Problem 20.
For all real numbers x and y, $\lceil xy \rceil = \lceil x \rceil \cdot \lceil y \rceil$
False

Proof. Let x = 1.2 and y = 1.3

$$\begin{aligned}
\lceil x \cdot y \rceil &= \lceil 1.2 \cdot 1.3 \rceil \\
\lceil 1.56 \rceil &= 2 \\
\lceil x \rceil + \lceil y \rceil &= \lceil 1.2 \rceil + \lceil 1.3 \rceil \\
2 + 2 &= 4
\end{aligned}$$

$2 \neq 4$, therefore the statement is false

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Problem 24.
For any integer m and any real number x, if x is not an integer, then $\lfloor x \rfloor + \lfloor m - x \rfloor = m - 1$
True

Proof. Let $m \in \mathbb{Z}$ and $x \in \mathbb{R} \ni x \notin \mathbb{Z}$
Let n = $\lfloor x \rfloor$. Then, by definition of floor, $n \in \mathbb{Z} \ni n \leq x < n + 1$
Since $x \notin \mathbb{Z}, x \neq n$, so $n < x < n + 1$

$$m - n > m - x > m - n - 1 \qquad \text{by algebra}$$

$-1, n, m \in \mathbb{Z}$ by closure of integers, so $m - n, m - n - 1 \in \mathbb{Z}$.
Since $m - n$, $m - n - 1$ are consecutive integers,

$$\begin{aligned}
\lfloor m - x \rfloor &= m - n - 1 && \text{by definition of floor} \\
\lfloor x \rfloor + \lfloor m - x \rfloor &= n + m - n - 1 && \text{by substitution} \\
\lfloor x \rfloor + \lfloor m - x \rfloor &= m - 1 && \text{by algebra}
\end{aligned}$$

Therefore, $\lfloor x \rfloor + \lfloor m - x \rfloor = m - 1$

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Problem 29.
For any odd integer n,

$$\left\lceil \frac{n^2}{4} \right\rceil = \frac{n^2+3}{4}$$

True

Proof. Let n be odd
$\exists\, k \in \mathbb{Z} \ni n = 2k+1$ by the definition of odd

$$n^2 = (2k+1)^2 \qquad \text{by substitution}$$

$$n^2 = 4k^2+4k+1 \qquad \text{by algebra}$$

$$\frac{n^2}{4} = \frac{4k^2+4k+1}{4}$$

$$\frac{n^2}{4} = k^2+k+\frac{1}{4}$$

$$k^2+k < k^2+k+\frac{1}{4} < k^2+k+1 \qquad \text{by substitution}$$

By definition of ceiling, since $k^2+k+1 \in \mathbb{Z}$ and $k^2+k \in \mathbb{Z}$,
k^2+k+1 and k^2+k are consecutive integers

$$\left\lceil \frac{n^2}{4} \right\rceil = k^2+k+1 \qquad \left[k^2+k+1 \text{ is the integer above } \frac{n^2}{4}\right]$$

$$\begin{aligned} \frac{n^2+3}{4} &= \frac{(4k^2+4k+1)+3}{4} && \text{by substitution} \\ &= \frac{4k^2+4k+4}{4} && \text{by algebra} \\ &= k^2+k+1 \end{aligned}$$

So $\left\lceil \frac{n^2}{4} \right\rceil = \frac{n^2+3}{4}$ by the definition of ceiling

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