MATH: Operations Research **2014-15 First Term**

Handout 3: 单纯形法的进一步讨论

Instructor: Junfeng Yang *September 17, 2014*

Contents

3.1 Slack variables

A basic feasible solution is sometimes immediately available for linear programming (LP for short) problems, e.g., in LP problems with constraints of the form

$$Ax \le b, \; x \ge 0,$$

where $b \ge 0$. A basic feasible solution is immediately available by introducing slack variables and transform the original problem to

$$\begin{array}{rl} \min & c^T x + 0^T y \\ s.t. & Ax + y = b \\ & x \ge 0, \; y \ge 0. \end{array}$$

However, an initial basic feasible solution is not always apparent.

3.2 Artificial variables

To find a basic feasible solution to

$$\begin{array}{rll} \min & c^T x & (\mathbf{LP}) \\ s.t. & Ax = b, \; x \ge 0, & \end{array}$$

where $b \ge 0$, we first solve an auxiliary LP:

$$\begin{array}{rll} \min & \sum_{i=1}^m y_i & (\mathbf{auxLP}) \\ s.t. & Ax + y = b, \; x \ge 0, \; y \ge 0. & \end{array}$$

Here $y = (y_1, \ldots, y_m)^T \in R^m$ is a vector of artificial variables. It is easy to see that (1) if **LP** has a feasible solution, then **auxLP** has a minimum value of zero with $y = 0$; and (2) if **LP** has no feasible solution, then the minimum value of **auxLP** is greater than 0.

Note that **auxLP** is already in canonical form with basic feasible solution $(x, y) = (0, b)$. Suppose **auxLP** is solved by the simplex method, a basic feasible solution is obtained at each step. If the minimum value of **auxLP** is zero, then the final basic solution will have $y = 0$. In this case, the x-part will give a basic feasible solution to the system $Ax = b, x \ge 0$.

In practice, one can first introduce slack variables and then introduce less number of artificial variables if necessary.

Example 3.2.1 *Find a basic feasible solution to*

$$\begin{array}{rcl} 2x_1 + x_2 + 2x_3 & = & 4 \\ 3x_1 + 3x_2 + x_3 & = & 3 \\ (x_1, x_2, x_3)^T & \ge & 0. \end{array}$$

Introduce artificial variables $x_4 \ge 0$, $x_5 \ge 0$ and an objective function $x_4 + x_5$. The initial tableau is

	x_1	x_2	x_3	x_4	x_5	b
	2	1	2	1	0	4
	3	3	1	0	1	3
c^T	0	0	0	1	1	0

To initiate the simplex procedure we must update the last row (via Gaussian reduction) so that it has zeros under the basic variables. This yields the first tableau:

$$\begin{array}{ccccccc} & x_1 & x_2 & x_3 & x_4 & x_5 & b \\ & 2 & 1 & 2 & 1 & 0 & 4 \\ & (3) & 3 & 1 & 0 & 1 & 3 \\ r^T & -5 & -4 & -3 & 0 & 0 & -7 \end{array}$$

Pivoting in the column having the most negative bottom row component as indicated, we obtain the second tableau:

$$\begin{array}{ccccccc} & x_1 & x_2 & x_3 & x_4 & x_5 & b \\ & 0 & -1 & (4/3) & 1 & -2/3 & 2 \\ & 1 & 1 & 1/3 & 0 & 1/3 & 1 \\ r^T & 0 & 1 & -4/3 & 0 & 5/3 & -2 \end{array}$$

Pivoting one more time, we obtain the next and also the final tableau:

$$\begin{array}{ccccccc} & x_1 & x_2 & x_3 & x_4 & x_5 & b \\ & 0 & -3/4 & 1 & 3/4 & -1/2 & 3/2 \\ & 1 & 5/4 & 0 & -1/4 & 1/2 & 1/2 \\ r^T & 0 & 0 & 0 & 1 & 1 & 0 \end{array}$$

Both of the artificial variables have been driven out of the basis, thus reducing the value of the objective function to zero and leading to the basic feasible solution to the original problem $(x_1, x_2, x_3) = (1/2, 0, 3/2)$.

3.3 Two-Phase Method

Using artificial variables, we solve a general LP problem by a two-phase method:

1. Phase I: Artificial variables are introduced and a basic feasible solution is found (or it is determined that no feasible solutions exist) for an auxiliary LP. Note that artificial variables need be introduced only in those equations that do not contain slack variables.

2. Phase II: Using the basic feasible solution resulting from phase I, the original LP can be solved by the simplex method.

Remark 3.3.1 *Note that in Phase I, we try to find a basic feasible solution via solving a LP. In this sense, finding a basic feasible solution is as difficult as solving the original LP. Indeed, this is true because solving LP is equivalent to solving a system of linear inequalities (will be made accurate later).*

Example 3.3.1 *Consider the problem*

$$\begin{aligned} \min \quad & 4x_1 + x_2 + x_3 \\ s.t. \quad & 2x_1 + x_2 + 2x_3 = 4 \\ & 3x_1 + 3x_2 + x_3 = 3 \\ & (x_1, x_2, x_3) \geq 0 \end{aligned}$$

A basic feasible solution has been found in Example 3.2.1. Deleting columns corresponds to artificial variables and replacing the last row by the new cost coefficients, we obtain the initial tableau

	x_1	x_2	x_3	b
	0	$-3/4$	1	$3/2$
	1	$5/4$	0	$1/2$
c^T	4	1	1	0

Transforming the last row so that zeros appear in the basic columns, we get the first tableau

	x_1	x_2	x_3	b
	0	$-3/4$	1	$3/2$
	1	$(5/4)$	0	$1/2$
r^T	0	$-13/4$	0	$-7/2$

Keep iterating, we arrive at

	x_1	x_2	x_3	b
	$3/5$	0	1	$9/5$
	$4/5$	1	0	$2/5$
r^T	$12/5$	0	0	$-11/5$

Thus the optimal solution is $(x_1, x_2, x_3) = (0, 2/5, 9/5)$.

Example 3.3.2 (A free variable problem)

$$\begin{aligned} \min \quad & -2x_1 + 4x_2 + 7x_3 + x_4 + 5x_5 \\ s.t. \quad & -x_1 + x_2 + 2x_3 + x_4 + 2x_5 = 7 \\ & -x_1 + 2x_2 + 3x_3 + x_4 + x_5 = 6 \\ & -x_1 + x_2 + x_3 + 2x_4 + x_5 = 4 \\ & x_1 \ \mathit{free}, (x_2, x_3, x_4, x_5) \geq 0. \end{aligned}$$

Since x_1 is free, it can be eliminated by solving for x_1 in terms of the other variables from the first equation and substituting everywhere else. This can all be done with the simplex tableau as follows:

	x_1	x_2	x_3	x_4	x_5	b
	(-1)	1	2	1	2	7
	-1	2	3	1	1	6
	-1	1	1	2	1	4
c^T	-2	4	7	1	5	0

Equivalent problem:

	x_1	x_2	x_3	x_4	x_5	b
	1	-1	-2	-1	-2	-7
	0	1	1	0	-1	-1
	0	0	-1	1	-1	-3
c^T	0	2	3	-1	1	-14

Multiplying the two equality constraints by -1 so that the right hand side become positive and introducing x_6 and x_7, we obtain the initial tableau of phase I:

$$\begin{array}{cccccccc} & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & b \\ & -1 & -1 & 0 & 1 & 1 & 0 & 1 \\ & 0 & 1 & -1 & 1 & 0 & 1 & 3 \\ c^T & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}$$

First tableau of phase I:

$$\begin{array}{cccccccc} & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & b \\ & -1 & -1 & 0 & (1) & 1 & 0 & 1 \\ & 0 & 1 & -1 & 1 & 0 & 1 & 3 \\ r^T & 1 & 0 & 1 & -2 & 0 & 0 & -4 \end{array}$$

Second tableau of phase I:

$$\begin{array}{cccccccc} & -1 & -1 & 0 & 1 & 1 & 0 & 1 \\ & (1) & 2 & -1 & 0 & -1 & 1 & 2 \\ r^T & -1 & -2 & 1 & 0 & 2 & 0 & -2 \end{array}$$

Final tableau of phase I:

$$\begin{array}{cccccccc} & 0 & 1 & -1 & 1 & 0 & 1 & 3 \\ & 1 & 2 & -1 & 0 & -1 & 1 & 2 \\ r^T & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}$$

Now we go back to the equivalent reduced problem

$$\begin{array}{cccccc} & x_2 & x_3 & x_4 & x_5 & b \\ & 0 & 1 & -1 & 1 & 3 \\ & 1 & 2 & -1 & 0 & 2 \\ c^T & 2 & 3 & -1 & 1 & -14 \end{array}$$

First tableau of phase II:

$$\begin{array}{cccccc} & 0 & 1 & -1 & 1 & 3 \\ & 1 & (2) & -1 & 0 & 2 \\ r^T & 0 & -2 & 2 & 0 & -21 \end{array}$$

Final tableau of phase II:

$$\begin{array}{cccccc} & -1/2 & 0 & -1/2 & 1 & 2 \\ & 1/2 & 1 & -1/2 & 0 & 1 \\ r^T & 1 & 0 & 1 & 0 & -19 \end{array}$$

Optimal solution: $(x_2, x_3, x_4, x_5) = (0, 1, 0, 2)$. The free variable x_1 and the optimal function value can then be computed.

3.4 Big-M Method

It is possible to combine the two phases of the two-phase method into a single procedure by the big-M method. Given the linear program in standard form

$$\begin{aligned} \min \quad & c^T x \\ s.t. \quad & Ax = b \\ & x \geq 0, \end{aligned}$$

one forms the approximating problem

$$\begin{array}{rl} \min & c^T x + M\sum_{i=1}^m y_i \\ s.t. & Ax + y = b \\ & x \geq 0 \\ & y \geq 0. \end{array}$$

In this problem $y = (y_1, y_2, \ldots, y_m)^T$ is a vector of artificial variables and $M > 0$ is a large constant. The term $M\sum_{i=1}^m y_i$ serves as a penalty term for nonzero y_i's. If this auxiliary LP problem is solved by the simplex method, the following conclusions are true:

1. If an optimal solution is found with $y = 0$, then the corresponding x is an optimal basic feasible solution to the original problem.
2. If for every $M > 0$ an optimal solution is found with $y \neq 0$, then the original problem is infeasible. Or equivalently, if the original problem is feasible, then the modified LP must have optimal solution with $y = 0$ for some $M > 0$.
3. If for every $M > 0$ the approximating problem is unbounded, then the original problem is either unbounded or infeasible.
4. Suppose that the original problem has a finite optimal value $V(\infty)$. Let $V(M)$ be the optimal value of the approximating problem. Then $V(M) \leq V(\infty)$.
5. For $M_1 \leq M_2$ we have $V(M_1) \leq V(M_2)$.
6. There exists $M_0 > 0$ such that for $M \geq M_0$, $V(M) = V(\infty)$, and hence the big-M method will produce the right solution for large enough values of M.

3.5 Matrix form of the simplex method

Let $B \in R^{m\times m}$ be a basis matrix, i.e., B is a nonsingular submatrix of A. As usual, we assume that B consists of the first m columns of A. Then by partitioning A, x and c as

$$A = (B, D),\ x^T = (x_B^T, x_D^T),\ c^T = (c_B^T, c_D^T),$$

the standard LP can be rewritten as

$$\begin{array}{rl} \min & c_B^T x_B + c_D^T x_D \\ s.t. & Bx_B + Dx_D = b \\ & x_B \geq 0, x_D \geq 0. \end{array}$$

Letting $x_D = 0$, we obtain a basic solution $x = (B^{-1}b, 0)$ to $Ax = b$. If B is a feasible basis, then the basic solution $x = (B^{-1}b, 0)$ is also feasible. The basic and nonbasic variables are related by

$$x_B = B^{-1}b - B^{-1}Dx_D. \tag{3.1}$$

Deleting basic variables in the objective function yields

$$z = c_B^T(B^{-1}b - B^{-1}Dx_D) + c_D^T x_D = c_B^T B^{-1}b + (c_D^T - c_B^T B^{-1}D)x_D. \tag{3.2}$$

The equations (3.1) and (3.2) are the two most important relations in deriving the simplex method. The vector

$$r_D^T := c_D^T - c_B^T B^{-1} D$$

is the relative cost vector for nonbasic variables (for basic variables, the relative cost is always zero). The components of this vector will be used to determine which vector to bring into basis.

The initial simplex tableau takes the form

$$\begin{bmatrix} A & b \\ c^T & 0 \end{bmatrix} = \begin{bmatrix} B & D & b \\ c_B^T & c_D^T & 0 \end{bmatrix}$$

which is, in general, not in canonical form and does not correspond to a point in the simplex procedure. If the matrix B is used as a basis, then the corresponding tableau becomes

$$T = \begin{bmatrix} I & B^{-1}D & B^{-1}b \\ 0 & c_D^T - c_B^T B^{-1} D & -c_B^T B^{-1} b \end{bmatrix}$$

which is the matrix form of the simplex method.

3.6 The revised simplex method

Experience based on extensive computation: the simplex method converges to an optimal basic feasible solution in around $m \sim 1.5m$ pivot operations. In particular, if $m \ll n$, i.e., if the matrix A has far fewer rows than columns, pivots will occur in only a small fraction of the columns during the course of optimization. Since the other columns are not explicitly used, it appears that the work expended in calculating the elements in these columns after each pivot is, in some sense, wasted effort. The revised simplex method aims to avoid unnecessary calculations.

Given the inverse B^{-1} of the current basis, and the current solution $x_B = y_0 = B^{-1}b$, do the following:

1. Calculate the current relative cost coefficients $r_D^T = c_D^T - c_B^T B^{-1} D$. This can best be done by first calculating $\lambda^T = c_B^T B^{-1}$ and then the relative cost vector $r_D^T = c_D^T - \lambda^T D$. If $r_D \geq 0$, the current solution is optimal.
2. Determine which vector a_q is to enter the basis by selecting, say, the most negative cost coefficient; and calculate $y_q = B^{-1} a_q$ which gives the vector a_q expressed in terms of the current basis.
3. If no y_{iq} (the ith component of y_q) is greater than 0, stop; the problem is unbounded. Otherwise, calculate the ratios y_{i0}/y_{iq} for $y_{iq} > 0$ to determine which vector is going to leave the basis.
4. Update B^{-1} and the current solution $B^{-1}b$. Return to Step 1.

The update of B^{-1} can be done by the usual pivot operations applied to an array consisting of B^{-1} and y_q, where the pivot is the appropriate element in y_q, as explained below. Suppose the current basis is $B = (a_1, a_2, \ldots, a_m)$, a_q is going to enter basis and a_p is going to leave.

$$\begin{aligned} A &= (B, D) \xrightarrow{B^{-1}} (I, B^{-1}D) \\ &= (B^{-1}a_1, \ldots, B^{-1}a_m, B^{-1}a_{m+1}, \ldots, B^{-1}a_n) \\ &= (e_1, \ldots, e_p, \ldots, e_m, y_{m+1}, \ldots, y_q, \ldots, y_n) \\ &\xrightarrow{Q} (Qe_1, \ldots, Qe_p, \ldots, Qe_m, Qy_{m+1}, \ldots, Qy_q, \ldots, Qy_n) \\ &= (e_1, \ldots, e_{p-1}, Qe_p, e_{p+1}, \ldots, e_m, \\ &\quad\; Qy_{m+1}, \ldots, Qy_{q-1}, e_p, Qy_{q+1}, \ldots, Qy_n). \end{aligned}$$

Now the new basis is $\bar{B} = (a_1, \ldots, a_{p-1}, a_{p+1}, \ldots, a_m, a_q)$. Clearly, it holds that

$$(QB^{-1})\bar{B} = (e_1, \ldots, e_{p-1}, e_{p+1}, \ldots, e_m, e_p),$$

which is a permutation of the identity matrix. So, we update B^{-1} by applying the same Gaussian reduction as we do for reducing y_q to e_p.

Example 3.6.1 (demonstrate the revised simplex method)

$$\begin{aligned} &\min -3x_1 - x_2 - 3x_3 + 0x_4 + 0x_5 + 0x_6 \\ &s.t. \begin{pmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_6 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix} \\ &\qquad\qquad\qquad\qquad\qquad x_i \geq 0,\, i = 1, \ldots, 6. \end{aligned}$$

The initial tableau of the simplex method:

	a_1	a_2	a_3	a_4	a_5	a_6	b
	2	1	1	1	0	0	2
	1	2	3	0	1	0	5
	2	2	1	0	0	1	6
c^T	-3	-1	-3	0	0	0	0

Let Ind_B and Ind_D be the set of indices of basic and nonbasic variables, respectively. The initial tableau of the the revised simplex method:

Ind_B	B^{-1}			x_B
4	1	0	0	2
5	0	1	0	5
6	0	0	1	6

$Ind_B = \{4, 5, 6\}$ *and* $Ind_D = \{1, 2, 3\}$. *Compute relative cost coefficients:*

$$\lambda^T = c_B^T B^{-1} = (0, 0, 0)B^{-1} = (0, 0, 0), \quad r_D^T = c_D^T - \lambda^T D = (-3, -1, -3).$$

We decide to bring a_2 into basis (to simplify the hand calculation). The representation of a_2 under the current basis B is given by

$$y_2 = B^{-1}a_2 = (1, 2, 2)^T.$$

Thus, we have

Ind_B	B^{-1}			x_B	y_2
4	1	0	0	2	(1)
5	0	1	0	5	2
6	0	0	1	6	2

Select (1) as pivot element (thus a_2 enters basis and a_4 leaves). Update B^{-1}, x_B, Ind_B and Ind_D:

Ind_B	B^{-1}			x_B
2	1	0	0	2
5	-2	1	0	1
6	-2	0	1	2

$Ind_B = \{2, 5, 6\}$ and $Ind_D = \{1, 3, 4\}$. Compute relative cost coefficients:

$$\lambda^T = c_B^T B^{-1} = (-1, 0, 0)B^{-1} = (-1, 0, 0), \quad r_D^T = c_D^T - \lambda^T D = (-1, -2, 1).$$

Bring a_3 into basis (corresponds to -2 in r_D). Compute the representation of a_3 under current basis B: $y_3 = B^{-1}a_3 = (1, 1, -1)^T$. Thus, we have

Ind_B	B^{-1}			x_B	y_3
2	1	0	0	2	1
5	-2	1	0	1	(1)
6	-2	0	1	2	-1

Select (1) as pivot element (thus a_3 enters basis and a_5 leaves). Update B^{-1}, x_B, Ind_B and Ind_D:

Ind_B	B^{-1}			x_B
2	3	-1	0	1
3	-2	1	0	1
6	-4	1	1	3

$Ind_B = \{2, 3, 6\}$ and $Ind_D = \{1, 4, 5\}$. Compute relative cost coefficients:

$$\lambda^T = c_B^T B^{-1} = (-1, -3, 0)B^{-1} = (3, -2, 0), \quad r_D^T = c_D^T - \lambda^T D = (-7, -3, 2).$$

Bring a_1 into basis (corresponds to -7 in r_D). Compute the representation of a_1 under current basis B: $y_1 = B^{-1}a_1 = (5, -3, -5)^T$. Thus, we have

Ind_B	B^{-1}			x_B	y_1
2	3	-1	0	1	(5)
3	-2	1	0	1	-3
6	-4	1	1	3	-5

Select (5) as pivot element (thus a_1 enters basis and a_2 leaves). Update B^{-1}, x_B, Ind_B and Ind_D:

Ind_B	B^{-1}			x_B
1	$3/5$	$-1/5$	0	$1/5$
3	$-1/5$	$2/5$	0	$8/5$
6	-1	0	1	4

$Ind_B = \{1, 3, 6\}$ and $Ind_D = \{2, 4, 5\}$. Compute relative cost coefficients:

$$\lambda^T = c_B^T B^{-1} = (-3, -3, 0)B^{-1} = (-6/5, -3/5, 0), \quad r_D^T = c_D^T - \lambda^T D = (7/5, 6/5, 3/5).$$

Since $r_D \geq 0$, we conclude that the solution $x = (1/5, 0, 8/5, 0, 0, 4)^T$ is optimal.

3.7 Homework

用两阶段法求解

$$\begin{aligned} \min \quad & -3x_1 + x_2 + 3x_3 - x_4 \\ s.t. \quad & x_1 + 2x_2 - x_3 + x_4 = 0 \\ & 2x_1 - 2x_2 + 3x_3 + 3x_4 = 9 \\ & x_1 - x_2 + 2x_3 - x_4 = 6 \\ & x_i \geq 0, \; i = 1, 2, 3, 4. \end{aligned}$$

References

[Luenberger-Ye] David G. Luenberger and Yinyu Ye Linear and nonlinear programming.