

All-pairs Shortest Path Problem

$W_{n \times n}$ $G \rightarrow n$ no. of vertices

$$w_{ij} = \begin{cases} 0, & \text{if } i=j \\ \text{the weight of directed edge } (i,j), & \text{if } i \neq j \text{ and } (i,j) \in E \\ \infty, & \text{if } i \neq j \text{ and } (i,j) \notin E \end{cases}$$

Negative-weighted edges are allowed, but the i/p graph will not contain any -ve-weighted cycle.

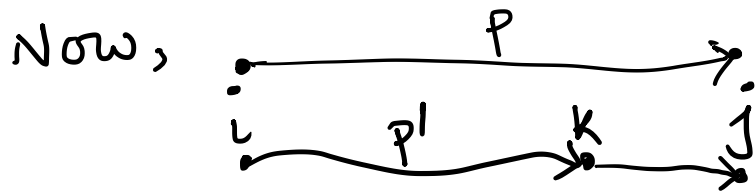
Structure of a shortest path for all-pairs shortest path problem:

$G(V, E)$ [All subpaths of a shortest path are also shortest paths]



$a-d \Rightarrow$ shortest path
 $a-b, b-c, c-d$

\Downarrow
 shortest paths



$P \rightarrow$ Shortest path between i & j .

\rightarrow Let, m edges (atmost)

Assuming, there is no -ve weighted cycles, m is finite.

Now, if $i=j$, then P has weight 0 and no edges. But, if $i \neq j$, we can decompose path P into

$i \xrightarrow{P'} k \rightarrow j$, so P' now contains atmost $m-1$ edges. So, P' is a shortest path from i to k ,
 shortest path, $\delta(i, j) = \delta(i, k) + w_{kj}$

-ve-weighted Floyd-Warshall Algorithm
edges may be present but All-pairs shortest path problem
(dynamic programming)
-ve-weighted cycles will not be there.

Structure of a shortest path:

path, $P = \langle v_1, v_2, \dots, v_k \rangle$

As this algorithm considers intermediate vertices of a shortest path, so, Any vertex of P , other than v_1 or v_k is called intermediate vertex.

$G \rightarrow V = \{1, 2, \dots, n\}$

Considering a subset of vertices $\{1, 2, \dots, k\}$

Considering a pair of vertices $i, j \in V$ and all paths from i to j whose intermediate vertices are from $\{1, 2, \dots, k\}$ and let P be a minimum-weight path among them. Depending on whether or not k is an intermediate vertex of path P , two possible cases.

- ① If k is not an intermediate vertex of path P , then all intermediate vertices of path P are in the set $\{1, 2, \dots, k-1\}$. Thus a shortest path from vertex i to vertex j with all intermediate vertices in the set $\{1, 2, \dots, k-1\}$ is also a shortest path from i to j with all intermediate vertices in the set $\{1, 2, \dots, k\}$

② If k is an intermediate vertex of path P , then we can decompose P like $i \xrightarrow{P_1} k \xrightarrow{P_2} j$. According to the theorem, P_1 is a shortest path from i to k with all intermediate vertices in the set $\{1, 2, \dots, k-1\}$. Also, P_2 is a shortest path from k to j with all intermediate vertices in the set $\{1, 2, \dots, k-1\}$.