Problem 9.
For every integer $n \geq 3$,
$4^3 + 4^4 + 4^5 + \cdots + 4^n = \dfrac{4(4^n - 16)}{3}$

Proof. (by induction)

$P(n) : 4^3 + 4^4 + 4^5 + \cdots + 4^n = \dfrac{4(4^n - 16)}{3}$

Base case; $P(3)$: " $4^3 \overset{?}{=} \dfrac{4 \cdot 48}{3}$ " $64 \overset{\checkmark}{=} 64$. True for $P(3)$

Inductive step: Let $k \in \mathbb{Z} \ni k \geq 3$

Assume $P(k)$. That is $4^3 + 4^4 + 4^5 + \cdots + 4^k = \dfrac{4(4^k - 16)}{3}$

[NTS $4^3 + 4^4 + 4^5 + \cdots + 4^k + 4^{k+1} = \dfrac{4(4^{k+1} - 16)}{3}$]

$$
\begin{aligned}
4^3 + \cdots + 4^k + 4^{k+1} &= \frac{4(4^k - 16)}{3} + 4^{k+1} && \text{by inductive hypothesis} \\
&= \frac{4^{k+1} - 64 + 3(4^{k+1})}{3} && \text{by algebra} \\
&= \frac{1(4^{k+1}) - 64 + 3(4^{k+1})}{3} \\
&= \frac{4(4^{k+1}) - 64}{3} \\
&= \frac{4(4^{k+1} - 16)}{3}
\end{aligned}
$$

$\therefore P(n)$ holds for $n = k + 1$ and the proof of the induction step is complete.
Conclusion: By the principle of induction, $P(n)$ is true for all $n \geq 3 \in \mathbb{Z}$.

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Problem 11.

$1^3 + 2^3 + \cdots + n^3 = \left[\dfrac{n(n+1)}{2}\right]^2$, for every integer $n \geq 1$.

Proof. (by induction)

$P(n) : 1^3 + 2^3 + \cdots + n^3 = \left[\dfrac{n(n+1)}{2}\right]^2$

Base case; $P(1)$: " $1^3 \stackrel{?}{=} \left[\dfrac{1(2)}{2}\right]^2$ " $1 \stackrel{\checkmark}{=} 1$. True for $P(1)$

Inductive step: Let $k \in \mathbb{Z} \ni k \geq 1$

Assume $P(k)$. That is $1^3 + 2^3 + \cdots + k^3 = \left[\dfrac{k(k+1)}{2}\right]^2$

[NTS $1^3 + \cdots + k^3 + (k+1)^3 = \left[\dfrac{(k+1)(k+2)}{2}\right]^2$

or equivalently $\dfrac{(k+1)(k+2)(k+1)(k+2)}{4}$]

$$\begin{aligned} 1^3 + \cdots + k^3 + (k+1)^3 &= \left[\frac{k(k+1)}{2}\right]^2 + (k+1)^3 && \text{by inductive hypothesis} \\ &= \frac{k^4 + 2k^3 + k^2 + 4(k^3 + 3k^2 + 3k + 1)}{4} && \text{by algebra} \\ &= \frac{k^4 + 6k^3 + 13k^2 + 12k + 4}{4} \end{aligned}$$

$\therefore P(n)$ holds for $n = k + 1$ and the proof of the induction step is complete.
Conclusion: By the principle of induction, $P(n)$ is true for all $n \geq 1 \in \mathbb{Z}$.

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Problem 17.

$$\prod_{i=0}^{n}\left(\frac{1}{2i+1}\cdot\frac{1}{2i+2}\right)=\frac{1}{(2n+2)!},\text{ for every integer } n\geq 0.$$

Proof. (by induction)

Base case; $P(0)$: " $\dfrac{1}{2(0)+1}\cdot\dfrac{1}{2(0)+2}\stackrel{?}{=}\dfrac{1}{(2(0)+2)!}$ " $\dfrac{1}{2}\stackrel{\checkmark}{=}\dfrac{1}{2}$. True for $P(0)$

Inductive step: Let $k\in\mathbb{Z}\ni k\geq 0$

Assume $P(k)$, that is $\displaystyle\prod_{i=0}^{k}\left(\frac{1}{2i+1}\cdot\frac{1}{2i+2}\right)=\frac{1}{(2k+2)!}$

[NTS $\displaystyle\prod_{i=0}^{k+1}\left(\frac{1}{2i+1}\cdot\frac{1}{2i+2}\right)=\frac{1}{(2(k+1)+2)!}$ or $\dfrac{1}{(2k+4)!}$]

$$\begin{aligned}
&=\prod_{i=1}^{k+1}\left(\frac{1}{2i+1}\cdot\frac{1}{2i+2}\right)\left(\frac{1}{2(k+1)+1}\cdot\frac{1}{2(k+1)+2}\right)\\
&=\left(\frac{1}{(2k+2)!}\right)\left(\frac{1}{2(k+1)+1}\cdot\frac{1}{2(k+1)+2}\right) && \text{by inductive hypothesis}\\
&=\left(\frac{1}{2k+2+2}\cdot\frac{1}{2k+2+1}\right)\left(\frac{1}{(2k+2)!}\right)\\
&=\frac{1}{(2k+2+2)!}\\
&=\frac{1}{(2k+4)!}
\end{aligned}$$

$\therefore P(n)$ holds for $n=k+1$ and the proof of the induction step is complete.
Conclusion: By the principle of induction, $P(n)$ is true for all $n\geq 0\in\mathbb{Z}$.

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