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Theoretische Informatik
Wintersemester 2024/2025
25.09.2024

Übungsblatt 0

(Besprechung am10.10.2024)

Don't be frustrated if you do not manage to solve all or most of the following exercises, but please try. This will take time and practice (that's why you should try). Nonetheless, it is important that we familiarize ourselves with proving mathematical statements and working with mathematical precision. Use the hints on the last page.

1. **Sets**
   (a) Let $A, B \subseteq \mathbb{N}$. We define $A + B$ to be $\{a + b \mid a \in A, b \in B\}$ and $A \cdot B$ to be $\{a \cdot b \mid a \in A, b \in B\}$. Thus, e.g., $\{2, 3, 6\} \cdot \{2, 4\} = \{4, 6, 8, 12, 24\}$. Which numbers do the following sets consist of (note that the complement is to be understood in relation to $\mathbb{N}$ as base set)?
      i. $\{2\} \cdot \mathbb{N}$
      ii. $\{1\} + \mathbb{N}$
      iii. $(\{2\} \cdot \mathbb{N}) + \{1\}$
      iv. $\overline{\{1\}} \cdot \overline{\{1\}}$
      v. $\overline{\overline{\{1\}} \cdot \overline{\{1\}}}$
      vi. $\overline{\overline{\{1\}} \cdot \overline{\{1\}}} \cap \overline{\{1\}}$
   (b) Prove that for all sets A it holds $\overline{\overline{A}} = A$ (i.e., $\overline{\overline{A}} \subseteq A$ and $\overline{\overline{A}} \supseteq A$). Consider the two inclusions $\subseteq$ and $\supseteq$ separately.
   (c) Prove $\overline{A \cup B} = \overline{A} \cap \overline{B}$ for all sets A and B. Again, argue for the two inclusions $\subseteq$ and $\supseteq$ separately.

2. **Properties of functions**
   Specify a function with the required properties in each case, or argue that such a function does not exist. For each function, also state its graph as a set.
   (a) surjective, non-injective $f_1 : \{1\} \to \{0, 1\}$
   (b) total, non-surjective $f_2 : \{0, 1, 2\} \to \{1, 3\}$
   (c) non-total $f_3 : \{5\} \to \{0\}$
   (d) bijective $f_4 : \{0, 1\} \to \{0, 1, 2\}$
   (e) total $f_5 : \emptyset \to \{0, 1\}$
   (f) surjective, non-injective $f_6 : \{0, 1, 2\} \to \{0, 1, 2\}$

3. **Summation formula for odd numbers**
   Prove the following statement using complete induction over n: For all $n \in \mathbb{N}^+$ it holds

$$\sum_{i=1}^{n} (2i - 1) = n^2.$$

4. **Fibonacci sequence**
   The Fibonacci sequence is defined as: $\mathrm{fib}(0) = 1$, $\mathrm{fib}(1) = 1$ and for all $n \geq 2$ we define

$\text{fib}(n) = \text{fib}(n-1) + \text{fib}(n-2)$. Prove the following statement using complete induction over $n \in \mathbb{N}$:

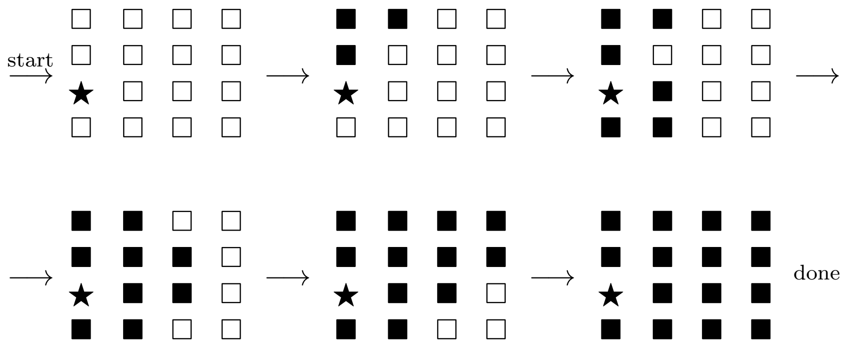
$$\text{fib}(n) = \frac{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}}{\sqrt{5} \cdot 2^{n+1}}.$$

5. **Property 2.3**
   Prove the following statement: Each $n \in \mathbb{N}^+$ can be represented in exactly one way as

$$n = \sum_{i=0}^{m} a_i \cdot 2^i$$

   with $m \in \mathbb{N}$, $a_m = 1$ and $a_0, \ldots, a_{m-1} \in \{0, 1\}$.

6. **Jigsaw puzzle**
   For $n \in \mathbb{N}^+$ let a quadratic grid with $2^n \times 2^n$ cells be given. A $\bigstar$ tile is placed on one of the cells. Is it now possible to fill the entire grid with tiles of the form ▄█? You have an infinite number of these tiles available and you can also rotate them. Example:



7. **Winning strategies for placing beer mats**
   Two players play the following game on an initially empty table (imagine it to be circular, oval, or rectangular):

   The players take turns to place beer mats (imagine these to be circular, oval, or rectangular) of uniform shape and size anywhere on the table. No two beer mats may overlap. The first player to be unable to place any more beer mats in the manner described above loses.

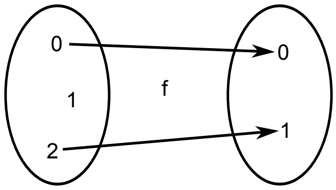
   Find a winning strategy for the player who starts.

8. **Hiking monk**
   A monk is hiking in the mountains. He climbs up a mountain on one day starting at precisely 8AM. He stays overnight and the next day he takes the very same way back also starting at 8AM. Is there any place on the way that the monk reaches at the very same time on both days?

Hints

Exercise 1:
Intuitively, $A + B$ is obtained as follows: take a number of A and ad each number of B to it. All numbers obtained this way are put into a set. Then continue with the next number of A. Note that sets to not contain duplicates.

Exercise 2:
You may describe functions with drawings as in the following. Also specify the function's graph formally.



Exercise 3:
You can orientate towards the example for the complete induction in the lecture notes. You also need a binomial formula.

Exercise 4:
When proving the induction step you should assume that the equation not only holds for $n = k$, but for $n = 0, \dots, k$. Think about why it is ok to make this assumption.

Exercise 5:
This is about binary representation of numbers. An example: 1010 represents $0 \cdot 2^0 + 1 \cdot 2^1 + 0 \cdot 2^2 + 1 \cdot 2^3 = 0+2+0+8 = 12$. You have to prove two things. First you have to show that each natural number can be represented in binary. Think about what appending a 0 or a 1 to a binary representation does with the associated number (compare: what does it do in decimal representation?). Prove the statement using induction.
Next you have to show that there is only one representation with the described properties. Prove the statement by contradiction, i.e., assume that the statement is wrong and show that this leads to contradictions. Assume that there is a number n that has two different binary representations. Take the least such number and then prove that there is an even lower number with two different binary representations.

Exercise 6:
Use induction. Think about how you can make the induction hypothesis applicable.

Exercise 7:
The winning strategy is easy and short to describe and once you know it, it will not be difficult to understand why it works. But finding it, is another matter...

Exercise 8:
This is an exercise that has nothing to do with the contents we will be dealing with in this lecture. It is just for fun and is a good example for typical mathematical thinking.

Extra tasks

1. **Enjoy**
   Enjoy the start of the semester!

Solutions

Solution for task 1:

(a) (i) $\{2n \mid n \in \mathbb{N}\}$
(ii) $\mathbb{N}^+$
(iii) $\{2n+1 \mid n \in \mathbb{N}\}$
(iv) $\mathbb{N} - (\mathbb{P} \cup \{1\})$
(v) $\mathbb{P} \cup \{1\}$
(vi) $\mathbb{P}$

(b) The step size of the individual steps when conducting a proof is always subjective. The second solution proceeds in even smaller steps and is very close to the definition.

Solution 1

$\overline{\overline{A}} \subseteq A$: Let $a \in \overline{\overline{A}}$. Then by the definition of the complement, $a \notin \overline{A}$. Applying the definition of the complement again, we obtain $a \in A$.

$\overline{\overline{A}} \supseteq A$: Let $a \in A$. Then by the definition of the complement, $a \notin \overline{A}$. Applying the definition of the complement again, we have $a \in \overline{\overline{A}}$.

Solution 2 If we go even more strictly over the definition, we can prove the statement as follows. Let's denote the base set by S.

$$\begin{aligned}
\overline{\overline{A}} &= S - \overline{A} \\
&= S - (S - A) \\
&= S - \{s \mid s \in S \text{ and } s \notin A\} \\
&\overset{(*)}{=} S - \{s \mid s \notin A\} \\
&= \{s \mid s \in S \text{ and } s \in A\} \\
&\overset{(*)}{=} \{s \mid s \in A\} \\
&= A
\end{aligned}$$

At the places $(*)$ we use that every element is in S (as S is the base set). Apart from that we only use the definition of the complement and the set difference.

(c) Again we provide two solutions. The second proof does not consider the two inclusions separately.

Solution 1 We prove (i) $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$ and (ii) $\overline{A \cup B} \supseteq \overline{A} \cap \overline{B}$.

(i) Let $x \in \overline{A \cup B}$. Then $x \notin A \cup B$.

If x was in A, then $x \in A \cup B$ (which it is not as we've just noted). Hence $x \notin A$.
Similarly, if x was in B, then $x \in A \cup B$ (which it is not as we've just noted). Hence $x \notin B$.

But then $x \in \overline{A}$ and $x \in \overline{B}$, i.e. $x \in \overline{A} \cap \overline{B}$.

(ii) Let $x \in \overline{A} \cap \overline{B}$. Then $x \in \overline{A}$ and $x \in \overline{B}$. Thus x is not in A and x is not in B. Consequently, $x \notin A \cup B$, i.e., $x \in \overline{A \cup B}$.

Solution 2 We prove that $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

First we prove that for two propositional formulas F, F' it holds

$$\neg(F \vee F') = \neg F \wedge \neg F'. \tag{1}$$

We do this with the following truth value table (note that the columns marked in red are the same row by row)

F	F'	$\neg F$	$\neg F'$	$\neg F \wedge \neg F'$	$F \vee F'$	$\neg(F \vee F')$
1	1	0	0	0	1	0
1	0	0	1	0	1	0
0	1	1	0	0	1	0
0	0	1	1	1	0	1

Then it holds:

$$\begin{aligned}
\overline{A \cup B} &= \overline{\{x \mid x \in A \vee x \in B\}} \\
&= \{x \mid \neg(x \in A \vee x \in B)\} \\
&\overset{(1)}{=} \{x \mid x \notin A \wedge x \notin B\} \\
&= \{x \mid x \notin A\} \cap \{x \mid x \notin B\} \\
&= \overline{A} \cap \overline{B}
\end{aligned}$$

Solution for task 2:

(a) Such a function does not exist: when a function f is non-injective, then there must be distinct a, b in the source set with $f(a) = f(b)$. However, as the source set has only one elment there aren't even two distinct elements in the source set.

   f_1 doesn't even exist if we don't demand it to be non-injective.

(b) $f_2(x) = 3$, $G_{f_2} = \{(0,1),(1,3),(2,3)\}$

(c) $f_3(x) = \text{n.d.}$, $G_{f_3} = \emptyset$

(d) such a function does not exist: the range can at most contain as many elements as the domain of definition (since a function can only map element of the source set to at most one element of the target set). As the target set contains more elements than the source set, no function $\{0,1\} \to \{0,1,2\}$ can be surjective.

(e) $f_5(x) = \text{n.d.}$, $G_{f_5} = \emptyset$

(f) such a function does not exist: if a function $\{0,1,2\} \to \{0,1,2\}$ is non-injective, then two distinct elements of the source set are mapped to the same element of the target set. Hence a non-injective function $\{0,1,2\} \to \{0,1,2\}$ has at most 2 elements in the range and thus is not surjective.

Solution for task 3:
Theorem: For all $n \in \mathbb{N}^+$ it holds $\sum_{i=1}^{n}(2i-1) = n^2$.

Proof: We prove the statement via induction over $n \in \mathbb{N}^+$.

(BC) $(n = 1)$ The statement is true for $n = 1$: $\sum_{i=1}^{1}(2i-1) = 1 = 1^2$

(IS) We assume that the statement holds for $n = k$ for some $k \in \mathbb{N}^+$, i.e., $\sum_{i=1}^{k}(2i-1) = k^2$ (we call this IH). We show that then the statement is also true for $k+1$, i.e. we have to prove $\sum_{i=1}^{k+1}(2i-1) = (k+1)^2$.

$$\begin{aligned}
\sum_{i=1}^{k+1}(2i-1) &= \sum_{i=1}^{k}(2i-1) + 2(k+1) - 1 \\
&\overset{\text{(IH)}}{=} k^2 + 2(k+1) - 1 \\
&= k^2 + 2k + 1 \\
&= (k+1)^2
\end{aligned}$$

$\square$

Solution for task 4:
Theorem: For all $n \geq 0$ it holds:

$$fib(n) = \frac{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}}{\sqrt{5} \cdot 2^{n+1}}$$

Proof: Induction over $n \geq 0$:
(IA) For $n = 0$ it holds: $\frac{(1+\sqrt{5})^1 - (1-\sqrt{5})^1}{\sqrt{5} \cdot 2^1} = 1$. For $n = 1$: $\frac{(1+\sqrt{5})^2 - (1-\sqrt{5})^2}{\sqrt{5} \cdot 2^2} = 1$. (IH)
(IS) We assume that for some $k \geq 1$ the theorem holds for all $n = 0, 1, \ldots, k$.
Then the theorem also holds for $n = k+1$:

$$\begin{aligned}
fib(k+1) &= fib(k) + fib(k-1) \\
&\overset{(IH)}{=} \frac{(1+\sqrt{5})^{k+1} - (1-\sqrt{5})^{k+1}}{\sqrt{5} \cdot 2^{k+1}} + \frac{(1+\sqrt{5})^{k} - (1-\sqrt{5})^{k}}{\sqrt{5} \cdot 2^{k}} \\
&= \frac{2 \cdot (1+\sqrt{5})^{k+1} - 2 \cdot (1-\sqrt{5})^{k+1} + 4 \cdot (1+\sqrt{5})^{k} - 4 \cdot (1-\sqrt{5})^{k}}{\sqrt{5} \cdot 2^{k+2}} \\
&= \frac{(1+\sqrt{5})^{k}(4 + 2 \cdot (1+\sqrt{5})) - (1-\sqrt{5})^{k}(4 + 2 \cdot (1-\sqrt{5}))}{\sqrt{5} \cdot 2^{k+2}} \\
&= \frac{(1+\sqrt{5})^{k+2} - (1-\sqrt{5})^{k+2}}{\sqrt{5} \cdot 2^{k+2}}
\end{aligned}$$

where we used that

$$\begin{aligned}
4 + 2 \cdot (1+\sqrt{5}) = 1 + 2\sqrt{5} + 5 = (1+\sqrt{5})^2 \\
4 + 2 \cdot (1-\sqrt{5}) = 1 - 2\sqrt{5} + 5 = (1-\sqrt{5})^2
\end{aligned}$$

$\square$

Solution for task 5:

to be proven: Each $n \in \mathbb{N}^+$ is in exactly one way representable as
$$n = \sum_{i=0}^{m} a_i \cdot 2^i \text{ with } m \in \mathbb{N},\ a_m = 1,\ a_0, \dots, a_{m-1} \in \{0,1\}.$$

a) Existence: we prove that for each $n \in \mathbb{N}^+$ there exists such a representation
Induction over $n \in \mathbb{N}^+$

BC $n=0$: $n = \sum_{i=0}^{0} 1 \cdot 2^i = 1$

IS Let $k \in \mathbb{N}^+$
We assume that for all $1, \dots, k$ a representation with the above properties exists. (IH)

Consider $k+1$. $k+1 = 2 \cdot \ell + a$ for $\ell = \lfloor \frac{k+1}{2} \rfloor$ and $a = (k+1 \bmod$

Due to $1 \le \ell \le k$ we can apply IH for ℓ, i.e.

there are $m \in \mathbb{N}$ and $a_0, \dots, a_m \in \{0,1\}$ with $a_m = 1$ and
$$\ell = \sum_{i=0}^{m} a_i \cdot 2^i.$$

$$\Rightarrow k+1 = 2 \cdot \ell + a = 2 \sum_{i=0}^{m} a_i \cdot 2^i + a$$
$$= \sum_{i=1}^{m+1} a_{i-1} \cdot 2^i + a$$
$$= \sum_{i=0}^{m+1} a'_i \cdot 2^i \quad \text{for } a'_0 = a,\ a'_i = a_{i-1} \text{ for } i = 1, \dots, m+1$$

(b) Uniqueness: indirect proof.
Assume there ex. $n \in \mathbb{N}^+$ for which there is more than one representation with the above properties. Let n be the minimal such number (clearly: $n \ge 2$)

$$n = \sum_{i=0}^{m} a_i \cdot 2^i \quad \text{with } a_0, \dots, a_m \in \{0,1\},\ a_m = 1,\ m \in \mathbb{N}$$
$$n = \sum_{i=0}^{m'} a'_i\, 2^i \quad \text{with } a'_0, \dots, a'_m \in \{0,1\},\ a'_{m'} = 1,\ m' \in \mathbb{N}$$

and there is j with $a_j \neq a'_j$. $\Rightarrow \underbrace{\sum_{i=1}^{m} a_i \cdot 2^i}_{\text{even}} + a_0 = n = \underbrace{\sum_{i=1}^{m'} a'_i \cdot 2^i}_{\text{even}} + a'_0$

$\Rightarrow a_0 = a'_0$ and $\sum_{i=1}^{m} a_i 2^i = \sum_{i=1}^{m'} a'_i 2^i \Rightarrow \sum_{i=0}^{m-1} a_{i+1} 2^i = \sum_{i=0}^{m'-1} a'_{i+1} 2^i =: n'$
$\Rightarrow n' < n$ and n' has no unique binary representation
$\Rightarrow$ contradiction to the choice of n $\square$

Solution for task 6:
Yes, it is possible.
Please note that it is not sufficient to prove that the number of cells to be filled after placing a tile on one cell is divisible by 3: For example the following grid cannot be filled although the number of cells to be filled is 24 and thus divisible by

3.

$$\begin{array}{ccccc}
\square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square \\
\square & \square & \bigstar & \square & \square \\
\square & \square & \square & \square & \square
\end{array}$$

One can prove via induction that a $2^n \times 2^n$ grid can be filled after placing a $\bigstar$-tile on an arbitrary cell.
For a 2×2-Feld this is possible. For the induction step one divides the grid horizontally and vertically in the middle and obtains four $2^{n-1} \times 2^{n-1}$-subgrids:

$$\left.\begin{array}{cccc|cccc}
\bigstar & \square & \cdots & \square & \square & \cdots & \square & \square \\
\square & \square & \cdots & \square & \square & \cdots & \square & \square \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\
\square & \square & \cdots & \square & \square & \cdots & \square & \square \\
\hline
\square & \square & \cdots & \square & \square & \cdots & \square & \square \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\
\square & \square & \cdots & \square & \square & \cdots & \square & \square \\
\square & \square & \cdots & \square & \square & \cdots & \square & \square
\end{array}\right\} 2^n$$

In exactly one of these subgrids there is the $\bigstar$-tile (wlog in the upper left subgrid, as is indicated above; otherwise, we could turn the grid; the position within this subgrid is arbitrary even if the picture above suggests the opposite). We now place a tile of the form $\begin{smallmatrix} & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}$ in the middle of the grid so that it does not cover a cell of the subgrid with the $\bigstar$-tile.

$$\left.\begin{array}{cccc|cccc}
\bigstar & \square & \cdots & \square & \square & \cdots & \square & \square \\
\square & \square & \cdots & \square & \square & \cdots & \square & \square \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\
\square & \square & \cdots & \square & \blacksquare & \cdots & \square & \square \\
\hline
\square & \square & \cdots & \blacksquare & \blacksquare & \cdots & \square & \square \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\
\square & \square & \cdots & \square & \square & \cdots & \square & \square \\
\square & \square & \cdots & \square & \square & \cdots & \square & \square
\end{array}\right\} 2^n$$

Now in each of the four subgrids there is exactly one cell that is covered. By the induction hypothesis we obtain that each of the subgrids can be filled completely and thus the whole grid can be filled (the proof even yields a concrete method for filling the grid).

Solution for task 7:
Prove inductively that the player that moves first can ensure that after each of his/her moves the table is point-symmetric relative to its middle point. For that the first mat must be placed in the middle

Solution for task 8:
Imagine that not the same monk is climbing down the next day, but another monk is climbing down the same day starting at 8AM. They will meet on the way. This is the point where both are at the very same time.

Solution for extra task 1:
I did.