Problem 22.
Prove the statement by the methods of exhaustion
For each integer n with $1 \leq n \leq 10$, $n^2 - n + 11$ is a prime number

Proof. Let $n \in \mathbb{Z}$ with $1 \leq n \leq 10$

$$\begin{aligned} &1^2 - 1 + 11 = 11 && 6^2 - 6 + 11 = 41 \\ &2^2 - 2 + 11 = 13 && 7^2 - 7 + 11 = 53 \\ &3^2 - 3 + 11 = 17 && 8^2 - 8 + 11 = 67 \\ &4^2 - 4 + 11 = 23 && 9^2 - 9 + 11 = 83 \\ &5^2 - 5 + 11 = 31 && 10^2 - 10 + 11 = 101 \end{aligned}$$

Therefore, for every integer n, if n is between 1 and 10, inclusive, then $n^2 - n + 11$ is a prime number.

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Problem 31b.
Fill in the blanks in the proof of the theorem

Theorem: Whenever n is an odd integer, $5n^2 + 7$ is even.
Proof: Suppose n is any *[particular but arbitrarily chosen]* odd integer.
[We must show that $5n^2 + 7$ *is even]*
By definition of odd, $n = \underline{2k+1}$ for some integer k.

Then

$$\begin{aligned} 5n^2 + 7 &= \underline{5(2k+1)^2 + 7} && \text{by substitution} \\ &= 5(4k^2 + 4k + 1) + 7 \\ &= 20k^2 + 20k + 12 \\ &= 2(10k^2 + 10k + 6) && \text{by algebra} \end{aligned}$$

Let $t = \underline{10k^2 + 10k + 6}$. Then t is an integer because products and sums of integers are integers.
Hence $5n^2 + 7 = 2t$, where t is an integer, and thus $\underline{5n^2 + 7 \text{ is even}}$ by definition of even *[as was to be shown]*.

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