Answers 3

Q1.
(a) Flow is incompressible if **flow-induced** changes of pressure and temperature do not cause significant density changes. The approximation is OK for flows at much less than the speed of sound and without large heat input.

(b) The inviscid approximation is acceptable (at least for the pressure distribution and overall flow structure) if boundary layers are thin (high Re) and do not separate, and the user does not require calculation of drag forces.

Q2.

$$\rho U \frac{\partial U}{\partial s} = -\frac{\partial p}{\partial s} - \rho g \sin\alpha$$

The $U\dfrac{\partial U}{\partial s}$ term can be written as $\dfrac{\partial}{\partial s}(\tfrac{1}{2}U^2)$ since, using the chain rule,

$$\frac{\partial}{\partial s}(\tfrac{1}{2}U^2) = \frac{\mathrm{d}}{\mathrm{d}U}(\tfrac{1}{2}U^2)\times\frac{\partial U}{\partial s} \quad = U\frac{\partial U}{\partial s}$$

If incompressible, then density is constant along a streamline ($\partial\rho/\partial s = 0$), so that ρ can be taken through the s derivative. Thus,

$$\frac{\partial}{\partial s}(\tfrac{1}{2}\rho U^2) \quad = \rho U\frac{\partial U}{\partial s}$$

Also,

$$\sin\alpha = \frac{\textit{change in height}}{\textit{distance}} = \frac{\partial z}{\partial s}$$

Hence,

$$\frac{\partial}{\partial s}(\tfrac{1}{2}\rho U^2) = -\frac{\partial p}{\partial s} - \rho g\frac{\partial z}{\partial s}$$

$$\Rightarrow \qquad \frac{\partial}{\partial s}(p + \rho g z + \tfrac{1}{2}\rho U^2) = 0$$

Since the bracketed term is independent of distance along a streamline, s, we have Bernoulli's equation:

$$p + \rho g z + \tfrac{1}{2}\rho U^2 = \textit{constant, along a streamline}$$

Q3.
(a)

$$u = \frac{\partial \phi}{\partial x} \quad = 2x$$
$$v = \frac{\partial \phi}{\partial y} \quad = -2y$$

(b)

$$a_x \equiv \frac{\mathrm{D}u}{\mathrm{D}t} \quad = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \quad = 0 + (2x)\times 2 + 0 + 0 \quad = 4x$$
$$a_y \equiv \frac{\mathrm{D}v}{\mathrm{D}t} \quad = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \quad = 0 + 0 + (-2y)\times(-2) + 0 \quad = 4y$$

(c) In general, the streamfunction is related to the velocity field by

$$\frac{\partial \psi}{\partial x} = -v, \qquad \frac{\partial \psi}{\partial y} = u$$

Substituting the velocity components:

$$\frac{\partial \psi}{\partial x} = 2y \quad \Rightarrow \quad \psi = 2xy + f(y)$$
$$\frac{\partial \psi}{\partial y} = 2x \quad \Rightarrow \quad \psi = 2xy + g(x)$$

where f and g are arbitrary functions of y and x, respectively. The only way these can simultaneously hold is if both f and g are equal to a constant. Hence,

$$\psi = 2xy + constant$$

(d) Streamlines correspond to ψ = constant (see Section 9). In this case, this implies

$$xy = constant$$

This is a rectangular hyperbola.

Examples include flow in a rectangular corner or an impinging 2-d jet.

Q4.

(a) Since $\nabla z = \mathbf{e}_z$, the combination of pressure and gravitational forces is

$$-\nabla p - \rho_0 g \mathbf{e}_z = -\nabla(p + \rho_0 gz)$$

$p + \rho_0 gz$ is the *piezometric pressure*.

(b) *Boussinesq approximation*: ignore density variations in all terms except the buoyancy force $(\rho - \rho_0)g$.

The approximation is often applied in environmental flows, where the density-determining scalars are salinity (oceans) or temperature (atmosphere or oceans).

(c) Momentum equation, with Boussinesq approximation in mass × acceleration and Coriolis terms and with the ρg term split into constant part (lumped with pressure) and remainder:

$$\rho_0 \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla(p + \rho_0 gz) - (\rho - \rho_0) g \mathbf{e}_z - 2\rho_0 \mathbf{\Omega} \wedge \mathbf{u} + \mu \nabla^2 \mathbf{u} \qquad (1)$$

Assume that density fluctuations are related to scalar fluctuations by

$$\frac{\rho - \rho_0}{\rho_0} = -\alpha(\theta - \theta_0) \qquad \text{or} \qquad \rho - \rho_0 = -\alpha\rho_0(\theta - \theta_0)$$

(The minus sign is appropriate for temperature, where increase in θ causes reduction in ρ).

Define non-dimensional variables by:

$$\mathbf{x} = L_0 \mathbf{x}^*$$

$$t = (L_0/U_0) t^*$$

$$\mathbf{u} = U_0 \mathbf{u}^*$$

$$p + \rho_0 gz = p_{ref} + \rho_0 U_0^2 p^*$$

$$\theta - \theta_0 = \Delta\theta\, \theta^*, \qquad \text{so that} \qquad \rho - \rho_0 = -\rho_0 \alpha \Delta\theta\, \theta^* = -\Delta\rho\, \theta^*$$

(In the last, the **magnitude** of change in density, $\Delta\rho = \rho_0 \alpha \Delta\theta$.)

Substitute in (1) to obtain:

$$\frac{\rho_0 U_0^2}{L_0} \frac{\mathrm{D}\mathbf{u}^*}{\mathrm{D}t^*} = -\frac{\rho_0 U_0^2}{L_0} \nabla^* p^* + \Delta\rho\, g \theta^* \mathbf{e}_z - 2\rho_0 \Omega U_0 \mathbf{e}_\Omega \wedge \mathbf{u}^* + \frac{\mu U_0}{L_0^2} \nabla^{*2} \mathbf{u}^*$$

($\mathbf{e}_z$ and $\mathbf{e}_\Omega$ are unit vectors in the direction of the z axis and rotation axis, respectively).

Multiply through by $\dfrac{L_0}{\rho_0 U_0^2}$:

$$\frac{\mathrm{D}\mathbf{u}^*}{\mathrm{D}t^*} = -\nabla^* p^* + \frac{\Delta\rho}{\rho} \frac{gL_0}{U_0^2} \theta^* \mathbf{e}_z - 2\frac{\Omega L_0}{U_0} \mathbf{e}_\Omega \wedge \mathbf{u}^* + \frac{\mu}{\rho_0 U_0 L_0} \nabla^{*2} \mathbf{u}^*$$

Finally, rename the dimensionless groups and drop the asterisks for convenience:

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p + \frac{\theta}{\mathrm{Fr}^2} \mathbf{e}_z - \frac{2}{\mathrm{Ro}} \mathbf{e}_\Omega \wedge \mathbf{u} + \frac{1}{\mathrm{Re}} \nabla^2 \mathbf{u}$$

Q5.

(a) Consider a control volume of dimensions $\Delta x \times \Delta y \times h$.

On the faces in the x and $-x$ directions (e and w) the mass fluxes are given by

$$\pm \rho u \, \Delta y \mathrm{d}z \, .$$

On the faces in the y and $-y$ directions (n and s) the mass fluxes are given by

$$\pm \rho v \, \Delta x \mathrm{d}z \, .$$

On the free surface and bed the *relative* mass fluxes are zero (because the free surface is a material surface, whilst the bed is a solid boundary). Hence we have the following.

<u>Continuity</u>

$$\frac{\mathrm{d}}{\mathrm{d}t}(mass) + net \;\; mass \;\; flux = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho \bar{h} \Delta x \Delta y) + (\rho u h \Delta y)_e - (\rho u h \Delta y)_w + (\rho v h \Delta x)_n - (\rho v h \Delta x)_s = 0$$

(where the overbar on h indicates an average over a cell).

Dividing by $\rho \Delta x \Delta y$,

$$\frac{\mathrm{d}\bar{h}}{\mathrm{d}t} + \frac{(hu)_e - (hu)_w}{\Delta x} + \frac{(hv)_n - (hv)_s}{\Delta y} = 0$$

Letting Δx and $\Delta y \to 0$:

$$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} + \frac{\partial (hv)}{\partial y} = 0$$

<u>x-Momentum</u>

$$\frac{\mathrm{d}}{\mathrm{d}t}(momentum) + net \;\; momentum \;\; flux = pressure \;\; force + viscous \;\; force$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho \bar{h} \Delta x \Delta y u) + (\rho u h \Delta y \times u)_e - (\rho u h \Delta y \times u)_w + (\rho v h \Delta x \times u)_n - (\rho v h \Delta x \times u)_s$$

$$= (\bar{p} h \Delta y)_w - (\bar{p} h \Delta y)_e + \bar{p}_{surf} \Delta z_{surf} \Delta y - \bar{p}_{bed} \Delta z_{bed} \Delta y$$

$$+ \bar{\tau}_{13}(surf) \Delta x \Delta y - \bar{\tau}_{13}(bed) \Delta x \Delta y$$

where $\bar{p}$ and $\bar{\tau}$ denote averages over their respective faces. Note that there are pressure forces on the projected free-surface and bed faces if these are not flat (i.e. if Δz_{surf} and Δz_{bed} are non-zero).

Dividing by $\rho \Delta x \Delta y$ and noting that, since the pressure is hydrostatic and varies linearly between 0 at the surface and $\rho g h$ at the bed, $\bar{p}_{surf} = 0$, $\bar{p}_{bed} = \rho g h$ and $\bar{p}_{e,w} = \frac{1}{2} \rho g h$:

$$\frac{\mathrm{d}(\bar{h} u)}{\mathrm{d}t} + \frac{(hu^2)_e - (hu^2)_w}{\Delta x} + \frac{(hvu)_n - (hvu)_s}{\Delta y}$$

$$= -\frac{(\frac{1}{2} g h^2)_e - (\frac{1}{2} g h^2)_w}{\Delta x} - gh \frac{\Delta z_{bed}}{\partial x} + \frac{\bar{\tau}_{13}(surf) - \bar{\tau}_{13}(bed)}{\rho}$$

Letting Δx and $\Delta y \to 0$:

$$\frac{\partial(hu)}{\partial t} + \frac{\partial(hu^2)}{\partial x} + \frac{\partial(hvu)}{\partial y} = -\frac{\partial(\frac{1}{2} gh^2)}{\partial x} - gh\frac{\partial z_{bed}}{\partial x} + \frac{\tau_{13}(surf) - \tau_{13}(bed)}{\rho}$$

$$= -gh\frac{\partial(h + z_{bed})}{\partial x} + \frac{\tau_{13}(surf) - \tau_{13}(bed)}{\rho}$$

$$= -gh\frac{\partial z_{surf}}{\partial x} + \frac{\tau_{13}(surf) - \tau_{13}(bed)}{\rho}$$

(b)

<u>Continuity</u>

Start with

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Use subscripts b and s for bed and surface, respectively. Integrate from $z = z_b$ to $z = z_s$:

$$\int_{z_b}^{z_s} \frac{\partial u}{\partial x}\ \mathrm{d}z + \int_{z_b}^{z_s} \frac{\partial v}{\partial y}\ \mathrm{d}z + w(z_s) - w(z_b) = 0$$

Bring space-derivative operators outside integral signs by (rearranging) Leibniz' Theorem:

$$\left(\frac{\partial}{\partial x}\int_{z_b}^{z_s} u\ \ \mathrm{d}z - u(z_s)\frac{\partial z_s}{\partial x} + u(z_b)\frac{\partial z_b}{\partial x}\right) + \left(\frac{\partial}{\partial y}\int_{z_b}^{z_s} v\ \ \mathrm{d}z - v(z_s)\frac{\partial z_s}{\partial y} + v(z_b)\frac{\partial z_b}{\partial y}\right)$$
$$+ w(z_s) - w(z_b) = 0$$

From the upper and lower boundary conditions:

$$w(z_s) - u(z_s)\frac{\partial z_s}{\partial x} - v(z_s)\frac{\partial z_s}{\partial y} = \frac{\partial z_s}{\partial t}$$

$$w(z_b) - u(z_b)\frac{\partial z_b}{\partial x} - v(z_b)\frac{\partial z_b}{\partial y} = \frac{\partial z_b}{\partial t}$$

Hence

$$\frac{\partial}{\partial t}(z_s - z_b) + \frac{\partial}{\partial x}\int_{z_b}^{z_s} u\ \ \mathrm{d}z + \frac{\partial}{\partial y}\int_{z_b}^{z_s} v\ \ \mathrm{d}z = 0$$

Using $h = z_s - z_b$ and the depth-averaged velocity components:

$$\frac{\partial h}{\partial t} + \frac{\partial(hu)}{\partial x} + \frac{\partial(hv)}{\partial y} = 0$$

<u>*x*-momentum</u>

Start with

$$\frac{\partial(\rho u)}{\partial t}+\frac{\partial(\rho u^2)}{\partial x}+\frac{\partial(\rho v u)}{\partial y}+\frac{\partial(\rho w u)}{\partial z}=-\frac{\partial p}{\partial x}+\frac{\partial \tau_{13}}{\partial z}$$

Divide by ρ (which is constant) and integrate from $z = z_b$ to $z = z_s$:

$$\int_{z_b}^{z_s}\frac{\partial u}{\partial t}\,\mathrm{dz}+\int_{z_b}^{z_s}\frac{\partial(u^2)}{\partial x}\,\mathrm{dz}+\int_{z_b}^{z_s}\frac{\partial(vu)}{\partial y}\,\mathrm{dz}+wu(z_s)-wu(z_b)$$

$$=-\frac{1}{\rho}\int_{z_b}^{z_s}\frac{\partial p}{\partial x}\,\mathrm{dz}+\frac{\tau_{13}(z_s)-\tau_{13}(z_b)}{\rho}$$

Apply Leibniz' Theorem to bring time and space derivatives outside the integrals:

$$\left(\frac{\partial}{\partial t}\int_{z_b}^{z_s}u\;\;\mathrm{dz}-u(z_s)\frac{\partial z_s}{\partial t}+u(z_b)\frac{\partial z_b}{\partial t}\right)$$

$$+\left(\frac{\partial}{\partial x}\int_{z_b}^{z_s}u^2\;\;\mathrm{dz}-u^2(z_s)\frac{\partial z_s}{\partial x}+u^2(z_b)\frac{\partial z_b}{\partial x}\right)$$

$$+\left(\frac{\partial}{\partial y}\int_{z_b}^{z_s}vu\;\;\mathrm{dz}-vu(z_s)\frac{\partial z_s}{\partial y}+vu(z_b)\frac{\partial z_b}{\partial y}\right)+wu(z_s)-wu(z_b)$$

$$=-\frac{1}{\rho}\left(\frac{\partial}{\partial x}\int_{z_b}^{z_s}p\;\;\mathrm{dz}-p(z_s)\frac{\partial z_s}{\partial x}+p(z_b)\frac{\partial z_b}{\partial x}\right)+\frac{\tau_{13}(z_s)-\tau_{13}(z_b)}{\rho}$$

If p is hydrostatic then

$$\int_{z_b}^{z_s}p\;\;\mathrm{dz}=\overline{p}h\quad=\tfrac{1}{2}\rho g h^2$$

whilst $p(z_s) = 0$ and $p(z_b) = \rho g h$. Hence, collecting terms:

$$\frac{\partial}{\partial t}\int_{z_b}^{z_s}u\;\;\mathrm{dz}+\frac{\partial}{\partial x}\int_{z_b}^{z_s}u^2\;\;\mathrm{dz}+\frac{\partial}{\partial y}\int_{z_b}^{z_s}vu\;\;\mathrm{dz}$$

$$+u(z_s)\underbrace{\left(w(z_s)-\frac{\partial z_s}{\partial t}-u(z_s)\frac{\partial z_s}{\partial x}-v(z_s)\frac{\partial z_s}{\partial y}\right)}_{=0}$$

$$-u(z_b)\underbrace{\left(w(z_b)-\frac{\partial z_b}{\partial t}-u(z_b)\frac{\partial z_b}{\partial x}-v(z_b)\frac{\partial z_b}{\partial y}\right)}_{=0}$$

$$=-\frac{\partial}{\partial x}(\tfrac{1}{2}gh^2)-g\frac{\partial z_b}{\partial x}+\frac{\tau_{13}(z_s)-\tau_{13}(z_b)}{\rho}$$

Replacing velocities by their depth averages, applying the boundary conditions and using

$$\frac{\partial}{\partial x}(\tfrac{1}{2}gh^2)+gh\frac{\partial z_b}{\partial x}\quad=gh\frac{\partial}{\partial x}(h+z_b)\quad=gh\frac{\partial z_s}{\partial x}$$

gives, finally,

$$\frac{\partial(hu)}{\partial t}+\frac{\partial(hu^2)}{\partial x}+\frac{\partial(hvu)}{\partial y}=-gh\frac{\partial z_s}{\partial x}+\frac{\tau_{13}(z_s)-\tau_{13}(z_b)}{\rho}$$