

# Problem 585

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## 1 Problem

My dog Fido is tethered to a pole at a point  $P$  inside a square yard  $ABCD$ . The distance from  $P$  to  $A$  is 13 feet, to  $B$  is 5 feet, and to  $C$  is 17 feet. City ordinance requires she have an area of at least 300 square feet to roam. If her rope is 21 feet long, will the ordinance be satisfied?

## 2 Solution

Let  $s$  be the length of each side of the square  $ABCD$ . Place the square in  $\mathbb{R}^2$  as follows:  $A(0, s)$ ,  $B(0, 0)$ ,  $C(s, 0)$ , and  $D(s, s)$ . With the pole at the point  $(x, y)$  such that  $AP = 13$  ft,  $BP = 5$  ft, and  $CP = 17$  ft. We can construct a circle of radius 13 ft centered at  $A$ , a circle of radius 5 ft centered at  $B$ , and a circle of radius 17 ft centered at  $C$ , such that all three circles intersect at point  $P$ . The equations of these three circles are:

$$x^2 + (y - s)^2 = 13^2 \tag{1}$$

$$x^2 + y^2 = 5^2 \tag{2}$$

$$(x - s)^2 + y^2 = 17^2 \tag{3}$$

By substituting (2) into (1) and (3), we find

$$x = \frac{s^2 - 264}{2s} \text{ and } y = \frac{s^2 - 144}{2s}$$

Substituting both into equation (2) gives us  $s^4 - 458s^2 + 45216 = 0$ . By quadratic formula  $s^2 = 144$  or  $s^2 = 314$ . Because city ordinance requires a square field of at least 300 square feet only the second option is considered. Using the x and y equations, we find  $P = (\frac{25}{\sqrt{314}}, \frac{85}{\sqrt{314}})$ . By graphing a circle of radius 21 feet centered at  $P$  it can be seen that the circle encompasses the area of the entire square. Through calculus optimization it can be found that the closest the circle gets to the square is a distance of .191 ft. Therefore Fido has a total roaming area of 314 square feet satisfying the ordinance.