3. APPROXIMATIONS AND SIMPLIFIED EQUATIONS SPRING 2015

Fluid dynamics is governed by equations for mass, momentum and energy. The momentum equation for a viscous fluid is called the *Navier-Stokes* equation; it is based upon:

- continuum mechanics;
- the momentum principle;
- shear stress proportional to velocity gradient.

A fluid for which the last is true is called a *Newtonian* fluid; this is the case for most fluids in civil engineering. However, there are important non-Newtonian fluids; e.g. mud, cement slurries, blood, paint, polymer solutions. CFD can still usefully be applied for these.

The full equations are time-dependent, 3-dimensional, viscous, compressible, non-linear and highly coupled. However, in most cases it is possible to simplify the analysis by adopting a reduced equation set. Some common approximations are listed below.

Reduction of dimension:

- steady-state;
- two-dimensional.

Neglect of some fluid property:

- incompressible;
- inviscid.

Simplified forces:

- hydrostatic;
- Boussinesq approximation for density.

Approximations based upon averaging:

- depth-averaging (shallow-water equations);
- Reynolds-averaging (turbulent flows).

3.1 Steady-State *vs* Time-Dependent Flow

Many flows are naturally **time-dependent**. Examples include waves, tides, reciprocating pumps and internal combustion engines.

Other flows have stationary boundaries but become time-dependent because of an **instability**. An important example is vortex shedding around cylindrical objects. Depending on the Reynolds number the instability may or may not go on to fully-developed turbulence.

Some computational solution procedures rely on a time-stepping method to **march to steady state**; examples are transonic flow and open-channel flow (where the mathematical nature of the governing equations is different for Mach or Froude numbers less than or greater than 1).

Thus, there are three major reasons for using the full time-dependent equations:
- time-dependent problem;
- time-dependent instability;
- time-marching to steady state.

Time-dependent methods will be addressed in Section 6.

3.2 Two- Dimensional *vs* Three-Dimensional Flow

Geometry and boundary conditions may dictate that the flow is two-dimensional (at least in the mean). Two-dimensional calculations require considerably less computer resources.

"Two-dimensional" may include "axisymmetric". This is actually easier to achieve in the laboratory than Cartesian 2-dimensionality because of wall boundary layers.

3.3 Incompressible *vs* Compressible Flow

A **flow** (not a **fluid**, note) is said to be *incompressible* if **flow-induced** pressure or temperature changes do not cause significant density changes. Compressibility is most important: (a) in high-speed flow; (b) where there is significant heat input.

Liquid flows are usually treated as incompressible (water hammer being an important exception), but gas flows can also be regarded as incompressible at speeds much less than the speed of sound (a common rule of thumb being Mach number < 0.3).

Density variations within fluids can occur for other reasons, notably from salinity variations (oceans) and temperature variations (atmosphere). These lead to buoyancy forces. Because the density variations are not flow-induced these flows can still be treated as incompressible; i.e. **"incompressible" does not necessarily mean "uniform density"**.

The key differences in CFD between compressible and incompressible flow concern:
(1) whether there is a need to solve a separate energy equation;
(2) how pressure is determined.

Compressible Flow

First Law of Thermodynamics:

$$\textit{change of energy} = \textit{heat input} + \textit{work done on fluid}$$

A transport equation has to be solved for an energy-related variable (e.g. internal energy e or enthalpy $h = e + p/\rho$) in order to obtain the absolute temperature T; for an ideal gas:

$$e = c_v T \qquad \text{or} \qquad h = c_p T$$

c_v and c_p are specific heat capacities at constant volume and constant pressure respectively.

Mass conservation provides a transport equation for ρ, whilst pressure is derived from an *equation of state*; e.g. the ideal-gas law:

$$p = \rho R T$$

For compressible flow it is necessary to solve an energy equation.

In *density-based methods* for compressible CFD:
- mass equation → ρ;
- energy equation → T;
- equation of state → p.

Incompressible Flow

In incompressible flow pressure changes (by definition) cause negligible changes to density. Temperature is not involved and so a separate energy equation is not necessary. The Mechanical Energy Principle:

change of kinetic energy = work done

is equivalent to, and readily derived from, the momentum equation. In the inviscid case it is often expressed as Bernoulli's equation (see the Examples).

Incompressibility implies that density is constant along a streamline:

$$\frac{D\rho}{Dt} = 0$$

but may vary between streamlines (e.g. due to salinity differences). Conservation of mass is then replaced by conservation of volume:

$$\sum_{faces} (\textit{volume flux}) = 0 \qquad \text{or} \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Pressure is not derived from a thermodynamic relation but from the requirement that solutions of the momentum equation be mass-consistent (Section 5).

In incompressible flow it is <u>not</u> necessary to solve a separate energy equation.

In *pressure-based methods* for incompressible CFD:
- incompressibility → volume is conserved;
- requiring solutions of momentum equation to be mass-consistent → equation for p.

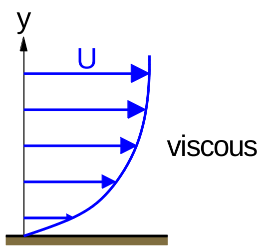
3.4 Inviscid *vs* Viscous Flow

If viscosity is neglected, the *Navier-Stokes equations* become the *Euler equations*.

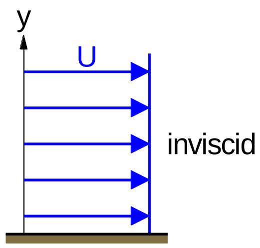
Consider streamwise momentum in a developing 2-d boundary layer:

$$\rho(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y})=-\frac{\partial p}{\partial x}+\mu\frac{\partial^2 u}{\partial y^2}$$

mass × *acceleration* = *pressure force* + *viscous force*



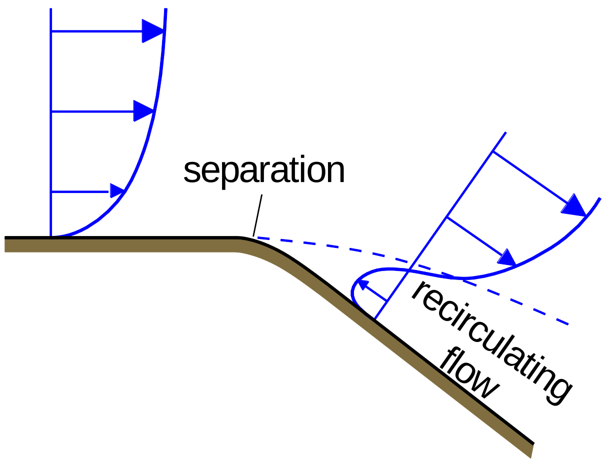
Dropping the viscous term reduces the order of the highest derivative from 2 to 1 and hence one less boundary condition is required.



- Viscous (real) flows require a **no-slip** (zero-velocity) condition at rigid walls – the *dynamic* boundary condition.
- Inviscid (ideal) flows require only the velocity component normal to the wall to be zero – the *kinematic* boundary condition. The wall shear stress is zero.

Although its magnitude is small, and consequently its direct influence via the shear stress is tiny, viscosity can have a global influence out of all proportion to its size. The most important effect is *flow separation*, where the viscous boundary layer required to satisfy the non-slip condition is first slowed and then reversed by an adverse pressure gradient. Boundary-layer separation has two important consequences:

- major disturbance to the flow;
- a large increase in pressure drag.



3.4.1 Inviscid Approximation: Potential Flow

Velocity Potential, ϕ

In **inviscid** flow it may be shown[1] that the velocity components can be written as the gradient of a single scalar variable, the *velocity potential* ϕ:

$$u = \frac{\partial \phi}{\partial x}, \qquad v = \frac{\partial \phi}{\partial y}, \qquad w = \frac{\partial \phi}{\partial z} \qquad \text{(concisely written: } \mathbf{u} = \nabla \phi\text{)}$$

Substituting these into the continuity equation for incompressible flow:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad \text{(concisely written: } \nabla \bullet \mathbf{u} = 0\text{)}$$

gives

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

which is often written

$$\nabla^2 \phi = 0 \qquad (\textit{Laplace's equation}) \qquad (1)$$

Stream Function, ψ

In **2-d incompressible flow** we will see in Section 9 that there exists a function ψ called the *stream function* such that

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}$$

If the flow is also **inviscid** then it may be shown that the fluid is irrotational and

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0$$

Substituting the expressions for u and v into this gives an equation for ψ:

$$\nabla^2 \psi = 0 \qquad (\textit{Laplace's equation}) \qquad (2)$$

In both cases above the entire flow is completely determined by a **single scalar** field (ϕ or ψ) satisfying Laplace's equation. Moreover, since Laplace's equation occurs in many branches of physics (electrostatics, heat conduction, gravitation, optics, ...) many good solvers exist.

Velocity components u, v and w are obtained by differentiating ϕ or ψ. Pressure is then recoverable from Bernoulli's equation:

$$p + \tfrac{1}{2}\rho U^2 = \textit{constant} \text{ (along a streamline)}$$

where U is the magnitude of velocity.

The potential-flow assumption often gives an adequate description of the flow and pressure fields for real fluids, except very close to solid surfaces where viscous forces are significant. It is useful, for example, in calculating the lift force on aerofoils and in wave theory (Hydraulics 3). However, in ignoring viscosity it implies that there are neither tangential stresses on boundaries nor flow separation, which leads to the erroneous conclusion (*D'Alembert's Paradox*) that an object moving through a fluid experiences no drag.

[1] Since pressure acts perpendicularly to a surface and cannot impart rotation, an **inviscid** fluid can be regarded as irrotational ($\nabla \wedge \mathbf{u} = 0$), and so the velocity field can be written as the gradient of a scalar function.

3.5 Hydrostatic *vs* Non-Hydrostatic Flow

The equation for the vertical component of momentum can be written:

$$\rho \frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{\partial p}{\partial z} - \rho g + viscous\ \ forces$$

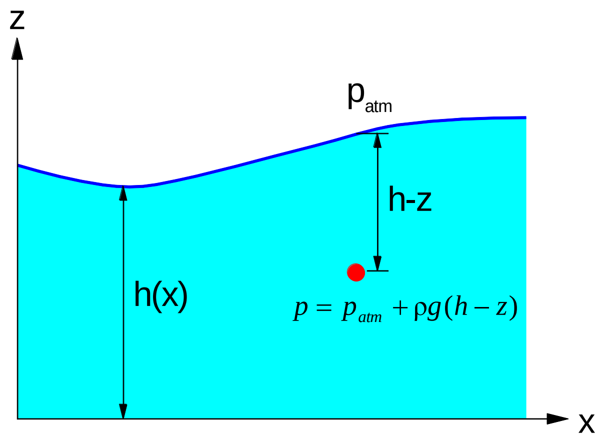
For large horizontal scales the vertical acceleration Dw/Dt is much less than g and the viscous forces are small. The balance of terms is the same as the hydrostatic law in a stationary fluid:

$$\frac{\partial p}{\partial z} \approx -\rho g \qquad \text{i.e. pressure forces balance weight.}$$

With this approximation, in constant-density flows with a free surface the pressure is determined everywhere by the position of the free surface:

$$p = p_{atm} + \rho g(h - z)\,, \qquad \text{where } h = h(x, y)$$

This results in a huge saving in computational time because the position of the surface automatically determines the pressure field without the need for a separate pressure equation.



The hydrostatic approximation is widely used in conjunction with the depth-averaged shallow-water equations (see Section 3.7 below and Dr Rogers' part of the course). However, it is not used in general-purpose flow solvers because it limits the type of flow that can be computed.

3.6 Boussinesq Approximation for Density[2]

Since for constant-density flows pressure and gravitational forces in the vertical momentum equation can be combined through the *piezometric pressure* $p^* = p + \rho g z$, gravity need not explicitly enter the flow equations unless:

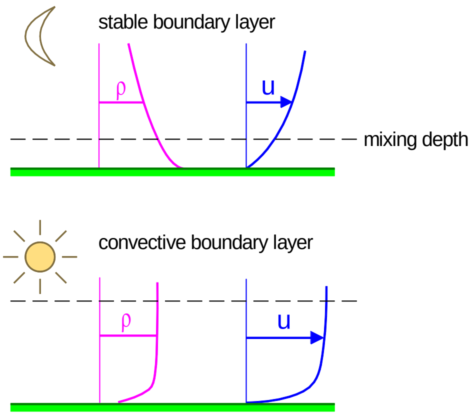
- pressure enters the boundary conditions (e.g. at a free surface, $p = p_{atm}$);
- there is variable density.

Density variations may arise even at low speeds because of changes in temperature or humidity (atmosphere), or salinity (water) which give rise to *buoyancy forces*.

Temperature variations in the atmosphere, brought about by surface (or cloud-top) heating or cooling, are responsible for significant changes in airflow and turbulence.

- On a cold night the atmosphere is *stable*. Cool, dense air collects near the surface and vertical motions are suppressed; the boundary-layer depth is 100 m or less.
- On a warm day the atmosphere is *unstable*. Surface heating produces warm; light air near the ground and convection occurs; the boundary layer may be 2 km deep.

Vertical dispersion of pollution is very different in the two cases.



If density ρ is a function of some scalar θ (typically, temperature or, in the oceans, salinity), then the **relative** change in density is proportional to the change in θ; i.e.

$$\frac{\rho - \rho_0}{\rho_0} = -\alpha\,(\theta - \theta_0) \qquad \text{or} \qquad \rho = \rho_0 - \rho_0 \alpha(\theta - \theta_0)$$

where θ_0 and ρ_0 are reference scalar and density and α is the coefficient of expansion. (The sign adopted here is that for temperature, where an increase in temperature leads to a reduction in density – the opposite would be true for salinity-driven density changes.)

The *Boussinesq approximation for density* amounts to retaining density variations in the gravitational term (which gives rise to buoyancy forces) but disregarding them in the inertial (mass × acceleration) term; i.e. in the vertical momentum equation:

$$\begin{aligned} \rho \frac{\mathrm{D}w}{\mathrm{D}t} &= -\frac{\partial p}{\partial z} - \rho g \\ &= -\frac{\partial p}{\partial z} - \rho_0 g - (\rho - \rho_0) g \end{aligned}$$

replace ρ on the LHS by ρ_0. The part of the weight resulting from the constant reference density ρ_0 is usually subsumed into a modified pressure p^*, whilst the varying part of the density is usually written in terms of whatever scalar θ (temperature, salinity, …) is causing the density difference:

$$\rho_0 \frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{\partial p^*}{\partial z} + \underbrace{\rho_0 \alpha(\theta - \theta_0) g}_{buoyancy\,force} \qquad \text{where} \qquad p^* = p + \rho_0 g z$$

The approximation is justified if relative density variations are not too large; i.e. $\Delta\rho/\rho_0 \ll 1$. This condition is usually satisfied in the atmosphere and oceans.

[2] Note that several other very-different approximations are also referred to as *the* Boussinesq approximation in different contexts – e.g. shallow-water equations or eddy-viscosity turbulence models.

Actually, whilst the Boussinesq approximation may be necessary to obtain exact theoretical solutions, it is not particularly important in general-purpose CFD because the momentum and scalar transport equations are solved iteratively and any scalar-dependent density variations are easily incorporated in the iterative procedure.

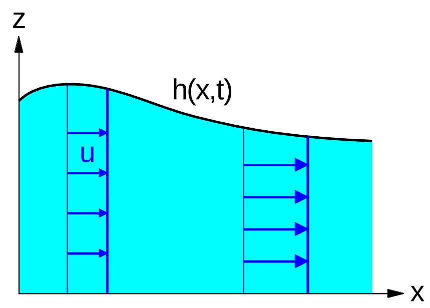
3.7 Depth-Averaged (Shallow-Water) Equations

This approximation is used for flow of a constant-density fluid with a free surface, where the depth of fluid is small compared with typical horizontal scales.

In this "hydraulic" approximation, the fluid can be regarded as quasi-2d with:

- horizontal velocity components *u*, *v*;
- depth of water, *h*.

Note that depth *h* may vary due to changes in level of free-surface, bed or both.



By applying mass and momentum principles to a vertical column of constant-density fluid of variable depth *h*, the depth-integrated equations governing the motion can be written (for the one-dimensional case and in conservative form) as:

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0$$

$$\frac{\partial (uh)}{\partial t} + \frac{\partial (u^2 h)}{\partial x} = -\frac{\partial (\frac{1}{2} g h^2)}{\partial x} + \frac{1}{\rho}(\tau_{surface} - \tau_{bed})$$

The $\frac{1}{2} g h^2$ term comes from ($1/\rho$ times) the hydrostatic pressure force per unit width on a water column of height *h*; i.e. average hydrostatic pressure ($\frac{1}{2}\rho g h$) × area ($h \times 1$). The final term is the net effect of surface stress (due to wind) and the bed shear stress (due to friction). These equations are derived in the Examples and in Dr Rogers' part of this course.

The resulting *shallow-water* (or *Saint-Venant*) *equations* are mathematically similar to those for a compressible gas. There are direct analogies between

- discontinuities: hydraulic jumps (shallow flow) and shocks (compressible flow);
- critical flow over a weir (shallow) or gas flow through a throat (compressible).

In both cases there is a characteristic wave speed ($c = \sqrt{gh}$ in the hydraulic case; $c = \sqrt{\gamma p/\rho}$ in compressible flow). Depending on whether this is greater or smaller than the flow velocity determines whether disturbances can propagate upstream and hence the nature of the flow. The ratio of flow speed to wave speed is known as:

Froude number: $\mathrm{Fr} = \dfrac{u}{\sqrt{gh}}$ in shallow flow

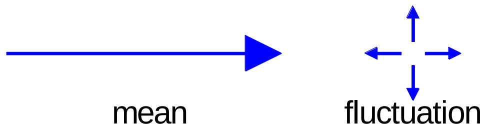
Mach number: $\mathrm{Ma} = \dfrac{u}{c}$ in compressible flow

3.8 Reynolds-Averaged Equations (Turbulent Flow)

The majority of flows encountered in engineering are turbulent. Most, however, can be regarded as time-dependent, three-dimensional *fluctuations* superimposed on a much simpler *mean* flow. Usually, we are only interested in the mean quantities, rather than details of the time-dependent flow.

The process of *Reynolds-averaging* (named after Osborne Reynolds[3], first Professor of Engineering at Owens College, later to become the University of Manchester) starts by decomposing each flow variable into mean and turbulent parts:

$$u = \underset{\textit{mean}}{\bar{u}} + \underset{\textit{fluctuation}}{u'}$$



The "mean" may be a *time average* (this is usually what is measured in the laboratory) or an *ensemble average* (a probabilistic mean over a large number of identical experiments).

When the Navier-Stokes equation is averaged, the result is (see Section 7):

- an equivalent equation for the *mean* flow,
  except for
- turbulent fluxes, $-\rho\overline{u'v'}$ etc. (called the *Reynolds stresses*) which provide a net transport of momentum.

For example, the viscous shear stress

$$\tau_{visc} = \mu\frac{\partial\bar{u}}{\partial y}$$

is supplemented by an additional turbulent stress:

$$\tau_{turb} = -\rho\overline{u'v'}$$

In order to solve the mean-flow equations, a *turbulence model* is required to supply these turbulent stresses. Popular models exploit an analogy between viscous and turbulent transport and employ an *eddy viscosity* μ_t to supplement the molecular viscosity. Thus,

$$\tau = \mu\frac{\partial\bar{u}}{\partial y} - \rho\overline{u'v'} \quad\rightarrow\quad (\mu + \mu_t)\frac{\partial\bar{u}}{\partial y}$$

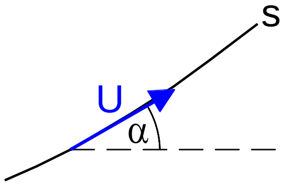
This is readily incorporated into the mean momentum equation via a (position-dependent) *effective viscosity*. However, actually specifying μ_t is by no means trivial – see the lectures on turbulence modelling (Sections 7 and 8).

[3] Osborne Reynolds' experimental apparatus – including that used in his famous pipe-flow experiments – is on display in the basement of the George Begg building at the University of Manchester. A modern replica is in the George Begg foyer.

Examples

Q1.
Discuss the circumstances under which a fluid flow can be approximated as:
(a) incompressible;
(b) inviscid.

Q2.
By resolving forces along a streamline, the steady-state momentum equation for an inviscid fluid can be written



$$\rho U \frac{\partial U}{\partial s} = -\frac{\partial p}{\partial s} - \rho g \sin \alpha$$

where U is the velocity magnitude, s is the distance along a streamline and α is the angle between local velocity and the horizontal. Assuming incompressible flow, derive Bernoulli's equation. (This question demonstrates that, for incompressible flow, the mechanical energy principle can be derived directly from the momentum equation.)

Q3.
A velocity field is given by the velocity potential $\phi = x^2 - y^2$.
(a) Calculate the velocity components u and v.
(b) Calculate the acceleration.
(c) Calculate the corresponding streamfunction, ψ.
(d) Sketch the streamlines and suggest a geometry in which one might expect this flow.

Q4.
For incompressible flow in a rotating reference frame the force per unit volume, **f**, is the sum of pressure, gravitational, Coriolis and viscous forces:

$$\mathbf{f} = -\nabla p - \rho g \mathbf{e}_z - 2\rho \mathbf{\Omega} \wedge \mathbf{u} + \mu \nabla^2 \mathbf{u}$$

where $\mathbf{e}_z$ is a unit vector in the z direction and $\mathbf{\Omega}$ is the angular velocity of the rotating frame.

(a) If the density is uniform, show that pressure and gravitational forces can be combined in a *piezometric* pressure (which should be defined).

(b) If density variations arise from a scalar field, describe the "Boussinesq" approximation in this context and give an application in which it is used.

(c) Show how the momentum equation (with Boussinesq approximation for density) can be non-dimensionalised in terms of densimetric Froude number, Rossby number and Reynolds number:

$$\mathrm{Fr} = \frac{U_0}{\sqrt{(\Delta\rho/\rho_0) g L_0}}, \qquad \mathrm{Ro} = \frac{U_0}{\Omega L_0}, \qquad \mathrm{Re} = \frac{\rho_0 U_0 L_0}{\mu}$$

where ρ_0, L_0, U_0 are characteristic density, length and velocity scales, respectively, and $\Delta\rho$ is a typical magnitude of density variation.

Q5.

(a) In flow with a free surface, by taking a control volume as a column of (time-varying) height $h(x,y,t)$ and horizontal cross-section $\Delta x \times \Delta y$, assuming that density is constant, τ_{13} and τ_{23} are the only significant stress components, the horizontal velocity field may be replaced by the depth-averaged velocity (u, v) and the pressure is hydrostatic, derive the shallow-water equations for continuity and x-momentum in the form

$$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} + \frac{\partial (hv)}{\partial y} = 0$$

$$\frac{\partial (hu)}{\partial t} + \frac{\partial (hu^2)}{\partial x} + \frac{\partial (hvu)}{\partial y} = -gh\frac{\partial z_s}{\partial x} + \frac{\tau_{13}(surface) - \tau_{13}(bed)}{\rho}$$

(b) Provide an alternative derivation by integrating the continuity and horizontal momentum equations for incompressible flow:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial (\rho u)}{\partial t} + \frac{\partial (\rho u^2)}{\partial x} + \frac{\partial (\rho v u)}{\partial y} + \frac{\partial (\rho w u)}{\partial z} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{13}}{\partial z}$$

over a depth $h = z_s - z_b$.

For part (b) you will need the boundary condition that the top and bottom surfaces $z = z_s(x, y)$ and $z = z_b(x, y)$ are material surfaces:

$$\frac{\mathrm{D}}{\mathrm{D}t}(z - z_s) = 0 \qquad \text{or} \qquad w - \frac{\partial z_s}{\partial t} - u\frac{\partial z_s}{\partial x} - v\frac{\partial z_s}{\partial y} = 0 \quad \text{on } z = z_s$$

and similarly for z_b, together with Leibniz' Theorem for differentiating an integral:

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{a(x)}^{b(x)} f(x') \ \mathrm{d}x' = \int_{a(x)}^{b(x)} \frac{\partial f}{\partial x'} \ \mathrm{d}x' + f(b)\frac{\mathrm{d}b}{\mathrm{d}x} - f(a)\frac{\mathrm{d}a}{\mathrm{d}x}$$

Note: this is easily extended to consider additional forces such as Coriolis forces and other stress terms (“*horizontal diffusion*”). *Dr Rogers will cover this in the second part of the course*.