MATH: Operations Research **2014-15 First Term**

Handout 2: 解线性规划的单纯形法

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Contents

2.1 Simplex (单纯形)

Definition 2.1 (general position) *A set of $d+1$ points $\{x_1, x_2, \ldots, x_{d+1}\}$ in R^d is said to be in general position if*

$$\begin{vmatrix} 1 & 1 & \ldots & 1 \\ x_1 & x_2 & \ldots & x_{d+1} \end{vmatrix} \neq 0.$$

Definition 2.2 (d-dimensional simplex, d 维单纯形) *In R^d, the convex hull of $d+1$ points in general position is called a d-dimensional simplex.*

Example 2.1.1 *A zero-dimensional simplex is a point; a one-dimensional simplex is a line segment; a two-dimensional simplex is a triangle and its interior; and a three-dimensional simplex is a tetrahedron and its interior.*

2.2 Pivots (旋转运算)

2.2.1 First interpretation

The linear equality constraints of LP (in standard form) are

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n & = & b_2 \\ \ldots & \ldots & \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n & = & b_m. \end{array}$$

In matrix form, they appear as $Ax = b$. Denote the ith row rector of A by a^i, $i = 1, 2, \ldots, m$. These constraints can be rewritten as

$$\begin{array}{rcl} a^1x & = & b_1 \\ a^2x & = & b_2 \\ & \ldots & \\ a^mx & = & b_m. \end{array}$$

Recall that we assume $m < n$ and rank$(A) = m$. Without loss of generality, we assume that the first m columns of A are linearly independent. Through Gaussian reduction, the linear equality constraints become the so-called *canonical form* (典范形式):

$$\begin{array}{cccccccl} x_1 & & & & + & y_{1,m+1}x_{m+1} + y_{1,m+2}x_{m+2} + \cdots + y_{1,n}x_n & = & y_{10} \\ & x_2 & & & + & y_{2,m+1}x_{m+1} + y_{2,m+2}x_{m+2} + \cdots + y_{2,n}x_n & = & y_{20} \\ & & \ddots & & & \vdots & \vdots & \vdots \\ & & & x_m & + & y_{m,m+1}x_{m+1} + y_{m,m+2}x_{m+2} + \cdots + y_{m,n}x_n & = & y_{m0}. \end{array}$$

Corresponding to this canonical form, $x_1, \ldots, x_m$ are called basic variables, the other variables are nonbasic, and the basic solution is

$$x_1 = y_{10}, \ldots, x_m = y_{m0}, x_{m+1} = 0, \ldots, x_n = 0,$$

or in vector form $x = (y_0, 0)^T$, where $y_0 = (y_{10}, \ldots, y_{m0})^T$. The system is called in canonical form if by some reordering of the equations and the variables its coefficients (and right-hand side) take the form

$$\begin{array}{cccccccc}
1 & & & & y_{1,m+1} & y_{1,m+2} & \cdots & y_{1,n} & y_{10} \\
& 1 & & & y_{2,m+1} & y_{2,m+2} & \cdots & y_{2,n} & y_{20} \\
& & \ddots & & \vdots & \vdots & \ddots & \vdots & \vdots \\
& & & 1 & y_{m,m+1} & y_{m,m+2} & \cdots & y_{m,n} & y_{m0}
\end{array}$$

The question solved by pivoting is this: given a system in canonical form, suppose a basic variable is to be made nonbasic and a nonbasic variable is to be made basic, what is the new canonical form corresponding to the new set of basic variables? Suppose in the canonical system we wish to replace the basic variable x_p, $1 \le p \le m$, by the nonbasic variable x_q, $q > m$. This can be done if and only if y_{pq} is nonzero; it is accomplished by dividing row p by y_{pq} to get a unit coefficient for x_q in the pth equation, and then subtracting suitable multiples of row p from each of the other rows in order to get a zero coefficient for x_q in all other equations.

Denoting the coefficients of the new system in canonical form by y'_{ij}, we have explicitly

$$y'_{ij} = \begin{cases} \frac{y_{ij}}{y_{pq}}, & i = p; \\ y_{ij} - \frac{y_{pj}}{y_{pq}} y_{iq}, & i \neq p, \end{cases} \quad j = 1, 2, \ldots, n.$$

The above equations are the pivot equations that frequently arise in LP. The element y_{pq} in the original system is said to be the pivot element (旋转元).

Example 2.2.1 *Consider the system in canonical form:*

$$\begin{array}{cccccc}
1 & & & 1 & 1 & -1 & 5 \\
& 1 & & 2 & -3 & 1 & 3 \\
& & 1 & -1 & 2 & -1 & -1
\end{array}$$

Find the basic solution having basic variables x_4, x_5, x_6.

2.2.2 Second interpretation

Denote the columns of A by $a_1, a_2, \ldots, a_n$. Suppose that the system is already in canonical form:

a_1	a_2	$\ldots$	a_m	a_{m+1}	a_{m+2}	$\ldots$	a_n	b
1	0	$\ldots$	0	$y_{1,m+1}$	$y_{1,m+2}$	$\ldots$	$y_{1,n}$	y_{10}
0	1	$\ldots$	0	$y_{2,m+1}$	$y_{2,m+2}$	$\ldots$	$y_{2,n}$	y_{20}
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
0	0	$\ldots$	1	$y_{m,m+1}$	$y_{m,m+2}$	$\ldots$	$y_{m,n}$	y_{m0}

Clearly, $a_j = y_{1j}a_1 + y_{2j}a_2 + \ldots + y_{mj}a_m$ for all j from 1 to n. Suppose we wish to replace the basic variable x_p, $1 \le p \le m$, by the nonbasic variable x_q, $q > m$ (note that this can be done if and only if $y_{pq} \neq 0$). It follows from $a_q = \sum_{i=1}^m y_{iq} a_i$ that

$$a_p = \frac{1}{y_{pq}} \left(a_q - \sum_{i=1, i \neq p}^{m} y_{iq} a_i \right).$$

Thus, for all j, it holds that

$$a_j = \sum_{i=1}^{m} y_{ij} a_i = \sum_{i=1, i\neq p}^{m} \left(y_{ij} - \frac{y_{iq}}{y_{pq}} y_{pj} \right) a_i + \frac{y_{pj}}{y_{pq}} a_q.$$

This again implies the same pivoting equations

$$y'_{ij} = \begin{cases} \frac{y_{pj}}{y_{pq}}, & i = p; \\ y_{ij} - \frac{y_{pj}}{y_{pq}} y_{iq}, & i \neq p, \end{cases} \quad j = 1, 2, \ldots, n.$$

2.3 Determination of vector to leave basis

Recall that the feasible region of LP is $\mathcal{F} = \{x \in R^n : Ax = b, x \geq 0\}$. Starting from a basic feasible solution, how can we generate a new basic solution that maintains feasibility?

In general, it is impossible to arbitrarily specify a pair of variables whose roles are to be interchanged and expect to maintain feasibility. However, it is possible to first arbitrarily specify which nonbasic variable is to become basic and then determine (according to certain rule) which basic variable should become nonbasic.

Assumption 2.3.1 *Every basic feasible solution is nondegenerate.*[1]

Suppose $x = (x_1, \ldots, x_m, 0, \ldots, 0)^T$ is a basic feasible solution (thus, $x_i > 0$, $i = 1, 2, \ldots, m$). It holds

$$x_1 a_1 + \ldots + x_m a_m = b.$$

Suppose we have decided to bring a_q ($q > m$) into basis, and in terms of the current basis, a_q can be represented as

$$a_q = y_{1q} a_1 + \ldots + y_{mq} a_m.$$

Let $\varepsilon \geq 0$. It follows that

$$(x_1 - \varepsilon y_{1q}) a_1 + \ldots + (x_m - \varepsilon y_{mq}) a_m + \varepsilon a_q = b.$$

- If $y_{iq} \leq 0$ for $i = 1, 2, \ldots, m$, it follows that $\mathcal{F}$ is unbounded.
- If otherwise, setting $\varepsilon = \min_i \{x_i / y_{iq} : y_{iq} > 0\}$ will give us a new basis with a_q replacing a_p, where p corresponds to the minimizing index defining ε. We note that p is uniquely determined under the nondegenerate assumption. The basic solution corresponding to the new basis maintains feasibility.

Suppose we start with a basic feasible solution determined by

a_1	a_2	$\ldots$	a_m	a_{m+1}	a_{m+2}	$\ldots$	a_n	b
1	0	$\ldots$	0	$y_{1,m+1}$	$y_{1,m+2}$	$\cdots$	$y_{1,n}$	y_{10}
0	1	$\ldots$	0	$y_{2,m+1}$	$y_{2,m+2}$	$\cdots$	$y_{2,n}$	y_{20}
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
0	0	$\ldots$	1	$y_{m,m+1}$	$y_{m,m+2}$	$\cdots$	$y_{m,n}$	y_{m0}

[1]This assumption is only for convenience of discussion. If it is violated, the simplex method can be amended properly.

Feasibility (and nondegeneracy) implies that $y_{i0} > 0$ for $i = 1, 2, \ldots, m$. We have determined to bring a_q into basis. Note that $x_i/y_{iq} = y_{i0}/y_{iq}$, $i = 1, 2, \ldots, m$, and

$$p = \arg\min_i \{x_i/y_{iq} = y_{i0}/y_{iq} : y_{iq} > 0\}.$$

a_1	a_2	$\ldots$	a_m	a_{m+1}	$\ldots$	a_q	$\ldots$	a_n	b
1	0	$\ldots$	0	$y_{1,m+1}$	$\ldots$	$y_{1,q}$	$\ldots$	$y_{1,n}$	y_{10}
0	1	$\ldots$	0	$y_{2,m+1}$	$\ldots$	$y_{2,q}$	$\ldots$	$y_{2,n}$	y_{20}
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
0	0	$\ldots$	1	$y_{m,m+1}$	$\ldots$	$y_{m,q}$	$\ldots$	$y_{m,n}$	y_{m0}

Example 2.3.1 *If we bring a_4 into basis, which one should leave?*

a_1	a_2	a_3	a_4	a_5	a_6	b
1	0	0	2	4	6	4
0	1	0	1	2	3	3
0	0	1	-1	2	1	1

2.4 Determination of vector to enter basis

Till far, we have not taken into account the objective function $c^T x$. The objective function determines which nonbasic vector is to become basic at the current basic feasible solution.

Suppose the current basic feasible solution is $x = (x_B, 0)^T = (y_{10}, \ldots, y_{m0}, 0, \ldots, 0)^T$, and correspondingly we have a canonical form tableau:

a_1	a_2	$\ldots$	a_m	a_{m+1}	a_{m+2}	$\ldots$	a_n	b
1	0	$\ldots$	0	$y_{1,m+1}$	$y_{1,m+2}$	$\ldots$	$y_{1,n}$	y_{10}
0	1	$\ldots$	0	$y_{2,m+1}$	$y_{2,m+2}$	$\ldots$	$y_{2,n}$	y_{20}
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
0	0	$\ldots$	1	$y_{m,m+1}$	$y_{m,m+2}$	$\ldots$	$y_{m,n}$	y_{m0}

The basic and nonbasic variables are linked by

$$x_i = y_{i0} - \sum_{j=m+1}^{n} y_{ij} x_j, \; i = 1, 2, \ldots, m.$$

Therefore, the objective function reads

$$\begin{aligned} z = c^T x &= \sum_{i=1}^{m} c_i \left(y_{i0} - \sum_{j=m+1}^{n} y_{ij} x_j \right) + \sum_{j=m+1}^{n} c_j x_j \\ &= \sum_{i=1}^{m} c_i y_{i0} + \sum_{j=m+1}^{n} \left(c_j - \sum_{i=1}^{m} c_i y_{ij} \right) x_j \\ &\triangleq z_0 + \sum_{j=m+1}^{n} (c_j - z_j) x_j, \end{aligned}$$

where $z_0 := \sum_{i=1}^{m} c_i y_{i0}$ and $z_j := \sum_{i=1}^{m} c_i y_{ij}$. This is the fundamental relation required to determine the pivot column, i.e., the nonbasic vector to become basic. The most important relations in deriving the simplex method:

$$\begin{cases} x_i = y_{i0} - \sum_{j=m+1}^{n} y_{ij} x_j, & \text{for } i = 1, 2, \ldots, m; \\ z = z_0 + \sum_{j=m+1}^{n} (c_j - z_j)\, x_j, & \text{where } z_j = \sum_{i=1}^{m} c_i y_{ij}. \end{cases}$$

Q: Starting at a basic feasible solution, which variable should enter basis?

A: Since our objective is minimization, we choose nonbasic variable x_j to enter basis if and only if $c_j - z_j < 0$, in which case increase x_j from 0 to a positive value will decrease the value of the objective function.

Q: What if more than one nonbasic variables are such that $c_j - z_j < 0$?

A: Various strategies can be utilized. The simplest one is to choose the one with the most negative $c_j - z_j$.

Q: What if $c_j - z_j \geq 0$ for all nonbasic variables?

A: In this case, the current basic feasible solution is already optimal.

Q: If x_q is to enter basis ($c_q - z_q < 0$), which one should leave?

A: x_p should leave, where $p = \arg\min_i \{y_{i0}/y_{iq} : y_{iq} > 0\}$. Under the nondegeneracy assumption, p is uniquely determined. The resulting new basic feasible solution will have objective function value smaller than z_0.

Q: What if $\arg\min_i \{y_{i0}/y_{iq} : y_{iq} > 0\}$ is not a singleton?

A: Degeneracy happens. If the simplex method is not amended properly, cycle may happen. In many applications, anticycling procedures are unnecessary (since cycle may not happen). However, many codes incorporate anticycling strategies for safety.

Q: What happens if $c_q - z_q < 0$ but $y_{iq} \leq 0$ for all $i \in \{1, 2, \ldots, m\}$?

A: $\mathcal{F}$ is unbounded, and the objective function can be made arbitrarily small (towards minus infinity).

Theorem 2.3 (Improvement of basic feasible solution) *(1) Given a nondegenerate basic feasible solution with corresponding objective value z_0, suppose that for some j there holds $c_j < z_j$. Then there is a feasible solution with objective value $z < z_0$. (2) If a_j can be substituted for some vector in the original basis to yield a new basic feasible solution, this new solution will have $z < z_0$. (3) If a_j cannot be substituted to yield a basic feasible solution, then $\mathcal{F}$ is unbounded and $c^T x$ can be made arbitrarily small (towards minus infinity) in $\mathcal{F}$.*

Theorem 2.4 (Optimality condition) *Any basic feasible solution satisfying $c_j \geq z_j$ for all j is optimal.*

The quantities $r_j \triangleq c_j - z_j = c_j - \sum_{i=1}^{m} c_i y_{ij}$, $j = m+1, \ldots, m$, are referred to as *relative cost coefficients*, which measure the cost of a variable relative to the current basis.

Example 2.4.1 *Take the diet problem as an example in which the nutritional requirements must be met exactly.*

a_1	a_2	$\ldots$	a_m	a_{m+1}	$\ldots$	a_q	$\ldots$	a_n	b
1	0	$\ldots$	0	$y_{1,m+1}$	$\ldots$	$y_{1,q}$	$\ldots$	$y_{1,n}$	y_{10}
0	1	$\ldots$	0	$y_{2,m+1}$	$\ldots$	$y_{2,q}$	$\ldots$	$y_{2,n}$	y_{20}
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
0	0	$\ldots$	1	$y_{m,m+1}$	$\ldots$	$y_{m,q}$	$\ldots$	$y_{m,n}$	y_{m0}

In this example, q represents a certain food. The food q can be synthetically replaced by the foods in the basis, i.e.,

$$a_q = y_{1,q}a_1 + y_{2,q}a_2 + \ldots + y_{m,q}a_m.$$

$r_q = c_q - z_q = c_q - \sum_{i=1}^m c_i y_{iq} < 0$ *implies that the price c_q of food q is cheaper than the price $\sum_{i=1}^m c_i y_{iq}$ of the* synthetic food q. *In this case, bring food q into basis will reduce the total cost. In this sense, for each j, $j = m+1, \ldots, n$, $z_j = \sum_{i=1}^m c_i y_{ij}$ is also called* synthetic price.

2.5 Simplex tableau

The standard form LP is equivalent to an augmented problem of the form

$$\min_{x,z} \left\{ z : s.t. \begin{pmatrix} A & 0 \\ c^T & -1 \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix}, \, x \geq 0 \right\}$$

An initial simplex tableau to this problem is given by

a_1	a_2	$\ldots$	a_m	a_{m+1}	$\ldots$	a_q	$\ldots$	a_n	a_{n+1}	b
1	0	$\ldots$	0	$y_{1,m+1}$	$\ldots$	$y_{1,q}$	$\ldots$	$y_{1,n}$	0	y_{10}
0	1	$\ldots$	0	$y_{2,m+1}$	$\ldots$	$y_{2,q}$	$\ldots$	$y_{2,n}$	0	y_{20}
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$	$\vdots$
0	0	$\ldots$	1	$y_{m,m+1}$	$\ldots$	$y_{m,q}$	$\ldots$	$y_{m,n}$	0	y_{m0}
c_1	c_2	$\ldots$	c_m	c_{m+1}	$\ldots$	c_q	$\ldots$	c_n	-1	0

Through Gaussian reduction, the last row can be transformed to

$$(0, \ldots, 0, r_{m+1}, \ldots, r_q, \ldots, r_n, -1, -z_0).$$

The second last column will not change if the exchange of basis vectors happen among the first n columns and can be deleted. The initial simplex tableau takes the form

a_1	a_2	$\ldots$	a_m	a_{m+1}	$\ldots$	a_q	$\ldots$	a_n	b
1	0	$\ldots$	0	$y_{1,m+1}$	$\ldots$	$y_{1,q}$	$\ldots$	$y_{1,n}$	y_{10}
0	1	$\ldots$	0	$y_{2,m+1}$	$\ldots$	$y_{2,q}$	$\ldots$	$y_{2,n}$	y_{20}
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
0	0	$\ldots$	1	$y_{m,m+1}$	$\ldots$	$y_{m,q}$	$\ldots$	$y_{m,n}$	y_{m0}
0	0	$\ldots$	0	r_{m+1}	$\ldots$	r_q	$\ldots$	r_n	$-z_0$

The basic feasible solution corresponding to this tableau is

$$x_i = \begin{cases} y_{i0}, & 1 \leq i \leq m; \\ 0, & m+1 \leq i \leq n. \end{cases}$$

By nondegeneracy assumption, we have $y_{i0} > 0$, $i = 1, 2, \ldots, m$. The corresponding objective value is z_0. Suppose we have selected $y_{p,q}$ to be the next pivot element (a_q to leave and a_p to enter basis), i.e.,

$$r_q < 0 \text{ and } y_{p,0}/y_{p,q} = \min_{i=1,2,\ldots,m} \{y_{i,0}/y_{i,q} : y_{i,q} > 0\}.$$

Recall that the current simplex tableau is

$$
\begin{array}{ccccc|ccccc}
a_1 & \dots & a_p & \dots & a_m & \dots & a_j & \dots & a_q & \dots \\
\hline
1 & & & & & \dots & y_{1,j} & \dots & y_{1,q} & \dots \\
 & \ddots & & & & & \vdots & & \vdots & \\
 & & 1 & & & \dots & y_{p,j} & \dots & (y_{p,q}) & \dots \\
 & & & \ddots & & & \vdots & & \vdots & \\
 & & & & 1 & \dots & y_{m,j} & \dots & y_{m,q} & \dots \\
\hline
0 & \dots & 0 & \dots & 0 & \dots & r_j & \dots & r_q & \dots
\end{array}
$$

Via Gaussian reduction, it is transformed to

$$
\begin{array}{ccccc|ccccc}
a_1 & \dots & a_p & \dots & a_m & \dots & a_j & \dots & a_q & \dots \\
\hline
1 & \dots & -\frac{1}{y_{p,q}} y_{1,q} & \dots & 0 & \dots & y_{1,j} - \frac{y_{p,j}}{y_{p,q}} y_{1,q} & \dots & 0 & \dots \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & \\
0 & \dots & \frac{1}{y_{p,q}} & \dots & 0 & \dots & \frac{y_{p,j}}{y_{p,q}} & \dots & 1 & \dots \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & \\
0 & \dots & -\frac{1}{y_{p,q}} y_{m,q} & \dots & 1 & \dots & y_{m,j} - \frac{y_{p,j}}{y_{p,q}} y_{m,q} & \dots & 0 & \dots \\
\hline
0 & \dots & -\frac{1}{y_{p,q}} r_q & \dots & 0 & \dots & r_j - \frac{y_{p,j}}{y_{p,q}} r_q & \dots & 0 & \dots
\end{array}
$$

The new basis is $\{a_1, \dots, a_{p-1}, a_{p+1}, \dots, a_m, a_q\}$. It can be shown that elements in the last row corresponding to nonbasic vectors are equal to the new relative cost coefficients. In fact, for $j \in \{m+1, \dots, n\}$, $j \neq q$, it holds that

$$
\begin{aligned}
r_j - \frac{y_{p,j}}{y_{p,q}} r_q &= c_j - \sum_{i=1}^{m} c_i y_{ij} - \frac{y_{p,j}}{y_{p,q}} \left(c_q - \sum_{i=1}^{m} c_i y_{iq} \right) \\
&= c_j - \sum_{i=1, i \neq p}^{m} c_i \left(y_{ij} - \frac{y_{p,j}}{y_{p,q}} y_{iq} \right) - c_q \frac{y_{p,j}}{y_{p,q}}.
\end{aligned}
$$

While for the new nonbasic vector a_p, it holds that

$$
-\frac{1}{y_{p,q}} r_q = -\frac{1}{y_{p,q}} \left(c_q - \sum_{i=1}^{m} c_i y_{iq} \right) = c_p - \sum_{i=1, i \neq p}^{m} c_i \left(-\frac{y_{iq}}{y_{p,q}} \right) - c_q \frac{1}{y_{p,q}}.
$$

2.6 The simplex method

The computational procedure of the simplex method is as follows.

1. Form an initial simplex tableau corresponding to a basic feasible solution. The relative cost coefficients can be found by row reduction and should also be appended as a last row.
2. If each $r_j \geq 0$, stop; the current basic feasible solution is optimal.
3. Select q such that $r_q < 0$ to determine which nonbasic variable is to become basic.
4. Calculate the ratios y_{i0}/y_{iq} for $y_{iq} > 0$, $i = 1, \dots, m$. If no $y_{iq} > 0$, stop; the problem is unbounded. Otherwise, select p as the index i corresponding to the minimum ratio.
5. Pivot on the pqth element, updating all rows including the last by Gaussian reduction. Return to step 2.

Remark 2.6.1 *(1) The process terminates only if optimality is achieved or unboundedness is discovered. (2) If neither condition is discovered at a given basic feasible solution, then the objective can be strictly decreased. (3) Since there are only a finite number of possible basic feasible solutions, and no basis repeats because of the strictly decreasing objective, the algorithm must reach a basis satisfying one of the two terminating conditions.*

Example 2.6.1 *Solve the following LP by simplex method.*

$$\begin{aligned} \max 3x_1 + x_2 + 3x_3 & & \\ s.t.\ 2x_1 + x_2 + x_3 &\leq& 2 \\ x_1 + 2x_2 + 3x_3 &\leq& 5 \\ 2x_1 + 2x_2 + x_3 &\leq& 6 \\ x_1, x_2, x_3 &\geq& 0. \end{aligned}$$

First, we transform it to standard form

$$\min -3x_1 - x_2 - 3x_3 + 0x_4 + 0x_5 + 0x_6$$
$$s.t. \begin{pmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_6 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix}$$
$$x_i \geq 0,\ i = 1, \ldots, 6.$$

The initial simplex tableau is

	a_1	a_2	a_3	a_4	a_5	a_6	b
	2	1	1	1	0	0	2
	1	2	3	0	1	0	5
	2	2	1	0	0	1	6
r^T	-3	-1	-3	0	0	0	0

Elements can be selected to pivot:

	a_1	a_2	a_3	a_4	a_5	a_6	b
	(2)	(1)	1	1	0	0	2
	1	2	(3)	0	1	0	5
	2	2	1	0	0	1	6
r^T	-3	-1	-3	0	0	0	0

For hand calculation, we select (1). Thus a_2 will enter basis and a_4 will leave. Through Gaussian reduction, we get the second simplex tableau

	a_1	a_2	a_3	a_4	a_5	a_6	b
	2	1	1	1	0	0	2
	-3	0	1	-2	1	0	1
	-2	0	-1	-2	0	1	2
r^T	-1	0	-2	1	0	0	2

Elements can be selected to pivot:

	a_1	a_2	a_3	a_4	a_5	a_6	b
	(2)	*1*	*1*	*1*	*0*	*0*	*2*
	-3	*0*	*(1)*	*-2*	*1*	*0*	*1*
	-2	*0*	*-1*	*-2*	*0*	*1*	*2*
r^T	*-1*	*0*	*-2*	*1*	*0*	*0*	*2*

We select (1). Thus a_3 will enter basis and a_5 will leave. The third simplex tableau

	a_1	a_2	a_3	a_4	a_5	a_6	b
	5	*1*	*0*	*3*	*-1*	*0*	*1*
	-3	*0*	*1*	*-2*	*1*	*0*	*1*
	-5	*0*	*0*	*-4*	*1*	*1*	*3*
r^T	*-7*	*0*	*0*	*-3*	*2*	*0*	*4*

Elements can be selected to pivot:

	a_1	a_2	a_3	a_4	a_5	a_6	b
	(5)	*1*	*0*	*(3)*	*-1*	*0*	*1*
	-3	*0*	*1*	*-2*	*1*	*0*	*1*
	-5	*0*	*0*	*-4*	*1*	*1*	*3*
r^T	*-7*	*0*	*0*	*-3*	*2*	*0*	*4*

We select (5). Thus a_1 will enter basis and a_2 will leave. The fourth simplex tableau

	a_1	a_2	a_3	a_4	a_5	a_6	b
	1	*1/5*	*0*	*3/5*	*-1/5*	*0*	*1/5*
	0	*3/5*	*1*	*-1/5*	*2/5*	*0*	*8/5*
	0	*1*	*0*	*-1*	*0*	*1*	*4*
r^T	*0*	*7/5*	*0*	*6/5*	*3/5*	*0*	*27/5*

$r \geq 0$*! So, we have reached an optimal solution:*

$$x^* = (1/5, 0, 8/5, 0, 0, 4)^T.$$

The optimal objective function value (of the modified "min" problem) is $-27/5$.

2.7 Homework

用单纯形法求解

$$\begin{aligned} \min \quad & -2x_1 - 4x_2 - x_3 - x_4 \\ s.t. \quad & x_1 + 3x_2 + x_4 \leq 4 \\ & 2x_1 + x_2 \leq 3 \\ & x_2 + 4x_3 + x_4 \leq 3 \\ & x_i \geq 0,\ i = 1, 2, 3, 4. \end{aligned}$$

References

[Luenberger-Ye] David G. Luenberger and Yinyu Ye Linear and nonlinear programming.