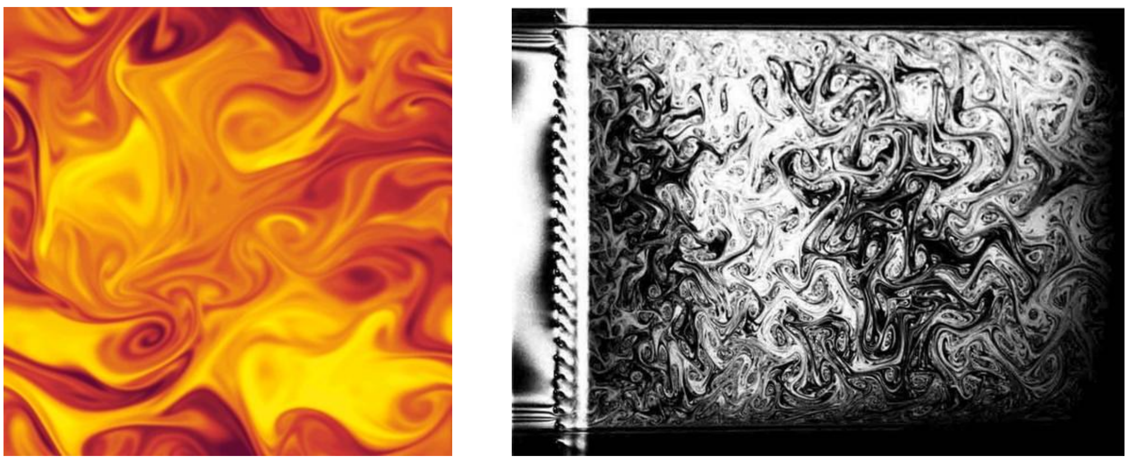
Essay 11

The RANS Equations ---- The Basis of Turbulence Modeling



(a) (b)

Fig. 11.1 Diagrams of turbulent flow. (a) Computer simulation of isotropic turbulence. (b) Experimental measurements of turbulent flow downstream of a turbulence promoter.

11.1 Introduction

Most fluid flows, especially those in pipes and ducts, are turbulent. A characteristic feature of a turbulent flow is a small-scale, high-frequency random fluctuation superimposed on a main flow which has an identifiable direction. Normally, the magnitude of the fluctuation is in the range of 5-10% of the magnitude of the main flow. It is remarkable that such a minor irregularity has such a profound impact on the character of the flow.

11.2 Characterization of Turbulence

To begin the characterization of turbulence, it is helpful to envision a simple experiment as displayed in Fig. 11.2. In that experiment, a stationary probe at a fixed location is positioned in a pipe flow. That probe might be a hot-wire anemometer, a laser-Doppler velocimeter (LDV), or a particle-imaging velocimeter (PIV). The velocity measurement provided by these instruments is in the form of a voltage. By suitable signal processing, the voltage can be converted into a velocity and displayed on the screen of a computer. A typical output of such a velocity measurement is pictured in Figs. 11.3(a) and (b). In the first of these figures, an actual trace of

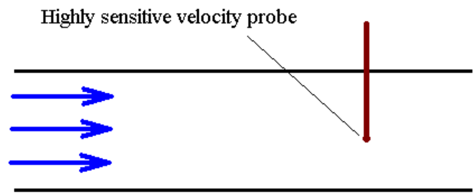


Fig. 11. 2 Test setup for measurement of the velocity of a turbulent flow

such an output is displayed, while the second exhibits an idealized version. Both diagrams show the instantaneous velocity as a function of time at a fixed position in pipe. In addition, in diagram (b), a horizontal line in brown color is shown to indicate the time independence of a laminar flow. Also, in the same diagram, the fluctuating flow is earmarked with a blue horizontal line which represents the time average of the fluctuating velocities. The averaging process is defined by:

$$\bar{u} = \frac{1}{\tau}\int_0^\tau u(t)dt \tag{11.1}$$

where τ is the averaging time.

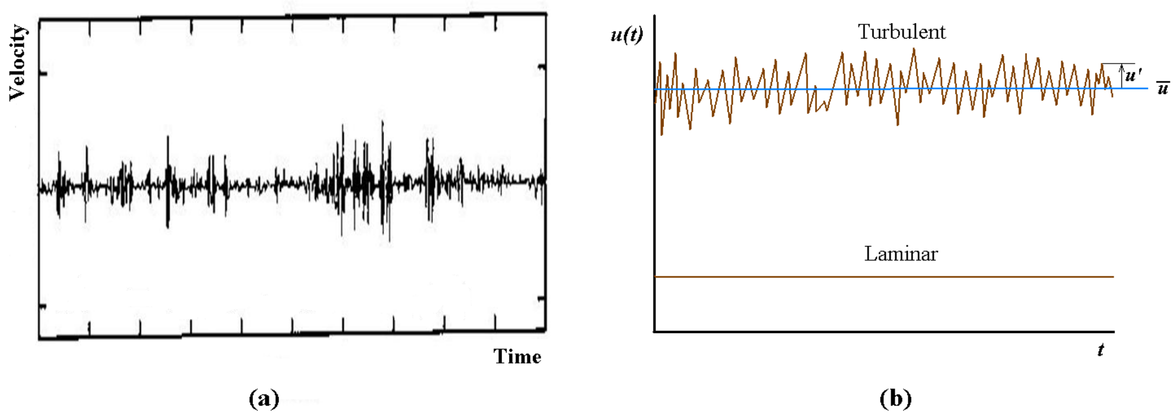


Fig. 11. 3 Instantaneous velocity measurements in a turbulent flow. (a) Actual measurement output. (b) Idealized version.

It is standard practice to describe a turbulent velocity field as a superposition of the time-averaged velocity $\bar{u}$ and the fluctuating component u' as follows:

$$u(x,y,z,t) = \bar{u}(x,y,z) + u'(x,y,z,t) \tag{11.2}$$

For a three-dimensional flow, the other velocity components are described in a similar manner, so that:

$$v(x,y,z,t) = \bar{v}(x,y,z) + v'(x,y,z,t) \tag{11.3}$$

$$w(x, y, z, t) = \bar{w}(x, y, z) + w'(x, y, z, t) \tag{11.4}$$

11.3 The time-dependent Navier-Stokes Equations

The Navier-Stokes equations have been set forth at the end of Essay 8. The x-direction equation, Eq. (8.14), taken from that assemblage of equations is reproduced here for convenience.

$$\rho\left[\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(u^2) + \frac{\partial}{\partial y}(uv) + \frac{\partial}{\partial z}(uw)\right] = -\frac{\partial p}{\partial x} + \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right] \tag{11.5}$$

Note that a time-dependent term $\partial u/\partial t$ has been appended to that equation to take account of the fact that a turbulent flow is inherently unsteady. The various velocities that appear in Eq. (11.5) are unsteady and may be expressed in the decomposition form stated in Eqs. (11.2-4).

Attention may first be focused on leftmost term of Eq. (11.5). After substitution of Eq. (11.2), there is obtained:

$$\frac{\partial u}{\partial t} = \frac{\partial \bar{u}}{\partial t} + \frac{\partial u'}{\partial t} = \frac{\partial u'}{\partial t} \tag{11.6}$$

In arriving at the rightmost term of this equation, advantage has been taken of the fact that the time-independent quantity $\bar{u}$ has a zero time-derivative.

Next, focus is shifted to the second term on the left-hand side of Eq. (11.5). Upon substitution of Eq. (11.2), there results:

$$\frac{\partial}{\partial x}(u^2) = \frac{\partial}{\partial x}[(\bar{u} + u')^2] = \frac{\partial}{\partial x}\left(\bar{u}^2 + 2\bar{u}u' + u'^2\right) \tag{11.7}$$

It is useful to examine a graphical representation of the respective terms that appear within the parentheses on the right-hand side of Eq. (11.7). Such a representation is displayed in Fig. 11.4.

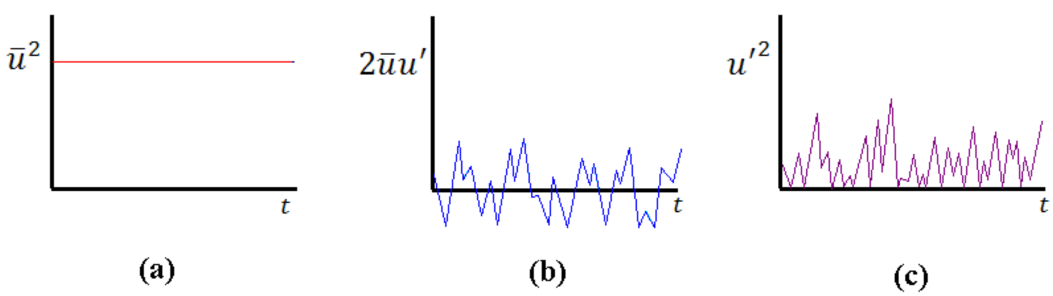


Fig. 11. 4 Graphical representations of the respective terms that appear on the right-hand side of Eq. (11.7)

The crucial step in deriving the Reynolds-Averaged Navier-Stokes Equations from the unsteady form of those equations, Eq. (11.5), is to time-average the latter. That time averaging can best be understood by viewing the graphical representations (a), (b), and (c) of Fig. 11.4. The (a) part of the figure represents a quantity that is independent of time, so that its time-average value is equal to the value of the time-independent quantity. The (b) part shows a time-varying quantity which is a multiple of u'. Since the time average of u' is zero, so is the time average of $2\bar{u}u'$. The (c) part of Fig. 11.4 shows an oscillating function whose value is always positive. Its time average is $\overline{u'^2}$.

In light of the foregoing discussion, the time average of $\frac{\partial u}{\partial t}$ as represented by Eq. (11.6) is zero since the time average of u' is zero. Then, if the focus is shifted to Eq. (11.7), the time average of the quantities in that equation is:

$$\overline{\frac{\partial}{\partial x}(u^2)} = \overline{\frac{\partial}{\partial x}\left(\bar{u}^2 + 2\bar{u}u' + u'^2\right)} = \frac{\partial}{\partial x}\left(\bar{u}^2 + \overline{u'^2}\right) \tag{11.8}$$

The next term in Eq. (11.5) to be considered is $\frac{\partial}{\partial y}(uv)$. After substitution of the decomposed representations for u and v, Eqs. (11.2-3), there is obtained:

$$\frac{\partial}{\partial y}(uv) = \frac{\partial}{\partial y}[(\bar{u} + u') \times (\bar{v} + v')] = \frac{\partial}{\partial y}(\bar{u}\bar{v} + \bar{u}v' + \bar{v}u' + u'v') \tag{11.9}$$

which, upon time averaging, gives rise to:

$$\overline{\frac{\partial}{\partial y}(uv)} = \overline{\frac{\partial}{\partial y}[(\bar{u} + u') \times (\bar{v} + v')]} = \frac{\partial}{\partial y}\left(\bar{u}\bar{v} + \overline{u'v'}\right) \tag{11.10}$$

The same treatment can be accorded to the last term on the left-hand side of Eq. (11.5). If that term is subjected to the same manipulations as its predecessors, there results:

$$\overline{\frac{\partial}{\partial z}(uw)} = \frac{\partial}{\partial z}\left(\bar{u}\bar{w} + \overline{u'w'}\right) \tag{11.11}$$

Attention is now shifted to the right-hand side of Eq. (11.5). If each of the terms which constitute that side are evaluated using the decompositions given in Eqs. (11.2-4) and subsequently time averaged, the outcome is:

$$-\frac{\partial \bar{p}}{\partial x} + \mu\left[\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial y^2} + \frac{\partial^2 \bar{u}}{\partial z^2}\right] \tag{11.12}$$

11.4 The Assembled RANS Equations

The components of the time-averaged Navier-Stokes equations may now be brought together from Eqs. (11.8, 11.10, 11.11, and 11.12) to yield,

$$\rho\left[\frac{\partial}{\partial x}\left(\bar{u}^2 + \overline{u'^2}\right) + \frac{\partial}{\partial y}\left(\bar{u}\bar{v} + \overline{u'v'}\right) + \frac{\partial}{\partial z}\left(\bar{u}\bar{w} + \overline{u'w'}\right)\right] = -\frac{\partial \bar{p}}{\partial x} + \mu\left[\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial y^2} + \frac{\partial^2 \bar{u}}{\partial z^2}\right] \tag{11.13}$$

This equation appears to be very similar to the steady-state, x-direction Navier-Stokes equation aside from terms involving the fluctuating velocities. It is conventional to transport these terms to the right-hand side so as to make the after-shift left-hand side be congruent with the steady-state, Navier-Stokes x-direction equation.

$$\rho\left[\frac{\partial}{\partial x}(\bar{u}^2)+\frac{\partial}{\partial y}(\bar{u}\bar{v})+\frac{\partial}{\partial z}(\bar{u}\bar{w})\right]$$
$$=-\frac{\partial \bar{p}}{\partial x}+\mu\left[\frac{\partial^2\bar{u}}{\partial x^2}+\frac{\partial^2\bar{u}}{\partial y^2}+\frac{\partial^2\bar{u}}{\partial z^2}\right]-\left[\frac{\partial}{\partial x}\left(\rho\overline{u'^2}\right)+\frac{\partial}{\partial y}\left(\rho\overline{u'v'}\right)+\frac{\partial}{\partial z}\left(\rho\overline{u'w'}\right)\right] \qquad (11.14)$$

Note that in the shifted terms, the density ρ has been placed within the derivative operators since it is a constant. The terms that appear in the last bracket of Eq. (11.14) are called *Reynolds Stresses*. This designation, while being traditional, is somewhat illogical because the terms in question arise from momentum flow rate terms.

These “stresses” are the source that creates the turbulent disturbance of the otherwise steady Navier-Stokes equations.

An often-cited form of the RANS equation, Eq. (11.14), is:

$$\rho\left[\frac{\partial}{\partial x}(\bar{u}^2)+\frac{\partial}{\partial y}(\bar{u}\bar{v})+\frac{\partial}{\partial z}(\bar{u}\bar{w})\right]$$
$$=-\frac{\partial \bar{p}}{\partial x}+\left[\frac{\partial}{\partial x}\left(\mu\frac{\partial\bar{u}}{\partial x}-\rho\overline{u'^2}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial\bar{u}}{\partial y}-\rho\overline{u'v'}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial\bar{u}}{\partial z}-\rho\overline{u'w'}\right)\right] \qquad (11.15)$$

With this rearrangement, the terms involving the fluctuations take on an appearance which supports their designation as stresses. For example, since $\mu\frac{\partial\bar{u}}{\partial x}$ is the viscous normal stress in a laminar flow, the quantity $-\rho\overline{u'^2}$ may be termed the normal stress due to turbulence.