Network flow optimization

- minimum cost network flows
- total unimodularity
- examples

Networks

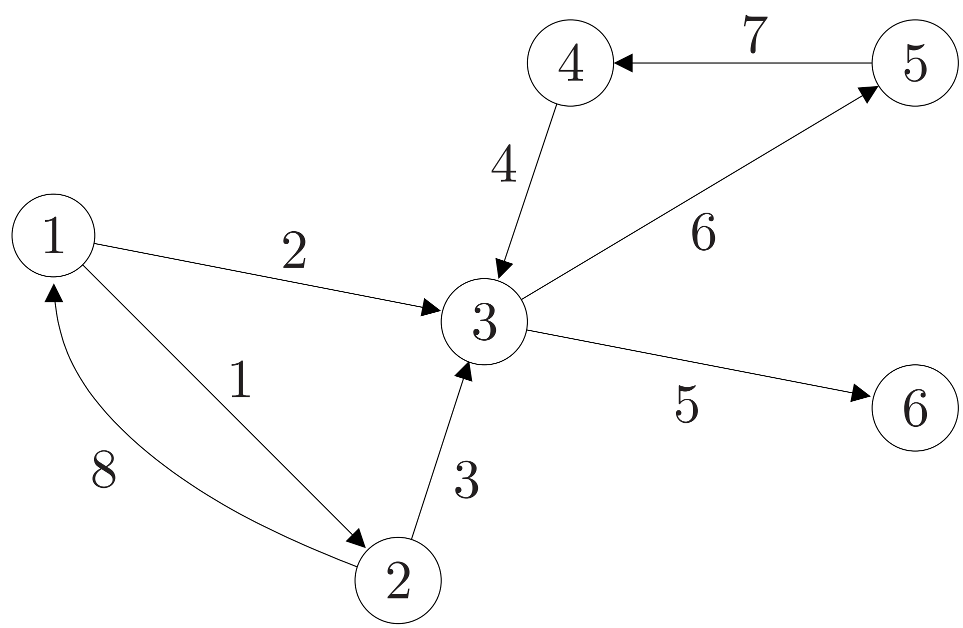
network (directed graph, digraph): m nodes connected by n directed arcs

- arcs are ordered pairs (i, j) of nodes
- we assume there is at most one arc from node i to node j
- there are no loops (arcs (i, i))

arc-node incidence matrix: $m \times n$ matrix A with entries

$$A_{ij} = \begin{cases} 1 & \text{if arc } j \text{ starts at node } i \\ -1 & \text{if arc } j \text{ ends at node } i \\ 0 & \text{otherwise} \end{cases}$$

Example



$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & -1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}$$

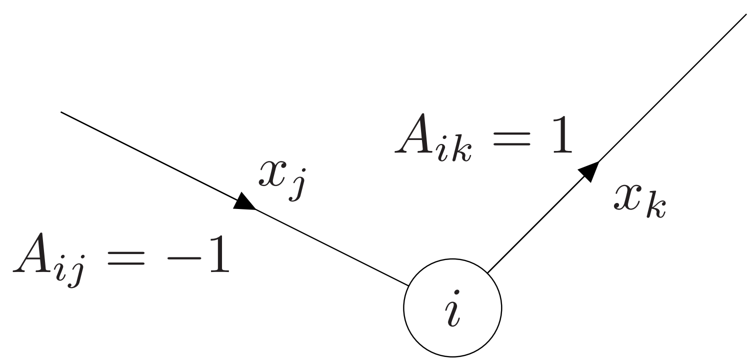
Network flow

flow vector $x \in \mathbf{R}^n$

- x_j: flow (of material, traffic, charge, information, . . .) through arc j
- positive if in direction of arc; negative otherwise

total flow leaving node i:

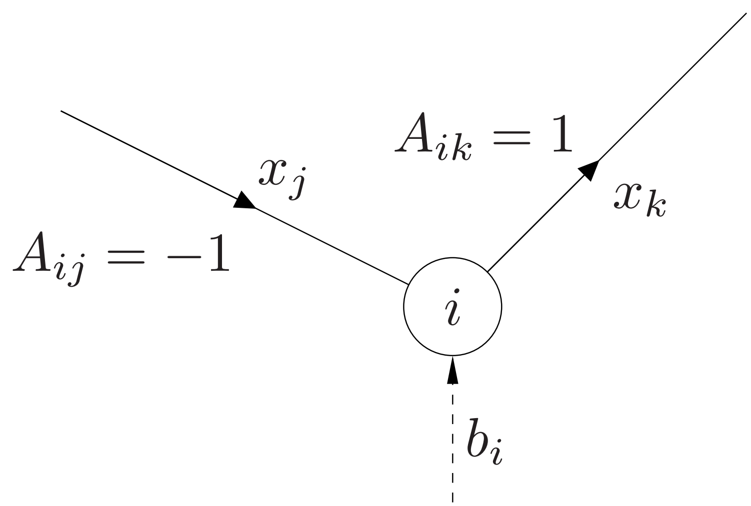
$$\sum_{j=1}^{n} A_{ij} x_j = (Ax)_i$$



External supply

supply vector $b \in \mathbf{R}^m$

- b_i is external supply at node i (negative b_i represents external demand)
- must satisfy $\mathbf{1}^T b = 0$ (total supply = total demand)



balance equations:

$$Ax = b$$

Minimum cost network flow problem

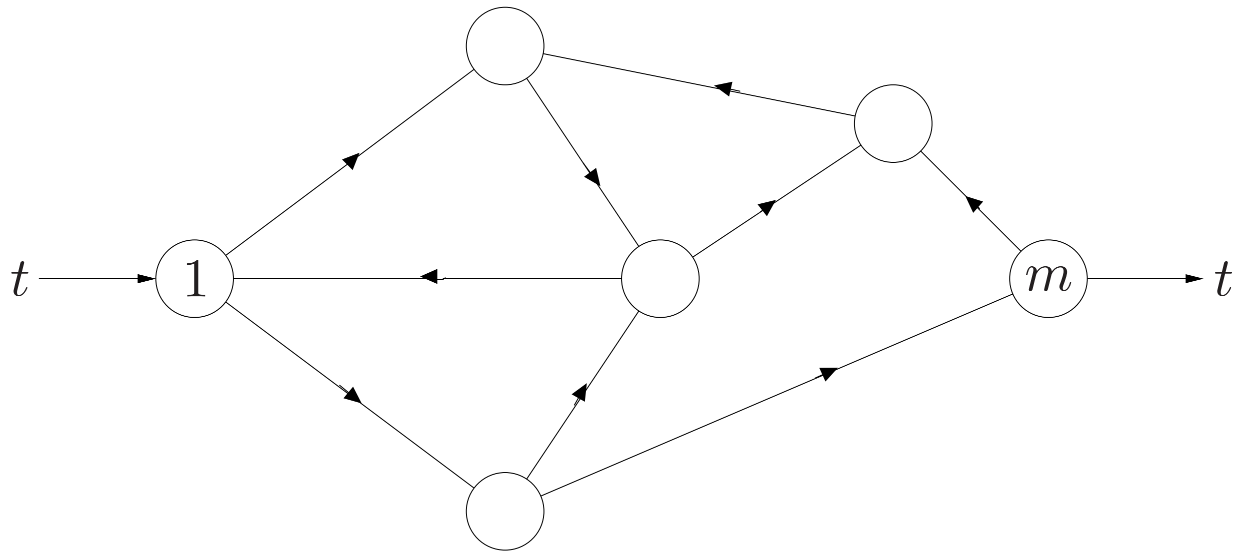
$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & l \leq x \leq u \end{array}$$

- c_i is unit cost of flow through arc i
- l_j and u_j are limits on flow through arc j (typically, $l_j \leq 0$, $u_j \geq 0$)
- we assume $l_j < u_j$, but allow $l_j = -\infty$ and $u_j = \infty$ to simplify notation

includes many network optimization problems as special cases

Maximum flow problem

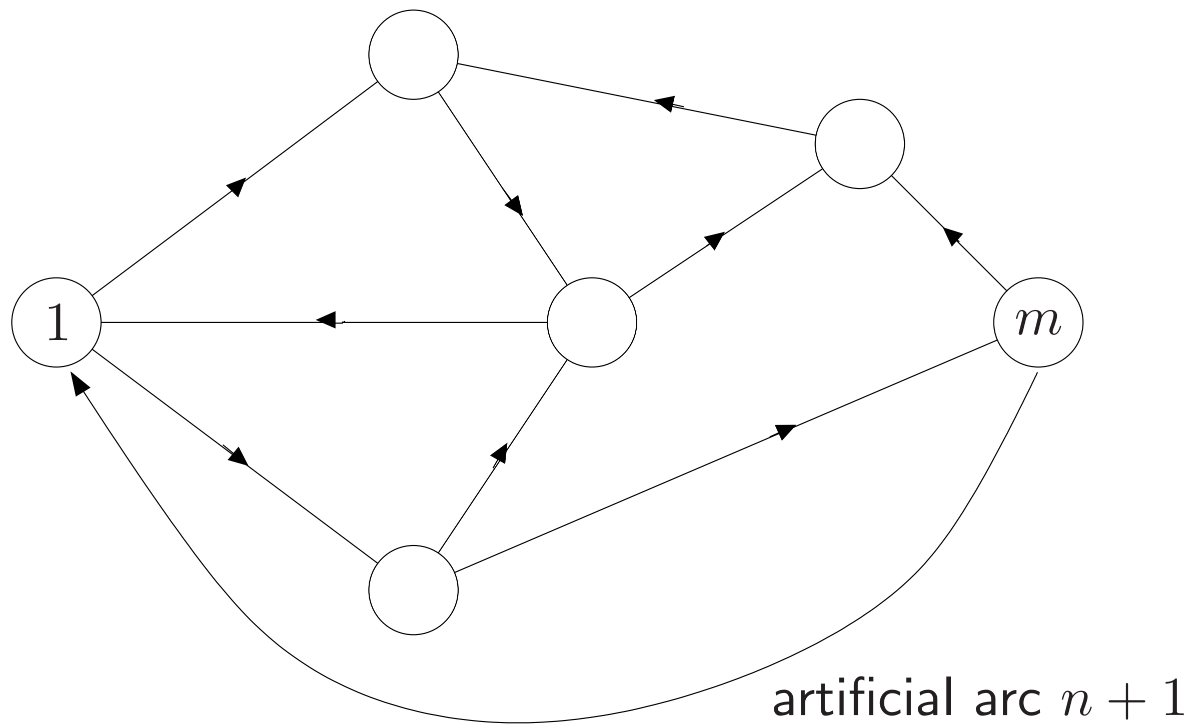
maximize flow from node 1 (source) to node m (sink) through the network



$$
\begin{array}{ll}
\text{maximize} & t \\
\text{subject to} & Ax = te \\
& l \leq x \leq u
\end{array}
$$

where $e = (1, 0, \ldots, 0, -1)$

Formulation as minimum cost flow problem



$$\begin{array}{ll} \text{minimize} & -t \\ \text{subject to} & \begin{bmatrix} A & -e \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} = 0 \\ & l \leq x \leq u \end{array}$$

Outline

- minimum cost network flows
- **total unimodularity**
- examples

Totally unimodular matrix

a matrix is **totally unimodular** if all its minors are -1, 0, or 1

(a minor is the determinant of a square submatrix)

examples

- the matrix

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & -1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

- node-arc incidence matrix of a directed graph (proof on next page)

properties of a totally unimodular matrix A

- the entries A_{ij} ($i.e.$, its minors of order 1) are -1, 0, or 1
- the inverse of any nonsingular square submatrix of A has entries ± 1, 0

proof: let A be an $m \times n$ node-arc incidence matrix

- the entries of A are -1, 0, or 1
- A has exactly two nonzero entries (-1 and 1) per column

consider a $k \times k$ submatrix B of A

- if B has a zero column, its determinant is zero
- if all columns of B have two nonzero entries, then $\mathbf{1}^T B = 0$, $\det B = 0$
- otherwise B has a column, say column j, with one nonzero entry B_{ij}, so

$$\det B = (-1)^{i+j} B_{ij} \det C$$

  C is square of order $k-1$, obtained by deleting row i and column j of B

hence, can show by induction on k that all minors of A are ± 1 or 0

Integrality of extreme points

let P be a polyhedron in $\mathbf{R}^n$ defined by

$$Ax = b, \qquad l \leq x \leq u$$

where

- A is totally unimodular
- b is an integer vector
- the finite lower bounds l_k and finite upper bounds u_k are integers

then all the extreme points of P are integer vectors *proof:* apply rank test to determine whether $\hat{x} \in P$ is an extreme point

- partition $\{1, 2, \ldots, n\}$ in three sets J_0, J_-, J_+ with

$$\begin{array}{ll} l_k < \hat{x}_k < u_k & \text{for } k \in J_0 \\ \hat{x}_k = l_k & \text{for } k \in J_- \\ \hat{x}_k = u_k & \text{for } k \in J_+ \end{array}$$

  let A_0, A_-, A_+ be the submatrices of A with columns in J_0, J_-, J_+

- $\hat{x}$ is an extreme point if and only if

$$\operatorname{rank} \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & -I \\ A_0 & A_- & A_+ \end{bmatrix} = n \qquad \Longleftrightarrow \qquad A_0 \text{ has full column rank}$$

integrality of $\hat{x}$ then follows from $A_0 \hat{x}_{J_0} = b - A_- \hat{x}_{J_-} - A_+ \hat{x}_{J_+}$

- right-hand side is an integral vector ($\hat{x}_k$ is integer for $k \in J_- \cup J_+$)
- inverse of any nonsingular submatrix of A_0 has integer entries

Implications for combinatorial optimization

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & Ax = b \\
& l \le x \le u \\
& x \in \mathbf{Z}^n
\end{array}
$$

- an integer linear program, very difficult in general
- equivalent to its linear program relaxation

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & Ax = b \\
& l \le x \le u
\end{array}
$$

if A is totally unimodular and b, l, u are integer vectors

(extreme optimal solution of the relaxation is optimal for the integer LP)

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Shortest path problem

shortest path in directed graph with node-arc incidence matrix A

- (forward) paths from node 1 to m can be represented by vectors x with

$$Ax = (1, 0, \ldots, 0, -1), \qquad x \in \{0, 1\}^n$$

- shortest path is solution of

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T x \\ \text{subject to} & Ax = (1, 0, \ldots, 0, -1) \\ & x \in \{0, 1\}^n \end{array}$$

LP formulation

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T x \\ \text{subject to} & Ax = (1, 0, \ldots, 0, -1) \\ & 0 \preceq x \preceq \mathbf{1} \end{array}$$

extreme optimal solutions satisfy $x_i \in \{0, 1\}$

Birkhoff theorem

doubly stochastic matrix: $N \times N$ matrices X with $0 \le X_{ij} \le 1$ and

$$\sum_{i=1}^{N} X_{ij} = 1, \quad j = 1, \ldots, N, \qquad \sum_{j=1}^{N} X_{ij} = 1, \quad i = 1, \ldots, N$$

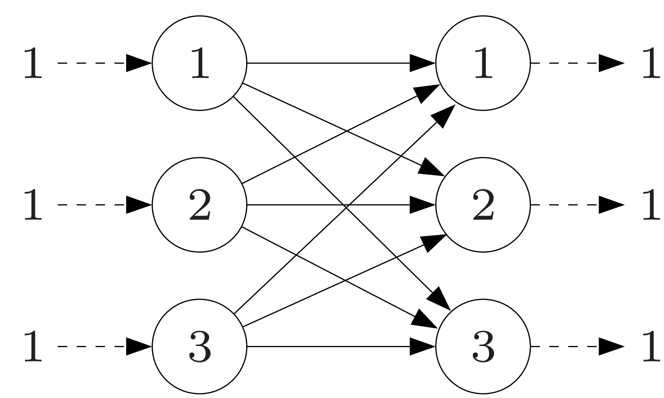
set of doubly stochastic matrices is a polyhedron P in $\mathbf{R}^{N \times N}$

theorem: the extreme points of P are the permutation matrices

proof: interpret X as network flow

- N input nodes, N output nodes
- X_{ij} is flow from input i to output j

hence extreme X has integer entries



example ($N = 3$)

Weighted bipartite matching

- match N persons to N tasks
- each person assigned to one task; each task assigned to one person
- cost of matching person i to task j is A_{ij}

LP formulation

$$
\begin{array}{ll}
\text{minimize} & \sum_{i,j=1}^{N} A_{ij} X_{ij} \\
\text{subject to} & \sum_{i=1}^{N} X_{ij} = 1, \quad j = 1, \ldots, N \\
& \sum_{j=1}^{N} X_{ij} = 1, \quad i = 1, \ldots, N \\
& 0 \leq X_{ij} \leq 1, \quad i, j = 1, \ldots, N
\end{array}
$$

integrality: extreme optimal solution X has entries $X_{ij} \in \{0, 1\}$