
2.1 Introduction

Fluid dynamics is governed by conservation equations for:

- mass;
- momentum;
- energy;
- (for a non-homogenous fluid) other constituents.

Equations for these can be expressed mathematically in many ways: notably as

- *integral* (*control-volume*) equations;
- *differential* equations.

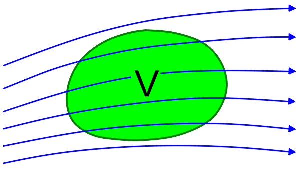
This course will focus on the integral (control-volume) approach because it is easier to relate to the real world, is naturally conservative and forms the basis of the finite-volume method. However, the equivalent differential equations (of which there are several forms) are often easier to write down, manipulate and, in a few cases, solve analytically.

Although there are different fluid-flow variables, most of them satisfy a single generic equation: the *scalar-transport* or *advection-diffusion* equation. You will have met a 1-d version of this in your Water Engineering course.

As discussed in Section 1, the rate of change of any physical quantity within an arbitrary control volume *V* is determined by:

- the net rate of transport through the bounding surface ("*flux*");
- the net rate of production within that control volume ("*source*").

$$\begin{pmatrix}\textbf{RATE OF CHANGE}\\ \textit{inside } V\end{pmatrix} + \begin{pmatrix}\textbf{NET FLUX}\\ \textit{through boundary of } V\end{pmatrix} = \begin{pmatrix}\textbf{SOURCE}\\ \textit{inside } V\end{pmatrix} \qquad (1)$$



The flux through the bounding surface can be divided into:

- *advection*[1]: transport with the flow;
- *diffusion*: net transport by molecular or turbulent fluctuations.

$$\boxed{\begin{pmatrix}\textbf{RATE OF CHANGE}\\ \textit{inside } V\end{pmatrix} + \begin{pmatrix}\textbf{ADVECTION} + \textbf{DIFFUSION}\\ \textit{through boundary of } V\end{pmatrix} = \begin{pmatrix}\textbf{SOURCE}\\ \textit{inside } V\end{pmatrix} \qquad (2)}$$

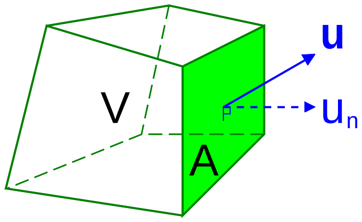
The **finite-volume method** is a natural discretisation of this.

[1] Some authors – but not this one – prefer the term *convection* to *advection*.

2.2 Conservative Forms of the Flow Equations

2.2.1 Mass (Continuity)

Physical principle (*mass conservation*): mass is neither created nor destroyed.



Consider an arbitrary control volume or *cell*.

Mass of fluid in the cell: ρV

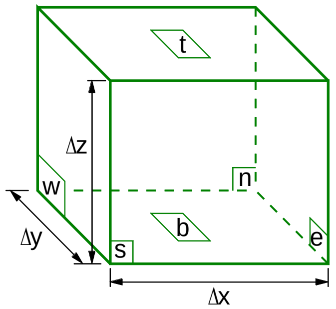
Mass flux through one face: $C = \rho u_n A \quad = \rho \mathbf{u} \bullet \mathbf{A}$

V is the cell volume, *A* the area of a typical face and u_n is the velocity component along the (outward) normal to that face

$$\textit{rate of change of mass in cell} + \textit{net outward mass flux} = 0 \tag{3}$$

Integral equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\textit{mass}) \quad + \quad \sum_{\textit{faces}}(\textit{mass flux}) \quad = \quad 0 \tag{4}$$



An equivalent **conservative differential** equation for mass conservation can be derived by considering a small Cartesian control volume with sides Δx, Δy, Δz as shown.

$$\underbrace{\frac{\mathrm{d}(\rho V)}{\mathrm{d}t}}_{\textit{rate of change of mass}} \quad + \quad \underbrace{(\rho u A)_e - (\rho u A)_w \quad + \quad (\rho v A)_n - (\rho v A)_s \quad + \quad (\rho w A)_t - (\rho w A)_b}_{\textit{net outward mass flux}} \quad = 0$$

where density and velocity are averages over cell volume or cell face as appropriate.

Noting that volume $V = \Delta x \Delta y \Delta z$ and areas $A_w = A_e = \Delta y \Delta z$ etc,

$$\frac{\mathrm{d}(\rho \Delta x \Delta y \Delta z)}{\mathrm{d}t} \quad + \quad [(\rho u)_e - (\rho u)_w]\Delta y \Delta z + [(\rho v)_n - (\rho v)_s]\Delta z \Delta x + [(\rho w)_t - (\rho w)_b]\Delta x \Delta y = 0$$

Dividing by the volume, $\Delta x \Delta y \Delta z$:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \frac{(\rho u)_e - (\rho u)_w}{\Delta x} + \frac{(\rho v)_n - (\rho v)_s}{\Delta y} + \frac{(\rho w)_t - (\rho w)_b}{\Delta z} = 0$$

Proceeding to the limit $\Delta x, \Delta y, \Delta z \to 0$:

(Conservative) differential equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \tag{5}$$

This analysis is analogous to the finite-volume procedure, except that in the latter the control volume does not shrink to zero; i.e. it is a **finite**-volume, not **infinitesimal**-volume, approach.

(*** *Advanced* ***)

More mathematically, for an arbitrary volume V with closed surface ∂V:

$$\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\int_V \rho \;\mathrm{d}V}_{mass\,in\,cell} + \underbrace{\oint_{\partial V} \rho\mathbf{u}\bullet\mathrm{d}\mathbf{A}}_{net\;mass\;flux} = 0 \qquad (6)$$

For a fixed control volume take d/dt under the integral sign and apply the divergence theorem to turn the surface integral into a volume integral:

$$\int_V \left\{\frac{\partial\rho}{\partial t} + \nabla\bullet(\rho\mathbf{u})\right\}\mathrm{d}V = 0$$

Since V is arbitrary, the integrand must be identically zero. Hence,

$$\frac{\partial\rho}{\partial t} + \nabla\bullet(\rho\mathbf{u}) = 0 \qquad (7)$$

Incompressible Flow

For incompressible flow, volume as well as mass is conserved, so that:

$$\underbrace{(uA)_e - (uA)_w \quad + \quad (vA)_n - (vA)_s \quad + \quad (wA)_t - (wA)_b}_{net\;outward\,VOLUME\;flux} \quad = 0$$

Substituting for face areas, dividing by volume and proceeding to the limit as above produces

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0} \qquad (8)$$

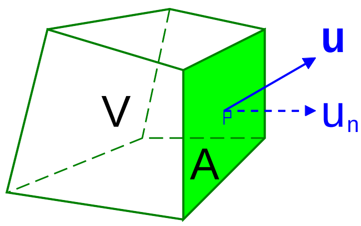
which is usually taken as the continuity equation in incompressible flow.

2.2.2 Momentum

Physical principle (*Newton's Second Law*): *rate of change of momentum = force*

The total rate of change of momentum for fluid passing through a control volume consists of:

- **rate of change** of momentum inside the control volume; plus
- **net outward momentum flux** through boundary.



For a cell with volume V and a typical face with area A:

momentum of fluid in the cell $= mass \times \mathbf{u} \qquad = (\rho V)\mathbf{u}$

momentum flux through one face $= mass\ \ flux \times \mathbf{u} \qquad = (\rho \mathbf{u} \bullet \mathbf{A})\mathbf{u}$

Rate of change of momentum in cell + net outward momentum flux = force (9)

Integral equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}(mass \times \mathbf{u}) + \sum_{faces}(mass\ \ flux \times \mathbf{u}) = \mathbf{F} \qquad (10)$$

Momentum, velocity and force are vectors, giving (in principle) 3 component equations.

<u>Fluid Forces</u>

There are two main types:

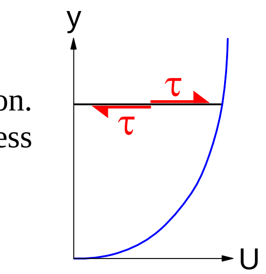
- *surface forces* (proportional to area; act on control-volume faces)
- *body forces* (proportional to volume)

(i) *Surface forces* are usually expressed in terms of *stress*:

$$stress = \frac{force}{area} \qquad \text{or} \qquad force = stress \times area$$

The main surface forces are:

- *pressure p*: always acts normal to a surface;

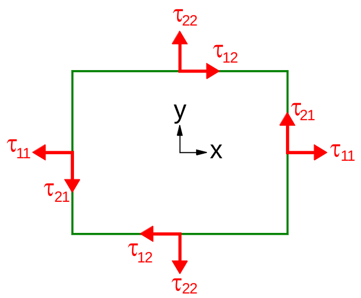


- *viscous stresses* $\boldsymbol{\tau}$: frictional forces arising from relative motion. For a simple shear flow there is only one non-zero stress component:

$$\tau \equiv \tau_{12} = \mu \frac{\partial u}{\partial y}$$

but, in general, $\boldsymbol{\tau}$ is a symmetric tensor (the components of stress imparted by external fluid on individual faces of a volume of fluid are shown right) and has a more complex expression for its components. In incompressible flow,

$$\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

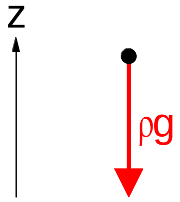


(ii) *Body forces*

The main body forces are:

- *gravity*: the force per unit volume is

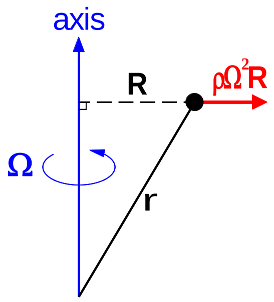
$$\rho \mathbf{g} = \rho(0,0,-g)$$



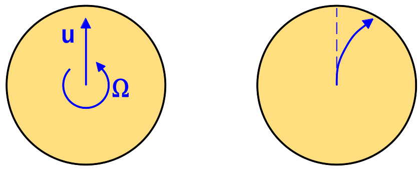
(For constant-density fluids, pressure and weight can be combined as a piezometric pressure $p^* = p + \rho g z$; when gravity is incorporated into a modified pressure it no longer appears explicitly in the flow equations – see the Examples.)

- *centrifugal* and *Coriolis* forces (apparent forces in a rotating reference frame)

centrifugal force:
$-\rho \mathbf{\Omega} \wedge (\mathbf{\Omega} \wedge \mathbf{r}) \quad = \rho \Omega^2 \mathbf{R}$

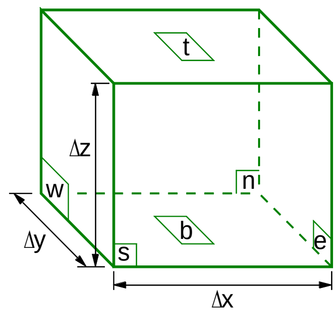


Coriolis force:
$-2\rho \mathbf{\Omega} \wedge \mathbf{u}$



In inertial frame   In rotating frame

(Because the centrifugal force can be written as the gradient of some quantity – in this case $\frac{1}{2}\rho\Omega^2 R^2$ – it can also be absorbed into a modified pressure and hence removed from the momentum equation; see the Examples).



Differential Equation For Momentum

Once again, a conservative differential equation can be derived by considering a fixed Cartesian control volume with sides Δx, Δy and Δz.

For the x-component of momentum:

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}(\rho V u)}_{\textit{rate of change of momentum}} + \underbrace{(\rho u A)_e u_e - (\rho u A)_w u_w + (\rho v A)_n u_n - (\rho v A)_s u_s + (\rho w A)_t u_t - (\rho w A)_b u_b}_{\textit{net outward momentum flux}}$$

$$= \underbrace{(p_w A_w - p_e A_e)}_{\textit{pressure force in x direction}} + \textit{viscous and other forces}$$

Substituting cell dimensions:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho \Delta x \Delta y \Delta z\, u) + [(\rho u)_e u_e - (\rho u)_w u_w]\Delta y \Delta z + [(\rho v)_n u_n - (\rho v)_s u_s]\Delta z \Delta x + [(\rho w)_t u_t - (\rho w)_b u_b]\Delta x \Delta y$$

$$= (p_w - p_e)\Delta y \Delta z \quad + \textit{viscous and other forces}$$

Dividing by volume $\Delta x \Delta y \Delta z$ (and changing the order of p_e and p_w):

$$\frac{\mathrm{d}(\rho u)}{\mathrm{d}t} + \frac{(\rho uu)_e - (\rho uu)_w}{\Delta x} + \frac{(\rho vu)_n - (\rho vu)_s}{\Delta y} + \frac{(\rho wu)_t - (\rho wu)_b}{\Delta z} = -\frac{(p_e - p_w)}{\Delta x}$$

$$+ \textit{viscous and other forces}$$

In the limit as Δx, Δy, $\Delta z \to 0$:

(Conservative) differential equation:

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho uu)}{\partial x} + \frac{\partial(\rho vu)}{\partial y} + \frac{\partial(\rho wu)}{\partial z} = -\frac{\partial p}{\partial x} + \mu \nabla^2 u + \textit{other forces} \qquad (11)$$

Notes.

(1) The viscous term is given without proof above (but you can read the notes below). ∇^2 is the *Laplacian* operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

(2) The pressure force per unit volume in the x direction is given by (minus) the pressure gradient in that direction.

(3) The y and z-momentum equations can be obtained by inspection / pattern-matching.

(*** *Advanced* ***)

Separating surface forces (determined by a stress tensor σ_{ij}) and body forces (f_i per unit volume), the control-volume equation for the i component of momentum may be written

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho u_i \;\mathrm{d}V}_{\textit{momentum in cell}} + \underbrace{\oint_{\partial V} \rho u_i u_j \;\mathrm{d}A_j}_{\textit{net momentum flux}} = \underbrace{\oint_{\partial V} \sigma_{ij} \;\mathrm{d}A_j}_{\textit{surface forces}} + \underbrace{\int_V f_i \;\mathrm{d}V}_{\textit{body forces}} \qquad (12)$$

The stress tensor has pressure and viscous parts:

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij}, \qquad \tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right) \qquad (13)$$

For a fixed volume, take d/dt under the integral sign and apply the divergence theorem to the surface integrals:

$$\int_V \left\{ \frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho u_i u_j)}{\partial x_j} - \frac{\partial \sigma_{ij}}{\partial x_j} - f_i \right\} \mathrm{d}V = 0$$

As V is arbitrary, the integrand vanishes identically. Hence, for arbitrary forces:

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho u_i u_j)}{\partial x_j} = \frac{\partial \sigma_{ij}}{\partial x_j} + f_i \qquad (14)$$

Splitting the stress tensor into pressure and viscous terms:

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho u_i u_j)}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + f_i \qquad (15)$$

If the fluid is incompressible and viscosity is uniform then the viscous term simplifies to give

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho u_i u_j)}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu \nabla^2 u_i + f_i$$

2.2.3 General Scalar

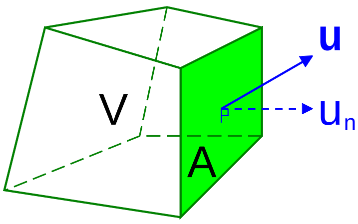
A similar equation may be derived for **any** physical quantity that is advected and diffused in a fluid flow. Examples include salt, sediment and chemical pollutants. For each such quantity an equation is solved for the *concentration* (i.e. amount per unit mass of fluid) ϕ.

Diffusion occurs when concentration varies with position and leads to net transport from regions of high concentration to regions of low concentration. For many scalars this rate of transport is proportional to area and concentration gradient and may be quantified by *Fick's diffusion law*:

$$\begin{aligned} \textit{rate of diffusion} &= -\textit{diffusivity} \times \textit{gradient} \times \textit{area} \\ &= -\Gamma \frac{\partial \phi}{\partial n} A \end{aligned}$$

This is often referred to as *gradient diffusion*. A common example is heat conduction.

For an arbitrary control volume:



amount in cell:	$\rho V \phi$	(mass × concentration)
advective flux:	$(\rho \mathbf{u} \bullet \mathbf{A})\phi$	(mass flux × concentration)
diffusive flux:	$-\Gamma \dfrac{\partial \phi}{\partial n} A$	(–diffusivity × gradient × area)
source:	$S = sV$	(source density × volume)

Balancing the rate of change, the net flux through the boundary and rate of production yields the *scalar-transport* or (*advection-diffusion*) *equation*:

$$\textit{rate of change} \quad + \quad \textit{net outward flux} \quad = \quad \textit{source}$$

Integral equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\textit{mass} \times \phi) + \sum_{\textit{faces}} (\textit{mass flux} \times \phi \;\; - \Gamma \frac{\partial \phi}{\partial n} A) = S \qquad (16)$$

(Conservative) differential equation:

$$\frac{\partial(\rho\phi)}{\partial t} + \frac{\partial}{\partial x}(\rho u \phi - \Gamma \frac{\partial \phi}{\partial x}) + \frac{\partial}{\partial y}(\rho v \phi - \Gamma \frac{\partial \phi}{\partial y}) + \frac{\partial}{\partial z}(\rho w \phi - \Gamma \frac{\partial \phi}{\partial z}) = s \qquad (17)$$

(*** *Advanced* ***)

This may be expressed more mathematically as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\phi \,\mathrm{d}V \;\; + \oint_{\partial V} (\rho \mathbf{u} \phi - \Gamma \nabla \phi) \bullet \mathrm{d}\mathbf{A} \;\; = \int_V s \,\mathrm{d}V \qquad (18)$$

For a fixed control volume, taking the time derivative under the integral sign and using Gauss's divergence theorem as before gives a corresponding differential equation

$$\frac{\partial(\rho\phi)}{\partial t} + \nabla \bullet (\rho \mathbf{u} \phi - \Gamma \nabla \phi) = s \qquad (19)$$

2.2.4 Momentum Components as Transported Scalars

In the momentum equation, if the viscous force $\tau A = \mu(\partial u / \partial n)A$ is transferred to the LHS it looks like a diffusive flux; e.g. for the *x*-component:

$$\frac{\mathrm{d}}{\mathrm{d}t}(mass \times u) + \sum_{faces}(mass\ \ flux \times u - \mu \frac{\partial u}{\partial n} A) = other\ \ forces$$

Compare this with the generic scalar-transport equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}(mass \times \phi) + \sum_{faces}(mass\ \ flux \times \phi\ - \Gamma \frac{\partial \phi}{\partial n} A) = S$$

Each component of momentum satisfies its own scalar-transport equation, with

concentration, ϕ	$\leftarrow$	**velocity (*u*, *v* or *w*)**
diffusivity, Γ	$\leftarrow$	**viscosity μ**
source, *S*	$\leftarrow$	**other forces**

Consequently, only one generic scalar-transport equation need be considered.

In Section 5 we shall see, however, that the momentum components differ from *passive* scalars (those not affecting the flow), because they are:

- equations are nonlinear (mass flux involves the velocity component being solved for);
- equations are coupled (mass fluxes involve the other velocity components);
- also constrained to be mass-consistent.

2.2.5 Non-Gradient Diffusion

The analysis above assumes that all non-advective flux is simple gradient diffusion:

$$-\Gamma \frac{\partial \phi}{\partial n} A$$

Actually, the real situation is a little more complex. For example, in the *u*-momentum equation the full expression for the 1-component of viscous stress through the 2-face is

$$\tau_{12} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

The $\partial u/\partial y$ part is gradient diffusion of *u*, but the $\partial v/\partial x$ term is not. In general, non-advective fluxes F' that can't be represented by gradient diffusion are discretised conservatively (i.e. worked out for particular cell faces) but are transferred to the RHS as a source term:

$$\frac{\mathrm{d}}{\mathrm{d}t}(mass \times \phi) + \sum_{faces}[mass\ \ flux \times \phi\quad - \mu \frac{\partial \phi}{\partial n} A\quad + F'] = S$$

2.2.6 Moving Control Volumes

Control-volume equations are also applicable to **moving** control volumes, provided the normal velocity component in the mass flux is that **relative to the mesh**; i.e.

$$u_n = (\mathbf{u} - \mathbf{u}_{mesh}) \bullet \mathbf{n}$$

The finite-volume method can thus be used for calculating flows with moving boundaries[2].

[2] See, for example: Apsley, D.D. and Hu, W., 2003, CFD Simulation of two- and three-dimensional free-surface flow, International Journal for Numerical Methods in fluids, 42, 465-491.

2.3 Non-Conservative Forms of the Flow Equations

Two equivalent differential forms of the flow equations may be derived from the control-volume equations in the limit as the control volume shrinks to a point.

- From **fixed** control volumes we obtain governing equations in **conservative** form as in Section 2.2 above; this is called the *Eulerian* approach.
- Using control volumes **moving with the fluid** we obtain the governing equations in **non-conservative** form; this is called the *Lagrangian* approach.

These forms can be derived from each other by mathematical manipulation.

The *conservative* differential equations are so-called because they can be integrated directly to give an equivalent integral form involving the net change in a flux, with the flux leaving one cell equal to that entering an adjacent cell. To do so, all terms involving derivatives of dependent variables must have differential operators "on the outside"; e.g. in one dimension:

$$\frac{\mathrm{d}f}{\mathrm{d}x} = g(x) \quad \longleftrightarrow \quad f(x_2) - f(x_1) = \int_{x_1}^{x_2} g(x)\ \mathrm{d}x$$

$$(\textit{differential form}) \qquad\qquad (\textit{integral form})$$

i.e. $\quad \mathit{flux}_{out} - \mathit{flux}_{in} = \mathit{source}$

> (*** *Advanced* ***)
> The three-dimensional version uses partial derivatives and the divergence theorem to change the differentials to surface flux integrals.

As an example of how essentially the same equation can appear in conservative and non-conservative forms consider a simple 1-d example:

$$\frac{\mathrm{d}}{\mathrm{d}x}(y^2) = g(x) \qquad \text{(conservative form – can be integrated directly)}$$

$$2y\frac{\mathrm{d}y}{\mathrm{d}x} = g(x) \qquad \text{(non-conservative form, obtained by applying the chain rule)}$$

<u>Material Derivatives</u>

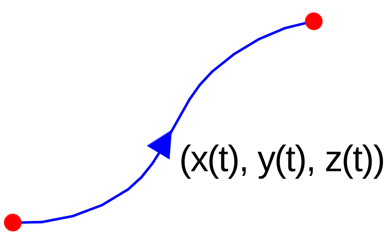
The rate of change of some property in a fluid element **moving with the flow** is called the *material* (or *substantive*) derivative. It is denoted by $\mathrm{D}\phi/\mathrm{D}t$ and worked out as follows.

Every field variable ϕ is a function of both time and position; i.e.

$$\phi = \phi(t, x, y, z)$$

As one follows a path through space ϕ will change with time because:

- it changes with time t at each point, **and**
- it changes with position (x, y, z) as it moves with the flow.



Thus, the **total** time derivative following an **arbitrary** path $(x(t), y(t), z(t))$ is

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{\partial\phi}{\partial t} + \frac{\partial\phi}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial\phi}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial\phi}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}$$

The *material derivative* is for the **particular path following the flow** ($\mathrm{d}x/\mathrm{d}t = u$, etc.):

$$\frac{\mathrm{D}\phi}{\mathrm{D}t} \equiv \frac{\partial\phi}{\partial t} + u\frac{\partial\phi}{\partial x} + v\frac{\partial\phi}{\partial y} + w\frac{\partial\phi}{\partial z} \qquad (20)$$

Using this definition, it is possible to write a non-conservative but more compact form of the governing equations. For a general scalar ϕ the sum of time-dependent and advective terms in its transport equation is

$$\frac{\partial(\rho\phi)}{\partial t} + \frac{\partial(\rho u\phi)}{\partial x} + \frac{\partial(\rho v\phi)}{\partial y} + \frac{\partial(\rho w\phi)}{\partial z}$$

$$= \left[\frac{\partial\rho}{\partial t}\phi + \rho\frac{\partial\phi}{\partial t}\right] + \left[\frac{\partial(\rho u)}{\partial x}\phi + \rho u\frac{\partial\phi}{\partial x}\right] + \left[\frac{\partial(\rho v)}{\partial y}\phi + \rho v\frac{\partial\phi}{\partial y}\right] + \left[\frac{\partial(\rho w)}{\partial z}\phi + \rho w\frac{\partial\phi}{\partial z}\right]$$

(by the product rule)

$$= \underbrace{\left[\frac{\partial\rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right]}_{=0 \text{ by continuity}}\phi \quad + \quad \rho\underbrace{\left[\frac{\partial\phi}{\partial t} + u\frac{\partial\phi}{\partial x} + v\frac{\partial\phi}{\partial y} + w\frac{\partial\phi}{\partial z}\right]}_{=\mathrm{D}\phi/\mathrm{D}t \text{ by definition}}$$

(by collecting similar terms)

$$= \rho\frac{\mathrm{D}\phi}{\mathrm{D}t} \qquad (21)$$

Using the material derivative, the time-dependent and advection terms in a scalar-transport equation can be combined as the much more compact (but non-conservative) form $\rho\mathrm{D}\phi/\mathrm{D}t$.

The material derivative of velocity ($\mathrm{D}u/\mathrm{D}t$) is the *acceleration*. The momentum equation can be written

$$\underbrace{\rho\frac{\mathrm{D}u}{\mathrm{D}t}}_{mass\times acceleration} = -\frac{\partial p}{\partial x} + \mu\nabla^2 u + other\ forces \qquad (22)$$

This form is simpler to write and is used both for convenience and to derive theoretical results in special cases (see the Examples). However, in the finite-volume method it is the conservative form which is discretised directly.

(*** *Advanced* ***)

The material derivative can be defined in suffix or vector notation respectively as:

$$\frac{\mathrm{D}\phi}{\mathrm{D}t} = \frac{\partial\phi}{\partial t} + u_i\frac{\partial\phi}{\partial x_i} \qquad \text{or} \qquad \frac{\mathrm{D}\phi}{\mathrm{D}t} = \frac{\partial\phi}{\partial t} + \mathbf{u}\bullet\nabla\phi$$

Simplify the derivation of (21) above using the summation convention or vector notation.

2.4 Non-Dimensionalisation

Although it is possible to work entirely in dimensional quantities, there are good theoretical reasons for working in non-dimensional variables. These include the following.

- All *dynamically-similar* problems (same Re, Fr etc.) can be solved with a single computation.
- The number of relevant parameters (and hence the number of graphs needed to convey results) is reduced.
- It indicates the relative size of different terms in the governing equations and, in particular, which might conveniently be neglected.
- Computational variables are of a similar order of magnitude (ideally of order unity), yielding better numerical accuracy.

2.4.1 Non-Dimensionalising the Governing Equations

For incompressible flow the governing equations are:

continuity: $$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \qquad (23)$$

momentum: $$\rho\frac{\mathrm{D}u}{\mathrm{D}t}=-\frac{\partial p}{\partial x}+\mu\nabla^2 u$$ (and similar equations in y, z directions) (24)

Adopting reference scales U_0, L_0 and ρ_0 for velocity, length and density, respectively, and derived scales L_0/U_0 for time and $\rho_0 U_0^2$ for pressure, each fluid quantity can be written as a product of a dimensional scale and a non-dimensional variable (indicated by an asterisk *):

$$\mathbf{x}=L_0\mathbf{x}^*, \qquad t=\frac{L_0}{U_0}t^*, \qquad \mathbf{u}=U_0\mathbf{u}^*, \qquad \rho=\rho_0\rho^*, \qquad p-p_{ref}=(\rho_0U_0^2)p^*, \qquad etc.$$

Substituting into mass and momentum equations (23) and (24) yields, after simplification:

$$\frac{\partial u^*}{\partial x^*}+\frac{\partial v^*}{\partial y^*}+\frac{\partial w^*}{\partial z^*}=0 \qquad (25)$$

$$\rho^*\frac{\mathrm{D}u^*}{\mathrm{D}t^*}=-\frac{\partial p^*}{\partial x^*}+\frac{1}{\mathrm{Re}}\nabla^{*2}u^* \qquad \text{where} \qquad \mathrm{Re}=\frac{\rho_0U_0L_0}{\mu} \qquad (26)$$

From this, it is seen that the key dimensionless group is the *Reynolds number* Re. If Re is large then viscous forces would be expected to be negligible in much of the flow.

Having derived the non-dimensional equations it is usual to drop the asterisks and simply declare that you are "working in non-dimensional variables".

Note.
The objective is that non-dimensional quantities (e.g. p^*) should be of order of magnitude unity, so the scale for a quantity should reflect its **range** of values and not its absolute value. For example, in incompressible flow it is **differences** in pressure that are important, not absolute values; since flow-induced pressures are usually much smaller than the absolute

pressure one usually works in terms of the departure from a reference pressure p_{ref}. Similarly, in Section 3 when we look at small variations in density due to temperature or salinity we shall use an alternative non-dimensionalisation:

$$\rho = \rho_0 + \Delta\rho\,\theta^*$$

2.4.2 Common Dimensionless Groups

If other types of fluid force are included then each introduces another non-dimensional group. For example, gravitational forces lead to a Froude number (Fr) and Coriolis forces to a Rossby number (Ro). Some of the most important dimensionless groups are given below.

If U and L are representative velocity and length scales, respectively, then:

$$\mathrm{Re} \equiv \frac{\rho UL}{\mu} \equiv \frac{UL}{\nu}$$ Reynolds number (viscous flow; μ = dynamic viscosity)

$$\mathrm{Fr} \equiv \frac{U}{\sqrt{gL}}$$ Froude number (open-channel flow; g = gravity)

$$\mathrm{Ma} \equiv \frac{U}{c}$$ Mach number (compressible flow; c = speed of sound)

$$\mathrm{Ro} \equiv \frac{U}{\Omega L}$$ Rossby number (rotating flows; Ω = angular velocity of frame)

$$\mathrm{We} \equiv \frac{\rho U^2 L}{\sigma}$$ Weber number (interfacial flows; σ = surface tension)

Note: for flows with buoyancy forces caused by a change in density, rather than open-channel flows, we sometimes use a *densimetric Froude number* instead; this is defined by

$$\mathrm{Fr} \equiv \frac{U}{\sqrt{(\Delta\rho/\rho)gL}}$$

i.e. g is replaced in the usual formula for Froude number by $(\Delta\rho/\rho)g$, sometimes called the *reduced gravity* g'.

Summary

- Fluid dynamics is governed by conservation equations for mass, momentum, energy (and, for a non-homogeneous fluid, the amount of individual constituents).

- The governing equations can be written in equivalent integral (control-volume) or differential forms.

- The *finite-volume* method is a direct discretisation of the control-volume equations.

- Differential forms of the flow equations may be *conservative* (i.e. can be integrated directly to something of the form "$flux_{out} - flux_{in} = source$") or *non-conservative*.

- A particular control-volume equation takes the form:
  rate of change + *net outward flux* = *source*

- There are really just two canonical equations to discretise and solve:
  mass conservation (*continuity*):
  $$\frac{\mathrm{d}}{\mathrm{d}t}(mass) \quad + \quad \sum_{faces}(mass\ \ flux) = 0$$
  scalar-transport (or *advection-diffusion*) *equation*:
  $$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}(mass \times \phi)}_{rate\ of\ change} \quad + \quad \sum_{faces}(\quad \underbrace{mass\ \ flux \times \phi}_{advection} \quad - \quad \underbrace{\Gamma \frac{\partial \phi}{\partial n} A}_{diffusion} \quad) \quad = \quad \underbrace{S}_{source}$$

- Each Cartesian velocity component (u, v, w) satisfies its own scalar-transport equation. However, these equations differ from those for a passive scalar because they are non-linear, are coupled through the advective fluxes and pressure forces and their solutions are also required to be mass-consistent.

- Non-dimensionalising the governing equations, allows dynamically-similar flows (those with the same values of Reynolds number, etc.) to be solved with a single calculation, reduces the overall number of parameters, indicates whether certain terms in the governing equations are significant or negligible and ensures that the main computational variables are of similar magnitude.

Examples

Q1.
In 2-d flow, the continuity and x-momentum equations can be written in conservative differential form as

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0$$

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho uu) + \frac{\partial}{\partial y}(\rho vu) = -\frac{\partial p}{\partial x} + \mu \nabla^2 u$$

respectively.

(a) Show that these can be written in the equivalent non-conservative forms:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) = 0$$

$$\rho \frac{\mathrm{D}u}{\mathrm{D}t} = -\frac{\partial p}{\partial x} + \mu \nabla^2 u$$

where the material derivative is given (in 2 dimensions) by $\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$.

(b) Define carefully what is meant by the statement that a flow is *incompressible*. To what does the continuity equation reduce in incompressible flow?

(c) Write down conservative forms of the 3-d equations for mass and x-momentum.

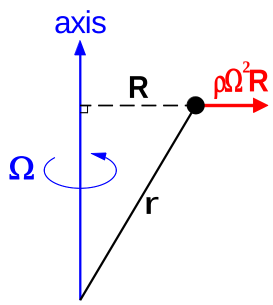
(d) Write down the z-momentum equation, including gravitational forces.

(e) Show that, for constant-density flows, pressure and gravity can be combined in the momentum equations via the *piezometric pressure* $p + \rho g z$.

(f) In a rotating reference frame there are additional apparent forces (per unit volume):

centrifugal force: $-\rho \mathbf{\Omega} \wedge (\mathbf{\Omega} \wedge \mathbf{r})$ or $\rho \Omega^2 \mathbf{R}$

Coriolis force: $-2\rho \mathbf{\Omega} \wedge \mathbf{u}$



where $\mathbf{\Omega}$ is the angular velocity of the reference frame, $\mathbf{u}$ is the fluid velocity in that frame, $\mathbf{r}$ is the position vector (relative to a point on the axis of rotation) and $\mathbf{R}$ is its projection perpendicular to the axis of rotation. By writing the centrifugal force as the gradient of some quantity show that it can be subsumed into a modified pressure. Also, find the components of the Coriolis force if rotation is about the z axis.

(*** *Advanced* ***)

(g) Write the conservative mass and momentum equations in vector notation.

(h) Write the conservative mass and momentum equations in suffix notation using the summation convention.

Q2.

The x-component of the momentum equation is given by

$$\rho\frac{\mathrm{D}u}{\mathrm{D}t} = -\frac{\partial p}{\partial x} + \mu\nabla^2 u$$

Using this equation derive the velocity profile in fully-developed, laminar flow for:

(a) pressure-driven flow between stationary parallel planes (“*Poiseuille flow*”);

(b) constant-pressure flow between stationary and moving planes (“*Couette flow*”).

Q3. (*** *Advanced* ***)

By applying Gauss’s divergence theorem deduce the conservative and non-conservative differential equations corresponding to the general integral scalar-transport equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\phi\,\mathrm{d}V \quad + \oint_{\partial V} (\rho\mathbf{u}\phi - \Gamma\nabla\phi)\bullet \mathrm{d}\mathbf{A} \quad = \int_V s\,\mathrm{d}V$$

Q4.

In each of the following cases, state which of (i), (ii), (iii) is a valid dimensionless number. Carry out research to find the name and physical significance of these numbers.

(L = length; u = velocity; z = height; p = pressure; ρ = density; μ = dynamic viscosity; ν = kinematic viscosity; g = gravitational acceleration; Ω = angular velocity).

(a) (i) $\dfrac{p - p_{ref}}{\rho U}$; (ii) $\dfrac{p - p_{ref}}{\frac{1}{2}\rho U^2}$; (iii) $\rho U^2 (p - p_{ref})$

(b) (i) $\dfrac{\rho UL}{\nu}$; (ii) $\dfrac{\rho UL}{\mu}$; (iii) μUL

(c) (i) $\dfrac{\left(-\dfrac{g}{\rho}\dfrac{\mathrm{d}\rho}{\mathrm{d}z}\right)^{1/2}}{\left|\dfrac{\mathrm{d}u}{\mathrm{d}z}\right|}$; (ii) $\dfrac{U}{gL}$; (iii) $\dfrac{p - p_{ref}}{\rho g}$

(d) (i) $\dfrac{U\Omega}{L}$; (ii) $\dfrac{\rho g L}{U\Omega}$; (iii) $\dfrac{U}{\Omega L}$

Q5. (Exam 2008; part (h) depends on later sections of this course)

The momentum equation for a viscous fluid in a rotating reference frame is

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p + \mu \nabla^2 \mathbf{u} - 2\rho \mathbf{\Omega} \wedge \mathbf{u} \qquad (*)$$

where ρ is density, $\mathbf{u} = (u,v,w)$ is velocity, p is pressure, μ is dynamic viscosity and $\mathbf{\Omega}$ is the angular-velocity vector of the reference frame. The symbol $\wedge$ denotes a vector product.

(a) If $\mathbf{\Omega} = (0,0,\Omega)$ write down the x and y components of the Coriolis force ($-2\rho\mathbf{\Omega} \wedge \mathbf{u}$).

(b) Hence write the x- and y-components of equation (*).

(c) Show how Equation (*) can be written in non-dimensional form in terms of a Reynolds number Re and Rossby number Ro (both of which should be defined).

(d) Define the terms *conservative* and *non-conservative* when applied to the differential equations describing fluid flow.

(e) Define (mathematically) the material derivative operator D/Dt. Then, noting that the continuity equation can be written

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0,$$

show that the x-momentum equation can be written in an equivalent conservative form.

(f) If the x-momentum equation were to be regarded as a special case of the general scalar-transport (or advection-diffusion) equation, identify the quantities representing:
- (i) concentration;
- (ii) diffusivity;
- (iii) source.

(g) Explain why the three equations for the components of momentum cannot be treated as independent scalar equations.

(h) Explain (briefly) how pressure can be derived in a CFD simulation of:
- (i) high-speed compressible gas flow;
- (ii) incompressible flow.

Q6.

(a) In a rotating reference frame (with angular velocity vector $\boldsymbol{\Omega}$) the non-viscous forces on a fluid are, per unit volume,

$$-\nabla p \quad + \quad \rho \mathbf{g} \quad + \quad \rho \Omega^2 \mathbf{R} \quad - \quad 2\rho \boldsymbol{\Omega} \wedge \mathbf{u}$$
$$\text{(I)} \qquad \text{(II)} \qquad \text{(III)} \qquad \text{(IV)}$$

where p is pressure, $\mathbf{g} = (0,0,-g)$ is the gravity vector and $\mathbf{R}$ is the vector from the closest point on the axis of rotation to a point. Show that, in a constant-density fluid, force densities (II) and (III) can be absorbed into a modified pressure.

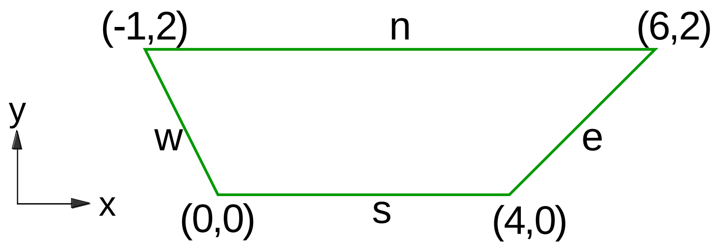
(b) Consider a closed cylindrical can of radius 5 cm and depth 15 cm. The can is completely filled with fluid of density 1100 kg m^{-3} and is rotating steadily about its axis (which is vertical) at 600 rpm. Where do the maximum and minimum pressures in the can occur, and what is the difference in pressure between them?

Q7. (Exam 2011 – part)

The figure below depicts a 2-d cell in a finite-volume CFD calculation. Vertices are given in the figure, and velocity in the adjacent table. At this instant $\rho = 1.0$ everywhere.

(a) Calculate the volume flux out of each face. (Assume unit depth.)

(b) Show that the flow is *not* incompressible and find the time derivative of density.



Face	Velocity (u,v)	
	u	v
e	4	10
n	1	8
w	2	2
s	1	4