8. Turbulence Modelling

Objective

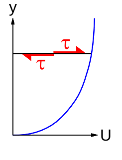
Model the **Reynolds stresses**

$-\rho\overline{uv}, \quad -\rho\overline{uu},$ etc.

in order to close the **mean-flow** equations.

Eddy-Viscosity Models

Total stress
(in simple shear): $\tau = \underbrace{\mu\frac{\partial U}{\partial y}}_{\text{viscous}} \underbrace{-\rho\overline{uv}}_{\text{turbulent}}$



Eddy-viscosity model: $\boxed{-\rho\overline{uv} = \mu_t\frac{\partial U}{\partial y}}$

Total stress: $\tau = (\mu + \mu_t)\frac{\partial U}{\partial y}$

Effective viscosity: $\mu_{eff} = \mu + \mu_t$

Eddy-Viscosity Models

$$-\rho\overline{uv} = \mu_t \frac{\partial U}{\partial y}$$

- This is a **model**!
- μ is a **physical** property of the **fluid**
  μ_t is a **hypothetical** property of the **flow**
- μ_t varies with position
- At high Reynolds numbers, $\mu_t >> \mu$

Eddy-Viscosity Models

For

- Easy to implement
- Extra viscosity aids stability
- Theoretical basis in simple flows

Against

- Little foundation in complex flows
- Turbulence modelling reduced to a single scalar, μ_t

Consistency With the Log Law

Logarithmic velocity profile: $\frac{\partial U}{\partial y} = \frac{u_\tau}{\kappa y}$

Total stress constant and dominated by turbulent stress:

$$\tau^{(turb)} = \tau_w \equiv \rho u_\tau^2$$

Definition of eddy viscosity: $\mu_t \equiv \frac{\tau^{(turb)}}{\partial U / \partial y} \quad = \frac{\rho u_\tau^2}{u_\tau / \kappa y} \quad = \rho(\kappa u_\tau y)$

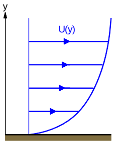
$$\mu_\tau = \rho(\kappa u_\tau y)$$

$$\nu_\tau = \kappa u_\tau y$$

General Stress-Strain Relationship

In **simple shear** $(U(y),0,0)$:

$$-\rho\overline{uv} = \mu_t \frac{\partial U}{\partial y}$$



In **arbitrary flow**:

$$-\rho\overline{uv} = \mu_t\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right)$$

$$-\rho\overline{uu} = 2\mu_t \frac{\partial U}{\partial x} - \frac{2}{3}\rho k$$

$$-\rho\overline{u_i u_j} = \mu_t\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) - \frac{2}{3}\rho k \delta_{ij}$$

Reynolds' Analogy

Similar **gradient-diffusion** model for other turbulent fluxes

$$-\rho\overline{v\phi} = \Gamma_t \frac{\partial \Phi}{\partial y} \qquad \Gamma_t = \frac{\mu_t}{\sigma_t}$$

turbulent **Prandtl number**

eddy diffusivity

The Eddy Viscosity

Kinematic eddy viscosity $\nu_t = \frac{\mu_t}{\rho}$

$\nu_t = u_0 l_0$

u_0 is a typical turbulent **velocity**
l_0 is a typical turbulent eddy **size**

e.g. In the **log layer**: $\nu_t = \kappa u_\tau y \quad = u_\tau \times (\kappa y)$

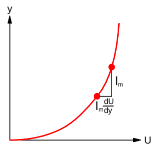
Types of Eddy-Viscosity Model

Based on the number of additional scalar-transport equations.

- **Zero-equation models:**
  - constant eddy-viscosity models
  - **mixing-length** models
- **One-equation models:**
  - l_0 specified algebraically; transport equation for u_0
- **Two-equation models:**
  - transport equations for 2 scales (***k*-ε**, k-ω, ...)

Mixing-Length Models

Eddy viscosity: $\nu_t = u_0 l_m$



Mixing length l_m specified algebraically

Velocity scale $u_0 = l_m \left| \frac{\partial U}{\partial y} \right|$ from the **mean velocity gradient**

Turbulent shear stress: $\tau^{(turb)} = \mu_t \frac{\partial U}{\partial y} \quad = \rho u_0 l_m \frac{\partial U}{\partial y} \quad = \rho l_m^2 \left(\frac{\partial U}{\partial y} \right)^2$

Consistency With the Log Law

$$\tau^{(turb)} = \rho l_m^2 \left(\frac{\partial U}{\partial y} \right)^2$$

Log-law region: $\frac{\partial U}{\partial y} = \frac{u_\tau}{\kappa y}$

$$\tau^{(turb)} = \tau_w = \rho u_\tau^2$$

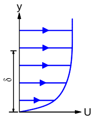
$$\rho u_\tau^2 = \rho l_m^2 \left(\frac{u_\tau}{\kappa y} \right)^2$$

$$l_m = \kappa y$$

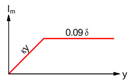
Mixing Length

Wall-bounded flows

l_m related to distance from the wall

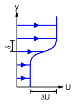


$l_m = \min(\kappa y, 0.09\delta)$



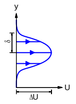
Free shear flows

l_m proportional to width of layer



mixing layer

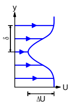
$l_m = 0.071\delta$



jet

$l_m = 0.098\delta$ (plane)

$l_m = 0.080\delta$ (round)



wake

$l_m = 0.180\delta$

The k-ε Model

1. Eddy-viscosity model: $\tau^{(turb)} = \mu_t \dfrac{\partial U}{\partial y}$
2. Formula for the eddy viscosity: $\mu_t = C_\mu \rho \dfrac{k^2}{\varepsilon}$ $\quad \nu_t = C_\mu \dfrac{k^2}{\varepsilon}$
3. Scalar-transport equations for:
   - k – turbulent kinetic energy
   - ε – rate of dissipation of k

The k and ε Transport Equations

$$\frac{\partial}{\partial t}(\rho k) + \frac{\partial}{\partial x_i}\left(\rho U_i k - \Gamma^{(k)}\frac{\partial k}{\partial x_i}\right) = \rho(P^{(k)} - \varepsilon)$$

$$\frac{\partial}{\partial t}(\rho \varepsilon) + \frac{\partial}{\partial x_i}\left(\rho U_i \varepsilon - \Gamma^{(\varepsilon)}\frac{\partial \varepsilon}{\partial x_i}\right) = \rho(C_{\varepsilon 1} P^{(k)} - C_{\varepsilon 2}\varepsilon)\frac{\varepsilon}{k}$$

rate of change *advection* *diffusion* *source*

Diffusivities: $\Gamma^{(k)} = \mu + \dfrac{\mu_t}{\sigma_k}, \quad \Gamma^{(\varepsilon)} = \mu + \dfrac{\mu_t}{\sigma_\varepsilon}$

Production: $P^{(k)} = -\overline{u_i u_j}\dfrac{\partial U_i}{\partial x_j}$

Constants: $C_\mu = 0.09, \quad C_{\varepsilon 1} = 1.44, \quad C_{\varepsilon 2} = 1.92, \quad \sigma_k = 1.0, \quad \sigma_\varepsilon = 1.3$

The k and ε Transport Equations

- Heavily modelled (especially the ε equation)
- Source-dominated, with a balance between:

  production by mean flow: $P^{(k)} = -\overline{u_i u_j}\dfrac{\partial U_i}{\partial x_j}$

  dissipation by viscosity: $\varepsilon = \nu \sum_{i,j} \overline{\left(\dfrac{\partial u_i}{\partial x_j}\right)^2}$

- Turbulence said to be in **local equilibrium** if $P^{(k)} = \varepsilon$
- Many variants, including low-Reynolds-number models
- Other choices of dimensional scales (e.g k-ω)

Consistency With the Log Law

(Constant) stress: $\tau \equiv -\rho\overline{uv} \quad \approx \tau_w = \rho u_\tau^2$ $\qquad \boxed{-\overline{uv} = u_\tau^2}$

Production of k: $P^{(k)} = -\overline{uv}\dfrac{\partial U}{\partial y} \quad = u_\tau^2 \times \dfrac{u_\tau}{\kappa y}$

Local equilibrium: $P^{(k)} = \varepsilon$ $\qquad \boxed{\varepsilon = P^{(k)} = \dfrac{u_\tau^3}{\kappa y}}$

Eddy viscosity: $\qquad \boxed{\nu_\tau = \kappa u_\tau y}$

$$\nu_t = \frac{u_\tau^4}{\varepsilon}$$

$$C_\mu = \frac{u_\tau^4}{k^2} \qquad \boxed{C_\mu = \left(\frac{-\overline{uv}}{k}\right)^2} \qquad \boxed{u_\tau = C_\mu^{1/4} k^{1/2}}$$

***k*-ε model**: $\nu_t = C_\mu \dfrac{k^2}{\varepsilon}$

Example

In the k-ε turbulence model, k is turbulent kinetic energy and ε is its dissipation rate. A (kinematic) eddy viscosity is defined by

$$\nu_t = C_\mu \frac{k^2}{\varepsilon}$$

where C_μ is a constant. A modeled scalar-transport equation for ε is

$$\frac{\mathrm{D}\varepsilon}{\mathrm{D}t} = \frac{\partial}{\partial x_i}\left(\frac{\nu_t}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial x_i}\right) + (C_{\varepsilon 1} P^{(k)} - C_{\varepsilon 2}\varepsilon)\frac{\varepsilon}{k}$$

where $\mathrm{D}/\mathrm{D}t$ is the derivative following the flow, $P^{(k)}$ is the rate of production of k and the summation convention is implied by the repeated index i. σ_ε, $C_{\varepsilon 1}$ and $C_{\varepsilon 2}$ are constants.

In the log-law region of a turbulent boundary layer,

$$P^{(k)} = \varepsilon = \frac{u_\tau^3}{\kappa y} \qquad k = C_\mu^{-1/2} u_\tau^2$$

where κ is von Karman's constant, u_τ is the friction velocity and y is the distance from the boundary. Using the scalar-transport equation for ε and the eddy-viscosity formulation, show that this implies the following relationship between coefficients:

$$(C_{\varepsilon 2} - C_{\varepsilon 1})\sigma_\varepsilon \sqrt{C_\mu} = \kappa^2$$

Limitations of Eddy-Viscosity Models

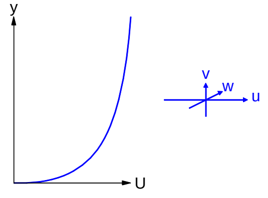
For

- Simple to code
- Extra viscosity aids stability
- Supported theoretically in some simple but common types of flow
- Effective in many engineering flows

Against

- Only capable of predicting one Reynolds-stress component accurately
- Lack justification in complex flows
- Fail to predict common properties such as anisotropy

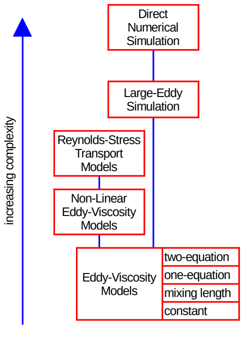
Turbulence Anisotropy



Experiment: $\overline{u^2} : \overline{v^2} : \overline{w^2} = 1.0 : 0.4 : 0.6$

Eddy-viscosity model: $\overline{u^2} = \overline{v^2} = \overline{w^2} = \frac{2}{3} k$

Advanced Turbulence Models



Reynolds-Stress Transport Models

- Solve **individual transport equations** for **each Reynolds stress**

$$-\rho\overline{uv}, \ -\rho\overline{uu}, \ etc.$$

- "Source" is a balance between:
  - **production** by mean-velocity gradients
  - **redistribution** by pressure fluctuations
  - **dissipation** by viscous action.

For

- Advection and production are exact and don't need modelling

Against

- Very complex: many terms need modelling
- Computational expense:
  - 6 extra transport equations
  - lack of a stabilising diffusion term

Non-Linear Eddy-Viscosity Models

Simple eddy-viscosity model:

stress ∝ velocity gradient $\tau_{turb} = \mu_t \dfrac{\partial U}{\partial y}$

Non-linear eddy-viscosity model:

stress = C_1(velocity gradient) + C_2(velocity gradient)^2 +...

For

- Qualitatively correct response of turbulence to certain types of strain
- Little more computationally expensive than linear models

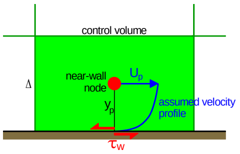
Against

- Doesn't accurately reflect the real physical processes
- Little justification in complex geometries

Wall Boundary Conditions

- Near a solid boundary:
  - there are very large flow gradients
  - wall-normal fluctuations are selectively damped
  - viscous and turbulent stresses are comparable
- Alternative approaches:
  - **low-Reynolds-number** turbulence models
  - **wall functions**

Wall Functions



Equilibrium (log law):

$$\frac{U_P}{u_\tau} = \frac{1}{\kappa}\ln(y_P^+) + B = \frac{1}{\kappa}\ln(Ey_P^+) \qquad y_P^+ = \frac{y_P u_\tau}{\nu}$$

$$\nu_t = \kappa u_\tau y \qquad u_\tau = C_\mu^{1/4} k_P^{1/2}$$

Non-equilibrium:

$$\nu_t = \kappa u_0 y \qquad u_0 = C_\mu^{1/4} k_P^{1/2}$$

$$\tau_w = \rho(\kappa u_0 y)\frac{\partial U}{\partial y}$$

$$\frac{\partial U}{\partial y} = \frac{\tau_w/\rho}{\kappa u_0}\frac{1}{y}$$

$$U = \frac{\tau_w/\rho}{\kappa u_0}\ln(Cy) \qquad \boxed{\tau_w = \rho\frac{\kappa u_0 U_p}{\ln(E\frac{y_p u_0}{\nu})}}$$

Implemented via an **effective wall viscosity**: $\boxed{\tau_w = \mu_w \frac{U_P}{y_P}}$ $\qquad \mu_w = \frac{\rho(\kappa u_0 y_p)}{\ln(E\frac{y_p u_0}{\nu})}$

Summary (1)

- A turbulence model or turbulence closure is a means of specifying Reynolds stresses (and other turbulent fluxes)
- The most popular type is an eddy-viscosity model
- The eddy viscosity may be specified:
  - geometrically (e.g. mixing-length models)
  - by solving transport equations (e.g. k-ε model)

Summary (2)

- Advanced turbulence models include:
  - Reynolds-stress transport models (RSTM)
  - non-linear eddy-viscosity models (NLEVM)
  - large-eddy simulation (LES)
- Wall boundary conditions require either:
  - low-Reynolds-number modifications (fine grid)
  - wall functions (coarse grid)