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Übungsblatt 0

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Don't be frustrated if you do not manage to solve all or most of the following exercises, but please try. This will take time and practice (that's why you should try). Nonetheless, it is important that we familiarize ourselves with proving mathematical statements and working with mathematical precision. Use the hints on the last page.

1. **Sets**
   (a) Let $A, B \subseteq \mathbb{N}$. We define $A + B := \{a + b \mid a \in A, b \in B\}$ and $A \cdot B := \{a \cdot b \mid a \in A, b \in B\}$. Thus, e.g., $\{2, 3, 6\} \cdot \{2, 4\} = \{4, 6, 8, 12, 24\}$. Which numbers do the following sets consist of?
      i. $\{2\} \cdot \mathbb{N}$
      ii. $\{1\} + \mathbb{N}$
      iii. $(\{2\} \cdot \mathbb{N}) + \{1\}$
      iv. $\overline{\{1\}} \cdot \overline{\{1\}}$
      v. $\overline{\overline{\{1\}} \cdot \overline{\{1\}}}$
      vi. $\overline{\overline{\{1\}} \cdot \overline{\{1\}}} \cap \overline{1}$
   (b) Prove that for all sets A it holds $\overline{\overline{A}} = A$ (i.e., $\overline{\overline{A}} \subseteq A$ and $\overline{\overline{A}} \supseteq A$). Consider the two inclusions $\subseteq$ and $\supseteq$ separately.
   (c) Prove $\overline{A \cup B} = \overline{A} \cap \overline{B}$ for all sets A and B. Again, argue for the two inclusions $\subseteq$ and $\supseteq$ separately.

2. **Properties of functions**
   Specify a function with the required properties in each case, or argue that such a function does not exist. For each function, also state its graph as a set.
   (a) surjective, non-injective $f_1 : \{1\} \to \{0, 1\}$
   (b) total, non-surjective $f_2 : \{0, 1, 2\} \to \{1, 3\}$
   (c) non-total $f_3 : \{5\} \to \{0\}$
   (d) bijective $f_4 : \{0, 1\} \to \{0, 1, 2\}$
   (e) total $f_5 : \emptyset \to \{0, 1\}$
   (f) surjective, non-injective $f_6 : \{0, 1, 2\} \to \{0, 1, 2\}$

3. **Summation formula for odd numbers**
   Prove the following statement using complete induction over n: For all $n \in \mathbb{N}^+$ it holds

$$\sum_{i=1}^{n} (2i - 1) = n^2.$$

4. **Fibonacci sequence**
   The Fibonacci sequence is defined as: $\mathrm{fib}(0) = 1$, $\mathrm{fib}(1) = 1$ and $\mathrm{fib}(n) = \mathrm{fib}(n-1) + \mathrm{fib}(n-2)$ for $n \geq 2$. Prove the following statement using complete induction over $n \in \mathbb{N}$:

$$\mathrm{fib}(n) = \frac{(1 + \sqrt{5})^{n+1} - (1 - \sqrt{5})^{n+1}}{\sqrt{5} \cdot 2^{n+1}}.$$

5. **Property 2.3**

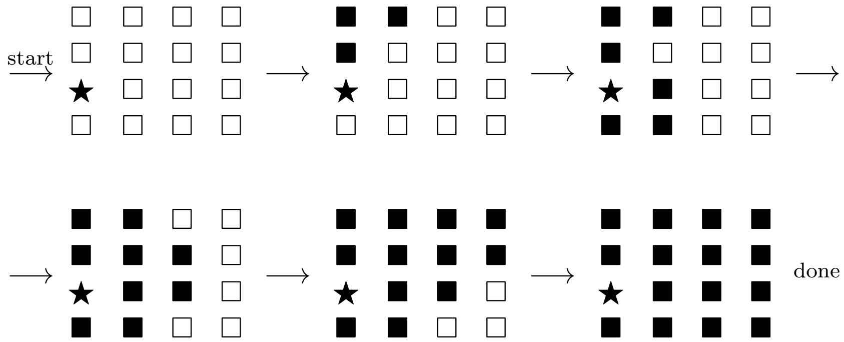
Prove the following statement: Each $n \in \mathbb{N}^+$ can be represented in exactly one way as

$$n = \sum_{i=0}^{m} a_i \cdot 2^i$$

with $m \in \mathbb{N}$, $a_m = 1$ and $a_0, \ldots, a_{m-1} \in \{0, 1\}$.

6. **Jigsaw puzzle**

For $n \in \mathbb{N}^+$ let a quadratic grid with $2^n \times 2^n$ cells be given. A ★ tile is placed on one of the cells. Is it now possible to fill the entire grid with tiles of the form ? You have an infinite number of these tiles available and you can also rotate them. Example:



7. **Winning strategies for placing beer mats**

Two players play the following game on an initially empty table (imagine it to be circular, oval, or rectangular):

The players take turns to place beer mats (imagine these to be circular, oval, or rectangular) of uniform shape and size anywhere on the table. No two beer mats may overlap. The first player to be unable to place any more beer mats in the manner described above loses.

Find a winning strategy for the player who starts.

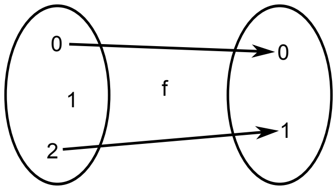
8. **Hiking monk**

A monk is hiking in the mountains. He climbs up a mountain on one day starting at precisely 8AM. He stays overnight and the next day he takes the very same way back also starting at 8AM. Is there any place on the way that the monk reaches at the very same time on both days?

Hints

Exercise 1:
Intuitively, $A + B$ is obtained as follows: take a number of A and ad each number of B to it. All numbers obtained this way are put into a set. Then continue with the next number of A. Note that sets to not contain duplicates.

Exercise 2:
You may describe functions with drawings as in the following. Also specify the function's graph formally.



Exercise 3:
You can orientate towards the example for the complete induction in the lecture notes. You also need a binomial formula.

Exercise 4:
When proving the induction step you should assume that the equation not only holds for $n = k$, but for $n = 0, \ldots, k$. Think about why it is ok to make this assumption.

Exercise 5:
This is about binary representation of numbers. An example: 1010 represents $0 \cdot 2^0 + 1 \cdot 2^1 + 0 \cdot 2^2 + 1 \cdot 2^3 = 0+2+0+8 = 12$. You have to prove two things. First you have to show that each natural number can be represented in binary. Think about what appending a 0 or a 1 to a binary representation does with the associated number (compare: what does it do in decimal representation?). Prove the statement using induction.
Next you have to show that there is only one representation with the described properties. Prove the statement by contradiction, i.e., assume that the statement is wrong and show that this leads to contradictions. Assume that there is a number n that has two different binary representations. Take the least such number and then prove that there is an even lower number with two different binary representations.

Exercise 6:
Use induction. Think about how you can make the induction hypothesis applicable.

Exercise 7:
The winning strategy is easy and short to describe and once you know it, it will not be difficult to understand why it works. But finding it is another matter...

Exercise 8:
This is an exercise that has nothing to do with the contents we will be dealing with in this lecture. It is just for fun and is a good example for typical mathematical thinking.

Extra tasks

1. **Enjoy**
   Enjoy the start of the semester!