Problem 13a.
Prove that for every integer $n \geq 1$, $10^n \equiv (-1)^n \pmod{11}$.

Proof. (by induction):
Let $P(n)$ be the statement "$10^n \equiv (-1)^n \pmod{11} \, \forall \, n \in \mathbb{Z} \ni n \geq 1$"

$P(n) \longrightarrow \qquad 10^n \equiv (-1)^n \pmod{11}$

Basis step: $P(1) : 10^1 \stackrel{?}{\equiv} -1^1 (\text{mod } 11)$. True since $11 \mid 10 + 1$
Inductive step: Let $k \in \mathbb{Z} \ni k \geq 1$. Assume $P(k)$: $10^k \equiv (-1)^k \pmod{11}$
By definition of divisibility, $\exists \, r \in \mathbb{Z} \ni 11r = 10^k - (-1)^k$ or $11r = 10^k + 1^k$
[**NTS:** $10^{k+1} \equiv (-1)^{k+1} \pmod{11}$ or $11 \mid 10^{k+1} + 1^{k+1}$]

$$\begin{aligned} 11r &= 10^k - (-1)^k \\ 11r \cdot 10 &= 10^k \cdot 10 - (-1)^k \cdot 10 && \text{by algebra} \\ 110r &= 10^{k+1} + (-1)^{k+1} \cdot 10 \end{aligned}$$

Let $t = 10^k - r$. $t \in \mathbb{Z}$ since $10^k, -1, r \in \mathbb{Z}$.
By substitution, $11t = 10^{k+1} - (-1)^{k+1}$ and by
definition of divisibility, $11 \mid (10^{k+1} - (-1)^{k+1})$

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