ROLE OF LUB AXIOM IN REAL ANALYSIS

SYNOPSIS:

- WHAT IS LUB AXIOM
- THEOREMS AND THEIR PROOFS USING LUB AXIOM

LUB PROPERTY:

Let A be a non empty set of real. Assume that A is bounded above, that is there is an $M \in R$ such that $a \leq M$ for all $a \in A$. Then there exists an $\alpha \in R$ with the property that

1. $a \leq \alpha$ for all $a \in A$.
2. If β is any upper bound for A then $\alpha \leq \beta$.

This α is unique and called the least upper bound for A. We shall denote it by sup A.

In this case , α is sup A if and only if, α is an upper bound for A and if $\varepsilon > 0$ is given then there is an $a \in A$ such that $\alpha - \varepsilon < a$.

THEOREMS:

1.NESTED INTERVAL THEOREM:

Let $J_n = [a_n, b_n]$ be intervals in R such that J_{n+1} subset of J_n for all n € N. Then $\cap J_n \neq \Phi$.

Proof: Let E={ a € R: a=a_n for some n} . E is non empty. We claim that b_k is an upper bound for E for each k€N, that is $a_n \leq b_k$ for all n and k. If k≤n, then $[a_n, b_n]$ is a subset of $[a_k, b_k]$ and hence $a_n \leq b_n \leq b_k$. If k>n, then $a_n \leq a_k \leq b_k$. Thus the claim proved. By the LUB axiom, there exists c€R such that c=sup E. We claim that c€J_n for all n. Since each b_n is an upper bound of E and c is the LUB for E we see that $c \leq b_n$. Thus we conclude $a_n \leq c \leq b_n$ or c€J_n for all n. Hence c€$\cap J_n$.

2. Any sequence of real numbers bounded above is convergent. That is if (x_n) is a sequence in R such that $x_n \leq x_{n+1}$ and there exists M€R such that $x_n \leq M$ for all n € N, then lim x_n exists.

Proof: Let E={x € R: x=x_n for some n € N} be the image of the sequence. By our assumption E is non empty and bounded above by M. By the LUB axiom there exists l € R which is sup E. We shall show that lim x_n =l. Let ε>0 be given. As l-ε is not an upper bound for E there exists an N such that l-ε<x_N. As the sequence is increasing we have $x_N \leq x_n$ for all n≥N. We see that l-ε<x_n≤l<l+ε for all n≥N. Thus x_n€(l-ε,l+ε) or lim x_n = l.

3. Any cauchy sequence in R converges.

Idea of the proof: We need show that every real cauchy sequence converges. For that we choose real cauchy sequence (x_n). From that we observe that x_n € $(x_N-\delta\backslash 2, x_N+\delta\backslash 2)$ for some arbitrary $\delta>0$. We define a set S that consists of x such that $x_n \geq x$ and claim that S is non empty and bounded above and sup S is the limit of S. Thus the cauchy sequence converges to the LUB of the set S. Thus we proved that every cauchy real sequence converges.

4. The interval [0,1] is uncountable.

Proof : If [0,1] is countable, since [0,1] is infinite, there exists a bijection f: N→[0,1]. Let z_n=f(n). We define two sequences (x_n) and (y_n) whose terms are defined recursively. Let x_1 be the z_r where r is the first integer such that $0<z_r<1$. let y_1 be z_s where s is the first integer such that $x_1<z_s<1$. Assume that we have chosen (x_i) and (y_i) i=1to n with the property,
$0=x_0<x_1<x_2<\ldots<x_n<y_n<y_{n-1}<\ldots<y_2<y_1<y_0=1$
We choose x_{n+1} be the z_r, $x_n<z_r<y_n$. Let y_{n+1} be z_s in the same way. Clearly $\{x_n\}\subset[0,1]$ is non empty and bounded above. Let x= sup$\{x_n\}$. Then it is clear that x€[0,1] and $x\neq z_n$ for n€N.

5.INTERMEDIATE VALUE THEOREM:

Let $f:[a,b]\subset R\rightarrow R$ be continous. Assume that $f(a)<0<f(b)$. Then there exists $c€R$ such that $f(c)=0$.

Proof : Define $E=\{x€[a,b]: f(y)\leq 0$ for $y€[a,x]$. Using the continuity of f at a for $\varepsilon= -f(a)\backslash 2$, we can find a $f(x)€(3f(a)\backslash 2,f(a)\backslash 2)$ for all $x€[a,a+\delta)$. This shows that $a+\delta/2€E$. Since E is bounded by b there is $c€R$ such that $c=\sup E$. Clearly we have $a+\delta/2\leq c\leq b$ and hence $c€(a,b]$. We claim that $c€E$ and that $f(c)=0$. since $c-1/n$ is not an upper bound for E, there is an $x_n€E$ such that $c-1/n <x_n\leq c$. By sandwich lemma, $\lim x_n=c$. By continuity of f at c, we have $f(x_n)\rightarrow f(c)$. As $f(x_n)\leq 0$ for all n we conclude that $f(c)\leq 0$. this implies that $c<b$.

Hence c€(a,b). If $f(c)\neq 0$, then $f(c)<0$. Arguing using the fact that $a<c<b$, we can find a small $\eta>0$ such that $(c-\eta,c+\eta)\subset(a,b)$ and such that for x€$(c-\delta/2,c+\delta/2)$ we have $f(x)$€$(3f(c)/2,f(c)/2)$. As $\lim x_n=c$, there is an N such that x_N€$(c-\eta,c+\eta)$. But we see that $f(x)<0$ for x€$[a,x_N]U(c-\eta,c+\eta/2)$. Hence $c+\eta/2$€E. This contradicts the fact that c=sup E. Hence we conclude that $f(c)=0$.

6.Heine borel theorem: If a closed and bounded interval in R is covered by a family of open intervals, then it is covered by finitely many open intervals from the given family.

Proof : Let $\{G_\alpha / \alpha \in I\}$ be a family of open sets in R such that $U_{\alpha \in I}$ [a,b] is a subset of $U_{\alpha \in I} G_\alpha$. Let $S=\{x/x\in[a,b]$ and $[a,x]$ can be covered by a finite number of G_α's. clearly $a\in S$ and S is non empty. Also S is bounded above by b. then let c be the lub of S. Clearly $c\in S$. Then $c\in G_{\alpha 1}$ for some $\alpha 1\in I$. $G_{\alpha 1}$ open, then there exists $\varepsilon>0$ such that $(c-\varepsilon,c+\varepsilon)\subset G_{\alpha 1}$. Choose $x_1\in[a,b]$, $x_1<c$ and $[x_1,c]\subset G_{\alpha 1}$.Then $[a,x_1]$ can be covered by finite number of G_α's. we claim that c=b. suppose $c\neq b$,then choose $x_2\in[a,b]$, $x_2>c$ and $[c,x_2]\subset G_{\alpha 1}$ and it can be covered by finite number of G_α's. Hence $x_2\in S$. But c is the lub of S, a contradiction. c=b. then [a,b] can be covered by finitely many open intervals from the given family

7.Bolzano Weierstrass theorem : Let A be an infinite bounded subset of R.Then there is a cluster point of A in R.

Proof: Let E={x€R : x≤a for infinitely many a€A}.Let M€R that -M≤a≤M for all a€A. It is obvious that-M€E. We can easily show that E is bounded by M. Hence there exists l€R such that l=sup E. We claim that l is a cluster point of E. That is, we need to show that for any given ε>0 there exists a point a€(l-ε,l+ε)∩A other than l itself. Since l-ε is not an upper bound for E there is an x€E such that l-ε<x. since x€E there exist infinitely many elements a€A, such that x≤a. Hence there exist infinitely many elements a€A such that l-ε<a. Also except for finitely many a€A we have a<l+ε

For , otherwise, for infinitely many elements a€A we have $a \geq l+\varepsilon$. But then $l+\varepsilon$€E. This contradicts the fact that $l=$ sup E. Thus there exist infinitely many a€A such that $l-\varepsilon<a<l+\varepsilon$. In particular there is atleast one a€A$\cap(l-\varepsilon,l+\varepsilon)$ which is different from l.

8. Archimedian property:

i. The set of natural numbers is not bounded above in R.

ii. Given two real numbers x, y with $x>0$, there exist a positive integer n such that $nx>y$.

Proof : If $\mathbb{N}$ is bounded above , then $\alpha \in R$ be the LUB for $\mathbb{N}$. That is, we have

$$n \le \alpha \text{ for all } n \in \mathbb{N}$$

$$n+1 \le \alpha \text{ for all } n \in \mathbb{N}$$

$$n \le \alpha-1 \text{ for all } n \in \mathbb{N}$$

We conclude that $\alpha-1$ is an upper bound for $\mathbb{N}$. This contradicts our assumption that α is the LUB for $\mathbb{N}$

ii. If no such n exists, then n≤y/x for all n. In other words, ℕ is bounded above by y/x, contradicting (i).

10. Density of ℚ in R: Given x,y €ℝ with x<y, there exists an rational number r such that x<r<y.

Proof: Assuming the existence of such an r,we write it as r=m/n with n >0.so, we have x<m/n<y, that is nx<y<ny.thus we are claiming that the interval[nx,ny]cointains an integer.

Since y-x>0,by archimedian property,there exist n€ℕ such that n(y-x)>1.we consider the set S={k€ℤ:k≤nx}.This is a not empty subset of R.Thus k>nx for all K€ℤ.from this,we get –k<-nx for all k€Z.In particular –nx is an upper bound for N.This contradiction shows that S is non empty.S is bounded above by nx .Let α€R be the LUB for S.since α-1<α and α is the LUB of S,there exists k€S such that k>α-1.Hence α<k+1.Let m=k+1.we claim that m>nx.

For otherwise $m \leq nx$ and hence $m=k+1 \in S$. Since α is an upper bound for S, we see that $k+1<\alpha$. it contradicts our choice of k. This proves $m>nx$. We also claim that $m<ny$. If false , then $m \geq ny$. Thus the interval $[nx,ny]$ of length greater than 1 is contained in $[k,k+1]$.

$$1=(k+1)-k=m-k \geq ny-nx=n(y-x)>1$$

Thus we conclude that $nx<m<ny$. Dividing the inequalities by n, we get the required result.

CONCLUSION:

From all these theorems and proofs we observe that LUB axiom plays a major role in real analysis. And the most important proof about LUB axiom is that, it is not true if the field is rational numbers inspite of real numbers. By using LUB axiom , we proved the existence of n^{th} roots, that is irrational numbers.