Essay 12

Turbulence Modeling Based on the RANS Equations

12.1 Introduction

To begin the presentation of turbulence modeling based on the RANS equations, it is appropriate to recall and reprint the last line in the derivation of the RANS equations proper. That equation, Eq. (11.15), is presented here as Eq. (12.1).

$$\rho\left[\frac{\partial}{\partial x}(\bar{u}^2)+\frac{\partial}{\partial y}(\bar{u}\bar{v})+\frac{\partial}{\partial z}(\bar{u}\bar{w})\right]$$

$$=-\frac{\partial\bar{p}}{\partial x}+\left[\frac{\partial}{\partial x}\left(\mu\frac{\partial\bar{u}}{\partial x}-\rho\overline{u'^2}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial\bar{u}}{\partial y}-\rho\overline{u'v'}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial\bar{u}}{\partial z}-\rho\overline{u'w'}\right)\right] \quad (12.1)$$

This equation contains the three temporal-mean velocity components $\bar{u}$, $\bar{v}$, and $\bar{w}$ as well as the corresponding fluctuating components u', v', and w'. The pressure is also an unknown. Clearly there is a significant imbalance between the number of unknowns and the number of equations that are available for determining those unknowns.

There are several additional equations that can be utilized to help in solving for the unknowns. In particular, upon recognizing that Eq. (12.1) is derived from the x-component of Newton's Second Law, it is clear that similar equations can be derived for the y- and z-components. This realization adds two additional equations. Two more equations can be obtained by making use of mass conservation. That conservation equation for the instantaneous velocities $u(x,y,z,t)$, $v(x,y,z,t)$, and $w(x,y,z,t)$ is:

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=0 \quad (12.2)$$

This equation can be tailored to apply to a turbulent flow described by Eqs. (11.2, 3, and 4). Furthermore, the accepted turbulence models continues to regard ρ as a constant. The substitution of Eqs. (11.2, 3, and 4) into the instantaneous mass conservation equation, Eq. (12.2) leads to:

$$\frac{\partial(\bar{u}+u')}{\partial x}+\frac{\partial(\bar{v}+v')}{\partial y}+\frac{\partial(\bar{w}+w')}{\partial z}=0 \quad (12.3)$$

or

$$\frac{\partial\bar{u}}{\partial x}+\frac{\partial\bar{v}}{\partial y}+\frac{\partial\bar{w}}{\partial z}+\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}=0 \qquad (12.4)$$

If this equation is Reynolds-averaged, the last three terms drop out, leaving:

$$\frac{\partial\bar{u}}{\partial x}+\frac{\partial\bar{v}}{\partial y}+\frac{\partial\bar{w}}{\partial z}=0 \qquad (12.5)$$

which states that mass conservation holds for the time-averaged flow. Then, by consideration of Eqs. (12.3 and 5), there follows:

$$\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}=0 \qquad (12.6)$$

At this point, it is appropriate to review the number of equations that are available to solve for the seven unknowns that were identified at the beginning of this section. An equation tally leads to five. Therefore, despite taking account of all possible conservation laws, the problem, as formulated via the RANS model, defies solution.

This dead end was recognized many decades ago, and numerous investigators have sought to create means of closing the gap between equations and unknowns. The common terminology that was in vogue decades ago was to characterize the closure of the gap as a seeking of *closure conditions*. In this search, primarily carried out by applied mathematicians, many unphysical closure conditions were invented.

12.2 Reynolds Stresses

One approach to closure is to work with the turbulent "stresses" that were identified in Essay 11 following Eq. (11.15). Those stresses are defined as:

$$\rho\overline{u'^2}=\mu_t\frac{\partial\bar{u}}{\partial x}=\tau'_{xx}$$

$$\rho\overline{u'v'}=\mu_t\frac{\partial\bar{u}}{\partial y}=\tau'_{xy} \qquad (12.7)$$

$$\rho\overline{u'w'}=\mu_t\frac{\partial\bar{u}}{\partial z}=\tau'_{xz}$$

and are commonly designated as *Reynolds stresses* as indicated in Essay 11.

The quantity μ_t is illogically defined as the *turbulent viscosity*. In fact, μ_t has nothing to do with viscosity but is more closely connected with turbulent momentum. Of particular note in the definitions set forth in Eqs. (12.7) is that the connection between the stresses and the velocity gradients is by means of a single quantity μ_t. This assumption is equivalent to stating that the turbulence is *isotropic*; that is, that the turbulent transport is independent of direction. Such an assumption may have merit for locations that are distant from walls. On the other hand, in the neighborhood of a wall, the assumption of isotropy is very difficult to defend.

Nevertheless, despite the absence of strict logic, Eqs. (12.7) are substituted into the x-direction RANS equation, Eq. (12.1), with the result:

$$\rho\left[\frac{\partial}{\partial x}(\bar{u}^2)+\frac{\partial}{\partial y}(\bar{u}\bar{v})+\frac{\partial}{\partial z}(\bar{u}\bar{w})\right]$$

$$= -\frac{\partial \bar{p}}{\partial x}+\left[\frac{\partial}{\partial x}\left[(\mu+\mu_t)\frac{\partial \bar{u}}{\partial x}\right]+\frac{\partial}{\partial y}\left[(\mu+\mu_t)\frac{\partial \bar{u}}{\partial y}\right]+\frac{\partial}{\partial z}\left[(\mu+\mu_t)\frac{\partial \bar{u}}{\partial z}\right]\right] \tag{12.8}$$

The true complexity of this equation is hidden in the quantity μ_t. An even more effective hiding is achieved by defining the effective viscosity μ_{eff} as,

$$\mu_{eff} = \mu + \mu_t \tag{12.9}$$

The dilemma of solving the trio of equations represented by Eq. (12.8) and its counterparts in the y and z directions is unchanged by the use of μ_{eff}. This puzzlement prompted the creation of models from which μ_t could be calculated. There are two groups of such models. One group of models is closely related to the RANS equations. A second group has another basis. Here, consideration will be confined to RANS-based models.