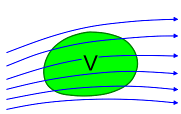
2. Fluid-Flow Equations

Governing Equations

- **Conservation equations for:**
  - mass
  - momentum
  - energy
  - (other constituents)
- **Alternative forms:**
  - integral (control-volume) equations
  - differential equations

Integral (Control-Volume) Approach

Consider the budget of **any physical quantity** in a **control volume** *V*



$$\begin{pmatrix}\text{RATE OF CHANGE}\\ \textit{inside } V\end{pmatrix} + \begin{pmatrix}\text{NET FLUX}\\ \textit{through boundary of } V\end{pmatrix} = \begin{pmatrix}\text{SOURCE}\\ \textit{inside } V\end{pmatrix}$$

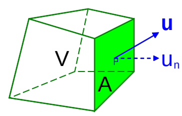
$$\begin{pmatrix}\text{RATE OF CHANGE}\\ \textit{inside } V\end{pmatrix} + \begin{pmatrix}\text{ADVECTION} + \text{DIFFUSION}\\ \textit{through boundary of } V\end{pmatrix} = \begin{pmatrix}\text{SOURCE}\\ \textit{inside } V\end{pmatrix}$$

→ **Finite-volume** method for CFD

Mass Conservation (Continuity)

Physical principle: mass is neither created nor destroyed

Change of mass = net mass that has entered



Rate form:

$$\frac{d}{dt}(\text{mass}) = \text{net } \textit{inward} \text{ mass flux}$$

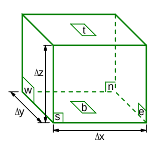
$$\frac{d}{dt}(\text{mass}) + \text{net } \textit{outward} \text{ mass flux} = 0$$

Mass in cell: ρV

Mass flux through a face: $C = \rho u_n A \;= \rho \mathbf{u} \bullet \mathbf{A}$

$$\frac{d}{dt}(\text{mass}) \;+\; \sum_{faces}(\text{mass flux}) \;=\; 0 \qquad\qquad \frac{d}{dt}\int_V \rho \, dV \;+ \oint_{\partial V} \rho \mathbf{u} \bullet d\mathbf{A} \;= 0$$

Mass Conservation - Differential Form



$$\frac{d}{dt}(\text{mass}) + \text{net outward mass flux} = 0$$

$$\frac{d}{dt}(\rho V) \;+\; (\rho u A)_e - (\rho u A)_w \;+\; (\rho v A)_n - (\rho v A)_s \;+\; (\rho w A)_t - (\rho w A)_b \;\; = 0$$

$$\frac{d}{dt}(\rho \Delta x \Delta y \Delta z) \;+\; [(\rho u)_e - (\rho u)_w]\Delta y \Delta z \;+\; [(\rho v)_n - (\rho v)_s]\Delta z \Delta x \;+\; [(\rho w)_t - (\rho w)_b]\Delta x \Delta y \;=\; 0$$

$$\frac{d\rho}{dt} \;+\; \frac{(\rho u)_e - (\rho u)_w}{\Delta x} \;+\; \frac{(\rho v)_n - (\rho v)_s}{\Delta y} \;+\; \frac{(\rho w)_t - (\rho w)_b}{\Delta z} \;=\; 0$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \qquad\qquad \frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \mathbf{u}) = 0$$

Differential Form (Faster Derivation)

$$\underbrace{\frac{d}{dt}\int_V \rho \;\; dV}_{mass\ in\ cell} + \underbrace{\oint_{\partial V} \rho \mathbf{u} \bullet d\mathbf{A}}_{net\ mass\ flux} = 0$$

fixed control volume

divergence theorem

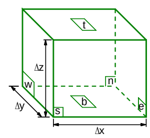
$$\int_V \left\{ \frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \mathbf{u}) \right\} dV = 0$$

volume *V* is arbitrary

$$\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \mathbf{u}) = 0$$

Continuity in Incompressible Flow

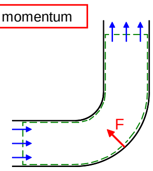
$\cancel{\frac{d}{dt}(\text{volume}) + \text{net outward volume flux} = 0}$



$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad \nabla \bullet \mathbf{u} = 0$$

Momentum Equation

Momentum Principle: Force = rate of change of momentum



If **steady**:

force = (momentum flux)$_{out}$ – (momentum flux)$_{in}$

If **unsteady**:

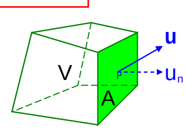
force = d/dt(momentum inside control volume)

+ (momentum flux)$_{out}$ – (momentum flux)$_{in}$

Momentum Equation

Physical principle: rate of change of momentum = force

$$\frac{d}{dt}(\text{momentum}) + \text{net outward momentum flux} = \text{force}$$



Momentum of fluid in the cell = mass × **u** $= (\rho V)\mathbf{u}$

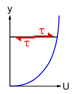
Momentum flux through a face = mass flux × **u** $= (\rho \mathbf{u} \bullet \mathbf{A})\mathbf{u}$

$$\frac{d}{dt}(\text{mass} \times \mathbf{u}) + \sum_{faces}(\text{mass flux} \times \mathbf{u}) = \mathbf{F}$$

Fluid Forces

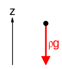
Surface forces (proportional to **area**): $\text{stress} = \dfrac{\text{force}}{\text{area}}$

- pressure
- viscous force: $\tau = \mu \dfrac{\partial u}{\partial y}$

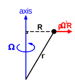


Body forces (proportional to **volume**): $\text{force density} = \dfrac{\text{force}}{\text{volume}}$

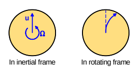
- gravity: $(0, 0, -\rho g)$



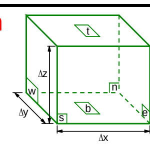
- centrifugal force: $\rho \Omega^2 \mathbf{R}$



- Coriolis force: $-2\rho \boldsymbol{\Omega} \wedge \mathbf{u}$



Momentum Equation



$$\frac{\mathrm{d}}{\mathrm{d}t}(\text{momentum}) + \text{net momentum flux} = \text{force}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho V u) \quad + \quad (\rho u A)_e u_e - (\rho u A)_w u_w \quad + \quad (\rho v A)_n u_n - (\rho v A)_s u_s \quad + \quad (\rho w A)_t u_t - (\rho w A)_b u_b$$
$$= \quad p_w A_w - p_e A_e \quad + \textit{viscous and other forces}$$

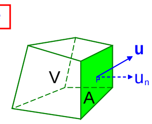
$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho \Delta x \Delta y \Delta z\, u) \quad + \quad [(\rho u)_e u_e - (\rho u)_w u_w]\Delta y \Delta z \quad + \quad [(\rho v)_n u_n - (\rho v)_s u_s]\Delta z \Delta x \quad + \quad [(\rho w)_t u_t - (\rho w)_b u_b]\Delta x \Delta y$$
$$= (p_w - p_e)\Delta y \Delta z \quad + \textit{viscous and other forces}$$

$$\frac{\mathrm{d}(\rho u)}{\mathrm{d}t} \quad + \quad \frac{(\rho u u)_e - (\rho u u)_w}{\Delta x} \quad + \quad \frac{(\rho v u)_n - (\rho v u)_s}{\Delta y} \quad + \quad \frac{(\rho w u)_t - (\rho w u)_b}{\Delta z}$$
$$= \quad -\frac{(p_e - p_w)}{\Delta x} \quad + \quad \textit{viscous and other forces}$$

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u u)}{\partial x} + \frac{\partial(\rho v u)}{\partial y} + \frac{\partial(\rho w u)}{\partial z} = -\frac{\partial p}{\partial x} + \mu \nabla^2 u + \textit{other forces}$$

General Scalar

Rate of change + net outward flux = source



ϕ = **concentration** (amount per unit mass)

Amount in cell: $\rho V \phi$ (mass × concentration)

Flux through a face:

- **advection**: $(\rho \mathbf{u} A)\phi$ (mass flux × concentration)
- **diffusion**: $-\Gamma \dfrac{\partial \phi}{\partial n} A$ (diffusivity × gradient × area)

Source: $S = sV$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\text{mass} \times \phi) \quad + \quad \sum_{\textit{faces}} (\text{mass flux} \times \phi \quad - \Gamma \frac{\partial \phi}{\partial n} A) \quad = \quad sV$$

$$\frac{\partial(\rho\phi)}{\partial t} + \frac{\partial}{\partial x}(\rho u \phi - \Gamma \frac{\partial \phi}{\partial x}) + \frac{\partial}{\partial y}(\rho v \phi - \Gamma \frac{\partial \phi}{\partial y}) + \frac{\partial}{\partial z}(\rho w \phi - \Gamma \frac{\partial \phi}{\partial z}) = s$$

Momentum Components as General Scalars

General scalar-transport equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\text{mass}\times\phi) \;+\; \sum_{faces}(\text{mass flux}\times\phi \;-\Gamma\frac{\partial\phi}{\partial n}A) \;=\; S$$

Momentum equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\text{mass}\times u) \;+\; \sum_{faces}\text{mass flux}\times u \;=\; \underbrace{\sum_{faces}(\mu\frac{\partial u}{\partial n}A)}_{viscous\ forces}+\text{other forces}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\text{mass}\times u) \;+\; \sum_{faces}(\text{mass flux}\times u \;-\mu\frac{\partial u}{\partial n}A) \;=\; \text{other forces}$$

- **Velocity components u, v, w satisfy individual scalar-transport equations:**
  - concentration, ϕ ← velocity
  - diffusivity, Γ ← viscosity
  - source, S ← other forces
- **Differences:**
  - non-linear
  - coupled
  - also have to be mass-consistent

Differential Equations For Fluid Flow

Forms of the equations in **primitive** variables may be:

- **Conservative**
  - can be integrated directly to give a control-volume equation
- **Non-conservative**
  - material derivative following the flow

Other forms of the equations include those for:

- **Derived variables**
  - e.g. velocity potential; stream function.

Example

$$\frac{\mathrm{d}}{\mathrm{d}x}(y^2) = g(x)$$ **conservative**

$$2y\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)$$ **non-conservative**

Same equation! – but only the first can be integrated directly

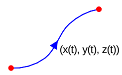
Rate of Change Following the Flow

(**Non-conservative** or **Lagrangian** equations)

$$\phi \equiv \phi(t, \mathbf{x})$$

Total derivative (following <u>any</u> path $\mathbf{x}(t)$)

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} \equiv \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial \phi}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial \phi}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}$$



Material derivative (following the flow):

$$\frac{\mathrm{D}\phi}{\mathrm{D}t} \equiv \frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} + v\frac{\partial \phi}{\partial y} + w\frac{\partial \phi}{\partial z}$$

conservative form → **non-conservative form**

$$\frac{\partial}{\partial t}(\rho\phi) + \frac{\partial}{\partial x}(\rho u\phi) + \frac{\partial}{\partial y}(\rho v\phi) + \frac{\partial}{\partial z}(\rho w\phi) \quad \underset{(mass\ conservation)}{\rightarrow} \quad \rho\frac{\mathrm{D}\phi}{\mathrm{D}t}$$

e.g. momentum equation: $\underbrace{\rho\frac{\mathrm{D}u}{\mathrm{D}t}}_{mass \times acceleration} = \underbrace{-\frac{\partial p}{\partial x} + \mu\nabla^2 u}_{forces}$

Example, Q1

In 2-d flow, the continuity and x-momentum equations can be written in conservative form as

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \qquad \frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u u) + \frac{\partial}{\partial y}(\rho v u) = -\frac{\partial p}{\partial x} + \mu\nabla^2 u$$

(a) Show that these can be written in the equivalent non-conservative forms:

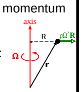
$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \qquad \rho\frac{\mathrm{D}u}{\mathrm{D}t} = -\frac{\partial p}{\partial x} + \mu\nabla^2 u$$

(b) Define carefully what is meant by the statement that a flow is *incompressible*. To what does the continuity equation reduce in incompressible flow?

(c) Write down conservative forms of the 3-d equations for mass and x-momentum.

(d) Write down the z-momentum equation, including gravitational forces;

(e) Show that, for constant-density flows, pressure and gravity can be combined in the momentum equations via the *piezometric pressure* $p + \rho g z$.



(f) In a rotating reference frame there are additional apparent forces (per unit volume):
centrifugal force: $-\rho\mathbf{\Omega}\wedge(\mathbf{\Omega}\wedge\mathbf{r})$ or $\rho\Omega^2\mathbf{R}$
Coriolis force: $-2\rho\mathbf{\Omega}\wedge\mathbf{u}$
where $\mathbf{\Omega}$ is the angular velocity of the reference frame, $\mathbf{u}$ is the fluid velocity in that frame, $\mathbf{r}$ is the position vector and $\mathbf{R}$ is its projection perpendicular to the axis of rotation. By writing the centrifugal force as the gradient of some quantity show that it can be subsumed into a modified pressure. Also, find the components of the Coriolis force if rotation is about the z axis.

Example, Q2

The x-component of the momentum equation is given by

$$\rho\frac{\mathrm{D}u}{\mathrm{D}t} = -\frac{\partial p}{\partial x} + \mu\nabla^2 u$$

Using this equation, derive the velocity profile in fully-developed, laminar flow for:

(a) pressure-driven flow between stationary parallel planes ("*Poiseuille flow*");

(b) constant-pressure flow between stationary and moving planes ("*Couette flow*").

Non-Dimensionalisation

Form non-dimensional variables using length (L_0), velocity (U_0) and density (ρ_0) scales:

$$\mathbf{x} = L_0\mathbf{x}^*, \quad t = \frac{L_0}{U_0}t^*, \quad \mathbf{u} = U_0\mathbf{u}^*, \quad \rho = \rho_0\rho^*, \quad p = p_{ref} + \rho_0 U_0^2 p^*, \quad etc.$$

Substitute into the governing equations:

$$\rho\frac{\mathrm{D}u}{\mathrm{D}t} = -\frac{\partial p}{\partial x} + \mu\nabla^2 u \longrightarrow \frac{\rho_0 U_0^2}{L_0}\rho^*\frac{\mathrm{D}u^*}{\mathrm{D}t^*} = -\frac{\rho_0 U_0^2}{L_0}\frac{\partial p^*}{\partial x^*} + \frac{\mu U_0}{L_0^2}\nabla^{*2}u^*$$

$$\longrightarrow \rho^*\frac{\mathrm{D}u^*}{\mathrm{D}t^*} = -\frac{\partial p^*}{\partial x^*} + \frac{\mu}{\rho_0 U_0 L_0}\nabla^{*2}u^*$$

$$\longrightarrow \rho^*\frac{\mathrm{D}u^*}{\mathrm{D}t^*} = -\frac{\partial p^*}{\partial x^*} + \frac{1}{\mathrm{Re}}\nabla^{*2}u^*$$

Identify important dimensionless groups: $\mathrm{Re} = \frac{\rho_0 U_0 L_0}{\mu}$ $\quad \rho\frac{\mathrm{D}u}{\mathrm{D}t} = -\frac{\partial p}{\partial x} + \frac{1}{\mathrm{Re}}\nabla^2 u$

Common Dimensionless Groups

$\mathrm{Re} \equiv \frac{\rho UL}{\mu}$ **Reynolds number** (viscous flow; μ = dynamic viscosity)

$\mathrm{Fr} \equiv \frac{U}{\sqrt{gL}}$ **Froude number** (open-channel flow; g = gravity)

$\mathrm{Ma} \equiv \frac{U}{c}$ **Mach number** (compressible flow; c = speed of sound)

$\mathrm{Ro} \equiv \frac{U}{\Omega L}$ **Rossby number** (rotating flows; Ω = angular velocity of frame)

$\mathrm{We} \equiv \frac{\rho U^2 L}{\sigma}$ **Weber number** (free-surface flows; σ = surface tension)

Advantages of Non-Dimensionalisation

- All **dynamically-similar** problems (same Re etc.) can be solved with a single computation
- The **number of parameters** is reduced
- It indicates the **relative size** of different terms in the governing equations: in particular, which might be neglected
- Computational **variables are similar size**, yielding better numerical accuracy

Summary (1)

- The fluid-flow equations are **conservation equations** for:
  - mass
  - momentum
  - energy
  - (additional constituents)
- The equations can be written in equivalent **integral** (**control-volume**) or **differential** forms
- The **finite-volume** method is a direct discretisation of the control-volume equations
- Differential forms of the flow equations may be **conservative** or **non-conservative**
- For any conserved property and arbitrary control volume:

  rate of change + net outward flux = source

Summary (2)

- There are really just two canonical equations to solve:
  - **mass conservation** (continuity)
  - a **generic scalar-transport** equation
- Each Cartesian velocity component satisfies its own scalar-transport equation
- However, the momentum equations are:
  - non-linear
  - coupled
  - also required to be mass-consistent
- **Non-dimensionalisation:**
  - solves dynamically-similar (Re, Fr, Ro, …) flows with a single computation
  - reduces number of relevant parameters
  - identifies relative importance of terms in governing equations
  - maintains numerical variables of similar size