Shortest Path Problem/Algorithm

$G(V, E) \rightarrow$ weighted graph
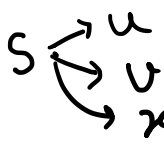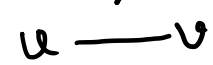
Weight of a path P, $\omega(P) \rightarrow$ is the sum of weights of its constituent edges

$$P = \langle v_0, v_1, \ldots, v_K \rangle$$
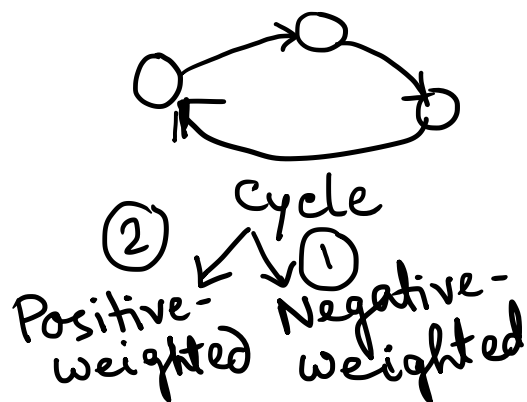
$$\omega(P) = \sum_{i=1}^{k} \omega(v_{i-1}, v_i)$$

Shortest path weight $\delta(u, v)$ from u to v is defined as
$$\delta(u, v) = \begin{cases} \min\{\omega(P) : u \xrightarrow{P} v\}, & \text{if there is a path from } u \text{ to } v \\ \infty, & \text{otherwise} \end{cases}$$

Edge weights can represent cost, time, ...

Variants of shortest path problem:

1. Single source shortest-path problem $S \begin{smallmatrix} \nearrow u \\ \rightarrow v \\ \searrow x \end{smallmatrix}$

2. Single destination " " " $\begin{smallmatrix} u \\ v \\ x \end{smallmatrix} \rightarrow t$

3. Single pair " " " $u \longrightarrow v$

4. All-pairs " " "

If a graph is having a cycle, then what will happen to its shortest path?

② ← ↗ ↘ ①

Cycle

② Positive-weighted   ① Negative-weighted

① Negative - weighted cycle :

- Source, S vertex

If a graph contains negative weighted cycle in its shortest path, then the shortest path weight is not well-defined.

i.e. the graph will not have negative weighted cycle in its shortest path.

② Positive - weighted cycle : A shortest path can't contain a positive - weighted cycle also. Removing the cycle from the path produces a path with same source and destination vertices and a lower path weight.

Conclusion: A graph can't contain a cycle (positive/ negative - weighted) in its shortest-path.