Essay 10

Generalization of Laminar Flow Results by Non-dimensionalization

10.1 Introduction

Although it has been the practice up to now to specify the actual values of the fluid properties and of the dimensional and fluid flow parameters when using CFX, it is more efficient to work with non-dimensional equations, variables, and input parameters. In this essay, it will be demonstrated how to work with the dimensionless forms and to show the great generalization that can be achieved by using these forms. For the present, focus will be directed to fluid flows that are laminar. In a later essay, a detailed analysis will be made to demonstrate that turbulent flows can also be analyzed non-dimensionally.

10.2 The Dimensionless Governing Equations

The equations that govern laminar fluid flows encompass the three equations that result from the application of Newton's Second Law and another equation that guarantees that mass is conserved. It will be sufficient for the present analysis to focus on one of the three equations from Newton's Second Law. Since the mainflow direction is normally referred to as the x direction, it is appropriate to demonstrate the non-dimensional approach for the x-component of Newton's Second Law, Eq. (8.16).

$$\rho\left[\frac{\partial}{\partial x}(u^2)+\frac{\partial}{\partial y}(uv)+\frac{\partial}{\partial z}(uw)\right]=-\frac{\partial p}{\partial x}+\mu\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right] \tag{10.1}$$

To begin the analysis, it is appropriate to select certain known parameters which will serve as the normalizations that create the non-dimensional variables and parameters. With respect to the velocity, it is appropriate to select the mean velocity U as the normalization quantity when consideration is being given to internal flows. By the same token, if external flows are being considered, the freestream velocity U_∞ is the proper scale for the non-dimensionalization. For simplicity, let U stand for both U and U_∞. Therefore, if primes are used for the dimensionless velocities, then:

$$u' = \frac{u}{U}, \qquad v' = \frac{v}{U}, \qquad w' = \frac{w}{U} \tag{10.2}$$

Next, a scale dimension will be chosen to non-dimensionalize the coordinates. For an internal flow, a reasonable scale is the diameter D for a round pipe and D_H for a non-circular duct. In the case of an external flow around an object, a suitable scale could be the length of the object or its thickness. For simplicity, the length scale will be denoted by $\mathcal{L}$. Therefore,

$$x^{'} = \frac{x}{\mathcal{L}}, \qquad y^{'} = \frac{y}{\mathcal{L}}, \qquad z^{'} = \frac{z}{\mathcal{L}} \tag{10.3}$$

For the diameter of a round pipe (where it is natural to take $\mathcal{L} = D$), the non-dimensional diameter $D^{'} = 1$. Furthermore, the dimensionless length of the pipe $L^{'} = L/D$. A similar approach will be used to normalize the dimensions of an object in an external flow.

These non-dimensional quantities will now be introduced into Eq. (10.1). The end result of these substitutions is:

$$\frac{\rho U^2}{\mathcal{L}}\left[\frac{\partial}{\partial x'}(u'^2) + \frac{\partial}{\partial y'}(u'v') + \frac{\partial}{\partial z'}(u'w')\right] = -\frac{1}{\mathcal{L}}\frac{\partial p}{\partial x'} + \frac{\mu U}{\mathcal{L}^2}\left[\frac{\partial^2 u'}{\partial x'^2} + \frac{\partial^2 u'}{\partial y'^2} + \frac{\partial^2 u'}{\partial z'^2}\right] \tag{10.4}$$

To simplify this equation, it is convenient to multiply through by $\frac{\mathcal{L}}{\rho U^2}$, which leads to:

$$\frac{\partial}{\partial x'}(u'^2) + \frac{\partial}{\partial y'}(u'v') + \frac{\partial}{\partial z'}(u'w') = -\frac{1}{\mathcal{L}}\frac{\mathcal{L}}{\rho U^2}\frac{\partial p}{\partial x'} + \frac{\mu U}{\mathcal{L}^2}\frac{\mathcal{L}}{\rho U^2}\left[\frac{\partial^2 u'}{\partial x'^2} + \frac{\partial^2 u'}{\partial y'^2} + \frac{\partial^2 u'}{\partial z'^2}\right] \tag{10.5}$$

For the pressure term, the $\mathcal{L}$ cancels out, and it is appropriate to redefine the pressure as:

$$p^{'} = \frac{p}{\rho U^2} \tag{10.6}$$

Furthermore, the parameters that multiply the bracketed term at the very right of the equation can be simplified to read:

$$\frac{\mu}{\rho U \mathcal{L}} = \frac{\rho U \mathcal{L}}{\mu} = \frac{1}{Re} \tag{10.7}$$

where Re is the Reynolds number of the flow. With these definitions, Eq. (10.5) becomes:

$$\frac{\partial}{\partial x'}(u'^2) + \frac{\partial}{\partial y'}(u'v') + \frac{\partial}{\partial z'}(u'w') = -\frac{\partial p'}{\partial x'} + \frac{1}{Re}\left[\frac{\partial^2 u'}{\partial x'^2} + \frac{\partial^2 u'}{\partial y'^2} + \frac{\partial^2 u'}{\partial z'^2}\right] \tag{10.8}$$

It is useful, at this point, to repeat Eq. (10.1) to enable a convenient comparison with Eq. (10.8).

$$\rho\left[\frac{\partial}{\partial x}(u^2) + \frac{\partial}{\partial y}(uv) + \frac{\partial}{\partial z}(uw)\right] = -\frac{\partial p}{\partial x} + \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right] \tag{10.9}$$

With regard to the software, it has no recognition of whether (u and x) or (u' and x') appear in the governing equation. The software will solve the governing equations regardless of what symbols the investigator may use for the independent and dependent variables. In this light, it is useful to compare Eqs. (10.8 and 9) without considering the primed and un-primed variables as different. What the software **does** require are the numerical values of ρ and μ. If Eq. (10.8) were to be used, those values are:

$$\rho = 1, \qquad \mu = \frac{1}{Re} \tag{10.10}$$

The use of Eq. (10.8) means that the actual values of ρ and μ need not be used as input to the software. Instead, all fluids will be characterized by a common density $\rho = 1$, and the viscosity of all fluids will be equal to $1/Re$.

The impact of the non-dimensionalization goes well beyond the aforementioned simplification of the fluid property information. With regard to dimensions, they, too, are significantly simplified. For example, all pipes have $D = 1$ as their diameter. Pipe lengths need not be specified in terms of physical dimensions. Rather, the lengths are characterized by the value of L/D.

The non-dimensionalization also has to be examined with respect to its possible effects on the boundary conditions. At the wall that bounds the flow, the physical conditions are that the velocities parallel and normal to the surface are zero. When those velocities are made dimensionless, the dimensionless velocities are also zero. At the downstream end of a pipe, the boundary conditions are unchanged, provided that the gauge pressure is taken to be zero. The only boundary condition that will change when non-dimensionalized is the inlet velocity. If the inlet velocity is uniform, then that uniform velocity becomes $U' = 1$. On the other hand, if the inlet velocity is not uniform, then the actual velocity profile has to be normalized by dividing it by the mean velocity U.

It is relevant to examine the effect of non-dimensionalization on the wall shear stress τ_w. For the simplest situation,

$$\tau_w = \mu \frac{\partial u}{\partial y}\bigg|_{wall} \tag{10.11}$$

In this equation, u denotes the velocity that is parallel to the wall, and y is the direction perpendicular to the wall. If this equation is made dimensionless, there follows:

$$\frac{\tau_w}{\rho U^2}\rho U^2 = \frac{\mu U}{D}\frac{\partial(u/U)}{\partial(y/D)}\bigg|_{wall} \tag{10.12}$$

If τ_w' is defined as $\tau_w/\rho U^2$, then:

$$\tau_w' = \frac{1}{Re}\frac{\partial u'}{\partial y'}\bigg|_{wall} \tag{10.13}$$