Problem 2 WebAssign.
Suppose that $f_0, f_1, f_2, \ldots$ is a sequence as follows.

$f_0 = 5,\ f_1 = 16,$
$f_k = 7f_{k-1} - 10f_{k-2}$ for every integer $k \geq 2$

Prove that $f_n = 3 \cdot 2^n + 2 \cdot 5^n$ for each integer $n \geq 0$

Proof. (by strong induction)
Let the property $P(n)$ be the sentence "$f_n = 3 \cdot 2^n + 2 \cdot 5^n$".

$P(n) \longrightarrow \qquad f_n = 3 \cdot 2^n + 2 \cdot 5^n.$

Basis step; $P(0) : 3 \cdot 2^0 + 2 \cdot 5^0$. True, since $5 = 3 + 2$
$P(1) : 3 \cdot 2^1 + 2 \cdot 5^1$. True, since $16 = 6 + 10$

Inductive step: Let $k \in \mathbb{Z} \ni k \geq 1$.
Assume $P(i)$ is true $\forall i \in \mathbb{Z} \ni 0 \leq i \leq k$. That is $f_i = 3 \cdot 2^i + 2 \cdot 5^i$
Since $k + 1 \geq 2$, $f_{k+1} = 7(f_k) - 10(f_{k-1})$ by the sequence.
[**NTS:** $P(k+1)$ is true, that is $f_{k+1} = 3 \cdot 2^{k+1} + 2 \cdot 5^{k+1}$]

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\begin{aligned}
f_{k+1} &= 7(f_k) - 10(f_{k-1}) \\
&= 7(3 \cdot 2^k + 2 \cdot 5^k) - 10(3 \cdot 2^{k-1} + 2 \cdot 5^{k-1}) && \text{by substitution} \\
&= 21(2^k) + 14(5^k) - 30(2^{k-1}) - 20(5^{k-1}) && \text{by algebra} \\
&= 21(2^k) + 14(5^k) - 15 \cdot 2(2^{k-1}) - 4 \cdot 5(5^{k-1}) \\
&= 21(2^k) + 14(5^k) - 15 \cdot 2^k - 4 \cdot 5^k \\
&= 2^k(21 - 15) + 5^k(14 - 4) \\
&= 2^k(6) + 5^k(10) \\
&= 2^k(2 \cdot 3) + 5^k(5 \cdot 2) \\
&= 3 \cdot 2^{k+1} + 2 \cdot 5^{k+1}
\end{aligned}
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Problem 8a.
Suppose that $h_0, h_1, h_2, \ldots$ is a sequence defined as follows:

$h_0 = 1, h_1 = 2, h_2 = 3,$
$h_k = h_{k-1} + h_{k-2} + h_{k-3}$ for each integer $k \geq 3$.

Prove that $h_n \leq 3^n$ for every integer $n \geq 0$.

Proof. (by strong induction)
Let the property $P(n)$ be the sentence "$h_n \leq 3^n$"

$P(n) \longrightarrow \qquad h_n \leq 3^n$

Basis step; $P(0) : h_0 \leq 3^0$. True, since $1 \leq 3^0$
$P(1) : h_1 \leq 3^1$. True, since $2 \leq 3^1$
$P(2) : h_2 \leq 3^2$. True, since $3 \leq 3^2$

Inductive step: Let $k \in \mathbb{Z} \ni k \geq 2$
Assume $P(i)$ is true for all $0 \leq i \leq k$, that is $h_i \leq 3^i$ for $0 \leq i \leq k$
Since $k + 1 \geq 3$, $h_{k+1} = h_k + h_{k-1} + h_{k-2}$ by the sequence.
[**NTS:** $P(k+1)$ is true, that is $h_{k+1} \leq 3^{k+1}$ since $k \geq 2$ then $k + 1 \geq 3$]

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\begin{aligned}
h_{k+1} &\leq 3^k + 3^{k-1} + 3^{k-2} && \text{by substitution} \\
&\leq 3^3 \cdot 3^{k-3} + 3^2 \cdot 3^{k-3} + 3 \cdot 3^{k-3} && \text{by algebra} \\
&\leq (3^{k-3})(3^3 + 3^2 + 3) \\
&\leq (3^{k-3})(39) \leq 3^{k+1} \\
&\therefore h_{k+1} \leq 3^{k+1}
\end{aligned}
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