{hL}{k}\theta(1,t) \; ; \; \frac{\partial \theta(0,t)}{\partial X} = 0 \quad \textbf{(4–11)}$$

- Therefore, the dimensionless time is $\tau = \alpha t / L^2$, which is called the **Fourier number** (Fo).
- hL/k is the **Biot number** (Bi).
- The one-dimensional transient heat conduction problem in a plane wall can be expressed in **nondimensional** form as

Differential equation:

$$\boxed{\frac{\partial^2 \theta}{\partial X^2} = \frac{\partial \theta}{\partial \tau}} \qquad \textbf{(4–12a)}$$

Boundary conditions:

$$\begin{cases} \dfrac{\partial \theta(0,\tau)}{\partial X} = 0 \\ \dfrac{\partial \theta(1,\tau)}{\partial X} = -Bi\,\theta(1,\tau) \end{cases} \qquad \textbf{(4–12b)}$$

Initial condition:

$$\theta(X, 0) = 1 \qquad \textbf{(4–12c)}$$

Exact Solution

- Several analytical and numerical techniques can be used to solve Eq. 4-12.
- We will use the method of **separation of variables**.
- The dimensionless temperature function $\theta(X,\tau)$ is expressed as a product of a function of X only and a function of τ only as

$$\theta(X,\tau) = F(X)G(\tau) \qquad \textbf{(4–14)}$$

- Substituting Eq. 4–14 into Eq. 4–12a and dividing by the product FG gives

$$\frac{1}{F}\frac{d^2F}{dX^2} = \frac{1}{G}\frac{dG}{d\tau} \qquad \textbf{(4–15)}$$

- *Since* X and τ can be varied independently, the equality in Eq. 4–15 can hold for any value of X and τ only if Eq. 4–15 is equal to a constant.
- It must be a *negative* constant that we will indicate by $-\lambda^2$ since a positive constant will cause the function $G(\tau)$ to increase indefinitely with time.
- Setting Eq. 4–15 equal to $-\lambda^2$ gives

$$\frac{d^2F}{dX^2}+\lambda^2 F=0 \quad ; \quad \frac{dG}{d\tau}+\lambda^2 F=0 \qquad \textbf{(4–16)}$$

- whose general solutions are

$$\begin{cases} F=C_1\cos\left(\lambda X\right)+C_2\sin\left(\lambda X\right) \\ \mathrm{G}=\mathrm{C}_3 e^{-\lambda^2\tau} \end{cases} \qquad \textbf{(4–17)}$$

$$\theta = FG = \mathrm{C}_3 e^{-\lambda^2\tau}\left[C_1\cos(\lambda X) + C_2\sin(\lambda X)\right]$$

$$= e^{-\lambda^2\tau}\left[A\cos(\lambda X) + B\sin(\lambda X)\right] \qquad \textbf{(4–18)}$$

- where $A=C_1C_3$ and $B=C_2C_3$ are arbitrary constants.
- Note that we need to determine only A and B to obtain the solution of the problem.
- Applying the boundary conditions in Eq. 4–12b gives

$$\frac{\partial\theta(0,\tau)}{\partial X} = 0 \rightarrow -e^{-\lambda^2\tau}\left(A\lambda\sin 0 + B\lambda\cos 0\right) = 0$$

$$\rightarrow B = 0 \rightarrow \theta = Ae^{-\lambda^2\tau}\cos(\lambda X)$$

$$\frac{\partial\theta(1,\tau)}{\partial X} = -Bi\theta(1,\tau) \rightarrow -Ae^{-\lambda^2\tau}\lambda\sin\lambda = -BiAe^{-\lambda^2\tau}\cos\lambda$$

$$\boxed{\rightarrow \lambda\tan\lambda = Bi}$$

- But tangent is a periodic function with a period of π, and the equation $\lambda\tan(\lambda)=\mathrm{Bi}$ has the root λ_1 between 0 and π, the root λ_2 between π and 2π, the root λ_n between $(n\text{-}1)\pi$ and $n\pi$, etc.
- To recognize that the transcendental equation $\lambda\tan(\lambda)=\mathrm{Bi}$ has an infinite number of roots, it is expressed as

$$\lambda_n \tan \lambda_n = Bi \qquad \textbf{(4–19)}$$

- Eq. 4–19 is called the **characteristic equation** or **eigenfunction**, and its roots are called the **characteristic values** or **eigenvalues**.
- It follows that there are an infinite number of solutions of the form $\theta = Ae^{-\lambda^2\tau}\cos(\lambda X)$, and the solution of this linear heat conduction problem is a linear combination of them,

$$\theta = \sum_{n=1}^{\infty} A_n e^{-\lambda_n^2\tau} \cos(\lambda_n X) \qquad \textbf{(4–20)}$$

- The constants A_n are determined from the initial condition, Eq. 4–12c,

$$\theta(X,0) = 1 \rightarrow 1 = \sum_{n=1}^{\infty} A_n \cos(\lambda_n X) \qquad \textbf{(4–21)}$$

- Multiply both sides of Eq. 4–21 by $\cos(\lambda_m X)$, and integrating from X=0 to X=1

$$\int_{X=0}^{X=1} \cos(\lambda_m X) = \int_{X=0}^{X=1} \cos(\lambda_m X) \sum_{n=1}^{\infty} A_n \cos(\lambda_n X)$$

- The right-hand side involves an infinite number of integrals of the form

$$\int_{X=0}^{X=1} \cos(\lambda_m X) \cos(\lambda_n X)\, dX$$

- It can be shown that all of these integrals vanish except when n=m, and the coefficient A_n becomes

$$\int_{X=0}^{X=1} \cos(\lambda_n X)\, dX = A_n \int_{X=0}^{X=1} \cos^2(\lambda_n X)\, dX$$

$$\rightarrow \boxed{A_n = \frac{4 \sin \lambda_n}{2\lambda_n + \sin(2\lambda_n)}} \quad \textbf{(4–22)}$$

- Substituting Eq. 4-22 into Eq. 20a gives

$$\boxed{\theta = \sum_{n=1}^{\infty} \frac{4\sin\lambda_n}{2\lambda_n + \sin(2\lambda_n)} e^{-\lambda_n^2 \tau} \cos(\lambda_n X)}$$

- Where λ_n is obtained from Eq. 4-19.
- As demonstrated in Fig. 4–14, the terms in the summation decline rapidly as n and thus λ_n increases.
- Solutions in other geometries such as a long cylinder and a sphere can be determined using the same approach and are given in Table 4-1.

$$\theta_n = A_n e^{-\lambda_n^2 \tau} \cos(\lambda_n X)$$

$$A_n = \frac{4 \sin \lambda_n}{2\lambda_n + \sin(2\lambda_n)}$$

$$\lambda_n \tan \lambda_n = \mathrm{Bi}$$

For Bi = 5, $X = 1$, and $t = 0.2$:

n	λ_n	A_n	θ_n
1	1.3138	1.2402	0.22321
2	4.0336	−0.3442	0.00835
3	6.9096	0.1588	0.00001
4	9.8928	−0.876	0.00000

FIGURE 4-14

$$\theta_n = A_n \, e^{-\lambda_n^2 \tau} \cos(\lambda_n X)$$

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Summary of the Solutions for One-Dimensional Transient Conduction

TABLE 4–1

Summary of the solutions for one-dimensional transient conduction in a plane wall of thickness $2L$, a cylinder of radius r_o and a sphere of radius r_o subjected to convention from all surfaces.*

Geometry	Solution	λ_n's are the roots of
Plane wall	$\theta = \sum_{n=1}^{\infty} \frac{4 \sin \lambda_n}{2\lambda_n + \sin(2\lambda_n)} e^{-\lambda_n^2 \tau} \cos(\lambda_n x / L)$	$\lambda_n \tan \lambda_n = \mathrm{Bi}$
Cylinder	$\theta = \sum_{n=1}^{\infty} \frac{2}{\lambda_n} \frac{J_1(\lambda_n)}{J_0^2(\lambda_n) + J_1^2(\lambda_n)} e^{-\lambda_n^2 \tau} J_0(\lambda_n r / r_o)$	$\lambda_n \frac{J_1(\lambda_n)}{J_0(\lambda_n)} = \mathrm{Bi}$
Sphere	$\theta = \sum_{n=1}^{\infty} \frac{4(\sin \lambda_n - \lambda_n \cos \lambda_n)}{2\lambda_n - \sin(2\lambda_n)} e^{-\lambda_n^2 \tau} \frac{\sin(\lambda_n x / L)}{\lambda_n x / L}$	$1 - \lambda_n \cot \lambda_n = \mathrm{Bi}$

*Here $\theta = (T - T_i)/(T_\infty - T_i)$ is the dimensionless temperature, $\mathrm{Bi} = hL/k$ or hr_o/k is the Biot number, $\mathrm{Fo} = \tau = \alpha t / L^2$ or $\alpha\tau / r_o^2$ is the Fourier number, and J_0 and J_1 are the Bessel functions of the first kind whose values are given in Table 4–3.

Approximate Analytical and Graphical Solutions

- The series solutions of Eq. 4-20 and in Table 4–1 converge rapidly with increasing time, and for $\tau > 0.2$, keeping the first term and neglecting all the remaining terms in the series results in an error under 2 percent.
- Thus for $\tau > 0.2$ the **one-term approximation** can be used

Plane wall: $$\theta_{wall} = \frac{T(x,t)-T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau} \cos\left(\lambda_1 x / L\right), \quad \tau > 0.2 \qquad \textbf{(4–23)}$$

Cylinder: $$\theta_{cyl} = \frac{T(r,t)-T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau} J_0\left(\lambda_1 r / r_0\right), \quad \tau > 0.2 \qquad \textbf{(4–24)}$$

Sphere: $$\theta_{sph} = \frac{T(r,t)-T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau} \frac{\sin\left(\lambda_1 r / r_0\right)}{\lambda_1 r / r_0}, \quad \tau > 0.2 \qquad \textbf{(4–25)}$$

- The constants A_1 and λ_1 are functions of the Bi number only, and their values are listed in Table 4–2 against the Bi number for all three geometries.
- The function J_0 is the zeroth-order Bessel function of the first kind, whose value can be determined from Table 4–3.

TABLE 4–2

Coefficients used in the one-term approximate solution of transient one-dimensional heat conduction in plane walls, cylinders, and spheres (Bi = hL/k for a plane wall of thickness $2L$, and Bi = hr_o/k for a cylinder or sphere of radius r_o)

	Plane Wall		*Cylinder*		*Sphere*	
Bi	λ_1	A_1	λ_1	A_1	λ_1	A_1
0.01	0.0998	1.0017	0.1412	1.0025	0.1730	1.0030
0.02	0.1410	1.0033	0.1995	1.0050	0.2445	1.0060
0.04	0.1987	1.0066	0.2814	1.0099	0.3450	1.0120
0.06	0.2425	1.0098	0.3438	1.0148	0.4217	1.0179
0.08	0.2791	1.0130	0.3960	1.0197	0.4860	1.0239
0.1	0.3111	1.0161	0.4417	1.0246	0.5423	1.0298
0.2	0.4328	1.0311	0.6170	1.0483	0.7593	1.0592
0.3	0.5218	1.0450	0.7465	1.0712	0.9208	1.0880
0.4	0.5932	1.0580	0.8516	1.0931	1.0528	1.1164
0.5	0.6533	1.0701	0.9408	1.1143	1.1656	1.1441
0.6	0.7051	1.0814	1.0184	1.1345	1.2644	1.1713
0.7	0.7506	1.0918	1.0873	1.1539	1.3525	1.1978
0.8	0.7910	1.1016	1.1490	1.1724	1.4320	1.2236
0.9	0.8274	1.1107	1.2048	1.1902	1.5044	1.2488
1.0	0.8603	1.1191	1.2558	1.2071	1.5708	1.2732
2.0	1.0769	1.1785	1.5995	1.3384	2.0288	1.4793
3.0	1.1925	1.2102	1.7887	1.4191	2.2889	1.6227
4.0	1.2646	1.2287	1.9081	1.4698	2.4556	1.7202
5.0	1.3138	1.2403	1.9898	1.5029	2.5704	1.7870
6.0	1.3496	1.2479	2.0490	1.5253	2.6537	1.8338
7.0	1.3766	1.2532	2.0937	1.5411	2.7165	1.8673
8.0	1.3978	1.2570	2.1286	1.5526	2.7654	1.8920
9.0	1.4149	1.2598	2.1566	1.5611	2.8044	1.9106
10.0	1.4289	1.2620	2.1795	1.5677	2.8363	1.9249
20.0	1.4961	1.2699	2.2880	1.5919	2.9857	1.9781
30.0	1.5202	1.2717	2.3261	1.5973	3.0372	1.9898
40.0	1.5325	1.2723	2.3455	1.5993	3.0632	1.9942
50.0	1.5400	1.2727	2.3572	1.6002	3.0788	1.9962
100.0	1.5552	1.2731	2.3809	1.6015	3.1102	1.9990
∞	1.5708	1.2732	2.4048	1.6021	3.1416	2.0000

TABLE 4–3

The zeroth- and first-order Bessel functions of the first kind

η	$J_0(\eta)$	$J_1(\eta)$
0.0	1.0000	0.0000
0.1	0.9975	0.0499
0.2	0.9900	0.0995
0.3	0.9776	0.1483
0.4	0.9604	0.1960
0.5	0.9385	0.2423
0.6	0.9120	0.2867
0.7	0.8812	0.3290
0.8	0.8463	0.3688
0.9	0.8075	0.4059
1.0	0.7652	0.4400
1.1	0.7196	0.4709
1.2	0.6711	0.4983
1.3	0.6201	0.5220
1.4	0.5669	0.5419
1.5	0.5118	0.5579
1.6	0.4554	0.5699
1.7	0.3980	0.5778
1.8	0.3400	0.5815
1.9	0.2818	0.5812
2.0	0.2239	0.5767
2.1	0.1666	0.5683
2.2	0.1104	0.5560
2.3	0.0555	0.5399
2.4	0.0025	0.5202
2.6	−0.0968	−0.4708
2.8	−0.1850	−0.4097
3.0	−0.2601	−0.3391
3.2	−0.3202	−0.2613

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0.02	0.1410	1.0033	0.1995	1.0050	0.2445	1.0060
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0.08	0.2791	1.0130	0.3960	1.0197	0.4860	1.0239
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0.7	0.7506	1.0918	1.0873	1.1539	1.3525	1.1978
0.8	0.7910	1.1016	1.1490	1.1724	1.4320	1.2236
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1.0	0.8603	1.1191	1.2558	1.2071	1.5708	1.2732
2.0	1.0769	1.1785	1.5995	1.3384	2.0288	1.4793
3.0	1.1925	1.2102	1.7887	1.4191	2.2889	1.6227
4.0	1.2646	1.2287	1.9081	1.4698	2.4556	1.7202
5.0	1.3138	1.2403	1.9898	1.5029	2.5704	1.7870
6.0	1.3496	1.2479	2.0490	1.5253	2.6537	1.8338
7.0	1.3766	1.2532	2.0937	1.5411	2.7165	1.8673
8.0	1.3978	1.2570	2.1286	1.5526	2.7654	1.8920
9.0	1.4149	1.2598	2.1566	1.5611	2.8044	1.9106
10.0	1.4289	1.2620	2.1795	1.5677	2.8363	1.9249
20.0	1.4961	1.2699	2.2880	1.5919	2.9857	1.9781
30.0	1.5202	1.2717	2.3261	1.5973	3.0372	1.9898
40.0	1.5325	1.2723	2.3455	1.5993	3.0632	1.9942
50.0	1.5400	1.2727	2.3572	1.6002	3.0788	1.9962
100.0	1.5552	1.2731	2.3809	1.6015	3.1102	1.9990
∞	1.5708	1.2732	2.4048	1.6021	3.1416	2.0000

TABLE 4–3

The zeroth- and first-order Bessel functions of the first kind

η	$J_0(\eta)$	$J_1(\eta)$
0.0	1.0000	0.0000
0.1	0.9975	0.0499
0.2	0.9900	0.0995
0.3	0.9776	0.1483
0.4	0.9604	0.1960
0.5	0.9385	0.2423
0.6	0.9120	0.2867
0.7	0.8812	0.3290
0.8	0.8463	0.3688
0.9	0.8075	0.4059
1.0	0.7652	0.4400
1.1	0.7196	0.4709
1.2	0.6711	0.4983
1.3	0.6201	0.5220
1.4	0.5669	0.5419
1.5	0.5118	0.5579
1.6	0.4554	0.5699
1.7	0.3980	0.5778
1.8	0.3400	0.5815
1.9	0.2818	0.5812
2.0	0.2239	0.5767
2.1	0.1666	0.5683
2.2	0.1104	0.5560
2.3	0.0555	0.5399
2.4	0.0025	0.5202
2.6	−0.0968	−0.4708
2.8	−0.1850	−0.4097
3.0	−0.2601	−0.3391
3.2	−0.3202	−0.2613

The solution at the center of a plane wall, cylinder, and sphere:

Center of plane wall (x=0): $$\theta_{0,wall} = \frac{T_0 - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau}$$ **(4–26)**

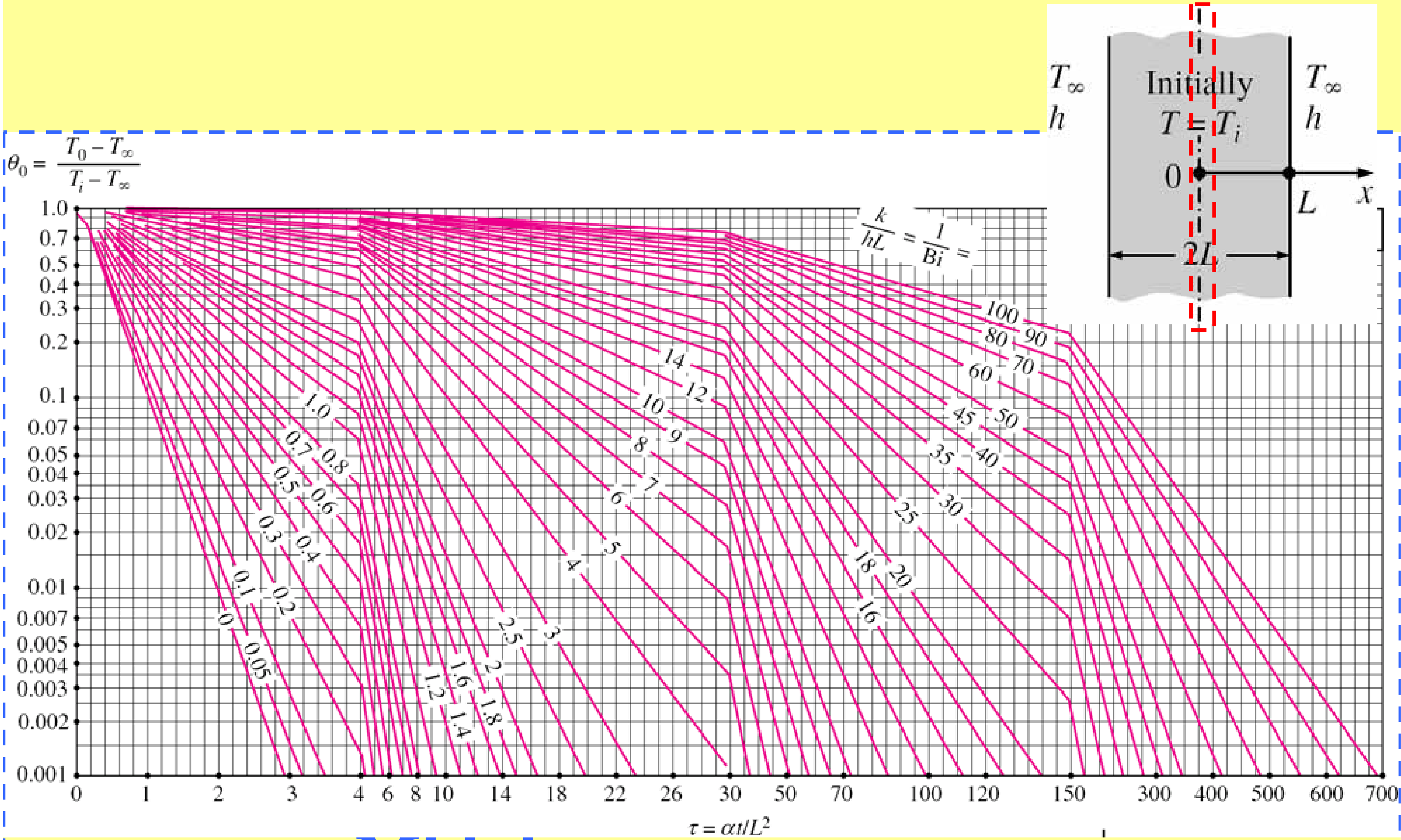
Center of cylinder (r=0): $$\theta_{0,cyl} = \frac{T_0 - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau}$$ **(4–27)**

Center of sphere (r=0): $$\theta_{sph} = \frac{T_0 - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau}$$ **(4–28)**

Heisler Charts

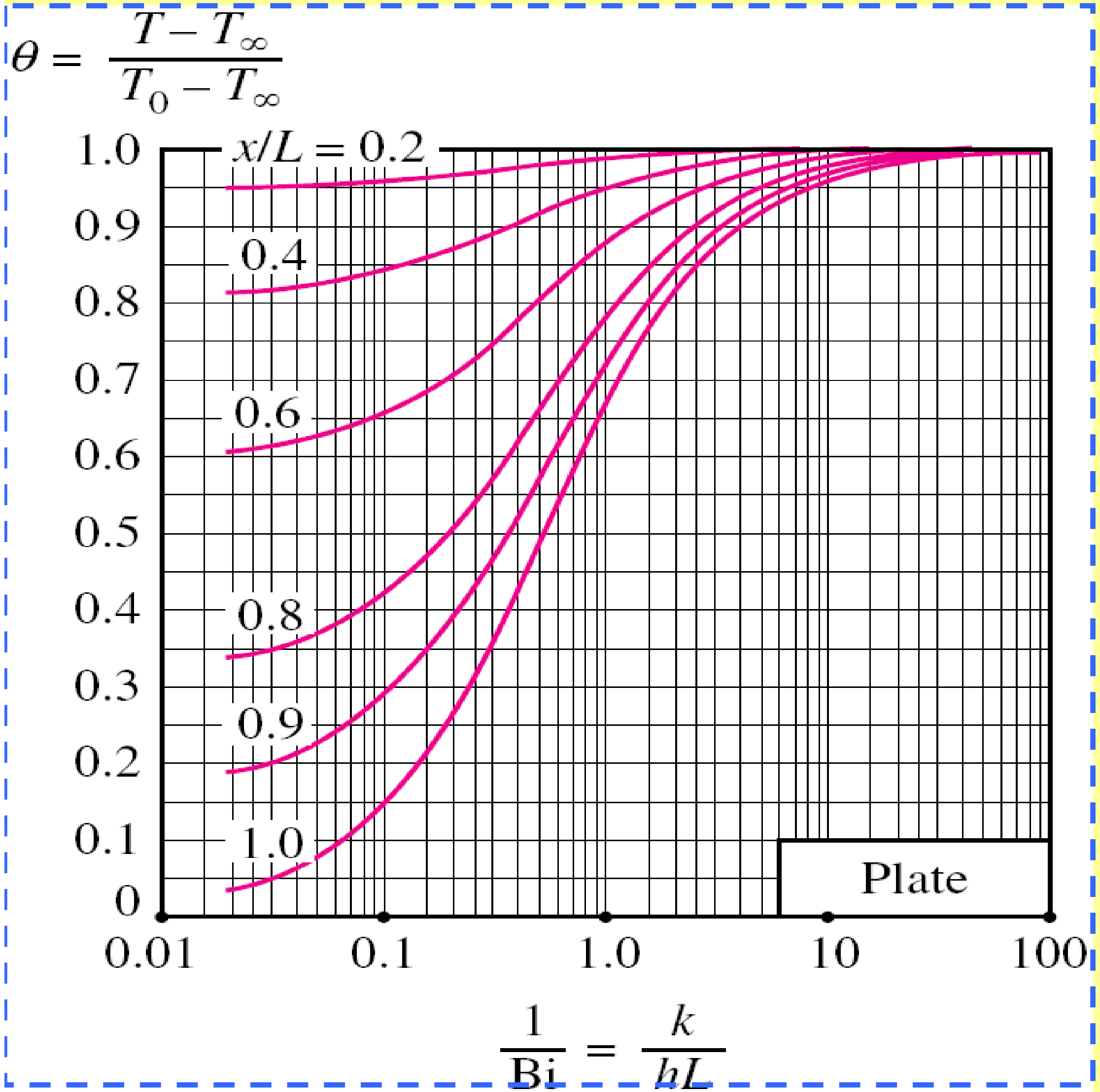
- The solution of the transient temperature for a large plane wall, long cylinder, and sphere are also presented in graphical form for $\tau > 0.2$, known as the *transient temperature charts* (also known as the Heisler Charts).
- There are *three* charts associated with each geometry:
  - the temperature T_0 at the *center* of the geometry at a given time t.
  - the temperature at *other locations* at the same time in terms of T_0.
  - the total amount of *heat transfer* up to the time t.

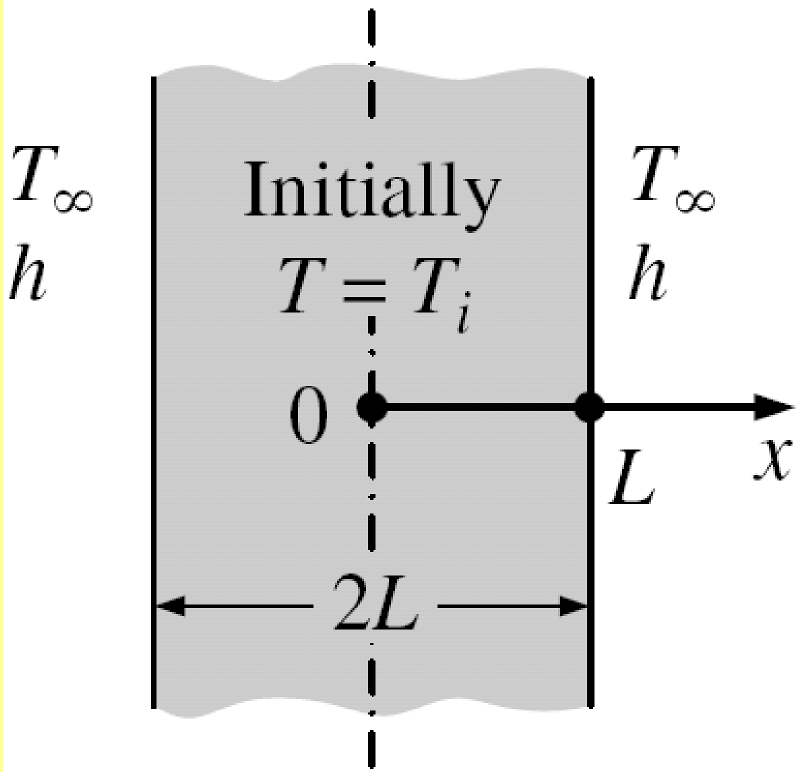
Heisler Charts – Plane Wall



Midplane temperature

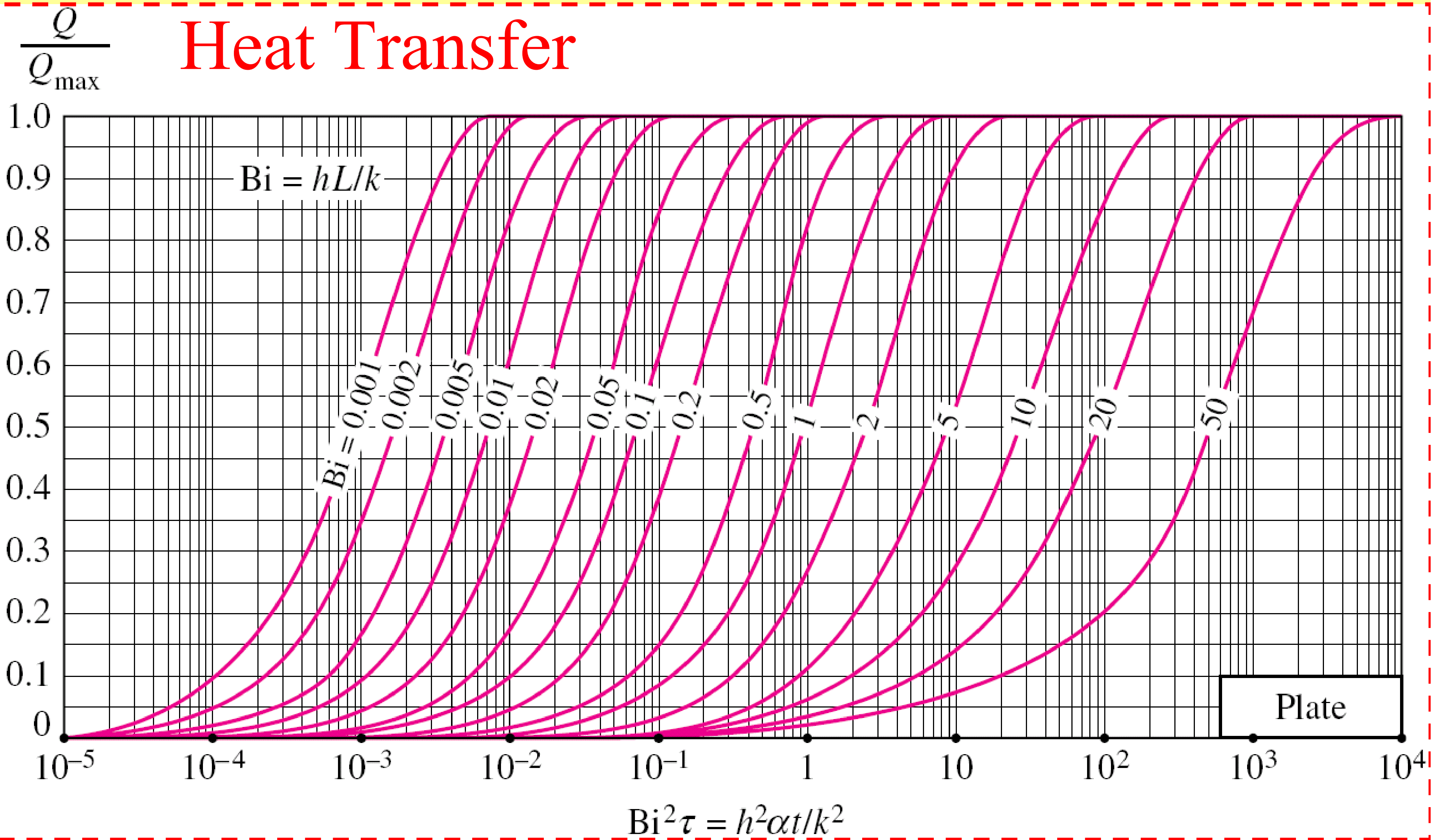
Heisler Charts – Plane Wall





Temperature distribution

Heat Transfer



Heat Transfer

- The *maximum* amount of heat that a body can gain (or lose if $T_i = T_\infty$) occurs when the temperature of the body is changes from the initial temperature T_i to the ambient temperature

$$Q_{\max} = mc_p\left(T_\infty - T_i\right) = \rho V c_p\left(T_\infty - T_i\right) \quad \text{(kJ)} \qquad \textbf{(4–30)}$$

- The amount of heat transfer Q at a finite time t is can be expressed as

$$Q = \int_V \rho c_p\left[T\left(x,t\right) - T_i\right] dV \qquad \textbf{(4–31)}$$

- Assuming constant properties, the ratio of Q/Q_{max} becomes

$$\frac{Q}{Q_{\max}} = \frac{\int_V \rho c_p \left[T(x,t) - T_i\right] dV}{\rho c_p \left(T_\infty - T_i\right) V} = \frac{1}{V}\int_V \left(1 - V\right) dV \quad \textbf{(4–32)}$$

- The following relations for the fraction of heat transfer in those geometries:

Plane wall:

$$\left(\frac{Q}{Q_{\max}}\right)_{wall} = 1 - \theta_{0,wall} \frac{\sin \lambda_1}{\lambda_1} \quad \textbf{(4–33)}$$

Cylinder:

$$\left(\frac{Q}{Q_{\max}}\right)_{cyl} = 1 - 2\theta_{0,cyl} \frac{J_1(\lambda_1)}{\lambda_1} \quad \textbf{(4–34)}$$

Sphere:

$$\left(\frac{Q}{Q_{\max}}\right)_{sph} = 1 - 3\theta_{0,sph} \frac{\sin \lambda_1 - \lambda_1 \cos \lambda_1}{\lambda_1^3} \quad \textbf{(4–35)}$$

Remember, the Heisler charts are not generally applicable

The Heisler Charts can only be used when:

- the body is initially at a *uniform* temperature,
- the **temperature** of the medium **surrounding the body** is *constant* and *uniform*.
- the **convection heat transfer coefficient** is *constant* and *uniform*, and there is no *heat generation* in the body.

Fourier number

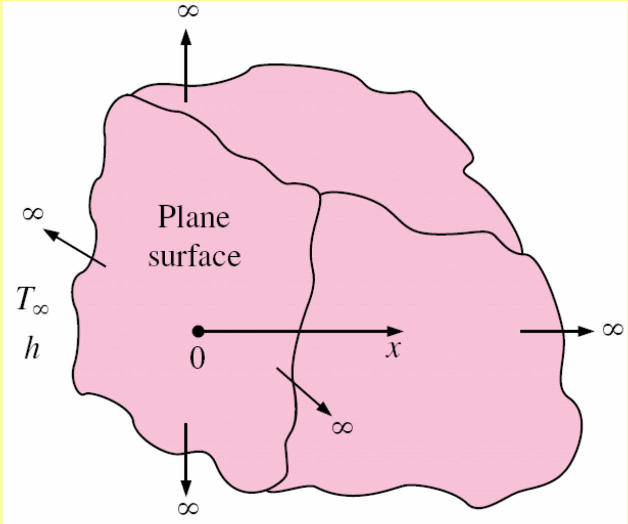
$$\tau = \frac{\alpha t}{L^2} = \frac{kL^2\left(1/L\right)}{\rho c_p L^3/t}\frac{\Delta T}{\Delta T} = \frac{\left(\begin{array}{c}\textbf{The rate at which heat is } \textbf{\textit{conducted}}\\ \textbf{across } \boldsymbol{L} \textbf{ of a body of volume } \boldsymbol{L^3}\end{array}\right)}{\left(\begin{array}{c}\textbf{The rate at which heat is } \textbf{\textit{stored}}\\ \textbf{in a body of volume } \boldsymbol{L^3}\end{array}\right)}$$

- The Fourier number is a measure of *heat conducted* through a body relative to *heat stored*.

- A large value of the Fourier number indicates faster propagation of heat through a body.

Transient Heat Conduction in Semi-Infinite Solids

- A semi-infinite solid is an idealized body that has a *single plane surface* and extends to infinity in all directions.



- Assumptions:
  - constant thermophysical properties
  - no internal heat generation
  - uniform thermal conditions on its exposed surface
  - initially a uniform temperature of T_i throughout.
- Heat transfer in this case occurs only in the direction normal to the surface (the x direction)

⇨ one-dimensional problem.

- Eq. 4–10*a* for one-dimensional transient conduction in Cartesian coordinates applies

Differential equation: $$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$ **(4–10a)**

Boundary conditions: $$\begin{cases} T(0,t) = T_s \\ T(x \to \infty, t) = T_i \end{cases}$$ **(4–37b)**

Initial condition: $$T(x,0) = T_i$$ **(4–10c)**

- The separation of variables technique does not work in this case since the medium is infinite.
- The partial differential equation can be converted into an ordinary differential equation by combining the two independent variables x and t into a single variable η, called the **similarity variable**.

Similarity Solution

- For transient conduction in a semi-infinite medium

  Similarity variable: $\eta = \dfrac{x}{\sqrt{4\alpha t}}$

- Assuming $T=T(\eta)$ (to be verified) and using the chain rule, all derivatives in the heat conduction equation can be transformed into the new variable

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \quad \text{and} \quad \eta = \frac{x}{\sqrt{4\alpha t}}$$

$$\frac{\partial T}{\partial t} = \frac{dT}{d\eta}\frac{\partial \eta}{\partial t} = -\frac{x}{2t\sqrt{4\alpha t}}\frac{dT}{d\eta}$$

$$\frac{\partial T}{\partial x} = \frac{dT}{d\eta}\frac{\partial \eta}{\partial x} = \frac{1}{\sqrt{4\alpha t}}\frac{dT}{d\eta}$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{d}{d\eta}\left(\frac{\partial T}{\partial x}\right)\frac{\partial \eta}{\partial x} = \frac{1}{4\alpha t}\frac{d^2 T}{d\eta^2}$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

(4–39a)

$$\boxed{\frac{\partial^2 T}{\partial \eta^2} = -2\eta\frac{\partial T}{\partial \eta}}$$

$$\frac{\partial^2 T}{\partial \eta^2} = -2\eta \frac{\partial T}{\partial \eta} \qquad \textbf{(4–39a)}$$

- Noting that η=0 at x=0 and $\eta \rightarrow \infty$ as $x \rightarrow \infty$ (and also at t=0) and substituting into Eqs. 4–37b (BC) give, after simplification

$$T(0) = T_s \quad ; \quad T(\eta \rightarrow \infty) = T_i \qquad \textbf{(4–39b)}$$

- Note that the second boundary condition and the initial condition result in the same boundary condition.
- Both the transformed equation and the boundary conditions depend on h only and are independent of x and t. Therefore, transformation is successful, and η is indeed a similarity variable.

- To solve the 2nd order ordinary differential equation in Eqs. 4–39, we define a new variable $\boldsymbol{w}$ as $\boldsymbol{w}=dT/d\eta$. This reduces Eq. 4–39*a* into a first order differential equation than can be solved by separating variables,

$$\frac{dw}{d\eta} = -2\eta w \rightarrow \frac{dw}{w} = -2\eta d\eta \rightarrow \ln(w) = -\eta^2 + C_0$$

$$\rightarrow w = C_1 e^{-\eta^2}$$

- where $C_1=\ln(C_0)$.
- Back substituting $w=dT/d\eta$ and integrating again,

$$T = C_1 \int_0^\eta e^{-u^2} du + C_2 \qquad \textbf{(4–40)}$$

- where u is a dummy integration variable. The boundary condition at $\eta=0$ gives $C_2=T_s$, and the one for $\eta\rightarrow\infty$ gives

$$T_i = C_1 \int_0^\infty e^{-u^2} du + C_2 = C_1 \frac{\sqrt{\pi}}{2} + T_s \rightarrow C_1 = \frac{2(T_i - T_s)}{\sqrt{\pi}} \qquad \textbf{(4–41)}$$

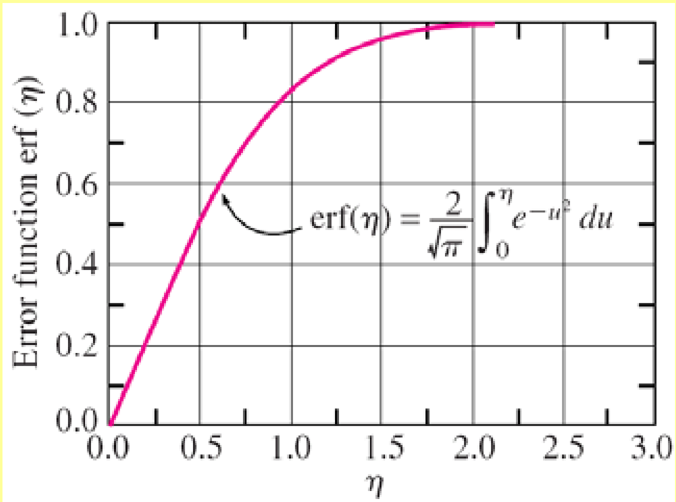
- Substituting the C_1 and C_2 expressions into Eq. 4–40 and rearranging,

$$\frac{T-T_s}{T_i-T_s}=\frac{2}{\sqrt{\pi}}\int_0^{\eta} e^{-u^2}\,du = erf\left(\eta\right)=1-erfc\left(\eta\right) \qquad \textbf{(4–42)}$$

- Where

$$erf\left(\eta\right)=\frac{2}{\sqrt{\pi}}\int_0^{\eta} e^{-u^2}\,du \quad ; \quad erfc\left(\eta\right)=1-\frac{2}{\sqrt{\pi}}\int_0^{\eta} e^{-u^2}\,du \qquad \textbf{(4–43)}$$

- are called the **error function** and the **complementary error function**, respectively, of argument η.



- Knowing the temperature distribution, the **heat flux** at the **surface** can be determined from the Fourier's law to be

$$\dot{q}_s = -k\frac{\partial T}{\partial x}\bigg|_{x=0} = -k\frac{\partial T}{\partial \eta}\frac{\partial \eta}{\partial x}\bigg|_{\eta=0} = -kC_1 e^{-\eta^2}\frac{1}{\sqrt{4\alpha t}}\bigg|_{\eta=0} = \frac{k\left(T_s - T_i\right)}{\sqrt{\pi \alpha t}}$$

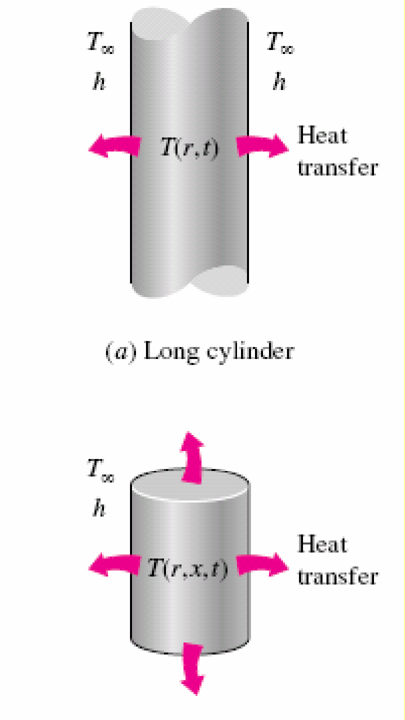
(4–44)

Other Boundary Conditions

- The solutions in Eqs. 4–42 and 4–44 correspond to the case when the temperature of the exposed surface of the medium is suddenly raised (or lowered) to T_s at $t=0$ and is maintained at that value at all times.
- Analytical solutions can be obtained for other boundary conditions on the surface and are given in the book
  - Specified Surface Temperature, T_s = constant.
  - Constant and specified surface heat flux.
  - Convection on the Surface,
  - Energy Pulse at Surface.

Transient Heat Conduction in Multidimensional Systems

- Using a superposition approach called the **product solution**, the *one-dimensional* heat conduction solutions can also be used to construct solutions for some two-dimensional (and even three-dimensional) transient heat conduction problems.
- Provided that *all* surfaces of the solid are subjected to convection to the *same* fluid at temperature, the *same* heat transfer coefficient *h*, and the body involves no heat generation.

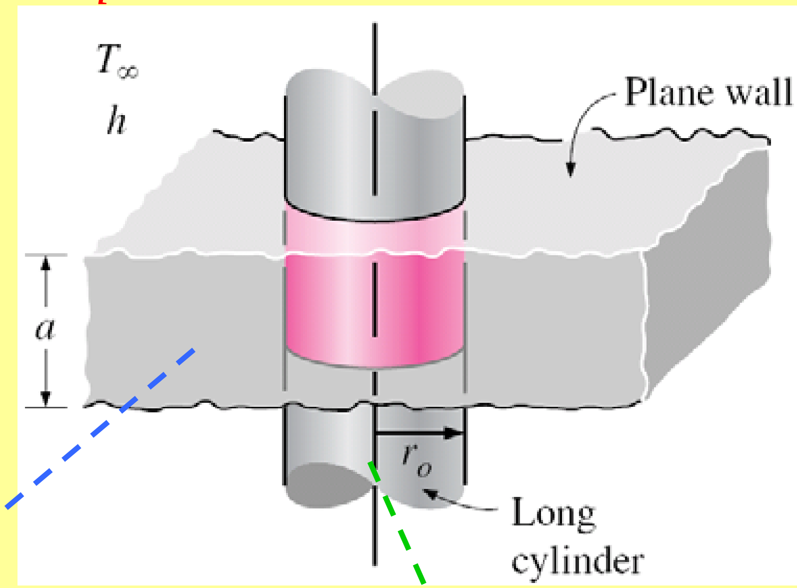


(*a*) Long cylinder

(*b*) Short cylinder (two-dimensional)

Example — short cylinder

- Height a and radius r_o.
- Initially uniform temperature T_i.
- No heat generation
- At time $t=0$:
  - convection T_∞
  - heat transfer coefficient h
- The solution:



$$\left(\frac{T(r,x,t)-T_\infty}{T_i-T_\infty}\right)\Bigg|_{\substack{\text{Short}\\\text{Cylinder}}} = \left(\frac{T(x,t)-T_\infty}{T_i-T_\infty}\right)\Bigg|_{\substack{\text{plane}\\\text{wall}}} \text{X} \left(\frac{T(r,t)-T_\infty}{T_i-T_\infty}\right)\Bigg|_{\substack{\text{infinite}\\\text{cylinder}}} \quad \textbf{(4–50)}$$

- The solution can be generalized as follows: *the solution for a multidimensional geometry is the product of the solutions of the one-dimensional geometries whose intersection is the multidimensional body.*
- For convenience, the one-dimensional solutions are denoted by

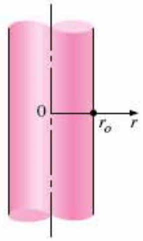
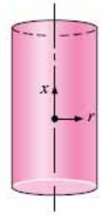
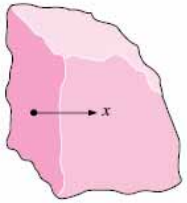
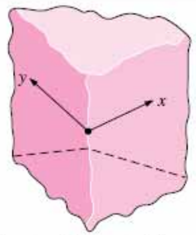
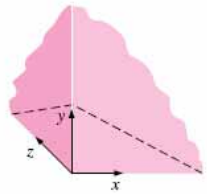
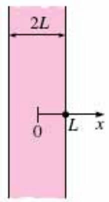
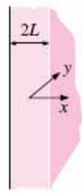
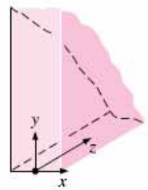
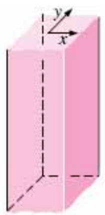
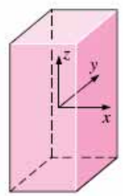
$$\theta_{\text{wall}}(x,t) = \left(\frac{T(x,t)-T_\infty}{T_i - T_\infty}\right)\Bigg|_{\substack{\text{plane}\\ \text{wall}}}$$

$$\theta_{\text{cyl}}(r,t) = \left(\frac{T(r,t)-T_\infty}{T_i - T_\infty}\right)\Bigg|_{\substack{\text{infinite}\\ \text{cylinder}}} \qquad \textbf{(4–51)}$$

$$\theta_{\text{semi-inf}}(x,t) = \left(\frac{T(x,t)-T_\infty}{T_i - T_\infty}\right)\Bigg|_{\substack{\text{semi-infinite}\\ \text{solid}}}$$

TABLE 4–5

Multidimensional solutions expressed as products of one-dimensional solutions for bodies that are initially at a uniform temperature T_i and exposed to convection from all surfaces to a medium at T_∞

 $\theta(r,t) = \theta_{cyl}(r,t)$ **Infinite cylinder**	 $\theta(x,r,t) = \theta_{cyl}(r,t)\,\theta_{semi\text{-}inf}(x,t)$ **Semi-infinite cylinder**	 $\theta(x,r,t) = \theta_{cyl}(r,t)\,\theta_{wall}(x,t)$ **Short cylinder**
 $\theta(x,t) = \theta_{semi\text{-}inf}(x,t)$ **Semi-infinite medium**	 $\theta(x,y,t) = \theta_{semi\text{-}inf}(x,t)\,\theta_{semi\text{-}inf}(y,t)$ **Quarter-infinite medium**	 $\theta(x,y,z,t) =$ $\theta_{semi\text{-}inf}(x,t)\,\theta_{semi\text{-}inf}(y,t)\,\theta_{semi\text{-}inf}(z,t)$ **Corner region of a large medium**
 $\theta(x,t) = \theta_{wall}(x,t)$ **Infinite plate (or plane wall)**	 $\theta(x,y,t) = \theta_{wall}(x,t)\,\theta_{semi\text{-}inf}(y,t)$ **Semi-infinite plate**	 $\theta(x,y,z,t) =$ $\theta_{wall}(x,t)\,\theta_{semi\text{-}inf}(y,t)\,\theta_{semi\text{-}inf}(z,t)$ **Quarter-infinite plate**
 $\theta(x,y,t) = \theta_{wall}(x,t)\,\theta_{wall}(y,t)$ **Infinite rectangular bar**	 $\theta(x,y,z,t) =$ $\theta_{wall}(x,t)\,\theta_{wall}(y,t)\,\theta_{semi\text{-}inf}(z,t)$ **Semi-infinite rectangular bar**	 $\theta(x,y,z,t) =$ $\theta_{wall}(x,t)\,\theta_{wall}(y,t)\,\theta_{wall}(z,t)$ **Rectangular parallelepiped**

Total Transient Heat Transfer

- The transient heat transfer for a two dimensional geometry formed by the intersection of two one-dimensional geometries 1 and 2 is:

$$\left(\frac{Q}{Q_{\max}}\right)_{total,\ 2D} = \left(\frac{Q}{Q_{\max}}\right)_1 + \left(\frac{Q}{Q_{\max}}\right)_2 \left[1 - \left(\frac{Q}{Q_{\max}}\right)_1\right] \qquad \textbf{(4–53)}$$

- Transient heat transfer for a three-dimensional (intersection of three one-dimensional bodies 1, 2, and 3) is:

$$\left(\frac{Q}{Q_{\max}}\right)_{total,\ 3D} = \left(\frac{Q}{Q_{\max}}\right)_1 + \left(\frac{Q}{Q_{\max}}\right)_2 \left[1 - \left(\frac{Q}{Q_{\max}}\right)_1\right] + \left(\frac{Q}{Q_{\max}}\right)_3 \left[1 - \left(\frac{Q}{Q_{\max}}\right)_1\right]\left[1 - \left(\frac{Q}{Q_{\max}}\right)_2\right] \qquad \textbf{(4–54)}$$