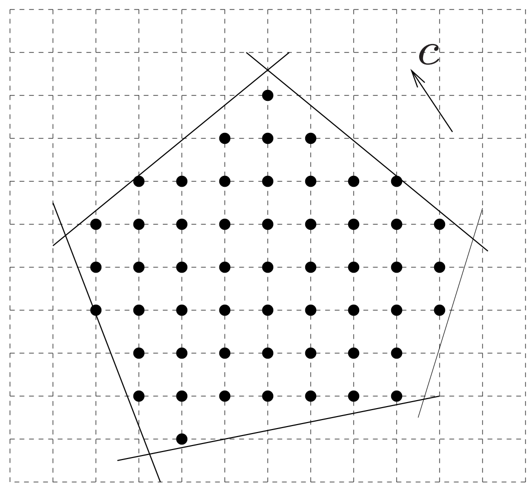
Integer linear programming

- a few basic facts
- branch-and-bound

Definitions

integer linear program (ILP)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \\ & x \in \mathbf{Z}^n \end{array}$$



mixed integer linear program: only some of the variables are integer

0-1 (Boolean) linear program: variables take values 0 or 1

Example: facility location problem

- n potential facility locations, m clients
- c_i: cost of opening a facility at location i
- d_{ij}: cost of serving client i from location j

Boolean LP formulation

$$
\begin{array}{ll}
\text{minimize} & \sum_{j=1}^{n} c_j y_j + \sum_{i=1}^{m} \sum_{j=1}^{n} d_{ij} x_{ij} \\
\text{subject to} & \sum_{j=1}^{n} x_{ij} = 1, \quad i = 1, \ldots, m \\
& x_{ij} \leq y_j, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n \\
& x_{ij}, y_j \in \{0, 1\}
\end{array}
$$

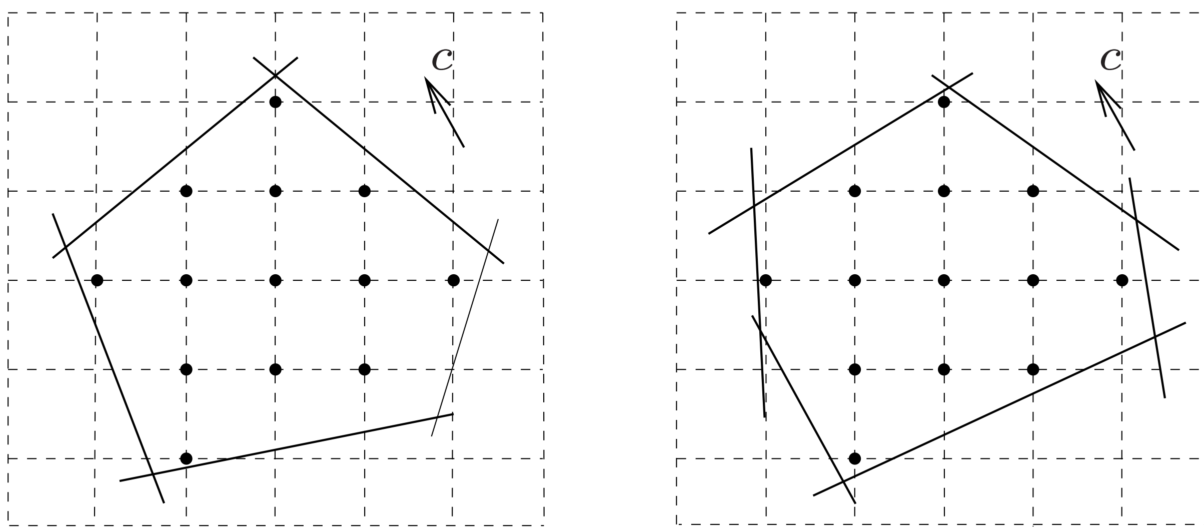
variables y_j, x_{ij}:

$y_j = 1$	location j is selected	$x_{ij} = 1$	location j serves client i
$y_j = 0$	otherwise	$x_{ij} = 0$	otherwise

Linear programming relaxation

relaxation: remove the constraints $x \in \mathbf{Z}^n$

- provides a lower bound on the optimal value of the integer LP
- if solution of relaxation is integer, then it solves the integer LP



equivalent ILP formulations can have different LP relaxations

Branch-and-bound algorithm

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x \in \mathcal{P} \end{array}$$

where $\mathcal{P}$ is a finite set

general idea

- recursively partition $\mathcal{P}$ in smaller sets $\mathcal{P}_i$ and solve subproblems

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x \in \mathcal{P}_i \end{array}$$

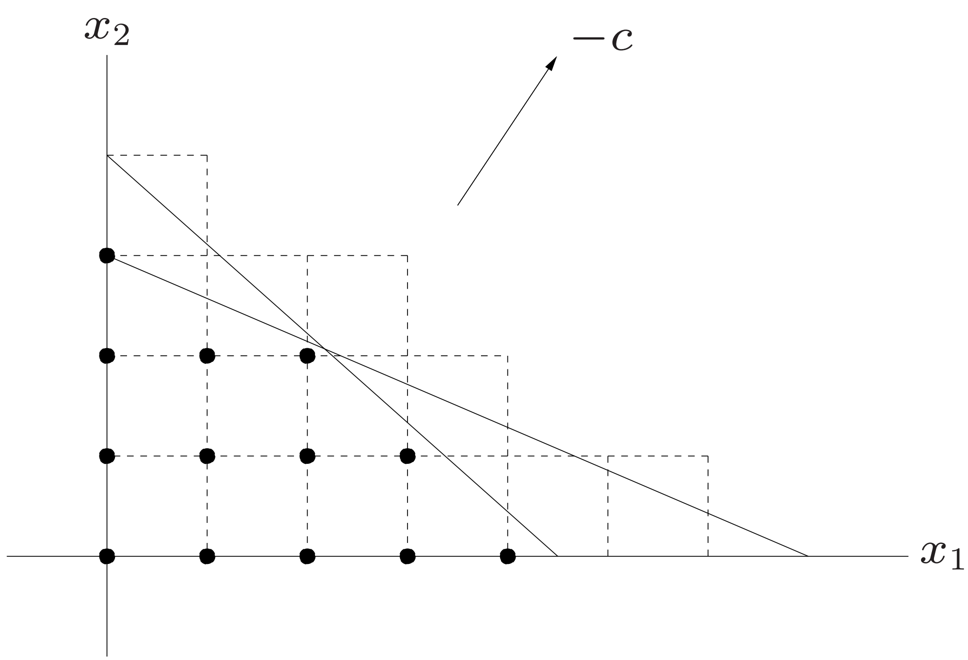
- use LP relaxations to discard subproblems that don't lead to a solution

Example

$$\begin{array}{ll} \text{minimize} & -2x_1 - 3x_2 \\ \text{subject to} & (x_1, x_2) \in \mathcal{P} \end{array}$$

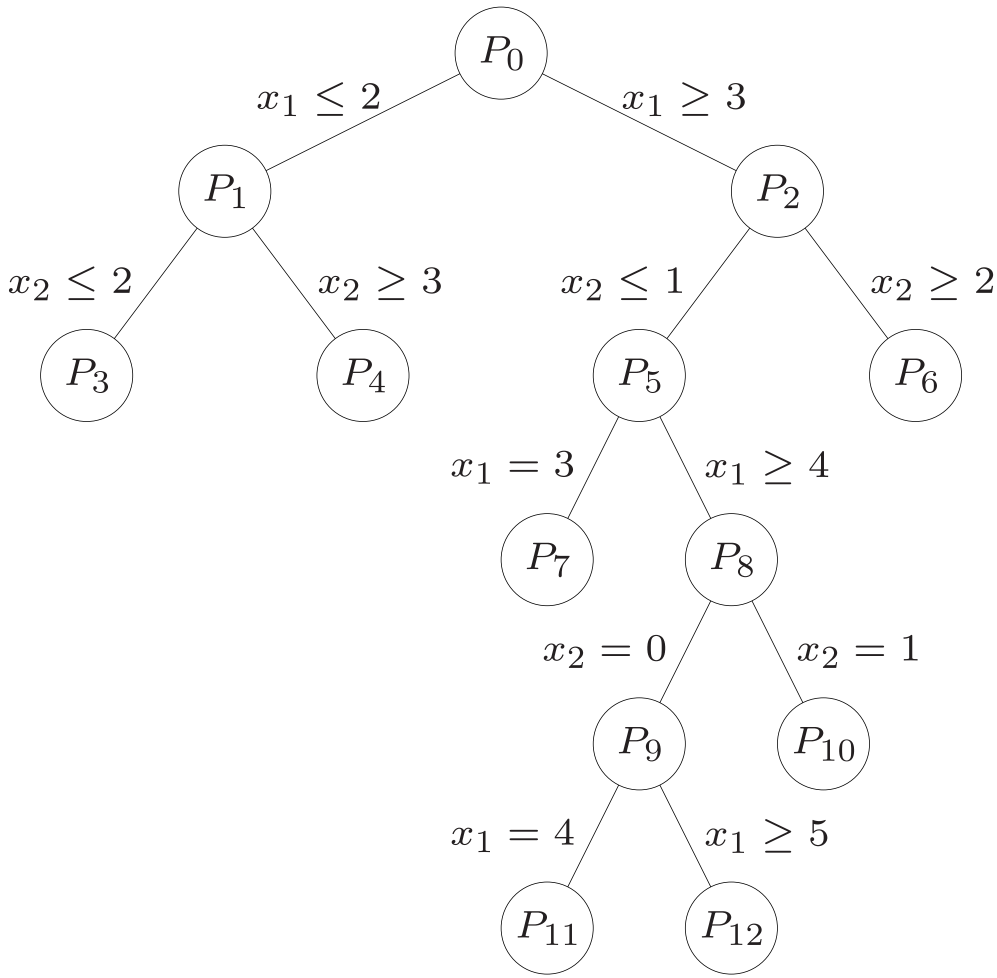
where

$$\mathcal{P} = \{x \in \mathbf{Z}_+^2 \mid \frac{2}{9}x_1 + \frac{1}{4}x_2 \leq 1, \quad \frac{1}{7}x_1 + \frac{1}{3}x_2 \leq 1\}$$



optimal point: $(2, 2)$

tree of subproblems and results of LP relaxations



	$x^\star$	$p^\star$
P_0	$(2.17, 2.07)$	-10.56
P_1	$(2.00, 2.14)$	-10.43
P_2	$(3.00, 1.33)$	-10.00
P_3	$(2.00, 2.00)$	-10.00
P_4	$(0.00, 3.00)$	-9.00
P_5	$(3.38, 1.00)$	-9.75
P_6		$+\infty$
P_7	$(3.00, 1.00)$	-9.00
P_8	$(4.00, 0.44)$	-9.33
P_9	$(4.50, 0.00)$	-9.00
P_{10}		$+\infty$
P_{11}	$(4.00, 0.00)$	-8.00
P_{12}		$+\infty$

conclusions from relaxed subproblems

- P_2: minimize c^Tx subject to $x \in \mathcal{P}$, $x_1 \geq 3$

  optimal value of subproblem is greater than or equal to -10.00
- P_3: minimize c^Tx subject to $x \in \mathcal{P}$, $x_1 \leq 2$, $x_2 \leq 2$

  solution of subproblem is $x = (2, 2)$
- P_6: minimize c^Tx, subject to $x \in \mathcal{P}$, $x_1 \leq 3$, $x_2 \geq 2$

  subproblem is infeasible

after solving the relaxations for subproblems

$$P_0, \quad P_1, \quad P_2, \quad P_3, \quad P_4$$

we can conclude that $(2, 2)$ is the optimal solution of the integer LP