Answers 9

Classroom Example 1

(a) Taking care with the outward directions (for which you will have to sketch the tetrahedron):

$$\mathbf{A}_{BCD} = \frac{1}{2}\overrightarrow{BD} \wedge \overrightarrow{BC} = \frac{1}{2}\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \wedge \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -2 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

$$\mathbf{A}_{ACD} = \frac{1}{2}\overrightarrow{AC} \wedge \overrightarrow{AD} = \frac{1}{2}\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \wedge \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$$\mathbf{A}_{ABD} = \frac{1}{2}\overrightarrow{AD} \wedge \overrightarrow{AB} = \frac{1}{2}\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \wedge \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -2 \\ -2 \\ -4 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -2 \end{pmatrix}$$

$$\mathbf{A}_{ABC} = \frac{1}{2}\overrightarrow{AB} \wedge \overrightarrow{AC} = \frac{1}{2}\begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 \\ 2 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

(b) **Method 1:** use the scalar-triple-product formula for volume of a tetrahedron:

$$V = \frac{1}{6}\overrightarrow{AD} \bullet \overrightarrow{AC} \wedge \overrightarrow{AB} = \frac{1}{6}\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \wedge \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} -2 \\ -2 \\ 4 \end{pmatrix} = \frac{4}{3}$$

(If you choose a left-handed, rather than right-handed, order of vectors in the triple product you will end up with a negative V; better to sketch the tetrahedron first.)

Method 2: use the general formula:

$$V = \frac{1}{3}\oint_{\partial v} \mathbf{r} \bullet \mathrm{d}\mathbf{A} = \frac{1}{3}\sum_{faces} \mathbf{r}_f \bullet \mathbf{A}_f$$

where $\mathbf{r}_f$ is any position vector on a cell face. WLOG one can (i) refer all positions vectors to, say, A; (ii) take the representative position vector for three of the faces at A. Then, taking B as the reference point on the remaining face,

$$V = \frac{1}{3}\overrightarrow{AB} \bullet \mathbf{A}_{BCD} = \frac{1}{3}\begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \frac{4}{3}$$

Answer: volume = 4/3.

(c) Use

$$(\nabla\phi)_{av} = \frac{1}{V}\oint_{\partial V} \phi \ \mathrm{d}\mathbf{A} = \frac{1}{V}\sum_{faces} \phi_f \mathbf{A}_f$$

Hence,

$$\begin{pmatrix} \partial\phi/\partial x \\ \partial\phi/\partial y \\ \partial\phi/\partial z \end{pmatrix}_{av} = \frac{1}{V}(\phi_{BCD}\mathbf{A}_{BCD} + \phi_{ACD}\mathbf{A}_{ACD} + \phi_{ABD}\mathbf{A}_{ABD} + \phi_{ABC}\mathbf{A}_{ABC})$$

$$= \frac{3}{4}\left\{ 5\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + 3\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + 4\begin{pmatrix} -1 \\ -1 \\ -2 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \right\}$$

$$= \begin{pmatrix} -3 \\ 0 \\ 3 \end{pmatrix}$$

Answer: $\left(\frac{\partial\phi}{\partial x}\right)_{av} = -3$, $\left(\frac{\partial\phi}{\partial y}\right)_{av} = 0$, $\left(\frac{\partial\phi}{\partial z}\right)_{av} = 3$.

Classroom Example 2

(a) The area of the pentagon could, in principle, be found by breaking it up into right triangles and trapezia. However, in the spirit of the rest of the question it can be found by summing the vector areas of triangles with common vertices at, say, the bottom point. Then:

$$\mathbf{A} = \frac{1}{2}\begin{pmatrix}3\\5\\0\end{pmatrix}\wedge\begin{pmatrix}-1\\7\\0\end{pmatrix}+\frac{1}{2}\begin{pmatrix}-1\\7\\0\end{pmatrix}\wedge\begin{pmatrix}-5\\4\\0\end{pmatrix}+\frac{1}{2}\begin{pmatrix}-5\\4\\0\end{pmatrix}\wedge\begin{pmatrix}-4\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\34\end{pmatrix}$$

Answer: area of the pentagon = 34 units.

(b) The volume-averaged derivatives (for a 3-dimensional cell) are given by:

$$(\nabla\phi)_{av} = \frac{1}{V}\oint_{\partial V}\phi \;\, \mathrm{d}\mathbf{A} \quad = \frac{1}{V}\sum_{faces}\phi_f \mathbf{A}_f$$

For a 2-dimensional cell one may consider a prism of unit depth in the z-direction. The contributions from top and bottom faces cancel. For each side the outward face area vector corresponding to an edge $(\Delta x,\Delta y)$ (traversed anticlockwise) is given by

$$\mathbf{A}_f = \begin{pmatrix}\Delta y\\-\Delta x\\0\end{pmatrix}$$

For the 5 sides of the pentagon:

$$\mathbf{A}_a = \begin{pmatrix}5\\-3\\0\end{pmatrix},\quad \mathbf{A}_b = \begin{pmatrix}2\\4\\0\end{pmatrix},\quad \mathbf{A}_c = \begin{pmatrix}-3\\4\\0\end{pmatrix},\quad \mathbf{A}_d = \begin{pmatrix}-3\\-1\\0\end{pmatrix},\quad \mathbf{A}_e = \begin{pmatrix}-1\\-4\\0\end{pmatrix}$$

(Check: outward vector areas add up to 0).

Then

$$\begin{pmatrix}\partial\phi/\partial x\\ \partial\phi/\partial y\\ \partial\phi/\partial z\end{pmatrix}_{av} = \frac{1}{V}\{\phi_1\mathbf{A}_1+\phi_2\mathbf{A}_2+\phi_3\mathbf{A}_3+\phi_4\mathbf{A}_4+\phi_5\mathbf{A}_5\}$$

$$= \frac{1}{34}\left\{-7\begin{pmatrix}5\\-3\\0\end{pmatrix}+8\begin{pmatrix}2\\4\\0\end{pmatrix}-2\begin{pmatrix}-3\\4\\0\end{pmatrix}+5\begin{pmatrix}-3\\-1\\0\end{pmatrix}+0\begin{pmatrix}-1\\-4\\0\end{pmatrix}\right\}$$

$$= \frac{1}{34}\begin{pmatrix}-28\\40\\0\end{pmatrix} = \begin{pmatrix}-14/17\\20/17\\0\end{pmatrix}$$

Answer: $\left(\frac{\partial\phi}{\partial x}\right)_{av} = -\frac{14}{17}$, $\left(\frac{\partial\phi}{\partial y}\right)_{av} = \frac{20}{17}$.

Classroom Example 3

Let $\phi*$ be the exact solution. Then, by the assumption of second-order accuracy:

$$0.74 - \phi* = C(\Delta x)^2$$

$$0.78 - \phi* = C(2\Delta x)^2$$

where Δx is the grid spacing in mesh A and C is a constant.

(a)
Eliminating C:

$$(4 \times 0.74 - 0.78) - (4 - 1)\phi* = 0$$

$$\Rightarrow \qquad \phi* = 0.727$$

Answer: improved solution = 0.727.

(b)
The error in the finer mesh, A, is estimated to be

$$C(\Delta x)^2 = 0.74 - \phi* \quad = 0.0133$$

Answer: estimate of error = 0.0133.

Classroom Example 4

(a) Between any two points:

change in ψ = volume flux (clockwise, per unit depth) any line joining two points.

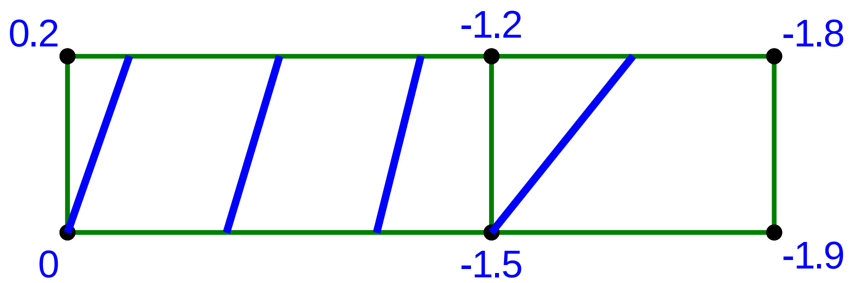
$\Rightarrow \qquad \Delta\psi = u\Delta y - v\Delta x$

Working in metre-second units:

$$\psi_B = \psi_A - 5\times 0.3 \quad = -1.5$$
$$\psi_D = \psi_A + 2\times 0.1 \quad = 0.2$$
$$\psi_E = \psi_B + 3\times 0.1 \quad = -1.2$$
$$\psi_F = \psi_E - 3\times 0.2 \quad = -1.8$$
$$\psi_C = \psi_F - 1\times 0.1 \quad = -1.9$$

Answer: (ψ_A, ψ_B, ψ_C, ψ_D, ψ_E, ψ_F) = (0, –1.5, –1.9, 0.2, –1.2, –1.8) $m^2\ s^{-1}$.

(b) By contouring the streamfunction at ψ = 0, –0.5, –1.0, –1.5 the streamlines (drawn as line segments) are as shown below.



Q1. (a)
(i) “Structured” – indexable as ϕ_{ijk} for $i = 1,\ldots N_i$, $j = 1,\ldots N_j$, $k = 1,\ldots N_k$.

(ii) “Multi-block structured” – comprised of a number of separate regions, in each of which the mesh is structured.

(iii) “Unstructured” – cannot be put in either of the above two forms.

(iv) “Chimera” – overlapping.

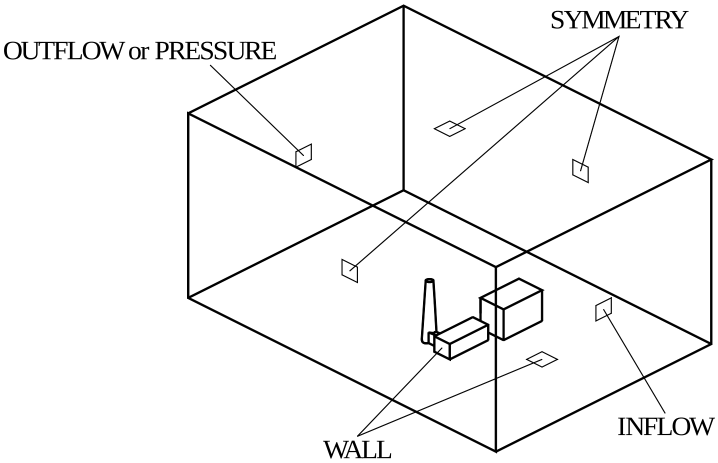
(b)
(i) “Cartesian” – coordinate lines are straight and cross at 90°; (i.e. the basis vectors are perpendicular and the same everywhere).

(ii) “Curvilinear” – not Cartesian!

(iii) “Orthogonal” – coordinate lines cross at 90°. (They don’t have to be straight; e.g. cylindrical polars).

Q2.

(a)



INFLOW: values of all transported variables (velocity, etc) specified; zero normal gradient for pressure.

SYMMETRY: zero normal velocity; zero normal gradient for all other variables.

OUTFLOW: zero normal gradient for all variables.
PRESSURE: as OUTFLOW, except fixed pressure.

WALL: zero velocity; zero gradient of pressure normal to the wall; wall functions to predict wall stress and turbulent source terms.

(b) The drag coefficient is the drag force non-dimensionalised by (dynamic pressure) × area:

$$c_D = \frac{F_x}{\frac{1}{2}\rho U_0^2 A_0}$$

Here:

F_x is the streamwise component of force on the object;
ρ is density;
U_0 is an appropriate reference velocity;
A_0 is an appropriate reference area (usually the projected area for a bluff body).

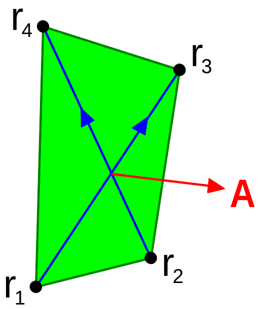
The drag force is calculated by summing elemental pressure × area and stress × area for all finite-volume faces making up the object (taking account of direction); i.e.

$$F_x = -\sum pA_x + \sum \tau_x A$$

where A_x is the streamwise component of the outward area vector (i.e. the pressure force is "into" the object).

Q3.

Since the vector area of a complete closed surface is zero, the vector area is the same whichever set of triangles is used to construct the cell face. Taking the triangles as Δ123 and Δ134 for example, the sum of the two triangular areas is



$$\begin{aligned}\mathbf{A} &= \tfrac{1}{2}(\mathbf{r}_2-\mathbf{r}_1)\wedge(\mathbf{r}_3-\mathbf{r}_1)+\tfrac{1}{2}(\mathbf{r}_3-\mathbf{r}_1)\wedge(\mathbf{r}_4-\mathbf{r}_1)\\ &= \tfrac{1}{2}(\mathbf{r}_3-\mathbf{r}_1)\wedge[-(\mathbf{r}_2-\mathbf{r}_1)+(\mathbf{r}_4-\mathbf{r}_1)]\\ &= \tfrac{1}{2}[(\mathbf{r}_3-\mathbf{r}_1)\wedge(\mathbf{r}_4-\mathbf{r}_2)]\\ &= \tfrac{1}{2}\mathbf{d}_{13}\wedge\mathbf{d}_{24}\end{aligned}$$

Q4.

(a) $$\nabla \bullet \mathbf{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} \quad = 3$$

Integrating over an arbitrary control volume *V*:

$$\int_V \nabla \bullet \mathbf{r} \quad \mathrm{d}V = 3V$$

Using the divergence theorem for the LHS to convert from a volume to a surface integral:

$$\oint_{\partial V} \mathbf{r} \bullet \mathrm{d}\mathbf{A} = 3V$$

Hence,

$$V = \frac{1}{3} \oint_{\partial V} \mathbf{r} \bullet \mathrm{d}\mathbf{A}$$

(b) The volume-averaged derivative is

$$\left(\frac{\partial \phi}{\partial x} \right)_{av} \equiv \frac{1}{V} \int_V \frac{\partial \phi}{\partial x} \mathrm{d}V$$

$$= \frac{1}{V} \int_V \nabla \bullet (\phi \mathbf{e}_x) \quad \mathrm{d}V$$

Hence, using the divergence theorem,

$$\left(\frac{\partial \phi}{\partial x} \right)_{av} = \frac{1}{V} \oint_{\partial V} \phi \mathbf{e}_x \bullet \mathrm{d}\mathbf{A}$$

$$= \frac{1}{V} \oint_{\partial V} \phi \quad \mathrm{d}A_x$$

Q5.
(a) The area of a quadrilateral is half the cross-product of its diagonals.
The diagonals are (e.g.):

$$\begin{pmatrix}3\\3\\0\end{pmatrix}-\begin{pmatrix}-2\\2\\0\end{pmatrix}=\begin{pmatrix}5\\1\\0\end{pmatrix}, \qquad \begin{pmatrix}0\\4\\0\end{pmatrix}-\begin{pmatrix}2\\0\\0\end{pmatrix}=\begin{pmatrix}-2\\4\\0\end{pmatrix}$$

Hence, the (vector area) of this quadrilateral is

$$\mathbf{A}=\frac{1}{2}\begin{pmatrix}5\\1\\0\end{pmatrix}\wedge\begin{pmatrix}-2\\4\\0\end{pmatrix}=\frac{1}{2}\begin{pmatrix}0\\0\\22\end{pmatrix}=\begin{pmatrix}0\\0\\11\end{pmatrix}$$

Answer: A = 11 units.

(b) This 2-d problem can be treated as a 3-d cell with unit depth. The "volume" of the cell is then, from part (a), V = 11 units.

The *outward* area vectors can be found from their projections:

$$A=\begin{pmatrix}\Delta y\\-\Delta x\\0\end{pmatrix}$$

Hence:

$$\mathbf{A}_e=\begin{pmatrix}3\\-1\\0\end{pmatrix}, \quad \mathbf{A}_n=\begin{pmatrix}1\\3\\0\end{pmatrix}, \quad \mathbf{A}_w=\begin{pmatrix}-2\\2\\0\end{pmatrix}, \quad \mathbf{A}_s=\begin{pmatrix}-2\\-4\\0\end{pmatrix}$$

Check: the total vector area is 0.

The cell-face values are, by averaging the values either side of a face:

$$\phi_e=3.5, \quad \phi_n=2.5, \quad \phi_w=1, \quad \phi_s=1.5$$

Hence, as top and bottom faces contribute nothing overall to the derivatives,

$$\begin{pmatrix}\partial\phi/\partial x\\\partial\phi/\partial y\\\partial\phi/\partial z\end{pmatrix}_{av}=\frac{1}{V}\oint_{\partial V}\phi\ \mathrm{d}\mathbf{A}=\frac{1}{V}\sum_{faces}\phi_f\mathbf{A}_f$$

$$=\frac{1}{V}(\phi_e\mathbf{A}_e+\phi_n\mathbf{A}_n+\phi_w\mathbf{A}_w+\phi_s\mathbf{A}_s)$$

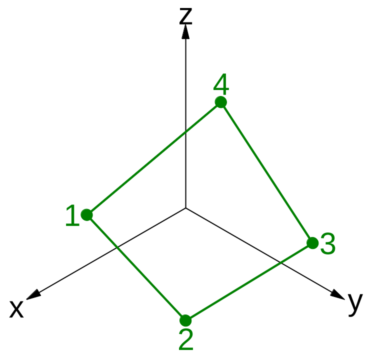
$$=\frac{1}{11}\left\{3.5\begin{pmatrix}3\\-1\\0\end{pmatrix}+2.5\begin{pmatrix}1\\3\\0\end{pmatrix}+1\begin{pmatrix}-2\\2\\0\end{pmatrix}+1.5\begin{pmatrix}-2\\-4\\0\end{pmatrix}\right\}=\frac{1}{11}\begin{pmatrix}8\\0\\0\end{pmatrix}$$

Selecting components,

$$\left(\frac{\partial\phi}{\partial x}\right)_{av}=\frac{8}{11}, \qquad \left(\frac{\partial\phi}{\partial y}\right)_{av}=0$$

Q6.

The cell face is sketched right. Anticipating the means of answering part (b) it is best to find the individual vector areas of triangles Δ_{123} and Δ_{134} first. The total area will be the sum of these two. The vertices will be coplanar if they are parallel.



(a)

$$\mathbf{A}_{123} = \tfrac{1}{2}(\mathbf{r}_2 - \mathbf{r}_1) \wedge (\mathbf{r}_3 - \mathbf{r}_1)$$
$$= \frac{1}{2}\begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix} \wedge \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix}$$
$$= \frac{1}{2}\begin{pmatrix} 6 \\ 4 \\ 4 \end{pmatrix} \quad = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$$

$$\mathbf{A}_{134} = \tfrac{1}{2}(\mathbf{r}_3 - \mathbf{r}_1) \wedge (\mathbf{r}_4 - \mathbf{r}_1)$$
$$= \frac{1}{2}\begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix} \wedge \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$$
$$= \frac{1}{2}\begin{pmatrix} 3 \\ 2 \\ 9 \end{pmatrix} \quad = \begin{pmatrix} 3/2 \\ 1 \\ 9/2 \end{pmatrix}$$

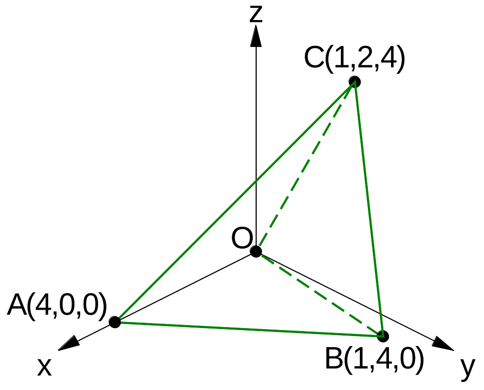
Total vector area:

$$\mathbf{A} = \mathbf{A}_{123} + \mathbf{A}_{134} \quad = \begin{pmatrix} 9/2 \\ 3 \\ 13/2 \end{pmatrix}$$

(b) $\mathbf{A}_{123}$ and $\mathbf{A}_{134}$ are not parallel (one is not a multiple of the other), so the vertices are not coplanar.

Q7.

(a)



$$\mathbf{A}_{OAC} = \tfrac{1}{2}\overrightarrow{OA} \wedge \overrightarrow{OC} = \frac{1}{2}\begin{pmatrix}4\\0\\0\end{pmatrix} \wedge \begin{pmatrix}1\\2\\4\end{pmatrix} = \frac{1}{2}\begin{pmatrix}0\\-16\\8\end{pmatrix} = \begin{pmatrix}0\\-8\\4\end{pmatrix}$$

$$\mathbf{A}_{OCB} = \tfrac{1}{2}\overrightarrow{OC} \wedge \overrightarrow{OB} = \frac{1}{2}\begin{pmatrix}1\\2\\4\end{pmatrix} \wedge \begin{pmatrix}1\\4\\0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-16\\4\\2\end{pmatrix} = \begin{pmatrix}-8\\2\\1\end{pmatrix}$$

$$\mathbf{A}_{OBA} = \tfrac{1}{2}\overrightarrow{OB} \wedge \overrightarrow{OA} = \frac{1}{2}\begin{pmatrix}1\\4\\0\end{pmatrix} \wedge \begin{pmatrix}4\\0\\0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}0\\0\\-16\end{pmatrix} = \begin{pmatrix}0\\0\\-8\end{pmatrix}$$

$$\mathbf{A}_{ABC} = \tfrac{1}{2}\overrightarrow{BC} \wedge \overrightarrow{BA} = \frac{1}{2}\begin{pmatrix}0\\-2\\4\end{pmatrix} \wedge \begin{pmatrix}3\\-4\\0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}16\\12\\6\end{pmatrix} = \begin{pmatrix}8\\6\\3\end{pmatrix}$$

(Check: $\sum \mathbf{A}_f = 0$; OK)

$$V = \frac{1}{3}\sum \mathbf{r}_f \bullet \mathbf{A}_f$$

For three faces we can conveniently take the reference point $\mathbf{r}_f$ as the origin. For the final face take it as the position vector of A (but B or C would do equally well). Then

$$V = \frac{1}{3}\overrightarrow{OA} \bullet \mathbf{A}_{ABC} = \frac{1}{3}\begin{pmatrix}4\\0\\0\end{pmatrix} \bullet \begin{pmatrix}8\\6\\3\end{pmatrix} = \frac{32}{3}$$

(b) The largest face (compare magnitudes of the face-area vectors) is ABC and the smallest is OBA. Hence,

$$(\nabla\phi)_{av} = \frac{1}{V}\sum_{faces} \phi_f \mathbf{A}_f = \frac{3}{32}\left\{6\begin{pmatrix}8\\6\\3\end{pmatrix} + 2\begin{pmatrix}0\\0\\-8\end{pmatrix} + 3\begin{pmatrix}0\\-8\\4\end{pmatrix} + 3\begin{pmatrix}-8\\2\\1\end{pmatrix}\right\} = \frac{3}{32}\begin{pmatrix}24\\18\\17\end{pmatrix}$$

Hence,

$$\left(\frac{\partial \phi}{\partial x}\right)_{av} = \frac{72}{32} = \frac{9}{4}$$

$$\left(\frac{\partial \phi}{\partial y}\right)_{av} = \frac{54}{32} = \frac{27}{16}$$

$$\left(\frac{\partial \phi}{\partial z}\right)_{av} = \frac{51}{32}$$

(c) The point is inside the cell if and only if it lies on the same side of each face as any point in the interior. This can be judged for each face by looking at $(\mathbf{r}-\mathbf{r}_f)\bullet\mathbf{A}_f$, where $\mathbf{r}_f$ is any point on that face and $\mathbf{A}_f$ is the *outward* face vector from part (a). This quantity is always negative for an interior point.

For face OAC:

$$\mathbf{r}_f = 0 \text{ (say)}$$

$$(\mathbf{r}-\mathbf{r}_f)\bullet\mathbf{A}_f = \begin{pmatrix}1\\2\\1\end{pmatrix}\bullet\begin{pmatrix}0\\-8\\4\end{pmatrix} = -12 \quad \text{(negative)}$$

For face OCB:

$$\mathbf{r}_f = 0$$

$$(\mathbf{r}-\mathbf{r}_f)\bullet\mathbf{A}_f = \begin{pmatrix}1\\2\\1\end{pmatrix}\bullet\begin{pmatrix}-8\\2\\1\end{pmatrix} = -3 \quad \text{(negative)}$$

For face OBA (this is actually obvious, but here goes):

$$\mathbf{r}_f = 0$$

$$(\mathbf{r}-\mathbf{r}_f)\bullet\mathbf{A}_f = \begin{pmatrix}1\\2\\1\end{pmatrix}\bullet\begin{pmatrix}0\\0\\-8\end{pmatrix} = -8 \quad \text{(negative)}$$

For face ABC:

$$\mathbf{r}_f = \overrightarrow{OA} = \begin{pmatrix}4\\0\\0\end{pmatrix}$$

$$(\mathbf{r}-\mathbf{r}_f)\bullet\mathbf{A}_f = \begin{pmatrix}-3\\2\\1\end{pmatrix}\bullet\begin{pmatrix}8\\6\\3\end{pmatrix} = -9 \quad \text{(negative)}$$

Hence, for every face the point lies on the same side of the face as points in the interior of the cell. Thus, the point *is* inside the cell.

Q8.
$\Delta\psi = u\Delta y - v\Delta x$

Going from (*sw*) to (*se*):

$$\psi_{se} - 0 = 0\times 0 - 2\times 2 = -4$$

$\Rightarrow \quad \psi_{se} = -4$

Going from (*se*) to (*ne*):

$$\psi_{ne} - \psi_{se} = 3\times 4 - 1\times 1 = 11$$

$\Rightarrow \quad \psi_{ne} = 7$

Going from (*sw*) to (*nw*):

$$\psi_{nw} - 0 = 3\times 2 - (-3)\times 0 = 6$$

$\Rightarrow \quad \psi_{nw} = 6$

If we now go from (*nw*) to (*ne*), at both of which we know ψ, then we can deduce v_n:

$$\psi_{ne} - \psi_{nw} = 1\times 2 - v_n\times 3$$

$\Rightarrow \quad 1 = 2 - 3v_n$

$\Rightarrow \quad v_n = \dfrac{1}{3}$

Q9.

(a) If the cell boundary is traversed anticlockwise then the pressure-force components (per unit depth) are

pressure × (inward) projected area

i.e.

$$\begin{pmatrix} -p\Delta y \\ p\Delta x \end{pmatrix}$$

The pressure-force components are then as set out in the following table.

Face	p_{face}	Δx	Δy	$\boldsymbol{F_x}$	$\boldsymbol{F_y}$
N	11	–4	–1	**11**	**–44**
w	7	0	–3	**21**	**0**
S	12	5	–1	**12**	**60**
E	13	–1	5	**-65**	**–13**

Summing the last two columns, the net pressure force on the cell is

$$\begin{pmatrix} -21 \\ 3 \end{pmatrix} \text{ units}$$

Answer: $\mathbf{F}_n$ = (11,–44), $\mathbf{F}_w$ = (21,0), $\mathbf{F}_s$ = (12,60), $\mathbf{F}_e$ = (–65,–13), $\mathbf{F}_{total}$ = (–21,3).

(b) The (vector) area is half the cross product of the diagonals; i.e.

$$\mathbf{A} = \frac{1}{2}\begin{pmatrix} 4 \\ 4 \\ 0 \end{pmatrix} \wedge \begin{pmatrix} -5 \\ 4 \\ 0 \end{pmatrix} \quad = \frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ 36 \end{pmatrix} \quad = \begin{pmatrix} 0 \\ 0 \\ 18 \end{pmatrix}$$

Answer: Area of the cell is 18 units.

The force per unit volume (or unit area in 2d) is minus the pressure gradient. Hence,

$$\begin{pmatrix} \partial p/\partial x \\ \partial p/\partial y \end{pmatrix} = -\frac{1}{V}\begin{pmatrix} F_x \\ F_y \end{pmatrix} \quad = -\frac{1}{18}\begin{pmatrix} -21 \\ 3 \end{pmatrix} \quad = \begin{pmatrix} 7/6 \\ -1/6 \end{pmatrix}$$

(Alternatively, you could use the formulae for cell-averaged gradients from Section 9 – but this amounts to recalculating pressure × area that you have already done in part (a)).

Answer: Cell-averaged pressure gradients are $(\partial p/\partial x)_{av}$ = 7/6, $(\partial p/\partial y)_{av}$ = –1/6.

(c)

If the cell boundary is traversed anticlockwise then the volume flow rates (per unit depth) are

velocity × projected area

i.e.

$$Q_{face} = u\Delta y - v\Delta x$$

The volume flow rates are then as set out in the following table.

Face	u	v	Δx	Δy	$\boldsymbol{Q_{face}}$
n	8	4	–4	–1	**8**
w	7	2	0	–3	**–21**
s	9	0	5	–1	**–9**
e	u	2	–1	5	**5*u* + 2**

The net outflow must be 0; hence

$$8-21-9+(5u+2)=0$$

$\Rightarrow$ $5u=20$

$\Rightarrow$ $u=4$

The resulting flow rate on this face is

$$Q_e=5\times 4+2 \quad =22$$

Answer: $Q_n = 8$, $Q_w = -21$, $Q_s = -9$, $Q_e = 22$; $u = 4$.

(d) The change in streamfunction from one vertex to the next is (with the commoner sign convention) the net flow clockwise across the line joining them. Hence,

$$\psi_A=3 \quad \text{(given)}$$

$$\psi_B=\psi_A+Q_n \quad =3+8 \quad =11$$

$$\psi_C=\psi_B+Q_w \quad =11-21 \quad =-10$$

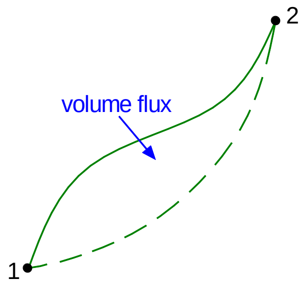
$$\psi_D=\psi_C+Q_s \quad =-10-9 \quad =-19$$

Answer: $\psi_A = 3$, $\psi_B = 11$, $\psi_C = -10$, $\psi_D = -19$.

Q10.

(a) The *change* in stream function between 2 points is given by

$\Delta\psi$ = net volume flux (per unit depth) across any curve connecting two points

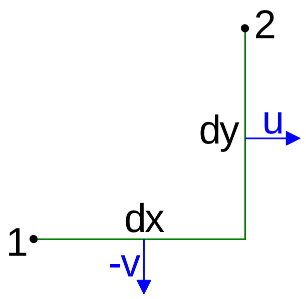


$$\psi_2 - \psi_1 = \int_1^2 \mathbf{u} \bullet \mathbf{n} \;\; \mathrm{d}s$$

By the incompressibility assumption (conservation of volume) this is well-defined (i.e. independent of the curve joining the points)

Evaluating the volume flux from components:

$$\mathrm{d}\psi = u\,\mathrm{d}y - v\,\mathrm{d}x$$



Hence,

$$u = \frac{\partial\psi}{\partial y}, \qquad v = -\frac{\partial\psi}{\partial x}$$

(**Note: the alternative sign convention for ψ is perfectly legitimate**).

(b) The vorticity is defined by $\boldsymbol{\omega} = \nabla \wedge \mathbf{u}$. If the flow is 2-dimensional then only the z component is non-zero and is given by

$$\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

Substituting for the velocity components in terms of ψ:

$$\omega_z = -\frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial y^2} \quad = -\nabla^2\psi$$

If the flow is irrotational then $\omega_z = 0$ and

$\nabla^2\psi = 0$ (Laplace's equation)

Otherwise, ψ and ω_z are related by

$$\omega_z = -\nabla^2\psi$$

(c) Use

$$\Delta\psi = u\Delta y - v\Delta x$$

Going from (*sw*) to (*se*):

$$\psi_{se} - 0 = 0 \times 0 - 2 \times 2 = -4$$

$\Rightarrow \qquad \psi_{se} = -4 \qquad\qquad (*)$

Going from (*se*) to (*ne*):

$$\psi_{ne} - \psi_{se} = 3 \times 3 - (-1) \times 1 = 10$$

$\Rightarrow \qquad \psi_{ne} = 6 \qquad\qquad (*)$

Going from (*sw*) to (*nw*):

$$\psi_{nw} - 0 = 3 \times 2 - 2 \times (-1) = 8$$

$$\Rightarrow \qquad \psi_{nw} = 8 \qquad (*)$$

(d) If we now go from (*nw*) to (*ne*), at both of which we know ψ, then we can deduce v_n:

$$\psi_{ne} - \psi_{nw} = 1 \times 1 - v_n \times 4$$

$$\Rightarrow \qquad -2 = 1 - 4v_n$$

$$\Rightarrow \qquad v_n = \frac{3}{4}$$

(e) By Stokes' Theorem,

$$\oint \mathbf{u} \bullet \mathrm{d}\mathbf{s} = \int_A (\nabla \wedge \mathbf{u}) \bullet \mathrm{d}\mathbf{A} \quad \rightarrow \quad \omega_z A$$

For the line integral, going round edges *w*-*s*-*e*-*n*:

$$\oint \mathbf{u} \bullet \mathrm{d}\mathbf{s} = [3 \times 1 + 2 \times (-2)] + [0 \times 2 + 2 \times 0] + [3 \times 1 + (-1) \times 3] + [1 \times (-4) + 0.75 \times (-1)]$$

$$= -1 + 0 + 0 - 4.75$$

$$= -5.75$$

Since the vector area is half the vector cross product of the diagonals (e.g. *sw* to *ne*, *se* to *nw*),

$$\mathbf{A} = \frac{1}{2}\begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} \wedge \begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix} \quad = \frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ 15 \end{pmatrix} \quad = \begin{pmatrix} 0 \\ 0 \\ 7.5 \end{pmatrix}$$

Hence,

$$A = 7.5$$

Then,

$$\omega_z = \frac{-5.75}{7.5} \quad = -\frac{23}{30}$$

Q11.

(a) Vector area = half the cross product of the diagonals. Hence,

$$\mathbf{A} = \frac{1}{2}\begin{pmatrix}4.5\\2\\0\end{pmatrix} \wedge \begin{pmatrix}-1\\4\\0\end{pmatrix} \quad = \frac{1}{2}\begin{pmatrix}0\\0\\20\end{pmatrix} \quad = \begin{pmatrix}0\\0\\10\end{pmatrix}$$

Answer: A = 10 units.

(b) The volume V of the 2-d cell is just the plan area. Hence, using the fact that the outward "area" of each side "face" is $(\Delta y, -\Delta x)$ when the cell is traversed anticlockwise, and taking faces in the order e, n, s, w:

$$\left(\frac{\partial u}{\partial x}\right)_{av} = \frac{1}{V}\sum u\Delta y \quad = \frac{1}{10}\big(10\times 3 + 13\times 1 + 9\times(-3) + 1\times(-1)\big) \quad = 1.5$$

$$\left(\frac{\partial u}{\partial y}\right)_{av} = \frac{1}{V}\sum u(-\Delta x) \quad = \frac{1}{10}\big(10\times(-2.5) + 13\times 3.5 + 9\times 1 + 1\times(-2)\big) \quad = 2.75$$

$$\left(\frac{\partial v}{\partial x}\right)_{av} = \frac{1}{V}\sum v\Delta y \quad = \frac{1}{10}\big(12\times 3 + 6\times 1 + (-6)\times(-3) + 0\times(-1)\big) \quad = 6$$

$$\left(\frac{\partial v}{\partial y}\right)_{av} = \frac{1}{V}\sum v(-\Delta x) \quad = \frac{1}{10}\big(12\times(-2.5) + 6\times 3.5 + (-6)\times 1 + 0\times(-2)\big) \quad = -1.5$$

(c) The line integral discretises as

$$\oint \mathbf{u}\bullet \mathrm{d}\mathbf{s} = \sum_{faces}(u\Delta x + v\Delta y)$$

$$= (10\times 2.5 + 12\times 3) + (13\times(-3.5) + 6\times 1) + (9\times(-1) + (-6)\times(-3)) + (1\times 2 + 0\times(-1))$$

$$= 32.5$$

Then, from Stokes' Law:

$$(\omega_z)_{av} = \frac{1}{A}\oint \mathbf{u}\bullet \mathrm{d}\mathbf{s} \quad = \frac{1}{10}\times 32.5 \quad = 3.25$$

Alternatively, from the cell-averaged velocity derivatives,

$$(\omega_z)_{av} = \left(\frac{\partial v}{\partial x}\right)_{av} - \left(\frac{\partial u}{\partial y}\right)_{av} \quad = 6 - 2.75 \quad = 3.25$$

Both methods give the same answer. (Using the formulae for the velocity derivatives, this is readily seen to be true in general).