"The Four Strongest" at the National Museum of Mongolia

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Does your hometown have any mathematical tourist attractions such as statues, plaques, graves, the café where the famous conjecture was made, the desk where the famous initials are scratched, birthplaces, houses, or memorials? Have you encountered a mathematical sight on your travels? If so, we invite you to submit an essay to this column. Be sure to include a picture, a description of its mathematical significance, and either a map or directions so that others may follow in your tracks.

➢ Submissions should be uploaded to http://tmin.edmgr.com or sent directly to **Ma. Louise Antonette N. De Las Peñas**, mathtourist1@gmail.com.

In February 2019, the National Museum of Mongolia (Figure 1) opened an exhibition on Mongolian traditional games and puzzles. Since it was offering a price reduction for family tickets, we took the opportunity to visit the exhibition with our two children. We enjoyed it immensely, though perhaps each of us in his or her own fashion. However, as with any group decision, there were some conflicts as we passed through the exhibit, since I wanted to look carefully and spend some time and thought on each item, while the younger members wanted to jump from one display to the next and see as much as possible.

The majority of items were artifacts from ancient bone games, board games (some of which are still very popular today), and interlocking puzzles, mostly made of wood. Among those puzzles, one item, which stood at the center of the hall, attracted considerable attention, for it was well preserved and beautifully designed. This was the game called the "The Four Strongest" (Figures 2 and 3). Its name comes from four legendary animals: a dragon, a tiger, a lion, and a bird originating in Hindu, Buddhist, and Jain mythology that we call Khangarid or Khan-Garuda or just Garuda.

The Puzzle

The task of the Four Strongest is to arrange the four animals so that they fit in their wooden container. It is thus a combinatorial puzzle with a geometric flavor. In fact, when I saw the shape of the container, a hexagon, it triggered some speculation and aroused some curiosity, since the connection between hexagons and efficient packings in the plane are well known (see, e.g., [1]).

I spoke briefly with the guide of the museum and asked whether she knew any additional details about the puzzle. She told me that the puzzle had been transferred from a local museum to the National Museum in 1961. She added that *finding its solution is difficult*! However, I was not allowed to touch it, and then my kids began to grow impatient, and we left.

The Conference

Eight months later, the National Museum made another announcement (Figure 4), this time a call for a national research conference on its collections. I decided to answer the call and investigate a little further the curious packing puzzle. I immediately contacted the organizers and asked for their support.

The research division of the museum and administrative staff at the Asia Research Center of my university helped me get through the official procedure, and I received permission to play with the puzzle. After some initial measure-

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Figure 1. The National Museum of Mongolia.

Figure 2. "The four strongest" puzzle. (Courtesy of the National Museum of Mongolia.)

Figure 3. Bottoms-up view: Garuda (left) and Lion (right). Garuda is holding and eating a snake and the craftsman of the puzzle was assiduous in decorating the underside of the lion as well.

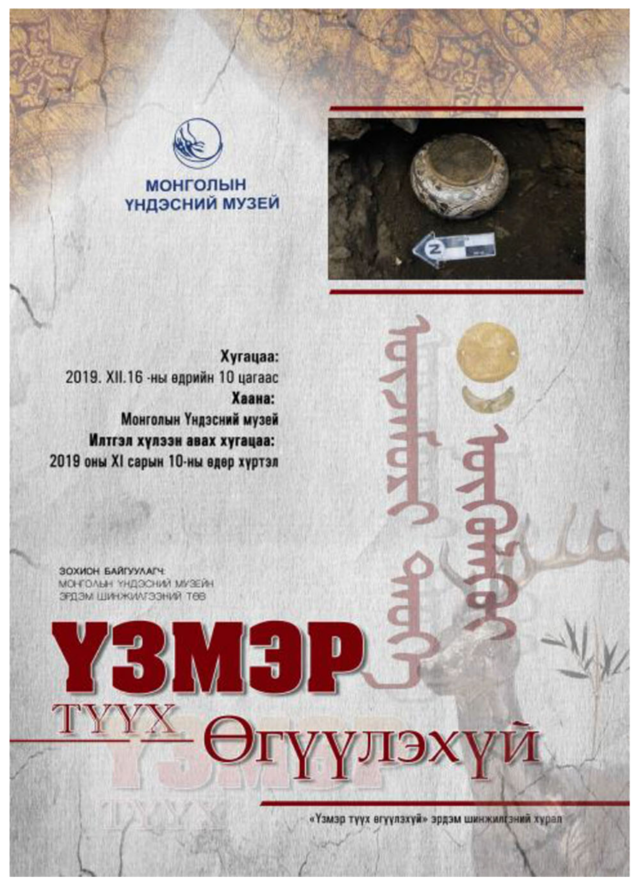

Figure 4. The conference announcement from the National Museum of Mongolia. (Courtesy of the National Museum of Mongolia.)

ments, I decided to formulate the following three mathematical questions:

1. How many possible arrangements, or candidate solutions, are there? That is, in how many ways can a player reasonably try to configure the pieces?
2. How many correct solutions are there?
3. Is the container the smallest in area or perimeter among those in which the items can be packed?

My goal was to give a definitive answer to at least some of these questions.

Before discussing my answers, let me give some details about the puzzle. First, the upper edge of the container forming its perimeter is (modulo a small error of 1 degree) a hexagon that can be thought of as comprising a 10.4 cm $\times$ 25.5 cm rectangle and two isosceles triangles with side lengths 7 cm, 7 cm, 10.4 cm, with the triangles glued to either end of the rectangle by their sides with a common length (Figure 5).

As for the animals, we projected them onto the plane by their bases and made measurements, as shown in Figure 6.

Based on the shapes of the four creatures, I decided to make the following definition.

Definition A *candidate solution* of the puzzle is a linear arrangement of the four animals such that each has two possible positions, either the head or the tail of each animal faces the player. One such solution is pictured in Figure 7.

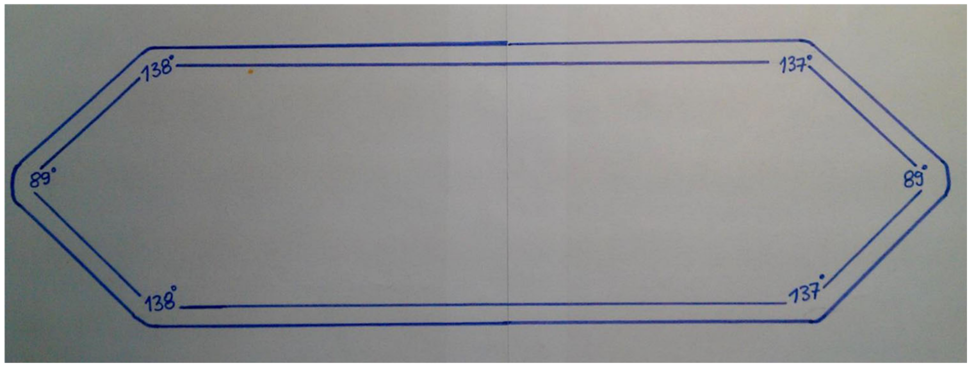

Figure 5. The projection of the frame.

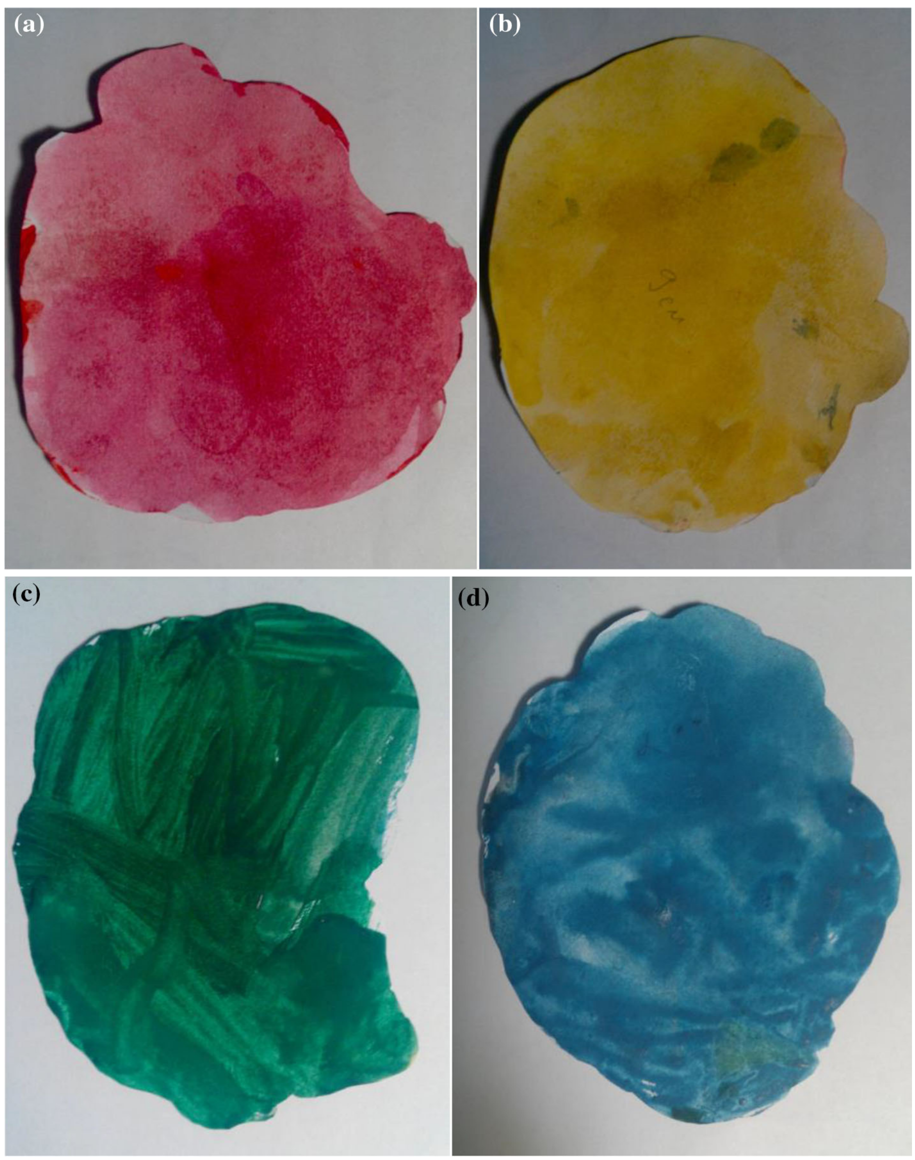

Figure 6. (a) Dragon has a base of diameter 9.8 cm. (b) Tiger has a base with diameter of 9.0 cm and a somewhat elliptical shape. (c) Lion has a base of diameter 9 cm and a rectangular shape. (d) Garuda has a base of diameter 9 cm and an ellipsoidal shape.

Figure 7. An "almost" correct solution of the puzzle.

Table 1. 192 candidate solutions are coded.

АБЛГ	А̄БЛГ	АБ̄ЛГ	АБЛ̄Г	АБЛГ̄	А̄Б̄ЛГ	А̄БЛ̄Г	А̄БЛГ̄	АБ̄Л̄Г
АБ̄ЛГ̄	АБЛ̄Г̄	АБ̄Л̄Г̄	А̄БЛ̄Г̄	А̄Б̄ЛГ̄	А̄Б̄Л̄Г	А̄Б̄Л̄Г̄	БАЛГ	Б̄АЛГ
БА̄ЛГ	БАЛ̄Г	БАЛГ̄	Б̄А̄ЛГ	Б̄АЛ̄Г	Б̄АЛГ̄	БА̄Л̄Г	БА̄ЛГ̄	БАЛ̄Г̄
БА̄Л̄Г̄	Б̄АЛ̄Г̄	Б̄А̄ЛГ̄	Б̄А̄Л̄Г	Б̄А̄Л̄Г̄	ЛАБГ	Л̄АБГ	ЛА̄БГ	ЛАБ̄Г
ЛАБГ̄	Л̄А̄БГ	Л̄АБ̄Г	Л̄АБГ̄	ЛА̄Б̄Г	ЛА̄БГ̄	ЛАБ̄Г̄	ЛА̄Б̄Г̄	Л̄АБ̄Г̄
Л̄А̄БГ̄	Л̄А̄Б̄Г	Л̄А̄Б̄Г̄	АЛБГ	А̄ЛБГ	АЛ̄БГ	АЛБ̄Г	АЛБГ̄	А̄Л̄БГ
А̄ЛБ̄Г	А̄ЛБГ̄	АЛ̄Б̄Г	АЛ̄БГ̄	АЛБ̄Г̄	АЛ̄Б̄Г̄	А̄ЛБ̄Г̄	А̄Л̄БГ̄	А̄Л̄Б̄Г
А̄Л̄Б̄Г̄	БЛАГ	Б̄ЛАГ	БЛ̄АГ	БЛА̄Г	БЛАГ̄	Б̄Л̄АГ	Б̄ЛА̄Г	Б̄ЛАГ̄
БЛ̄А̄Г	БЛ̄АГ̄	БЛА̄Г̄	БЛ̄А̄Г̄	Б̄ЛА̄Г̄	Б̄Л̄АГ̄	Б̄Л̄А̄Г	Б̄Л̄А̄Г̄	ЛБАГ
Л̄БАГ	ЛБ̄АГ	ЛБА̄Г	ЛБАГ̄	Л̄Б̄АГ	Л̄БА̄Г	Л̄БАГ̄	ЛБ̄А̄Г	ЛБ̄АГ̄
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Б̄Л̄ГА	Б̄ЛГ̄А	Б̄ЛГА̄	БЛ̄Г̄А	БЛ̄ГА̄	БЛГ̄А̄	БЛ̄Г̄А̄	Б̄ЛГ̄А̄	Б̄Л̄ГА̄
Б̄Л̄Г̄А	Б̄Л̄Г̄А̄	ЛГБА	Л̄ГБА	ЛГ̄БА	ЛГБ̄А	ЛГБА̄	Л̄Г̄БА	Л̄ГБ̄А
Л̄ГБА̄	ЛГ̄Б̄А	ЛГ̄БА̄	ЛГБ̄А̄	ЛГ̄Б̄А̄	Л̄ГБ̄А̄	Л̄Г̄БА̄	Л̄Г̄Б̄А	Л̄Г̄Б̄А̄
БГЛА	Б̄ГЛА	БГ̄ЛА	БГЛ̄А	БГЛА̄	Б̄Г̄ЛА	Б̄ГЛ̄А	Б̄ГЛА̄	БГ̄Л̄А
БГ̄ЛА̄	БГЛ̄А̄	БГ̄Л̄А̄	Б̄ГЛ̄А̄	Б̄Г̄ЛА̄	Б̄Г̄Л̄А	Б̄Г̄Л̄А̄	ЛГАБ	Л̄ГАБ
ЛГ̄АБ	ЛГА̄Б	ЛГАБ̄	Л̄Г̄АБ	Л̄ГА̄Б	Л̄ГАБ̄	ЛГ̄А̄Б	ЛГ̄АБ̄	ЛГА̄Б̄
ЛГ̄А̄Б̄	Л̄ГА̄Б̄	Л̄Г̄АБ̄	Л̄Г̄А̄Б	Л̄Г̄А̄Б̄	ЛАГБ	Л̄АГБ	ЛА̄ГБ	ЛАГ̄Б
ЛАГБ̄	Л̄А̄ГБ	Л̄АГ̄Б	Л̄АГБ̄	ЛА̄Г̄Б	ЛА̄ГБ̄	ЛАГ̄Б̄	ЛА̄Г̄Б̄	Л̄АГ̄Б̄
Л̄А̄ГБ̄	Л̄А̄Г̄Б	Л̄А̄Г̄Б̄						

The letters А, Б, Л, Г represent Lion, Tiger, Dragon, and Garuda, respectively. A letter with an overbar means that the corresponding animal is reversed (i.e., rotated by 180 degrees). The red entry denotes the only clearly correct solution, while the 14 blue entries denote solutions with some degree of cheating.

With this assumption I was ready to give an answer to the first question.

Theorem 1 *There are* 192 *candidate solutions for the puzzle of the four strongest.*

Proof This is an easy combinatorial problem that can be solved directly, or via Pólya's counting technique. Note that in the first (say, from the left) position there are eight choices (four choices of animal, then two choices of position). There are then six in the second, four in the third, and two in the last. So the number of choices is their product, or 384. But for each solution, an equivalent solution can be obtained by rotating the container through an angle of 180°, so there are actually only half as many inequivalent solutions, that is, 192. ■

However, I was not able to give a definite answer for the second and third questions. The second question requires a careful and detailed analysis of the shapes of the four animals and the container. I felt that an attempt to answer this question presented me with the risk of repeating the anecdote of a mathematician who tried to rearrange his

Figure 8. The author after conducting the exhaustive search at the National Museum of Mongolia's research division.

furniture, but after several unsuccessful attempts sat down and decided to prove that such a rearrangement was impossible by building a mathematical model, but then his wife came and moved it to the desired location. So, instead of trying to solve the puzzle in the abstract mathematical way, I suggested solving it via a systematic method of trial and error, namely to work on a list in which all 192 candidate solutions are coded. I provided one such list to the museum researchers (see Table 1).

Following the conference, I made another visit to the National Museum to conduct the exhaustive search mentioned above (Figure 8) and seek an answer to my second question. The outcome of the experiment is given also in Table 1. To my surprise, there was a unique clearly correct solution, with the code $\overline{\text{Л}}\text{А}\overline{\text{Б}}\text{Г}$, corresponding to the arrangement Dragon, Lion, Tiger, Garuda.

There were 14 other "almost" solutions with varying degrees of cheating, i.e., an overlap between two animals or with part of an animal with an edge of the frame, or the arrangement required some force. For example, the solution given in Figure 7 is a good one (it has a code $\overline{\text{Л}}\overline{\text{Б}}\text{А}\overline{\text{Г}}$ in Table 1), but it still has some overlap, either on the edge or between two of the animals. In the distinctly correct solution, I was able to move the items vertically by a very small amount. So, what the museum guide told me on my first visit regarding to the difficulty of the puzzle was justified.

The third question is more closely related to finite packing theory. Sphere packing theory is both old and new and full of nice theorems as well as intriguing open problems and conjectures. The most intensively studied cases are packings of identical convex bodies such as spheres, either freely (i.e., to minimize the volume of their convex hull), or in a smallest container of a specific kind—perhaps a circle, a square, or a triangle (see [1]). In the latter case, the exact solutions are known when the number of items is not very large, i.e., up to 20.

Our third question deals with packing four nonidentical shapes in three-dimensional space, which is a rather unexplored territory for packing theory. Even after reducing the problem to two dimensions by replacing the shapes with their bases, the problem is a tricky one. However, together with one of our master's students, Orgil-Erdene Erdenebaatar, we were able solve a simpler problem to gain some insight.

Theorem 2 *The convex hexagon of smallest area that contains four disks of radius $r > 0$ arranged linearly (i.e., like a sausage) is obtained as follows. Take a rectangle of size $2r \times 6r$ and a regular hexagon with side length $\frac{2r}{\sqrt{3}}$. Divide the hexagon into two equal halves by a line segment connecting the midpoints of two opposite sides. Glue each half to the rectangle by the side of common length. See Figure 9.*

The main tool that we used in proving this result was the following result, proofs of which can be found in [2, 11].

Theorem 3 *Let $C \subseteq \mathbb{R}^2$ be a convex disk and P the convex polygon of smallest area containing it. Then the midpoints of the sides of P lie on the boundary of C.*

Last December, I made a presentation at the conference explaining the above results, and by doing so, I tried to promote the potential role of mathematics in the study of museum collections. The reception was positive, and my presentation at least provided some knowledge to the audience, adding some color and diversity to the conference.

Conclusion

Most, if not all, good puzzles have some mathematical content. For example, Ernst Zermelo proved a theorem about chess that later become a fundamental result in the theory of two-person zero-sum games with perfect information. Édouard Lucas's analysis of the Hanoi tower puzzle led to a new direction of mathematical research (see, e.g., [4]), and the first group-theoretic analysis of the famous 15 puzzle was given around 1879 [5], which also led to a rich literature (see, e.g., [6]). Even today, the study of puzzles remains an active research area; see, e.g., [8, 9].

In this short article, I have tried to introduce a tiny part of Mongolian tradition and culture, an arrangement puzzle related to discrete geometry. In the mathematics literature, one occasionally encounters Mongolian history and

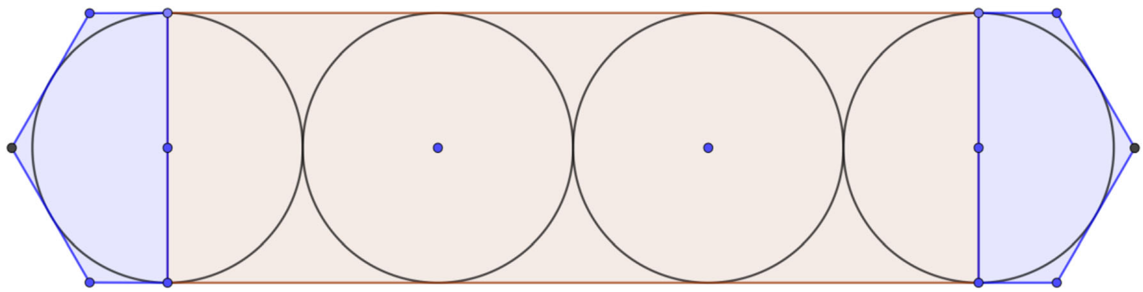

Figure 9. The smallest hexagon containing four linearly packed disks.

culture, for example in Chapter 1 of [3] and in [7, 10]. This article is an attempt to add another piece to this list. If you, dear reader, have been intrigued, then perhaps you should visit my country. In that case, I strongly recommend that you make another "mathematical visit" to the National Museum of Mongolia, which is located at the center of the capital city, and perhaps add another piece to the list of mathematical studies of Mongolian culture.

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