Answers 8

Q1.
(a)

$$\tau_w = \rho u_\tau^2 \qquad \text{or} \qquad u_\tau = \sqrt{\tau_w / \rho}$$

(b) Near to the wall the *turbulent* stress can be neglected, whilst the stress is effectively equal to that at the wall, $\tau = \tau_w$. Hence,

$$\tau_w = \mu \frac{\partial U}{\partial y}, \qquad U(0) = 0$$

Rearrange as

$$\frac{\partial U}{\partial y} = \frac{\tau_w}{\mu}$$

which integrates to give

$$U = \frac{\tau_w}{\mu} y$$

(In order that $U = 0$ at $y = 0$, the constant of integration is 0.)

(c) At larger distances the *viscous* stress can be ignored, but τ is still approximately equal to τ_w. Hence,

$$\tau_w = -\rho \overline{uv} = \mu_t \frac{\partial U}{\partial y}$$

Using the definition of friction velocity u_τ and eddy viscosity μ_t:

$$\begin{aligned} \rho u_\tau^2 &= \rho u_0 l_m \frac{\partial U}{\partial y} \\ &= \rho l_m \left| \frac{\partial U}{\partial y} \right| l_m \frac{\partial U}{\partial y} \end{aligned}$$

$\partial U / \partial y$ is positive, so that

$$u_\tau^2 = \left(l_m \frac{\partial U}{\partial y} \right)^2$$

Hence,

$$\frac{\partial U}{\partial y} = \frac{u_\tau}{l_m} \quad = \frac{u_\tau}{\kappa y}$$

Integrating:

$$U = \frac{u_\tau}{\kappa} (\ln y + constant)$$

Absorbing the constant of integration into the logarithm,

$$\frac{U}{u_\tau} = \frac{1}{\kappa} \ln E \frac{y u_\tau}{\nu}$$

(d) In the viscous sublayer,

$$U = \frac{\tau_w}{\mu} y$$

$$\Rightarrow \qquad U = \frac{\rho u_\tau^2}{\mu} y$$

$$\Rightarrow \qquad \frac{U}{u_\tau} = \frac{u_\tau y}{\nu}$$

Hence, in the viscous sublayer:

$$U^+ = y^+$$

In the log layer,

$$U^+ = \frac{1}{\kappa} \ln E y^+$$

The non-dimensional velocity and distance from the boundary are given by

$$U^+ \equiv \frac{U}{u_\tau}, \qquad y^+ = \frac{y u_\tau}{\nu}$$

(e) In the log-law region,

$$-\overline{uv} = \frac{\tau_w}{\rho} = u_\tau^2 \qquad \text{and} \qquad \frac{\partial U}{\partial y} = \frac{u_\tau}{\kappa y}$$

Hence,

$$-\overline{uv} \frac{\partial U}{\partial y} = u_\tau^2 \times \frac{u_\tau}{\kappa y} \quad = \frac{u_\tau^3}{\kappa y}$$

Local equilibrium implies that rate of production and dissipation are equal; i.e.

$$P^{(k)} = \varepsilon$$

Q2.

On dimensional grounds, $u_0 \propto k^{1/2}$. If we take

$$u_0 = k^{1/2}$$

then

$$l_0 = C_\mu \frac{k^{3/2}}{\varepsilon}$$

Actually, the constant of proportionality can be factored between u_0 and l_0 in various ways and I (personally) prefer to split C_μ so that

$$u_0 = C_\mu^{1/4} k^{1/2}, \qquad l_0 = \frac{u_0^3}{\varepsilon}$$

This has the advantage of reducing to $u_0 = u_\tau$ and $l_0 = \kappa y$ in an equilibrium boundary layer.

Q3.

(a) Dimensions:

$$[\mu_t] = \frac{[\,force/area]}{[velocity/length]} \quad = \frac{\text{MLT}^{-2}\,/\,\text{L}^2}{\text{LT}^{-1}\,/\,\text{L}} \quad = \text{ML}^{-1}\text{T}^{-1}$$

$$[\rho] = \text{ML}^{-3}$$

$$[k] = [velocity]^2 = \text{L}^2\text{T}^{-2}$$

$$[\varepsilon] = \frac{[k]}{[time]} = \text{L}^2\text{T}^{-3}$$

Since there are 3 fundamental dimensions (M, L and T) and 4 variables there can be only one dimensionless Π group (remember Dimensional Analysis in Hydraulics 2!), which must, therefore, be a constant. Choosing the 3 variables ρ, k and ε as scaling variables, and non-dimensionalising μ_t:

$$\Pi = \mu_t \rho^a k^b \varepsilon^c$$

for some a, b and c.

In terms of the dimensions involved:

$$\text{M}^0\text{L}^0\text{T}^0 = \text{ML}^{-1}\text{T}^{-1}(\text{ML}^{-3})^a(\text{L}^2\text{T}^{-2})^b(\text{L}^2\text{T}^{-3})^c$$

Equating powers:

M: $0 = 1 + a$

L: $0 = -1 - 3a + 2b + 2c$

T: $0 = -1 - 2b - 3c$

The first gives $a = -1$. Adding the second and third:

$$0 = -2 - 3a - c$$

$\Rightarrow \quad c = 1$

Finally,

$$b = \frac{-1-3c}{2} = -2$$

Hence,

$$\mu_t \rho^{-1} k^{-2} \varepsilon = constant$$

or

$$\mu_t = constant \times \rho \frac{k^2}{\varepsilon}$$

(b)

As in part (a) there are 4 variables and 3 independent dimensions, and hence we may assume a single dimensionless variable of the form

$$\mu_t \rho^a k^b \omega^c$$

for some a, b and c.

In terms of the dimensions involved:

$$\text{M}^0\text{L}^0\text{T}^0 = \text{ML}^{-1}\text{T}^{-1}(\text{ML}^{-3})^a(\text{L}^2\text{T}^{-2})^b(\text{T}^{-1})^c$$

Equating powers:

M: $0 = 1 + a$

L: $0 = -1 - 3a + 2b$
T: $0 = -1 - 2b - c$

The first gives $a = -1$. The second gives $b = -1$, and the third gives $c = -1 - 2b = 1$. Hence, as this is the only non-dimensional group,

$$\mu_t \rho^{-1} k^{-1} \omega = constant$$

or

$$\mu_t = constant \times \rho \frac{k}{\omega}$$

Note:

In practice, the constant is usually absorbed into the definition of ω, so that $\mu_t = \rho \frac{k}{\omega}$.

Comparing the expressions for eddy viscosity from parts (a) and (b) gives ω in terms of ε:

$$\omega = \frac{\varepsilon}{C_\mu k}$$

Q4.
For fully-developed flow,

$$\frac{\mathrm{D}\varepsilon}{\mathrm{D}t} = 0 \qquad \text{(no variation in the flow direction)}$$

$$\frac{\partial\varepsilon}{\partial x} = \frac{\partial\varepsilon}{\partial z} = 0 \qquad \text{(variation only in the } y \text{ direction)}$$

$$P^{(k)} = \varepsilon \qquad \text{(given)}$$

The ε transport equation then reduces to:

$$0 = \frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{\nu_t}{\sigma_e}\frac{\mathrm{d}\varepsilon}{\mathrm{d}y}\right) + (C_{\varepsilon 1} - C_{\varepsilon 2})\frac{\varepsilon^2}{k}$$

(dy may replace ∂y since quantities are only functions of y here.)

Swapping the last term to the LHS and substituting the eddy-viscosity formula for ν_t:

$$(C_{\varepsilon 2} - C_{\varepsilon 1})\frac{\varepsilon^2}{k} = \frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{C_\mu}{\sigma_e}\frac{k^2}{\varepsilon}\frac{\mathrm{d}\varepsilon}{\mathrm{d}y}\right)$$

or, since k is constant:

$$(C_{\varepsilon 2} - C_{\varepsilon 1})\frac{\varepsilon^2}{k} = \frac{C_\mu k^2}{\sigma_\varepsilon}\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{1}{\varepsilon}\frac{\mathrm{d}\varepsilon}{\mathrm{d}y}\right) \qquad (*)$$

Now,

$$\varepsilon = \frac{u_\tau^3}{\kappa y}$$

$$\Rightarrow \qquad \frac{\mathrm{d}\varepsilon}{\mathrm{d}y} = -\frac{u_\tau^3}{\kappa y^2}$$

$$\Rightarrow \qquad \frac{1}{\varepsilon}\frac{\mathrm{d}\varepsilon}{\mathrm{d}y} = -\frac{1}{y}$$

$$\Rightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{1}{\varepsilon}\frac{\mathrm{d}\varepsilon}{\mathrm{d}y}\right) = \frac{1}{y^2}$$

Substituting in (*) for k and ε:

$$(C_{\varepsilon 2} - C_{\varepsilon 1})\frac{u_\tau^6}{\kappa^2 y^2}\frac{1}{C_\mu^{-1/2}u_\tau^2} = \frac{C_\mu C_\mu^{-1}u_\tau^4}{\sigma_\varepsilon} \times \frac{1}{y^2}$$

Hence,

$$(C_{\varepsilon 2} - C_{\varepsilon 1})\frac{\sqrt{C_\mu}}{\kappa^2} = \frac{1}{\sigma_\varepsilon}$$

which rearranges to

$$(C_{\varepsilon 2} - C_{\varepsilon 1})\sigma_e\sqrt{C_\mu} = \kappa^2$$

Q5.
From the given quantities for the log-law region:

$$k = C_\mu^{-1/2} u_\tau^2 \qquad \text{and} \qquad \omega = \frac{1}{C_\mu k}\frac{u_\tau^3}{\kappa y} = C_\mu^{-1/2}\frac{u_\tau}{\kappa y}$$

whilst

$$P^{(k)} = \frac{u_\tau^3}{\kappa y} \qquad \text{and} \qquad \nu_t = \frac{k}{\omega} = \kappa u_\tau y$$

Substituting these into the transport equation for ω, and noting that for a fully-developed boundary layer D/D$t \to 0$, whilst $\partial/\partial y$ is the only non-zero derivative,

$$0 = \frac{\mathrm{d}}{\mathrm{d}y}\left[\frac{\kappa u_\tau y}{\sigma_\omega}\frac{\mathrm{d}}{\mathrm{d}y}(C_\mu^{-1/2}\frac{u_\tau}{\kappa y})\right] + \alpha\frac{u_\tau^3/\kappa y}{\kappa u_\tau y} - \beta C_\mu^{-1}\frac{u_\tau^2}{(\kappa y)^2}$$

Simplifying and multiplying through by $C_\mu^{1/2}\sigma_\omega / u_\tau^2$,

$$0 = \frac{\mathrm{d}}{\mathrm{d}y}\left[y\frac{\mathrm{d}}{\mathrm{d}y}(y^{-1})\right] + (\alpha - \frac{\beta}{C_\mu})\frac{\sigma_\omega\sqrt{C_\mu}}{\kappa^2 y^2}$$

The first term on the RHS is

$$\frac{\mathrm{d}}{\mathrm{d}y}\left[y\frac{\mathrm{d}}{\mathrm{d}y}(y^{-1})\right] = \frac{\mathrm{d}}{\mathrm{d}y}\left[y\times(-y^{-2})\right] = \frac{\mathrm{d}}{\mathrm{d}y}(-y^{-1}) = y^{-2} = \frac{1}{y^2}$$

Hence,

$$0 = \frac{1}{y^2} + (\alpha - \frac{\beta}{C_\mu})\frac{\sigma_\omega\sqrt{C_\mu}}{\kappa^2 y^2}$$

or

$$(\frac{\beta}{C_\mu} - \alpha)\frac{\sigma_\omega\sqrt{C_\mu}}{\kappa^2 y^2} = \frac{1}{y^2}$$

Multiplying through by $\kappa^2 y^2$:

$$(\frac{\beta}{C_\mu} - \alpha)\sigma_\omega\sqrt{C_\mu} = \kappa^2$$

Q6.
(a)

$$P_{22} = -2(\overline{vu}\frac{\partial V}{\partial x} + \overline{vv}\frac{\partial V}{\partial y} + \overline{vw}\frac{\partial V}{\partial z})$$

(b) *Anisotropy* means that the mean-square fluctuations $\overline{u^2}$, $\overline{v^2}$, $\overline{w^2}$ in the different coordinate directions are distinct.

$\overline{u^2}$ tends to be bigger than $\overline{v^2}$ (in flow along a plane wall y = constant) because:

- the production term of $\overline{u^2}$ is bigger ($P_{11} > P_{22}$);
- wall-normal fluctuations $\overline{v^2}$ are selectively damped because the presence of the solid boundary limits vertical movement.

(c)
Eddy viscosity models:
These are based on the assumption that Reynolds stress is proportional to rate of strain. The constant of proportionality is called an *eddy viscosity*.

e.g in simple shear:

$$\tau \equiv -\rho\overline{uv} = \mu_t \frac{\partial U}{\partial y}$$

Advantages:
- Simple to implement in existing viscous solvers (just modify the effective viscosity).
- Additional viscosity aids numerical stability.
- Some theoretical justification in simple flows.

Disadvantages:
- Merely a model!
- Doesn't include key physics; in particular, doesn't model production or advection of different components of stress and hence takes no account of anisotropy.
- Can predict at most one Reynolds stress accurately (because there is only one free parameter, μ_t); hence, unjustified in complex flows where more than one Reynolds-stress component is dynamically significant.

Reynolds-stress transport models
Solve scalar-transport equations for each individual stress component $\overline{u_i u_j}$.
Advantages:
- Key turbulence physics (notably production and advection) is exact, without any need for modelling.

Disadvantages:
- Numerical expense of 6 turbulent transport equations.
- No extra diffusive term to aid numerical stability.
- Many important terms in the stress-transport equations need modelling.

Q7.
(a) $u_\tau = \sqrt{\tau_w / \rho}$

(b) Differentiating (*) with respect to y:

$$\frac{1}{u_\tau}\frac{\partial U}{\partial y} = \frac{1}{\kappa y}$$

Hence,

$$\frac{\partial U}{\partial y} = \frac{u_\tau}{\kappa y}$$

The stress and mean-velocity gradient are related by

$$\tau_w = \rho \nu_t \frac{\partial U}{\partial y}$$

Hence,

$$\nu_t = \frac{\tau_w / \rho}{\partial U / \partial y} \quad = \frac{u_\tau^2}{u_\tau / \kappa y} \quad = \kappa u_\tau y$$

Answer: $\nu_t = \kappa u_\tau y$.

(c) For a mixing-length model,

$$\nu_t = l_m^2 \left| \frac{\partial U}{\partial y} \right|$$

Here,

$$\kappa u_\tau y = l_m^2 \frac{u_\tau}{\kappa y}$$

$\Rightarrow \qquad (\kappa y)^2 = l_m^2$

$\Rightarrow \qquad l_m = \kappa y$

Answer: $l_m = \kappa y$.

(d) If the turbulent fluctuations of velocity about the ensemble mean are denoted (u', v', w') then the turbulent kinetic energy k is given by

$$k = \frac{1}{2}(\overline{u'^2} + \overline{v'^2} + \overline{w'^2})$$

(e)

$$\tau_{23} = \mu_t \left(\frac{\partial V}{\partial z} + \frac{\partial W}{\partial y} \right), \qquad \tau_{31} = \mu_t \left(\frac{\partial W}{\partial x} + \frac{\partial U}{\partial z} \right)$$

$$\tau_{11} = 2\mu_t \frac{\partial U}{\partial x} - \frac{2}{3}\rho k\,, \qquad \tau_{22} = 2\mu_t \frac{\partial V}{\partial y} - \frac{2}{3}\rho k\,, \qquad \tau_{33} = 2\mu_t \frac{\partial W}{\partial z} - \frac{2}{3}\rho k$$

The added term $-\frac{2}{3}\rho k$ in the last three stresses is because $\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = 0$ in incompressible flow, but the sum of the normal stresses is $-2\rho k$; as all directions are equally important, the same term must be added to each stress to give the correct sum.

(f) The dimensions of μ_t, ρ, k and T are as follows:

$$[\mu_t] = \frac{[\tau]}{[\partial U/\partial y]} \quad = \frac{\mathrm{MLT^{-2}/L^2}}{(\mathrm{LT^{-1}})/\mathrm{L}} \quad = \mathrm{ML^{-1}T^{-1}}$$

$$[\rho] = \mathrm{ML^{-3}}$$

$$[k] = [velocity]^2 \quad = \mathrm{L^2T^{-2}}$$

$$[T] = \mathrm{T}$$

If

$$\mu_t = C\rho^\alpha k^\beta T^\gamma$$

then, balancing dimensions,

$$\mathrm{ML^{-1}T^{-1}} = \mathrm{M^\alpha L^{-3\alpha+2\beta} T^{-2\beta+\gamma}}$$

M: $1 = \alpha \quad \Rightarrow \quad \alpha = 1$

L: $-1 = -3\alpha + 2\beta \quad \Rightarrow \quad \beta = 1$

T: $-1 = -2\beta + \gamma \quad \Rightarrow \quad \gamma = 1$

Q8.
(a)

$$\tau_{ij}^{(turb)} = -\rho \overline{u_i' u_j'}$$

where a prime denotes a fluctuating quantity and an overbar an averaged quantity.

(b) Index form:

$$\tau_{ij}^{(turb)} = \mu_t \left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i} \right) - \frac{2}{3}\rho k \delta_{ij}$$

Alternatively, typical normal and shear stresses are

$$\tau_{11}^{(turb)} = 2\mu_t \left(\frac{\partial \overline{u}}{\partial x} \right) - \frac{2}{3}\rho k$$

$$\tau_{12}^{(turb)} = \mu_t \left(\frac{\partial \overline{u}}{\partial y} + \frac{\partial \overline{v}}{\partial x} \right)$$

from which other components follow by permutation.

(c) k is the turbulent kinetic energy (per unit mass), whilst ε is its dissipation rate. They are determined by solving modelled transport equations.

(d)

$$\tau_w = \rho u_\tau^2 \qquad \text{or} \qquad u_\tau = \sqrt{\tau_w / \rho}$$

(e) Eddy-viscosity:

$$\mu_t = \rho C_\mu \frac{k^2}{\varepsilon} \quad = \rho C_\mu \times C_\mu^{-1} u_\tau^4 \times \frac{\kappa y}{u_\tau^3} \quad = \rho(\kappa u_\tau y)$$

Fully turbulent ($\tau = \tau^{(turb)}$), constant-stress ($\tau = \tau_w$) and simple shear imply:

$$\tau_w = \mu_t \frac{\mathrm{d}\overline{u}}{\mathrm{d}y}$$

$$\Rightarrow \qquad \rho u_\tau^2 = \rho(\kappa u_\tau y) \frac{\mathrm{d}\overline{u}}{\mathrm{d}y}$$

$$\Rightarrow \qquad \frac{\mathrm{d}\overline{u}}{\mathrm{d}y} = \frac{u_\tau}{\kappa} \times \frac{1}{y}$$

This integrates to give

$$\overline{u} = \frac{u_\tau}{\kappa} \ln y + constant$$

(f) Reynolds-stress transport models solve separate transport equations for each individual Reynolds stress, rather than relating them to the mean-velocity field.

Advantages of Reynolds-stress models:

- more flow physics (modelled exact equations, rather than just a model);
- exact form of production and history terms in model equations, so should be able to get a better representation of turbulence anisotropy in complex flows.

Disadvantages:

- computationally much more expensive (more transport equations);
- less stable; (no stability-enhancing viscosity term);
- many terms in the stress-transport equations require modelling.

Q9.

(a) Consider the *i*-momentum flux through a face area A in the j direction:

$$momentum\ \ flux = mass\ \ flux \times velocity \quad = (\rho u_i A) u_j \quad = \rho u_i u_j A$$

Using the rules for averaging a product the average momentum flux is

$$\rho \bar{u}_i \bar{u}_j A + \rho \overline{u_i' u_j'} A$$

The first term corresponds to the flux of momentum by the mean velocity field. The additional term $\rho \overline{u_i' u_j'} A$ is the net momentum flux from "lower" (smaller x_j) to "upper" side of the face or, equivalently, $-\rho \overline{u_i' u_j'} A$ represents the net rate of transfer from upper to lower sides. This has the same effect (on the mean transfer of momentum) as a force of the same size, or a stress (force per unit area) of

$$\tau_{ij} = -\rho \overline{u_i' u_j'}$$

(b)

$$-\rho \overline{u'v'} = \mu_t \left(\frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x} \right)$$

$$-\rho \overline{u'^2} = 2\mu_t \frac{\partial \bar{u}}{\partial x} - \frac{2}{3} \rho k$$

where $k = \frac{1}{2}(\overline{u^2} + \overline{v^2} + \overline{w^2})$ is the turbulent kinetic energy.

From the expression for the normal stress,

$$\overline{u'^2} = 2\left(\frac{k}{3} - \nu_t \frac{\partial \bar{u}}{\partial x} \right)$$

where $\nu_t = \mu_t/\rho$. Hence, for a velocity gradient sufficient large that

$$\frac{\partial \bar{u}}{\partial x} > \frac{k}{3\nu_t}$$

then such a model predicts $\overline{u'^2} < 0$. It is physically impossible for a squared (and hence, mean squared) quantity to be negative.

(c) In a fully-developed, zero-pressure-gradient flow there are no changes to mean quantities in the x direction and thus the net force in the x direction is zero:

$$\frac{\partial \tau}{\partial y} = 0$$

or

$$\tau = constant$$

In simple shear, then,

$$\tau = \mu_{eff} \frac{\partial \bar{u}}{\partial y}$$

or

$$\frac{\partial \bar{u}}{\partial y} = \frac{\tau}{\mu_{eff}}$$

(i) If μ_{eff} is constant then the RHS is constant and hence this integrates to give

$$\bar{u} = \frac{\tau}{\mu_{eff}} y + \text{constant}$$

(with the constant of integration being zero if $\bar{u} = 0$ at $y = 0$).

(ii) If $\mu_{eff} = Cy$ (where C must have dimensions of velocity) then

$$\frac{\partial \bar{u}}{\partial y} = \frac{\tau}{C} \frac{1}{y}$$

which integrates to give

$$\bar{u} = \frac{\tau}{C} \ln y + constant$$

i.e. the mean velocity profile is logarithmic.

(d)

(i) k is turbulent kinetic energy (per unit mass), defined by $k = \frac{1}{2}(\overline{u^2} + \overline{v^2} + \overline{w^2})$;
ε is the rate of dissipation of turbulent kinetic energy.

(ii) $\mu_t = C_\mu \rho \frac{k^2}{\varepsilon}$, where C_μ is a constant.

(iii) modelled transport equations are solved for k and ε.

(e) Near a solid boundary:

- mean gradients are very large;
- turbulence becomes very anisotropic (wall-normal components selectively damped);
- molecular viscous forces becomes of comparable importance to momentum transport by turbulent eddies, conflicting with a key premise of the log-law assumptions on which many turbulence models are calibrated.

Two techniques for handling this boundary condition are:

- *wall functions*, with a relatively coarse grid, using simple theory to parameterise the unresolved flow between near-wall node and the boundary;
- *low-Reynolds-number turbulence models*, to resolve flow profiles with a fine grid, including viscosity-dependent modifications to the k and ε equations and eddy-viscosity formulation.