Problem 13.
S = The product of any irrational number and any nonzero rational number is irrational
a. Write a negation for S.
b. Prove S by contradiction.

$\exists\, d \notin \mathbb{Q}$ and $f \in \mathbb{Q} \ni d \cdot f \in \mathbb{Q}$

Proof. (by contradiction) Suppose not.
Then, there exists an irrational number, call it d and a nonzero rational number, call it f such that the product of d and f is a rational.
By definition of rational,
$f = h/j$ and $d \cdot f = k/l$ for some integers h, j, k, and l with $h = j \neq 0$ and $l \neq 0$ since f is a nonzero rational number.

$$\begin{aligned} d \cdot \frac{h}{j} &= \frac{k}{l} && \text{by substitution} \\ d &= \frac{kj}{lh} && \text{by algebra} \end{aligned}$$

$lh \neq 0$ by zero-product property so d is a ratio of integers with a nonzero denominator.
So $d \in \mathbb{Q}$, hence the supposition is false and the given statement is true

$\Rightarrow\Leftarrow$ ∎

Problem 22.
Theoerm: For every real number r, if r^2 is irrational then r is irrational.
Write what you would suppose and what you would need to show to prove this statement by:

a. contradiction
Suppose not.
That is suppose $\exists\, r \in \mathbb{R} \ni r^2 \notin \mathbb{Q}$ and $r \in \mathbb{Q}$.
We need to show that $r \in \mathbb{Q}$.

b. contraposition
Suppose $r \in \mathbb{Q}$.
We need to show that $r^2 \in \mathbb{Q}$.

Problem 24.
Prove by contraposition and contradiction
Theoerm: The reciprocal of any irrational number is irrational.

Proof. (by contradiction) Suppose not.
Then $\exists\, k \notin \mathbb{Q} \ni \dfrac{1}{k} \in \mathbb{Q}$
By definition of a rational, $1/k = a/b \ni a, b \in \mathbb{Z}$ with $b \neq 0$

$$\begin{aligned} \frac{1}{k}(b) &= \frac{a}{b}(b) && \text{by algebra} \\ \frac{b}{k} &= a \\ k &= \frac{b}{a} \end{aligned}$$

Since b and k are both non zeroes, this implies that a is non zero
a and b are both integers and k is a ratio of integers with a nonzero denominator.
Hence k is a rational.
$\therefore$ the supposition is false and the given statement is true

$\Rightarrow\!\Leftarrow$ ■

Proof. (by contraposition)
Let $r \in \mathbb{R} \ni \frac{1}{r} \in \mathbb{Q}$
By definition of rational, $r = \dfrac{a}{b}$ for some integer a and b with $b \neq 0$

$$\begin{aligned} \frac{1}{r} &= \frac{1}{a/b} \\ r &= \frac{b}{a} && \text{by algebra} \end{aligned}$$

a and b are both integers and r is a ratio of integers with a nonzero denominator,
Hence r is a rational.

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