Problem 19.
Theorem: For all integers m and n, if m and n have the same parity, then $5m+7n$ is even. Divide into two cases: m and n are both even and m and n are both odd.

Proof. (by cases) Let $m, n \in \mathbb{Z} \ni m, n$ have the same parity
$\exists\, k, l \in \mathbb{Z} \ni m = 2k$ and $n = 2l$ or
$\exists\, k, l \in \mathbb{Z} \ni m = 2k+1$ and $n = 2l+1$

Case I ($m = 2k$ and $n = 2l$):

$$\begin{aligned} 5m + 7n &= 5(2k) + 7(2l) && \text{by substitution} \\ &= 10k + 14l && \text{by algebra} \\ &= 2(5k + 7l) \end{aligned}$$

Let $t = 5k + 7l$.
$t \in \mathbb{Z}$ since $k, l, 5, 7 \in \mathbb{Z}$ and the set of integers are closed under sums of products.
So $5m + 7n = 2t$, where $t \ni \mathbb{Z}$ is even by definition of even.

Case II ($m = 2k+1$ and $n = 2l+1$):

$$\begin{aligned} 5m + 7n &= 5(2k+1) + 7(2l+1) && \text{by substitution} \\ &= 10k + 5 + 14l + 7 && \text{by algebra} \\ &= 10k + 14l + 12 \\ &= 2(5k + 7l + 6) \end{aligned}$$

Let $t = 5k + 7l + 6$.
$t \in \mathbb{Z}$ since $k, l, 5, 6, 7 \in \mathbb{Z}$ and the set of integers are closed under sum of products.
So $5m + 7n = 2t$, where $t \ni \mathbb{Z}$ is even by definition of even.

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Problem 24.
Theorem: For all integers m and n, if $m \mod 5 = 2$ and $n \mod 5 = 1$ then $mn \mod 5 = 2$.

Proof. Let $m, n \in \mathbb{Z} \ni m \mod 5 = 2$ and $n \mod 5 = 1$.
By Q.R. Theorem, $\exists\, d, f \in \mathbb{Z} \ni m = 5d + 2$ and $n = 5f + 1$

$$\begin{aligned} m \times n &= (5d+2)(5f+1) && \text{by substitution} \\ &= 25df + 5d + 10f + 2 && \text{by algebra} \\ &= 5(5df + d + 2f) + 2 \end{aligned}$$

Let $t = 5df + d + 2f$. $t \in \mathbb{Z}$ since $d, f, 5 \in Z$
The set of integers are closed under sums & products
Therefore, $mn = 5t + 2$ where $t \in \mathbb{Z}$ and $0 \leq 2 < 5$.
So $mn \mod 5 = 2$ by Q.R Theorem.

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