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1. Turbulence Models For General-Purpose CFD

Turbulence models for general-purpose CFD must be *frame-invariant* – i.e. independent of any particular coordinate system – and hence must be expressed in tensor form. This rules out, for example, simpler models of boundary-layer type.

Turbulent flows are computed either by solving the Reynolds-averaged Navier-Stokes equations with suitable models for turbulent fluxes or by computing the fluctuating quantities directly. The main types are summarised below.

Reynolds-Averaged Navier-Stokes (RANS) Models

- **Linear eddy-viscosity models (EVM)**
  – (deviatoric) turbulent stress proportional to mean strain;
  – eddy viscosity based on turbulence scalars (usually k + one other), determined by solving transport equations.

- **Non-linear eddy-viscosity models (NLEVM)**
  – turbulent stress is a non-linear function of mean strain and vorticity;
  – coefficients depend on turbulence scalars (usually k + one other), determined by solving transport equations;
  – mimic response of turbulence to certain important types of strain.

- **Differential stress models (DSM)**
  – aka Reynolds-stress transport models (RSTM) or second-order closure (SOC);
  – solve (modelled) transport equations for all turbulent fluxes.

Models That Compute Fluctuating Quantities

- **Large-eddy simulation (LES)**
  – compute time-varying fluctuations, but model sub-grid-scale motion.

- **Direct numerical simulation (DNS)**
  – no modelling; resolve the smallest scales of the flow.

2. Linear Eddy-Viscosity Models

2.1 General Form

Stress-strain constitutive relation:

$$-\rho\overline{u_i u_j} = \mu_t\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right)-\frac{2}{3}\rho k\delta_{ij}\,, \qquad \mu_t = \rho\nu_t \qquad (1)$$

The *eddy viscosity* μ_t is derived from turbulent quantities such as the turbulent kinetic energy k and its dissipation rate ε. These quantities are themselves determined by solving scalar-transport equations (see below).

A typical shear stress and normal stress are given by

$$-\rho\overline{uv} = \mu_t\left(\frac{\partial U}{\partial y}+\frac{\partial V}{\partial x}\right)$$

$$-\rho\overline{u^2} = 2\mu_t\frac{\partial U}{\partial x}-\frac{2}{3}\rho k$$

From these the other stress components are easily deduced by inspection/cyclic permutation.

Most eddy-viscosity models in general-purpose CFD codes are of the 2-equation type; (i.e. scalar-transport equations are solved for 2 turbulent scales). The commonest types are k-ε and k-ω models, for which specifications are given below.

2.2 *k*-ε Models

Eddy viscosity:

$$\nu_t = C_\mu\frac{k^2}{\varepsilon} \qquad (2)$$

Scalar-transport equations (non-conservative form):

$$\begin{array}{cccc} \rho\dfrac{\mathrm{D}k}{\mathrm{D}t} = & \dfrac{\partial}{\partial x_i}(\Gamma^{(k)}\dfrac{\partial k}{\partial x_i}) & +\rho(P^{(k)} & -\varepsilon) \\ \rho\dfrac{\mathrm{D}\varepsilon}{\mathrm{D}t} = & \dfrac{\partial}{\partial x_i}(\Gamma^{(\varepsilon)}\dfrac{\partial \varepsilon}{\partial x_i}) & +\rho(C_{\varepsilon 1}P^{(k)} & -C_{\varepsilon 2}\varepsilon)\dfrac{\varepsilon}{k} \\ \substack{rate\ of\ change \\ +\ advection} & diffusion & production & dissipation \end{array} \qquad (3)$$

The diffusivities of k and ε are related to the eddy-viscosity:

$$\Gamma^{(k)} = \mu+\frac{\mu_t}{\sigma^{(k)}}\,, \qquad \Gamma^{(\varepsilon)} = \mu+\frac{\mu_t}{\sigma^{(\varepsilon)}}$$

The rate of production of turbulent kinetic energy (per unit mass) is

$$P^{(k)} \equiv -\overline{u_i u_j}\frac{\partial U_i}{\partial x_j} \qquad (4)$$

In the standard k-ε model (Launder and Spalding, 1974) the coefficients take the values

$$C_\mu = 0.09, \quad C_{\varepsilon 1} = 1.92, \quad C_{\varepsilon 2} = 1.44, \quad \sigma^{(k)} = 1.0, \quad \sigma^{(\varepsilon)} = 1.3 \qquad (5)$$

Other important variants include RNG k-ε (Yakhot et al., 1992) and low-Re models such as Launder and Sharma (1974), Lam and Bremhorst (1981), and Lien and Leschziner (1993).

Modifications are employed in low-Re models (see later) to incorporate effects of molecular viscosity. Specifically, C_μ, $C_{\varepsilon 1}$ and $C_{\varepsilon 2}$ are multiplied by viscosity-dependent factors f_μ, f_1 and f_2 respectively, and an additional source term $S^{(\varepsilon)}$ may be required in the ε equation. The damping factor f_μ is necessary because $\nu_t \propto y^3$ as $y \to 0$, but $k \propto y^2$ and $\varepsilon \sim$ *constant*, so that k^2/ε yields the wrong power of y.

Some models (notably Launder and Sharma, 1974) solve for the *homogeneous* dissipation rate $\tilde{\varepsilon}$ which vanishes at solid boundaries and is related to ε by

$$\varepsilon = \tilde{\varepsilon} + D, \qquad D = 2\nu(\nabla k^{1/2})^2 \tag{6}$$

This reflects the theoretical near-wall behaviour of ε (i.e. $\varepsilon \sim 2\nu k / y^2$) in a form which avoids using a geometric distance y explicitly.

2.3 *k*-ω Models

ω (nominally equal to $\dfrac{\varepsilon}{C_\mu k}$) is sometimes known as the *specific dissipation rate* and has dimensions of frequency or (time)$^{-1}$.

Eddy viscosity:

$$\nu_t = \frac{k}{\omega} \tag{7}$$

Scalar-transport equations:

$$\begin{aligned}
\rho \frac{\mathrm{D}k}{\mathrm{D}t} &= \frac{\partial}{\partial x_i}\left(\Gamma^{(k)} \frac{\partial k}{\partial x_i}\right) + \rho(P^{(k)} - \beta^* \omega k) \\
\rho \frac{\mathrm{D}\omega}{\mathrm{D}t} &= \frac{\partial}{\partial x_i}\left(\Gamma^{(\omega)} \frac{\partial \omega}{\partial x_i}\right) + \rho\left(\frac{\alpha}{\nu_t} P^{(k)} - \beta\omega^2\right)
\end{aligned} \tag{8}$$

Again, the diffusivities of k and ω are related to the eddy-viscosity:

$$\Gamma^{(k)} = \mu + \frac{\mu_t}{\sigma^{(k)}}, \qquad \Gamma^{(\omega)} = \mu + \frac{\mu_t}{\sigma^{(\omega)}}$$

The original k-ω model was that of Wilcox (1988a) where the coefficients take the values

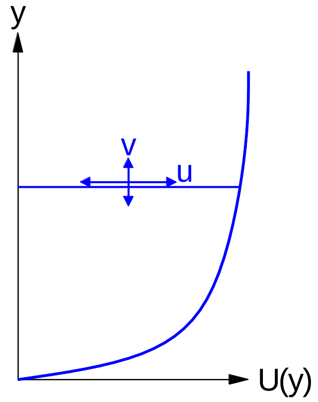
$$\beta^* = \frac{9}{100}, \quad \alpha = \frac{5}{9}, \quad \beta = \frac{3}{40}, \quad \sigma^{(k)} = 2.0, \quad \sigma^{(\omega)} = 2.0 \tag{9}$$

but in later versions of the model the coefficients become functions of $k^2/\nu\varepsilon$ (see Wilcox, 1998).

Menter (1994) devised a *shear-stress-transport* (SST) model. The model blends the k-ω model (which is – allegedly – superior in the near-wall region), with the k-ε model (which is less sensitive to the level of turbulence in the free stream) using wall-distance-dependent blending functions. Transport equations are solved for k and ω, but this is an odd choice because in a *free* flow with no wall boundaries (e.g. a jet) the model is simply a transformed k-ε model.

All models of k-ω type suffer from a problematic wall boundary condition ($\omega \to \infty$ as $y \to 0$) which is routinely fudged!

2.4 Behaviour of Linear Eddy-Viscosity Models in Simple Shear



In simple shear flow the shear stress is

$$-\rho \overline{uv} = \mu_t \frac{\partial U}{\partial y}$$

The three normal stresses are predicted to be equal:

$$\overline{u^2} = \overline{v^2} = \overline{w^2} = \frac{2}{3} k$$

whereas, in practice, there is considerable *anisotropy*; e.g. in the log-law region:

$$\overline{u^2} : \overline{v^2} : \overline{w^2} \approx 1.0 : 0.4 : 0.6$$

Actually, in *simple shear* flows, this is not a problem, since only the gradient of the shear stress $-\rho \overline{uv}$ plays a dynamically-significant role in the mean momentum balance. However, it is a warning of more serious problems in *complex* flows.

3. Non-Linear Eddy-Viscosity Models

3.1 General Form

The stress-strain relationship for linear eddy-viscosity models gives for the *deviatoric* Reynolds stress (i.e. subtracting the trace):

$$\overline{u_i u_j} - \frac{2}{3} k\delta_{ij} = -\nu_t \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right)$$

Dividing by k and writing $\nu_t = C_\mu k^2/\varepsilon$ gives

$$\frac{\overline{u_i u_j}}{k} - \frac{2}{3}\delta_{ij} = -C_\mu \frac{k}{\varepsilon}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \qquad (10)$$

We define the LHS of (10) as the *anisotropy tensor* a_{ij}, the dimensionless and traceless form of the Reynolds stress:

$$a_{ij} \equiv \frac{\overline{u_i u_j}}{k} - \frac{2}{3}\delta_{ij} \qquad (11)$$

For the RHS of (10), the symmetric and antisymmetric parts of the mean-velocity gradient are called the *mean strain* and *mean vorticity* tensors, respectively:

$$S_{ij} \equiv \frac{1}{2}(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}), \qquad \Omega_{ij} \equiv \frac{1}{2}(\frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i}) \qquad (12)$$

Their dimensionless forms, scaled by the turbulent timescale k/ε, are written in lower case:

$$s_{ij} \equiv \frac{k}{\varepsilon} S_{ij}, \qquad \omega_{ij} \equiv \frac{k}{\varepsilon}\Omega_{ij} \qquad (13)$$

Then equation (10) can be written

$$a_{ij} = -2C_\mu s_{ij}$$

or,

$$\mathbf{a} = -2C_\mu \mathbf{s} \qquad (14)$$

Hence, for a linear eddy-viscosity model the *anisotropy tensor is proportional to the dimensionless mean strain*.

The main idea of non-linear eddy-viscosity models is to generalise this to a *non-linear* relationship between the anisotropy tensor and the mean strain and vorticity:

$$\mathbf{a} = -2C_\mu \mathbf{s} + \mathbf{NL}(\mathbf{s}, \boldsymbol{\omega}) \qquad (15)$$

Additional non-linear components cannot be completely arbitrary, but must be symmetric and traceless. For example, a quadratic NLEVM must be of the form

$$\begin{aligned} \mathbf{a} &= -2C_\mu \mathbf{s} \\ &\quad + \beta_1(\mathbf{s}^2 - \tfrac{1}{3}\{\mathbf{s}^2\}\mathbf{I}) + \beta_2(\boldsymbol{\omega}\mathbf{s} - \mathbf{s}\boldsymbol{\omega}) + \beta_3(\boldsymbol{\omega}^2 - \tfrac{1}{3}\{\boldsymbol{\omega}^2\}\mathbf{I}) \end{aligned} \qquad (16)$$

where {.} denotes a trace and $\mathbf{I}$ is the identity matrix; i.e.

$$\{\mathbf{M}\} \equiv trace(\mathbf{M}) \equiv M_{ii}, \qquad (\mathbf{I})_{ij} \equiv \delta_{ij} \qquad (17)$$

We shall see below that an appropriate choice of the coefficients β_1, β_2 and β_3 allows the model to reproduce the correct anisotropy in simple shear.

Theory (based on the Cayley-Hamilton Theorem – a matrix satisfies its own characteristic equation) predicts that the most general possible relationship involves ten independent tensor bases and includes terms up to the 5th power in $\mathbf{s}$ and $\boldsymbol{\omega}$:

$$\mathbf{a} = \sum_{\alpha=1}^{10} C_\alpha \mathbf{T}_\alpha(\mathbf{s}, \boldsymbol{\omega}) \tag{18}$$

where all $\mathbf{T}_\alpha$ are linearly-independent, symmetric, traceless, second-rank tensor products of $\mathbf{s}$ and $\boldsymbol{\omega}$ and the C_α are, in general, functions of their invariants. One possible choice of bases (but by no means the only one) is

Linear:	$\mathbf{T}_1 = \mathbf{s}$
Quadratic:	$\mathbf{T}_2 = \mathbf{s}^2 - \frac{1}{3}\{\mathbf{s}^2\}\mathbf{I}$
	$\mathbf{T}_3 = \boldsymbol{\omega}\mathbf{s} - \mathbf{s}\boldsymbol{\omega}$
	$\mathbf{T}_4 = \boldsymbol{\omega}^2 - \frac{1}{3}\{\boldsymbol{\omega}^2\}\mathbf{I}$
Cubic:	$\mathbf{T}_5 = \boldsymbol{\omega}^2\mathbf{s} + \mathbf{s}\boldsymbol{\omega}^2 - \{\boldsymbol{\omega}^2\}\mathbf{s} - \frac{2}{3}\{\boldsymbol{\omega}\mathbf{s}\boldsymbol{\omega}\}\mathbf{I}$
	$\mathbf{T}_6 = \boldsymbol{\omega}\mathbf{s}^2 - \mathbf{s}^2\boldsymbol{\omega}$
Quartic:	$\mathbf{T}_7 = \boldsymbol{\omega}^2\mathbf{s}^2 + \mathbf{s}^2\boldsymbol{\omega}^2 - \frac{2}{3}\{\mathbf{s}^2\boldsymbol{\omega}^2\}\mathbf{I} - \{\boldsymbol{\omega}^2\}(\mathbf{s}^2 - \frac{1}{3}\{\mathbf{s}^2\}\mathbf{I})$
	$\mathbf{T}_8 = \mathbf{s}^2\boldsymbol{\omega}\mathbf{s} - \mathbf{s}\boldsymbol{\omega}\mathbf{s}^2 - \frac{1}{2}\{\mathbf{s}^2\}(\boldsymbol{\omega}\mathbf{s} - \mathbf{s}\boldsymbol{\omega})$
	$\mathbf{T}_9 = \boldsymbol{\omega}\mathbf{s}\boldsymbol{\omega}^2 - \boldsymbol{\omega}^2\mathbf{s}\boldsymbol{\omega} - \frac{1}{2}\{\boldsymbol{\omega}^2\}(\boldsymbol{\omega}\mathbf{s} - \mathbf{s}\boldsymbol{\omega})$
Quintic:	$\mathbf{T}_{10} = \boldsymbol{\omega}\mathbf{s}^2\boldsymbol{\omega}^2 - \boldsymbol{\omega}^2\mathbf{s}^2\boldsymbol{\omega}$

Exercise. (i) Prove that all these bases are symmetric and traceless.
(ii) Show that the bases $\mathbf{T}_5 - \mathbf{T}_{10}$ vanish in 2-d incompressible flow.

The first base corresponds to a linear eddy-viscosity model and the next three to the quadratic extension in equation (16). $\mathbf{T}_5$, $\mathbf{T}_7$, $\mathbf{T}_8$, $\mathbf{T}_9$ clearly contain multiples of earlier bases and hence could be replaced by simpler forms; however, the bases chosen here ensure that they vanish in 2-d incompressible flow.

A number of routes have been taken in devising such NLEVMs, including:

- assuming the form of the series expansion to quadratic or cubic order and then calibrating against important flows (e.g. Speziale, 1987; Craft, Launder and Suga, 1996);
- simplifying a differential stress model by an explicit solution (e.g. Speziale and Gatski, 1993) or by successive approximation (e.g. Apsley and Leschziner, 1998);
- renormalisation group methods (e.g. Rubinstein and Barton, 1990);
- direct interaction approximation (e.g. Yoshizawa, 1987).

In devising such NLEVMs, model developers have sought to incorporate such physically-significant properties as *realisability*:

$$\begin{aligned} &\overline{u_\alpha^2} \geq 0 && \text{(positive normal stresses)} \\ &\overline{u_\alpha u_\beta}^2 \leq \overline{u_\alpha^2}\;\overline{u_\beta^2} && \text{(Cauchy – Schwartz inequality)} \end{aligned} \tag{19}$$

3.2 Cubic Eddy-Viscosity Models

The preferred level of modelling in MACE is a *cubic* eddy viscosity model, which can be written in the form

$$\begin{aligned}\mathbf{a} = \; & -2C_\mu f_\mu \mathbf{s} \\ & +\beta_1(\mathbf{s}^2 - \tfrac{1}{3}\{\mathbf{s}^2\}\mathbf{I}) + \beta_2(\boldsymbol{\omega}\mathbf{s} - \mathbf{s}\boldsymbol{\omega}) + \beta_3(\boldsymbol{\omega}^2 - \tfrac{1}{3}\{\boldsymbol{\omega}^2\}\mathbf{I}) \\ & -\gamma_1\{\mathbf{s}^2\}\mathbf{s} - \gamma_2\{\boldsymbol{\omega}^2\}\mathbf{s} - \gamma_3(\boldsymbol{\omega}^2\mathbf{s} + \mathbf{s}\boldsymbol{\omega}^2 - \{\boldsymbol{\omega}^2\}\mathbf{s} - \tfrac{2}{3}\{\boldsymbol{\omega}\mathbf{s}\boldsymbol{\omega}\}\mathbf{I}) - \gamma_4(\boldsymbol{\omega}\mathbf{s}^2 - \mathbf{s}^2\boldsymbol{\omega})\end{aligned} \tag{20}$$

Note the following properties (some of which will be developed further below and in the Examples).

(i) A *cubic* stress-strain relationship is the minimum order with at least the same number of independent coefficients as the anisotropy tensor (i.e. 5). In this case it will be precisely 5 if we assume $\beta_3 = 0$ (see (vi) below) and note that the γ_1 and γ_2 terms are tensorially similar to the linear term (see (iv) below).

(ii) The first term on the RHS corresponds to a linear eddy-viscosity model.

(iii) The various non-linear terms evoke sensitivities to specific types of strain:
– the quadratic (β_1, β_2, β_3) terms evoke sensitivity to *anisotropy*;
– the cubic γ_1 and γ_2 terms evoke sensitivity to *curvature*;
– the cubic γ_4 term evokes sensitivity to *swirl*.

(iv) The γ_1 and γ_2 terms are tensorially proportional to the linear term; however they (or rather their difference) provide a sensitivity to curvature, so have been kept distinct.

(v) The γ_3 and γ_4 terms vanish in 2-d incompressible flow.

(vi) Theory and experiment indicate that pure rotation generates no turbulence. This implies that β_3 should either be zero or tend to 0 in the limit $\bar{S} \to 0$.

As an example of such a model we cite the Craft et al. (1996) model in which coefficients are functions of the mean-strain invariants and turbulent Reynolds number:

$$C_\mu = \frac{0.3[1-\exp(-0.36e^{0.75\eta})]}{1+0.35\eta^{3/2}}$$

$$f_\mu = 1 - \exp[-(\frac{R_t}{90})^{1/2} - (\frac{R_t}{400})^2], \qquad R_t = \frac{k^2}{\nu\tilde{\varepsilon}} \tag{21}$$

where

$$\bar{S} = \sqrt{2S_{ij}S_{ij}}\,, \quad \bar{\Omega} = \sqrt{2\Omega_{ij}\Omega_{ij}}\,, \quad \eta = \frac{k}{\tilde{\varepsilon}}\max(\bar{S}, \bar{\Omega}) \tag{22}$$

The coefficients of non-linear terms are (in the present notation):

$$(\beta_1, \beta_2, \beta_3) = (-0.4,\ 0.4,\ -1.04)C_\mu f_\mu$$

$$(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = (40,\ 40,\ 0,\ -80)C_\mu^3 f_\mu \tag{23}$$

Non-linearity is built into both tensor products and strain-dependent coefficients – notably C_μ. The model is completed by transport equations for k and $\tilde{\varepsilon}$ where

$$\tilde{\varepsilon} = \varepsilon - 2\nu(\nabla k^{1/2})^2$$

$\tilde{\varepsilon}$ is called the homogeneous dissipation rate, which vanishes at solid boundaries because of the near-wall behaviour of k and ε (see later):

$$k \sim k_0 y^2\,, \qquad \varepsilon \sim 2\nu k_0 \sim 2\nu k / y^2$$

In this model, mean strain and vorticity are also non-dimensionalised using $\tilde{\varepsilon}$ rather than ε.

3.3 General Properties of Non-Linear Eddy-Viscosity Models

(i) 2-d Incompressible Flow

The non-linear combinations of $\mathbf{s}$ and $\boldsymbol{\omega}$ have particularly simple forms in 2-d incompressible flow. In such a flow:

$$\mathbf{s} = \begin{pmatrix} s_{11} & s_{12} & 0 \\ s_{21} & s_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \boldsymbol{\omega} = \begin{pmatrix} 0 & \omega_{12} & 0 \\ \omega_{21} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Incompressibility ($s_{11} = -s_{22}$) and the symmetry and antisymmetry properties of s_{ij} and ω_{ij} ($s_{21} = s_{12}$, $\omega_{21} = -\omega_{12}$) reduce these to

$$\mathbf{s} = \begin{pmatrix} s_{11} & s_{12} & 0 \\ s_{12} & -s_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \boldsymbol{\omega} = \begin{pmatrix} 0 & \omega_{12} & 0 \\ -\omega_{12} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

From these we find

$$\mathbf{s}^2 = (s_{11}^2 + s_{12}^2)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \boldsymbol{\omega}^2 = -\omega_{12}^2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\boldsymbol{\omega}\mathbf{s} - \mathbf{s}\boldsymbol{\omega} = 2\omega_{12}s_{12}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} - 2\omega_{12}s_{11}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{24}$$

PROPERTY 1
In 2-d incompressible flow:

$$\begin{aligned} \mathbf{s}^2 &= (s_{11}^2 + s_{12}^2)\mathbf{I}_2 &&= \tfrac{1}{2}\{\mathbf{s}^2\}\mathbf{I}_2 \\ \boldsymbol{\omega}^2 &= -\omega_{12}^2\mathbf{I}_2 &&= \tfrac{1}{2}\{\boldsymbol{\omega}^2\}\mathbf{I}_2 \end{aligned} \tag{25}$$

where, $\mathbf{I}_2$ = diag(1,1,0). In particular, taking tensor products of $\mathbf{s}^2$ or $\boldsymbol{\omega}^2$ with matrices whose third row and third column are all zero has the same effect as multiplication by the scalars $\tfrac{1}{2}\{\mathbf{s}^2\}$ or $\tfrac{1}{2}\{\boldsymbol{\omega}^2\}$ respectively.

PROPERTY 2

$$\frac{P^{(k)}}{\varepsilon} = -a_{ij}s_{ij} = -\{\mathbf{as}\} \tag{26}$$

Moreover, in 2-d incompressible flow the quadratic terms do not contribute to the production of turbulent kinetic energy.

Proof.

$$P^{(k)} = -\overline{u_i u_j}\frac{\partial U_i}{\partial x_j} \quad = -k(a_{ij} + \tfrac{2}{3}\delta_{ij})(S_{ij} + \Omega_{ij})$$

Now $(a_{ij} + \frac{2}{3}\delta_{ij})\Omega_{ij} = 0$ since the first factor is symmetric whilst the Ω_{ij} is antisymmetric. Also, incompressibility implies $\delta_{ij}S_{ij} = S_{ii} = 0$. Hence,

$$P^{(k)} = -ka_{ij}S_{ij}$$

or

$$\frac{P^{(k)}}{\varepsilon} = -a_{ij}s_{ij} = -\{\mathbf{as}\}$$

This is true for any incompressible flow, but, in the 2-d case, multiplying (20) by $\mathbf{s}$, taking the trace and using the results (25) it is found that the contribution of the quadratic terms to $\{\mathbf{as}\}$ is 0.

PROPERTY 3
In 2-d incompressible flow the γ_3- and γ_4-related terms of the non-linear expansion (20) vanish.

Proof. Substitute the results (25) for $\mathbf{s}^2$ and $\boldsymbol{\omega}^2$ into (20).

(ii) Particular Types of Strain

The non-linear constitutive relationship (20) allows the model to mimic the response of turbulence to particular important types of strain.

PROPERTY 4
The quadratic terms yield turbulence *anisotropy* in simple shear:

$$\frac{\overline{u^2}}{k} = \frac{2}{3} + (\beta_1 + 6\beta_2 - \beta_3)\frac{\sigma^2}{12}$$

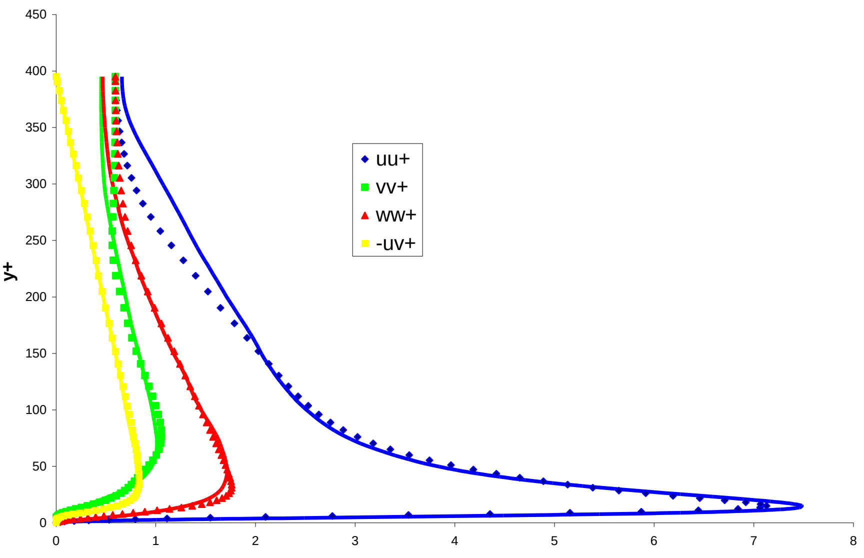
$$\frac{\overline{v^2}}{k} = \frac{2}{3} + (\beta_1 - 6\beta_2 - \beta_3)\frac{\sigma^2}{12} \qquad \text{where} \qquad \sigma = \frac{k}{\varepsilon}\frac{\partial U}{\partial y} \tag{27}$$

$$\frac{\overline{w^2}}{k} = \frac{2}{3} - (\beta_1 - \beta_3)\frac{\sigma^2}{6}$$

This may be deduced by substituting the results (24) into (20), noting that $s_{11} = 0$, whilst

$$s_{12} = \omega_{12} = \frac{1}{2}\frac{k}{\varepsilon}\frac{\partial U}{\partial y} = \frac{1}{2}\sigma$$

As an example the figure right shows application of the Apsley and Leschziner (1998) model to computing the Reynolds stresses in channel flow.



PROPERTY 5
The γ_1 and γ_2-related cubic terms yield the correct sensitivity to *curvature*.

In curved shear flow, $\dfrac{\partial U}{\partial y}=\dfrac{\partial U_s}{\partial R}$, $\dfrac{\partial V}{\partial x}=-\dfrac{U_s}{R_c}$, where R_c is radius of curvature. From (24),

$$\{\mathbf{s}^2\}+\{\boldsymbol{\omega}^2\}\equiv 2(s_{12}^2-\omega_{12}^2)$$

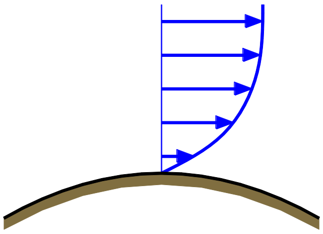
where

$$s_{12}=\frac{1}{2}\left(\frac{\partial U_s}{\partial R}-\frac{U_s}{R_c}\right), \qquad \omega_{12}=\frac{1}{2}\left(\frac{\partial U_s}{\partial R}+\frac{U_s}{R_c}\right)$$

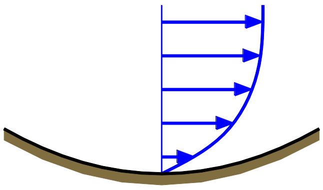
Hence,

$$\{\mathbf{s}^2\}+\{\boldsymbol{\omega}^2\}\equiv -2(\frac{k}{\varepsilon})^2\frac{\partial U_s}{\partial R}\frac{U_s}{R_c}$$

Inspection of the production terms in the stress-transport equations (Section 4) shows that curvature is stabilising (reducing turbulence) if U_s increases in the direction away from the centre of curvature ($\partial U_s/\partial R > 0$) and destabilising (increasing turbulence) if U_s decreases in the direction away from the centre of curvature ($\partial U_s/\partial R < 0$). In the constitutive relation (20) the response is correct if γ_1 and γ_2 are both positive.

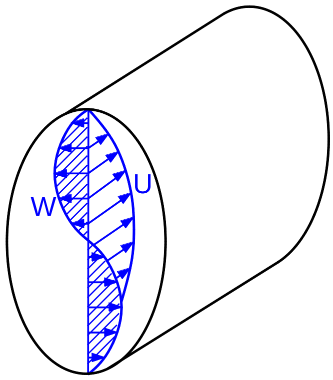


'stable' curvature
(reducing turbulence)



'unstable' curvature
(increasing turbulence)

PROPERTY 6
In 3-d flows, the γ_4-related term evokes the correct sensitivity to *swirl*.



4. Differential Stress Modelling

Differential stress models (aka *Reynolds-stress transport models* or *second-order closure*) solve separate scalar-transport equations for each stress component. By writing a momentum equation for the fluctuating u_i component, multiplying by u_j, adding a corresponding term with i and j reversed and then adding we obtain (for a full derivation see the Appendix):

$$\frac{\partial}{\partial t}(\overline{u_i u_j}) + U_k \frac{\partial}{\partial x_k}(\overline{u_i u_j}) = \frac{\partial}{\partial x_k}[\nu \frac{\partial}{\partial x_k}(\overline{u_i u_j}) - \frac{1}{\rho}\overline{p(u_i \delta_{jk} + u_j \delta_{ik})} - \overline{u_i u_j u_k}]$$

$$-(\overline{u_i u_k}\frac{\partial U_j}{\partial x_k} + \overline{u_j u_k}\frac{\partial U_i}{\partial x_k}) + (\overline{u_i f_j} + \overline{u_j f_i}) + \overline{\frac{p}{\rho}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})} - 2\nu \overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}}$$

or, multiplying by ρ:

$$\rho \frac{\mathrm{D}(\overline{u_i u_j})}{\mathrm{D}t} = \frac{\partial d_{ijk}}{\partial x_k} + \rho(P_{ij} + F_{ij} + \Phi_{ij} - \varepsilon_{ij}) \tag{28}$$

Term	Name and role	Exact or Model
$\rho \frac{\mathrm{D}(\overline{u_i u_j})}{\mathrm{D}t}$	**RATE OF CHANGE (time derivative + advection)** Transport with the mean flow.	**EXACT** $\frac{\partial}{\partial t}(\rho\overline{u_i u_j}) + \frac{\partial}{\partial x_k}(\rho U_k \overline{u_i u_j})$
P_{ij}	**PRODUCTION (mean strain)** Generation by mean-velocity gradients.	**EXACT** $P_{ij} \equiv -\overline{u_i u_k}\frac{\partial U_j}{\partial x_k} - \overline{u_j u_k}\frac{\partial U_i}{\partial x_k}$
F_{ij}	**PRODUCTION (body forces)** Generation by body forces.	**EXACT** (usually) $F_{ij} \equiv \overline{u_i f_j} + \overline{u_j f_i}$
d_{ijk}	**DIFFUSION** Spatial redistribution.	**MODEL** $d_{ijk} = (\mu\delta_{kl} + C_s \frac{\rho k \overline{u_k u_l}}{\varepsilon})\frac{\partial}{\partial x_l}(\overline{u_i u_j})$
Φ_{ij}	**PRESSURE-STRAIN** Redistribution between components.	**MODEL** $\Phi_{ij} = \Phi_{ij}^{(1)} + \Phi_{ij}^{(2)} + \Phi_{ij}^{(w)}$ $\Phi_{ij}^{(1)} = -C_1 \frac{\varepsilon}{k}(\overline{u_i u_j} - \frac{2}{3}k\delta_{ij})$ $\Phi_{ij}^{(2)} = -C_2(P_{ij} - \frac{1}{3}P_{kk}\delta_{ij})$ $\Phi_{ij}^{(w)} = (\tilde{\Phi}_{kl} n_k n_l \delta_{ij} - \frac{3}{2}\tilde{\Phi}_{ik} n_j n_k - \frac{3}{2}\tilde{\Phi}_{jk} n_i n_k) f$ $\tilde{\Phi}_{ij} = C_1^{(w)} \frac{\varepsilon}{k}\overline{u_i u_j} + C_2^{(w)}\Phi_{ij}^{(2)}, \quad f = \frac{C_\mu^{3/4} k^{3/2}/\varepsilon}{\kappa y_n}$
ε_{ij}	**DISSIPATION** Removal by viscosity	**MODEL** $\varepsilon_{ij} = \frac{2}{3}\varepsilon\delta_{ij}$

Typical values of the constants are:

$$C_1 = 1.8\,, \qquad C_2 = 0.6\,, \qquad C_1^{(w)} = 0.5\,, \qquad C_2^{(w)} = 0.3 \tag{29}$$

The stress-transport equations must be supplemented by a means of specifying ε – typically by its own transport equation, or a transport equation for a related quantity such as ω.

The most significant term requiring modelling is the pressure-strain correlation. This term is traceless (the sum of the diagonal terms $\Phi_{11} + \Phi_{22} + \Phi_{33} = 0$) and its accepted role is to promote isotropy – hence models for $\Phi_{ij}^{(1)}$ and $\Phi_{ij}^{(2)}$. Near walls this isotropising tendency must be over-ridden, necessitating a "wall-correction" term $\Phi_{ij}^{(w)}$.

General Assessment of DSMs

For:

- Include more turbulence physics than eddy-viscosity models.
- Advection and production terms (the "energy-in" terms) are exact and do not need modelling.

Against:

- Models are very complex and many important terms (particularly the redistribution and dissipation terms) require modelling.
- Models are very expensive computationally (6 stress-transport equations in 3 dimensions) and tend to be numerically unstable (only the small molecular viscosity contributes to any sort of gradient diffusion term).

DSMs of Interest

The nearest thing to a standard model is a high-Re closure based on that of Launder et al. (1975), with wall-reflection terms from Gibson and Launder (1978).

Other models of interest include:

- Speziale et al. (1991) – non-linear Φ_{ij} formulation, eliminating wall-correction terms;
- Craft (1998) – low-Re DSM, attempting to eliminate wall-dependent parameters;
- Jakirlić and Hanjalić (1995) – low-Re DSM admitting anisotropic dissipation;
- Wilcox (1988b) – low-Re DSM, with ω rather than ε as additional turbulent scalar.

5. Implementation of Turbulence Models in CFD

5.1 Transport Equations

The implementation of a turbulence model in CFD requires:

(1) a means of specifying the turbulent stresses by either:
– a constitutive relation (eddy-viscosity models), or
– individual transport equations (differential stress models);

(2) the solution of additional scalar-transport equations.

Special Considerations for the Mean Flow Equations

- Only part of the stress is diffusive. $\rho\overline{u_i u_j}$ represents a turbulent flux of U_i-momentum in the x_j direction. For linear or non-linear eddy-viscosity models only a part of this can be treated implicitly as a diffusion-like term; e.g. for the U equation through a face normal to the y direction:

$$-\rho\overline{uv} = \underbrace{\mu_t(\frac{\partial U}{\partial y}}_{\substack{diffusive \\ part}} + \underbrace{\frac{\partial V}{\partial x}) + (non-linear\ \ terms)}_{transferred\ to\ source}$$

  The remainder of the flux is treated as part of the source term for that control volume. Nevertheless, it is still treated in a conservative fashion; i.e. the mean momentum lost by one cell is equal to that gained by the adjacent cell.

- The lack of a turbulent viscosity in differential stress models leads to numerical instability. This can be addressed by the use of "effective viscosities" – see below.

Special Considerations for the Turbulence Equations

- They are usually source-dominated; i.e. the most significant terms are production, redistribution and dissipation (this is sometimes invoked as an excuse to use a low-order advection scheme).

- Variables such as k and ε must be non-negative. This demands:
  – care in discretising the source term (see below);
  – use of an unconditionally-bounded advection scheme.

Source-Term Linearisation For Non-Negative Quantities

The general discretised scalar-transport equation for a control volume centred on node P is

$$a_P\phi_P - \sum_F a_F\phi_F = b_P + s_P\phi_P$$

For stability one requires

$$s_P \le 0$$

To ensure non-negative ϕ one requires, in addition,

$$b_P \geq 0$$

You should, by inspection of the k and ε transport equations (3), be able to identify how the source term is linearised in this way.

If $b_P < 0$ for a quantity such as k or ε which is always non-negative (e.g. due to transfer of non-linear parts of the advection term or non-diffusive fluxes to the source term) then, to ensure that the variable doesn't become negative, the source term should be rearranged as

$$s_P \to s_P + \frac{b_P}{\phi_P^*} \qquad (30)$$

$$b_P \to 0$$

where * denotes the current value of a variable.

5.2 Wall Boundary Conditions

At walls the no-slip boundary condition applies, so that both mean and fluctuating velocities vanish. At high Reynolds numbers this presents three problems:

- there are very large flow gradients;
- wall-normal fluctuations are suppressed (i.e., selectively damped);
- viscous and turbulent stresses are of comparable magnitude.

There are two main ways of handling this in turbulent flow:

- **low-Reynolds-number turbulence models**
  – resolve the flow right up to the wall with a very fine grid and viscous modifications to the turbulence equations;
- **wall functions**
  – use a coarser grid and assume theoretical profiles in the unresolved near-wall region.

5.2.1 Low-Reynolds-Number Turbulence Models

- Aim to resolve the flow right up to the boundary.

- Have to include effects of molecular viscosity in the coefficients of the eddy-viscosity formula and ε (or ω) transport equations.

- Try to ensure the theoretical near-wall behaviour (see the Examples):

$$k \propto y^2, \qquad \varepsilon \sim \frac{2\nu k}{y^2} \sim \text{constant}, \qquad \nu_t \propto y^3 \qquad (y \to 0) \qquad (31)$$

- Full resolution of the flow requires the near-wall node to satisfy $y^+ \leq 1$, where

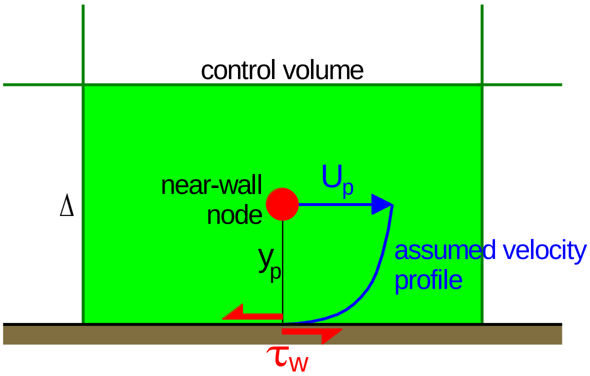
$$y^+ \equiv \frac{u_\tau y}{\nu}, \qquad u_\tau = \sqrt{\tau_w/\rho} \qquad (32)$$

  This can be very computationally demanding, particularly for high-speed flows.

5.2.2 High-Reynolds-Number Turbulence Models

- Bridge the near-wall region with *wall functions*; i.e., assume profiles (based on boundary-layer theory) between near-wall node and boundary.

- OK if the equilibrium assumption is reasonable (e.g. slowly-developing boundary layers), but dodgy in highly non-equilibrium regions (particularly near impingement, separation or reattachment points).



- The near-wall node should ideally (if seldom in practice) be placed in the log-law region ($30 < y^+ < 150$). This means that numerical meshes cannot be arbitrarily refined close to solid boundaries.

In the finite-volume method, various quantities are required from the wall-function approach. Values may be fixed on the wall (w) itself or by forcing a value at the near-wall node (P).

Variable	**Wall boundary condition**	**Required from wall function**
Mean velocity (U,V,W)	(relative) velocity =0 at the wall	Wall shear stress
k, $\overline{u_i u_j}$	$\overline{u_i u_j} = k = 0$ at the wall; zero flux	Cell-averaged production and dissipation
ε	ε_P fixed at near-wall node	Value at the near-wall node

The means of deriving these quantities are set out below.

<u>Mean-Velocity Equation: Wall Shear Stress</u>

Method 1

The friction velocity u_τ is defined in terms of the wall shear stress:

$$\tau_w = \rho u_\tau^2$$

If the near-wall node lies in the logarithmic region then

$$\frac{U_P}{u_\tau} = \frac{1}{\kappa}\ln(E y_P^+) , \qquad y_P^+ = \frac{y_P u_\tau}{\nu} \tag{33}$$

where subscript P denotes the near-wall node. Given the value of U_P this could be solved (iteratively) for u_τ and hence the wall stress τ_w. This works well only for near-equilibrium boundary layers.

Method 2

A better approach when the turbulence is far from equilibrium (e.g. near separation or reattachment points) is to estimate an "equivalent" friction velocity *from the turbulent kinetic energy* at the near-wall node:

$$u_0 = C_\mu^{1/4} k_P^{1/2}$$

and integrate the mean-velocity profile assuming an eddy viscosity ν_t. If we adopt the log-law version:

$$\nu_t = \kappa u_0 y$$

and solve for U from

$$\tau_w = \rho \nu_t \frac{\partial U}{\partial y}$$

we get

$$\tau_w / \rho = \frac{\kappa u_0 U_P}{\ln(E \dfrac{u_0 y_P}{\nu})} \qquad (34)$$

(If the turbulence were genuinely in equilibrium, then u_0 would equal u_τ and (33) and (34) would be equivalent).

Method 3

An even more advanced approach which depends less on the assumption that the near-wall node is well within the log layer is to assume a total viscosity (molecular + eddy) which matches both the viscous ($\nu_{eff} = \nu$) and log-layer ($\nu_{eff} \sim \kappa u_\tau y$) limits:

$$\nu_{eff} = \nu + \max\{0, \kappa u_0 (y - y_\nu)\} \qquad (35)$$

where y_ν is a matching height. Similar integration to that above (see the examples) leads to a complete mean-velocity profile satisfying both linear viscous sublayer and log-law limits:

$$\frac{U}{u_0} = \frac{\tau_w}{\rho u_0^2} \times \begin{cases} y^+, & y^+ \le y_\nu^+ \\ y_\nu^+ + \dfrac{1}{\kappa} \ln\{1 + \kappa(y^+ - y_\nu^+)\}, & y^+ \ge y_\nu^+ \end{cases}, \qquad y^+ \equiv \frac{y u_0}{\nu} \qquad (36)$$

where we note that y^+ is based on u_0 rather than the unknown u_τ. A similar approach can be applied for rough-wall boundary layers (Apsley, 2007). A typical value of the non-dimensional matching height (for smooth walls) is $y_\nu^+ = 7.37$.

As far as the computational implementation is concerned the required output for a finite-volume calculation is the wall shear stress in terms of the mean velocity at the near-wall node, y_p, not vice versa. To this end, (36) is conveniently rearranged in terms of an *effective wall viscosity* $\nu_{eff,wall}$ such that

$$\tau_w = \rho \nu_{eff,wall} \frac{U_p}{y_p} \qquad (37)$$

where

$$\nu_{eff,wall} = \nu \times \begin{cases} 1, & y_P^+ \le y_\nu^+ \\ \dfrac{y_P^+}{y_\nu^+ + \dfrac{1}{\kappa} \ln\{1 + \kappa(y_P^+ - y_\nu^+)\}}, & y_P^+ \ge y_\nu^+ \end{cases} \qquad (38)$$

(The reason for using the form (37) is that the code will "see" the mean velocity gradient as U_p/y_p.)

k Equation: Cell-Averaged Production and Dissipation

Because, in a finite-volume calculation, it represents the total production and dissipation for the near wall cell the source term of the k transport equation requires *cell-averaged* values of production $P^{(k)}$ and dissipation rate ε. These are derived by integrating assumed profiles for these quantities:

$$P^{(k)} \equiv -\overline{uv}\frac{\partial U}{\partial y} = \begin{cases} 0 & y \le y_\nu \\ \nu_t (\dfrac{\partial U}{\partial y})^2 & y > y_\nu \end{cases} \qquad \text{where } \nu_t = \nu_{eff} - \nu \tag{39}$$

$$\varepsilon = \begin{cases} \varepsilon_w & (y \le y_\varepsilon) \\ \dfrac{u_0^3}{\kappa(y - y_d)} & (y > y_\varepsilon) \end{cases} \tag{40}$$

For *smooth* walls, the matching height y_ε and offset y_d are given in wall units by (see Apsley, 2007):

$$y_\varepsilon^+ = 27.4, \qquad y_d^+ = 4.9$$

Integration over a cell (see the examples) leads to cell averages

$$P_{av}^{(k)} \equiv \frac{1}{\Delta}\int_0^\Delta P^{(k)} \ \mathrm{d}y = \frac{(\tau_w/\rho)^2}{\kappa u_0 \Delta}\left\{ \ln[1 + \kappa(\Delta^+ - y_\nu^+)] - \frac{\kappa(\Delta^+ - y_\nu^+)}{1 + \kappa(\Delta^+ - y_\nu^+)} \right\} \tag{41}$$

$$\varepsilon_{av} \equiv \frac{1}{\Delta}\int_0^\Delta \varepsilon \ \mathrm{d}y = \frac{u_0^3}{\kappa\Delta}\left[\ln(\frac{\Delta - y_d}{y_\varepsilon - y_d}) + \frac{y_\varepsilon}{y_\varepsilon - y_d} \right] \tag{42}$$

ε Equation: Boundary Condition on ε

ε_P is fixed from its assumed profile (equation (40)) at the near-wall node. A particular value at a cell centre can be forced in a finite-volume calculation by modifying the source coefficients:

$$s_P \to -\gamma, \qquad b_P \to \gamma\varepsilon_P$$

where γ is a large number (e.g. 10^{30}). The matrix equations for that cell then become

$$(\gamma + a_P)\phi_P - \sum a_F \phi_F = \gamma\varepsilon_P$$

or

$$\phi_P = \frac{\sum a_F \phi_F}{\gamma + a_P} + \frac{\gamma}{\gamma + a_P}\varepsilon_P$$

Since γ is a large number this effectively forces ϕ_P to take the value ε_P.

Reynolds-Stress Equations

For the Reynolds stresses, one method is to fix the values at the near-wall node from the near-wall value of k and the *structure functions* $\overline{u_i u_j}/k$, the latter being derived from the differential stress-transport equations on the assumption of local equilibrium. For the standard model this gives (see the examples):

$$
\begin{aligned}
\frac{\overline{v^2}}{k} &= \frac{2}{3}\left(\frac{-1+C_1+C_2-2C_2^{(w)}C_2}{C_1+2C_1^{(w)}}\right)\\
\frac{\overline{u^2}}{k} &= \frac{2}{3}\left(\frac{2+C_1-2C_2+C_2^{(w)}C_2}{C_1}\right)+\frac{C_1^{(w)}}{C_1}\frac{\overline{v^2}}{k}\\
\frac{\overline{w^2}}{k} &= \frac{2}{3}\left(\frac{-1+C_1+C_2+C_2^{(w)}C_2}{C_1}\right)+\frac{C_1^{(w)}}{C_1}\frac{\overline{v^2}}{k}\\
-\frac{\overline{uv}}{k} &= \sqrt{\left(\frac{1-C_2+\frac{3}{2}C_2^{(w)}C_2}{C_1+\frac{3}{2}C_1^{(w)}}\right)\frac{\overline{v^2}}{k}}
\end{aligned}
\qquad (43)
$$

With the values for C_1, C_2, etc. from the standard model this gives

$$
\frac{\overline{u^2}}{k}=1.098\,,\qquad \frac{\overline{v^2}}{k}=0.248\,,\qquad \frac{\overline{w^2}}{k}=0.654\,,\qquad \frac{-\overline{uv}}{k}=0.255 \qquad (44)
$$

When the near-wall flow and wall-normal direction are not conveniently aligned in the *x* and *y* directions respectively, the actual structure functions can be obtained by rotation. However, for 3-dimensional and separating/reattaching flow the *flow-oriented* coordinate system is not fixed *a priori* and can swing round significantly between iterations if the mean velocity is small, making convergence difficult to obtain. A second – and now my preferred – approach is to use cell-averaged production and dissipation in the Reynolds-stress equation in the same manner as the *k*-equation.

5.3 Effective Viscosity for Differential Stress Models

DSMs contain no turbulent viscosity and have a reputation for numerical instability.

An artificial means of promoting stability is to add and subtract a gradient-diffusion term to the turbulent flux:

$$
\overline{u_\alpha u_\beta} = (\overline{u_\alpha u_\beta} + \nu_{\alpha\beta}\frac{\partial U_\alpha}{\partial x_\beta}) - \nu_{\alpha\beta}\frac{\partial U_\alpha}{\partial x_\beta} \qquad (45)
$$

with the first part averaged between nodal values and the last part discretised across a cell face and treated implicitly. (This is analogous to the Rhie-Chow algorithm for pressure-velocity coupling in the momentum equations).

The simplest choice for the *effective viscosity* $\nu_{\alpha\beta}$ is just

$$
\nu_{\alpha\beta} = \nu_t = C_\mu \frac{k^2}{\varepsilon} \qquad (46)
$$

A better choice is to make use of a natural linkage between individual stresses and the corresponding mean-velocity gradient which arise from the actual stress-transport equations.

Assuming that the stress-transport equations (with no body forces) are source-dominated then

$$
P_{ij} + \Phi_{ij} - \varepsilon_{ij} \approx 0
$$

or, with the basic DSM (without wall-reflection terms),

$$P_{ij} - C_1\varepsilon\left(\frac{\overline{u_i u_j}}{k} - \tfrac{2}{3}\delta_{ij}\right) - C_2(P_{ij} - \tfrac{1}{3}P_{kk}\delta_{ij}) - \tfrac{2}{3}\varepsilon\delta_{ij} \approx 0$$

Expand this, identifying the terms which contain only $\overline{u_\alpha u_\beta}$ or $\dfrac{\partial U_\alpha}{\partial x_\beta}$ as follows.

For the normal stresses $\overline{u_\alpha^2}$:

$$P_{\alpha\alpha} - C_1\frac{\varepsilon}{k}(\overline{u_\alpha^2} - \ldots) - C_2\tfrac{2}{3}P_{\alpha\alpha} + \ldots = 0$$

Hence,

$$\overline{u_\alpha^2} = \frac{(1-\frac{2}{3}C_2)}{C_1}\frac{k}{\varepsilon}P_{\alpha\alpha} + \ldots \quad = \frac{(1-\frac{2}{3}C_2)}{C_1}\frac{k}{\varepsilon}\left(-2\overline{u_\alpha^2}\frac{\partial U_\alpha}{\partial x_\alpha} + \ldots\right)$$

Similarly for the shear stresses $\overline{u_\alpha u_\beta}$:

$$P_{\alpha\beta} - C_1\frac{\varepsilon}{k}\overline{u_\alpha u_\beta} - C_2 P_{\alpha\beta} + \ldots = 0$$

whence

$$\overline{u_\alpha u_\beta} = \frac{(1-C_2)}{C_1}\frac{k}{\varepsilon}P_{\alpha\beta} + \ldots \quad = \frac{(1-C_2)}{C_1}\frac{k}{\varepsilon}\left(-\overline{u_\beta^2}\frac{\partial U_\alpha}{\partial x_\beta} + \ldots\right)$$

Hence, from the stress-transport equations,

$$\begin{aligned} \overline{u_\alpha^2} &= -\nu_{\alpha\alpha}\frac{\partial U_\alpha}{\partial x_\alpha} + \ldots \\ \overline{u_\alpha u_\beta} &= -\nu_{\alpha\beta}\frac{\partial U_\alpha}{\partial x_\beta} + \ldots \end{aligned} \qquad (47)$$

where the effective viscosities (both for the U_α component of momentum) are:

$$\nu_{\alpha\alpha} = 2\left(\frac{1-\frac{2}{3}C_2}{C_1}\right)\frac{k\overline{u_\alpha^2}}{\varepsilon}, \qquad \nu_{\alpha\beta} = \left(\frac{1-C_2}{C_1}\right)\frac{k\overline{u_\beta^2}}{\varepsilon} \qquad (48)$$

Note that the effective viscosities are anisotropic, being linked to particular normal stresses.

A more detailed analysis can accommodate wall-reflection terms in the pressure-strain model, but the extra complexity is not justified.

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Appendix: Derivation of the Reynolds-Stress Transport Equations (*For Reference and Hangover-Cure Only*!)

Restrict to constant-density fluids for simplicity. f_i are the components of problem-dependent body forces (buoyancy, Coriolis forces, ...). Use prime/overbar notation (e.g. $u = \bar{u} + u'$) and summation convention throughout.

<u>Continuity</u>

Instantaneous: $$\frac{\partial u_k}{\partial x_k} = 0 \qquad \text{(A1)}$$

Average: $$\frac{\partial \bar{u}_k}{\partial x_k} = 0 \qquad \text{(A2)}$$

Subtract: $$\frac{\partial u'_k}{\partial x_k} = 0 \qquad \text{(A3)}$$

Result (I): *Both mean and fluctuating quantities satisfy the incompressibility equation*.

<u>Momentum</u>

Instantaneous: $$\frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_k \partial x_k} + f_i \qquad \text{(A4)}$$

Average: $$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_k \frac{\partial \bar{u}_i}{\partial x_k} + \overline{u'_k \frac{\partial u'_i}{\partial x_k}} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \nu \frac{\partial^2 \bar{u}_i}{\partial x_k \partial x_k} + \bar{f}_i \qquad \text{(A5)}$$

Rearrange: $$\frac{\mathrm{D}\bar{u}_i}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_k}\left(\nu \frac{\partial \bar{u}_i}{\partial x_k} - \overline{u'_i u'_k}\right) + \bar{f}_i \qquad \text{(A6)}$$

(Note: u'_k can be "taken through" the $\partial/\partial x_k$ derivative whenever required, due to the incompressibility condition)

Result (II): *The mean momentum equation is the <u>same</u> as the corresponding instantaneous equation except for additional <u>apparent</u> stresses* $-\overline{u'_i u'_j}$.

Subtract: $$\frac{\partial u'_i}{\partial t} + \bar{u}_k \frac{\partial u'_i}{\partial x_k} + u'_k \frac{\partial \bar{u}_i}{\partial x_k} + u'_k \frac{\partial u'_i}{\partial x_k} - \overline{u'_k \frac{\partial u'_i}{\partial x_k}} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_i} + \nu \frac{\partial^2 u'_i}{\partial x_k \partial x_k} + f'_i \qquad \text{(A7)}_i$$

Similarly *j*: $$\frac{\partial u'_j}{\partial t} + \bar{u}_k \frac{\partial u'_j}{\partial x_k} + u'_k \frac{\partial \bar{u}_j}{\partial x_k} + u'_k \frac{\partial u'_j}{\partial x_k} - \overline{u'_k \frac{\partial u'_j}{\partial x_k}} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_j} + \nu \frac{\partial^2 u'_j}{\partial x_k \partial x_k} + f'_j \qquad \text{(A7)}_j$$

Form $\overline{u'_i (\mathrm{A7})_j + u'_j (\mathrm{A7})_i}$:

$$\frac{\partial}{\partial t}(\overline{u_i'u_j'}) + \bar{u}_k \frac{\partial}{\partial x_k}(\overline{u_i'u_j'}) + \overline{u_i'u_k'}\frac{\partial \bar{u}_j}{\partial x_k} + \overline{u_j'u_k'}\frac{\partial \bar{u}_i}{\partial x_k} + \frac{\partial}{\partial x_k}(\overline{u_i'u_j'u_k'})$$

$$= -\frac{1}{\rho}(\overline{u_i'\frac{\partial p'}{\partial x_j} + u_j'\frac{\partial p'}{\partial x_i}}) + \nu(\overline{u_i'\frac{\partial^2 u_j'}{\partial x_k \partial x_k} + u_j'\frac{\partial^2 u_i'}{\partial x_k \partial x_k}}) + (\overline{u_i' f_j' + u_j' f_i'})$$

Rewrite the pressure terms and rearrange:

$$\frac{\partial}{\partial t}(\overline{u_i'u_j'}) + \bar{u}_k \frac{\partial}{\partial x_k}(\overline{u_i'u_j'}) = \frac{\partial}{\partial x_k}[\nu\frac{\partial}{\partial x_k}(\overline{u_i'u_j'}) - \frac{1}{\rho}\overline{p'(u_i'\delta_{jk} + u_j'\delta_{ik})} - \overline{u_i'u_j'u_k'}]$$

$$-(\overline{u_i'u_k'}\frac{\partial \bar{u}_j}{\partial x_k} + \overline{u_j'u_k'}\frac{\partial \bar{u}_i}{\partial x_k}) + (\overline{u_i' f_j'} + \overline{u_j' f_i'}) + \overline{\frac{p'}{\rho}(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i})} - 2\nu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}}$$

Result (III): *Reynolds-stress transport equation*:

$$\boxed{\frac{\mathrm{D}}{\mathrm{D}t}(\overline{u_i'u_j'}) = \frac{\partial d_{ijk}}{\partial x_k} + P_{ij} + F_{ij} + \Phi_{ij} - \varepsilon_{ij} \qquad \text{(A8)}}$$

where:

$$\frac{\mathrm{D}}{\mathrm{D}t}(\overline{u_i'u_j'}) = \frac{\partial}{\partial t}(\overline{u_i'u_j'}) + \bar{u}_k \frac{\partial}{\partial x_k}(\overline{u_i'u_j'})$$ *advection* (*by mean flow*)

$$P_{ij} = -(\overline{u_i'u_k'}\frac{\partial \bar{u}_j}{\partial x_k} + \overline{u_j'u_k'}\frac{\partial \bar{u}_i}{\partial x_k})$$ *production* (*by mean strain*)

$$F_{ij} = \overline{u_i' f_j' + u_j' f_i'}$$ *production* (*by body forces*)

$$\Phi_{ij} = \overline{\frac{p'}{\rho}(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i})}$$ *pressure-strain correlation*

$$\varepsilon_{ij} = 2\nu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}}$$ *dissipation*

$$d_{ijk} = \nu\frac{\partial(\overline{u_i'u_j'})}{\partial x_k} - \frac{1}{\rho}\overline{p'(u_i'\delta_{jk} + u_j'\delta_{ik})} - \overline{u_i'u_j'u_k'}$$ *diffusion*

Contract (A8), then divide by 2. Change summation subscript *k* to *i* to minimise confusion.

Result (IV): *turbulent kinetic energy equation*:

$$\boxed{\frac{\mathrm{D}k}{\mathrm{D}t} = \frac{\partial d_i^{(k)}}{\partial x_i} + P^{(k)} + F^{(k)} - \varepsilon \qquad \text{(A9)}}$$

where:

$$P^{(k)} = -\overline{u_i'u_j'}\frac{\partial \bar{u}_i}{\partial x_j}$$ *production* (*by mean strain*)

$$F^{(k)} = \overline{u_i' f_i'}$$ *production* (*by body forces*)

$$\varepsilon = \nu\overline{(\frac{\partial u_i'}{\partial x_j})^2}$$ *dissipation*

$$d_i^{(k)} = \nu\frac{\partial k}{\partial x_i} - \frac{1}{\rho}\overline{p'u_i'} - \tfrac{1}{2}\overline{u_j'u_j'u_i'}$$ *diffusion*

Examples

RANS Models - General

Q1.

(a) What is meant by a "Reynolds-averaged Navier-Stokes" solver?

(b) Describe the main principles of:

- (i) eddy-viscosity models;
- (ii) non-linear eddy-viscosity models;
- (iii) differential stress models;
- (iv) large-eddy simulation.

Q2.

Write down the dimensions of ρ, μ_t, k, ε and ω in terms of the fundamental physical dimensions M (mass), L (length) and T (time). Hence, show that any expression for μ_t in terms of ρ, k and ε must be of the form

$$\mu_t = constant \times \rho \frac{k^2}{\varepsilon}$$

whilst any expression for μ_t in terms of ρ, k and ω must be of the form

$$\mu_t = constant \times \rho \frac{k}{\omega}$$

Summation Convention

Q3.

$\frac{\partial U_i}{\partial x_i}$ is shorthand for $\frac{\partial U_1}{\partial x_1} + \frac{\partial U_2}{\partial x_2} + \frac{\partial U_3}{\partial x_3}$ or $\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z}$. Expand the following.

(a) $\frac{\mathrm{D}U_i}{\mathrm{D}t} (\equiv \frac{\partial U_i}{\partial t} + U_k \frac{\partial U_i}{\partial x_k})$ when i = 1 and when i = 2.

(b) $P^{(k)} = -\overline{u_i u_j} \frac{\partial U_i}{\partial x_j}$

(c) $P_{ij} = -(\overline{u_i u_k} \frac{\partial U_j}{\partial x_k} + \overline{u_j u_k} \frac{\partial U_i}{\partial x_k})$ when $i = 1, j = 1$ and when $i = 1, j = 2$.

(d) For a general matrix **M**, what quantities are represented by M_{ii}, $M_{ij}M_{ji}$ and $M_{ij}M_{jk}M_{ki}$?

Q4.

What do the following quantities reduce to in a simple shear flow (in which $\partial U/\partial y$ is the only non-zero velocity component)?

(a) $P^{(k)}$ (use the definition in Question 3(b))

(b) P_{ij} (for each combination of i and j; use the definition in Question 3(c))

(c) $S = (2S_{ij}S_{ij})^{1/2}$

(d) $\Omega = (2\Omega_{ij}\Omega_{ij})^{1/2}$

Production of Turbulent Kinetic Energy

Q5.
The rate of production of turbulent kinetic energy (per unit mass) is given by

$$P^{(k)} = -\overline{u_i u_j}\,\frac{\partial U_i}{\partial x_j}$$

Show that, in an incompressible flow:

(a) only the symmetric part S_{ij} of the mean velocity gradient affects production;

(b) only the anisotropic part of the turbulent stress tensor $k a_{ij} = \overline{u_i u_j} - \frac{2}{3} k \delta_{ij}$ affects production;

(c) with the eddy-viscosity hypothesis, $P^{(k)}$ is greater than or equal to zero.

Linear Eddy-Viscosity Models

Q6.
Two-equation turbulence models require the solution of two scalar-transport equations, typically for the turbulent kinetic energy k and a second dimensionally-independent variable $\phi = k^m \varepsilon^n$, where ε is the rate of dissipation of turbulent kinetic energy and m and n are constants. These transport equations take the form:

$$\frac{\mathrm{D}k}{\mathrm{D}t} = \nabla \bullet (\Gamma^{(k)} \nabla k) + P^{(k)} - \varepsilon$$

$$\frac{\mathrm{D}\phi}{\mathrm{D}t} = \nabla \bullet (\Gamma^{(\phi)} \nabla \phi) + (C_{\phi 1} P^{(k)} - C_{\phi 2} \varepsilon)\frac{\phi}{k} + S^{(\phi)}$$

where $P^{(k)}$ is the rate of production of turbulent kinetic energy.

(a) If the kinematic eddy viscosity ν_t is given by

$$\nu_t = C_\mu \frac{k^2}{\varepsilon}$$

find an equivalent expression for ν_t in terms of k and ϕ.

(b) Show that the transport equation for ϕ can be transformed into a transport equation for ε in the form

$$\frac{\mathrm{D}\varepsilon}{\mathrm{D}t} = \nabla \bullet (\Gamma^{(\varepsilon)} \nabla \varepsilon) + (C_{\varepsilon 1} P^{(k)} - C_{\varepsilon 2} \varepsilon)\frac{\varepsilon}{k} + S^{(\varepsilon)}$$

where

$$S^{(\varepsilon)} = \frac{\varepsilon}{n}\left\{ \frac{S^{(\phi)}}{\phi} + \frac{m}{k}\nabla \bullet \left[(\Gamma^{(\phi)} - \Gamma^{(k)}) \nabla k \right] + \Gamma^{(\phi)} \left[m(m-1)\left(\frac{\nabla k}{k}\right)^2 + n(n-1)\left(\frac{\nabla \varepsilon}{\varepsilon}\right)^2 + 2mn\frac{\nabla k}{k} \bullet \frac{\nabla \varepsilon}{\varepsilon} \right] \right\}$$

and find expressions for $\Gamma^{(\varepsilon)}$, $C_{\varepsilon 1}$ and $C_{\varepsilon 2}$.

(*A personal view: if the transport equation for any second variable can always be transformed into a transport equation for* ε *– or vice versa – then it seems that there is no fundamental reason for preferring one particular pair of turbulence variables over any other. Rather, there are pragmatic reasons for the choice, such as the ease of setting boundary conditions or the relationship to measurable physical quantities*.)

Non-Linear Eddy-Viscosity Models

Q7.
A quadratic stress-strain relationship can be written in the form

$$\begin{aligned}\mathbf{a} = \; & -2C_\mu \mathbf{s} \\ & +\beta_1(\mathbf{s}^2 - \tfrac{1}{3}\{\mathbf{s}^2\}\mathbf{I}) + \beta_2(\boldsymbol{\omega}\mathbf{s} - \mathbf{s}\boldsymbol{\omega}) + \beta_3(\boldsymbol{\omega}^2 - \tfrac{1}{3}\{\boldsymbol{\omega}^2\}\mathbf{I})\end{aligned}$$

where the anisotropy **a,** non-dimensional mean-strain **s** and vorticity **ω** are defined by

$$a_{ij} \equiv \frac{\overline{u_i u_j}}{k} - \frac{2}{3}\delta_{ij},$$

$$S_{ij} = \frac{1}{2}(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}), \qquad \Omega_{ij} = \frac{1}{2}(\frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i}), \qquad s_{ij} = \frac{k}{\varepsilon}S_{ij}, \qquad \omega_{ij} = \frac{k}{\varepsilon}\Omega_{ij}$$

Prove that with this model the non-zero stresses in a simple shear flow are given by

$$\frac{\overline{uv}}{k} = -C_\mu \sigma$$

$$\frac{\overline{u^2}}{k} = \frac{2}{3} + (\beta_1 + 6\beta_2 - \beta_3)\frac{\sigma^2}{12}$$

$$\frac{\overline{v^2}}{k} = \frac{2}{3} + (\beta_1 - 6\beta_2 - \beta_3)\frac{\sigma^2}{12} \qquad \text{where } \sigma = \frac{k}{\varepsilon}\frac{\partial U}{\partial y}.$$

$$\frac{\overline{w^2}}{k} = \frac{2}{3} - (\beta_1 - \beta_3)\frac{\sigma^2}{6}$$

Q8.
Find similar results to those of Q7 for the cases of:

(a) plane strain ($\frac{\partial V}{\partial y} = -\frac{\partial U}{\partial x}, \qquad W = 0$)

(b) axisymmetric strain ($\frac{\partial V}{\partial y} = \frac{\partial W}{\partial z} = -\frac{1}{2}\frac{\partial U}{\partial x}$)

In each case, use $\sigma = \frac{k}{\varepsilon}\frac{\partial U}{\partial x}$.

Differerential Stress Models

Q9.
A basic differential stress model (DSM) for turbulence closure in wall-bounded incompressible fluid flow is given (in standard suffix notation) by

$$\frac{\mathrm{D}}{\mathrm{D}t}(\overline{u_i u_j}) = \frac{\partial d_{ijk}}{\partial x_k} + P_{ij} + \Phi_{ij} - \varepsilon_{ij}$$

where

$$P_{ij} \equiv -\overline{u_i u_k}\frac{\partial U_j}{\partial x_k} - \overline{u_j u_k}\frac{\partial U_i}{\partial x_k}$$

$$\Phi_{ij} = \Phi_{ij}^{(1)} + \Phi_{ij}^{(2)} + \Phi_{ij}^{(w)}$$

$$\Phi_{ij}^{(1)} = -C_1\varepsilon\left(\frac{\overline{u_i u_j}}{k} - \tfrac{2}{3}\delta_{ij}\right), \qquad \Phi_{ij}^{(2)} = -C_2(P_{ij} - \tfrac{1}{3}P_{kk}\delta_{ij})$$

$$\Phi_{ij}^{(w)} = (\tilde{\Phi}_{kl}n_k n_l \delta_{ij} - \tfrac{3}{2}\tilde{\Phi}_{ik}n_j n_k - \tfrac{3}{2}\tilde{\Phi}_{jk}n_i n_k)f,$$

$$\tilde{\Phi}_{ij} = C_1^{(w)}\varepsilon\frac{\overline{u_i u_j}}{k} + C_2^{(w)}\Phi_{ij}^{(2)}, \qquad f = \frac{C_\mu^{3/4}k^{3/2}/\varepsilon}{\kappa y_n}$$

$$\varepsilon_{ij} = \tfrac{2}{3}\varepsilon\delta_{ij}$$

d_{ijk} is a diffusive flux, n_i are the components of a unit wall-normal vector and y_n is the distance to the nearest wall. k is the turbulent kinetic energy (per unit mass) and ε is its dissipation rate. C_1, C_2, $C_1^{(w)}$ and $C_2^{(w)}$ are model constants.

(a) What are the accepted *physical* roles of the terms denoted P_{ij}, Φ_{ij} and ε_{ij}?

(b) What is the purpose of the wall correction $\Phi_{ij}^{(w)}$ and what is the value of f in an equilibrium wall boundary layer?

(c) Write down expressions for all components of P_{ij}, Φ_{ij} and ε_{ij} in simple shear flow (where $\partial U_1/\partial x_2 \equiv \partial U/\partial y$ is the only non-zero mean-velocity derivative). Assume that $\mathbf{n} = (0,1,0)$.

(d) Show that, in an equilibrium turbulent flow (where $P_{ij} + \Phi_{ij} - \varepsilon_{ij} = 0$),

$$\frac{\overline{v^2}}{k} = \frac{2}{3}\left(\frac{-1 + C_1 + C_2 - 2C_2^{(w)}C_2}{C_1 + 2C_1^{(w)}}\right)$$

$$\frac{\overline{u^2}}{k} = \frac{2}{3}\left(\frac{2 + C_1 - 2C_2 + C_2^{(w)}C_2}{C_1}\right) + \frac{C_1^{(w)}}{C_1}\frac{\overline{v^2}}{k}$$

$$\frac{\overline{w^2}}{k} = \frac{2}{3}\left(\frac{-1 + C_1 + C_2 + C_2^{(w)}C_2}{C_1}\right) + \frac{C_1^{(w)}}{C_1}\frac{\overline{v^2}}{k}$$

$$-\frac{\overline{uv}}{k} = \sqrt{\left(\frac{1 - C_2 + \tfrac{3}{2}C_2^{(w)}C_2}{C_1 + \tfrac{3}{2}C_1^{(w)}}\right)\frac{\overline{v^2}}{k}}$$

Q10.

(a) In the transport equation for $\overline{u_i u_j}$ the production term is given in suffix notation by

$$P_{ij} \equiv -\overline{u_i u_k}\frac{\partial U_j}{\partial x_k} - \overline{u_j u_k}\frac{\partial U_i}{\partial x_k}$$

Write this out in full for P_{11}, P_{22} and P_{12}.

(b) In a simple shear flow, $\partial U/\partial y$ is the only non-zero mean-velocity gradient and the production term simplifies dramatically. Write expressions for all P_{ij} in this case.

(c) Give two reasons why the streamwise velocity variance $\overline{u^2}$ tends to be larger than the wall-normal component $\overline{v^2}$ in flow along a plane wall $y = 0$.

Q11.

The Navier-Stokes equations for the instantaneous velocity in a constant-density fluid may be written

$$\nabla \bullet \mathbf{u} = 0\,, \qquad \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u}\,.$$

Use these to derive the equation for turbulent kinetic energy (per unit mass), k, in the form

$$\frac{\mathrm{D}k}{\mathrm{D}t} = \frac{\partial d_j^{(k)}}{\partial x_j} + P^{(k)} - \varepsilon\,,$$

where $\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + \overline{\mathbf{u}} \bullet \nabla$ is the material derivative based on mean velocity. State carefully the form of terms $d_j^{(k)}$, $P^{(k)}$ and ε, and give their physical interpretation.

Near-Wall Behaviour

Q12.

By expanding the fluctuating velocities in the form

$$\begin{aligned} u &= a_1 + b_1 y + c_1 y^2 + \cdots \\ v &= a_2 + b_2 y + c_2 y^2 + \cdots \\ w &= a_3 + b_3 y + c_3 y^2 + \cdots \end{aligned}$$

show that $\overline{u^2} = \overline{b_1^2}\, y^2 + \cdots$ and derive similar expressions for $\overline{v^2}$, $\overline{w^2}$, $\overline{uv}$, k and ν_t.

Q13.

Use the turbulent kinetic energy equation and the near-wall behaviour of k from Q12 to show that the near-wall behaviour of ε is

$$\varepsilon \sim \frac{2\nu k}{y^2} \sim constant \qquad (y \to 0)$$

(Note that this implies that some modification is required in the ε equation as $y \to 0$).

Wall Functions

Q14. (Cell-averaged production and dissipation)
An effective viscosity profile is given by

$$\nu_{eff} = \nu + \max\{0, \kappa u_0 (y - y_\nu)\}$$

where y_ν is the zero-eddy-viscosity height.

(a) Show that the mean velocity profile and cell-averaged rate of production of turbulent kinetic energy are given by

$$\frac{U}{u_0} = \frac{\tau_w}{\rho u_0^2} \times \begin{cases} y^+, & y^+ \le y_\nu^+ \\ y_\nu^+ + \frac{1}{\kappa} \ln\{1 + \kappa(y^+ - y_\nu^+)\}, & y^+ \ge y_\nu^+ \end{cases}$$

$$P_{av}^{(k)} = \frac{(\tau_w / \rho)^2}{\kappa u_0 \Delta} \left\{ \ln[1 + \kappa(\Delta^+ - y_\nu^+)] - \frac{\kappa(\Delta^+ - y_\nu^+)}{1 + \kappa(\Delta^+ - y_\nu^+)} \right\}$$

respectively, where $y^+ \equiv \frac{y u_0}{\nu}$ and Δ (assumed greater than y_ν) is the depth of cell.

(b) If the dissipation rate ε is given by

$$\varepsilon = \begin{cases} \varepsilon_w & (y \le y_\varepsilon) \\ \dfrac{u_0^3}{\kappa (y - y_d)} & (y > y_\varepsilon) \end{cases}$$

where ε_w is such as to make the profile continuous at $y = y_\varepsilon$, show that the cell-averaged dissipation rate is given by

$$\varepsilon_{av} = \frac{u_0^3}{\kappa \Delta} \left[\ln\left(\frac{\Delta - y_d}{y_\varepsilon - y_d}\right) + \frac{y_\varepsilon}{y_\varepsilon - y_d} \right]$$

Q15. (Rough and smooth walls)

A generalised form of the log-law mean-velocity profile which satisfies both smooth- and rough-wall limits can be written in wall units as

$$U^+ = \frac{1}{\kappa}\ln\left(\frac{y^+}{1+ck_s^+}\right) + B \qquad (*)$$

where κ, B and c are constants and

$$U^+ = \frac{U}{u_\tau}, \qquad y^+ = \frac{u_\tau y}{\nu}, \qquad k_s^+ = \frac{u_\tau k_s}{\nu}$$

y is the distance from the wall, u_τ is the friction velocity and k_s is the Nikuradse roughness.

(a) Assuming that (*) holds from the wall to the centreline of a pipe of diameter D, integrate to find an implicit relationship between the skin-friction coefficient c_f, pipe Reynolds number Re and relative roughness k_s/D.

(b) By comparing your results with the Colebrook-White formula (in terms of skin-friction coefficient c_f rather than friction factor $\lambda = 4c_f$):

$$\frac{1}{\sqrt{c_f}} = -4.0\log_{10}\left(\frac{k_s}{3.7D} + \frac{1.26}{\mathrm{Re}\sqrt{c_f}}\right)$$

deduce values for κ, B and c.

(c) Show that (*) can also be written in the form

$$U^+ = \frac{1}{\kappa}\ln y^+ + \tilde{B}(k_s^+) \qquad (**)$$

and deduce the functional form of $\tilde{B}(k_s^+)$.

(d) In the fully-rough limit ($k_s^+ >> 1$) (**) can be written as

$$U^+ = \frac{1}{\kappa}\ln\frac{y}{k_s} + B_k$$

Use your answers to parts (b) and (c) to deduce a value for B_k.