First order optimality conditions for constrained optimization

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Outline

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Problems with convex feasible set

Feasible set and active set

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Different feasible directions

First order necessary optimality conditions

- A geometric necessary condition
- KKT optimality conditions
- Constraint qualifications

First order sufficient conditions

Problems with convex feasible set

Let $f : R^n \to R$ be continuously differentiable and $\Omega \subset R^n$ be convex. Consider generic optimization of the form

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and the set of feasible directions at $x \in \Omega$ by

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It can be shown that

$$\mathcal{D}_{\text{feasible}}(x) := \{\beta(y - x) \mid y \in \Omega, \beta > 0\}.$$

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$$x^* \in \Omega \quad \text{and} \quad \mathcal{D}_{\text{feasible}}(x^*) \cap \mathcal{D}_{\text{descent}}(x^*) = \emptyset,$$

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It can be shown that x^* satisfies

$$x^* = \text{Proj}_\Omega \left(x^* - \nabla f(x^*)\right).$$

Therefore, solving the underlying optimization problem reduces to solving nonlinear equations

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More general problem is variational inequality:

$$\text{find } x^* \in \Omega \quad \text{such that} \quad (x - x^*)^T F(x^*) \geq 0, \quad \forall x \in \Omega,$$

where $F : R^n \to R^n$ and $\Omega \subset R^n$.

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Though a geometric viewpoint for

$$\min\{f(x) : s.t.\ x \in \Omega\}$$

is possible, we first derive optimality conditions for the concrete problem

$$\begin{aligned} \min \quad & f(x) \\ s.t. \quad & c_i(x) = 0, \qquad i \in \mathcal{E}, \\ & c_i(x) \geq 0, \qquad i \in \mathcal{I}, \end{aligned}$$

where f and c_i ($i \in \mathcal{E} \cup \mathcal{I}$) are all smooth, real-valued functions defined on R^n, and $\mathcal{E}$ and $\mathcal{I}$ are finite sets of indices.

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Feasible set and active set

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Definition (active set)

Let $x \in \mathcal{F}$ be a feasible point. An inequality constraint $c_i(x) \geq 0$ is said to be active at x if $c_i(x) = 0$ and inactive at x if $c_i(x) > 0$; all equality constraints are said to be active at x. The active set at $x \in \mathcal{F}$ is

$$\mathcal{A}(x) := \mathcal{E} \cup \{i \in \mathcal{I} : c_i(x) = 0\}.$$

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- However, constraints can also make things more difficult. Example

$$\begin{array}{ll} \min & 0.01x_1^2 + (x_2 + 100)^2 \\ s.t. & x_2 - \cos x_1 \geq 0. \end{array}$$

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With the constraint, there are local solutions near

$$(x_1, x_2) = (k\pi, -1), \quad k = \pm 1, \pm 3, \pm 5, \ldots$$

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- It ensures that the objective function and the constraints all behave in a reasonably predictable way and therefore allows algorithms to make good choices for search directions.
- Graphs of nonsmooth functions contain “kinks" or “jumps" where the smoothness breaks down.
- The feasible region for constrained optimization problem may contain many kinks and sharp edges, and this, in general, does not mean that the constraint functions are nonsmooth.

- For example, the set

$$\Omega := \{x = (x_1, x_2) \mid \|x\|_1 \leq 1\}$$

can also be described by

$$\Omega := \{x = (x_1, x_2) \mid x_1 \pm x_2 \leq 1, -x_1 \pm x_2 \leq 1\}.$$

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- On the other hand, nonsmooth unconstrained problems can sometimes be reformulated as smooth constrained problems, e.g.,

$$\min_{x \in \Omega} \{f(x) := \max(x^2, x)\}$$

is equivalent to

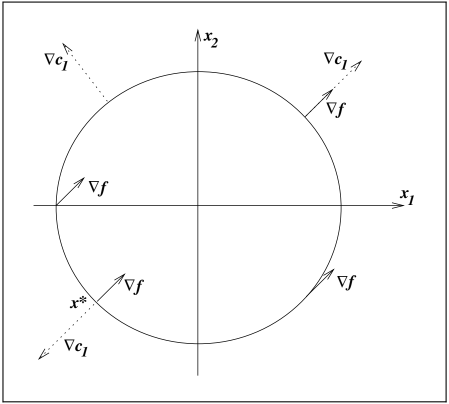
$$\min_{x,t} \{t : \ s.t. \ t \geq x, t \geq x^2, x \in \Omega\}.$$

Example: A single equality constraint

$$\min\{x_1 + x_2 : \ s.t. \ x_1^2 + x_2^2 - 2 = 0\},$$

i.e., $n = 2$, $\mathcal{I} = \emptyset$, $\mathcal{E} = \{1\}$,

$$f(x) = x_1 + x_2 \quad \text{and} \quad c_1(x) = x_1^2 + x_2^2 - 2.$$



- The solution is $x^* = (-1, -1)^T$;
- From other points on the circle, can find a way to move that stays feasible while decreasing f;
- It holds that $\nabla f(x^*) = \lambda_1^* \nabla c_1(x^*)$, where $\lambda_1^* = -\frac{1}{2}$;

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Suppose that $c_1(x) = 0$. We require $c_1(x + d) = 0$, i.e.,

$$0 = c_1(x + d) \approx c_1(x) + \nabla c_1(x)^T d = \nabla c_1(x)^T d.$$

Hence, the direction d retains feasibility w.r.t. c_1, to first order, when it satisfies $\nabla c_1(x)^T d = 0$.

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If there is no direction d such that

$$\nabla c_1(x)^T d = 0 \quad \text{and} \quad \nabla f(x)^T d < 0,$$

then is it likely that x is a local minimizer. Easy to check that such direction d does not exist only when $\nabla f(x)$ and $\nabla c_1(x)$ are parallel, i.e., $\nabla f(x) = \lambda_1 \nabla c_1(x)$ for some scalar λ_1.

Define Lagrangian function

$$\mathcal{L}(x, \lambda) := f(x) - \lambda_1 c_1(x).$$

For this example, x^* is optimal and there exists $\lambda_1^* \in R$ such that

$$\begin{aligned} \nabla_x \mathcal{L}(x^*, \lambda_1^*) &= 0, \\ c_1(x^*) &= 0. \end{aligned}$$

(will be made precise later.)

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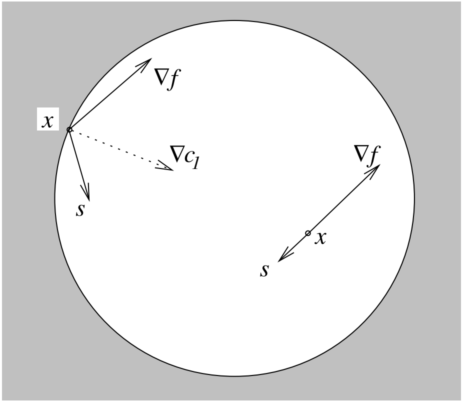
- Similar condition holds at $(1, 1)$, which indicates that this condition is not sufficient to be a minimum;
- By replacing c_1 by $-c_1$, we see that the sign of λ_1^* is not essential (in this case, λ_1^* changes from $-1/2$ to $1/2$).

Example: A single inequality constraint

$$\min\{x_1 + x_2 : \ s.t.\ 2 - x_1^2 - x_2^2 \geq 0\}.$$



- The solution is still $x^* = (-1, -1)^T$;
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Suppose that $c_1(x) \geq 0$. We require $c_1(x + d) \geq 0$, i.e.,

$$0 \leq c_1(x + d) \approx c_1(x) + \nabla c_1(x)^T d.$$

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- If $c_1(x) > 0$, then can find d such that

  $$c_1(x) + \nabla c_1(x)^T d \geq 0 \quad \text{and} \quad \nabla f(x)^T d < 0,$$

  unless $\nabla f(x) = 0$. Thus, if $c_1(x) > 0$ and $\nabla f(x) = 0$, then x is likely a local minimizer.

- Suppose $c_1(x) = 0$. The conditions

$$\nabla c_1(x)^T d \geq 0 \quad \text{and} \quad \nabla f(x)^T d < 0$$

cannot be simultaneously satisfied by some $d \in R^n$ only when $\nabla f(x)$ and $\nabla c_1(x)$ point in the same direction, i.e.,

$$\nabla f(x) = \lambda_1 \nabla c_1(x), \quad \text{for some } \lambda_1 \geq 0.$$

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Define Lagrangian function

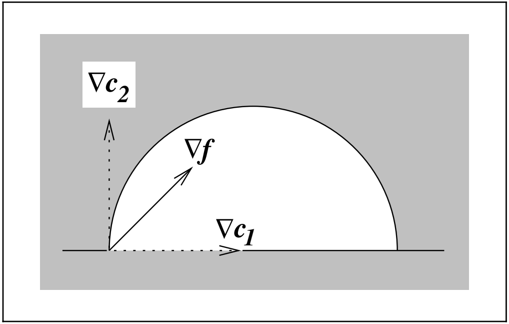
$$\mathcal{L}(x, \lambda) := f(x) - \lambda_1 c_1(x).$$

For this example, x^* is optimal and there exists $\lambda_1^* \in R$ such that

$$\begin{aligned} \nabla_x \mathcal{L}(x^*, \lambda_1^*) &= 0, \\ \lambda_1^* c_1(x^*) &= 0, \\ \lambda_1^* &\geq 0. \end{aligned}$$

Example: Two inequality constraints

$$\min\{x_1 + x_2 : \ s.t.\ 2 - x_1^2 - x_2^2 \geq 0, x_2 \geq 0\}.$$



- The solution is $x^* = (-\sqrt{2}, 0)^T$, a point at which both constraints are active. If both constraints are active at a feasible point x, then at this point a direction d is a feasible descent direction, to first-order, if it satisfies

$$\nabla c_i(x)^T d \geq 0,\ i \in \mathcal{I} = \{1, 2\}, \quad \nabla f(x)^T d < 0.$$

  Easy to see that at $(-\sqrt{2}, 0)^T$ no such direction exists.

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  Easy to see that at $(-\sqrt{2}, 0)^T$ no such direction exists.
- Define Lagrangian function by

$$\mathcal{L}(x, \lambda) = f(x) - \lambda_1 c_1(x) - \lambda_2 c_2(x).$$

  For this example, $x^* = (-\sqrt{2}, 0)^T$ is optimal and there exists λ^* such that

$$\begin{aligned} \nabla_x \mathcal{L}(x^*, \lambda^*) &= 0, \\ \lambda_i^* c_i(x^*) &= 0, \quad i \in \mathcal{I}, \\ \lambda_i^* &\geq 0, \quad i \in \mathcal{I}. \end{aligned}$$

  In fact, $\lambda^* = (\sqrt{2}/4, 1)^T$.

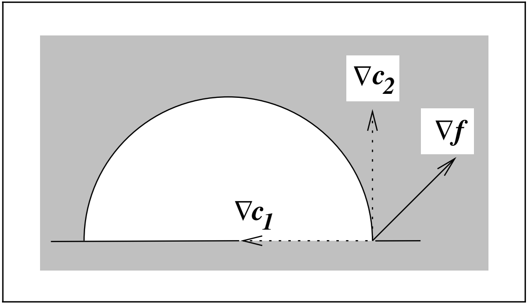
- At $x = (\sqrt{2}, 0)^T$, where both constraints are active, it is easy to find d such that

$$\nabla c_i(x)^T d \geq 0,\ i \in \mathcal{I} = \{1, 2\}, \quad \nabla f(x)^T d < 0.$$

Thus, $x = (\sqrt{2}, 0)^T$ is not a solution. Easy to show that at this point

$$\nabla_x \mathcal{L}(x, \lambda) = 0$$

holds only when $\lambda = (-\sqrt{2}/4, 1)^T$. Thus the three conditions given in the last slide cannot be satisfied simultaneously.



- At $x = (1, 0)^T$ only the constraint $c_2(x) = x_2 \geq 0$ is active. Since $c_1(x) > 0$ at this point, stepping forward in any direction d will maintain $c_1(x + d) > 0$ as long as d is sufficiently small. Thus, no need to take care of the first constraint. At $(1, 0)$, a direction d is feasible and descent, to first order, if it satisfies

$$\nabla c_2(x)^T d \geq 0, \quad \nabla f(x)^T d < 0.$$

Clearly such d exists, and thus $x = (1, 0)^T$ is not a solution. In this case, easy to show that the following conditions cannot be satisfied simultaneously:

$$\begin{aligned} \nabla_x \mathcal{L}(x^*, \lambda^*) &= 0, \\ \lambda_i^* c_i(x^*) = 0, \quad i &\in \mathcal{I}, \\ \lambda^* &\geq 0. \end{aligned}$$

- The previous examples show that it might be possible to derive necessary optimality conditions for $x^* \in \mathcal{F}$ to be a local optimal solution by using

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- However, this is not trivial since a geometric set can be described in different ways, and this could cause problem. For example, consider

$$\min\{f(x) := x_1 + x_2 :\ s.t.\ c_1(x) := (x_1^2 + x_2^2 - 2)^2 = 0\}.$$

  In this case, at the solution $x^* = (-1, -1)^T$, the following conditions cannot be fulfilled anymore:

$$\begin{aligned} \nabla_x \mathcal{L}(x^*, \lambda_1^*) &= 0 \quad \text{for some } \lambda_1^* \in R, \\ c_1(x^*) &= 0. \end{aligned}$$

  (can show that $\nabla c_1(x^*)^T d = 0$ for all $d \in R^n$.)

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Three types of feasible directions

Recall that the feasible set is defined by

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Definition (feasible directions)

Let $x \in \mathcal{F}$. A direction $d \in R^n$ is said to be a feasible direction at $x \in \mathcal{F}$ if there exists $\delta > 0$ such that

$$x + \alpha d \in \mathcal{F}, \quad \forall \alpha \in [0, \delta).$$

The set of all feasible directions at a feasible point $x \in \mathcal{F}$ is denoted by $FD(x, \mathcal{F})$.

Definition (linearized feasible directions)

Let $x \in \mathcal{F}$. Suppose that all functions c_i, $i \in \mathcal{I}$, are differentiable. A direction $d \in R^n$ is said to be a linearized feasible direction if it satisfies

$$\nabla c_i(x)^T d = 0, \quad i \in \mathcal{E}; \quad \nabla c_i(x)^T d \geq 0, \quad i \in \mathcal{I} \cap \mathcal{A}(x).$$

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Definition (sequential feasible directions)

Let $x \in \mathcal{F}$. A direction $d \in R^n$ is said to be a sequential feasible direction at $x \in \mathcal{F}$ if there exists $\{d_k \in R^n : k = 1, 2, \ldots\}$ and $\{\delta_k > 0 : k = 1, 2, \ldots\}$ such that

$$d_k \to d, \quad \delta_k \to 0 \quad \text{and} \quad x + \delta_k d_k \in \mathcal{F}, \quad \forall\, k.$$

The set of all sequential feasible directions at a feasible point $x \in \mathcal{F}$ is denoted by $SFD(x, \mathcal{F})$.

Remarks

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- $FD(x, \mathcal{F})$ *is simple, but mainly used when* $\mathcal{F}$ *is convex.*
- $SFD(x, \mathcal{F})$ *does not depend on how* $\mathcal{F}$ *is described and is completely determined by the geometry of* $\mathcal{F}$*. However, it is not easy to characterize and thus not easy to use. This set is also called the tangent cone at* $x \in \mathcal{F}$ *(elements in which are called tangent directions).*

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- *$LFD(x, \mathcal{F})$ is easy to use, but it depends on how $\mathcal{F}$ is described.*

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First order necessary optimality conditions

- A geometric necessary condition
- KKT optimality conditions
- Constraint qualifications

First order sufficient conditions

A geometric necessary condition

Theorem (Geometric necessary optimality condition)

Let x^* be a local optimal solution. Suppose f and c_i, $i \in \mathcal{I}$, are all differentiable. Then, it holds that

$$\nabla f(x^*)^T d \geq 0, \quad \forall d \in SFD(x^*, \mathcal{F}).$$

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Theorem

Let $x \in \mathcal{F}$. Suppose all c_i's are differentiable at x. It holds that

$$FD(x, \mathcal{F}) \subseteqq SFD(x, \mathcal{F}) \subseteqq LFD(x, \mathcal{F}).$$

Constraint qualifications

Note that $SFD(x, \mathcal{F}) \subsetneqq LFD(x, \mathcal{F})$ can happen. E.g., the singleton $\mathcal{F} = \{x^* = (0,0)^T\}$ can be expressed as

$$\mathcal{F} = \{x \in R^2 : c_1(x) \geq 0, c_2(x) \geq 0\},$$

where

$$c_1(x) = 1 - x_1^2 - (x_2 - 1)^2 \quad \text{and} \quad c_2(x) = -x_2.$$

Easy to verify that $SFD(x^*, \mathcal{F}) = \{(0,0)^T\}$ and

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Constraint qualifications (约束规范) are conditions under which there holds

$$SFD(x, \mathcal{F}) = LFD(x, \mathcal{F}).$$

Lemma (Farkas lemma)

Let p, q be any nonnegative integers. Suppose

$$\{a_i : i = 1, 2, \dots, p\},\ \{b_j : j = 1, 2, \dots, q\} \text{ and } v$$

are vectors in R^n. Then, the system

$$\begin{array}{rcll} a_i^T d & = & 0, & i = 1, 2, \dots, p, \\ b_j^T d & \geq & 0, & j = 1, 2, \dots, q, \\ v^T d & < & 0, & \end{array}$$

has no solution if and only if there exist $\lambda_i \in R$, $i = 1, 2, \dots, p$, and $\lambda_j \geq 0$, $j = 1, 2, \dots, q$, such that

$$v = \sum_{i=1}^{p} \lambda_i a_i + \sum_{j=1}^{q} \lambda_j b_j.$$

Theorem (KKT optimality conditions)

Suppose that all functions defining the problem are C^1 and that x^* is a local minimizer. If $SFD(x^*, \mathcal{F}) = LFD(x^*, \mathcal{F})$, then there exist $\lambda_i^* \in R$, $i \in \mathcal{E} \cup \mathcal{I}$, such that the following conditions are satisfied:

$$\begin{aligned}
\nabla_x \mathcal{L}(x^*, \lambda^*) &= \nabla f(x^*) - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i^* \nabla c_i(x^*) = 0, \\
c_i(x^*) &= 0, \quad i \in \mathcal{E}, \\
c_i(x^*) &\geq 0, \quad i \in \mathcal{I}, \\
\lambda_i^* &\geq 0, \quad i \in \mathcal{I}, \\
\lambda_i^* c_i(x^*) &= 0, \quad i \in \mathcal{I},
\end{aligned}$$

where $\mathcal{L}(x, \lambda)$ is called the Lagrange function and is defined by

$$\mathcal{L}(x, \lambda) := f(x) - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i c_i(x).$$

The vector λ is usually referred to as Lagrange multipliers.

- The first order necessary optimality conditions presented in the last slide is implied by the Farkas lemma.

[1] H. W. Kuhn and A. W. Tucker, Nonlinear programming, in: J. Neyman, ed., Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability (University of California Press, Berkeley, California, 1951) 481-492.

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- A point that satisfies the conditions is referred to as a KKT point. Without convexity assumptions, a KKT point is the best we can expect to achieve in most cases.
- The function $\mathcal{L}(x, \lambda)$ can be traced back to Lagrange (1760) and thus is referred to as the Lagrange function.

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LP example

Consider standard form LP and its dual:

$$\min\{c^T x : \ s.t. Ax = b, x \geq 0\} \quad \text{and} \quad \max\{b^T y : \ s.t. \ A^T y \leq c\}.$$

The Lagrange function: $\mathcal{L}(x, y, s) = c^T x - y^T(Ax - b) - s^T x$.
The KKT conditions:

$$\begin{aligned} \nabla_x \mathcal{L}(x, y, s) = c - A^T y - s = 0, \\ Ax = b, \quad x \geq 0, \\ s_i \geq 0, \ s_i x_i = 0, \ \forall i, \end{aligned}$$

are equivalent to

$$\begin{aligned} A^T y + s = c, \\ Ax = b, \quad x \geq 0, \\ s \geq 0, \ s^T x = 0. \end{aligned}$$

For LP, we already know that these conditions are also sufficient for x (resp. (y, s)) to be primal (resp. dual) optimal.

If $SFD(x^*, \mathcal{F}) \subsetneq LFD(x^*, \mathcal{F})$, a local minimizer can fail to be a KKT point

$$\begin{aligned} \min_{x \in R^2} \quad & x_1 \\ s.t. \quad & x_1^3 - x_2 \geq 0, \\ & x_2 \geq 0. \end{aligned}$$

- Global solution $x^* = (0, 0)^T$ is not a KKT point.
- Note that

$$\begin{aligned} SFD(x^*, \mathcal{F}) &= \left\{ d \in R^2 \mid d = (d_1, 0)^T, d_1 \geq 0 \right\}, \\ LFD(x^*, \mathcal{F}) &= \left\{ d \in R^2 \mid d = (d_1, 0)^T, d_1 \in R \right\}. \end{aligned}$$

The condition $SFD(x^*, \mathcal{F}) = LFD(x^*, \mathcal{F})$ is sufficient but not necessary

$$\begin{aligned} \min_{x \in R^2} \quad & x_2 \\ s.t. \quad & x_1^2 + (x_2 - 1)^2 - 1 = 0, \\ & x_1^2 + (x_2 + 1)^2 - 1 = 0. \end{aligned}$$

- Global solution $x^* = (0, 0)^T$ is a KKT point.
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Other constraint qualifications: LFCQ

Definition (LFCQ)

Let x^* be a local minimizer. If all active constraint functions

$$\{c_i(x) : \ i \in \mathcal{A}(x^*)\}$$

are linear functions, we say that linear function constraint qualification (LFCQ) holds at x^*.

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Theorem

Let x_0 be a feasible point. If all active constraint functions

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are linear functions, then it holds that $SFD(x_0, \mathcal{F}) = LFD(x_0, \mathcal{F})$.

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Corollary

If LFCQ holds at a local minimizer x^*, then x^* is a KKT point.

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Corollary

If LFCQ holds at a local minimizer x^*, then x^* is a KKT point.

Corollary

If all constraint functions $\{c_i(x) : \; i \in \mathcal{E} \cup \mathcal{I}\}$ are linear functions, then any local solution x^* is a KKT point.

Other constraint qualifications: LICQ

Definition (LICQ)

Let x^* be a local minimizer. If all active constraint gradients

$$\{\nabla c_i(x^*) : \; i \in \mathcal{A}(x^*)\}$$

are linearly independent, we say that the linear independence constraint qualification (LICQ) holds at x^*.

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Let $x_0 \in \mathcal{F}$ be a feasible point. If $\{\nabla c_i(x_0) : \; i \in \mathcal{A}(x_0)\}$ are linearly independent, then $SFD(x_0, \mathcal{F}) = LFD(x_0, \mathcal{F})$.

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Corollary

Let x^* be a local minimizer. If LICQ holds at x^*, then x^* is a KKT point.

Other constraint qualifications: MFCQ

Definition (MFCQ)

Let x^* be a local minimizer. If $\{\nabla c_i(x^*) : \ i \in \mathcal{E}\}$ are linearly independent, and

$$\{d \mid \nabla c_i(x^*)^T d = 0, i \in \mathcal{E}; \nabla c_i(x^*)^T d > 0, i \in \mathcal{A}(x^*) \cap \mathcal{I}\} \neq \emptyset,$$

then we say that Mangasarian-Fromowitz CQ holds at x^*.

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then we say that Mangasarian-Fromowitz CQ holds at x^*.

It can be shown that if MFCQ holds at a feasible point x then $SFD(x, \mathcal{F}) = LFD(x, \mathcal{F})$. Also, LICQ implies MFCQ.

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It can be shown that if MFCQ holds at a feasible point x then $SFD(x, \mathcal{F}) = LFD(x, \mathcal{F})$. Also, LICQ implies MFCQ.

Theorem

Let x^* be a local minimizer. If MFCQ holds at x^*, then x^* is a KKT point.

Outline

Theorem (A first order sufficient condition)

Suppose all functions are C^1. Let x^* be a feasible point. If

$$\nabla f(x^*)^T d > 0, \forall 0 \neq d \in SFD(x^*, \mathcal{F}),$$

then x^* is a strict local minimizer.

Proof.

Refer to Theorem 8.2.16 in the book by Yuan-Sun (Chinese version). $\square$

Theorem

Consider convex optimization problem of the form

$$\begin{aligned} \min_{x \in R^n} \quad & f(x) \\ s.t. \quad & Ax = b, \\ & c_i(x) \leq 0, \\ & i = 1, 2, \ldots, q, \end{aligned}$$

where $f \in C^1(R^n)$ *and* $c_i \in C^1(R^n)$, $i = 1, 2, \ldots, q$, *are all convex functions,* $A \in R^{p \times n}$ *and* $b \in R^p$. *If* x^* *is a KKT point, then it is a global optimal solution.*

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Define the Lagrange function $\mathcal{L} : R^n \times R^p \times R^q \to R$ by

$$\mathcal{L}(x, \lambda, \eta) := f(x) - \lambda^T (Ax - b) + \sum_{i=1}^{q} \eta_i c_i(x).$$

Clearly, for any fixed λ and η, $\mathcal{L}$ is convex in x.

Proof

Suppose x^* is a KKT point, i.e., there exist $\lambda^* \in R^p$ and $\eta^* \in R^q$ such that

$$\begin{aligned} \nabla f(x^*) - A^T\lambda^* + \sum_{i=1}^{q} \eta_i^* \nabla c_i(x^*) &= 0, \\ Ax^* &= b, \\ c_i(x^*) \leq 0, \quad \forall i = 1, 2, \ldots, q&, \\ \eta^* &\geq 0, \\ \eta_i^* c_i(x^*) = 0, \quad \forall i = 1, 2, \ldots, q&. \end{aligned}$$

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\eta^* &\geq 0, \\
\eta_i^* c_i(x^*) = 0, \quad \forall i = 1, 2, \ldots, q&.
\end{aligned}$$

Since $\mathcal{L}$ is convex and differentiable in x, for any feasible x, it holds that

$$\mathcal{L}(x, \lambda^*, \eta^*) \geq \mathcal{L}(x^*, \lambda^*, \eta^*) + \langle \nabla_x \mathcal{L}(x^*, \lambda^*, \eta^*), x - x^* \rangle.$$

Proof continued

Or, equivalently,

$$\begin{aligned} & f(x) + \sum_{i=1}^{q} \eta_i^* c_i(x) \\ \geq \quad & f(x^*) + \sum_{i=1}^{q} \eta_i^* c_i(x^*) + \langle \nabla_x \mathcal{L}(x^*, \lambda^*, \eta^*), x - x^* \rangle \\ = \quad & f(x^*). \end{aligned}$$

Proof continued

Or, equivalently,

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\begin{aligned}
& f(x) + \sum_{i=1}^{q} \eta_i^* c_i(x) \\
\geq \quad & f(x^*) + \sum_{i=1}^{q} \eta_i^* c_i(x^*) + \langle \nabla_x \mathcal{L}(x^*, \lambda^*, \eta^*), x - x^* \rangle \\
= \quad & f(x^*).
\end{aligned}
$$

As a result, for all feasible x, it holds that

$$
f(x) \geq f(x^*) - \sum_{i=1}^{q} \eta_i^* c_i(x) \geq f(x^*).
$$