Essay 3

Mathematical Background of Heat Conduction

3.1 Fourier's Law of Heat Conduction

In a groundbreaking treatise published in 1822, *Théorie Analytique de la Chaleur*, Jean Baptiste Joseph Fourier set forth the mathematical theory of heat conduction. That treatise not only conveyed the basic law of heat conduction, but also included several other original concepts. Those concepts, primarily mathematical, are Fourier series and the Fourier integral. In classical heat conduction courses, those mathematical concepts are used to obtain solutions for a restricted class of heat conduction problems. While those solution methods possess a degree of elegance, they are of minimal use for solving the complex heat conduction problems that are encountered in engineering practice.

Fig. 3. 1 Jean Baptiste Joseph Fourier (1768-1830)

To introduce Fourier's law, it is convenient to focus attention on a plane plate as pictured in Fig. 3.2. The dimensions L and W are both much greater than the plate thickness t. Furthermore, the edges of the plate are assumed to be adiabatic. For this condition, the only possible heat flow is perpendicular to the principle faces of the plate, as indicated in the figure. Since the heat is flowing in only one direction, this situation is called *one-dimensional heat flow*. For such a situation, it is convenient to focus attention solely on the heat flow direction, and Fig. 3.3 has been prepared for this purpose.
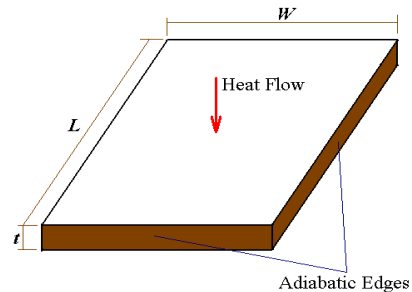

Fig. 3. 2 Geometry used in the introduction of Fourier's law

The figure displays the direction of heat flow, which is designated as the x direction. Since heat must flow from a higher to a lower temperature, it follows that T_1 is greater than T_2. The symbol to be used here for the rate of heat transfer (W or Btu/hr) is Q. The temperature difference $(T_1 - T_2) = \Delta T$ is the driving force which creates Q. In accordance with the Second Law of Thermodynamics, ΔT is a temperature drop.

Fig. 3. 3 One-dimensional model of heat conduction in a plane plate
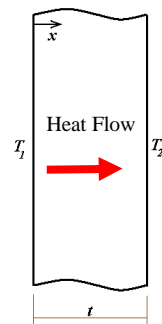
The thermal resistance R between locations 1 and 2 opposes the heat flow.

This trio of physical quantities, Q, ΔT, and R has a direct counterpart in the quantities I, ΔV, and R, where the latter trio respectively represents the electric current, the voltage drop, and the electrical resistance. The mathematical connection between the electrical quantities is the familiar Ohm's law:

$$I = \frac{\Delta V}{R} \tag{3.1}$$

The same mathematical connection is applicable to the three thermal quantities, so that:

$$Q = \frac{\Delta T}{R} \tag{3.2}$$

This equation is commonly called the *thermal Ohm's law*.

The thermal resistance can be deduced from physical reasoning. From Essay 2, the capability of transferring heat by means of conduction was related to the thermal conductivity k. It can be reasoned that the higher the thermal conductivity, the lower is the thermal resistance ($R \sim 1/k$). Another important contribution to R is the path length over which the heat must flow. For the situation depicted in Fig. 3.3, the path length is the thickness t. It is physically plausible that the longer the path length, the greater is the thermal resistance ($R \sim t$). The third component of the thermal resistance is the cross-sectional area perpendicular to the direction of heat flow. If the heat is forced to pass through a very small cross section, the heat would have to struggle. On the other hand, if the same amount of heat were to pass through a much larger cross section, the degree of struggle would be much diminished. This reasoning suggests that $R \sim 1/A$.

If these proportionalities are brought together, there results:

$$R = \frac{t}{kA} \tag{3.3}$$

The substitution of Eq. (3.3) into (3.2) yields:

$$Q = kA\frac{\Delta T}{t} \tag{3.4}$$

where ΔT is the temperature *drop*. It is convenient for the next steps in the analysis to let t be Δx, so that Eq. (3.4) becomes:

$$Q = kA\frac{\Delta T}{\Delta x} \tag{3.5}$$

Next, consider a succession of plates for which the thickness Δx is decreased systematically and approaches dx. Since the temperature difference across an infinitesimal thickness must also be infinitesimal, it follows that $\Delta T \rightarrow dT$. With these modifications, Fourier's law is obtained from Eq. (3.5) as:

$$Q = -kA\frac{\mathrm{d}T}{\mathrm{d}x} \tag{3.6}$$

The presence of the minus sign that has unceremoniously appeared in Eq. (3.6) requires explanation. One explanation is physical. Suppose that the temperature variation across the thickness of the plate is shown in Fig. 3.4. It is clear that for that case, $\frac{dT}{dx} < 0$. It is also evident that heat is flowing in the positive x-direction. The minus sign in Eq. (3.6), when taken together with the negative value of dT/dx, yields a positive value for Q. The presence of the minus sign is based on the definition that Q is positive when heat flows in the positive x-direction.
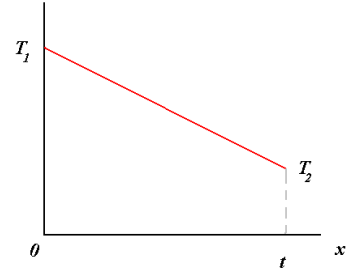
Fig. 3. 4 Temperature variation across the thickness of the plate depicted in Fig. 3.3

The other explanation for the presence of the minus sign in Eq. (3.6) is a mathematical one. Although it was convenient for the discussion of the thermal Ohm's law to define ΔT as the temperature drop, the mathematical quantity ΔT has a very specific, albeit mathematical definition. In mathematics, ΔT denotes the temperature at the larger value of x minus the temperature at the smaller value of x. If this mathematical rule is followed, the minus sign that appears in Eq. (3.6) emerges directly.

Whereas the rate of heat transfer Q is a very important physical quantity, there is a related quantity, the *heat flux q*, that has equal practical relevance. The word *flux* means per unit area. Therefore, it follows from Eq. (3.6) that the heat flux is given by:

$$q = -k\frac{\mathrm{d}T}{\mathrm{d}x} \tag{3.7}$$

The Fourier law equations, Eqs. (3.6) and (3.7) are called constitutive equations. Although they are the best known constitutive equations for heat conduction, they are by no means the only such equations. There are a number of other constitutive models which are focused either on very fast time variations or material properties that are different in different directions (non-isotropic). There is a special class of materials which possess non-isotropic properties that are called *orthotropic*. Those materials are characterized by a different value of the thermal conductivity in each coordinate direction. An example of a material of this type is wood, which has different conductivities along the grain and across the grain.

3.2 Conservation of Energy: The First Law of Thermodynamics

The constitutive equations discussed in the foregoing are only one ingredient in the physics of heat conduction. All heat transfer processes must obey the fundamental law of nature that is embodied in the First Law of Thermodynamics. It is well established that the First Law has separate forms when applied to a fixed-mass system or to a flowing fluid. Heat conduction occurs both in solids and unmoving fluids (fixed-mass systems) and in flowing fluids (control-

volume systems). Here, attention will be focused on solids and unmoving fluids. In reality, fluids tend to move, and special conditions are needed to prevent such motions. Therefore, when heat conduction is analyzed for fixed-mass systems, that analysis applies almost exclusively to solids.

For a fixed mass which is stationary (no kinetic energy and no changes in potential energy), the First Law of Thermodynamics is:

$$\Delta U = U_2 - U_1 = Q_{1 \to 2} - W_{1 \to 2} \tag{3.8}$$

This equation describes changes which occur between two instants of time, t_1 and t_2. During that period of time, heat is transferred to the mass in the amount $Q_{1 \to 2}$ and work $W_{1 \to 2}$ is extracted. These processes create a change in internal energy $U_2 - U_1$. Suppose, as a special case, there is no *net* work done and there is no *net* heat transfer in the time interval. In that case, there is no change in internal energy, and the system is in a steady state. This situation is not necessarily without thermal activity. The statement $Q_{1 \to 2} = 0$ does not mean that there are no heat transfers occurring during the time interval. The true meaning is that any heat flow into the system is precisely balanced by heat flow of the same magnitude out of the system. In the next part of this essay, attention will be focused on problems for which $Q_{net} = 0$ during all time intervals.

3.3 Steady-State Heat Conduction

The steady-state condition $Q_{net} = 0$ can be conveniently rephrased as:

$$Q_{inflow} = Q_{outflow} \tag{3.9}$$

Equivalently,

$$\left(\frac{dQ}{dt} \right)_{inflow} = \left(\frac{dQ}{dt} \right)_{outflow} \tag{3.10}$$

For the vast majority of heat conduction problems, the emphasis is on rates of heat transfer rather than on amounts of heat transferred. Therefore, Eq. (3.10) is the starting point of further analysis for steady-state heat transfer.

To implement Eq. (3.10), it is convenient to make reference to Fig. 3.5. That figure shows a solid of arbitrary shape through which heat is passing. Also seen in the figure is a very small volume of dimensions dx, dy, and dz. It is this volume, called a *control volume*, to which Eq. (3.10) will be applied.
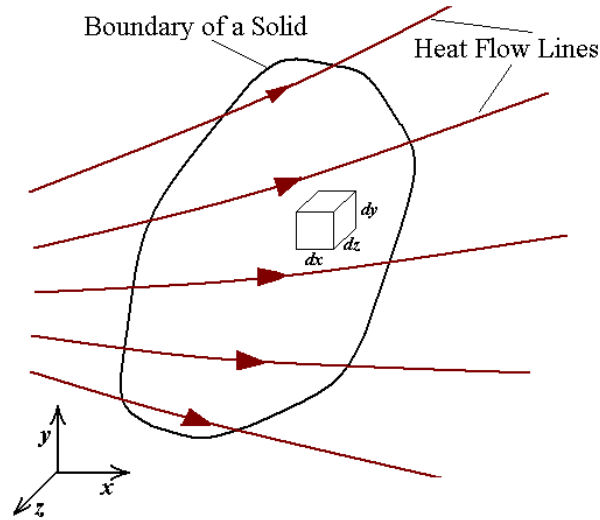
Fig. 3. 5 A solid of arbitrary shape across which heat is passing

For the application of this heat balance equation, it is useful to enlarge the control volume and to focus attention on its individual faces. Figure 3.6 shows enlarged views of left-hand and right-hand faces of the control volume.
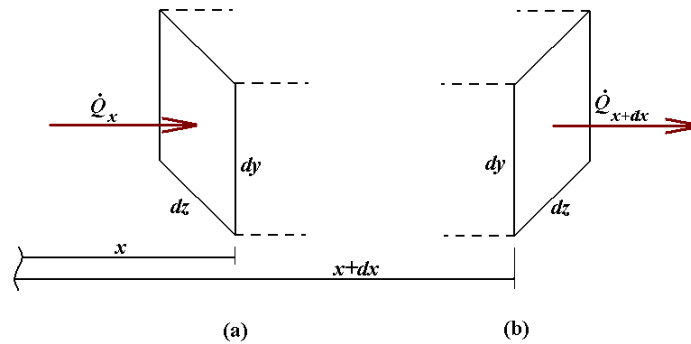

Fig. 3. 6 Enlarged views of the faces of the control volume. (a) left-hand face and (b) right-hand face

In the figure, the symbol $\dot{Q}_x$ is used to denote the derivative dQ_x/dt. The subscript x indicates a heat flow in the x direction.

The rate of heat transfer crossing the left-hand face follows from Eq. (3.6) as:

$$\dot{Q}_x = -k^x(dy \cdot dz)\left(\frac{\partial T}{\partial x}\right)_x \tag{3.11}$$

In this equation, the area across which heat is flowing is the face area $dy \cdot dz$. In addition, it should be noted that the derivative of the temperature in the x direction is now a *partial derivative*. The need for a partial derivative occurs because the temperature is now being considered to be a function of all three coordinates x, y, and z. The subscript x that is appended to the derivative indicates that it is to be evaluated at the location $x = x$. There is a superscript x

appended to the thermal conductivity k. This superscript is intended to indicate that the thermal conductivity is specific to the x direction, which means that an orthotropic solid is being considered.

Next, in a manner analogous to Eq. (3.11), the rate at which heat crosses the right-hand face is:

$$\dot{Q}_{x+dx} = -k^x(dy \cdot dz)\left(\frac{\partial T}{\partial x}\right)_{x+dx} \tag{3.12}$$

It is useful, at this point, to determine the net rate of heat outflow for the x direction. This quantity is obtained by differencing Eqs. (3.12) and (3.11), so that:

$$\dot{Q}_{x+dx} - \dot{Q}_x = -k^x(dy \cdot dz)\left[\left(\frac{\partial T}{\partial x}\right)_{x+dx} - \left(\frac{\partial T}{\partial x}\right)_x\right] \tag{3.13}$$

If use is made of the definition of the derivative:

$$\frac{\partial f}{\partial x} = \frac{f(x+dx,y,z)-f(x,y,z)}{dx} \tag{3.14}$$

then,

$$\dot{Q}_{x+dx} - \dot{Q}_x = -k^x(dx \cdot dy \cdot dz)\frac{\partial^2 T}{\partial x^2} \tag{3.15}$$

A similar derivation yields:

$$\dot{Q}_{y+dy} - \dot{Q}_y = -k^y(dx \cdot dy \cdot dz)\frac{\partial^2 T}{\partial y^2} \tag{3.16}$$

$$\dot{Q}_{z+dz} - \dot{Q}_z = -k^z(dx \cdot dy \cdot dz)\frac{\partial^2 T}{\partial z^2} \tag{3.17}$$

Equation (3.10) requires that for steady state, there is no net heat transfer at the chosen control volume. This means that the sum of Eqs. (3.15) to (3.17) be zero, which leads to:

$$k^x\frac{\partial^2 T}{\partial x^2} + k^y\frac{\partial^2 T}{\partial y^2} + k^z\frac{\partial^2 T}{\partial z^2} = 0 \tag{3.18}$$

If the thermal conductivity is independent of direction (isotropic), then Eq. (3.18) reduces to:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0 \tag{3.18}$$

This equation is very famous and is encountered in many physical processes beside heat conduction. It is called Laplace's equation. It is, in fact, the most investigated equation in all of mathematics. There are many elegant solution methods that have been employed to solve Laplace's equation. Those methods are taught in a graduate course on heat conduction. However, it is possible to obtain analytical solutions for the Laplace's equation only for the most simple shapes and boundary conditions.

Laplace's equation can also be written in vector form as:

$$\nabla^2 T = 0 \tag{9}$$

The operator ∇^2 takes different forms for different coordinate systems. For Cartesian coordinates:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{0}$$

In cylindrical coordinates, the expression for this operator is:

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \tag{3.21}$$

Fig. 3. 7 Pierre Simon de Laplace (1749-1827)

where r is the radial coordinate, θ is the angular coordinate, and z is the axial coordinate.

There is a large class of problems in which the temperature does not depend on the angular coordinate θ. Such problems are called *axisymmetric*. The steady-state heat conduction in axisymmetric situations is governed by:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial z^2} = 0 \tag{3.22}$$

In the next essay, attention will be focused on numerical means of solving the foregoing equations. It will be demonstrated that virtually any problem in heat conduction will be amenable to solution by means of the software to be set forth.

3.4 Unsteady Heat Conduction

When the temperature varies with time in a solid material, there will be a corresponding variation of the rates of heat transfer within the solid and at its boundary. In such situations, the change of internal energy in a given time interval will no longer be zero as it was in the steady state. If attention is redirected to the First Law, Eq. (3.8), and if the times t_1 and t_2 are taken to be:

$$t_2 = t_1 + dt \tag{3.23}$$

Then, the change of internal energy must be very small to correspond to the small time interval dt. In this light, Eq. (3.8) may be rewritten as:

$$dU = \delta Q \tag{3.24}$$

In writing this equation, the work transfer has been omitted. In addition, the quantity δQ represents a very small quantity of heat, but it cannot be represented by dQ since Q is not a state function. Since our focus is now on timewise variations, it is convenient to rewrite Eq. (3.24) as:

$$\frac{\partial U}{\partial t} = \delta \dot{Q} \tag{3.25}$$

In this equation, $\delta\dot{Q}$ is the net rate of heat inflow into the control volume.

Equations (3.15-17), when summed, represent the net rate of outflow from the control volume. By reversing the sum and using Eq. (3.25), the unsteady equation for heat conduction emerges as:

$$\frac{\partial U}{\partial t} = (dx \cdot dy \cdot dz)\left[k^x\frac{\partial^2 T}{\partial x^2} + k^y\frac{\partial^2 T}{\partial y^2} + k^z\frac{\partial^2 T}{\partial z^2}\right] \tag{3.26}$$

The internal energy U is equal to the specific internal energy u times the mass within the control volume. In turn, the mass is equal to the density of the material ρ times the volume $dx \cdot dy \cdot dz$. With this information:

$$\frac{\partial U}{\partial t} = \rho(dx \cdot dy \cdot dz)\frac{du}{dt} \tag{3.27}$$

From thermodynamics, for solid media,

$$\frac{\partial u}{\partial t} = c_v\frac{\partial T}{\partial t} \tag{3.28}$$

The introduction of Eqs. (3.27) and (3.28) into the First Law, Eq. (3.26), there is obtained:

$$\rho c_v\frac{\partial T}{\partial t} = k^x\frac{\partial^2 T}{\partial x^2} + k^y\frac{\partial^2 T}{\partial y^2} + k^z\frac{\partial^2 T}{\partial z^2} \tag{3.29}$$

This general equation for unsteady heat conduction can be specialized to solids ($c_v = c$) and to isotropic media, $k^x = k^y = k^z = k$, with the result:

$$\rho c\frac{\partial T}{\partial t} = k\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right] = k\nabla^2 T \tag{3.30}$$

where ∇^2 has been defined in Eqs. (3.20) and (3.21) for Cartesian and cylindrical coordinates, respectively.

In many textbooks and, in particular, mathematics texts on differential equations, Eq. (3.30) is written as:

$$\frac{\partial T}{\partial t} = \alpha\,\nabla^2 T \tag{3.31}$$

In this equation, α is the *thermal diffusivity* whose units are m^2/s or ft^2/s. The one-dimensional form of this equation:

$$\frac{\partial T}{\partial t} = \alpha\,\frac{\partial^2 T}{\partial x^2} \tag{3.32}$$

is called the *heat equation* by mathematicians.

Equations (3.31) and (3.32) will be solved in a highly effective manner by numerical means.