Handout 4: Linear programming duality theory – Weak duality

Instructor: Junfeng Yang *September 21, 2014*

Contents

4.1 Introduction

Associated with every LP (linear program), and intimately related to it, is a corresponding dual LP. The primal and the dual problems are constructed from the same underlying cost and constraint coefficients. If the primal is to minimize, then the dual is to maximize, and vice versa. The optimum function values of both the primal and the dual, if finite, are equal. The variables of the dual problem can be interpreted as prices associated with the constraints of the primal problem. The variables of the dual problem are also intimately related to the calculation of the relative cost coefficients in the simplex method. In summary, studying the dual LP sharpens our understanding.

4.2 Dual problem of LP and weak duality

It is convenient to assume that the minimum value of any real valued function over an empty set is $+\infty$, while the maximum value of any real valued function over an empty set is $-\infty$, i.e.,

$$+\infty = \min_x \{f(x) : \ s.t.\ x \in \emptyset\} \quad \text{and} \quad -\infty = \max_x \{f(x) : \ s.t.\ x \in \emptyset\}.$$

In this course, we use the above as convention.

4.2.1 Standard form

Notation 4.2.1

- *$A \in R^{m\times n}$, $b \in R^m$ and $c \in R^n$;*
- *$\mathcal{F}_1 := \{x \in R^n : Ax = b, x \geq 0\}$;*
- *$R^n_+ := \{x \in R^n : x \geq 0\}$.*

Consider LP in standard form:

$$p_1^* := \min_{x\in R^n} \{c^T x : s.t.\ Ax = b, x \geq 0\}. \tag{Primal-LP1}$$

The Lagrange function associated with Primal-LP1 is defined by

$$\mathcal{L}(x, \lambda, \mu) = c^T x - \lambda^T (Ax - b) - \mu^T x,$$

which is a function from $R^n \times R^m \times R^n$ to R. For any $x \in \mathcal{F}_1$, $\lambda \in R^m$ and $\mu \in R^n_+$, it is clear that

$$\mathcal{L}(x, \lambda, \mu) \leq c^T x.$$

Therefore, for any $\lambda \in R^m$ and $\mu \in R^n_+$, we have

$$\inf_{x\in R^n} \mathcal{L}(x, \lambda, \mu) \leq \inf_{x\in\mathcal{F}_1} \mathcal{L}(x, \lambda, \mu) \leq \inf_{x\in\mathcal{F}_1} c^T x = p_1^*,$$

i.e., $\inf_{x\in R^n} \mathcal{L}(x, \lambda, \mu)$ is a lower bound of the optimal function value p_1^*. It is natural to raise a question like what is the best low bound of p_1^* obtained in this way? In fact, the best lower bound of p_1^* that can be so obtained is

$$\begin{aligned}
& \sup\nolimits_{\lambda\in R^m, \mu\in R^n_+} \inf_{x\in R^n} \mathcal{L}(x, \lambda, \mu) \\
= \ & \sup\nolimits_{\lambda\in R^m, \mu\in R^n_+} \begin{cases} b^T\lambda, & \text{if } c - A^T\lambda - \mu = 0; \\ -\infty, & \text{o.w.} \end{cases} \\
= \ & \sup\nolimits_{\lambda\in R^m, \mu\in R^n_+} \{b^T\lambda : s.t.\ c - A^T\lambda - \mu = 0\} \\
= \ & \sup\nolimits_{\lambda\in R^m} \{b^T\lambda : s.t.\ A^T\lambda \leq c\}.
\end{aligned}$$

The following problem is called the *dual problem* of Primal-LP1:

$$d_1^* := \max_{\lambda \in R^m} \{b^T\lambda : s.t.\ A^T\lambda \le c\}. \tag{Dual-LP1}$$

Theorem 4.1 (Weak Duality) *Let p_1^* and d_1^* be, respectively, defined as above. Then, $d_1^* \le p_1^*$.*

Corollary 4.2 *1. If x and λ are feasible for Primal-LP1 and Dual-LP1, respectively, then $b^T\lambda \le d_1^* \le p_1^* \le c^T x$.*

2. *If x_0 and λ_0 are feasible for Primal-LP1 and Dual-LP1, respectively, and if $b^T\lambda_0 = c^T x_0$, then x_0 and λ_0 are optimal for their respective problems. Furthermore, $b^T\lambda_0 = d_1^* = p_1^* = c^T x_0$.*

3. *If $p_1^* = -\infty$, i.e., Primal-LP1 is feasible and unbounded below, then Dual-LP1 must be infeasible.*

4. *If $d_1^* = +\infty$, i.e., Dual-LP1 is feasible and unbounded above, then Primal-LP1 must be infeasible.*

4.2.2 Another form

Notation 4.2.2 $\mathcal{F}_2 := \{x \in R^n : Ax \ge b, x \ge 0\}$.

Consider LP in the form:

$$p_2^* := \min_{x \in R^n} \{c^T x : s.t.\ Ax \ge b, x \ge 0\}. \tag{Primal-LP2}$$

The Lagrange function associated with Primal-LP2 is defined by

$$\mathcal{L}(x, \lambda, \mu) = c^T x - \lambda^T(Ax - b) - \mu^T x,$$

which is a function from $R^n \times R^m \times R^n$ to R. For any $x \in \mathcal{F}_2$, $\lambda \in R_+^m$ and $\mu \in R_+^n$, it is clear that

$$\mathcal{L}(x, \lambda, \mu) \le c^T x.$$

Therefore, for any $\lambda \in R_+^m$ and $\mu \in R_+^n$, it holds that

$$\inf_{x \in R^n} \mathcal{L}(x, \lambda, \mu) \le \inf_{x \in \mathcal{F}_2} \mathcal{L}(x, \lambda, \mu) \le \inf_{x \in \mathcal{F}_2} c^T x = p_2^*,$$

i.e., $\inf_{x \in R^n} \mathcal{L}(x, \lambda, \mu)$ is a lower bound of the optimal function value p_2^*. The best lower bound of p_2^* that can be so obtained is

$$\begin{aligned}
& \sup_{\lambda \in R_+^m, \mu \in R_+^n} \inf_{x \in R^n} \mathcal{L}(x, \lambda, \mu) \\
= & \sup_{\lambda \in R_+^m, \mu \in R_+^n} \begin{cases} b^T\lambda, & \text{if } c - A^T\lambda - \mu = 0; \\ -\infty, & \text{o.w.} \end{cases} \\
= & \sup_{\lambda \in R_+^m, \mu \in R_+^n} \{b^T\lambda : s.t.\ c - A^T\lambda - \mu = 0\} \\
= & \sup_{\lambda \in R^m} \{b^T\lambda : s.t.\ A^T\lambda \le c, \lambda \ge 0\}.
\end{aligned}$$

The following problem is called the *dual problem* of Primal-LP2:

$$d_2^* := \max_{\lambda \in R^m} \{b^T\lambda : s.t.\ A^T\lambda \le c, \lambda \ge 0\}. \tag{Dual-LP2}$$

Theorem 4.3 (Weak Duality) *Let p_2^* and d_2^* be, respectively, defined as above for Primal-LP2 and Dual-LP2. Then, $d_2^* \le p_2^*$.*

Corollary 4.4
1. *If x and λ are, respectively, feasible for Primal-LP2 and Dual-LP2, then $b^T\lambda \le d_2^* \le p_2^* \le c^T x$.*
2. *If x_0 and λ_0 are, respectively, feasible for Primal-LP2 and Dual-LP2, and if $b^T\lambda_0 = c^T x_0$, then x_0 and λ_0 are optimal for their respective problems. Furthermore, $b^T\lambda_0 = d_2^* = p_2^* = c^T x_0$.*
3. *If $p_2^* = -\infty$, i.e., Primal-LP2 is feasible and unbounded below, then Dual-LP2 must be infeasible.*
4. *If $d_2^* = +\infty$, i.e., Dual-LP2 is feasible and unbounded above, then Primal-LP2 must be infeasible.*

4.2.3 More general form

Notation 4.2.3
- $A \in R^{m_1 \times n}$, $b \in R^{m_1}$, $c \in R^n$;
- $E \in R^{m_2 \times n}$, $f \in R^{m_2}$, $G \in R^{m_3 \times n}$, $h \in R^{m_3}$;
- $I, J, K \subset \{1, 2, \ldots, n\}$;
- $\mathcal{F}_3 := \{x \in R^n : Ax = b,\ Ex \ge f,\ Gx \le h,\ x_I \ge 0,\ x_J \le 0\}$.

Consider LP in the form:

$$p_3^* := \min_{x \in R^n} \left\{ c^T x \;\middle|\; \begin{array}{c} Ax = b,\ Ex \ge f,\ Gx \le h \\ x_I \ge 0,\ x_J \le 0,\ x_K \ \text{free} \end{array} \right\}. \qquad \text{(Primal-LP3)}$$

The associated Lagrange function is defined by

$$\mathcal{L}(x, u, v, w, \lambda, \mu) = c^T x - u^T(Ax - b) - v^T(Ex - f) - w^T(h - Gx) - \lambda^T x_I - \mu^T(-x_J),$$

which is a function from $R^n \times R^{m_1} \times R^{m_2} \times R^{m_3} \times R^{|I|} \times R^{|J|}$ to R. For any $x \in \mathcal{F}_3$, $u \in R^{m_1}, v \in R_+^{m_2}, w \in R_+^{m_3}, \lambda \in R_+^{|I|}$ and $\mu \in R_+^{|J|}$, it is clear that

$$\mathcal{L}(x, u, v, w, \lambda, \mu) \le c^T x.$$

Therefore, for any $u \in R^{m_1}, v \in R_+^{m_2}, w \in R_+^{m_3}, \lambda \in R_+^{|I|}$ and $\mu \in R_+^{|J|}$, it holds that

$$\inf_{x \in R^n} \mathcal{L}(x, u, v, w, \lambda, \mu) \le \inf_{x \in \mathcal{F}_3} \mathcal{L}(x, u, v, w, \lambda, \mu) \le \inf_{x \in \mathcal{F}_3} c^T x = p_3^*,$$

i.e., $\inf_{x \in R^n} \mathcal{L}(x, u, v, w, \lambda, \mu)$ is a lower bound of p_3^*. The best lower bound of p_3^* that can be so obtained is

$$\begin{array}{cll} & \sup_{u \in R^{m_1}, v \ge 0, w \ge 0, \lambda \ge 0, \mu \ge 0} & \inf_{x \in R^n} \mathcal{L}(x, u, v, w, \lambda, \mu) \\ = & \sup_{u \in R^{m_1}, v \in R_+^{m_2}, w \in R_+^{m_3}} & \left\{ \begin{array}{l} b^T u + f^T v - h^T w, \\ \text{if} \left\{ \begin{array}{l} (c - A^T u - E^T v + G^T w)_I \ge 0, \\ (c - A^T u - E^T v + G^T w)_J \le 0, \\ (c - A^T u - E^T v + G^T w)_K = 0. \end{array} \right. \\ -\infty, \text{o.w.} \end{array} \right. \end{array}$$

The following problem is called the *dual problem* of Primal-LP3:

$$d_3^* := \max_{u, v, w} \left\{ b^T u + f^T v - h^T w \;\middle|\; \begin{array}{c} (c - A^T u - E^T v + G^T w)_I \ge 0, \\ (c - A^T u - E^T v + G^T w)_J \le 0, \\ (c - A^T u - E^T v + G^T w)_K = 0, \\ u \in R^{m_1}, v \in R_+^{m_2}, w \in R_+^{m_3}. \end{array} \right\} \qquad \text{(Dual-LP3)}$$

Theorem 4.5 (Weak Duality) *Let p_3^* and d_3^* be, respectively, defined as above for Primal-LP3 and Dual-LP3. Then, $d_3^* \le p_3^*$.*

Corollary 4.6

1. *If x and (u, v, w) are feasible for Primal-LP3 and Dual-LP3, respectively, then*

$$b^T u + f^T v - h^T w \le d_3^* \le p_3^* \le c^T x.$$

2. *If x_0 and (u_0, v_0, w_0) are, resp., feasible for Primal-LP3 and Dual-LP3, and if $b^T u_0 + f^T v_0 - h^T w_0 = c^T x_0$, then x_0 and (u_0, v_0, w_0) are optimal for their respective problems. Furthermore, $d_3^* = p_3^* = c^T x_0$.*
3. *If $p_3^* = -\infty$, i.e., Primal-LP3 is feasible and unbounded below, then Dual-LP3 must be infeasible.*
4. *If $d_3^* = +\infty$, i.e., Dual-LP3 is feasible and unbounded above, then Primal-LP3 must be infeasible.*

4.2.4 Summary

$$\begin{array}{rl} \min & c^T x \\ \text{s.t.} & Ax = b \\ & x \ge 0. \end{array}$$

(a) Primal-LP1

$$\begin{array}{rl} \max & b^T \lambda \\ \text{s.t.} & \lambda \text{ free} \\ & A^T \lambda \le c. \end{array}$$

(b) Dual-LP1

$$\begin{array}{rl} \min & c^T x \\ \text{s.t.} & Ax \ge b \\ & x \ge 0. \end{array}$$

(c) Primal-LP2

$$\begin{array}{rl} \max & b^T \lambda \\ \text{s.t.} & \lambda \ge 0 \\ & A^T \lambda \le c. \end{array}$$

(d) Dual-LP2

$$\begin{array}{rl} \min & c^T x \\ \text{s.t.} & Ax = b \\ & Ex \ge f \\ & Gx \le h \\ & x_I \ge 0 \\ & x_J \le 0 \\ & x_K \text{ free.} \end{array}$$

(e) Primal-LP3

$$\begin{array}{rl} \max & b^T u + f^T v - h^T w \\ \text{s.t.} & u \in R^{m_1} \\ & v \in R_+^{m_2} \\ & w \in R_+^{m_3} \\ & (c - A^T u - E^T v + G^T w)_I \ge 0 \\ & (c - A^T u - E^T v + G^T w)_J \le 0 \\ & (c - A^T u - E^T v + G^T w)_K = 0. \end{array}$$

(f) Dual-LP3

(g) Relation of primal and dual values.

Figure 4.1: Summary on Primal and Dual LPs.

Variables in the dual programs are called dual variables of the primal programs. Now we focus on the most general form Primal-LP3 and its dual problem. It is easy to see that (1) if an inequality constraint in the primal problem is changed to equality, then the corresponding dual variable will be freed; (2) If some of the components of x_I or x_J in the primal problem are freed, then the corresponding inequalities in the dual problem will become equality; Similar remarks apply to other constraints in the primal and the counterparts in the dual.

Theorem 4.7 *For any LP, the dual of the dual is itself.*

Proof: Take the standard form LP for example. Its dual problem is

$$d_1^* := \max\{b^T\lambda : s.t.\ A^T\lambda \leq c\}.$$

Let $\mathcal{F}_D := \{\lambda \in R^m : A^T\lambda \leq c\}$ and $\mathcal{L} : R^m \times R^n \to R$ be defined by

$$\mathcal{L}(\lambda, x) = b^T\lambda - x^T(A^T\lambda - c).$$

For any $\lambda \in \mathcal{F}_D$ and $x \in R^n_+$, it holds that $b^T\lambda \leq \mathcal{L}(\lambda, x)$. Therefore, for any $x \in R^n_+$, it holds that

$$d_1^* = \sup_{\lambda \in \mathcal{F}_D} b^T\lambda \leq \sup_{\lambda \in \mathcal{F}_D} \mathcal{L}(\lambda, x) \leq \sup_{\lambda \in R^m} \mathcal{L}(\lambda, x),$$

i.e., $\sup_{\lambda \in R^m} \mathcal{L}(\lambda, x)$ is an upper bound of the dual optimal function value d_1^*. It can be shown that the best upper bound of d_1^* that can be so obtained is

$$\inf_{x \in R^n_+} \sup_{\lambda \in R^m} \mathcal{L}(\lambda, x) = \inf_{x \in R^n_+} \{c^Tx : s.t.\ Ax = b\},$$

which is exactly the primal LP in standard form. Thus, the dual problem of $\max\{b^T\lambda : s.t.\ A^T\lambda \leq c\}$ is

$$\min_x \{c^Tx : s.t.\ Ax = b, x \geq 0\}.$$

For other forms of LP, the derivations are similar. ■

Remark 4.2.1 *If the objective is to minimize, then construct lower bound of the optimal value, while if the objective is to maximize, then construct upper bound. This is the key in deriving the dual problems. Lagrange function which incorporates the constraints into the objective function plays the central role.*

4.3 Explanations of dual LP

4.3.1 Diet problem

The diet problem takes the form

$$\begin{array}{rc} \min & c_1x_1 + c_2x_2 + \ldots + c_nx_n \\ s.t. & a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \geq b_1 \\ & a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \geq b_2 \\ & \ldots \\ & a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \geq b_m \\ & x_1, x_2, \ldots, x_n \geq 0. \end{array}$$

Here c_j represents the unit price of food j, $j = 1, 2, \ldots, n$; each b_i represents the minimum requirement of nutrition i, $i = 1, 2, \ldots, m$; a_{ij} represents the quantity of nutrition i that can be provided by a unit of food j, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$; Each row of A corresponds to a certain nutrition, while each column of A corresponds to a certain food.

Suppose each nutrition is made into pill by a drug company. The unit price of the ith nutrition is λ_i, $i = 1, \ldots, m$. How to determine λ so that the m types of pills are competitive with the n types of real foods and meanwhile the revenue is maximized?

The dual of the diet problem:

$$\begin{array}{rc} \max & b_1\lambda_1 + b_2\lambda_2 + \ldots + b_m\lambda_m \\ s.t. & a_{11}\lambda_1 + a_{21}\lambda_2 + \ldots + a_{m1}\lambda_m \le c_1 \\ & a_{12}\lambda_1 + a_{22}\lambda_2 + \ldots + a_{m2}\lambda_m \le c_1 \\ & \ldots \\ & a_{1n}\lambda_1 + a_{2n}\lambda_2 + \ldots + a_{mn}\lambda_m \le c_n \\ & \lambda_1, \lambda_2, \ldots, \lambda_m \ge 0. \end{array}$$

Note that each row of A^T corresponds to a certain food, while each column of A^T corresponds to a certain nutrition.

4.3.2 Transportation problem

The transportation problem takes the form

$$\begin{array}{rc} \min & \sum_{i=1}^m \sum_{j=1}^n c_{ij}x_{ij} \\ s.t. & \sum_{j=1}^n x_{ij} = a_i,\ i = 1, 2, \ldots, m; \\ & \sum_{i=1}^m x_{ij} = b_j,\ j = 1, 2, \ldots, n; \\ & x_{ij} \ge 0,\ i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n. \end{array}$$

Here each c_{ij} represents the unit transportation cost from ith origin to jth destination, $i = 1, 2, \ldots, m$, $j = 1, \ldots, n$; each a_i represents the total amount of product needs to be shipped out at the ith origin, $i = 1, 2, \ldots, m$; each b_j represents the total amount of product needs to be shipped in at the jth destination, $j = 1, 2, \ldots, n$.

Suppose that an entrepreneur believes he can do better and plans to buy the product at all origins and sell it to all destinations. The price he is willing to buy the product at the ith origin is u_i, $i = 1, 2, \ldots, m$; The price he is willing to sell the product at the jth origin is v_j, $j = 1, 2, \ldots, n$; How to determine the prices u and v so that his offer is competitive and meanwhile the revenue is maximized?

The dual of the transportation problem:

$$\begin{array}{rc} \max & \sum_{j=1}^n b_jv_j - \sum_{i=1}^m a_iu_i \\ s.t. & v_j - u_i \le c_{ij} \\ & i = 1, 2, \ldots, m; \\ & j = 1, 2, \ldots, n. \end{array}$$

4.3.3 World Cup auction problem

Consider the World Cup auction problem. The data is give as below.

Order	#1	#2	#3	#4	#5	...
Argentina	1	0	1	1	0	...
Brazil	1	0	0	1	1	...
Italy	1	0	1	1	0	...
Germany	0	1	0	1	1	...
France	0	0	1	0	0	...
Bidding Price p	0.75	0.35	0.4	0.95	0.75	...
Quantity limit q	10	5	10	10	5	...
Order fill x	x_1	x_2	x_3	x_4	x_5	...

The LP model:

$$\max_{x,z}\{p^T x - z : \ s.t. \ Ax \le z\mathbf{1}, \ 0 \le x \le q\}.$$

The dual problem is

$$\min_{y,\lambda}\{q^T y : \ s.t. \ A^T\lambda + y \ge p, \ \mathbf{1}^T\lambda = 1, \lambda \ge 0, y \ge 0\}.$$

The dual variable λ can be interpreted as the price of the teams, i.e., λ_i is the price of the ith team. The following relations should be understandable:

1. $x_j > 0$ implies that $a_j^T\lambda \le p_j$
2. $0 < x_j < q_j$ implies that $a_j^T\lambda = p_j$
3. $a_j^T\lambda < p_j$ implies that $x_j = q_j$
4. $a_j^T\lambda > p_j$ implies that $x_j = 0$.

An optimal solution to the World Cup auction problem with the five players is given in the following table:

Order	#1	#2	#3	#4	#5	price λ
Argentina	1	0	1	1	0	0.2
Brazil	1	0	0	1	1	0.35
Italy	1	0	1	1	0	0.2
Germany	0	1	0	1	1	0.25
France	0	0	1	0	0	0
Bidding Price p	0.75	0.35	0.4	0.95	0.75	–
Quantity limit q	10	5	10	10	5	–
Order fill x	5	5	5	0	5	–

References

[Luenberger-Ye] David G. Luenberger and Yinyu Ye Linear and nonlinear programming.