3. Approximations and Simplified Equations

Common Approximations

- Reduction of **dimension**:
  - steady-state
  - 2-dimensional
- Neglect of some **fluid property**:
  - incompressible
  - inviscid
- Simplified **forces**:
  - hydrostatic
  - Boussinesq approximation for density
- **Averaging**:
  - depth-averaging (shallow-water flows)
  - Reynolds averaging (turbulent flows)

Time-Dependent vs Steady-State

- Use time-dependent calculations for:
  - **time-dependent** problems (waves, pumps, …)
  - flows with a natural **instability** (vortex shedding, ...)
  - time-marching to **steady state** (high-Mach or high-Froude flows)
- Time-dependent calculations:
  - "parabolic" or "hyperbolic" equations
  - 1st-order in time; solved by **forward-marching** in time
- Steady-state calculations:
  - "elliptic" equations
  - 2nd-order in space: **implicit**, **iterative** solution methods

2- or 3-Dimensional

- Determined by geometry and boundary conditions
- Significant computational implications
- 2-d flows difficult to achieve in the laboratory
- Axisymmetric flows also “2-d”

Incompressible Flow

- **Definition: flow-induced** pressure and temperature changes don’t cause significant density changes
- Usually requires velocity << speed of sound (Ma << 1)
- Doesn’t necessarily mean “uniform density”
  (Environmental flows driven by density differences in the atmosphere or ocean can still be regarded as incompressible)

Compressible vs Incompressible CFD

Compressible flow:

- density changes due to large pressure and/or temperature changes;
- requires equation for internal energy e (or enthalpy h):

  change in energy = heat input + work done

- mass equation $\rightarrow$ density ρ
  energy equation $\rightarrow$ temperature T
  equation of state $\rightarrow$ pressure p (e.g. ideal gas law, $p = \rho RT$)

Incompressible flow:

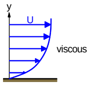
- density constant along a streamline; volume conserved
- mechanical energy equation:

  change in kinetic energy = work done

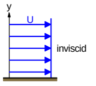
- equivalent to momentum equation: **no separate energy equation**
- pressure equation arises from the requirement that solutions of the momentum equation also be mass-consistent

Viscous vs Inviscid

- **Viscous** (Navier-Stokes) equations:
  - dynamic ("no-slip") boundary condition: $(u, v, w) = 0$



- **Inviscid** (Euler) equations:
  - no 2nd-order derivatives (one less b.c)
  - kinematic ("slip-wall") boundary condition: u_n=0



- Inviscid approximation OK only if boundary layer is:
  - thin (high Re)
  - attached (no flow separation)

- Inviscid approximation implies no drag, no heat transfer, no sediment transport, …

Potential Flow

- **Approximation:** inviscid, incompressible

- Velocity derived from a **velocity potential** ϕ:

$$\left.\begin{array}{l} \mathbf{u} = \nabla\phi \\ \nabla \bullet \mathbf{u} = 0 \end{array}\right\} \longrightarrow \nabla^2\phi = 0$$

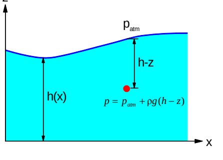
Laplace's Equation

- **Consequences:**
  - entire flow field determined by a single scalar
  - very common equation; plenty of good solvers around
  - ignores boundary-layer effects: no drag or flow separation

Hydrostatic Approximation

- **Approximation:**
  - pressure forces balance weight:

$$\frac{\partial p}{\partial z} = -\rho g \qquad \Delta p = -\rho g \Delta z$$



- **Validity:**
  - always true in stationary fluid
  - good approximation if vertical acceleration << g

- **Consequence:**
  - pressure is determined everywhere from the depth below the free-surface:

$$p = p_{atm} + \rho g(h - z)$$

Boussinesq Approximation for Density

- **Application:** variable-density environmental flows:
  - atmosphere (temperature);
  - oceans (salinity).

$$\frac{\rho-\rho_0}{\rho_0} = -\alpha\,(\theta-\theta_0) \quad \longleftrightarrow \quad \rho = \rho_0 - \rho_0\alpha(\theta-\theta_0)$$

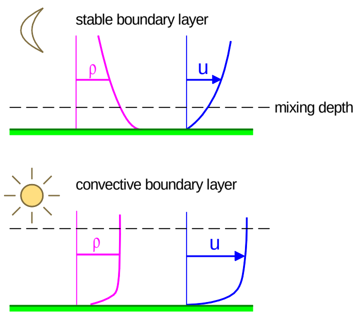
- **Approximation:**
  - retain density changes in buoyancy force
  - neglect density changes in mass × acceleration

$$\rho\frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{\partial p}{\partial z} - \rho g \longrightarrow \rho\frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{\partial p}{\partial z} - \rho_0 g - (\rho-\rho_0)g \longrightarrow \rho_0\frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{\partial p^*}{\partial z} + \underbrace{\rho_0\alpha(\theta-\theta_0)g}_{buoyancy force}$$

- **Comments:**
  - sometimes needed in theoretical work;
  - usually unnecessary in general-purpose CFD.

$$p^* = p + \rho_0 g z$$

Atmospheric Boundary Layer

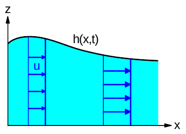


Fresh-Water Outfall

Shallow-Water Equations

- **Application:** open-channel hydraulics

- **Approximation:** depth-averaged
  - horizontal velocities u, v
  - water depth h



$$(mass) \qquad \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) = 0$$

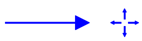
$$(momentum) \qquad \frac{\partial}{\partial t}(uh) + \frac{\partial}{\partial x}(u^2 h) = -\frac{\partial}{\partial x}(\tfrac{1}{2} g h^2) + \frac{1}{\rho}(\tau_{surface} - \tau_{bed})$$

- **Compressible-flow analogy:**
  - hydraulic jump $\leftrightarrow$ shock
  - wave speed: $c = \sqrt{gh} \quad \leftrightarrow \quad c = \sqrt{\gamma p / \rho}$
  - Froude number Fr $\leftrightarrow$ Mach number Ma

Reynolds Averaging

Application: turbulent flows

$$u = \underbrace{\bar{u}}_{mean} + \underbrace{u'}_{fluctuation}$$



Averaging produces an additional effective stress in the **mean** flow equations:

$$\tau_{turb} = -\rho \overline{u'v'} \qquad \textbf{Reynolds stress}$$

Common modelling practice:

$$\tau_{turb} = \mu_t \frac{\partial \bar{u}}{\partial y} \qquad \mu_t = \textbf{eddy viscosity}$$

Turbulent Jet