Maximos y Minimos, Multiplicadores de Lagrange

Calculo Multivariado

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Ejercicios:

- 26) e <
- 26) f <
- 29) d <
- 30) <

26.(e): $f(x,y) = 2x^4 + y^2 - x^2 - 2y$

Las primeras derivadas parciales son:

$$\frac{\partial f}{\partial x} = 8x^3 - 2x \qquad \frac{\partial f}{\partial y} = 2y - 2$$

Localizamos los puntos criticos:

$f_x = 8x^3 - 2x = 0$	$f_y = 2y - 2 = 0$
$\begin{aligned} 8x^3 - 2x &= 0 \\ 2x(2x+1)(2x-1) &= 0 \end{aligned}$ $x = 0,\ x = \frac{1}{2},\ x = -\frac{1}{2}$	$\begin{aligned} 2y - 2 &= 0 \\ 2y &= 2 \\ y &= \frac{2}{2} \\ y &= 1 \end{aligned}$

Puntos criticos:
$x = 0,\ x = \frac{1}{2},\ x = -\frac{1}{2},\ y = 1$ $P_1(0,1)$ $P_2\left(\frac{1}{2},1\right)$ $P_3\left(-\frac{1}{2},1\right)$

Las segundas derivadas parciales son:

$\frac{\partial^2 f}{\partial x^2} = 24x^2 - 2$	$\frac{\partial^2 f}{\partial y^2} = 2$
$\frac{\partial f}{\partial x \partial y} = 0$	$\frac{\partial f}{\partial y \partial x} = 0$

Prueba de las segundas derivadas parciales:

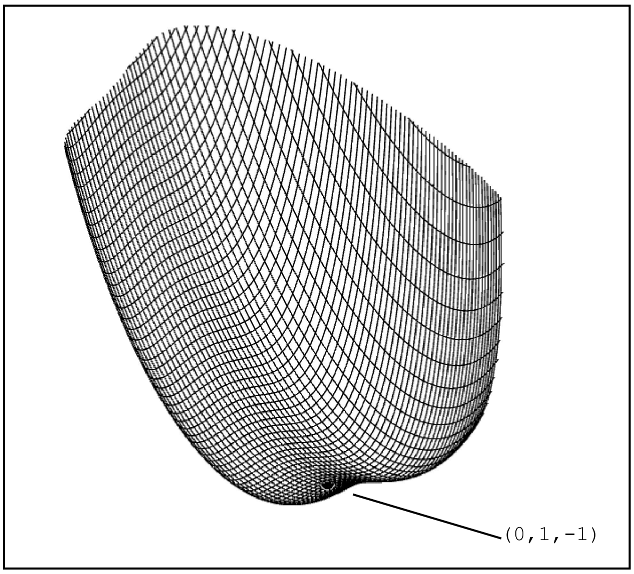
$$D(x,y) = \begin{vmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{vmatrix} = \begin{vmatrix} 24x^2 - 2 & 0 \\ 0 & 2 \end{vmatrix} = \left[\left(24x^2 - 2\right) \cdot 2\right] - [0 \cdot 0] = 48x^2 - 4$$

Para el punto critico $P_1(0,1)$ tenemos que:

$D(0,1) = 48 \cdot (0)^2 - 4 = -4$	$D(0,1) < 0$

Entonces:

Tenemos que $D(0,1) < 0$, por lo tanto $(0,1,f(0,1))$ *no es un extremo relativo*, es un **punto silla**.

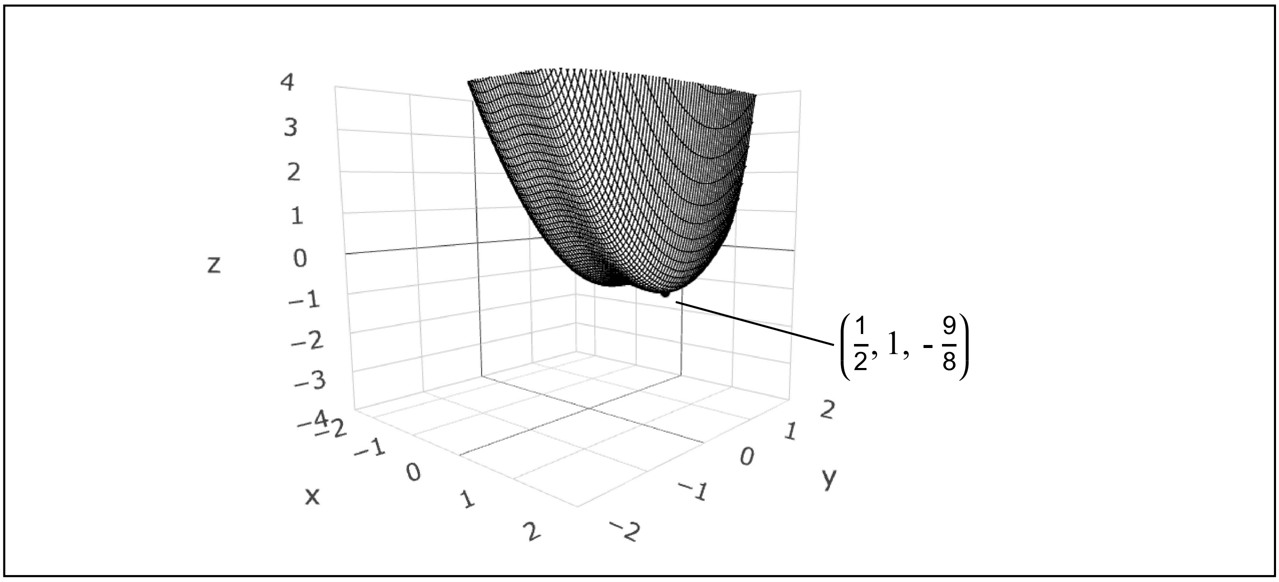


Para el punto critico $P_2\left(\frac{1}{2},1\right)$ tenemos que:

$D\left(\frac{1}{2},1\right) = 48\cdot\left(\frac{1}{2}\right)^2 - 4 = 8$	$D\left(\frac{1}{2},1\right) > 0$
$f_{xx}\left(\frac{1}{2},1\right) = 24\cdot\left(\frac{1}{2}\right)^2 - 2 = 4$	$f_{xx}\left(\frac{1}{2},1\right) > 0$

Entonces:

Tenemos que $D\left(\frac{1}{2},1\right) > 0$ y $f_{xx}\left(\frac{1}{2},1\right) > 0$, por lo tanto $f\left(\frac{1}{2},1\right)$ es un **minimo relativo**.

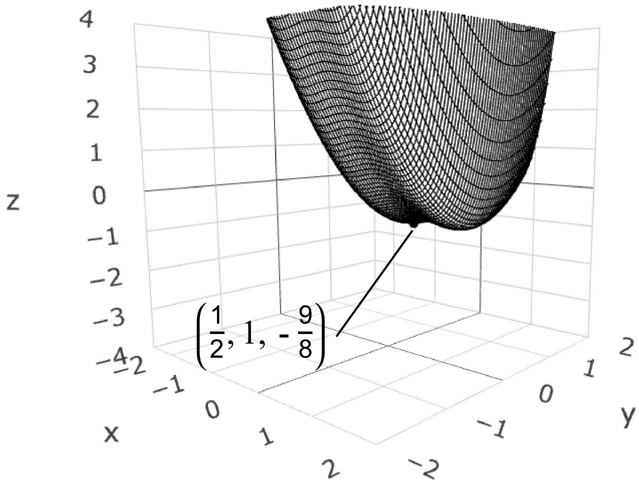


Para el punto critico $P_3\left(-\frac{1}{2},1\right)$ tenemos que:

$D\left(-\frac{1}{2},1\right) = 48\cdot\left(-\frac{1}{2}\right)^2 - 4 = 8$	$D\left(-\frac{1}{2},1\right) > 0$
$f_{xx}\left(-\frac{1}{2},1\right) = 24\cdot\left(-\frac{1}{2}\right)^2 - 2 = 4$	$f_{xx}\left(-\frac{1}{2},1\right) > 0$

Entonces:

Tenemos que $D\left(-\frac{1}{2},1\right) > 0$ y $f_{xx}\left(-\frac{1}{2},1\right) > 0$, por lo tanto $f\left(-\frac{1}{2},1\right)$ es un **minimo relativo.**



26.(f): $z = xy + \frac{50}{x} + \frac{20}{y}$ si $y > 0, x > 0$

Las primeras derivadas parciales son:

$$\frac{\partial z}{\partial x} = y - \frac{50}{x^2} \qquad \frac{\partial z}{\partial y} = x - \frac{20}{y^2}$$

Localizamos los puntos criticos:

$f_x = y - \frac{50}{x^2} = 0$	$f_y = x - \frac{20}{y^2} = 0$
$y - \frac{50}{x^2} = 0$ $y = \frac{50}{x^2}$ (1)	$x - \frac{20}{y^2} = 0$ $x = \frac{20}{y^2}$ (2)
Remplazo (1) en (2):	
$x = \frac{20}{y^2}$ $x = \frac{20}{\left(\frac{50}{x^2}\right)^2}$ $x = \frac{\frac{20}{1}}{\frac{2500}{x^4}}$ $x = \frac{20x^4}{2500}$ $x = \frac{x^4}{125}$ $x - \frac{x^4}{125} = 0$ $x = 5$, en $x = 0$ la funcion no esta definida.	Restricciones: $x > 0$ y $y > 0$ $x = 5,\ y = \frac{50}{x^2}$ entonces: $P_1(5, 2)$

Las segundas derivadas parciales son:

$\frac{\partial^2 z}{\partial x^2} = \frac{100}{x^3}$	$\frac{\partial^2 z}{\partial y^2} = \frac{40}{x^3}$
$\frac{\partial^2 z}{\partial x \partial y} = 1$	$\frac{\partial^2 z}{\partial y \partial x} = 1$

Prueba de las segundas derivadas:

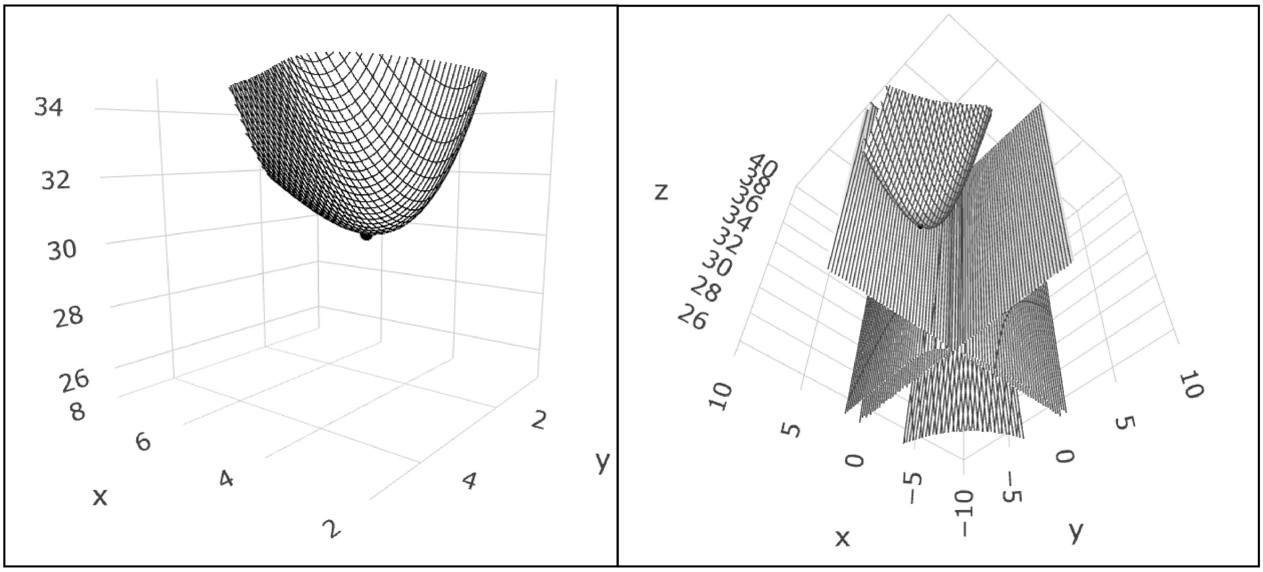
$$D(x,y) = \begin{vmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{vmatrix} = \begin{vmatrix} \frac{100}{x^3} & 1 \\ 1 & \frac{40}{x^3} \end{vmatrix} = \left(\frac{100}{x^3} \cdot \frac{40}{x^3}\right) - (1 \cdot 1) = \frac{4000}{x^6} - 1$$

Para el punto critico $P_1(5, 2)$ tenemos que:

$D(5,2) = \frac{4000}{(5)^6} - 1 = \frac{16}{625} \approx 0.025$	$D(5,2) > 0$
$f_{xx}(5,2) = \frac{100}{(5)^3} = \frac{4}{5}$	$f_{xx} > 0$

Entonces:

Tenemos que $D(5, 2) > 0$ y $f_{xx}(5, 2) > 0$, por lo tanto $f(5, 2)$ es un **minimo relativo**.



29.(d): $u = x - 2y + 2z$ sujeto a $x^2 + y^2 + z^2 = 1$

$\frac{\partial u}{\partial x} = 1$ $\frac{\partial u}{\partial y} = -2$ $\frac{\partial u}{\partial z} = 2$

$g(x,y,z) = x^2 + y^2 + z^2$

$\frac{\partial g}{\partial x} = 2x$ $\frac{\partial g}{\partial y} = 2y$ $\frac{\partial g}{\partial z} = 2z$

$u_x(x,y,z) = \lambda g_x(x,y,z)$	$u_y(x,y,z) = \lambda g_y(x,y,z)$	$u_z(x,y,z) = \lambda g_z(x,y,z)$
$1 = \lambda 2x$	$-2 = \lambda 2y$	$2 = \lambda 2z$
$\frac{1}{2x} = \lambda$	$-\frac{1}{y} = \lambda$	$\frac{1}{z} = \lambda$

$$x = \frac{1}{2\lambda}$$
$$y = -\frac{1}{\lambda}$$
$$z = \frac{1}{\lambda}$$
$$x^2 + y^2 + z^2 - 1 = 0$$

Remplazamos la ultima ecuacion para encontrar λ :

$$x^2 + y^2 + z^2 - 1 = 0$$
$$\left(\frac{1}{2\lambda}\right)^2 + \left(-\frac{1}{\lambda}\right)^2 + \left(\frac{1}{\lambda}\right)^2 - 1 = 0$$
$$\frac{1}{4\lambda^2} + \frac{1}{\lambda^2} + \frac{1}{\lambda^2} - 1 = 0$$
$$\frac{1}{\lambda^2}\left(\frac{1}{4} + 1 + 1\right) = 1$$
$$\frac{1}{\lambda^2} = \frac{4}{9}$$
$$\left(\lambda^2\right)\frac{1}{\lambda^2} = \left(\lambda^2\right)\frac{4}{9}$$
$$1 = \frac{4\lambda^2}{9}$$
$$9 = 4\lambda^2$$
$$\frac{9}{4} = \lambda^2$$

$$\lambda_1 = \frac{3}{2},\ \lambda_2 = -\frac{3}{2}$$

Remplazamos λ_1 para encontrar x_1, y_1, z_1:

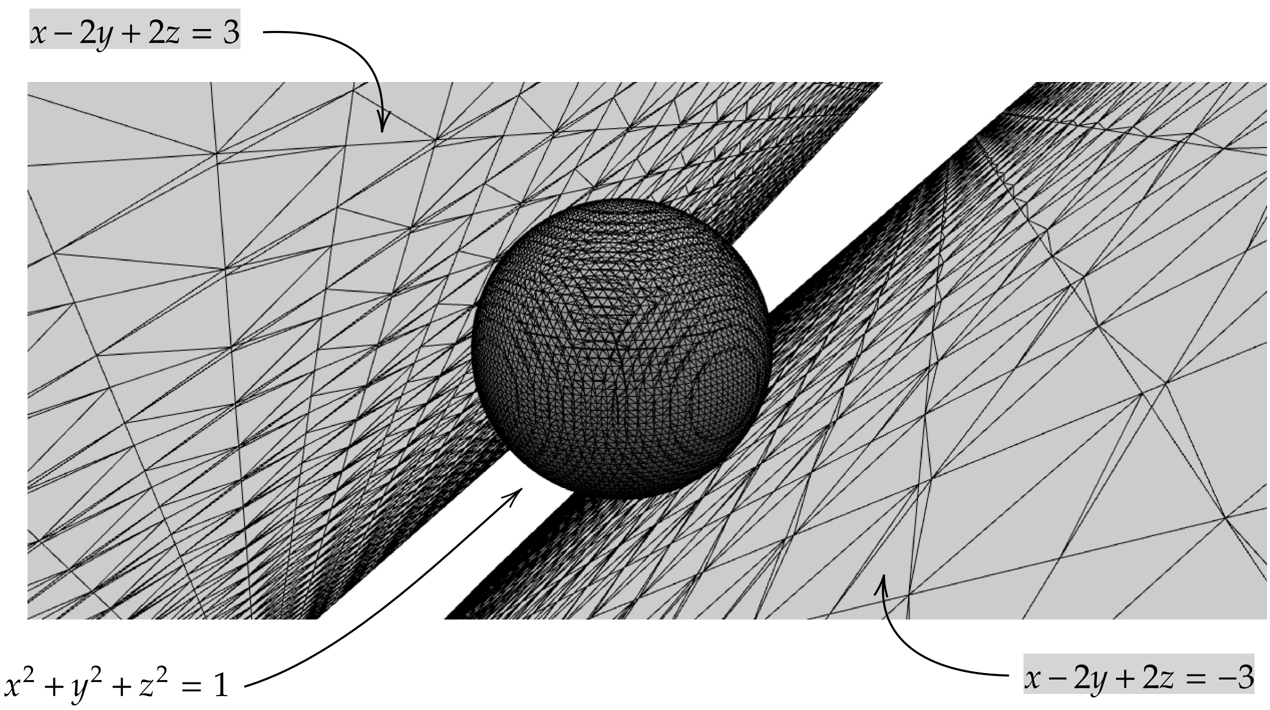
$x_1 = \frac{1}{2\lambda}$ $x_1 = \frac{1}{2\left(\frac{3}{2}\right)}$ $x_1 = \frac{1}{3}$	$y_1 = -\frac{1}{\lambda}$ $y_1 = -\frac{1}{\left(\frac{3}{2}\right)}$ $y_1 = -\frac{2}{3}$	$z_1 = \frac{1}{\lambda}$ $z_1 = \frac{1}{\left(\frac{3}{2}\right)}$ $z_1 = \frac{2}{3}$

Entonces $x_1 = \frac{1}{3},\ y_2 = -\frac{2}{3},\ z_2 = \frac{2}{3}$. De tal manera, un extremo con restricciones es: $u\left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right) = 3$

Remplazamos λ_2 para encontrar x_2, y_2, z_2:

$x_2 = \frac{1}{2\lambda}$ $x_2 = \frac{1}{2\left(-\frac{3}{2}\right)}$ $x_2 = -\frac{1}{3}$	$y_2 = -\frac{1}{\lambda}$ $y_2 = -\frac{1}{\left(-\frac{3}{2}\right)}$ $y_2 = \frac{2}{3}$	$z_2 = \frac{1}{\lambda}$ $z_2 = \frac{1}{\left(-\frac{3}{2}\right)}$ $z_2 = -\frac{2}{3}$

Entonces $x_2 = -\frac{1}{3},\ y_2 = \frac{2}{3},\ z_2 = -\frac{2}{3}$. De tal manera, un extremo con restricciones es: $u\left(-\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right) = -3$



30. Hallar el valor maximo de $f(x,y) = 9 - x^2 - y^2$ sobre la recta $x + 3y = 12$.

$\frac{\partial f}{\partial x} = -2x$

$\frac{\partial f}{\partial y} = -2y$

$g(x,y) = x + 3y - 12$

$\frac{\partial g}{\partial x} = 1$

$\frac{\partial g}{\partial y} = 3$

$f_x(x,y) = \lambda g_x(x,y)$	$f_y(x,y) = \lambda g_y(x,y)$
$-2x = \lambda 1$	$-2y = \lambda 3$
$-2x = \lambda$	$-\frac{2y}{3} = \lambda$

$$x = -\lambda\frac{1}{2}$$
$$y = -\lambda\frac{3}{2}$$
$$x + 3y - 12 = 0$$

Remplazamos la ultima ecuacion para encontrar λ :

$$\left(-\lambda\frac{1}{2}\right) + 3\left(-\lambda\frac{3}{2}\right) - 12 = 0$$
$$-\lambda\frac{1}{2} - \lambda\frac{9}{2} - 12 = 0$$
$$-\lambda\frac{1}{2} - \lambda\frac{9}{2} = 12$$
$$\lambda\left(-\frac{1}{2} - \frac{9}{2}\right) = 12$$
$$\lambda(-5) = 12$$
$$\lambda = -\frac{12}{5}$$

$x = -\lambda\frac{1}{2}$ $x = -\left(-\frac{12}{5}\right)\frac{1}{2}$ $x = \frac{6}{5}$	$y = -\lambda\frac{3}{2}$ $y = -\left(-\frac{12}{5}\right)\frac{3}{2}$ $y = \frac{18}{5}$

Entonces $x = \frac{6}{5}$, $y = \frac{18}{5}$. De tal manera, un extremo con restricciones es: $f\left(\frac{6}{5}, \frac{18}{5}\right) = -\frac{27}{5}$