

Problem 9.

For every integer $n \geq 3$,

$$4^3 + 4^4 + 4^5 + \cdots + 4^n = \frac{4(4^n - 16)}{3}$$

Proof. (by induction)

$$P(n) : 4^3 + 4^4 + 4^5 + \cdots + 4^n = \frac{4(4^n - 16)}{3}$$

Base case; $P(3)$: “ $4^3 \stackrel{?}{=} \frac{4 \cdot 48}{3}$ ” $64 \stackrel{\checkmark}{=} 64$. True for $P(3)$

Inductive step: Let $k \in \mathbb{Z} \ni k \geq 3$

Assume $P(k)$. That is $4^3 + 4^4 + 4^5 + \cdots + 4^k = \frac{4(4^k - 16)}{3}$

$$[\text{NTS } 4^3 + 4^4 + 4^5 + \cdots + 4^k + 4^{k+1} = \frac{4(4^{k+1} - 16)}{3}]$$

$$\begin{aligned} 4^3 + \cdots + 4^k + 4^{k+1} &= \frac{4(4^k - 16)}{3} + 4^{k+1} && \text{by inductive hypothesis} \\ &= \frac{4^{k+1} - 64 + 3(4^{k+1})}{3} && \text{by algebra} \\ &= \frac{1(4^{k+1}) - 64 + 3(4^{k+1})}{3} \\ &= \frac{4(4^{k+1}) - 64}{3} \\ &= \frac{4(4^{k+1} - 16)}{3} \end{aligned}$$

$\therefore P(n)$ holds for $n = k + 1$ and the proof of the induction step is complete.

Conclusion: By the principle of induction, $P(n)$ is true for all $n \geq 3 \in \mathbb{Z}$. ■

Problem 11.

$$1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2} \right]^2, \text{ for every integer } n \geq 1.$$

Proof. (by induction)

$$P(n) : 1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2} \right]^2$$

Base case; $P(1)$: “ $1^3 \stackrel{?}{=} \left[\frac{1(2)}{2} \right]^2$ ” $1 \stackrel{\checkmark}{=} 1$. True for $P(1)$

Inductive step: Let $k \in \mathbb{Z} \ni k \geq 1$

$$\text{Assume } P(k). \text{ That is } 1^3 + 2^3 + \cdots + k^3 = \left[\frac{k(k+1)}{2} \right]^2$$

$$[\text{NTS } 1^3 + \cdots + k^3 + (k+1)^3 = \left[\frac{(k+1)(k+2)}{2} \right]^2]$$

$$\text{or equivalently } \frac{(k+1)(k+2)(k+1)(k+2)}{4}]$$

$$\begin{aligned} 1^3 + \cdots + k^3 + (k+1)^3 &= \left[\frac{k(k+1)}{2} \right]^2 + (k+1)^3 && \text{by inductive hypothesis} \\ &= \frac{k^4 + 2k^3 + k^2 + 4(k^3 + 3k^2 + 3k + 1)}{4} && \text{by algebra} \\ &= \frac{k^4 + 6k^3 + 13k^2 + 12k + 4}{4} \end{aligned}$$

$\therefore P(n)$ holds for $n = k + 1$ and the proof of the induction step is complete.

Conclusion: By the principle of induction, $P(n)$ is true for all $n \geq 1 \in \mathbb{Z}$. ■

Problem 17.

$$\prod_{i=0}^n \left(\frac{1}{2i+1} \cdot \frac{1}{2i+2} \right) = \frac{1}{(2n+2)!}, \text{ for every integer } n \geq 0.$$

Proof. (by induction)

Base case; $P(0)$: “ $\frac{1}{2(0)+1} \cdot \frac{1}{2(0)+2} \stackrel{?}{=} \frac{1}{(2(0)+2)!}$ ” $\frac{1}{2} \stackrel{\checkmark}{=} \frac{1}{2}$. True for $P(0)$

Inductive step: Let $k \in \mathbb{Z} \ni k \geq 0$

Assume $P(k)$, that is $\prod_{i=0}^k \left(\frac{1}{2i+1} \cdot \frac{1}{2i+2} \right) = \frac{1}{(2k+2)!}$

$$[\text{NTS } \prod_{i=0}^{k+1} \left(\frac{1}{2i+1} \cdot \frac{1}{2i+2} \right) = \frac{1}{(2(k+1)+2)!} \text{ or } \frac{1}{(2k+4)!}]$$

$$= \prod_{i=1}^{k+1} \left(\frac{1}{2i+1} \cdot \frac{1}{2i+2} \right) \left(\frac{1}{2(k+1)+1} \cdot \frac{1}{2(k+1)+2} \right)$$

$$= \left(\frac{1}{(2k+2)!} \right) \left(\frac{1}{2(k+1)+1} \cdot \frac{1}{2(k+1)+2} \right)$$

by inductive hypothesis

$$= \left(\frac{1}{2k+2+2} \cdot \frac{1}{2k+2+1} \right) \left(\frac{1}{(2k+2)!} \right)$$

$$= \frac{1}{(2k+2+2)!}$$

$$= \frac{1}{(2k+4)!}$$

$\therefore P(n)$ holds for $n = k + 1$ and the proof of the induction step is complete.

Conclusion: By the principle of induction, $P(n)$ is true for all $n \geq 0 \in \mathbb{Z}$.

■