Answers 10

Q1.
(a)
RANS solver – a computer code which solves for average, not instantaneous, flow quantities. (For theoretical work "average" refers to a probabilistic or "ensemble" average).

(b)
(i) *Eddy-viscosity models*: (deviatoric) stress $\propto$ mean rate of strain.

(ii) *Non-linear eddy-viscosity models*: Reynolds stresses modelled as non-linear functions of the mean-velocity gradients.

(iii) *Differential stress models*: solve transport equations for all Reynolds stresses.

(iv) *Large-eddy simulation*: carry out a full time-dependent simulation of resolvable motions, with modelling only of subgrid (i.e., unresolvable) scales.

Q2.
Dimensions:

$[\rho] = ML^{-3}$
$[\mu_t] = ML^{-1}T^{-1}$
$[k] = L^2T^{-2}$
$[\varepsilon] = L^2T^{-3}$
$[\omega] = T^{-1}$

Since only μ_t and ρ contain M as a dimension, the expression for μ_t in terms of ρ, k and ε must be of the form

$$\mu_t = constant \times \rho k^a \varepsilon^b$$

Equating powers of the fundamental dimensions:

$$\text{L}: \quad -1 = \quad -3 + 2a + 2b$$
$$\text{T}: \quad -1 = \quad -2a - 3b$$

Solving gives $a = 2$, $b = -1$; i.e.

$$\mu_t = constant \times \rho \frac{k^2}{\varepsilon}$$

Similarly, the expression for μ_t in terms of ρ, k and ω must be of the form

$$\mu_t = constant \times \rho k^c \omega^d$$

Equating powers of the fundamental dimensions:

$$\text{L}: \quad -1 = \quad -3 + 2c$$
$$\text{T}: \quad -1 = \quad -2c - d$$

Solving gives $c = 1$, $d = -1$; i.e.

$$\mu_t = constant \times \rho \frac{k}{\omega}$$

Q3.

(a)

$$\frac{\mathrm{D}U_1}{\mathrm{D}t} = \frac{\partial U_1}{\partial t} + U_1 \frac{\partial U_1}{\partial x_1} + U_2 \frac{\partial U_1}{\partial x_2} + U_3 \frac{\partial U_1}{\partial x_3}$$

$$\Rightarrow \qquad \frac{\mathrm{D}U}{\mathrm{D}t} = \frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} + W \frac{\partial U}{\partial z}$$

Similarly, by expanding, or by inspection of the result above:

$$\frac{\mathrm{D}V}{\mathrm{D}t} = \frac{\partial V}{\partial t} + U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y} + W \frac{\partial V}{\partial z}$$

(b) It is required to sum over all i and all j. A good way is to separate cases where $i = j$ and $i \neq j$:

$$P^{(k)} = -\left(\overline{u^2} \frac{\partial U}{\partial x} + \overline{v^2} \frac{\partial V}{\partial y} + \overline{w^2} \frac{\partial W}{\partial z} \right)$$
$$-\overline{vw}\left(\frac{\partial V}{\partial z} + \frac{\partial W}{\partial y} \right) - \overline{wu}\left(\frac{\partial W}{\partial x} + \frac{\partial U}{\partial z} \right) - \overline{uv}\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} \right)$$

(c) When $i = j$ the two parts of the sum are the same, so

$$P_{11} = -2\overline{u_1 u_k} \frac{\partial U_1}{\partial x_k} \quad = -2\left(\overline{u_1 u_1} \frac{\partial U_1}{\partial x_1} + \overline{u_1 u_2} \frac{\partial U_1}{\partial x_2} + \overline{u_1 u_3} \frac{\partial U_1}{\partial x_3} \right)$$
$$= -2\left(\overline{uu} \frac{\partial U}{\partial x} + \overline{uv} \frac{\partial U}{\partial y} + \overline{uw} \frac{\partial U}{\partial w} \right)$$

When $i \neq j$ there are, unfortunately, twice as many terms:

$$P_{12} = -\overline{u_1 u_k} \frac{\partial U_2}{\partial x_k} - \overline{u_2 u_k} \frac{\partial U_1}{\partial x_k}$$
$$= -\left(\overline{u_1 u_1} \frac{\partial U_2}{\partial x_1} + \overline{u_1 u_2} \frac{\partial U_2}{\partial x_2} + \overline{u_1 u_3} \frac{\partial U_2}{\partial x_3} \right) - \left(\overline{u_2 u_1} \frac{\partial U_1}{\partial x_1} + \overline{u_2 u_2} \frac{\partial U_1}{\partial x_2} + \overline{u_2 u_3} \frac{\partial U_1}{\partial x_3} \right)$$
$$= -\left(\overline{uu} \frac{\partial V}{\partial x} + \overline{uv} \frac{\partial V}{\partial y} + \overline{uw} \frac{\partial V}{\partial z} \right) - \left(\overline{vu} \frac{\partial U}{\partial x} + \overline{vv} \frac{\partial U}{\partial y} + \overline{vw} \frac{\partial U}{\partial z} \right)$$

(d)

$$M_{ii} = M_{11} + M_{22} + M_{33} \quad = trace(\mathbf{M}) \qquad \text{(also written as } \{\mathbf{M}\}\text{)}$$

$$M_{ij} M_{ji} = (\mathbf{M}^2)_{ii} \qquad \text{(by definition of matrix multiplication)}$$
$$= (\mathbf{M}^2)_{11} + (\mathbf{M}^2)_{22} + (\mathbf{M}^2)_{33}$$
$$= trace(\mathbf{M}^2) \qquad \text{(also written as } \{\mathbf{M}^2\}\text{)}$$

Similarly,

$$M_{ij} M_{jk} M_{ki} = trace(\mathbf{M}^3)$$

Q4.

(a) By definition,

$$P^{(k)} \equiv -\overline{u_i u_j}\frac{\partial U_i}{\partial x_j}$$

This is a sum over all i and j. However, if $\frac{\partial U}{\partial y}$ is the only non-zero mean-velocity gradient then there is only a non-zero contribution when $i = 1$ and $j = 2$. Hence, in simple shear,

$$P^{(k)} \quad \rightarrow \quad -\overline{u_1 u_2}\frac{\partial U_1}{\partial x_2} \quad = -\overline{uv}\frac{\partial U}{\partial y}$$

(b) By definition,

$$P_{ij} \equiv -\left(\overline{u_i u_k}\frac{\partial U_j}{\partial x_k} + \overline{u_j u_k}\frac{\partial U_i}{\partial x_k}\right)$$

This is a sum over all k. However, if $\frac{\partial U}{\partial y}$ is the only non-zero mean-velocity gradient then there is only a non-zero contribution to the first term when $j = 1$ and $k = 2$ and to the second term when $i = 1$ and $k = 2$. Hence, in simple shear,

$$P_{11} = -2\overline{uv}\frac{\partial U}{\partial y}, \qquad P_{22} = P_{33} = 0$$

$$P_{12} = -\overline{v^2}\frac{\partial U}{\partial y}, \qquad P_{23} = P_{31} = 0 \qquad (P_{ji} = P_{ij} \text{ in any flow})$$

(c) By definition,

$$S_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)$$

In simple shear,

$$(S_{ij}) = \begin{pmatrix} 0 & \frac{1}{2}\frac{\partial U}{\partial y} & 0 \\ \frac{1}{2}\frac{\partial U}{\partial y} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$S^2 \equiv 2S_{ij}S_{ij} \quad = 2\times(\text{sum of elements})^2 \quad = 2\times 2\times\left(\frac{1}{2}\frac{\partial U}{\partial y}\right)^2 \quad = \left(\frac{\partial U}{\partial y}\right)^2$$

Hence,

$$S = \left|\frac{\partial U}{\partial y}\right|$$

(d) By definition,

$$\Omega_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i}\right)$$

In simple shear,

$$(\Omega_{ij}) = \begin{pmatrix} 0 & \frac{1}{2}\frac{\partial U}{\partial y} & 0 \\ -\frac{1}{2}\frac{\partial U}{\partial y} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Omega^2 = 2\Omega_{ij}\Omega_{ij} \quad = 2\times(\text{sum of elements})^2 \quad = 2\times 2\times\left(\frac{1}{2}\frac{\partial U}{\partial y}\right)^2 \quad = \left(\frac{\partial U}{\partial y}\right)^2$$

Hence,

$$\Omega = \left|\frac{\partial U}{\partial y}\right|$$

Q5.
(a)

$$P^{(k)} \equiv -\overline{u_i u_j}\frac{\partial U_i}{\partial x_k} \quad = -\overline{u_i u_j}(S_{ij} + \Omega_{ij}) \qquad (*)$$

where

$$S_{ij} \equiv \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_i}{\partial x_j}\right) \qquad = \text{mean strain (symmetric)}$$

$$\Omega_{ij} \equiv \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} - \frac{\partial U_i}{\partial x_j}\right) \qquad = \text{mean vorticity (antisymmetric)}$$

But $\overline{u_i u_j}$ is symmetric, whereas $\Omega_{ij} = -\Omega_{ji}$ so that all Ω-related contributions to (*) will cancel in pairs. Hence,

$$P^{(k)} = -\overline{u_i u_j} S_{ij}$$

(b) The anisotropy tensor is defined by

$$a_{ij} = \frac{\overline{u_i u_j}}{k} - \frac{2}{3}\delta_{ij}$$

Hence,

$$\overline{u_i u_j} = k a_{ij} + \frac{2}{3} k \delta_{ij}$$

Then,

$$\begin{aligned} P^{(k)} &= -\overline{u_i u_j} S_{ij} \\ &= -(k a_{ij} + \frac{2}{3} k \delta_{ij}) S_{ij} \end{aligned}$$

But for the second term in the summation:

$$\delta_{ij} S_{ij} = S_{ii} \quad = S_{11} + S_{22} + S_{33} \quad = \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} \quad = 0 \quad \text{(incompressibility)}$$

Hence, in incompressible flow,

$$P^{(k)} = -k a_{ij} S_{ij}$$

(c) The eddy-viscosity hypothesis gives

$$-\overline{u_i u_j} = \nu_t S_{ij} - \frac{2}{3} k \delta_{ij}$$

$$\Rightarrow \qquad -k a_{ij} = \nu_t S_{ij}$$

Hence, in incompressible flow,

$$\begin{aligned} P^{(k)} &= \nu_t S_{ij} S_{ij} \\ &= \nu_t \times (\text{ sum of elements of } S_{ij} \text{ squared}) \\ &\geq 0 \end{aligned}$$

Q6.
(a)

$$\phi = k^m \varepsilon^n \quad \Rightarrow \quad \varepsilon = \phi^{1/n} k^{-m/n}$$

Hence,

$$\begin{aligned} \nu_t &= C_\mu \frac{k^2}{\varepsilon} \\ &= C_\mu k^{2+m/n} \phi^{-1/n} \end{aligned}$$

(b) For ANY differential of ϕ,

$$\begin{aligned} \mathrm{d}\phi &= \mathrm{d}(k^m \varepsilon^n) \\ &= mk^{m-1}\varepsilon^n \mathrm{d}k + k^m n\varepsilon^{n-1}\mathrm{d}\varepsilon \end{aligned}$$

Hence,

$$\frac{\mathrm{d}\phi}{\phi} = m\frac{\mathrm{d}k}{k} + n\frac{\mathrm{d}\varepsilon}{\varepsilon}$$

In particular,

$$\frac{1}{\phi}\frac{\mathrm{D}\phi}{\mathrm{D}t} = \frac{m}{k}\frac{\mathrm{D}k}{\mathrm{D}t} + \frac{n}{\varepsilon}\frac{\mathrm{D}\varepsilon}{\mathrm{D}t} \qquad \text{and} \qquad \frac{1}{\phi}\nabla\phi = \frac{m}{k}\nabla k + \frac{n}{\varepsilon}\nabla\varepsilon$$

From the first of these,

$$\begin{aligned} \frac{n}{\varepsilon}\frac{\mathrm{D}\varepsilon}{\mathrm{D}t} &= \frac{1}{\phi}\frac{\mathrm{D}\phi}{\mathrm{D}t} - \frac{m}{k}\frac{\mathrm{D}k}{\mathrm{D}t} \\ &= \frac{1}{\phi}\left\{\nabla\bullet(\Gamma^{(\phi)}\nabla\phi) + (C_{\phi 1}P^{(k)} - C_{\phi 2}\varepsilon)\frac{\phi}{k} + S^{(\phi)}\right\} - \frac{m}{k}\left\{\nabla\bullet(\Gamma^{(k)}\nabla k) + P^{(k)} - \varepsilon\right\} \\ &= \frac{1}{\phi}\nabla\bullet(\Gamma^{(\phi)}\nabla\phi) - \frac{m}{k}\nabla\bullet(\Gamma^{(k)}\nabla k) + \left\{(C_{\phi 1} - m)P^{(k)} - (C_{\phi 2} - m)\varepsilon\right\}\frac{1}{k} + \frac{S^{(\phi)}}{\phi} \end{aligned} \qquad (*)$$

The first two terms on the RHS of (*) give:

$$\begin{aligned} &\frac{1}{\phi}\nabla\bullet(\Gamma^{(\phi)}\nabla\phi) - \frac{m}{k}\nabla\bullet(\Gamma^{(k)}\nabla k) \\ &= \frac{1}{\phi}\nabla\bullet\left[\Gamma^{(\phi)}(\frac{m}{k}\nabla k + \frac{n}{\varepsilon}\nabla\varepsilon)\phi\right] - \frac{m}{k}\nabla\bullet(\Gamma^{(k)}\nabla k) \\ &= \nabla\bullet\left[\Gamma^{(\phi)}(\frac{m}{k}\nabla k + \frac{n}{\varepsilon}\nabla\varepsilon)\right] + \Gamma^{(\phi)}(\frac{m}{k}\nabla k + \frac{n}{\varepsilon}\nabla\varepsilon)\bullet\frac{\nabla\phi}{\phi} - \frac{m}{k}\nabla\bullet(\Gamma^{(k)}\nabla k) \\ &= \frac{m}{k}\nabla\bullet(\Gamma^{(\phi)}\nabla k) - \Gamma^{(\phi)}\frac{m}{k^2}(\nabla k)^2 + \frac{n}{\varepsilon}\nabla\bullet(\Gamma^{(\phi)}\nabla\varepsilon) - \Gamma^{(\phi)}\frac{n}{\varepsilon^2}(\nabla\varepsilon)^2 + \Gamma^{(\phi)}(\frac{m}{k}\nabla k + \frac{n}{\varepsilon}\nabla\varepsilon)^2 - \frac{m}{k}\nabla\bullet(\Gamma^{(k)}\nabla k) \\ &= \frac{m}{k}\nabla\bullet\left[(\Gamma^{(\phi)} - \Gamma^{(k)})\nabla k\right] + \frac{n}{\varepsilon}\nabla\bullet(\Gamma^{(\phi)}\nabla\varepsilon) + \Gamma^{(\phi)}\left[(m\frac{\nabla k}{k} + n\frac{\nabla\varepsilon}{\varepsilon})^2 - m(\frac{\nabla k}{k})^2 - n(\frac{\nabla\varepsilon}{\varepsilon})^2\right] \\ &= \frac{m}{k}\nabla\bullet\left[(\Gamma^{(\phi)} - \Gamma^{(k)})\nabla k\right] + \frac{n}{\varepsilon}\nabla\bullet(\Gamma^{(\phi)}\nabla\varepsilon) + \Gamma^{(\phi)}\left[m(m-1)(\frac{\nabla k}{k})^2 + n(n-1)(\frac{\nabla\varepsilon}{\varepsilon})^2 + 2mn(\frac{\nabla k}{k})\bullet(\frac{\nabla\varepsilon}{\varepsilon})\right] \end{aligned}$$

Substituting into (*) and multiplying by ε/n gives

$$\frac{\mathrm{D}\varepsilon}{\mathrm{D}t} = \nabla \bullet (\Gamma^{(\phi)} \nabla \varepsilon) + \left\{ (\frac{C_{\phi 1} - m}{n}) P^{(k)} - (\frac{C_{\phi 2} - m}{n}) \varepsilon \right\} \frac{\varepsilon}{k}$$

$$+ \frac{\varepsilon}{n} \left\{ \frac{S^{(\phi)}}{\phi} + \frac{m}{k} \nabla \bullet \left[(\Gamma^{(\phi)} - \Gamma^{(k)}) \nabla k \right] + \Gamma^{(\phi)} \left[m(m-1)(\frac{\nabla k}{k})^2 + n(n-1)(\frac{\nabla \varepsilon}{\varepsilon})^2 + 2mn(\frac{\nabla k}{k}) \bullet (\frac{\nabla \varepsilon}{\varepsilon}) \right] \right\}$$

This is of the form:

$$\frac{\mathrm{D}\varepsilon}{\mathrm{D}t} = \nabla \bullet (\Gamma^{(\varepsilon)} \nabla \varepsilon) + \{ C_{\varepsilon 1} P^{(k)} - C_{\varepsilon 2} \varepsilon \} \frac{\varepsilon}{k} + S_\varepsilon$$

where S_ε is of the form given in the question and

$$\Gamma^{(\varepsilon)} = \Gamma^{(\phi)}$$

$$C_{\varepsilon 1} = \frac{C_{\phi 1} - m}{n}$$

$$C_{\varepsilon 2} = \frac{C_{\phi 2} - m}{n}$$

Q7.
In simple shear flow,

$$(s_{ij}) = \begin{pmatrix} 0 & \frac{1}{2}\sigma & 0 \\ \frac{1}{2}\sigma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \tfrac{1}{2}\sigma\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$(\omega_{ij}) = \begin{pmatrix} 0 & \frac{1}{2}\sigma & 0 \\ -\frac{1}{2}\sigma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \tfrac{1}{2}\sigma\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence,

$$\mathbf{s} = \frac{\sigma}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \boldsymbol{\omega} = \frac{\sigma}{2}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{s}^2 = \frac{\sigma^2}{4}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \{\mathbf{s}^2\} = \frac{\sigma^2}{2}, \qquad \mathbf{s}^2 - \tfrac{1}{3}\{\mathbf{s}^2\}\mathbf{I} = \frac{\sigma^2}{12}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$\boldsymbol{\omega}\mathbf{s} = \frac{\sigma^2}{4}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{s}\boldsymbol{\omega} = \frac{\sigma^2}{4}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \boldsymbol{\omega}\mathbf{s} - \mathbf{s}\boldsymbol{\omega} = \frac{\sigma^2}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\boldsymbol{\omega}^2 = \frac{\sigma^2}{4}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \{\boldsymbol{\omega}^2\} = -\frac{\sigma^2}{2}, \qquad \boldsymbol{\omega}^2 - \tfrac{1}{3}\{\boldsymbol{\omega}^2\}\mathbf{I} = -\frac{\sigma^2}{12}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Substituting these in the constitutive relationship,

$$\mathbf{a} = \left(\frac{\overline{u_i u_j}}{k} - \frac{2}{3}\delta_{ij}\right) = -C_\mu\sigma\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$+\beta_1\frac{\sigma^2}{12}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} + \beta_2\frac{\sigma^2}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \beta_3\frac{\sigma^2}{12}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Extracting components:

$$\frac{\overline{u^2}}{k} = \frac{2}{3} + (\beta_1 + 6\beta_2 - \beta_3)\frac{\sigma^2}{12}$$

$$\frac{\overline{v^2}}{k} = \frac{2}{3} + (\beta_1 - 6\beta_2 - \beta_3)\frac{\sigma^2}{12}$$

$$\frac{\overline{w^2}}{k} = \frac{2}{3} - (\beta_1 - \beta_3)\frac{\sigma^2}{6}$$

$$\frac{\overline{uv}}{k} = -C_\mu\sigma$$

$$\frac{\overline{vw}}{k} = \frac{\overline{wu}}{k} = 0$$

Q8.

(a) In plane strain,

$$\mathbf{s} = \sigma \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\boldsymbol{\omega} = \mathbf{0}$$

Hence,

$$\mathbf{s}^2 = \sigma^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \{\mathbf{s}^2\} = 2\sigma^2, \qquad \mathbf{s}^2 - \tfrac{1}{3}\{\mathbf{s}^2\}\mathbf{I} = \frac{\sigma^2}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$\boldsymbol{\omega}\mathbf{s} - \mathbf{s}\boldsymbol{\omega} \quad = \quad \boldsymbol{\omega}^2 - \tfrac{1}{3}\{\boldsymbol{\omega}^2\}\mathbf{I} \quad = \quad \mathbf{0}$$

Substituting these in the constitutive relationship,

$$\mathbf{a} = (\frac{\overline{u_i u_j}}{k} - \frac{2}{3}\delta_{ij}) = -2C_\mu \sigma \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \beta_1 \frac{\sigma^2}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Extracting components:

$$\frac{\overline{u^2}}{k} = \frac{2}{3} - 2C_\mu \sigma + \frac{\beta_1}{3}\sigma^2$$

$$\frac{\overline{v^2}}{k} = \frac{2}{3} + 2C_\mu \sigma + \frac{\beta_1}{3}\sigma^2$$

$$\frac{\overline{w^2}}{k} = \frac{2}{3} - \frac{2\beta_1}{3}\sigma^2$$

$$\frac{\overline{uv}}{k} = \frac{\overline{vw}}{k} = \frac{\overline{wu}}{k} = 0$$

(b) In axisymmetric strain,

$$\mathbf{s} = \frac{\sigma}{2} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\boldsymbol{\omega} = \mathbf{0}$$

Hence,

$$\mathbf{s}^2 = \frac{\sigma^2}{4} \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \{\mathbf{s}^2\} = \frac{6}{4}\sigma^2, \qquad \mathbf{s}^2 - \tfrac{1}{3}\{\mathbf{s}^2\}\mathbf{I} = \frac{\sigma^2}{4} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\boldsymbol{\omega}\mathbf{s} - \mathbf{s}\boldsymbol{\omega} \quad = \quad \boldsymbol{\omega}^2 - \tfrac{1}{3}\{\boldsymbol{\omega}^2\}\mathbf{I} \quad = \quad \mathbf{0}$$

Substituting these in the constitutive relationship,

$$\mathbf{a} = (\frac{\overline{u_i u_j}}{k} - \frac{2}{3}\delta_{ij}) = -C_\mu \sigma \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \beta_1 \frac{\sigma^2}{4} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Extracting components:

$$\frac{\overline{u^2}}{k} = \frac{2}{3} - 2C_\mu \sigma + \frac{\beta_1}{2}\sigma^2$$

$$\frac{\overline{v^2}}{k} = \frac{\overline{w^2}}{k} = \frac{2}{3} + C_\mu \sigma - \frac{\beta_1}{4}\sigma^2$$

$$\frac{\overline{uv}}{k} = \frac{\overline{vw}}{k} = \frac{\overline{wu}}{k} = 0$$

Q9.

(a)

P_{ij} is the rate of **production** of $\overline{u_i u_j}$ from the mean flow.

Φ_{ij} is the rate of **redistribution** of energy amongst components by pressure forces.

ε_{ij} is the rate of **dissipation** by viscous action.

(b)

The purpose of $\Phi_{ij}^{(w)}$ is to try to maintain the correct level of anisotropy near a boundary (the rest of Φ_{ij} is trying to isotropise the turbulence).

In an equilibrium turbulent boundary layer $f = 1$.

(c)

Production

$$P_{11} = -2\overline{uv}\frac{\partial U}{\partial y} \quad = 2P^{(k)}, \qquad P_{22} = P_{33} = 0$$

$$P_{12} = -\overline{v^2}\frac{\partial U}{\partial y} \quad = \frac{\overline{v^2}}{\overline{uv}}P^{(k)}, \qquad P_{23} = P_{31} = 0$$

$$\text{where } P^{(k)} = -\overline{uv}\frac{\partial U}{\partial y}$$

Dissipation

$$\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33} = \frac{2}{3}\varepsilon$$

$$\varepsilon_{23} = \varepsilon_{31} = \varepsilon_{12} = 0$$

Pressure-strain (return-to-isotropy part)

$$\Phi_{11}^{(1)} = -C_1\varepsilon(\frac{\overline{u^2}}{k} - \frac{2}{3}), \qquad \Phi_{22}^{(1)} = -C_1\varepsilon(\frac{\overline{v^2}}{k} - \frac{2}{3}), \qquad \Phi_{33}^{(1)} = -C_1\varepsilon(\frac{\overline{w^2}}{k} - \frac{2}{3})$$

$$\Phi_{23}^{(1)} = -C_1\varepsilon(\frac{\overline{vw}}{k}), \qquad \Phi_{31}^{(1)} = -C_1\varepsilon(\frac{\overline{wu}}{k}), \qquad \Phi_{12}^{(1)} = -C_1\varepsilon(\frac{\overline{uv}}{k})$$

Pressure-strain (rapid part)

$$\Phi_{11}^{(2)} = -\frac{4}{3}C_2 P^{(k)}, \qquad \Phi_{22}^{(2)} = \Phi_{33}^{(2)} = \frac{2}{3}C_2 P^{(k)}$$

$$\Phi_{23}^{(2)} = \Phi_{31}^{(2)} = 0, \qquad \Phi_{12}^{(2)} = -C_2\frac{\overline{v^2}}{\overline{uv}}P^{(k)}$$

Pressure-strain (wall-reflection)

$$\Phi_{11}^{(w)} = \tilde{\Phi}_{22}, \qquad \Phi_{22}^{(w)} = -2\tilde{\Phi}_{22}, \qquad \Phi_{33}^{(w)} = \tilde{\Phi}_{22}$$

$$\Phi_{23}^{(w)} = 0, \qquad \Phi_{12}^{(w)} = -\tfrac{3}{2}\tilde{\Phi}_{12}, \qquad \Phi_{31}^{(w)} = 0$$

where

$$\tilde{\Phi}_{22} = C_1^{(w)}\varepsilon\frac{\overline{vv}}{k} + \frac{2}{3}C_2^{(w)}C_2 P^{(k)}, \qquad \tilde{\Phi}_{12} = C_1^{(w)}\varepsilon\frac{\overline{uv}}{k} - C_2^{(w)}C_2\frac{\overline{v^2}}{\overline{uv}}P^{(k)}$$

(d)

Setting $P_{22}+\Phi_{22}-\varepsilon_{22}=0$:

$$0-C_1\varepsilon(\frac{\overline{v^2}}{k}-\frac{2}{3})+\frac{2}{3}C_2P^{(k)}-2\left[C_1^{(w)}\varepsilon\frac{\overline{vv}}{k}+\frac{2}{3}C_2^{(w)}C_2P^{(k)}\right]-\frac{2}{3}\varepsilon=0$$

Dividing through by $P^{(k)}(=\varepsilon)$:

$$-(C_1+2C_1^{(w)})\frac{\overline{v^2}}{k}+\frac{2}{3}C_1+\frac{2}{3}C_2-\frac{4}{3}C_2^{(w)}C_2-\frac{2}{3}=0$$

$$\Rightarrow \qquad \frac{2}{3}(-1+C_1+C_2-2C_2^{(w)}C_2)=(C_1+2C_1^{(w)})\frac{\overline{v^2}}{k}$$

$$\Rightarrow \qquad \frac{\overline{v^2}}{k}=\frac{2}{3}\left(\frac{-1+C_1+C_2-2C_2^{(w)}C_2}{C_1+2C_1^{(w)}}\right)$$

Setting $P_{11}+\Phi_{11}-\varepsilon_{11}=0$:

$$2P^{(k)}-C_1\varepsilon(\frac{\overline{u^2}}{k}-\frac{2}{3})-\frac{4}{3}C_2P^{(k)}+C_1^{(w)}\varepsilon\frac{\overline{vv}}{k}+\frac{2}{3}C_2^{(w)}C_2P^{(k)}-\frac{2}{3}\varepsilon=0$$

Dividing through by $P^{(k)}(=\varepsilon)$:

$$2-C_1(\frac{\overline{u^2}}{k}-\frac{2}{3})-\frac{4}{3}C_2+C_1^{(w)}\frac{\overline{vv}}{k}+\frac{2}{3}C_2^{(w)}C_2-\frac{2}{3}=0$$

$$\Rightarrow \qquad \frac{4}{3}+\frac{2}{3}C_1-\frac{4}{3}C_2+C_1^{(w)}\frac{\overline{vv}}{k}+\frac{2}{3}C_2^{(w)}C_2=C_1\frac{\overline{u^2}}{k}$$

$$\Rightarrow \qquad \frac{\overline{u^2}}{k}=\frac{2}{3}\left(\frac{2+C_1-2C_2+C_2^{(w)}C_2}{C_1}\right)+\frac{C_1^{(w)}}{C_1}\frac{\overline{vv}}{k}$$

The other results follow similarly.

Q10.

(a)

$$P_{11} = -2(\overline{uu}\frac{\partial U}{\partial x} + \overline{uv}\frac{\partial U}{\partial y} + \overline{uw}\frac{\partial U}{\partial z})$$

$$P_{22} = -2(\overline{vu}\frac{\partial V}{\partial x} + \overline{vv}\frac{\partial V}{\partial y} + \overline{vw}\frac{\partial V}{\partial z})$$

$$P_{12} = -(\overline{uu}\frac{\partial V}{\partial x} + \overline{uv}\frac{\partial V}{\partial y} + \overline{uw}\frac{\partial V}{\partial z}) - (\overline{vu}\frac{\partial U}{\partial x} + \overline{vv}\frac{\partial U}{\partial y} + \overline{vw}\frac{\partial U}{\partial z})$$

(b)

$$P_{11} = -2\overline{uv}\frac{\partial U}{\partial y}, \qquad P_{12} = -\overline{v^2}\frac{\partial U}{\partial y}, \qquad P_{22} = P_{33} = P_{23} = P_{31} = 0$$

(c)

- Only $\overline{u^2}$ has a non-zero production term (see above) – almost invariably positive;
- $\overline{v^2}$ is preferentially damped by the physical presence of the wall.

Q11.
<u>Continuity</u>

Instantaneous: $\dfrac{\partial u_j}{\partial x_j} = 0$

Average: $\dfrac{\partial \bar{u}_j}{\partial x_j} = 0$

Subtract: $\dfrac{\partial u'_j}{\partial x_j} = 0$

Hence, both mean and fluctuating quantities satisfy the incompressibility equation. Moreover, u'_j can commute with $\partial/\partial x_j$ (with an implied sum) whenever required.

<u>Momentum</u>

Instantaneous: $\dfrac{\partial u_i}{\partial t} + u_j \dfrac{\partial u_i}{\partial x_j} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x_i} + \nu \dfrac{\partial^2 u_i}{\partial x_j \partial x_j}$

Average: $\dfrac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \dfrac{\partial \bar{u}_i}{\partial x_j} + \overline{u'_j \dfrac{\partial u'_i}{\partial x_j}} = -\dfrac{1}{\rho}\dfrac{\partial \bar{p}}{\partial x_i} + \nu \dfrac{\partial^2 \bar{u}_i}{\partial x_j \partial x_j}$

Subtract: $$\frac{\partial u'_i}{\partial t} + \bar{u}_j \frac{\partial u'_i}{\partial x_j} + u'_j \frac{\partial \bar{u}_i}{\partial x_j} + u'_j \frac{\partial u'_i}{\partial x_j} - \overline{u'_j \frac{\partial u'_i}{\partial x_j}} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_i} + \nu \frac{\partial^2 u'_i}{\partial x_j \partial x_j} \qquad (*)$$

Form $\overline{u'_i \times (*)}$ and contract:

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\overline{u'_i u'_i}) + \bar{u}_j \frac{\partial}{\partial x_j}(\tfrac{1}{2}\overline{u'_i u'_i}) + \overline{u'_i u'_j}\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial}{\partial x_j}(\tfrac{1}{2}\overline{u'_i u'_i u'_j}) = -\frac{1}{\rho}(\overline{u'_i \frac{\partial p'}{\partial x_i}}) + \nu(\overline{u'_i \frac{\partial^2 u'_i}{\partial x_j \partial x_j}})$$

Using $k \equiv \tfrac{1}{2}\overline{u'_i u'_i}$ and rewriting the RHS:

$$\frac{\partial k}{\partial t} + \bar{u}_j \frac{\partial k}{\partial x_j} + \overline{u'_i u'_j}\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial}{\partial x_j}(\tfrac{1}{2}\overline{u'_i u'_i u'_j}) = -\frac{1}{\rho}\frac{\partial}{\partial x_i}(\overline{u'_i p'}) + \frac{\partial}{\partial x_j}\nu(\overline{u'_i \frac{\partial u'_i}{\partial x_j}}) - \nu\overline{\frac{\partial u'_i}{\partial x_j}\frac{\partial u'_i}{\partial x_j}}$$

Hence, collecting terms (and changing the summation index in the pressure term):

$$\frac{\mathrm{D}k}{\mathrm{D}t} = \frac{\partial}{\partial x_j}(\nu \frac{\partial k}{\partial x_j} - \frac{1}{\rho}\overline{u'_j p'} - \tfrac{1}{2}\overline{u'_i u'_i u'_j}) - \overline{u'_i u'_j}\frac{\partial \bar{u}_i}{\partial x_j} - \nu\overline{\frac{\partial u'_i}{\partial x_j}\frac{\partial u'_i}{\partial x_j}}$$

This is of the form

$$\frac{\mathrm{D}k}{\mathrm{D}t} = \frac{\partial d_j^{(k)}}{\partial x_j} + P^{(k)} - \varepsilon$$

where:

$d_i^{(j)} = \nu \dfrac{\partial k}{\partial x_j} - \dfrac{1}{\rho}\overline{p' u'_j} - \tfrac{1}{2}\overline{u'_i u'_i u'_j}$ *diffusive flux*

$P^{(k)} = -\overline{u'_i u'_j}\dfrac{\partial \bar{u}_i}{\partial x_j}$ *production* (*by mean strain*)

$\varepsilon = \nu\overline{(\dfrac{\partial u'_i}{\partial x_j})^2}$ *dissipation*

Q12.

Expand each velocity component in a Taylor series about $y = 0$:

$$u = a_1 + b_1 y + c_1 y^2 + \cdots$$

$$v = a_2 + b_2 y + c_2 y^2 + \cdots$$

$$w = a_3 + b_3 y + c_3 y^2 + \cdots$$

where the a_i, b_i, etc. are functions of x, z and t.

Non-slip condition:

$$u = v = w = 0 \text{ on } y = 0$$

$$\Rightarrow \qquad a_1 = a_2 = a_3 = 0$$

Continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\Rightarrow \qquad \frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} - \frac{\partial w}{\partial z} \quad = 0$$

$$\Rightarrow \qquad b_2 = 0$$

Hence, to leading order:

$$u = b_1 y + \cdots$$

$$v = c_2 y^2 + \cdots \quad (*)$$

$$w = b_3 y + \cdots$$

Hence,

$$\overline{u^2} = \overline{b_2^2} y^2 + O(y^3)$$

$$\overline{v^2} = \overline{c_2^2} y^4 + O(y^5)$$

$$\overline{w^2} = \overline{b_3^2} y^2 + O(y^3)$$

$$\overline{uv} = \overline{b_1 c_2} y^3 + O(y^4)$$

$$k \equiv \tfrac{1}{2}(\overline{u^2} + \overline{v^2} + \overline{w^2}) \quad = \tfrac{1}{2}(\overline{b_1^2} + \overline{b_3^2}) y^2 + O(y^3)$$

By definition,

$$\nu_t = \frac{-\overline{uv}}{\partial U / \partial y}$$

But

$$\tau_w \equiv \rho u_\tau^2 = \mu \frac{\partial U}{\partial y} - \rho \overline{uv}$$

$$\Rightarrow \qquad u_\tau^2 = \nu \frac{\partial U}{\partial y} + O(y^3)$$

$$\Rightarrow \qquad \frac{\partial U}{\partial y} = \frac{u_\tau^2}{\nu} + O(y^3)$$

Hence,

$$\nu_t = -\overline{b_1 c_2} \frac{\nu}{u_\tau^2} y^3 + O(y^4)$$

Q13.

The k equation is

$$\frac{\mathrm{D}k}{\mathrm{D}t} = \frac{\partial}{\partial x_j}\left\{(\nu + \frac{\nu_t}{\sigma^{(k)}})\frac{\partial k}{\partial x_j}\right\} + P^{(k)} - \varepsilon$$

As $y \to 0$ this reduces to

$$0 = \frac{\partial}{\partial y}\left(\nu \frac{\partial k}{\partial y}\right) - \varepsilon$$

$$\Rightarrow \qquad \varepsilon = \nu \frac{\partial^2 k}{\partial y^2} \qquad (*)$$

But, from Q12,

$$k = k_0 y^2 + O(y^3)$$

whence,

$$\frac{\partial^2 k}{\partial y^2} = 2k_0 + O(y)$$

$$= 2\frac{k}{y^2} + O(y)$$

Hence, from (*),

$$\varepsilon = \frac{2\nu k}{y^2} + O(y) \qquad (y \to 0)$$

Q14.

(a) Assuming constant shear stress (ρu_0^2) the mean-velocity profile can be derived from the viscosity law in viscous and turbulent regions, with the former providing a boundary condition for the latter at $y = y_\nu$.

<u>Viscous Sublayer</u> ($y < y_\nu$)

$$\frac{\tau_w}{\rho} = \nu \frac{\partial U}{\partial y}, \qquad U(0) = 0$$

This integrates immediately to give

$$U = \frac{\tau_w}{\rho \nu} y$$

or, in a non-dimensional form,

$$\frac{U}{u_0} = \frac{\tau_w}{\rho u_0^2} \frac{u_0 y}{\nu}$$

<u>Turbulent Region</u> ($y > y_\nu$)

$$\frac{\tau_w}{\rho} = [\nu + \kappa u_0 (y - y_\nu)] \frac{\partial U}{\partial y}, \qquad U(y_\nu) = \frac{\tau_w}{\rho \nu} y$$

Rearrange:

$$\frac{\partial U}{\partial y} = \frac{\tau_w / \rho}{\nu + \kappa u_0 (y - y_\nu)}$$

Integrate, using the velocity at $y = y_\nu$ as a lower limit:

$$\Big[U \Big]_{y_\nu}^{y} = \frac{\tau_w / \rho}{\kappa u_0} \Big[\ln\{\nu + \kappa u_0 (y - y_\nu)\} \Big]_{y_{gn}}^{y}$$

$$\Rightarrow \qquad U - \frac{\tau_w / \rho}{\nu} y_\nu = \frac{\tau_w / \rho}{\kappa u_0} \ln \frac{\nu + \kappa u_0 (y - y_\nu)}{\nu}$$

$$\Rightarrow \qquad \frac{U}{u_0} = \frac{\tau_w}{\rho u_0^2} \left\{ \frac{u_0 y_\nu}{\nu} + \frac{u_0}{\kappa} \ln \left[1 + \frac{\kappa u_0 (y - y_\nu)}{\nu} \right] \right\}$$

Hence, the complete mean velocity profile is:

$$\frac{U}{u_0} = \frac{\tau_w}{\rho u_0^2} \times \begin{cases} y^+ & y \le y_\nu \\ y_\nu^+ + \dfrac{1}{\kappa} \ln\left[1 + \kappa (y^+ - y_\nu^+)\right] & y > y_\nu \end{cases}$$

(b) The cell-averaged rate of production of turbulent kinetic energy is

$$P_{av}^{(k)} \equiv \frac{1}{\Delta}\int_0^\Delta P^{(k)} \quad \mathrm{d}y$$

$$= \frac{1}{\Delta}\int_{y_\nu}^\Delta \nu_t \left(\frac{\partial U}{\partial y}\right)^2 \mathrm{d}y$$

$$= \frac{1}{\Delta}\int_{y_\nu}^\Delta \kappa u_0 (y - y_\nu) \times \frac{(\tau_w/\rho)^2}{\{\nu + \kappa u_0 (y - y_\nu)\}^2} \mathrm{d}y$$

To simplify the integral, change variables to

$$Y = \frac{\kappa u_0 (y - y_\nu)}{\nu}, \qquad \mathrm{d}y = \frac{\nu}{\kappa u_0} \mathrm{d}Y$$

Then

$$P_{av}^{(k)} = \frac{1}{\Delta}\int_0^{\frac{\kappa u_0 (\Delta - y_\nu)}{\nu}} \nu Y \times \frac{(\tau_w/\rho)^2}{\{\nu + \nu Y\}^2} \frac{\nu}{\kappa u_0} \mathrm{d}Y$$

$$= \frac{(\tau_w/\rho)^2}{\kappa u_0 \Delta}\int_0^{\kappa(\Delta^+ - y_\nu^+)} \frac{Y}{(1+Y)^2} \mathrm{d}Y$$

$$= \frac{(\tau_w/\rho)^2}{\kappa u_0 \Delta}\int_0^{\kappa(\Delta^+ - y_\nu^+)} \left\{\frac{1}{1+Y} - \frac{1}{(1+Y)^2}\right\} \mathrm{d}Y$$

$$= \frac{(\tau_w/\rho)^2}{\kappa u_0 \Delta}\left[\ln(1+Y) + \frac{1}{1+Y}\right]_0^{\kappa(\Delta^+ - y_\nu^+)}$$

$$= \frac{(\tau_w/\rho)^2}{\kappa u_0 \Delta}\left\{\ln[1 + \kappa(\Delta^+ - y_\nu^+)] + \frac{1}{1 + \kappa(\Delta^+ - y_\nu^+)} - 1\right\}$$

$$= \frac{(\tau_w/\rho)^2}{\kappa u_0 \Delta}\left\{\ln[1 + \kappa(\Delta^+ - y_\nu^+)] - \frac{\kappa(\Delta^+ - y_\nu^+)}{1 + \kappa(\Delta^+ - y_\nu^+)}\right\}$$

(b) The cell-averaged dissipation rate is

$$\varepsilon_{av} \equiv \frac{1}{\Delta}\int_0^\Delta \varepsilon \quad \mathrm{d}y$$

$$= \frac{1}{\Delta}\left(\int_0^{y_\varepsilon} \varepsilon_w \quad \mathrm{d}y + \int_{y_\varepsilon}^\Delta \frac{u_0^3}{\kappa(y - y_d)} \mathrm{d}y\right) \qquad \text{where} \qquad \varepsilon_w = \frac{u_0^3}{\kappa(y_\varepsilon - y_d)}$$

$$= \frac{u_0^3}{\kappa\Delta}\left(\frac{y_\varepsilon}{y_\varepsilon - y_d} + \int_{y_\varepsilon}^\Delta \frac{\mathrm{d}y}{y - y_d}\right)$$

$$= \frac{u_0^3}{\kappa\Delta}\left[\frac{y_\varepsilon}{y_\varepsilon - y_d} + \ln\left(\frac{\Delta - y_d}{y_\varepsilon - y_d}\right)\right]$$

Q15.

(a) Equation (*) can be written conveniently as

$$U = \frac{u_\tau}{\kappa}\ln(E\frac{u_\tau y}{\nu}) \qquad \text{where} \qquad E = \frac{e^{\kappa B}}{1+ck_s^+}$$

The total quantity of flow in the pipe is

$$Q \equiv U_{av}\pi R^2 = \int_0^R U.2\pi r \quad dr$$

or, writing $r = R - y$, $dr = -dy$ and switching the integral limits:

$$U_{av}\pi R^2 = 2\pi\frac{u_\tau}{\kappa}\int_0^R \ln(E\frac{u_\tau y}{\nu})(R-y) \quad dy$$

Substituting the boundary-layer coordinate,

$$\eta = \frac{y}{R} \qquad \text{or } y = R\eta, \quad dy = Rd\eta$$

then

$$\frac{U_{av}}{u_\tau} = \frac{2}{\kappa}\int_0^1 \ln(E\frac{u_\tau R}{\nu}\eta)(1-\eta) \quad d\eta$$

$$= \frac{2}{\kappa}\left\{\left[\ln(E\frac{u_\tau R}{\nu}\eta)(\eta - \frac{\eta^2}{2})\right]_0^1 - \int_0^1 (1-\tfrac{1}{2}\eta) \quad d\eta\right\}$$

$$= \frac{2}{\kappa}\left\{\frac{1}{2}\ln(E\frac{u_\tau R}{\nu}) - \left[\eta - \tfrac{1}{4}\eta^2\right]_0^1\right\}$$

$$= \frac{1}{\kappa}\left\{\ln(E\frac{u_\tau R}{\nu}) - \frac{3}{2}\right\}$$

Hence

$$\frac{U_{av}}{u_\tau} = \frac{1}{\kappa}\ln\left(\frac{Ee^{-3/2}}{2}\times\frac{u_\tau}{U_{av}}\times\frac{U_{av}D}{\nu}\right)$$

But

$$\frac{u_\tau}{U_{av}} = \sqrt{\frac{c_f}{2}}$$

Hence

$$\sqrt{\frac{2}{c_f}} = \frac{1}{\kappa}\ln\left(\frac{Ee^{-3/2}}{2}\times \text{Re}\sqrt{\frac{c_f}{2}}\right)$$

or

$$\frac{1}{\sqrt{c_f}} = \frac{1}{\kappa\sqrt{2}}\ln\left(\frac{e^{\kappa B-3/2}}{2\sqrt{2}}\frac{\text{Re}\sqrt{c_f}}{1+ck_s^+}\right)$$

But

$$k_s^+ = \frac{u_\tau k_s}{\nu} = \frac{u_\tau}{U_{av}} \times \frac{k_s}{D} \times \frac{U_{av} D}{\nu}$$

$$= \sqrt{\frac{c_f}{2}} \times \frac{k_s}{D} \times \mathrm{Re}$$

Hence

$$\frac{1}{\sqrt{c_f}} = \frac{1}{\kappa\sqrt{2}} \ln\left(\frac{\mathrm{e}^{\kappa B - 3/2}}{2\sqrt{2}} \frac{\mathrm{Re}\sqrt{c_f}}{1 + c\sqrt{c_f/2}\,\mathrm{Re}(k_s/D)} \right)$$

or, inverting the argument of the logarithm and rearranging:

$$\frac{1}{\sqrt{c_f}} = \frac{-1}{\kappa\sqrt{2}} \ln\left(\frac{2\sqrt{2}\mathrm{e}^{3/2-\kappa B}}{\mathrm{Re}\sqrt{c_f}} + 2\mathrm{e}^{3/2-\kappa B} c \frac{k_s}{D} \right)$$

(b) To compare with the Colebrook-White formula change the base of logarithms:

$$\frac{1}{\sqrt{c_f}} = \frac{-1}{\kappa\sqrt{2}\log_{10}\mathrm{e}} \log_{10}\left(\frac{2\sqrt{2}\mathrm{e}^{3/2-\kappa B}}{\mathrm{Re}\sqrt{c_f}} + 2\mathrm{e}^{3/2-\kappa B} c \frac{k_s}{D} \right)$$

Comparing with the Colebrook-White formula:

$$\frac{1}{\sqrt{c_f}} = -4.0\log_{10}\left(\frac{1.26}{\mathrm{Re}\sqrt{c_f}} + \frac{k_s}{3.7D} \right)$$

gives

(i) $\quad \dfrac{1}{\kappa\sqrt{2}\log_{10}\mathrm{e}} = 4.0 \quad$ or $\quad \kappa = \dfrac{1}{4.0\sqrt{2}\log_{10}\mathrm{e}} = 0.4070$

(ii) $\quad 2\sqrt{2}\mathrm{e}^{3/2-\kappa B} = 1.26 \quad$ or $\quad B = \dfrac{1}{\kappa}\left(\dfrac{3}{2} - \ln(\dfrac{1.26}{2\sqrt{2}}) \right) = 5.672$

(iii) $\quad 2\mathrm{e}^{3/2-\kappa B} c = \dfrac{1}{3.7} \quad$ or $\quad c = \dfrac{\sqrt{2}}{3.7 \times 1.26} = 0.3033$

Answer: $\kappa = 0.41$, $B = 5.7$, $c = 0.30$

(c) Rewrite equation (*) as

$$U^+ = \frac{1}{\kappa}\ln y^+ + B - \frac{1}{\kappa}\ln(1 + ck_s^+)$$

Hence,

$$U^+ = \frac{1}{\kappa}\ln y^+ + \tilde{B}(k_s^+)$$

where

$$\tilde{B}(k_s^+) = B - \frac{1}{\kappa}\ln(1 + ck_s^+)$$

(d) The last can be further rearranged as

$$\tilde{B}(k_s^+) = B - \frac{1}{\kappa}\ln c - \frac{1}{\kappa}\ln(k_s^+ + \frac{1}{c})$$

The mean-velocity profile is then

$$U^+ = \frac{1}{\kappa}\ln\frac{y^+}{k_s^+ + \frac{1}{c}} + B - \frac{1}{\kappa}\ln c$$

In the limit as k_s^+ becomes very large, this asymptotes to

$$U^+ = \frac{1}{\kappa}\ln\frac{y}{k_s} + B_k$$

where

$$\begin{aligned} B_k &= B - \frac{1}{\kappa}\ln c \\ &= 8.60 \end{aligned}$$

Answer: $B_k = 8.6$.