

Problem 20.

For all real numbers x and y , $\lceil xy \rceil = \lceil x \rceil \cdot \lceil y \rceil$

False

Proof. Let $x = 1.2$ and $y = 1.3$

$$\lceil x \cdot y \rceil = \lceil 1.2 \cdot 1.3 \rceil$$

$$\lceil 1.56 \rceil = 2$$

$$\lceil x \rceil + \lceil y \rceil = \lceil 1.2 \rceil + \lceil 1.3 \rceil$$

$$2 + 2 = 4$$

$2 \neq 4$, therefore the statement is false

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Problem 24.

For any integer m and any real number x , if x is not an integer, then $\lfloor x \rfloor + \lfloor m - x \rfloor = m - 1$

True

Proof. Let $m \in \mathbb{Z}$ and $x \in \mathbb{R} \ni x \notin \mathbb{Z}$

Let $n = \lfloor x \rfloor$. Then, by definition of floor, $n \in \mathbb{Z} \ni n \leq x < n + 1$

Since $x \notin \mathbb{Z}$, $x \neq n$, so $n < x < n + 1$

$$m - n > m - x > m - n - 1 \quad \text{by algebra}$$

$-1, n, m \in \mathbb{Z}$ by closure of integers, so $m - n, m - n - 1 \in \mathbb{Z}$.

Since $m - n, m - n - 1$ are consecutive integers,

$$\lfloor m - x \rfloor = m - n - 1 \quad \text{by definition of floor}$$

$$\lfloor x \rfloor + \lfloor m - x \rfloor = n + m - n - 1 \quad \text{by substitution}$$

$$\lfloor x \rfloor + \lfloor m - x \rfloor = m - 1 \quad \text{by algebra}$$

Therefore, $\lfloor x \rfloor + \lfloor m - x \rfloor = m - 1$

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Problem 29.

For any odd integer n ,

$$\left\lceil \frac{n^2}{4} \right\rceil = \frac{n^2 + 3}{4}$$

True

Proof. Let n be odd

$\exists k \in \mathbb{Z} \ni n = 2k + 1$ by the definition of odd

$$n^2 = (2k + 1)^2 \quad \text{by substitution}$$

$$n^2 = 4k^2 + 4k + 1 \quad \text{by algebra}$$

$$\frac{n^2}{4} = \frac{4k^2 + 4k + 1}{4}$$

$$\frac{n^2}{4} = k^2 + k + \frac{1}{4}$$

$$k^2 + k < k^2 + k + \frac{1}{4} < k^2 + k + 1 \quad \text{since } 0 < \frac{1}{4} < 1$$

By definition of ceiling, since $k^2 + k + 1 \in \mathbb{Z}$ and $k^2 + k \in \mathbb{Z}$,
 $k^2 + k + 1$ and $k^2 + k$ are consecutive integers

$$\left\lceil \frac{n^2}{4} \right\rceil = k^2 + k + 1 \quad \left[k^2 + k + 1 \text{ is the integer above } \frac{n^2}{4} \right]$$

$$\frac{n^2 + 3}{4} = \frac{(4k^2 + 4k + 1) + 3}{4} \quad \text{by substitution}$$

$$= \frac{4k^2 + 4k + 4}{4} \quad \text{by algebra}$$

$$= k^2 + k + 1$$

$$\text{So } \left\lceil \frac{n^2}{4} \right\rceil = \frac{n^2 + 3}{4} \text{ by substitution}$$

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