Duality

- dual of an LP in inequality form
- variants and examples
- complementary slackness
- sensitivity analysis
- two-person zero-sum games

Dual of linear program in inequality form

we define two LPs with the same parameters $c \in \mathbf{R}^n$, $A \in \mathbf{R}^{m\times n}$, $b \in \mathbf{R}^m$

- an LP in 'inequality form'

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$

- an LP in 'standard form'

$$\begin{array}{ll} \text{maximize} & -b^T z \\ \text{subject to} & A^T z + c = 0 \\ & z \geq 0 \end{array}$$

  this problem is called the **dual** of the first LP

in the context of duality, the first LP is called the **primal** problem

Duality theorem

notation

- $p^\star$ is the primal optimal value; $d^\star$ is the dual optimal value
- $p^\star = +\infty$ if primal problem is infeasible; $d^\star = -\infty$ if dual is infeasible
- $p^\star = -\infty$ if primal problem is unbounded; $d^\star = \infty$ if dual is unbounded

duality theorem: if primal or dual problem is feasible, then

$$p^\star = d^\star$$

moreover, if $p^\star = d^\star$ is finite, then primal and dual optima are attained

note: only exception to $p^\star = d^\star$ occurs when primal *and* dual are infeasible

Weak duality

lower bound property: if x is primal feasible and z is dual feasible, then

$$c^T x \geq -b^T z$$

proof: if $Ax \leq b$, $A^T z + c = 0$, and $z \geq 0$, then

$$0 \leq z^T (b - Ax) = b^T z + c^T x$$

$c^T x + b^T z$ is the **duality gap** associated with primal and dual feasible x, z

weak duality: the lower bound property immediately implies that

$$p^\star \geq d^\star$$

(without exception)

Strong duality

if primal and dual problems are feasible, then there exist $x^\star$, $z^\star$ that satisfy

$$c^T x^\star = -b^T z^\star, \qquad A x^\star \le b, \qquad A^T z^\star + c = 0, \qquad z^\star \ge 0$$

combined with the lower bound property, this implies that

- $x^\star$ is primal optimal and $z^\star$ is dual optimal
- the primal and dual optimal values are finite and equal:

$$p^\star = c^T x^\star = -b^T z^\star = d^\star$$

(proof on next page)

proof: we show that there exist $x^\star$, $z^\star$ that satisfy

$$\begin{bmatrix} A & 0 \\ 0 & -I \\ c^T & b^T \end{bmatrix} \begin{bmatrix} x^\star \\ z^\star \end{bmatrix} \leq \begin{bmatrix} b \\ 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & -A^T \end{bmatrix} \begin{bmatrix} x^\star \\ z^\star \end{bmatrix} = c$$

- the lower-bound property implies that any solution necessarily satisfies

$$c^T x^\star + b^T z^\star = 0$$

- to prove a solution exists we show that the alternative system (p. 5–5)

$$u \geq 0, \quad t \geq 0, \qquad A^T u + tc = 0, \qquad Aw \leq tb, \qquad b^T u + c^T w < 0$$

has no solution

the alternative system has no solution because:

- if $t > 0$, defining $\tilde{x} = w/t$, $\tilde{z} = u/t$ gives

$$\tilde{z} \geq 0, \qquad A^T\tilde{z} + c = 0, \qquad A\tilde{x} \leq b, \qquad c^T\tilde{x} < -b^T\tilde{z}$$

  this contradicts the lower bound property

- if $t = 0$ and $b^T u < 0$, u satisfies

$$u \geq 0, \qquad A^T u = 0, \qquad b^T u < 0$$

  this contradicts feasibility of $Ax \leq b$ (page 5–2)

- if $t = 0$ and $c^T w < 0$, w satisfies

$$Aw \leq 0, \qquad c^T w < 0$$

  this contradicts feasibility of $A^T z + c = 0$, $z \geq 0$ (page 5–3)

Primal infeasible problems

if $p^\star = +\infty$ then $d^\star = +\infty$ or $d^\star = -\infty$

proof: if primal is infeasible, then from page 5–2, there exists w such that

$$w \geq 0, \qquad A^T w = 0, \qquad b^T w < 0$$

if the dual problem is feasible and z is any dual feasible point, then

$$z + tw \geq 0, \qquad A^T(z + tw) + c = 0 \qquad \text{for all } t \geq 0$$

therefore $z + tw$ is dual feasible for all $t \geq 0$; moreover, as $t \to \infty$,

$$-b^T(z + tw) = -b^T z - tb^T w \to +\infty$$

so the dual problem is unbounded above

Dual infeasible problems

if $d^\star = -\infty$ then $p^\star = -\infty$ or $p^\star = +\infty$

proof: if dual is infeasible, then from page 5–3, there exists y such that

$$Ay \le 0, \qquad c^T y < 0$$

if the primal problem is feasible and x is any primal feasible point, then

$$A(x + ty) \le b \text{ for all } t \ge 0$$

therefore $x + ty$ is primal feasible for all $t \ge 0$; moreover, as $t \to \infty$,

$$c^T(x + ty) = c^T x + tc^T y \to -\infty$$

so the primal problem is unbounded below

Exception to strong duality

an example that shows that $p^\star = +\infty$, $d^\star = -\infty$ is possible

primal problem (one variable, one inequality)

$$\begin{array}{ll} \text{minimize} & x \\ \text{subject to} & 0 \cdot x \leq -1 \end{array}$$

optimal value is $p^\star = +\infty$

dual problem

$$\begin{array}{ll} \text{maximize} & z \\ \text{subject to} & 0 \cdot z + 1 = 0 \\ & z \geq 0 \end{array}$$

optimal value is $d^\star = -\infty$

Summary

	$p^\star = +\infty$	$p^\star$ finite	$p^\star = -\infty$
$d^\star = +\infty$	primal inf. dual unb.		
$d^\star$ finite		optimal values equal and attained	
$d^\star = -\infty$	exception		primal unb. dual inf.

- upper-right part of the table is excluded by weak duality
- first column: proved on page 6–8
- bottom row: proved on page 6–9
- center: proved on page 6–5

Outline

- dual of an LP in inequality form
- **variants and examples**
- complementary slackness
- sensitivity analysis
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Variants

LP with inequality and equality constraints

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \le b \\ & Cx = d \end{array} \qquad\qquad \begin{array}{ll} \text{maximize} & -b^T z - d^T y \\ \text{subject to} & A^T z + C^T y + c = 0 \\ & z \ge 0 \end{array}$$

standard form LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \ge 0 \end{array} \qquad\qquad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y \le c \end{array}$$

- dual problems can be derived by converting primal to inequality form
- same duality results apply

Piecewise-linear minimization

$$\text{minimize} \quad f(x) = \max_{i=1,\ldots,m} (a_i^T x + b_i)$$

LP formulation (variables x, t; optimal value is $\min_x f(x)$)

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \begin{bmatrix} A & -\mathbf{1} \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \leq -b \end{array}$$

dual LP (same optimal value)

$$\begin{array}{ll} \text{maximize} & b^T z \\ \text{subject to} & A^T z = 0 \\ & \mathbf{1}^T z = 1 \\ & z \geq 0 \end{array}$$

Interpretation

- for any $z \geq 0$ with $\sum_i z_i = 1$,

$$f(x) = \max_i (a_i^T x + b_i) \geq z^T(Ax + b) \quad \text{for all } x$$

- this provides a lower bound on the optimal value of the PWL problem

$$\begin{aligned} \min_x f(x) &\geq \min_x z^T(Ax + b) \\ &= \begin{cases} b^T z & \text{if } A^T z = 0 \\ -\infty & \text{otherwise} \end{cases} \end{aligned}$$

- the dual problem is to find the best lower bound of this type
- strong duality tells us that the best lower bound is tight

ℓ_∞-Norm approximation

$$\text{minimize} \quad \|Ax - b\|_\infty$$

LP formulation

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \begin{bmatrix} A & -\mathbf{1} \\ -A & -\mathbf{1} \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \leq \begin{bmatrix} b \\ -b \end{bmatrix} \end{array}$$

dual problem

$$\begin{array}{ll} \text{maximize} & -b^T u + b^T v \\ \text{subject to} & A^T u - A^T v = 0 \\ & \mathbf{1}^T u + \mathbf{1}^T v = 1 \\ & u \geq 0, \ v \geq 0 \end{array} \tag{1}$$

simpler equivalent dual

$$\begin{array}{ll} \text{maximize} & b^T z \\ \text{subject to} & A^T z = 0, \quad \|z\|_1 \leq 1 \end{array} \tag{2}$$

proof of equivalence of the dual problems (assume A is $m \times n$)

- if u, v are feasible in (1), then $z = v - u$ is feasible in (2):

$$\|z\|_1 = \sum_{i=1}^{m} |v_i - u_i| \leq \mathbf{1}^T v + \mathbf{1}^T u = 1$$

  moreover the objective values are equal: $b^T z = b^T (v - u)$

- if z is feasible in (2), define vectors u, v by

$$u_i = \max\{-z_i, 0\} + \alpha, \; v_i = \max\{z_i, 0\} + \alpha, \; i = 1, \ldots, m$$

  with $\alpha = (1 - \|z\|_1)/(2m)$

  these vectors are feasible in (1) with objective value $b^T (v - u) = b^T z$

Interpretation

- lemma: $u^T v \leq \|u\|_1 \|v\|_\infty$ holds for all u, v
- therefore, for any z with $\|z\|_1 \leq 1$,

$$\|Ax - b\|_\infty \geq z^T(Ax - b)$$

- this provides a bound on the optimal value of the ℓ_∞-norm problem

$$\begin{aligned} \min_x \|Ax - b\|_\infty &\geq \min_x z^T(Ax - b) \\ &= \begin{cases} -b^T z & \text{if } A^T z = 0 \\ -\infty & \text{otherwise} \end{cases} \end{aligned}$$

- the dual problem is to find the best lower bound of this type
- strong duality tells us the best lower bound is tight

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- variants and examples
- **complementary slackness**
- sensitivity analysis
- two-person zero-sum games

Optimality conditions

primal and dual LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \le b \\ & Cx = d \end{array} \qquad\qquad \begin{array}{ll} \text{maximize} & -b^T z - d^T y \\ \text{subject to} & A^T z + C^T y + c = 0 \\ & z \ge 0 \end{array}$$

optimality conditions: x and (y, z) are primal, dual optimal if and only if

- x is primal feasible: $Ax \le b$ and $Cx = d$
- y, z are dual feasible: $A^T z + C^T y + c = 0$ and $z \ge 0$
- the duality gap is zero: $c^T x = -b^T z - d^T y$

Complementary slackness

assume A is $m \times n$ with rows a_i^T

- the duality gap at primal feasible x, dual feasible y, z can be written as

$$\begin{aligned} c^T x + b^T z + d^T y &= (b - Ax)^T z + (d - Cx)^T y \\ &= (b - Ax)^T z \\ &= \sum_{i=1}^{m} z_i (b_i - a_i^T x) \end{aligned}$$

- primal, dual feasible x, y, z are optimal if and only if

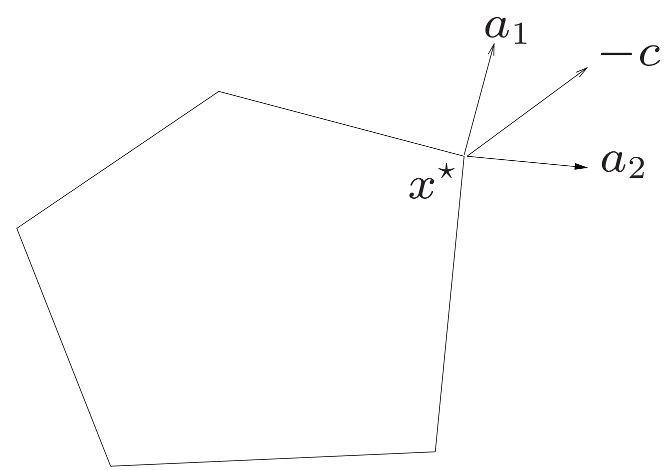
$$z_i (b_i - a_i^T x) = 0, \quad i = 1, \ldots, m$$

$i.e.$, at optimum, $b - Ax$ and z have a *complementary* sparsity pattern:

$$z_i > 0 \implies a_i^T x = b_i, \qquad a_i^T x < b_i \implies z_i = 0$$

Geometric interpretation

example in $\mathbf{R}^2$



- two active constraints at optimum: $a_1^T x^\star = b_1$, $a_2^T x^\star = b_2$
- optimal dual solution satisfies

$$A^T z + c = 0, \qquad z \geq 0, \qquad z_i = 0 \text{ for } i \notin \{1, 2\}$$

in other words, $-c = a_1 z_1 + a_2 z_2$ with $z_1 \geq 0$, $z_2 \geq 0$

- geometric interpretation: $-c$ lies in the cone generated by a_1 and a_2

Example

$$\begin{array}{ll} \text{minimize} & -4x_1 - 5x_2 \\ \text{subject to} & \begin{bmatrix} -1 & 0 \\ 2 & 1 \\ 0 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 3 \\ 0 \\ 3 \end{bmatrix} \end{array}$$

show that $x = (1, 1)$ is optimal

- second and fourth constraints are active at $(1, 1)$
- therefore any dual optimal z must be of the form $z = (0, z_2, 0, z_4)$ with

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} z_2 \\ z_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}, \qquad z_2 \geq 0, \quad z_4 \geq 0$$

  $z = (0, 1, 0, 2)$ satisfies these conditions

dual feasible z with correct sparsity pattern proves that x is optimal

Optimal set

primal and dual LP (A is $m \times n$ with rows a_i^T)

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & Ax \le b \\
& Cx = d
\end{array}
\qquad\qquad
\begin{array}{ll}
\text{maximize} & -b^T z - d^T y \\
\text{subject to} & A^T z + C^T y + c = 0 \\
& z \ge 0
\end{array}
$$

assume the optimal value is finite

- let $(y^\star, z^\star)$ be *any* dual optimal solution and define $J = \{i \mid z_i^\star > 0\}$
- x is optimal iff it is feasible and complementary slackness with $z^\star$ holds:

$$
a_i^T x = b_i \quad \text{for } i \in J, \qquad a_i^T x \le b_i \quad \text{for } i \notin J, \qquad Cx = d
$$

conclusion: optimal set is a face of the polyhedron $\{x \mid Ax \le b, Cx = d\}$

Strict complementarity

- primal and dual optimal solutions are not necessarily unique
- any combination of primal and dual optimal points must satisfy

$$z_i(b_i - a_i^T x) = 0, \qquad i = 1, \ldots, m$$

in other words, for all i,

$$a_i^T x < b_i,\ z_i = 0 \qquad \text{or} \qquad a_i^T x = b_i,\ z_i > 0 \qquad \text{or} \qquad a_i^T x = b_i,\ z_i = 0$$

- primal and dual optimal points are **strictly complementary** if for all i

$$a_i^T x < b_i, \quad z_i = 0 \qquad \text{or} \qquad a_i^T x = b_i, \quad z_i > 0$$

it can be shown that strictly complementary solutions exist for any LP with a finite optimal value (Goldman-Tucker Theorem)

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- **sensitivity analysis**
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Sensitivity analysis

purpose: extract from the solution of an LP information about the sensitivity of the solution with respect to changes in problem data

this lecture:

- sensitivity w.r.t. to changes in the right-hand side of the constraints
- we define $p^\star(u)$ as the optimal value of the modified LP (variables x)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b + u \end{array}$$

- we are interested in obtaining information about $p^\star(u)$ from primal, dual optimal solutions $x^\star$, $z^\star$ at $u = 0$

Global inequality

dual of modified LP

$$\begin{array}{ll} \text{maximize} & -(b+u)^T z \\ \text{subject to} & A^T z + c = 0 \\ & z \geq 0 \end{array}$$

global lower bound: if $z^\star$ is (any) dual optimal solution for $u = 0$, then

$$\begin{aligned} p^\star(u) &\geq -(b+u)^T z^\star \\ &= p^\star(0) - u^T z^\star \end{aligned}$$

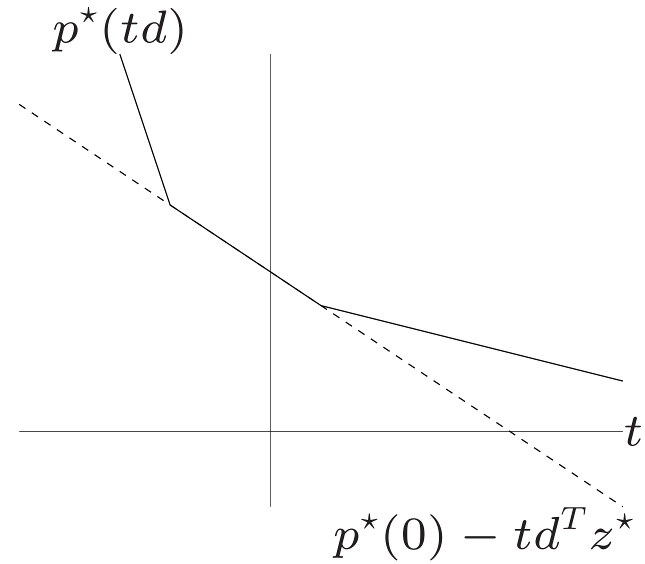
- follows from weak duality and feasibility of $z^\star$
- inequality holds for all u (not necessarily small)

Example (one varying parameter)

take $u = td$ with d fixed:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b + td \end{array}$$

$p^\star(td)$ is optimal value as a function of t



sensitivity information from lower bound (assuming $d^T z^\star > 0$):

- if $t < 0$ the optimal value increases (by a large amount of $|t|$ is large)
- if $t > 0$ optimal value may increase or decrease
- if t is positive and small, optimal value certainly does not decrease much

Optimal value function

$$p^\star(u) = \min\{c^T x \mid Ax \le b + u\}$$

properties (we assume $p^\star(0)$ is finite)

- $p^\star(u) > -\infty$ everywhere (this follows from the global lower bound)
- the domain $\{u \mid p^\star(u) < +\infty\}$ is a polyhedron
- $p^\star(u)$ is piecewise-linear on its domain

(proof on next page)

proof. let P be the dual feasible set, K the recession cone of P:

$$P = \{z \mid A^T z + c = 0,\ z \geq 0\}, \qquad K = \{w \mid A^T w = 0, w \geq 0\}$$

- $p^\star(u) = +\infty$ (modified primal is infeasible) iff there exists a w such that

$$A^T w = 0, \qquad w \geq 0, \qquad b^T w + u^T w < 0$$

  therefore $p^\star(u) < \infty$ if and only if

$$b^T w_k + u^T w_k \geq 0 \qquad \text{for all extreme rays } w_k \text{ of } K$$

  this is a finite set of linear inequalities in u

- if $p^\star(u)$ is finite,

$$p^\star(u) = \max_{z \in P} (-b^T z - u^T z) = \max_{k=1,\ldots,r} (-b^T z_k - u^T z_k)$$

  where z_1, . . . , z_r are the extreme points of P

Local sensitivity analysis

let $x^\star$ be optimal for the unmodified problem, with active constraint set

$$J = \{i \mid a_i^T x^\star = b_i\}$$

assume $x^\star$ is a **nondegenerate extreme point**, $i.e.$,

- an extreme point: A_J has full column rank $(\mathrm{rank}(A_J) = n)$
- nondegenerate: $|J| = n$ (n active constraints)

then, for u in a neighborhood of the origin, $x^\star(u)$ and $z^\star$ defined by

$$x^\star(u) = A_J^{-1}(b_J + u_J), \qquad z_J^\star = -A_J^{-T}c, \qquad z_i^\star = 0 \ \text{(for } i \notin J\text{)},$$

are primal, dual optimal for the modified problem

note: $x^\star(u)$ is affine in u and $z^\star$ is independent of u

proof

solution of original LP ($u = 0$)

- since A_J is square and nonsingular, we can express $x^\star$ as $x^\star = A_J^{-1} b_J$
- complementary slackness determines optimal $z^\star$ uniquely:

$$z_i^\star = 0 \quad i \notin J, \qquad A_J^T z_J^\star + c = 0$$

solution of modified LP (for sufficiently small u)

- $x^\star(u)$ satisfies inequalities indexed by J: $A_J x^\star(u) = b_J + u_J$ (for all u)
- $x^\star(u)$ satisfies the other inequalities ($i \notin J$) for sufficiently small u:

$$a_i^T x^\star(u) \leq b_i + u_i \quad \Longleftrightarrow \quad a_i^T A_J^{-1} u_J - u_i \leq b_i - a_i^T x^\star$$

  and $b_i - a_i^T x^\star > 0$
- $z^\star$ is dual feasible (for all u)
- $x^\star(u)$ and $z^\star$ satisfy complementary slackness conditions

Derivative of optimal value function

under the assumptions of the local analysis (page 6–32),

$$\begin{aligned} p^\star(u) &= c^T x^\star(u) \\ &= c^T x^\star + c^T A_J^{-1} u_J \\ &= p^\star(0) - z_J^{\star T} u_J \end{aligned}$$

for u in a neighborhood of the origin

- optimal value function is affine in u for small u
- $-z_i^\star$ is derivative of $p^\star(u)$ with respect to u_i at $u = 0$

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Two-person zero-sum game (matrix game)

- player 1 chooses a number in $\{1, \ldots, m\}$ (one of m possible actions)
- player 2 chooses a number in $\{1, \ldots, n\}$ (n possible actions)
- players make their choices independently
- if P1 chooses i and P2 chooses j, then P1 pays A_{ij} to P2

  (negative A_{ij} means P2 pays $-A_{ij}$ to P1)
- the $m \times n$-matrix A is called the **payoff matrix**

Mixed (randomized) strategies

players choose actions randomly according to some probability distribution

- P1 chooses randomly according to distribution x:

$$x_i = \text{probability that P1 selects action } i$$

- P2 chooses randomly according to distribution y:

$$y_j = \text{probability that P2 selects action } j$$

expected payoff (from P1 to P2), if they use mixed stragies x and y,

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_i y_j A_{ij} = x^T A y$$

Optimal mixed strategies

denote by $P_k = \{p \in \mathbf{R}^k \mid p \geq 0, \mathbf{1}^T p = 1\}$ the probability simplex in $\mathbf{R}^k$

- player 1: optimal strategy $x^\star$ is solution of the equivalent problems

$$\begin{array}{ll} \text{minimize} & \max_{y \in P_n} x^T A y \\ \text{subject to} & x \in P_m \end{array} \qquad \begin{array}{ll} \text{minimize} & \max_{j=1,\ldots,n} (A^T x)_j \\ \text{subject to} & x \in P_m \end{array}$$

- player 2: optimal strategy $y^\star$ is solution of

$$\begin{array}{ll} \text{maximize} & \min_{x \in P_m} x^T A y \\ \text{subject to} & y \in P_n \end{array} \qquad \begin{array}{ll} \text{maximize} & \min_{i=1,\ldots,m} (A y)_i \\ \text{subject to} & y \in P_n \end{array}$$

optimal strategies $x^\star$, $y^\star$ can be computed by linear optimization

Exercise: minimax theorem

prove that

$$\max_{y \in P_n} \min_{x \in P_m} x^T A y = \min_{x \in P_m} \max_{y \in P_n} x^T A y$$

some consequences

- if $x^\star$ and $y^\star$ are the optimal mixed strategies, then

$$\min_{x \in P_m} x^T A y^\star = \max_{y \in P_n} x^{\star T} A y$$

- if $x^\star$ and $y^\star$ are the optimal mixed strategies, then

$$x^T A y^\star \geq x^{\star T} A y^\star \geq x^{\star T} A y \qquad \forall x \in P_m, \ \forall y \in P_n$$

solution

- optimal strategy $x^\star$ is the solution of the LP (with variables x, t)

$$
\begin{array}{ll}
\text{minimize} & t \\
\text{subject to} & A^T x \le t\mathbf{1} \\
& x \ge 0 \\
& \mathbf{1}^T x = 1
\end{array}
$$

- optimal strategy $y^\star$ is the solution of the LP (with variables y, w)

$$
\begin{array}{ll}
\text{maximize} & w \\
\text{subject to} & Ay \ge w\mathbf{1} \\
& y \ge 0 \\
& \mathbf{1}^T y = 1
\end{array}
$$

- the two LPs can be shown to be duals

Example

$$A = \begin{bmatrix} 4 & 2 & 0 & -3 \\ -2 & -4 & -3 & 3 \\ -2 & -3 & 4 & 1 \end{bmatrix}$$

- note that

$$\min_i \max_j A_{ij} = 3 > -2 = \max_j \min_i A_{ij}$$

- optimal mixed strategies

$$x^\star = (0.37,\ 0.33,\ 0.3), \qquad y^\star = (0.4,\ 0,\ 0.13,\ 0.47)$$

- expected payoff is $x^{\star T} A y^\star = 0.2$