A Search Algorithm for Continuous Phase Modulated Gravitational Waves Originating from Binary Neutron Star Systems

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-we are given time series data. -we are looking for phase modulated gravitational waves. -form of phase modulation depends on the nature of the binary orbit. [5] [2] [1] [3] [4]

I.   INTRODUCTION

As a consequence of Einstein's Theory of General Relativity, it has been postulated that information about the spatial distribution of matter in the universe is transmitted via a gravitational field. From a theoretical standpoint, gravitational waves are manifested in the form of detectable ripples within the fabric of space and time, by changes in gravitational field and as a function of the movement of masses.

Recently, the Laser Interferometer Gravitational-Wave Observatory (LIGO) confirmed the discovery of gravitational waves. Neutron stars have been implicated as one of the prime sources of gravitational waves. These stars are expected to emit single frequency waves over very long periods of time. Additionally, it has also been proposed that an appreciable fraction of detectable gravitational waves are binary neutron star systems — two neutron starts orbiting around each other.

A fundamental challenge to efficient detection of gravitational waves is that orbiting binary neutron star systems invoke phase modulations which reduce the power of the carrier frequency and distributes it to higher and lower harmonics. In essence, the power of the signal is dispersed over a range of frequencies. Since the noise observed in the LIGO data is approximately Gaussian and of low signal-to-noise ratio, the extraction of reliable information about gravitational waves is arduous and in some cases meta-data can be tenuous.

The goal of this paper is to design an algorithm that extracts the ABC, DEF and GHI by BLAH BLAH BLAH:

$$Ge^{i(\omega_c t + \Gamma \cos(\Omega t + \phi))} \rightarrow G\delta(\omega_c)\delta(\Gamma)\delta(\Omega) \qquad (1)$$

Previous methods have utilized, where f(t) is the data.

$$Ge^{i(\omega_c t + \Gamma \cos(\Omega t + \phi))} = f(t) \qquad (2)$$

$$Ge^{i\omega_c t} = f(t)e^{-i\Gamma \cos(\Omega t + \phi)} \qquad (3)$$

$$F_t\left[Ge^{i\omega_c t}\right] = F_t\left[f(t)e^{-i\Gamma \cos(\Omega t + \phi)}\right] \qquad (4)$$

invoking the convolution theorem

$$\frac{G}{\sqrt{2\pi}}\delta(\omega_c - \omega) = F_t[f(t)] \star F_t\left[e^{-i\Gamma \cos(\Omega t + \phi)}\right] \qquad (5)$$

invoking the Jacobi-Anger Expansion which has the form:

$$e^{iz\cos(\theta)} = J_0(z) + 2\sum_{n=1}^{\infty} i^n J_n(z)\cos(n\theta) \qquad (6)$$

so equation (2) becomes

$$\frac{G}{\sqrt{2\pi}}\delta(\omega_c - \omega) = F_t[f(t)]$$

$$\star F_t\left[J_0(-\Gamma) + 2\sum_{n=1}^{\infty} i^n J_n(-\Gamma)\cos(n(\Omega t + \phi))\right] \qquad (7)$$

Bessel Functions have the property

$$J_n(-z) = (-1)^n J(z) \qquad (8)$$

so

$$F_t\left[J_0(-\Gamma) + 2\sum_{n=1}^{\infty} i^n J_n(-\Gamma)\cos(n(\Omega t + \phi))\right]$$

$$= F_t\left[J_0(\Gamma) + 2\sum_{n=1}^{\infty} e^{-in\pi/2} J_n(\Gamma)\cos(n(\Omega t + \phi))\right]$$

$$= J_0(\Gamma)F_t[1] + 2\sum_{n=1}^{\infty} e^{-in\pi/2} J_n(\Gamma)F_t[\cos(n(\Omega t + \phi))]$$

$$(9)$$

where we have used $(-1)^n i^n = (-i)^n = e^{-in\pi/2}$, and the linearity of the Fourier Transform $F_t[f+g] = F_t[f] + F_t[g]$. The Fourier transform of the this is

$$F_t\left[\cos(n(\Omega t + \phi))\right] = \frac{1}{2\sqrt{2\pi}}\left[e^{in\phi}\delta(\omega - n\Omega) + e^{-in\phi}\delta(\omega + n\Omega)\right] \qquad (10)$$

so equation (4) becomes

$$G\delta(\omega_c - \omega) = F_t[f(t)] \star \left[J_0(\Gamma)\delta(\omega) + \sum_{n=1}^{\infty} J_n(\Gamma)\left[e^{in(\phi - \pi/2)}\delta(\omega - n\Omega\right.\right. \qquad (11)$$

define $F_t[f(t)](\omega) = F(\omega)$ so performing the convolution becomes

$$G\delta(\omega_c-\omega) = J_0(\Gamma)F(\omega)+\sum_{n=1}^{\infty} J_n(\Gamma)\left[e^{in(\phi-\pi/2)}F(\omega-n\Omega) + e^{-in(\phi}\right.$$

(12)

This algorithm has been implemented in the past to demodulate the carrier frequency to recover the true amplitude.

II. CARSON'S RULE

The important thing to take away from this is that it takes an infinite bandwidth to transmit a phase/frequency modulated signal regardless of how smooth it is, but in practice only a finite bandwidth is needed for an accurate approximation.

III. VARIOUS COMPLECATIONS THAT NEED TO BE DEALT WITH (DAVID AND COLIN)

-The LIGO data is interupted (ie. it comes in chunks– it is not a continuous streem of data). How will this be dealt with?

-Estimating the likelyhood that any signal we see is truly a discovery and not noise. (Error Analysis)

-Negative frequencies imply the wave is traveling backwards in time. This is not physically realistic. How can we deal with this? The wave we see is real, so in order for the fourier series to be real we need negative frequencies to cancel the imaginary terms.

-Real data obays the law of causality. That means the signal starts at t=0 and f(t)=0 for t¡0. Putting this restriction on our waveform adds an infinite number of higher frequencies to the fourier transform, even though our waveform doesn't truly contain those frequencies. How should this be dealt with? Hardy functions? (Colin)

-Nyquist frequency and the effects that puts on our wave. What waves can we hope to detect with this restriction? What restriction does this put on the (gamma, Omega, and omegac)?

IV. NOISE: SOURCES AND ELIMINATION

A. Noise Sources

Since the signal being searched for is relatively stable but of little amplitude, the reliability of data obtained from the detector is related to the degree of noise therein. Among others, quantum, thermal, seismic and Newtonian noise have been implicated as prime drivers of LIGO data adulteration. Since the prime noise contributor is quantum, a summation of all the noise sources are approximately Gaussian. Thus, it suffices to use the properties of Gaussian noise to understand the volatility of the data.
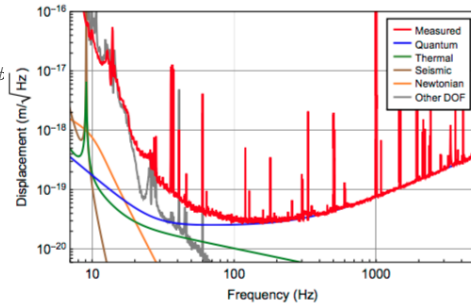
FIG. 1: This plot shows the theoretical noise sources, and the measured noise (red) in the interferometer.

B. Noise Characteristics

asdf

C. Noise Elimination

As mentioned before, most of the noise observed in data from LIGO is relatively Gaussian. By the same token, the proposed approach will assume the noise therein is Gaussian.

Let ξ be the recorded signal. Assume we have a phase-modulated signal X that has been adulterated with Gaussian noise ϵ, which give rise to ξ:

$$\xi = X + \epsilon \tag{13}$$

We first decompose the signal into intrinsic mode functions by empirical mode decomposition.

V. POSSIBLE OTHER AREAS TO REASEARCH

1) Instead of performing a Fourier Transform on the data f(t) maybe we could expand the data in terms of some other, cleverly chosen, set of orthogonal basis functions. Is this viable and what would be the set of functions? Can this reduce our search to a two dimentional search instead of a three dimentional search? (Task for Fourier Analysis Class MTH 490)

2) The LIGO interferometers sample the data at a finite frequency. This introduces a maximum possible frequency of Gravitational Waves that can be detected, up to the Niquest Frequency. This will, in tern, limit the maximum number of terms we can include in our sum (equation (9)). What is the biggest ω_c, Ω, and Γ that can be reasonably detected? (Task for Fourier Analysis Class MTH 490)

3) Obviously the duration of data is finite. This puts a limit on how finly the Fourier Transform can resolve frequencies, ie. how small frequency bins can be. This in tern tells us how small our step size in ω_c and Ω can

be. What are they? (Task for Fourier Analysis Class MTH 490)

4) Over very long periods of time $\omega_c \to 0$ because the neutron star loses energy/momentum to gravitational wave radation. Over relatively much much shorter time periods Ω and Γ change due to environmental reasons (not entirely sure on this). What limitations does this put on our search, especially if the explicit nature of the time evolution of these parameters is unknown? AN-SWER: it shortens the ammount of time we can take data over? (Task for Fourier Analysis Class MTH 490)

5) The Bessel Functions need to be calculated in a more efficient way. What is it? A basic investigation with Mathematica indicates we may only need to calculate the first Γ terms or less. That is if $\Gamma = 25.6830583$ then we may only need to take the sum in equation (10) to 25 or so. (Task for Computational Physics Class PHY 480)

6) For any given value of Γ there is a n such that $J_n(\Gamma)$ is maximized. What is that n? A basic investigation with a Mathematica script indicates $n \approx \Gamma$ is the answer. How accurate is this approximation? Can we show this analytically? How many terms in equation (9) do we need to go out from n Γ do we need to go to get an accurate answer? (Task for Computational Physics Class PHY 480)

7) There are multiple data channels (because there are multiple observatories). Could we compare data channels to help eliminate noise and pick out the exceptionally weak signals?

8) Some of the delta function type peaks on the noise spectrum (red) are known. For example the 60Hz noise from power lines. The LIGO scientists have ways of monitering this noise (they know the frequency, amplitude, and phase). Does the data we obtain have this already subtracted off? (Probably yes) Curtis will look into this.

VI. IMPLEMENTING THE GUSTAFSON ALGORITHM IN C++ (CURTIS RAU WILL DO THIS FOR ANOTHER CLASS)

Known problems with the algorithm include calculating large order bessel functions. The bessel functions of natural number order are given by

$$J_n(z) = \sum_{m=0}^{\infty} \frac{(-1)^n}{m!(m+n)!} \left(\frac{z}{2} \right)^{2m+n} \qquad (14)$$

so calculating the nth Bessel function involves calculating factorials greater than n!. The highest n for which we can store n! in an unsigned long integer type, the largest positive integer type, is 20:

$$2^{64} - 1 = 18,446,744,073,709,551,615 \qquad (15)$$
$$< 21! = 51,090,942,171,709,440,000 \qquad (16)$$

if we are willing to sacrafise some precision to truncation we can use a double precision floating point integer which allows values upto $1.797,693,134,862,315,7 \cdot 10^{308}$

$$170! \approx 7.25 \cdot 10^{306} \qquad (17)$$
$$< 1.797,693,134,862,315,7 \cdot 10^{308} < \qquad (18)$$
$$171! \approx 1.24 \cdot 10^{309} \qquad (19)$$

it has been shown that bessel functions up 250 or higher are necessary. This whole buiseness of wresteling with factorials can be avoided by calculating the Bessel Functions using Bessel's Equation 20 [3].

$$x^2 J_n''(x) + x J_n'(x) + (x^2 + n^2) J_n(x) = 0 \qquad (20)$$

To solve this second-order differential equation we will need two boundary conditions. The boundary condition at $x = 0$ is simple.

$$J_n(x) = \begin{cases} 1 & : n = 0 \\ 0 & : n \neq 0 \end{cases}$$

For the other boundary condition we will use the asymtotic form of the Bessel functions. Theorem 5.1 in [3] states:
For each $n \in N$ there is a constant $C_n \in R$ such that, if $x \geq 1$, then

$$\left\| J_n(x) - \sqrt{\frac{2}{\pi x}} \cos \left(x - \frac{\pi}{4}(2n+1) \right) \right\| \leq \frac{C_n}{x^{3/2}} \qquad (21)$$

The standard perscription for numerically solving differential equations is first to discretize the independent variable $x \to x_i = x_{min} + i \cdot \delta x$. Then use the limit definitions of the derivatives; we will use the three point definitions because their error goes as $O(\delta x^2)$ rather than using the two point definition with an error that goes as $O(\delta x)$. Also, because there are only one order of Bessel Functions in Bessel's Equation, the order is implied by the n that appears, so we will drop the subscript of n, $J_n \to J$, and we will further adopt the notation $J(x_i) = J_i$.

$$J_i' = \frac{J_{i+1} - J_{i-1}}{2\delta x} \qquad (22)$$
$$J_i'' = \frac{J_{i-1} - 2J_i + J_{i+1}}{\delta x^2} \qquad (23)$$

So the discretized form of Bessel's Equation is

$$x^2 \frac{J_{i-1} - 2J_i + J_{i+1}}{\delta x^2} + x \frac{J_{i+1} - J_{i-1}}{2\delta x} + (x^2 + n^2) J_i = 0 \qquad (24)$$

$$\left(\frac{x_i^2}{\delta x^2} - \frac{x_i}{2\delta x}\right) J_{i-1} + \left(\left(1 - \frac{2}{\delta x^2}\right) x_i^2 - n^2\right) J_i$$

$$+ \left(\frac{x_i^2}{\delta x^2} + \frac{x_i}{2\delta x}\right) J_{i+1} = 0 \quad (25)$$

Where we have collected like terms of J_i. For convenience we define coefficients to simplify the above equation.

$$a_i = \frac{x_i^2}{\delta x^2} - \frac{x_i}{2\delta x} \tag{26}$$

$$b_i = \left(1 - \frac{2}{\delta x^2}\right) x_i^2 - n^2 \tag{27}$$

$$c_i = \frac{x_i^2}{\delta x^2} + \frac{x_i}{2\delta x} \tag{28}$$

[1] Deanna Emery. A coherent three dimensional fft based search scheme for gravitational waves from binary neurton star systems. *LIGO DCC*, (T1500307-v2), September 2015.

[2] B.P. Abbott et al. (LIGO Scientific Collaboration and Virgo Collaboration). Observation of gravitational waves from a binary black hole merger. *Physical Review Letters*, 116(061102):1–16, 12 February 2016.

[3] Gerald B. Folland. *Fourier Analysis and its Applications*. Number ISBN 0-534-17097-3. Brooks and Cole Publishing Company, 1992.

[4] David J. Griffiths. *Introduction to Quantum Mechanics*. Number ISBN 0-13-111892-7. Pearson, second edition edition, 2005.

[5] Peter R. Saulson. *Fundamentals of Interferometric Gravitational Wave Detectors*. World Scientific Publishing Company, March 31, 1994.