1 Orthogonal matrices

A set of vectors $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_k\}$ in R^n is called an **orthogonal** set if all pairs of distinct vectors in the set are orthogonal that is if $\vec{v}_i \cdot \vec{v}_j = 0$ whenever $i \neq j$ for $i, j = 1, 2, ..., k$

Example 1: Show that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ is an orthogonal set in R^3 if

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

.

Solution:

$$\vec{v}_1 \cdot \vec{v}_2 = 2 \cdot 0 + 1 \cdot 1 + (-1) \cdot 1 = 0$$
$$\vec{v}_2 \cdot \vec{v}_3 = 0 \cdot 1 + 1 \cdot (-1) + 1 \cdot 1 = 0$$
$$\vec{v}_1 \cdot \vec{v}_3 = 2 \cdot 1 + 1 \cdot (-1) + (-1) \cdot 1 = 0$$

A set of vectors in R^n is an **orthonormal set** if it is an orthogonal set of *unit vectors.*

Example 2: Show that $\vec{q}_1, \vec{q}_2$ is an orthonormal set in R^3 if

$$\vec{q}_1 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \vec{q}_2 = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$$

.

Solution:

$$\vec{q}_1 \cdot \vec{q}_2 = \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{6}} + \left(-\frac{1}{\sqrt{3}}\right) \cdot \frac{2}{\sqrt{6}} + \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{6}} = 0$$
$$\vec{q}_1 \cdot \vec{q}_1 = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1, \quad \|\vec{q}_1\| = \sqrt{\vec{q}_1 \cdot \vec{q}_1} = \sqrt{\vec{q}_1^T \vec{q}_1} = 1$$
$$\vec{q}_2 \cdot \vec{q}_2 = \frac{1}{6} + \frac{4}{6} + \frac{1}{6} = 1, \quad \|\vec{q}_2\| = \sqrt{\vec{q}_2 \cdot \vec{q}_2} = \sqrt{\vec{q}_2^T \vec{q}_2} = 1$$

A $n \times n$ matrix Q whose columns form an orthonormal set is called an **orthogonal matrix**.

Theorem 1.1 *A square matrix Q is orthogonal if and only if $Q^{-1} = Q^T$.*

Proof:

1. Let Q be an orthogonal matrix. Let us show that $Q^TQ = I$. Let $\vec{q}_i$ be the ith column of Q (and ith row of Q^T). The (i, j) entry of Q^TQ is the dot product of the ith row of Q^T and jth column of Q:

$$(Q^TQ)_{ij} = \vec{q}_i \cdot \vec{q}_j = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} ; \Rightarrow Q^TQ = I$$

Similarly, $QQ^T = I$.

2. If $Q^TQ = I$ and $QQ^T = I$, then there exists the inverse $Q^{-1} = Q^T$.

3. If $Q^{-1} = Q^T$, then multiplying by Q we get $Q^{-1}Q = Q^TQ = I$ and $QQ^{-1} = QQ^T = I; \Rightarrow Q$ is an orthogonal matrix.

Theorem 1.2 *Let Q be an $n \times n$ matrix. The following statements are equivalent:*
(a) Q is orthogonal.
(b) $||Q\vec{x}|| = ||\vec{x}||$ for every $\vec{x}$ inR^n.
(c) $Q\vec{x} \cdot Q\vec{y} = \vec{x} \cdot \vec{y}$ for every $\vec{x}$ and $\vec{y}$ in R^n.

Proof:

1. Let us prove that (a) $\Rightarrow$ (c): Q is orthogonal; $\Rightarrow Q^TQ = I; \Rightarrow Q\vec{x} \cdot Q\vec{y} = (Q\vec{x})^TQ\vec{y} = \vec{x}^TQ^TQ\vec{y} = \vec{x}^TI\vec{y} = \vec{x}^T\vec{x} = \vec{x} \cdot \vec{y}$.

2. Let us prove that (c) $\Rightarrow$ (b): Assume $Q\vec{x} \cdot Q\vec{y} = \vec{x} \cdot \vec{y}$ for every $\vec{x} \in \mathbb{R}^n, \vec{y} \in \mathbb{R}^n$. Then taking $\vec{x} = \vec{y}$ we have $Q\vec{x} \cdot Q\vec{x} = \vec{x} \cdot \vec{x}$; $\Rightarrow$ for every $\vec{x} \in \mathbb{R}^n$.

3. Let us prove that (b) $\Rightarrow$ (a): Assume $\|Q\vec{x}\| = \|\vec{x}\|$ for any $\vec{x} \in \mathbb{R}^n$. Let $\vec{q_i}$ be ith column of Q. $\vec{x} \cdot \vec{y} = Q\vec{x} \cdot Q\vec{y}$ for all $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^n$. If $\vec{e_i}$ is the ith standard basis vector then $\vec{q_i} = Q\vec{e_i}$. $\Rightarrow \vec{q_i} \cdot \vec{q_j} = Q\vec{e_i} \cdot Q\vec{e_i} = \vec{e_i} \cdot \vec{e_j} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$ $; \Rightarrow Q$ is orthogonal.

Theorem 1.3 *If Q is an orthogonal matrix, then its rows form an orthonormal set.*

Proof:
From **Theorem 1.1** we know that $Q^{-1} = Q^T$. Therefore, $(Q^T)^{-1} = (Q^{-1})^{-1} = Q = (Q^T)^T; \Rightarrow Q^T$ is an orthogonal matrix. $\Rightarrow$ The columns of Q^T (which are the rows of Q) form an orthonormal set.

Theorem 1.4 *Let Q be an orthogonal matrix.*
(a) Q^{-1} is orthogonal.
(b) $det(Q) = \pm 1$.
(c) If λ is an eigenvalue of Q, then $|\lambda| = 1$.
(d) If Q_1 and Q_2 are orthogonal $n \times n$ matrices, then so is Q_1Q_2.

Proof:

1. Let us prove (c): Let λ be an eigenvalue of Q with corresponding eigenvector $\vec{v}$. Then $Q\vec{v} = \lambda\vec{v}, \Rightarrow$ using **Theorem 1.2**, $\|\vec{v}\| = \|Q\vec{v}\| = \|\lambda\vec{v}\| = |\lambda|\|\vec{v}\|$. Since, $\|\vec{v}\| \neq 0$, then $|\lambda| = 1$.

2. Let us prove (b): If Q is orthogonal, then $Q^TQ = I$ (see the Proof for **Theorem 1.1**). $det(Q^TQ) = det(Q^T)det(Q) = det(I) = 1; \Rightarrow (det(Q)^2) = 1; \Rightarrow det(Q) \pm 1$.

2 Orthogonal matrices. Diagonalization of symmetric matrices

A square matrix is **symmetric** if $A^T = A$ that is if A is equal to its own transpose.

Example 2: Show that matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$ is symmetric. Diagonalize matrix A.

Solution:

The characteristic polynomial of A is $\lambda^2 + \lambda - 6 = (\lambda + 3)(\lambda - 2) = 0$. So, the eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = 2$.

Solving $(A - \lambda_{1,2} I)\vec{v} = \vec{0}$, we find the corresponding eigenvectors $\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

So, A is diagonalizable, and if we set $S = [\vec{v}_1 \quad \vec{v}_2] = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$, then we know that $D = S^{-1}AS = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$.

Observe that $\vec{[v]}_1$ and $\vec{v}_2$ are orthogonal. So, we can normalize them to get unit vectors $\vec{u}_1 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{bmatrix}$ and $\vec{u}_2 = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$.

Let us take $Q = [\vec{u}_1 \quad \vec{u}_2] = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$. Then $D = Q^{-1}AQ$ also.

But now Q is an *orthogonal* matrix since $\{\vec{u}_1, \vec{u}_2\}$ is an orthogonal set of vectors. Therefore $Q^{-1} = Q^T$, and we have $D = Q^T AQ$.

A square matrix A is **orthogonally diagonalizable** if there exists an orthogonal matrix Q and a diagonal matrix D such that $Q^T AQ = D$.

Theorem 2.1 *If A is orthogonally diagonalizable, then A is symmetric.*

Proof:

If A is orthogonally diagonalizable, then there exists an orthogonal matrix D such that $Q^T AQ = D$. Since $Q^{-1} = Q^T$, we have $Q^T Q = QQ^T = I; \Rightarrow QDQ^T = Q(Q^T AQ)Q^T = (QQ^T)A(QQ^T) = IAI = A$. But then $A^T = (QDQ^T)^T = (Q^T)^T D^T Q^T = QDQ^T = A$ ($D^T = D$ every diagonal matrix is symmetric). $\Rightarrow A^T = A; \Rightarrow A$ is symmetric.

Theorem 2.2 *If A is a real symmetric matrix, then the eigenvalues of A are real.*

Proof:

Let λ be an eigenvalue of A with corresponding eigenvector $\vec{v}$; $\Rightarrow A\vec{v} = \lambda\vec{v}$. Take complex conjugates: $\overline{A\vec{v}} = \overline{\lambda\vec{v}}$. Since A is real, then $A\bar{\vec{v}} = \bar{A}\bar{\vec{v}} = \overline{A\vec{v}} = \overline{\lambda\vec{v}} = \bar{\lambda}\bar{\vec{v}}$.

Take transposes and use the fact that A is symmetric:

$$(A\bar{\vec{v}})^T = \bar{\vec{v}}^T A^T = (A\bar{\vec{v}})^T = (\bar{\lambda}\bar{\vec{v}})^T = \bar{\lambda}\bar{\vec{v}}^T$$

Therefore, $\lambda(\bar{\vec{v}}^T\vec{v}) = \bar{\vec{v}}^T(\lambda\vec{v}) = \bar{\vec{v}}^T(A\vec{v}) = (\bar{\vec{v}}^T A)\vec{v} = (\bar{\lambda}\bar{\vec{v}}^T)\vec{v} = \bar{\lambda}(\bar{\vec{v}}^T\vec{v})$; $\Rightarrow$ $\lambda(\bar{\vec{v}}^T\vec{v}) = \bar{\lambda}(\bar{\vec{v}}^T\vec{v})$ i.e. $(\lambda - \bar{\lambda})(\bar{\vec{v}}^T\vec{v}) = 0$ if $\vec{v} = \begin{bmatrix} a_1 + ib_1 \\ \vdots \\ a_n + ib_n \end{bmatrix}$ and $\bar{\vec{v}} = \begin{bmatrix} a_1 - ib_1 \\ \vdots \\ a_n - ib_n \end{bmatrix}$, then $\bar{\vec{v}}^T\vec{v} = (a_1^2 + b_1^2) + \cdots + (a_n^2 + b_n^2) \neq 0$. Since $\vec{v} \neq \vec{0}$ (it is an eigenvector); $\Rightarrow \lambda - \bar{\lambda} = 0$; $\Rightarrow \lambda = \bar{\lambda}$.

Theorem 2.3 *If A is a symmetric matrix, then any two eigenvectors corresponding to distinct eigenvalues of A are orthogonal.*

Proof:

Let $\vec{v}_1$ and $\vec{v}_2$ are eigenvectors corresponding to distinct eigenvalues λ_1 and λ_2 respectfully ($\lambda_1 \neq \lambda_2$). $\Rightarrow A\vec{v}_1 = \lambda_1\vec{v}_1, A\vec{v}_2 = \lambda_2\vec{v}_2$. Using that A is symmetric ($A^T = A$), we have $\lambda_1(\vec{v}_1 \cdot \vec{v}_2) = (\lambda_1\vec{v}_1) \cdot \vec{v}_2 = A\vec{v}_1 \cdot \vec{v}_2 = (A\vec{v}_1)^T\vec{v}_2 = \vec{v}_1^T A^T\vec{v}_2 = \vec{v}_1^T A\vec{v}_2 = \vec{v}_1^T\lambda_2\vec{v}_2 = \lambda_2\vec{v}_1^T\vec{v}_2 = \lambda_2(\vec{v}_1 \cdot \vec{v}_2)$.

So, $\lambda_1(\vec{v}_1 \cdot \vec{v}_2) = \lambda_2(\vec{v}_1 \cdot \vec{v}_2)$; $\Rightarrow (\lambda_1 - \lambda_2)(\vec{v}_1 \cdot \vec{v}_2) = 0$.

Since, $\lambda_1 - \lambda_2 \neq 0$ (distinct eigenvalues) , then $\vec{v}_1 \cdot \vec{v}_2$ (orthogonal eigenvectors).

Theorem 2.4 *(The Spectral Theorem): Let A be $n \times n$ real matrix. Then A is symmetric if and only if it is orthogonally diagonalizable.*

Spectral decomposition of A: $A = QDQ^T$, where D is diagonal matrix (it's entries are just the eigenvalues of A) and Q is orthonormal.

Example 3: Orthogonally diagonalize the matrix $A = \begin{bmatrix} 5 & 8 & -4 \\ 8 & 5 & -4 \\ -4 & -4 & -1 \end{bmatrix}$.

- First we find the eigenvalues: $\lambda_1 = -3$ (with multiplicity 2) and $\lambda_2 = 15$.

- Find a basis for the eigenspace of each eigenvector.

$$\lambda_1 = -3 \text{ has } \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\} \quad \text{and} \quad \lambda_2 = 15 \text{ has } \left\{ \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} \right\}$$

- Using the Gram–Schmidt process, find an orthogonal basis for the eigenspaces.

We have $\mathbf{w}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{w}_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, and $\mathbf{w}_3 = \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$. $\mathbf{w}_1$ and $\mathbf{w}_2$ are NOT orthogonal. We set $\mathbf{v}_1 = \mathbf{w}_1$. We find the component of $\mathbf{w}_2$ orthogonal to $\mathbf{v}_1$.

$$\text{proj}_{\mathbf{v}_1}\mathbf{w}_2 = \frac{\mathbf{w}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \frac{-1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ 0 \end{bmatrix}$$

Our next vector is $\mathbf{v}_2 = \mathbf{w}_2 - \text{proj}_{\mathbf{v}_1}\mathbf{w}_2 = \begin{bmatrix} 1/2 \\ 1/2 \\ 2 \end{bmatrix}$. $\mathbf{w}_3$ is orthogonal to $\mathbf{v}_1$ and $\mathbf{v}_2$, so we set $\mathbf{v}_3 = \mathbf{w}_3$.

- Normalizing the Vectors:

We now have orthogonal eigenvectors $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1/2 \\ 1/2 \\ 2 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$. Normalize each column vector to find possible columns of Q.

$$Q = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \sqrt{2}/6 & -2/3 \\ \frac{1}{\sqrt{2}} & \sqrt{2}/6 & -2/3 \\ 0 & 2\sqrt{2}/3 & 1/3 \end{bmatrix}$$

- We can check that $A = QDQ^T$:

$$A = \begin{bmatrix} 5 & 8 & -4 \\ 8 & 5 & -4 \\ -4 & -4 & -1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \sqrt{2}/6 & -2/3 \\ \frac{1}{\sqrt{2}} & \sqrt{2}/6 & -2/3 \\ 0 & 2\sqrt{2}/3 & 1/3 \end{bmatrix} \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 15 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \sqrt{2}/6 & \sqrt{2}/6 & 2\sqrt{2}/3 \\ -2/3 & -2/3 & 1/3 \end{bmatrix}$$

Example 4: Orthogonally diagonalize the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Answer: $Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \end{bmatrix}$, $D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Homework: Diagonalization of symmetric matrix

1. Orthogonally diagonalize the symmetric matrix A by finding an orthogonal matrix Q and a diagonal matrix D such that $Q^T AQ = D$.

$$A = \begin{bmatrix} 6 & -2 \\ -2 & 9 \end{bmatrix}$$

Answer: $Q = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$, $D = \begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix}$

2. Orthogonally diagonalize the symmetric matrix A by finding an orthogonal matrix Q and a diagonal matrix D such that $Q^T AQ = D$.

$$A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 3 & 1 \end{bmatrix}$$

Answer: $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$, $D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

3. Orthogonally diagonalize the symmetric matrix A by finding an orthogonal matrix Q and a diagonal matrix D such that $Q^T AQ = D$.

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Answer: $Q = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

4. Orthogonally diagonalize the matrix $A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$ by finding an orthogonal matrix Q and a diagonal matrix D such that $Q^T AQ = D$.

Answer: $Q = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -1/\sqrt{6} & 1/\sqrt{3} \\ \frac{1}{\sqrt{2}} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$

5. Orthogonally diagonalize the symmetric matrix A by finding an orthogonal matrix Q and a diagonal matrix D such that $Q^T AQ = D$.

$$A = \begin{bmatrix} 1 & -6 & 4 \\ -6 & 2 & -2 \\ 4 & -2 & -3 \end{bmatrix}$$

Answer: $Q = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$, $D = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 9 \end{bmatrix}$