Deconvoluting the approaches to the Deleglise-Rivat combinatorial prime counting algorithm

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$$\pi(x)=\phi(x,a)+a-1-P_2(x,a)$$
$$\textit{where } a=\pi(y)$$

After Deleglise and Rivat's report of an improvement to the Meissel-Lehmer-Lagarias-Miller-Odlyzko combinatorial approach for computing the above sum, there have been two papers describing additional approaches that follow the same lines. The methods of Tomas Oliveira e Silva and Xavier Gourdon differ from that of Deleglise-Rivat most notably in how this formula is partitioned into terms with the hope of simplifying the resulting implementations. In so doing, however, these authors have complicated the task of comparing these implementations, which I seek here to facilitate by deconvoluting the math used to arrive at them.

The nomenclature used is as follows:

$y: \ x^{1/3} \le y \le x^{1/2}$	$\phi(x,a)$: *The number of numbers* $\le x$ *that are not multiples of any of the first a primes*
$z: \ z=\frac{x}{y}$	
c: *a small non-negative integer*	$P_2(x,a)$: *The number of numbers* $\le x$ *with exactly two prime factors, neither of which is among the first a primes*
$\mu(m)$: *Mobius function*	
$\delta(m)$: *least prime factor*	

with the additional caveat that some simplifications are based on the use of large values of x, assuming that $c < z^{1/3}$.

The first task is the compuation of $P_2(x, a)$, which is described identically by Deleglise-Rivat and Oliveira e Silva, but broken up further by Gourdon:

$$P_2(x,a)=\sum_{y<p\le x^{1/2}}[\pi(\frac{x}{p})-\pi(p)+1]$$

$$P_2(x,a)=\frac{-\pi(x^{1/2})(\pi(x^{1/2})-1)}{2}+\frac{\pi(y)(\pi(y)-1)}{2}+(\sum_{y<p\le x^{1/2}}\pi(\frac{x}{p}))$$

$$\pi(y)-1-P_2(x,\pi(y))=\pi(y)-1+\frac{\pi(x^{1/2})(\pi(x^{1/2})+1)}{2}+\frac{\pi(y)(\pi(y)-1)}{2}-(\sum_{y<p\le x^{1/2}}\pi(\frac{x}{p}))$$

$$\underbrace{\pi(y)-1-P_2(x,\pi(y))}_{\textit{Deleglise-Rivat, Oliveira e Silva}}=\underbrace{\Sigma_0-B}_{\textit{Gourdon}}$$

Computation of the partial sieve function $\phi(x, a)$ is accomplished by repeated recurrence of the identity

$$\phi(x,b)=\phi(x,b-1)-\phi(\frac{x}{p_b},b-1)$$

which leads to the creation of a binary tree structure, whose growth is halted by applying two rules:

1. Do not split a node (*i.e.* apply this identity) if $b = c$ and $p_b \le y$, where c is a small constant such that $c \ge 0$,
2. Do not split a node if $p_b > y$.

This leads to the creation of two types of leaves. Those two meet the first criterion are known as *ordinary leaves*, while those that meet the second are *special leaves*. Ordinary leaves are computed separately in all works:

$$\textit{Ordinary Leaves}: \underbrace{S_0}_{\textit{Deleglise}-\textit{Rivat}} = \underbrace{S_1}_{\textit{Oliveira e Silva}} = \underbrace{\phi_0}_{\textit{Gourdon}} = \sum_{n \le y,\, \delta(n) > p_k} \mu(n)\phi(\frac{x}{n}, p_k)$$

The special leaves remain, and their computation is the main factor differentiating the three works discussed here. However, it is important to note that the constant c defines the parition between ordinary and special leaves, so the two classes are not completely independent.

$$\textit{Special Leaves}: \underbrace{S}_{\textit{Deleglise}-\textit{Rivat}} = \underbrace{\phi_1}_{\textit{Gourdon}} = -\sum_{p_c < p \le y} \sum_{\substack{\delta(m) > p \\ \frac{y}{p} < m \le y}} \mu(m)\phi(\frac{x}{mp}, \pi(p)-1)$$

These leaves are subdivided into trivial special leaves, easy special leaves, and hard special leaves by Oliveira e Silva, S_1, S_2, and S_3 by Deleglise and Rivat, and S_1 and S_2 by Gourdon (who further subdivides S_2). These classes are differentiated by the range over which the sum in S is conducted, and this division allows the computation of the easy and trivial leaves to be computed without directly calculating the partial sieve function for each term of the sum. The classes in the original algorithm are as follows:

$$S = S_1 + S_2 + S_3 = -\sum_{p_c < p \le y} \sum_{\substack{\delta(m) > p \\ m \le y < mp}} \mu(m)\phi(\frac{x}{mp}, \pi(p)-1)$$

$$S_1 = -\sum_{x^{1/3} < p \le y} \sum_{\substack{\delta(m) > p \\ \frac{y}{p} < m \le y}} \mu(m)\phi(\frac{x}{mp}, \pi(p)-1)$$

$$S_2 = -\sum_{x^{1/4} < p \le x^{1/3}} \sum_{\substack{\delta(m) > p \\ \frac{y}{p} < m \le y}} \mu(m)\phi(\frac{x}{mp}, \pi(p)-1)$$

$$S_3 = -\sum_{p_c < p \le x^{1/4}} \sum_{\substack{\delta(m) > p \\ \frac{y}{p} < m \le y}} \mu(m)\phi(\frac{x}{mp}, \pi(p)-1)$$

These classes *roughly* correspond to Oliveira e Silva's trivial, easy, and hard special leaves, respectively. Firstly, we can capture the trivial leaves with S_1 and part of S_2.

$$\underbrace{S_1}_{\textit{Deleglise}-\textit{Rivat}} = \underbrace{\Sigma_1}_{\textit{Gourdon}} = \frac{(\pi(y)-\pi(x^{1/3}))(\pi(y)-\pi(x^{1/3})-1)}{2}$$

$$Deleglise-Rivat:$$

$$S_2 = U+V$$

$$U = \sum_{x^{1/4}<p\le x^{1/3}} \sum_{\substack{p<q\le y \\ q>\frac{x}{p^2}}} \phi\left(\frac{x}{pq}, \pi(p)-1\right)$$

$$V = \sum_{x^{1/4}<p\le x^{1/3}} \sum_{\substack{p<q\le y \\ q\le\frac{x}{p^2}}} \phi\left(\frac{x}{pq}, \pi(p)-1\right)$$

$$U = \sum_{z^{1/2}<p\le x^{1/3}} \left[\pi(y)-\pi\left(\frac{x}{p^2}\right)\right] = \pi(y)\left(\pi(x^{1/3})-\pi(z^{1/2})\right) - \sum_{z^{1/2}<p\le x^{1/3}} \pi\left(\frac{x}{p^2}\right)$$

$$\ldots$$

$$\underbrace{S_1+U}_{Deleglise-Rivat} = \underbrace{S_{2trivial}}_{Oliveira\ e\ Silva}$$

In order to make a match with Gourdon's work, we have to subdivide one of his terms:

$$Gourdon:$$

$$\Sigma_2 = \pi(y)[\pi(x^{1/3})-\pi(z^{1/2})] - \pi(y)\frac{\pi(z^{1/2})(\pi(z^{1/2}) - 3)+\pi(x^{1/4})(\pi(x^{1/4})-3)}{2}$$

which we split into two parts:

$$\Sigma_2 = \Sigma_{2a}+\Sigma_{2b}$$

$$\Sigma_{2a} = \pi(y)[\pi(x^{1/3})-\pi(z^{1/2})+1]$$

$$\Sigma_{2b} = -\pi(y)\frac{\pi(z^{1/2})(\pi(z^{1/2}) - 3)+\pi(x^{1/4})(\pi(x^{1/4})-3)+1}{2}$$

Bringing in another term, we can complete the sum of the trivial leaves.

$$\Sigma_5 = \sum_{z^{1/2}<p\le x^{1/3}} \pi\left(\frac{x}{p^2}\right)$$

$$\ldots$$

$$\underbrace{S_1+U}_{Deleglise-Rivat} = \underbrace{S_{2trivial}}_{Oliveira\ e\ Silva} = \underbrace{\Sigma_1+\Sigma_{2a}-\Sigma_5}_{Gourdon}$$

Note, however, that we have subtracted Σ_5 which means that we will have to add 2 Σ_5 elsewhere.

The most significant difference is in the computation of the easy leaves. This class is defined by the fact that its computation can be simplified to a sum of a sum whose individual terms can be expressed in terms of the prime counting function $\pi(n)$ rather than requiring more onerous direct compuation of the partial sieve function. First, let us note that the easy leaves are entirely encompassed by the remainder of the term S_2, as well as the following:

$$\underbrace{S_3}_{Deleglise-Rivat} = \underbrace{\omega}_{Gourdon}$$

$$Gourdon: \quad \pi(x) = A - B + \omega + \phi_0 + \sum_{i=0}^{6} \Sigma_i$$

with the terms Σ_i indivually defined. Because we have already accounted for several terms in this sum, we can make an assertion that warrants further inspection:

$$\underbrace{V}_{Deleglise-Rivat} = \underbrace{A + \Sigma_{2b} + \Sigma_3 + \Sigma_4 + 2\Sigma_5 + \Sigma_6}_{Gourdon}$$

$$\ldots$$

$$\sum_{x^{1/4} < p \le x^{1/3}} \sum_{\substack{p < q \le y \\ q \le \frac{x}{p^2}}} \phi\left(\frac{x}{pq}, \pi(p) - 1\right) = \sum_{x^{1/4} < p \le x^{1/3}} \sum_{p < q \le (x/p)^{1/2}} \left[\pi\left(\frac{x}{pq}\right)\chi\left(\frac{x}{pq}\right)\right] + \Sigma_{2a} + \Sigma_3 + \Sigma_4 + 2\Sigma_5 + \Sigma_6$$

$$where$$

$$\chi\left(\frac{x}{pq}\right) = \begin{matrix} 2\, if\ x/pq < y \\ 1\, if\ x/pq \ge y \end{matrix}$$

$$\Sigma_3 = \frac{\pi(x^{1/3})(\pi(x^{1/3}) - 1)(2\pi(x^{1/3}) - 1)}{6} - \pi(x^{1/3}) - \frac{\pi(x^{1/4}(\pi(x^{1/4}) - 1)(2\pi(x^{1/4}) - 1))}{6} + \pi(x^{1/4})$$

$$\Sigma_4 = \pi(y) \sum_{x^{1/4} < p \le z^{1/2}} \pi\left(\frac{x}{py}\right)$$

$$\Sigma_6 = -\sum_{x^{1/4} < p \le x^{1/3}} \pi\left(\frac{x^{1/2}}{p^{1/2}}\right)$$

At this point, it is necessary to revisit the division between the easy and hard leaves in Oliveira e Silva's description of the algorithm. These are the terms that remain:

$$\underbrace{S_{2easy} + S_{2hard}}_{Oliveira\ e\ Silva} = \underbrace{V + S_3}_{Deleglise-Rivat} = \sum_{x^{1/4} < p \le x^{1/3}} \sum_{\substack{p < q \le y \\ q \le \frac{x}{p^2}}} \phi\left(\frac{x}{pq}, \pi(p) - 1\right) - \sum_{p_c < p \le x^{1/4}} \sum_{\substack{\delta(m) > p \\ \frac{y}{p} < m \le y}} \mu(m)\phi\left(\frac{x}{mp}, \pi(p) - 1\right)$$

$$S_{2hard} = -\sum_{p_c < p \le z^{1/2}} \sum_{\substack{\delta(m) > p \\ \frac{y}{p} < m \le min(y, \frac{z}{p})}} \mu(m)\phi\left(\frac{x}{mp}, \pi(p) - 1\right)$$

$$S_{2easy} = \sum_{max(p_c, y^{1/2}) < p \le x^{1/3}} \sum_{min(\frac{z}{p}, y) < q \le min(\frac{x^{1/2}}{p}, y)} \phi\left(\frac{x}{pq}, \pi(p) - 1\right)$$

S_3 must be further subdivided in order to draw a correspondence with S_{2hard}:

$$S_3 = S_{3a} + S_{3b} + S_{3c}$$

$$S_{3a} = -\sum_{p_c < p \le y^{1/2}} \sum_{\substack{\delta(m) > p \\ \frac{y}{p} < m \le y}} \mu(m) \phi\left(\frac{x}{mp}, \pi(p) - 1\right)$$

$$S_{3b} = \sum_{y^{1/2} < p \le x^{1/4}} \sum_{\frac{y}{p} < q \le \frac{z}{p}} \phi\left(\frac{x}{pq}, \pi(p) - 1\right)$$

$$S_{3c} = \sum_{y^{1/2} < p \le x^{1/4}} \sum_{\frac{z}{p} < q \le y} \phi\left(\frac{x}{pq}, \pi(p) - 1\right)$$

$S_{2\mathrm{hard}}$ clearly contains two of those terms:

$$\underbrace{S_{2\,hard}}_{Oliveira\, e\, Silva} = \underbrace{S_{3a} + S_{3b}}_{Deleglise-Rivat} + \lambda$$

$$where$$

$$\lambda = \sum_{x^{1/4} < p \le z^{1/2}} \sum_{p < q \le \frac{z}{p}} \phi\left(\frac{x}{pq}, \pi(p) - 1\right)$$

which means that the remaining terms fall under the umbrella of easy leaves:

$$\underbrace{S_{2\,easy}}_{Oliveira\, e\, Silva} = \underbrace{V + S_{3c}}_{Deleglise-Rivat} - \lambda$$

Unfortunately, λ does not directly correspond to any of Deleglise-Rivat's W terms that were used to partition V.

Importantly, Gourdon also breaks down the ω term to simplify its calculation such that:

$$\pi(x) = A - B + C + D + \phi_0 + \sum_{i=0}^{6} \Sigma_i$$

$$with$$

$$C = -\sum_{p_c < p \le x^{1/4}} \sum_{\substack{\frac{x}{p^3} < m \le y \\ \delta(m) > p}} \mu(m) \left(\pi\left(\frac{x}{pm}\right) - \pi(p) + 2\right)$$

$$D = -\sum_{p_c < p \le x^{1/4}} \sum_{\substack{\frac{y}{p} < m \le min(y, \frac{x}{p^3}) \\ \delta(m) > p}} \mu(m) \phi\left(\frac{x}{pm}, \pi(p) - 1\right)$$

With reasonably large x such that $p_c^3 \le z$, the sum C can be given a more specific range that only includes values of p where $x/p^3 \le y$.

$$C = -\sum_{z^{1/3} < p \le x^{1/4}} \sum_{\substack{\frac{x}{p^3} < m \le y \\ \delta(m) > p}} \mu(m) \left(\pi\left(\frac{x}{pm}\right) - \pi(p) + 2\right)$$

However, if that is not the case, another term for C is required to minimize the need for $\mu(m)$ and $\delta(m)$:

$$C_a = -\sum_{min(z^{1/3}, y^{1/2}) \le p < y^{1/2}} \sum_{\substack{\frac{x}{p^3} < m \le y \\ \delta(m) > p}} \mu(m) \phi\left(\frac{x}{mp}, \pi(p) - 1\right)$$

$$C_b = \sum_{y^{1/2} < p \le x^{1/4}} \sum_{\frac{x}{p^3} < q \le y} \left[\pi\left(\frac{x}{pq}\right) - \pi(p) + 2\right]$$

If $y \le x^{2/5}$ (*i.e.* $z^{1/3} \ge y^{1/2}$) then all values of m that fulfill $\delta(m) > p$ are prime, so C_a vanishes and C can be simplified to

$$C = \sum_{z^{1/3} < p \le x^{1/4}} \sum_{\frac{x}{p^3} < q \le y} \left[\pi\left(\frac{x}{pq}\right) - \pi(p) + 2\right]$$
$$\text{if } y \le x^{2/5}$$

Alternatively, $C + D$ (*i.e.* ω or S_3) can be expressed in such a way as to simplify implementations for calculating this sum.

$$\omega = \omega_1 + \omega_2 + \omega_3$$
$$\text{where}$$
$$\omega_1 = S_{3a} = -\sum_{p_c < p \le y^{1/2}} \sum_{\substack{\delta(m) > p \\ \frac{y}{p} < m \le y}} \mu(m) \phi\left(\frac{x}{mp}, \pi(p) - 1\right)$$
$$\omega_2 = \sum_{y^{1/2} < p \le x^{1/4}} \sum_{\frac{y}{p} < q \le min\left(y, \frac{x}{p^3}\right)} \phi\left(\frac{x}{pq}, \pi(p) - 1\right)$$
$$\omega_3 = C_b = \sum_{y^{1/2} < p \le x^{1/4}} \sum_{\frac{x}{p^3} < q \le y} \left[\pi\left(\frac{x}{pq}\right) - \pi(p) + 2\right]$$

This represents a general case and can serve as an efficient partitioning scheme for ω in parallel implementations of this algorithm.

I favor breaking up the sum as follows:

$$\pi(x) = \phi(x, \pi(y)) + \pi(y) - 1 - P_2(x, \pi(y))$$
$$\phi(x, \pi(y)) = \underbrace{S_0 + S_1 + S_2 + S_3}_{Deleglise-Rivat}$$

Thus, P_2 is calculated as in the original work, and phi is calculated by repartitioning Gourdon's formula to correspond to Deleglise-Rivat's description:

$$P_2(x,a) = \sum_{y < p \le x^{1/2}} [\pi(\frac{x}{p}) - \pi(p) + 1]$$

$$S_1 = \underbrace{\Sigma_1}_{Gourdon} = \frac{(\pi(y) - \pi(x^{1/3}))(\pi(y) - \pi(x^{1/3}) - 1)}{2}$$

$$S_2 = \underbrace{A + \Sigma_2 + \Sigma_3 + \Sigma_4 + \Sigma_5 + \Sigma_6}_{Gourdon}$$

$$S_3 = \underbrace{\omega_1 + \omega_2 + \omega_3}_{this\ work}$$

With minimal effort, $S_1 + S_2 + S_3$ can be changed to include Oliveira e Silva's trivial leaves:

$$\underbrace{S_1 + S_2 + S_3}_{Deleglise-Rivat} = \underbrace{S_{2\,trivial}}_{Oliveira\ e\ Silva} + \underbrace{V + S_3}_{Deleglise-Rivat}$$

$$S_{2\,trivial} = \underbrace{S_1 + U}_{Deleglise-Rivat} = \Sigma_1 + \Sigma_{2a} - \Sigma_5$$

$$V = \underbrace{A + \Sigma_{2b} + \Sigma_3 + \Sigma_5 + 2\Sigma_5 + \Sigma_6}_{Gourdon}$$

In this way, an implementation would give the same results for an Oliveira e Silva-based implementation on P_2, ordinary leaves, and trivial leaves. Additionally, $V + S_3$ would give the same sum as easy leaves + hard leaves, aiding comparison.

Hopefully this will aid others in comparing the previous works on this subject as well as facilitate correctness testing on implementations in progress.

References:

(1) Deleglise, M.; Rivat, J. *Mathematics of computation* **65 (**1996) 235-245.
(2) Oliveira e Silva, T. *Revista Do Detua* **4** (2006) 759-768.
(3) Gourdon, X. *Unpublished Work (http://numbers.computation.free.fr/Constants/Primes/Pix/piNalgorithm.ps)*