AMATYC J-4

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1 Problem

Start with a regular hexagon of side 1 as level 1. At each level $n-1$, form the regular hexagon of level n by joining consecutive midpoints of the one for level $n-1$. Find the lowest level at which the associated regular hexagon has a perimeter under 1.

2 Solution

Construct a regular hexagon with side length S_k, such that the perimeter $P_k = 6S_k$. Construct a second hexagon inscribed in the first with vertices intersecting the first's midpoints. Observe that six identical triangles are made between both hexagons with side lengths of the triangle of $\frac{S_k}{2}$ separated by an angle of 120° (the interior angle of the hexagon). Using law of cosines, the side length of the inscribed hexagon can be found to be $S_{k+1} = \frac{\sqrt{3}}{2}S_k$. We can construct an equation for P_k as such:

$$P_k = 6S_1 \cdot \left(\frac{\sqrt{3}}{2}\right)^{k-1}$$

Setting P_k to 1 and S_1 to 1, it can be found $k = 1 + \frac{\ln 6}{\ln \frac{2\sqrt{3}}{3}} \approx 13.45$. Therefore the smallest integer level n where $P_n \leq 1$ is $n = \lceil k \rceil = 14$ where $P_{14} \approx 0.92.\square$

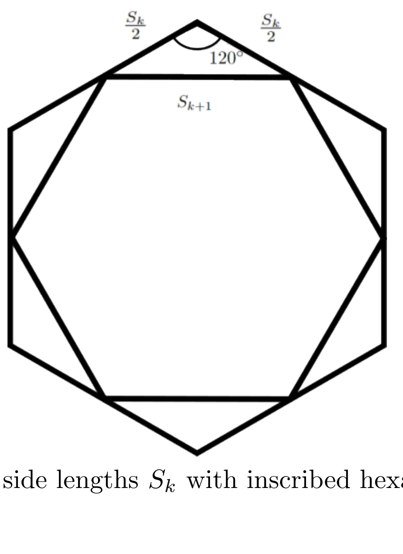

Diagram of hexagon with side lengths S_k with inscribed hexagon of side lengths S_{k+1}