Problem 18.
Prove if true and disprove if false.

a. Prove that for every integer a if a^3 is even then a is even.

Proof. (by contraposition) If a is odd then a^3 is odd.
Let a be odd, then $a = 2k+1$ for some $k \in \mathbb{Z}$ by definition of odd.

$$\begin{aligned} a^3 &= (2k+1)^3 && \text{by substitution} \\ a^3 &= 2(4k^3+6k^2+3k)+1 && \text{by algebra} \end{aligned}$$

Let $t = 4k^3 + 6k^2 + 3k$.
$t \in \mathbb{Z}$ since $4, 6, 3, \&\ k \in \mathbb{Z}$ because integers are closed under addition & multiplication.
$a^3 = 2t + 1$ is odd by definition of odd. Therefore, a^3 is odd.

■

b. Prove that $\sqrt[3]{2}$ is irrational

Proof. (by contradiction)
$\sqrt[3]{2}$ is rational.
By definition of rational, $\exists\, a, b \in \mathbb{Z}$ with $b \neq 0$ and WLOG, a and b have no common factor

$$\begin{aligned} \sqrt[3]{2} &= \frac{a}{b} && \text{by substitution} \\ 2 &= \frac{a^3}{b^3} && \text{by algebra} \\ 2(b^3) &= \frac{a^3}{b^3}(b^3) \\ 2(b^3) &= a^3 \end{aligned}$$

Let $h = b^3$. $h \in \mathbb{Z}$ because integers are closed under multiplication.
So a^3 is even by definition of even. By part (a), since a^3 is even, a is even.
Then $\exists\, k \in \mathbb{Z} \ni a = 2k$ by definition of even.

$$\begin{aligned} 2b^3 &= (2k)^3 && \text{by substitution} \\ b^3 &= 4k^3 && \text{by algebra} \\ b^3 &= 2(2k^3) && \text{by definition of even} \end{aligned}$$

Let $t = 2k^3$. $t \in \mathbb{Z}$ since $k, 2 \in \mathbb{Z}$ as integers are closed under multiplication.
So b^3 is even by definition of even. By part (a), since b^3 is even, b is even.
Therefore, this contradicts the assumption that a and b doesn't have a common factor since a and b are both even.
So this contradicts the assumption that $\sqrt[3]{2}$ is a rational

$\Rightarrow\!\Leftarrow$ ■

Problem 23.
Prove that for any integer a, $9 \nmid (a^2 - 3)$

Proof. (by contradiction)
Suppose not. $\exists\, a \in \mathbb{Z} \ni 9 \mid (a^2 - 3)$
By the definition of divisibility, $\exists\, d \in \mathbb{Z} \ni a^2 - 3 = 9d$

$$\begin{aligned} a^2 &= 9d + 3 && \text{by algebra} \\ a^2 &= 3(3d + 1) \end{aligned}$$

Let $k = 3d + 1$. $k \in \mathbb{Z}$ since $3, d, \& \, 1 \in \mathbb{Z}$ by the closure of addition & multiplication
So $a^2 = 3k$ by substitution, where $k \in \mathbb{Z}$. Therefore, $3 \mid a^2$ by definition of divisibility.
By 19.b, $\forall\, a \in \mathbb{Z}, \text{if } 3 \mid a^2 \to 3 \mid a$.
Then by definition of divisibility, $\exists\, f \in \mathbb{Z} \ni a = 3f$

$$\begin{aligned} (3f)^2 &= 3(3d + 1) && \text{by substitution} \\ \frac{9f^2}{3} &= \frac{3(3d + 1)}{3} && \text{by algebra} \\ 3f^2 - 3d &= 1 \\ 3(f^2 - d) &= 1 \end{aligned}$$

Let $x = f^2 - d$ and $x \in \mathbb{Z}$ because $f, d \;\&\; -1 \in \mathbb{Z}$ by the closure of integers of addition and multiplication.
So $3x = 1$ by substitution. This implies $3 \mid 1$ by definition of divisibility, which is false.
Therefore, $9 \mid (a^2 - 3)$ must be also be false.

$\Rightarrow\!\Leftarrow$ ■