Problem 15

Prove by mathematical induction.
$n(n^2+5)$ is divisible by 6, for each integer $n \geq 0$.

Proof. (by induction)
Let the property $P(n)$ be the sentence "$n(n^2+5)$ is divisible by 6".

$P(n) \longrightarrow$ $n(n^2+5)$ is divisible by 6.

Base case; $P(0)$: "$0(0^0+5)$ is divisible by 6". True, $6 \mid 0$ since $6 \cdot 0 = 0$.
Inductive hypothesis: Let $k \in \mathbb{Z} \ni k \geq 0$. Assume $P(k)$, that is $6 \mid k(k^2+5)$.
Then $\exists\, r \in \mathbb{Z} \ni 6r = k(k^2+5)$ or $6r = k^3+5$ by definition of divisibility.
[**NTS:** $6 \mid (k+1)((k+1)^2+5)$]

$$\begin{aligned}
(k+1)((k+1)^2+5) &= (k+1)(k^2+2k+1+5) && \text{by algebra}\\
&= k(k^2+2k+5+1)+1(k^2+2k+6)\\
&= k^3+5k+2k^2+k+k^2+2k+6\\
&= 6r+3k^2+3k+6 && \text{by substitution}\\
&= 6r+6+3k(k+1) && \text{by algebra}
\end{aligned}$$

$k(k+1)$ are consecutive integers, and by the Parity Property (Theorem 4.5.2) k and $k+1$ have opposite parity. Multiplying an odd and an even integer results in an even integer by Theorem 4.3.3. We let $S = k(k+1) \ni S$ is even and by definition of even $\exists\, a \in \mathbb{Z} \ni S = 2a$

$$\begin{aligned}
&= 6r+6+3 \cdot 2a\\
&= 6(r+1+a)
\end{aligned}$$

Let $t \in \mathbb{Z}$. $t \in \mathbb{Z}$ since $r,\, 1, a \in \mathbb{Z}$ by closure of the integers under sums.
$6t = k(k^2+5)$ by substitution, where $t \in \mathbb{Z}$.
Therefore $6 \mid k(k^2+5)$ by definition of divisibility.

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