Answers 2

Q1.
(a)
Mass
Conservative form (given in the question):

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)=0$$

Expanding the second and third derivatives using the product rule:

$$\frac{\partial \rho}{\partial t}+\left(\frac{\partial \rho}{\partial x}u+\rho\frac{\partial u}{\partial x}\right)+\left(\frac{\partial \rho}{\partial y}v+\rho\frac{\partial v}{\partial y}\right)=0$$

Collecting terms into those containing derivatives of ρ and those containing multiples of ρ:

$$\left(\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}\right)+\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0$$

Hence, using the definition of the material derivative:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t}+\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0$$

Momentum
Conservative form (given in the question):

$$\frac{\partial}{\partial t}(\rho u)+\frac{\partial}{\partial x}(\rho uu)+\frac{\partial}{\partial y}(\rho vu)=-\frac{\partial p}{\partial x}+\mu\nabla^2 u$$

(Partially) expanding the derivatives of products ρu, $\rho u\times u$ and $\rho v\times u$ on the LHS:

$$\mathrm{LHS}=\left(\frac{\partial \rho}{\partial t}u+\rho\frac{\partial u}{\partial t}\right)+\left(\frac{\partial(\rho u)}{\partial x}u+\rho u\frac{\partial u}{\partial x}\right)+\left(\frac{\partial(\rho v)}{\partial y}u+\rho v\frac{\partial u}{\partial y}\right)$$

Collecting terms (as either multiples of u or multiples of ρ):

$$\mathrm{LHS}=\underbrace{\left(\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}\right)}_{=0\ \textit{by mass conservation}}u+\rho\underbrace{\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)}_{=\mathrm{D}u/\mathrm{D}t}$$

Hence, the LHS of the momentum equation is just $\rho\mathrm{D}u/\mathrm{D}t$. The equation is then

$$\rho\frac{\mathrm{D}u}{\mathrm{D}t}=-\frac{\partial p}{\partial x}+\mu\nabla^2 u$$

(b) *Incompressible*: flow-induced changes to pressure (or temperature) do not cause significant changes in density; i.e. density does not change along a streamline:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t}=0$$

(Note that this **does not necessarily** imply that ρ is the same everywhere, although that is often the case.)

From part (a) the 2-d continuity equation then reduces to

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

(c)

Mass: $$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)=0$$

Momentum: $$\frac{\partial}{\partial t}(\rho u)+\frac{\partial}{\partial x}(\rho uu)+\frac{\partial}{\partial y}(\rho vu)+\frac{\partial}{\partial z}(\rho wu)=-\frac{\partial p}{\partial x}+\mu\nabla^2 u$$

where $$\nabla^2 \equiv \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$$

(d)

$$\frac{\partial}{\partial t}(\rho w)+\frac{\partial}{\partial x}(\rho uw)+\frac{\partial}{\partial y}(\rho vw)+\frac{\partial}{\partial z}(\rho ww)=-\frac{\partial p}{\partial z}-\rho g+\mu\nabla^2 w$$

Note the extra term on the RHS, corresponding to the gravitational force per unit volume. The LHS can also be written in the more compact non-conservative form as $\rho\frac{\mathrm{D}w}{\mathrm{D}t}$.

(e) The terms representing pressure and gravitational forces in the momentum equation can be combined as the gradient of a single scalar field:

$$\begin{aligned}
-\frac{\partial p}{\partial x} &= -\frac{\partial}{\partial x}(p+\rho g z)\\
-\frac{\partial p}{\partial y} &= -\frac{\partial}{\partial y}(p+\rho g z)\\
-\frac{\partial p}{\partial z}-\rho g &= -\frac{\partial}{\partial z}(p+\rho g z)
\end{aligned}$$

or, more concisely,

$$-\nabla p+\rho\mathbf{g}=-\nabla(p+\rho g z) \qquad \text{where} \qquad \mathbf{g}=(0,0,-g)$$

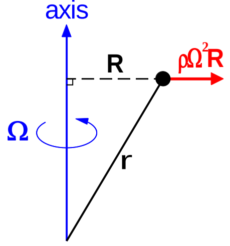
Hence, pressure and gravitational forces can be combined as the gradient of a single modified pressure $p + \rho g z$ (called the *piezometric pressure*).

(f)

Centrifugal Force

The centrifugal force (corresponding to centrifugal acceleration $\Omega^2 R$) is

$$\begin{aligned}
-\rho\boldsymbol{\Omega}\wedge(\boldsymbol{\Omega}\wedge\mathbf{r}) &= \rho\Omega^2\mathbf{R}\\
&= \nabla(\tfrac{1}{2}\rho\Omega^2 R^2)
\end{aligned}$$



where $\mathbf{R}$ is the part of the position vector $\mathbf{r}$ perpendicular to the axis of rotation.

Hence, in the momentum equations the pressure and centrifugal forces can again be combined as the gradient of a single scalar field:

$$-\nabla p+\rho\Omega^2\mathbf{R}=-\nabla[p-\tfrac{1}{2}\rho\Omega^2 R^2]$$

corresponding to a modified pressure $p-\frac{1}{2}\rho\Omega^2 R^2$.

In steady rotation, $p - \frac{1}{2}\rho\Omega^2 R^2$ will be a constant, meaning that pressure p itself will increase with radius R (essentially to provide the centripetal acceleration necessary to keep moving in a circle).

CFD codes usually solve internally for the modified pressure $p - \frac{1}{2}\rho\Omega^2 R^2$, but will need to recover the actual pressure p at boundaries in order to calculate forces on structures etc.

Coriolis Force

$$\begin{aligned} -2\rho\mathbf{\Omega} \wedge \mathbf{u} &= -2\rho \begin{pmatrix} 0 \\ 0 \\ \Omega \end{pmatrix} \wedge \begin{pmatrix} u \\ v \\ w \end{pmatrix} \\ &= -2\rho \begin{pmatrix} -\Omega v \\ \Omega u \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 2\rho\Omega v \\ -2\rho\Omega u \\ 0 \end{pmatrix} \end{aligned}$$

The x and y components are, therefore, $2\rho\Omega v$ and $-2\rho\Omega u$

This is perpendicular to the velocity and causes a turn to the right when looking down the axis of rotation. It is why the geostrophic winds on a weather map blow clockwise around a high-pressure region in the northern hemisphere and anticlockwise in the southern hemisphere; the Coriolis force (perpendicular to velocity) is then in balance with the pressure force (from high pressure to low pressure). Since the rotation about the high pressure centre is in the opposite sense to the earth's rotation this is known as an *anticyclone*.

(g)

Mass: $$\frac{\partial\rho}{\partial t} + \nabla \bullet (\rho\mathbf{u}) = 0$$

Momentum: $$\frac{\partial}{\partial t}(\rho\mathbf{u}) + \nabla \bullet (\rho\mathbf{u} \otimes \mathbf{u}) = -\nabla p + \mu\nabla^2\mathbf{u}$$

(h)

Mass: $$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) = 0$$

Momentum: $$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = -\frac{\partial p}{\partial x_i} + \mu\frac{\partial^2 u_i}{\partial x_j \partial x_j}$$

Q2.

Since the flow is fully developed, there is no acceleration: $\frac{\mathrm{D}u}{\mathrm{D}t} = 0$.

Since u is only a function of y, the Laplacian $\nabla^2 u$ reduces to $\frac{\partial^2 u}{\partial y^2}$.

Hence, the x-momentum equation reduces to

$$0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2}$$

Since u and its derivatives are at most functions of y, then $\partial p/\partial x$ is independent of x or z. The corresponding y-momentum equation reduces to

$$0 = -\frac{\partial p}{\partial y}$$

so that p (and hence $\partial p/\partial x$) are not functions of y either. Hence, the pressure gradient $\partial p/\partial x$ is a constant: call it $-G$; (negative since pressure must decrease in the direction of the flow).

Thus, the mean velocity profile satisfies

$$\frac{\mathrm{d}^2 u}{\mathrm{d}y^2} = -\frac{G}{\mu} \quad (= \mathit{constant})$$

(Partial derivatives may be replaced by ordinary derivatives since u is only a function of y here). This can be integrated twice to give

$$u = -\frac{1}{2}\frac{G}{\mu} y^2 + Ay + B$$

where A and B are constants of integration.

In both cases considered, $u = 0$ on $y = 0$, so that $B = 0$, and the profile may be written

$$u = y(A - \frac{1}{2}\frac{G}{\mu} y) \qquad (*)$$

The value of A depends on the boundary condition at the top of the channel, $y = h$.

(a) Plane Poiseuille flow: $u = 0$ at $y = h$ implies that

$$A = \frac{1}{2}\frac{G}{\mu} h$$

and the solution (*) becomes

$$u = \frac{G}{2\mu} y(h - y) \qquad \text{(a parabola)}$$

(b) Couette flow: with zero pressure gradient, $G = 0$, (*) becomes

$$u = Ay \quad \text{(linear)}$$

and the boundary condition $u = U_w$ on $y = h$ gives $A = U_w / h$, or

$$u = \frac{U_w y}{h} \qquad \text{(a straight line)}$$

A pressure gradient is not required to drive the flow in this instance as the fluid is "dragged along" by the upper wall.

Q3.

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$$\frac{\partial}{\partial t}(\rho\phi) + \nabla \bullet (\rho \mathbf{u}\phi - \Gamma\nabla\phi) = s \qquad \text{(conservative form)}$$

$$\rho\frac{\mathrm{D}\phi}{\mathrm{D}t} - \nabla \bullet (\Gamma\nabla\phi) = s \qquad \text{(non-conservative form)}$$

Q4.
(a) (ii) = pressure coefficient, c_P.

(b) (ii) = Reynolds number (viscous flows).

(c) (i) = gradient Richardson number (stratified or density-varying flows).

The numerator is the *buoyancy frequency* (aka *Brunt-Väisälä frequency*), the frequency of small-amplitude gravity waves in density-stratified flow, such as often occurs in the atmosphere or oceans. The denominator is a frequency based on the mean velocity gradient (aka *shear frequency*).

(d) (iii) = Rossby number (flow in a rotating reference frame).

Some authors refer to this as (Rossby number)$^{-1}$. C'est la vie!

Q5.

(a)

$$-2\rho\boldsymbol{\Omega}\wedge\mathbf{u} = -2\rho\begin{pmatrix}0\\0\\\Omega\end{pmatrix}\wedge\begin{pmatrix}u\\v\\w\end{pmatrix}$$

$$= -2\rho\begin{pmatrix}-\Omega v\\\Omega u\\0\end{pmatrix}$$

$$= \begin{pmatrix}2\rho\Omega v\\-2\rho\Omega u\\0\end{pmatrix}$$

The x and y components of the Coriolis force are $2\rho\Omega v$ and $-2\rho\Omega u$, respectively.

(b)

x-component: $\rho\dfrac{\mathrm{D}u}{\mathrm{D}t} = -\dfrac{\partial p}{\partial x} + \mu\nabla^2 u + 2\rho\Omega v$

y-component: $\rho\dfrac{\mathrm{D}v}{\mathrm{D}t} = -\dfrac{\partial p}{\partial y} + \mu\nabla^2 v - 2\rho\Omega u$

(c) Write variables in terms of their non-dimensional counterparts (denoted by an asterisk, *) and scales ρ_0 (density), U_0 (velocity) and L_0 (length), together with a reference pressure p_{ref}:

$$\rho = \rho_0\rho^*$$
$$\mathbf{u} = U_0\mathbf{u}^*$$
$$p = p_{ref} + \rho_0 U_0^2 p^*$$
$$\mathbf{x} = L_0\mathbf{x}^*$$
$$t = (L_0/U_0)t^*$$

Substituting in Equation (*),

$$\frac{\rho_0 U_0^2}{L_0}\rho^*\frac{\mathrm{D}\mathbf{u}^*}{\mathrm{D}t^*} = -\frac{\rho_0 U_0^2}{L_0}\nabla^* p^* + \frac{\mu U_0}{L_0^2}\nabla^{*2}\mathbf{u}^* - 2\rho_0\Omega U_0\rho^*\mathbf{e}_\Omega\wedge\mathbf{u}^*$$

where $\mathbf{e}_\Omega$ is a unit vector along the rotation axis.

Multiplying through by $\dfrac{L_0}{\rho_0 U_0^2}$:

$$\rho^*\frac{\mathrm{D}\mathbf{u}^*}{\mathrm{D}t^*} = -\nabla^* p^* + \frac{\mu}{\rho_0 U_0 L_0}\nabla^{*2}\mathbf{u}^* - 2\frac{\Omega L_0}{U_0}\rho^*\mathbf{e}_\Omega\wedge\mathbf{u}^*$$

Hence, in non-dimensional variables (dropping the asterisks), the momentum equation is

$$\rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p + \frac{1}{\mathrm{Re}}\nabla^2\mathbf{u} - \frac{2}{\mathrm{Ro}}\rho\mathbf{e}_\Omega \wedge \mathbf{u}$$

where:

Reynolds number: $\mathrm{Re} \equiv \dfrac{\rho_0 U_0 L_0}{\mu}$

Rossby number: $\mathrm{Ro} \equiv \dfrac{U_0}{\Omega L_0}$

(d) *Conservative* means that flux terms can be "integrated directly" – i.e. all derivatives of the dependent variable are "on the outside" of expressions.

For example, in one dimension a conservative differential equation for variable ϕ has the form

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) = s(x)$$

and can be integrated directly to give

$$\Big[f(x) \Big]_{x_1}^{x_2} = \int_{x_1}^{x_2} s \;\; \mathrm{d}x$$

or

$$f(x_2) - f(x_2) = \int_{x_1}^{x_2} s \;\; \mathrm{d}x$$

with the LHS corresponding to "flux$_{out}$ – flux$_{in}$".

Non-conservative means that flux terms are written in a form that cannot be integrated directly; i.e. are not total derivatives.

(e) The material derivative operator is defined by

$$\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

The x-momentum equation is

$$\rho\frac{\mathrm{D}u}{\mathrm{D}t} = -\frac{\partial p}{\partial x} + \mu\nabla^2 u + 2\rho\Omega v$$

Expanding the LHS and adding u times the continuity equation, the LHS is equivalent to

$$\underbrace{\left(\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} + \rho w\frac{\partial u}{\partial z}\right)}_{\rho\mathrm{D}u/\mathrm{D}t} \quad + u\underbrace{\left(\frac{\partial\rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right)}_{=0 \text{ by continuity}}$$

and combining corresponding terms in each bracket, by the product rule,

$$\mathrm{LHS} = \frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2)}{\partial x} + \frac{\partial(\rho uv)}{\partial y} + \frac{\partial(\rho uw)}{\partial z}$$

The x-momentum equation can therefore by written in a longer, but conservative, form:

$$\frac{\partial(\rho u)}{\partial t}+\frac{\partial(\rho u^2)}{\partial x}+\frac{\partial(\rho uv)}{\partial y}+\frac{\partial(\rho uw)}{\partial z}=-\frac{\partial p}{\partial x}+\mu\nabla^2 u+2\rho\Omega v$$

with all the flux terms on the LHS directly integrable (derivatives "on the outside").

(f)

concentration	←	*u* (velocity)
diffusivity	←	μ (viscosity)
source	←	non-viscous forces

(g) They can not be treated as independent because each velocity component appears in all the other equations (via the mass fluxes). They are said to be *coupled*.

(h)

(i) In high-speed flow of a compressible gas:

– the continuity equation gives the density, ρ;
– a transport equation for energy (or enthalpy) gives the temperature, T;
– pressure is then derived from an *equation of state* (e.g. ideal gas law, $p = \rho RT$).

(ii) In incompressible flow, pressure is derived from the condition that solutions of the momentum equation be mass-consistent. (The momentum equation gives a link between velocity and pressure which, when substituted into the continuity equation, gives an equation for pressure.)

Q6.

(a) Since

$$\rho\mathbf{g} = \rho(0,0,-g) \quad = -\rho g \nabla z$$

and

$$\rho\Omega^2\mathbf{R} = \tfrac{1}{2}\rho\Omega^2\nabla(R^2)$$

we have

$$-\nabla p + \rho\mathbf{g} + \rho\Omega^2\mathbf{R} = -\nabla(p + \rho g z - \tfrac{1}{2}\rho\Omega^2 R^2)$$

Thus, pressure, gravitational and centrifugal forces (per unit volume) can be combined as a single negative gradient:

$$-\nabla p^*$$

for a modified pressure

$$p^* = p + \rho g z - \tfrac{1}{2}\rho\Omega^2 R^2$$

A CFD code would solve for modified pressure, p^*, separating the individual parts for post-processing only where necessary to distinguish the separate forces.

(b) In steady rotation the fluid is at rest in the rotating reference frame and hence the net force in this frame, $-\nabla p^*$, is zero. The modified pressure p^* is then constant. Since

$$p = p^* - \rho g z + \tfrac{1}{2}\rho\Omega^2 R^2$$

the pressure p:

- increases with depth; (this is due to hydrostatic forces);
- increases with radius; (this is due to centrifugal forces, or, in a non-rotating reference frame, the need to supply a centripetal acceleration to maintain motion in a circle).

Hence, the minimum and maximum pressure p are at top-centre and bottom-outside, respectively, and the difference in pressure between them is

$$\rho g \times height + \tfrac{1}{2}\rho\Omega^2 \times radius^2$$

Here,

$$\Omega = \frac{600 \times 2\pi}{60} = 62.83 \text{ rad s}^{-1}$$

and hence the difference in pressure is

$$1100 \times 9.81 \times 0.15 + \tfrac{1}{2} \times 1100 \times 62.83^2 \times 0.05^2 \quad = 7047 \text{ Pa}$$

Answer: 7.05 kPa.

Q7.

(a) The *outward* volume flux through each face, Q_f, is the sum of

velocity component × projected area

or, if the cell is traversed anticlockwise:

$$Q_f = u\Delta y - v\Delta x$$

Volume fluxes are given in the final column of the table.

Face	u	v	Δx	Δy	Q_f
e	4	10	2	2	–12
n	1	8	–7	0	56
w	2	2	1	–2	–6
s	1	4	4	0	–16

(b) The net outward volume flux is

$$\sum Q_f = 22$$

This is non-zero; i.e. there is net outflow of volume. Hence, the flow is *not* incompressible.

Continuity for the whole cell gives:

$$\frac{\mathrm{d}}{\mathrm{d}t}(mass) + \sum_{faces}(outward\ mass\ flux) = 0$$

$$\Rightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}t}(\rho V) + \sum_{faces}\rho_f Q_f = 0$$

The “volume” of the cell (a trapezium in plan) is:

$$V = \tfrac{1}{2}(4+7)\times 2 \quad = 11$$

whilst the density everywhere at this instant is 1.0. Hence,

$$11\frac{\mathrm{d}\rho}{\mathrm{d}t} + 22 = 0$$

Hence,

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -2$$