5. PRESSURE AND VELOCITY SPRING 2015

5.1 The Scalar-Transport Equation for Momentum

Each momentum component satisfies its own scalar-transport equation. For one cell:

$$\underset{\textit{rate of change}}{\frac{\mathrm{d}}{\mathrm{d}t}(mass\times\phi)} + \sum_{faces}(\underset{\textit{advection}}{C\phi} \underset{\textit{diffusion}}{-\Gamma\frac{\partial\phi}{\partial n}A}) = \underset{\textit{source}}{S} \qquad (1)$$

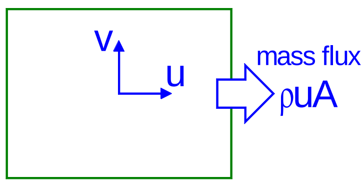
where C is the mass flux through a cell face.

For momentum:

concentration, ϕ	$\leftarrow$	velocity component ($\phi = u$, v or w)
diffusivity, Γ	$\leftarrow$	viscosity, μ
source	$\leftarrow$	non-viscous forces

However, the scalar-transport equations for momentum differ in three important ways from those for passive scalars because they are:

- **non-linear;**
- **coupled;**
- **required also to be mass-consistent.**



For example, the x-momentum flux through an x-directed face is

$$Cu = (\rho uA)u$$

The mass flux C is not constant but changes with u. The momentum equation is therefore **non-linear** and must be solved **iteratively**.

Similarly, the y-momentum flux through an x-directed face is

$$Cv = (\rho uA)v$$

The v equation depends on the solution of the u equation (and vice versa). Hence, the momentum equations are **coupled** and must be solved **together**.

Pressure also appears in each momentum equation. This further couples the equations and demands some means of determining pressure. This is very different in compressible or incompressible CFD.

- In **compressible** flow, continuity provides a transport equation for density (ρ). Pressure is obtained by solving an energy equation to find temperature (T) and then using an equation of state (e.g. $p = \rho RT$).
- In **incompressible** flow, density variations (if there are any) are, by definition, not determined by pressure. A pressure equation arises from the **requirement that the solutions of the momentum equations are also mass-consistent**. In other words, mass conservation actually leads to a pressure equation!

In most incompressible-flow solvers mass and momentum equations are solved sequentially and iteratively. We call this a *segregated* approach, as in the following pseudocode.

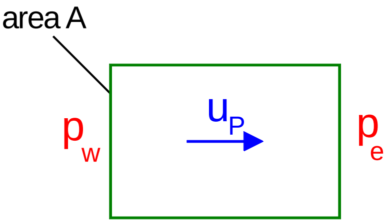
```
DO WHILE (not_converged)
        CALL SCALAR_TRANSPORT( u )
        CALL SCALAR_TRANSPORT( v )
        CALL SCALAR_TRANSPORT( w )

        CALL MASS_CONSERVATION( p )
END DO
```

In many compressible-flow codes the main fluid variables are assembled and solved as a vector (ρ, ρu, ρv, ρw, ρe). This is called a *coupled* approach, but it won't be pursued here.

5.2 Pressure-Velocity Coupling

Question 1. How are velocity and pressure linked?
Question 2. How does a pressure equation arise?
Question 3. Should velocity and pressure be *co-located* (stored at the same positions)?

5.2.1 Pressure-Velocity Linkage



In the momentum equation, pressure forces appear as a **source of momentum**; e.g. in the *x*-momentum equation:

$$net\ \ pressure\ force = (p_w - p_e)A$$

The discretised momentum equation is of the form

$$\underbrace{a_P u_P - \sum_F a_F u_F}_{net\ \ flux} = \underbrace{A(p_w - p_e)}_{pressure\,forces} + other\ forces \qquad (2)$$

Hence,

$$u_P = d_P(p_w - p_e) + \cdots \qquad \text{where} \qquad d_P = \frac{A}{a_P} \qquad (3)$$

<u>Answer 1</u>

(a) The force terms in the momentum equation provide a link between velocity and pressure.

(b) Velocity depends on the pressure gradient or, when discretised, on the difference between pressure values ½ cell either side.

$$momentum\ equation \quad \Rightarrow \quad u_P = d_P(p_w - p_e) + \cdots$$

or

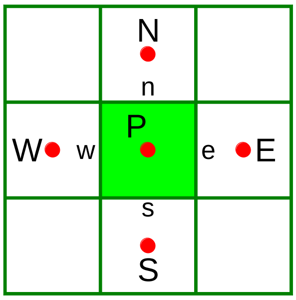
$$u = -d\Delta p + \cdots$$

where Δ indicates a centred difference ("right minus left").

Substituting for velocity in the continuity equation,

$$\begin{aligned} 0 &= (\rho u A)_e - (\rho u A)_w + \cdots \\ &= (\rho A d)_e (p_P - p_E) - (\rho A d)_w (p_W - p_P) + \cdots \\ &= -a_W p_W + a_p p_p - a_E p_E + \cdots \end{aligned}$$

This has the same algebraic form as the scalar-transport equations.



<u>Answer 2</u>

The momentum equation gives a link between velocity and pressure which, when substituted into the continuity equation, gives an equation for pressure.

i.e.

A pressure equation arises from the requirement that solutions of the momentum equation be mass-consistent.

(*** *Advanced* ***)

The Navier-Stokes equation for a constant-density fluid may be written

$$\frac{\partial u_i}{\partial t} + \frac{\partial (u_i u_j)}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \nabla^2 u_i$$

Velocity depends on the pressure **gradient**.

Taking the divergence (i.e. $\partial/\partial x_i$) and using continuity ($\partial u_i/\partial x_i = 0$) gives

$$\frac{\partial^2 (u_i u_j)}{\partial x_i \partial x_j} = -\frac{1}{\rho}\nabla^2 p$$

or

$$\nabla^2 p = -\rho \frac{\partial^2 (u_i u_j)}{\partial x_i \partial x_j}$$

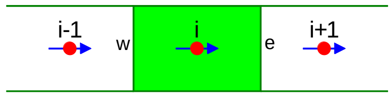
Hence, analytically, **the combination of momentum and continuity equations gives a Poisson equation for pressure**.

5.2.2 Co-located Storage of Variables

Initially, suppose that pressure and velocity are *co-located* (stored at the same positions) and that advective velocities (the cell-face velocities used to calculate mass fluxes) are calculated by linear interpolation.

In the **momentum** equation the net pressure force involves

$$p_w - p_e = \tfrac{1}{2}(p_{i-1} + p_i) - \tfrac{1}{2}(p_i + p_{i+1})$$
$$= \tfrac{1}{2}(p_{i-1} - p_{i+1})$$

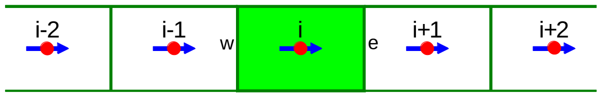


Hence, the discretised momentum equation has the form:

$$u_i = \tfrac{1}{2} d_i (p_{i-1} - p_{i+1}) + \cdots$$

In the **continuity** equation the net outward mass flux depends on

$$u_e - u_w = \tfrac{1}{2}(u_i + u_{i+1}) - \tfrac{1}{2}(u_{i-1} + u_i)$$
$$= \tfrac{1}{2}(u_{i+1} - u_{i-1})$$
$$= \tfrac{1}{4}[d_{i+1}(p_i - p_{i+2}) - d_{i-1}(p_{i-2} - p_i)] + \cdots$$

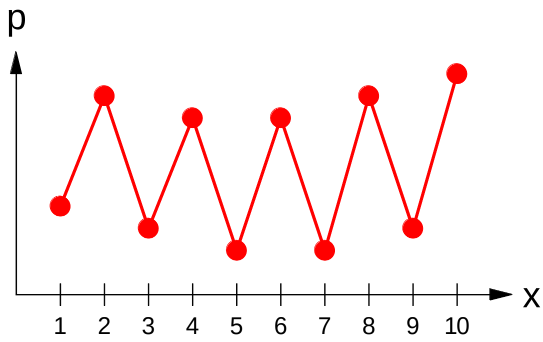


So, both mass and momentum equations only link pressures at alternate nodes.

Thus, the combination of:

- co-located u, p,
- linear interpolation for advective velocities,

leads to decoupling of odd nodal values p_1, p_3, p_5, … from even nodal values p_2, p_4, p_6, … . This *odd-even decoupling* or *checkerboard* effect leads to indeterminate oscillations in the pressure field and usually causes calculations to crash.



There are two common remedies:

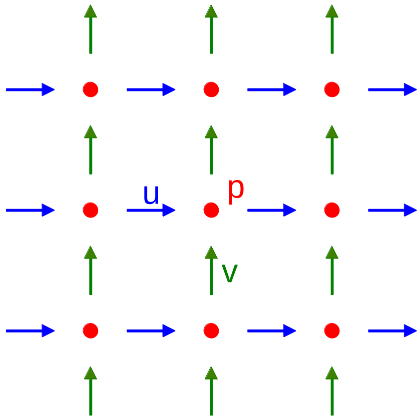
(1) use a *staggered* grid (velocity and pressure stored at different locations); or

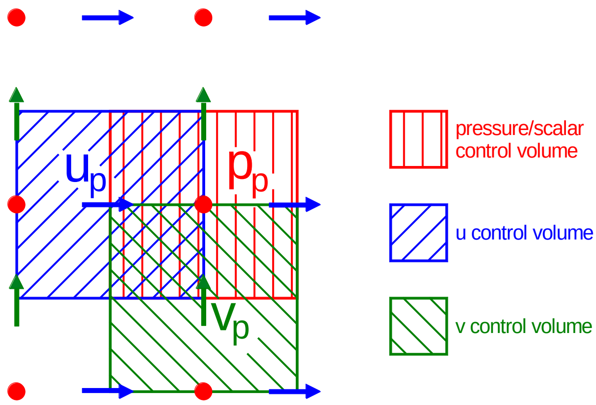
(2) use a *co-located* grid but *Rhie-Chow interpolation* for the advective velocities

Both provide a link between adjacent pressure nodes, preventing odd-even decoupling.

5.2.3 Staggered Grid (Harlow and Welch, 1965)

In the *staggered-grid* arrangement, velocity components are stored half-way between the pressure nodes that drive them.





This leads to different sets of control volumes.

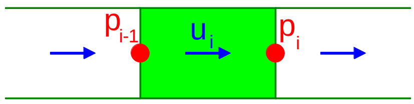
The usual convention is that each velocity node has the same index (*P* or *ijk*) as the pressure node to which it points.

Other scalars are stored at the same position as pressure.

On a **Cartesian** mesh …

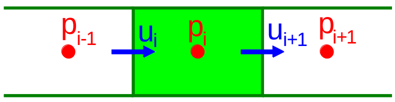
- In the momentum equation pressure is stored at precisely the points required to compute the pressure force.

$$u_i = d_i(p_{i-1} - p_i) + \cdots$$



- In the continuity equation velocity is stored at precisely the points required to compute mass fluxes. The net mass flux involves:

$$u_{i+1} - u_i + \cdots = d_{i+1}(p_i - p_{i+1}) - d_i(p_{i-1} - p_i) + \cdots$$
$$= -d_i p_{i-1} + (d_i + d_{i+1})p_i - d_{i+1}p_{i+1}$$



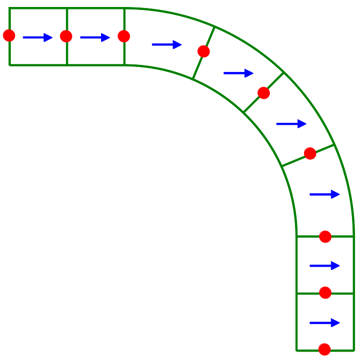
In both cases no interpolation is required for cell-face values and there is a strong linkage between successive, rather than alternate, pressure nodes, avoiding odd-even decoupling.

Advantages

- No interpolation required; variables are stored where they are needed.
- No problem of odd-even pressure decoupling

Disadvantages

- Added geometrical complexity from multiple sets of nodes and control volumes.
- If the mesh is **not Cartesian** then the velocity nodes may cease to lie between the pressure nodes that drive them (see right).

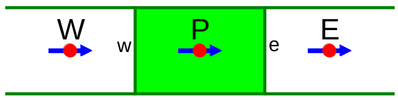


5.2.4 Rhie-Chow Velocity Interpolation (Rhie and Chow, 1983)

The alternative approach is to use co-located pressure and velocity but employ a different interpolation for *advective velocities* (the cell-face velocities used to calculate mass fluxes).

The momentum equation provides a connection between the nodal velocity and pressure:

$$u_P = \frac{\sum a_F u_F}{a_P} - d_P (p_e - p_w) + \ldots$$



In the Rhie-Chow algorithm the pressure and non-pressure parts of the velocity

$$u = \hat{u} - d\Delta p$$

are separately interpolated to the cell face.

(i) Invert to work out the non-pressure part ("*pseudovelocity*" $\hat{u}$) at nodes:

$$\hat{u} = u + d\Delta p$$ with centred difference Δp from interpolated face values

(ii) Then linearly interpolate $\hat{u}$ and d to the cell face:

$$u_{face} = \hat{u}_{face} - d_{face}\Delta p$$ with centred difference Δp taken from adjacent nodes

This amounts to adding and subtracting centred pressure differences worked out at different places; e.g. on the east face, with an overbar denoting linear interpolation to that face,

$$u_e = \overline{(u + d\Delta p)}_e - \overline{d}_e (p_E - p_P) \qquad (4)$$

Using this interpolative technique, mass conservation gives:

$$\begin{aligned} 0 &= (\rho A u)_e - (\rho A u)_w + \cdots \\ &= (\rho A \hat{u})_e - (\rho A \hat{u})_w \quad - \quad (\rho A d)_e (p_E - p_P) + (\rho A d)_w (p_P - p_W) + \cdots \qquad (5) \\ &= (\rho A \hat{u})_e - (\rho A \hat{u})_w \quad - \quad a_W p_W + a_P p_P - a_E p_E + \cdots \end{aligned}$$

where

$$a_W = (\rho A d)_w, \quad a_E = (\rho A d)_e, \quad a_P = a_W + a_E + \cdots$$

("…" denotes terms from other directions).

Notes.

- The central pressure value p_P does not cancel so there is no odd-even decoupling.

- This is a pressure equation. Moreover, on rearrangement it has the same algebraic form as that arising from the scalar-transport equation:

$$a_P p_P = \sum_F a_F p_F + b_P, \qquad \text{where } a_F \geq 0, \quad a_P = \sum a_F$$

- Actually, in practice one does not solve directly for pressure but iteratively for **pressure corrections** (see Section 5.3). Equation (5) then becomes

$$0 = (\rho A u^*)_e - (\rho A u^*)_w \quad - \quad a_W p'_W + a_P p'_P - a_E p'_E + \cdots$$

  where the asterisk * here denotes "current value of". This can be rearranged as:

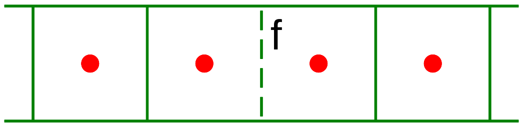
$$-a_W p'_W + a_P p'_P - a_E p'_E + \cdots = -\textit{current mass outflow}$$

Example

For the uniform Cartesian mesh shown below the momentum equation gives a velocity/pressure relationship

$$u = -4\Delta p + \cdots$$

for each cell, where Δ denotes a centred difference.



u =	1	2	3	3
p =	p_A	0.8	0.7	0.6

For the *u* and *p* values given, calculate the advective velocity on the cell face marked *f*:

(a) by linear interpolation;

(b) by Rhie-Chow interpolation, if $p_A = 0.6$ (local pressure maximum to the left of face);

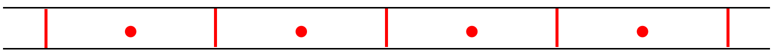
(c) by Rhie-Chow interpolation, if $p_A = 0.9$ (constant pressure gradient).

<u>Answer</u>

(a) By linear interpolation,

$$u_f = \tfrac{1}{2}(2+3) \quad = 2.5$$

(b) The working for Rhie-Chow may be set out as follows.



u	1		2		3		3	
p	0.6		0.8		0.7		0.6	
p_{face}		0.7		0.75		0.65		by interpolation of *p*.
$\hat{u}$			2.2		2.6			$\hat{u} = u + 4(p_e - p_w)$
$\hat{u}_{face}$				2.4				by interpolation of $\hat{u}$.
u_{face}				2.8				$u_{face} = \hat{u}_{face} - 4(p_E - p_W)$

Answer: $u_f = 2.8$

This is higher than that obtained from linear interpolation, showing the velocity field trying to alleviate the local pressure maximum just to the left of the face.

(c) If $p_A = 0.9$ then a similar Rhie-Chow analysis produces $u_f = 2.5$: the same as is obtained by linear interpolation. This is to be expected since the pressure gradient, and hence the successive pressure differences Δp, are constants.

Analysis of Rhie-Chow Interpolation

Parts (a) and (c) of the example above show that in a constant pressure gradient Rhie-Chow interpolation gives the same result as linear interpolation, whereas if (as in part(b)) there is a local pressure peak to one side of the face then the advective velocity rises to compensate.

By retaining general values of u_i and p_i and following the same analysis with a general (constant) coefficient *d* the above example gives

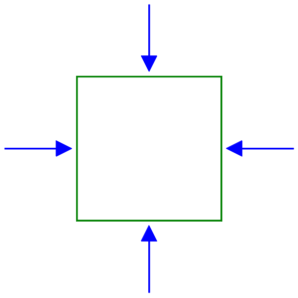
$$u_{face} = \frac{1}{2}(u_2 + u_3) + \frac{d}{4}(-p_1 + 3p_2 - 3p_3 + p_4)$$

Thus, Rhie-Chow interpolation is equivalent to the addition of a "pressure-smoothing" term to the velocity that would be obtained by linear interpolation. Expanding pressures about the cell face shows that the bracketed pressure term is, to leading order, proportional to $\Delta x^3(\partial^3 p / \partial x^3)$ on the cell face. Subtracting a similar term on the opposite cell face to give the net mass flux yields an additional term proportional to $\Delta x^4(\partial^4 p / \partial x^4)$, corresponding to 4th-order diffusion.

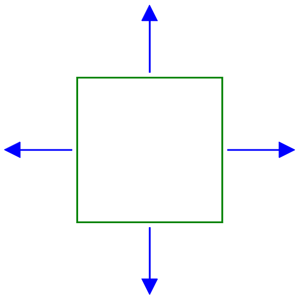
Pressure-velocity coupling is <u>the</u> dominant feature of the Navier-Stokes equations. Staggered grids are an effective way of handling it on **Cartesian** meshes. However, for **non-Cartesian** (or *curvilinear*) meshes, co-located grids are the norm and are employed in almost all general-purpose CFD codes.

5.3 Pressure-Correction Methods

Consider how changing pressure can be used to enforce mass conservation.



Net mass flux in;
increase cell pressure to drive mass out



Net mass flux out;
decrease cell pressure to suck mass in

What are *pressure-correction* methods?

- Iterative numerical schemes for pressure-linked equations.
- Derive velocity and pressure fields satisfying both mass and momentum equations.
- Consist of alternating updates of velocity and pressure:
  – solve the momentum equation for velocity with the current pressure;
  – use the link between velocity and pressure changes to rephrase continuity as a pressure-correction equation and solve for the pressure corrections necessary to "nudge" the velocity field towards mass conservation.
- There are two common schemes: SIMPLE and PISO.

Velocity and Pressure Corrections

The momentum equation connects velocity and pressure:

$$u = d(p_{-1/2} - p_{+1/2}) + \cdots$$

where –½ and +½ here indicate the relative location in terms of grid spacings.

One must correct velocity to satisfy continuity:

$$u \to u^* + u'$$

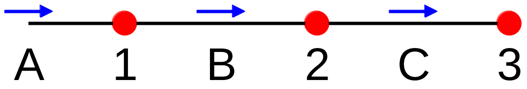
but simultaneously correct pressure so as to retain a solution of the momentum equation:

$$u' = d(p'_{-1/2} - p'_{+1/2}) + \cdots$$

The velocity-correction formula is, therefore,

$$u \quad \to \quad u^* + d(p'_{-1/2} - p'_{+1/2}) + \cdots$$

Classroom Example (Patankar, 1980)

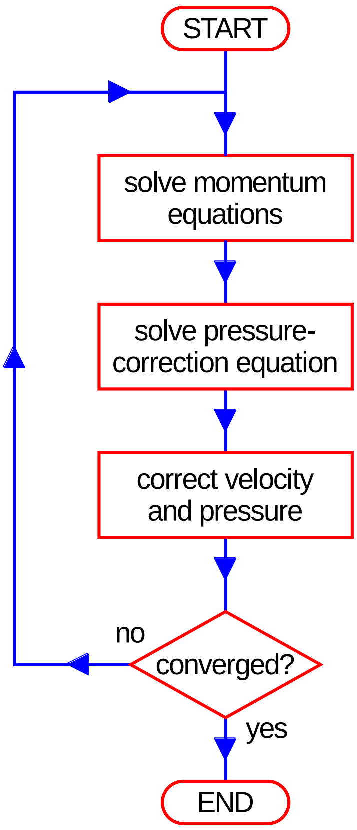


In the steady, one-dimensional, constant-density situation shown, the velocity u is calculated for locations A, B and C, whilst the pressure p is calculated for locations 1, 2 and 3. The velocity-correction formula is

$$u = u^* + u' , \quad \text{where} \quad u' = d(p'_{i-1} - p'_i)$$

where the locations $i-1$ and i lie on either side of the location for u. The value of d is 2 everywhere. The boundary condition is $u_A = 10$. If, at a given stage in the iteration process, the momentum equations give $u_B^* = 8$ and $u_C^* = 11$, calculate the values of p'_1, p'_2, p'_3 and the resulting velocity corrections.

5.3.1 SIMPLE: Semi-Implicit Method for Pressure-Linked Equations (Patankar and Spalding, 1972)



Stage 1. Solve the momentum equation with current pressure.

$$a_P u_p - \sum a_F u_F = \underbrace{A(p_w^* - p_e^*)}_{\text{pressure force}} + \underbrace{b_P}_{\text{other sources}}$$

The resulting velocity generally won't be mass-consistent.

Stage 2. Formulate the pressure-correction equation.

(i) Relate changes in u to changes in p:

$$u'_P = \frac{\sum a_F u'_F}{a_P} + d_P(p'_w - p'_e), \qquad d_P = \frac{A}{a_P}.$$

(ii) Make the SIMPLE approximation: neglect $\sum a_F u'_F$.

$$u'_P \approx d_P(p'_w - p'_e)$$

(Legitimate, since corrections vanish in the final solution.)

(iii) Apply mass conservation to a control volume centred on the pressure node. The net mass flux results from current ($\mathbf{u}^*$) plus correction ($\mathbf{u}'$) velocity fields:

$$\text{net mass flow out} \quad = \sum_{faces} \rho u_n^* A \quad + \sum_{faces} \rho u'_n A \quad = 0$$

Hence,

$$(\rho u' A)_e - (\rho u' A)_w + \cdots = -\dot{m}^* \qquad \text{(minus the current net mass flux)}$$

or, writing in terms of the pressure correction (staggered or non-staggered mesh):

$$(\rho A d)_e (p'_P - p'_E) - (\rho A d)_w (p'_W - p'_P) + \cdots = -\dot{m}^*$$

This results in a pressure-correction equation of the form:

$$a_P p'_P - \sum_F a_F p'_F = -\dot{m}^* \qquad (6)$$

Stage 3. Solve the pressure-correction equation

The discretised pressure-correction equation (6) is of precisely the same form as the discretised scalar equations, and hence the same algebraic solvers may be used.

Stage 4. Correct pressure and velocity:

$$\begin{aligned} p_P &\rightarrow p_P^* + p_P' \\ u_P &\rightarrow u_P^* + d_P(p_w' - p_e') \end{aligned} \tag{7}$$

(and similarly for the other velocity components).

Notes.

- **Staggered and unstaggered grids**. The distinction between staggered and unstaggered grids is subtle. In both cases, the expression for u_e' etc. at cell faces depends on the pressure corrections at adjacent nodes. However, for a (Cartesian) staggered grid the relevant normal velocities are actually stored on the faces of the pressure control volumes where they are required to establish mass conservation.

- **Source term for the pressure-correction equation**. That the "source" for the pressure-correction equation should be minus the current net mass flux $(-\dot{m}^*)$ is reasonable. If there is a net mass flow into a control volume then the pressure in that control volume must rise in order to "push" mass back out of the cell.

- **Under-relaxation**. In practice, substantial under-relaxation of the pressure update is needed to prevent divergence. In the correction step the pressure (but not the velocity) update is relaxed:

  $$p \rightarrow p^* + \alpha_p p'$$

  Typical values of α_p are in the range 0.1 – 0.3. Velocity is under-relaxed in the momentum transport equations, but the under-relaxation is generally less severe: $\alpha_u \approx 0.6 - 0.8$. (Note that under-relaxation in the momentum equations should be applied **after** setting the d coefficients for pressure; otherwise the Rhie-Chow algorithm will give advective velocities that depend on the velocity under-relaxation). There is apparently a "rule of thumb" that α_p and α_u should sum to unity. I have never understood why and I don't enforce it.

- **Iterative process**. Since the equations are non-linear and coupled, the matrix equations may change at each iteration. There is little to be gained by solving the matrix equations exactly at each stage, but only doing enough iterations of the matrix solver to reduce the residuals by a sufficient amount.

  Alternative strategies at each SIMPLE iteration are:

  m iterations of each u, v, w equation, followed by n iterations of the p' equation (typically $m = 1$, $n = 4$);

  or

  do enough iterations of each equation to reduce the residual error to a small fraction of the original (say 10%).

Classroom Example (Computational Hydraulics Exam, January 2006 – part)

In the conventional 2-d staggered-grid arrangement shown below, u and v (the x and y components of velocity), are stored at nodes indicated by arrows, whilst pressure p is stored at the intermediate nodes A, B, C, D. The grid spacing is uniform and the same in both directions. The velocity is fixed on the boundaries as shown in the figure. The velocity components at the interior nodes (u_B, u_D, v_C and v_D) are to be found.

At an intermediate stage of calculation the internal velocity values are found to be

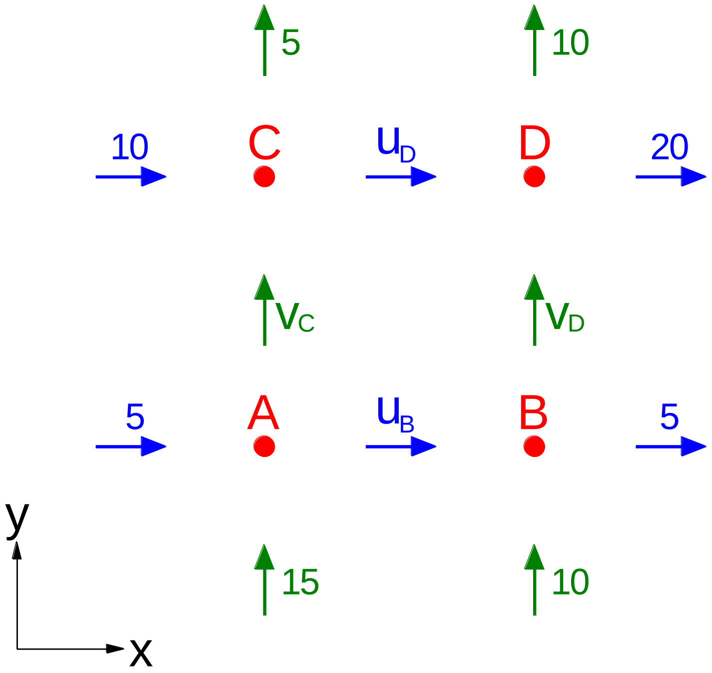
$$u_B = 11, \quad u_D = 14, \quad v_C = 8, \quad v_D = 5$$

whilst correction formulae derived from the momentum equation are

$$u' = 2(p'_w - p'_e), \qquad v' = 3(p'_s - p'_n)$$

with geographical (w,e,s,n) notation indicating the **relative** location of pressure nodes.

(a) Show that applying mass conservation to control volumes centred on pressure nodes leads to simultaneous equations for the pressure corrections. Solve for the pressure corrections and use them to generate a mass-consistent flow field.



(b) Explain why, in practice, it is necessary to solve for the pressure correction and not just the velocity corrections in order to satisfy mass conservation.

5.3.2 Variants of SIMPLE

The SIMPLE scheme can be inefficient and requires considerable pressure under-relaxation. This is because the corrected fields are good for updating velocity (since a mass-consistent flow field is produced) but not pressure (because of the inaccuracy of the approximation connecting velocity and pressure corrections).

To remedy this, a number of variants of SIMPLE have been produced, including:

SIMPLER (SIMPLE Revised – Patankar, 1980),
SIMPLEC (Van Doormaal and Raithby, 1984),
SIMPLEX (Raithby and Schneider, 1988).

SIMPLER

This variant acknowledges that the correction equation is good for updating velocity but not pressure and precedes the momentum and pressure-correction equations with the equation for the pressure itself (equation (5)).

SIMPLEC

This scheme seeks a more accurate relationship between velocity and pressure changes.

From the momentum equations, the velocity and pressure equations are related by

$$u'_P = \underbrace{\frac{1}{a_P}\sum a_F u'_F}_{(*)} + d_P(p'_w - p'_e) \qquad (8)$$

The SIMPLE approximation is to neglect the term (*). However, this is actually of comparable size to the LHS. In the SIMPLEC scheme, (8) is rewritten by subtracting $(1/a_P)\sum a_F u'_P$ from both sides:

$$(1 - \frac{1}{a_P}\sum a_F)u'_P = \underbrace{\frac{1}{a_P}\sum a_F(u'_F - u'_P)}_{(**)} + d_P(p'_w - p'_e)$$

Assuming that $|u'_F - u'_P| \ll |u'_P|$, it is more accurate to neglect the term (**), thus producing an alternative formula connecting velocity and pressure changes:

$$u'_P \approx \frac{d_P}{1 - \sum a_F/a_P}(p'_w - p'_e) \qquad (9)$$

Compare:

$$\begin{aligned} u'_P &\approx d_P(p'_w - p'_e) && \text{(SIMPLE)} \\ u'_P &\approx \frac{d_P}{1 - \sum a_F/a_P}(p'_w - p'_e) && \text{(SIMPLEC)} \end{aligned} \qquad (10)$$

Although conceptually appealing, there is a difficulty here. The "sum-of-the-neighbouring-coefficients" constraint on a_P means that, in steady-state calculations, $\sum a_F/a_P = 1$, and hence the denominator of (9) vanishes. This problem doesn't arise in time-varying calculations, where a time-dependent part is added to a_P, removing the singularity.

SIMPLEX

This scheme assumes that velocity and pressure corrections are linked by some general relationship

$$u'_P \approx \delta_P (p'_w - p'_e) \qquad (11)$$

This includes both SIMPLE and SIMPLEX as special cases (see equation (10)). However, the SIMPLEX scheme attempts to improve on this by actually **solving equations** for the δ_P. These equations are derived from (8) but we shall not go into the details here.

The author's experience is that SIMPLER and SIMPLEX offer substantial performance improvements over SIMPLE on **staggered** grids, but that the "advanced" schemes are difficult (impossible?) to formulate on co-located grids and offer little advantage there.

5.3.3 PISO

PISO – Pressure Implicit with Splitting of Operators (Issa, 1986).

This was originally proposed as a **time-dependent**, **non-iterative** pressure-correction method. Each timestep ($t^{old} \to t^{new}$) consists of a sequence of three stages:

(I) Solution of the time-dependent momentum equation with the t^{old} pressure in the source term.

(II) A pressure-correction equation and pressure/velocity update à la SIMPLE to produce a mass-consistent flow field.

(III) A second corrector step required to produce a second mass-consistent flow field but with time-advanced pressure.

Apart from time-dependence, steps (I) and (II) are essentially the same as SIMPLE. However, step (III) is designed to eliminate the need for iteration at each time step as would be the case with SIMPLE.

Evidence suggests that PISO is more efficient in **time-dependent** calculations, but SIMPLE and its variants are better in direct iteration to **steady state**.

Summary

- **Each component of momentum satisfies its own scalar-transport equation**, with the following correspondence:

  concentration, ϕ $\leftarrow$ velocity component (u, v or w)
  diffusivity, Γ $\leftarrow$ viscosity, μ
  source, S $\leftarrow$ forces

- However, the momentum equations are:

  non-linear;
  coupled;
  required also to be mass-consistent.

  and, as a consequence, have to be solved:

  together;
  iteratively;
  in a manner that also leads to **mass-conservation**.

- For incompressible flow the requirement that solutions of the momentum equation be mass-consistent generates a **pressure equation**.

- Pressure-gradient source terms can lead to an **odd-even decoupling** ("checkerboard") effect when all variables are co-located (stored at the same nodes). This may be remedied by using either:

  – a **staggered velocity grid;**
  – a non-staggered grid, but **Rhie-Chow interpolation** for advective velocities.

- **Pressure-correction methods** operate by making small corrections to pressure in order to "nudge" the velocity field towards mass conservation whilst still preserving a solution of the momentum equation.

- Widely-used pressure-correction algorithms are **SIMPLE** (and its variants) and **PISO**. The first is an iterative scheme; the second is a non-iterative, time-dependent scheme.

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Rhie, C.M. and Chow, W.L., 1983, A numerical study of the turbulent flow past an isolated airfoil with trailing edge separation, AIAA Journal, **21**, 1525-1532.

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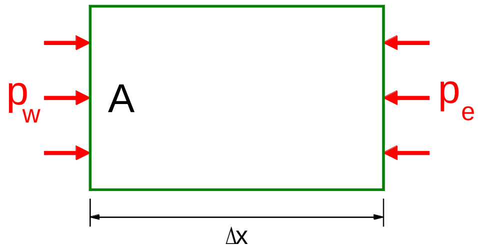
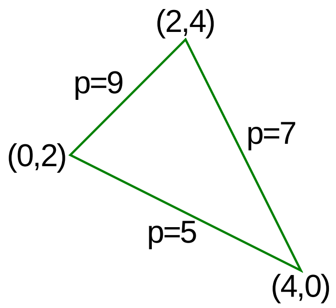
Examples

Q1.
For the rectangular control volume with surface pressures shown, what is:
(a) the net force in the x direction?
(b) the net force in the x direction, *per unit volume*?
(c) the average pressure gradient in the x direction?





Q2.
The figure shows a triangular cell in a 2-d unstructured mesh, together with the coordinates of its vertices and the average pressures on the cell faces. Calculate the x and y components of the net pressure force on the cell (per unit depth).

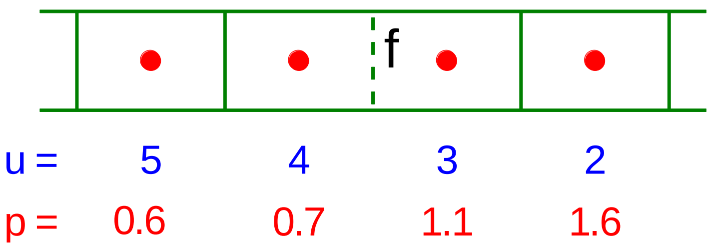
Q3.
(a) Explain (briefly) how an equation for pressure is derived in finite-volume calculations of incompressible fluid flow.

(b) Explain how the problem of "odd-even decoupling" in the discretised pressure field arises with co-located storage and linear interpolation for advective velocities, and detail the Rhie-Chow interpolation which can be used to overcome this.

(c) The figure below shows part of a Cartesian mesh with the velocity u and pressure p at the centre of 4 control volumes. If the momentum equation leads to a pressure-velocity linkage of the form

$$u = -3\Delta p + \cdots,$$

(where Δ represents a centred difference) use the Rhie-Chow procedure to find the advective velocity on the cell face marked f.



(d) Describe the SIMPLE pressure-correction method for the solution of coupled mass and momentum equations.

Q4. (Exam 2009)

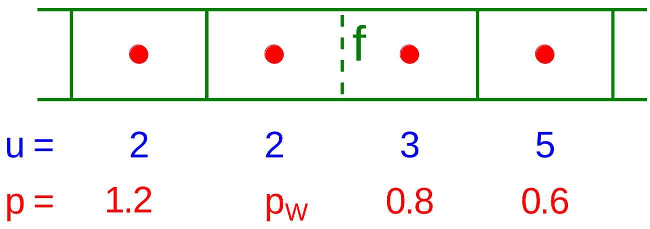
(a) Explain, briefly, the origin of the "odd-even decoupling" problem in the discrete pressure field that may occur with co-located storage of velocity and pressure in a finite-volume mesh.

(b) Describe, with the help of diagrams, the "staggered-grid" arrangement of velocity and pressure to overcome this problem and state its advantages and disadvantages.

(c) Describe the alternative Rhie-Chow approach to computing cell-face velocities on a co-located mesh.

(d) For the part of a uniform Cartesian mesh shown in the figure right the momentum equation gives a velocity vs pressure relationship of the form

$$u = -2\Delta p + \cdots$$

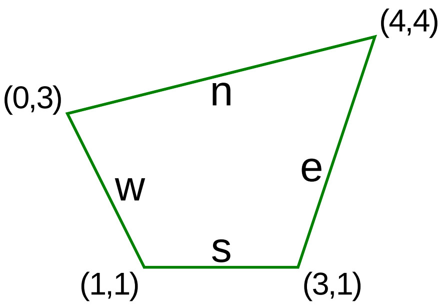


for each cell, where Δ denotes a centred difference, u is velocity, p is pressure and all quantities are expressed in consistent units.

If the values of u and p are as given in the figure, calculate the advective velocity on the cell face marked f by Rhie-Chow interpolation if:

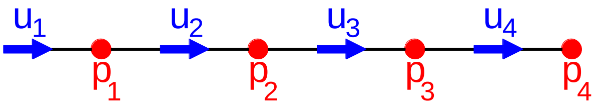
(i) $p_W = 1.2$;
(ii) $p_W = 1.0$.

(e) The figure below shows a quadrilateral 2-d cell (of unit depth) with given cell-vertex coordinates and cell-face pressures. Find the x and y components of the net pressure force on the cell.



face	pressure
e	3
n	9
w	5
s	4

Q5.
The figure defines the relative position of velocity (→) and pressure (•) nodes in a 1-d, *staggered-grid* arrangement.



Velocity $u_1 = 4$ is fixed as a boundary condition. After solving the momentum equation the velocities at the other nodes are found to be

$u_2 = 3, \quad u_3 = 5, \quad u_4 = 6.$

The relationship between velocity and pressure is found (from the discretised momentum equation) to be of the form

$$u = -4\Delta p + \cdots$$

at each node, where Δ denotes a centred difference in space.

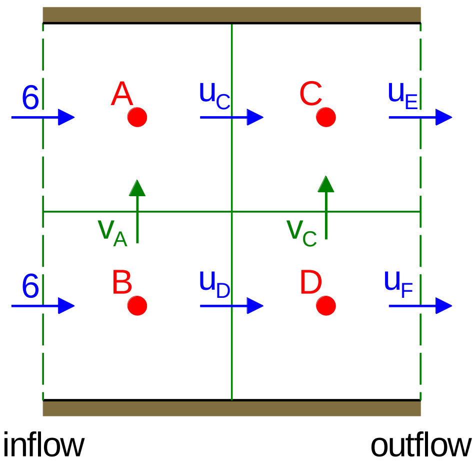
Apply mass conservation to cells centred on scalar nodes, calculate the pressure corrections necessary to enforce continuity, and confirm that a mass-consistent velocity field is obtained.

Q6. (Exam 2013)

(a) Explain, without mathematical detail, how pressure-correction methods may be used to solve the coupled equations of incompressible fluid flow.

(b) State the two main differences between the PISO and SIMPLE pressure-correction methods.

In the 2-d staggered-grid arrangement shown right, u and v (the x and y components of velocity), are stored at nodes indicated by arrows, whilst pressure p is stored at intermediate nodes A, B, C, D. Grid spacing is uniform and the same in both directions. Velocity is fixed at inflow. Upper and lower boundaries are impermeable.



At an intermediate stage of calculation the outflow velocities are

$u_E = 12, \quad u_F = 4,$

whilst the internal velocity components are

$u_C = 2, \quad u_D = 1, \quad v_A = 1, \quad v_C = -2.$

Velocity-pressure correction formulae are

$$u' = 3(p'_w - p'_e)\,, \qquad v' = 2(p'_s - p'_n)$$

with geographical (w,e,s,n) notation indicating the relative location of pressure nodes.

(c) Apply a uniform scale factor to the outflow velocities to enforce global mass conservation, stating the scale factor and outflow velocities after its application.

(d) Show that applying mass conservation to control volumes centred on pressure nodes leads to simultaneous equations for the pressure corrections. Solve for the pressure corrections and use them to generate a mass-consistent flow field.

Q7. (Exam 2011)

(a) In a finite-volume CFD calculation, flow variables are usually stored at the centre of computational cells. Explain briefly, and without mathematical detail,

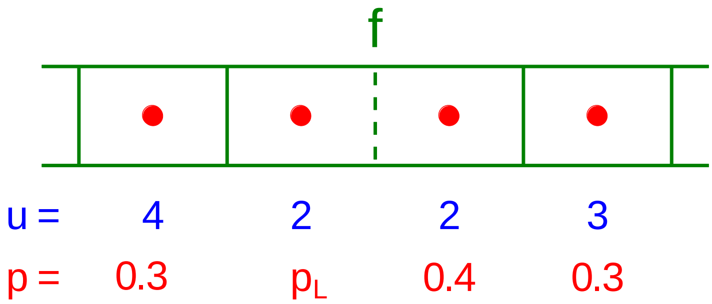
(i) how *pressure-correction* methods can be used to generate a mass-consistent velocity field;

(ii) why simple *linear* interpolation may cause difficulties with the coupling of pressure and velocity if these variables are stored at the same locations.

(b) Part of one line of cells in a uniform Cartesian mesh is shown in the figure below, together with the velocity u and pressure p at nodes. In each cell the momentum equation gives a relationship between velocity and pressure of the form

$$u = -4\Delta p + \cdots$$

where Δ denotes a centred difference.



Calculate the velocity on the cell face marked *f* by Rhie-Chow interpolation if:

(i) $p_L = 0.5$;

(ii) $p_L = 0.3$.

(c) The figure below shows a triangular cell in a 2-d mesh for an inviscid CFD calculation. The vertex coordinates and the average pressures on the cell edges are shown in the figure. The density $\rho = 1.0$ everywhere. Find the x and y components of:

(i) the net pressure force on the cell;

(ii) the fluid acceleration.