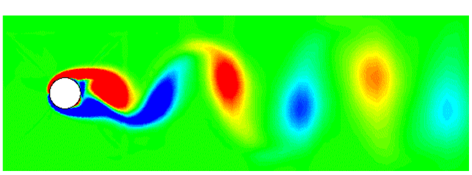
6. Time-Dependent Methods

Why Perform Time-Dependent Calculations?

- Solve a time-dependent problem

  or

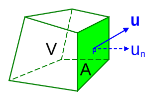
- Iterate toward steady state



Time-Dependent Scalar-Transport Equation

Conservation:

$$\frac{\mathrm{d}}{\mathrm{d}t}(amount) + net\ \ flux = source$$



$$amount \;=\; (\rho V)\phi_P$$

$$net\ \ flux - source \;=\; a_P\phi_P - \sum a_F\phi_F - b_P$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho V\phi_P) + a_P\phi_P - \sum a_F\phi_F = b_P$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = F(t,\phi)$$

1st-order in time; solve by **time-marching**

General Methods For 1st-Order Differential Equations

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = F(t,\phi)$$

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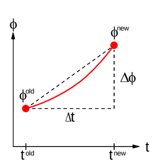
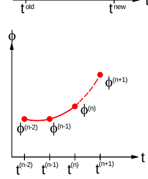
One-step methods:

- given $\phi^{(n)}$, find $\phi^{(n+1)}$



Multi-step methods:

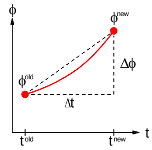
- given $\phi^{(n)}$, $\phi^{(n-1)}$, $\phi^{(n-2)}$, …, find $\phi^{(n+1)}$



One-Step Methods

$$\frac{d\phi}{dt} = F$$

F is the **gradient**.



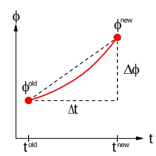
Replace infinitessimals by finite differences:

$$\frac{d\phi}{dt} = F \longrightarrow \frac{\Delta\phi}{\Delta t} = F^{av} \longrightarrow \Delta\phi = F^{av}\,\Delta t$$

$$\phi^{new} = \phi^{old} + F^{av}\Delta t$$

One-Step Methods

$$\frac{d\phi}{dt} = F \qquad \phi^{new} = \phi^{old} + F^{av}\Delta t$$



1. **Forward Differencing** (explicit): $F^{av} = F^{old}$
2. **Backward Differencing** (implicit): $F^{av} = F^{new}$
3. **Crank-Nicolson** (semi-implicit): $F^{av} = \frac{1}{2}(F^{old} + F^{new})$

Example

The following differential equation is to be solved on the interval $[0,1]$:

$$\frac{d\phi}{dt} = t - \phi\,, \qquad \phi(0) = 1$$

Solve this numerically, with a step size $\Delta t = 0.2$ using:

(a) forward differencing;
(b) backward differencing;
(c) Crank-Nicolson.

Solve the equation analytically and compare with the numerical approximations.

Forward Differencing

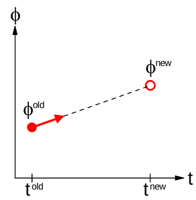
(**Explicit** or **Forward Euler** Method)

Estimate the average gradient from the **start** of the timestep

$$\frac{d\phi}{dt} = F \longrightarrow \frac{\Delta\phi}{\Delta t} = F^{old}$$

$$\Delta\phi = F^{old}\Delta t$$

$$\boxed{\phi^{new} = \phi^{old} + F^{old}\Delta t}$$



Forward Differencing: Assessment

$$\phi^{new} = \phi^{old} + F^{old}\Delta t$$

For

- Easy to implement (because explicit)

Against

- Only 1st-order accurate in Δt
- In CFD, timestep restrictions

Backward Differencing

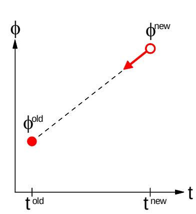
(**Implicit** or **Backward-Euler** Method)

Estimate the average gradient from the **end** of the timestep

$$\frac{d\phi}{dt} = F \longrightarrow \frac{\Delta\phi}{\Delta t} = F^{new}$$

$$\Delta\phi = F^{new}\Delta t$$

$$\boxed{\phi^{new} = \phi^{old} + F^{new}\Delta t}$$



Backward Differencing: Assessment

$$\phi^{new} = \phi^{old} + F^{new}\Delta t$$

For

- In CFD, no timestep restrictions

Against

- Implicit; usually requires iteration
- Only 1st-order accurate in Δt

Crank-Nicolson

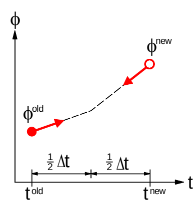
(Semi-Implicit or Time-Centred Method)

Use the **average** of the gradients at the start and end of the timestep

$$\frac{d\phi}{dt} = F \longrightarrow \frac{\Delta\phi}{\Delta t} = \frac{1}{2}(F^{old} + F^{new})$$

$$\Delta\phi = \frac{1}{2}(F^{old} + F^{new})\Delta t$$

$$\phi^{new} = \phi^{old} + \frac{1}{2}(F^{old} + F^{new})\Delta t$$



Crank-Nicolson: Assessment

$$\phi^{new} = \phi^{old} + \frac{1}{2}(F^{old} + F^{new})\Delta t$$

For

- 2nd-order accurate in Δt

Against

- Implicit; usually requires iteration
- In CFD, timestep restrictions

Example

The equation

$$\frac{d\phi}{dt} = t - \phi^4, \qquad \phi(0) = 2$$

is to be solved numerically, using a timestep $\Delta t = 0.1$. Solve this equation up to time $t = 0.4$ using the following approaches to time-marching:

(a) forward-differencing ("fully-explicit");
(b) backward-differencing ("fully-implicit");
(c) centred-differencing ("semi-implicit").

Note. Be very careful how you rearrange the implicit schemes for iteration.

Refined Methods

$$\frac{d\phi}{dt} = F(t, \phi) \longrightarrow \Delta\phi = \Delta t\, F$$

Modified Euler:

$$\begin{aligned}
\Delta\phi_1 &= \Delta t\, F(t^{old}, \phi^{old}) \\
\Delta\phi_2 &= \Delta t\, F(t^{old} + \Delta t, \phi^{old} + \Delta\phi_1) \\
\Delta\phi &= \tfrac{1}{2}(\Delta\phi_1 + \Delta\phi_2)
\end{aligned}$$

Runge-Kutta:

$$\begin{aligned}
\Delta\phi_1 &= \Delta t\, F(t^{old}, \phi^{old}) \\
\Delta\phi_2 &= \Delta t\, F(t^{old} + \tfrac{1}{2}\Delta t, \phi^{old} + \tfrac{1}{2}\Delta\phi_1) \\
\Delta\phi_3 &= \Delta t\, F(t^{old} + \tfrac{1}{2}\Delta t, \phi^{old} + \tfrac{1}{2}\Delta\phi_2) \\
\Delta\phi_4 &= \Delta t\, F(t^{old} + \Delta t, \phi^{old} + \Delta\phi_3) \\
\Delta\phi &= \tfrac{1}{6}(\Delta\phi_1 + 2\Delta\phi_2 + 2\Delta\phi_3 + \Delta\phi_4)
\end{aligned}$$

One-Step Methods in CFD

One-Step Methods in CFD

Scalar-transport equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}(amount) + net\ \ flux = source$$

$$\frac{(\rho V \phi_P)^{new} - (\rho V \phi_P)^{old}}{\Delta t} + (net\ \ flux - source)^{av} = 0$$

$$net\ \ flux - source = a_P \phi_P - \sum a_F \phi_F - b_P$$

Methods differ in the time level at which flux and source are evaluated

Forward Differencing

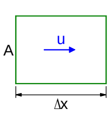
Conservation: $\frac{\mathrm{d}}{\mathrm{d}t}(amount) = source - net\ \ flux$

Forward differencing: $\frac{(\rho V \phi_P)^{new} - (\rho V \phi_P)^{old}}{\Delta t} = \left[b_P - a_P \phi_P + \sum a_F \phi_F\right]^{old}$

Rearrange (dropping “new”): $\frac{\rho V}{\Delta t}\phi_P = \left[(\frac{\rho V}{\Delta t} - a_P)\phi_P + b_P + \sum a_F \phi_F\right]^{old}$

- Fully explicit (easy to implement)
- Timestep restriction for boundedness: $\Delta t \le \frac{\rho V}{a_P}$

Case: 1-d Advection



$$\frac{\mathrm{d}}{\mathrm{d}t}(amount) + net\ \ flux = 0$$

$$\frac{(\rho V \phi_P)^{new} - (\rho V \phi_P)^{old}}{\Delta t} + C\phi_e^{old} - C\phi_w^{old} = 0 \qquad \text{(pure advection)}$$

$$\frac{\rho V}{\Delta t}(\phi_P - \phi_P^{old}) + C(\phi_P^{old} - \phi_W^{old}) = 0 \qquad \text{(upwind differencing)}$$

$$\frac{\rho A \Delta x}{\Delta t}(\phi_P - \phi_P^{old}) + \rho u A(\phi_P^{old} - \phi_W^{old}) = 0$$

$$\phi_P = (1 - \frac{u\Delta t}{\Delta x})\phi_P^{old} + \frac{u\Delta t}{\Delta x}\phi_W^{old}$$

Requirement for boundedness: **Courant number** $\frac{u\Delta t}{\Delta x} \le 1$

Backward Differencing

Conservation: $\frac{\mathrm{d}}{\mathrm{d}t}(amount) = source - net\ \ flux$

Backward-differencing: $\frac{(\rho V \phi_P)^{new} - (\rho V \phi_P)^{old}}{\Delta t} = \left[b_P - a_P\phi_P + \sum a_F\phi_F\right]^{new}$

Rearrange: $(\frac{\rho V}{\Delta t} + a_P)\phi_P - \sum a_F\phi_F = b_P + (\frac{\rho V}{\Delta t}\phi_P)^{old}$

- Implicit (but better than steady state)
- No timestep restrictions
- Implemented by modifying matrix coefficients

Crank-Nicolson

Conservation: $\frac{\mathrm{d}}{\mathrm{d}t}(amount) = source - net\ \ flux$

Time-centred-differencing:

$$\frac{(\rho V \phi_P)^{new} - (\rho V \phi_P)^{old}}{\Delta t} = \frac{1}{2}\left[b_P - a_P\phi_P + \sum a_F\phi_F\right]^{old} + \frac{1}{2}\left[b_P - a_P\phi_P + \sum a_F\phi_F\right]^{new}$$

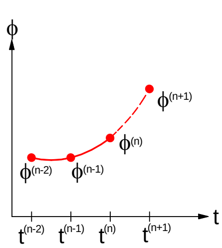
Rearrange:

$$(\frac{\rho V}{\Delta t} + \tfrac{1}{2}a_P)\phi_P - \tfrac{1}{2}\sum a_F\phi_F = \tfrac{1}{2}b_P + \left[(\frac{\rho V}{\Delta t} - \tfrac{1}{2}a_P)\phi_P + \tfrac{1}{2}(b_P + \sum a_F\phi_F)\right]^{old}$$

$$(2\frac{\rho V}{\Delta t} + a_P)\phi_P - \sum a_F\phi_F = b_P + \left[(2\frac{\rho V}{\Delta t} - a_P)\phi_P + (b_P + \sum a_F\phi_F)\right]^{old}$$

- Implicit
- Implemented by modifying coefficients
- Timestep restriction for boundedness: $\Delta t \le 2\frac{\rho V}{a_P}$

Multi-Step Methods in CFD



Problems:

- Storage
- Start-up

Gear's Scheme

$$\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^{(n)} = \frac{3\phi^{(n)} - 4\phi^{(n-1)} + \phi^{(n-2)}}{2\Delta t}$$

2nd-order accurate in Δt

Predictor-Corrector Methods

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = F \longrightarrow \Delta\phi = \Delta t\ F$$

Adams-Bashforth predictor:

$$\phi^{n+1}_{pred} = \phi^n + \tfrac{1}{24}\Delta t\,[-9F^{n-3} + 37F^{n-2} - 59F^{n-1} + 55F^n]$$

Adams-Moulton corrector:

$$\phi^{n+1} = \phi^n + \tfrac{1}{24}\Delta t[F^{n-2} - 5F^{n-1} + 19F^n + 9F^{n+1}_{pred}]$$

4th-order accurate in Δt

Uses of Time-Marching

- Solve a genuinely time-dependent problem:
  - need **accuracy**
  - require **global** timestep
- Iterate toward steady state:
  - need **boundedness** and **stability**, not accuracy
  - can use **local** timestep

Summary (1)

- The fluid-flow equations are 1st-order in time, and are solved by time-marching
- Time-marching methods can be either:
  - explicit (direct update)
  - implicit (require iteration)
- One-step methods:
  - forward differencing (1st-order, explicit)
  - backward differencing (1st-order, implicit)
  - Crank-Nicolson (2nd-order, semi-implicit)

Summary (2)

- One-step methods are easily implemented by modifying matrix coefficients
- Explicit schemes have time-step restrictions
  - Courant number, $c = u\Delta t/\Delta x$
- Multi-step methods are less common in CFD
- The timestep Δt may be global or local