

NOTES ON TOPOLOGICAL ASPECTS OF CONDENSED MATTER PHYSICS

ABSTRACT. These notes correspond to the lectures of Prof. Gian Michele Graf given in the fall semester of 2015 in ETH Zurich. They are based on the lecturer's handwritten notes by attempting to slightly elaborate with explanations, provide omitted proofs, and mention possible generalizations and qualifications to the given arguments. It should be noted that these notes are published on the course website without the lecturer reading them beforehand. So they are provided as is with no guarantees. Reporting of typos, inaccuracies and errors to jshapiro@itp.phys.ethz.ch would be greatly appreciated.

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NOTATION

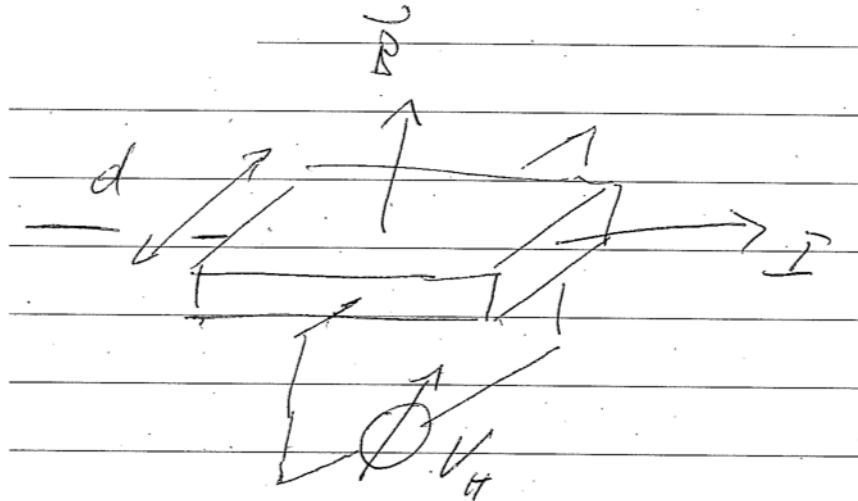
- The equation $A := B$ means we are now defining A to be equal to B . The equation $A \equiv B$ means that at some point earlier A has been defined to be equal to B . The equation $A = B$ means A turns out to be equal to B .
- A vector (an element of \mathbb{R}^3) will be denoted by boldface: \mathbf{B} , and when the same symbol appears without boldface, B , the meaning is, as usual, the magnitude of the corresponding vector which appeared previously: $B \equiv \|\mathbf{B}\|$. If $B \neq 0$ then we also denote by $\hat{\mathbf{B}}$ the unit vector $\hat{\mathbf{B}} \equiv \frac{\mathbf{B}}{B}$.
- $\hat{\mathbf{e}}_i$ is the i th unit vector of the standard basis of \mathbb{R}^n .
- c is the speed of light, h is Planck's constant.
- $\sigma(H)$ is the spectrum of the operator H (to be defined below).
- \mathcal{C} is a placeholder for a generic category.
- $\text{Obj}(\mathcal{C})$ are all the objects in the category \mathcal{C} .
- $\text{Mor}_{\mathcal{C}}(A, B)$ are all the morphisms in category \mathcal{C} from the object A to the object B .
- Grp is the category whose objects are groups and whose morphisms are group homomorphisms.
- Top is the category whose objects are topological spaces and whose morphisms are continuous maps.
- TVS is the category whose objects are topological vector spaces and whose morphisms are continuous linear maps.
- $\text{Vect}_{\mathbb{F}}$ is the category whose objects are vector spaces over the field \mathbb{F} and whose morphisms are \mathbb{F} -linear maps. $\text{Vect}_{\mathbb{F}}^n$ is the restriction to vector fields of dimension n for a given $n \in \mathbb{N}$.
- $\text{Vect}_{\mathbb{F}}(X)$ is the category whose objects are vector bundles over X and whose morphisms are vector bundle homomorphisms, with typical fiber in $\text{obj}(\text{Vect}_{\mathbb{F}})$.
- If $X \in \text{Obj}(\text{Top})$ and $x \in X$, then $\text{Nbhd}_X(x)$ is the set of all open sets in X which contain x .
- \wedge stands for "and", \vee stands for "or" and $\vee\!\vee$ stands for "xor".
- If $\prod_{\alpha \in A} X_\alpha$ is a product of spaces indexed by A , then for all $\beta \in A$, $\pi_\beta : \prod_{\alpha \in A} X_\alpha \rightarrow X_\beta$ is the natural projection map, which is by definition continuous. We will also use π_j with $j \in \mathbb{N}_{>0}$ when referring to products $X \times Y \times Z \times \dots$ which are implicitly labelled by $\{1, 2, \dots\}$.
- $\mathbb{T}^n \equiv (\mathbb{R}/2\pi\mathbb{Z})^n$ is the n -torus and $S^1 \equiv \mathbb{T}^1$ is the unit circle.

Part 1. The Integer Quantum Hall Effect

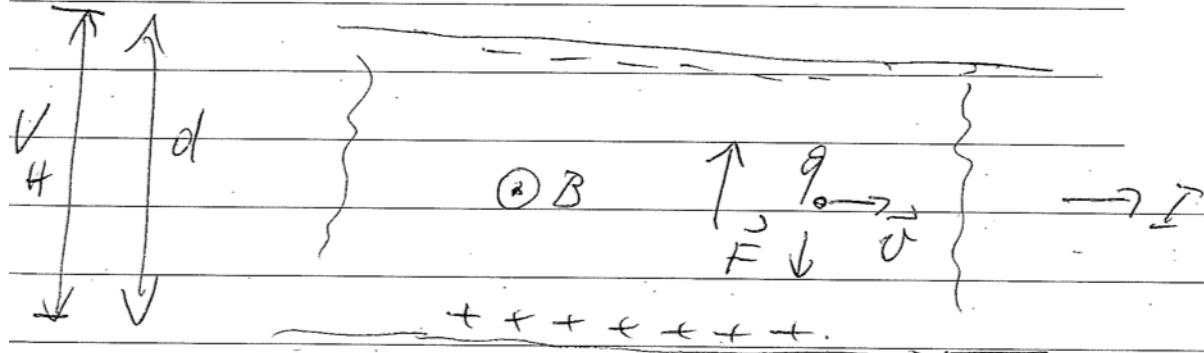
1. THE CLASSICAL HALL EFFECT

The material in this section may be found in the first article of [36].

1.0.1. Definition. The classical Hall effect refers to the phenomenon whereby a current carrying conductor (with current I) exposed to a transverse magnetic field \mathbf{B} develops a transverse potential difference V_H (and conversely).



A top view of the system:



1.0.2. Claim. The ratio $\frac{V_H}{I}$, called the Hall resistance, is equal to $\frac{B}{qnc}$ where q is the charge of the current carriers and n is the number of carriers per unit area.

Proof. We analyze the situation using Newton's second law. Assume the carriers have velocity \mathbf{v} and mass m . Then the Lorentz force is

$$\mathbf{F}_L = q \frac{\mathbf{v}}{c} \times \mathbf{B}$$

so that carriers of opposite charge will accumulate on both ends of the boundaries of the conductor along the $\hat{\mathbf{v}}$ direction, causing an electric field \mathbf{E} which is perpendicular to both \mathbf{v} and \mathbf{B} . At equilibrium, the forces are equal

$$q\mathbf{E} = \mathbf{F}_L$$

so that $\mathbf{E} = \frac{v}{c}\mathbf{B}$. Then the potential difference across the width of the sample d is given by $V_H = d \cdot E$. If the external current applied on the conductor is given by I , then we may write $I = d \cdot j$ where j is the two-dimensional current density, given by $j \equiv qnv$. Then the Hall resistance is defined and is equal to: $R_H := \frac{V_H}{I} = \frac{d \cdot E}{d \cdot j} = \frac{\frac{v}{c}B}{qnv} = \frac{B}{qnc}$ so

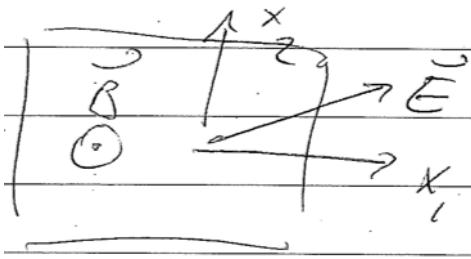
$$R_H = \frac{B}{qnc}$$

□

1.0.3. Remark. Note that:

- R_H is large when n is small, so that *thin* conductors generate *large* Hall resistance.
- By measuring R_H we can determine the *sign* of q and thus determine whether current is carried by electrons or holes.
- Hall's original motivation [21] stemmed from reading in [32] that a magnetic force acts on conducting media and not on the carriers inside it.

We now repeat the same exercise, taking friction into account:



The equations of motion of a carrier in the conductor are given by

$$m\dot{v} = q \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right) - q\mu^{-1}\mathbf{v}$$

where μ^{-1} is a newly introduced coefficient of friction¹. The stationary regime means $\dot{v} = 0$ (at least after averaging over some time) so that

$$\mathbf{E} = \mu^{-1}\mathbf{v} - \frac{\mathbf{B}}{c} \times \mathbf{B}$$

In matrix form this equation is

$$\begin{bmatrix} E_1 \\ E_2 \end{bmatrix} = \begin{bmatrix} \mu^{-1} & -\frac{B}{c} \\ \frac{B}{c} & \mu^{-1} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

The resistivity matrix $\rho \equiv \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix}$ is defined via the equation $\mathbf{E} = \rho\mathbf{j}$ so that

$$\rho = \frac{1}{qn} \begin{bmatrix} \mu^{-1} & -\frac{B}{c} \\ \frac{B}{c} & \mu^{-1} \end{bmatrix}$$

or written using the conductivity matrix (where σ , the conductivity, is defined via the equation $\mathbf{j} = \sigma\mathbf{E}$, so $\sigma \equiv \rho^{-1}$): we get

$$\sigma = \frac{qn}{\mu^{-2} + \frac{B^2}{c^2}} \begin{bmatrix} \mu^{-1} & \frac{B}{c} \\ -\frac{B}{c} & \mu^{-1} \end{bmatrix} =: \begin{bmatrix} \sigma_D & \sigma_H \\ -\sigma_H & \sigma_D \end{bmatrix}$$

where we have defined the Hall conductivity σ_H and the direct or dissipative conductivity σ_D (called dissipative because the dissipated power per unit area, is $\mathbf{j} \cdot \mathbf{E} = \mathbf{E} \cdot \sigma \mathbf{E}$ so that σ is like a quadratic form, and only the diagonal terms contribute to this quantity).

Note that according to our convention, σ_H is positive, so that a particle with positive charge under the influence of a magnetic field in the $\hat{\mathbf{e}}_3$ direction and an electric field in the $\hat{\mathbf{e}}_2$ direction will have drift velocity in the $\hat{\mathbf{e}}_1$ direction:

$$\begin{aligned} \mathbf{j} &= \begin{bmatrix} 0 & \sigma_H \\ -\sigma_H & 0 \end{bmatrix} \begin{bmatrix} 0 \\ E_2 \end{bmatrix} \\ &= \begin{bmatrix} E_2 \\ 0 \end{bmatrix} \end{aligned}$$

Let us consider the ideal case, where $\mu \rightarrow \infty$ (so that $\mu^{-1} \rightarrow 0$ and there is no friction):

- If $\mathbf{B} = 0$, then

$$\begin{aligned} \lim_{\mu \rightarrow \infty} \sigma_D &= \lim_{\mu \rightarrow \infty} \frac{qn\mu^{-1}}{\mu^{-2}} \\ &= \lim_{\mu \rightarrow \infty} qn\mu \\ &= \infty \end{aligned}$$

and the particle accelerates indefinitely in the direction of \mathbf{E} .

- If $\mathbf{B} \neq 0$ (however small), then:
 - The direct conductivity is

$$\begin{aligned} \lim_{\mu \rightarrow \infty} \sigma_D &= \lim_{\mu \rightarrow \infty} \frac{qn\mu^{-1}}{\mu^{-2} + \frac{B^2}{c^2}} \\ &= 0 \end{aligned}$$

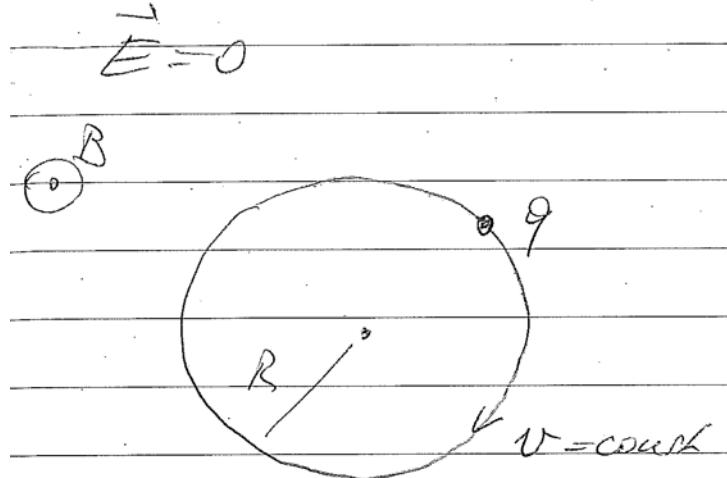
so that we have dissipationless medium, but we also have $\rho_D = 0$.

- For the Hall conductivity we have

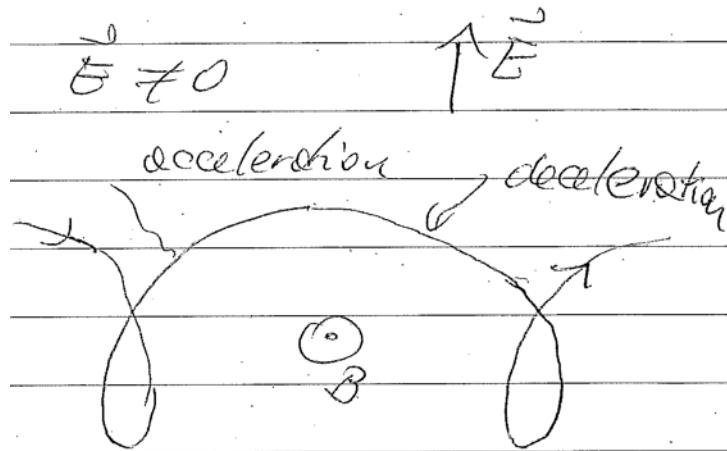
$$\begin{aligned} \lim_{\mu \rightarrow \infty} \sigma_H &= \frac{qn c}{B} \\ &= R_H^{-1} \end{aligned}$$

¹ μ is called the *mobility* and is a measure of the conductivity of the material. Indeed, if the magnetic field were zero, we'd have in a stationary state $v = \mu E$ so that the conductivity is $\sigma = \frac{j}{E} = \frac{qnv}{\mu^{-1}v} = qn\mu$. This is also related to the mean free time τ in the Drude model (see the corresponding chapter in [4]).

- If $\mathbf{B} \neq 0$ and $\mathbf{E} = 0$ then the particle moves in circles of constant speed with radius $R = \frac{cv}{qB}$.

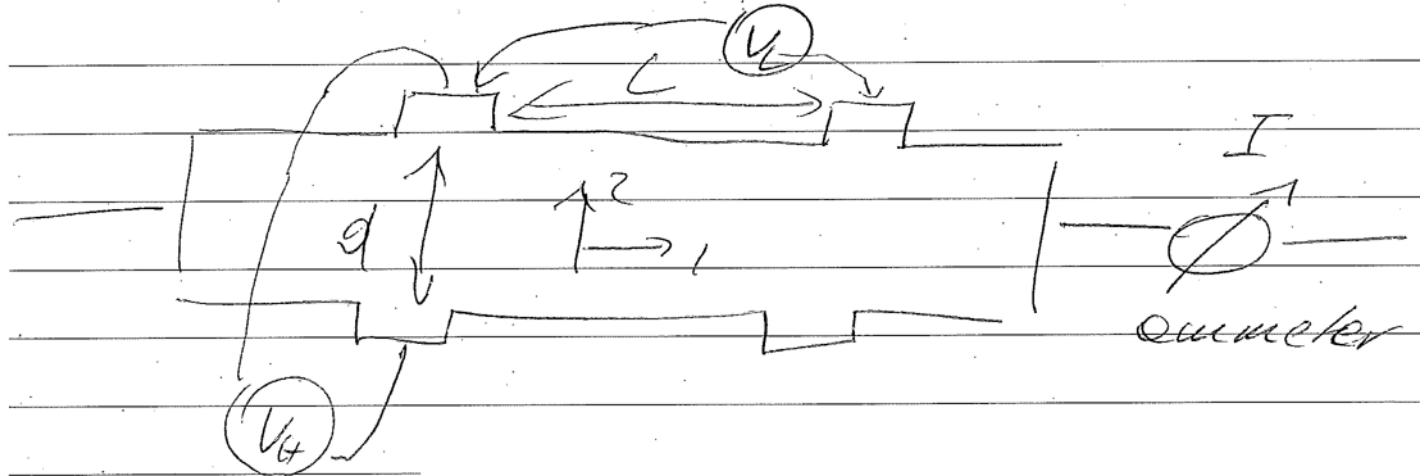


- If $\mathbf{B} \neq 0$ and $\mathbf{E} \neq 0$ then we observe drift motion.



(Prolate trochoidal path)

1.1. Measurement of Resistances.



In the laboratory one measures resistances. We have $R_L \equiv \frac{V_L}{I}$ and $R_H \equiv \frac{V_H}{I}$. Since we know \mathbf{j} is in the $\hat{\mathbf{e}}_1$ direction, we may write $\begin{bmatrix} j_1 \\ j_2 \end{bmatrix} = \frac{I}{d} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and so since $\mathbf{E} = \rho \mathbf{j}$ we obtain

$$\begin{aligned} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} &= \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix} \frac{I}{d} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{I}{d} \begin{bmatrix} \rho_{11} \\ \rho_{21} \end{bmatrix} \end{aligned}$$

so that

$$\begin{aligned} V_L &= E_1 \cdot L \\ &= \rho_{11} \frac{L}{d} I \end{aligned}$$

and thus $R_L = \rho_{11} \frac{L}{d}$ and

$$\begin{aligned} V_H &= E_2 \cdot d \\ &= \rho_{21} \frac{I}{d} d \\ &= \rho_{21} I \end{aligned}$$

and thus $R_H = \rho_{21} =: \rho_H$.

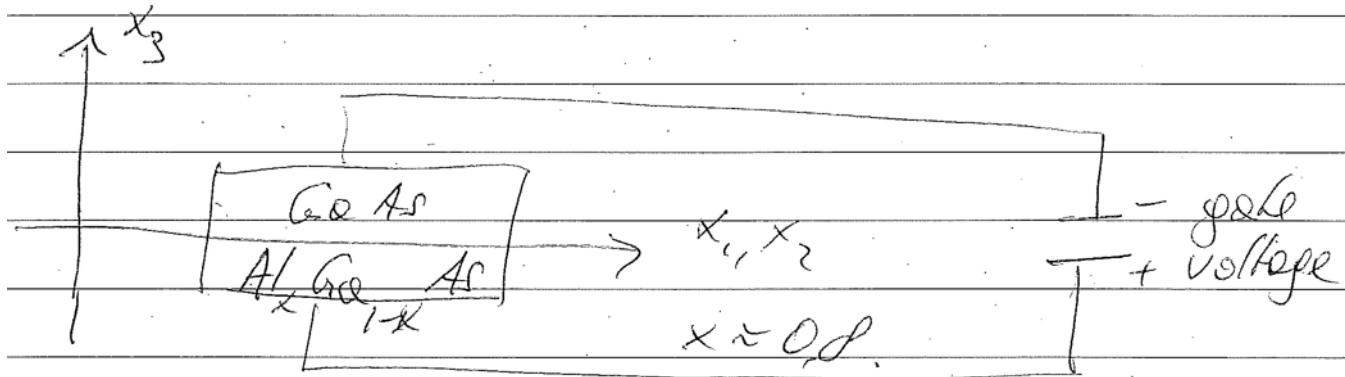
1.1.1. *Remark.* It should be noted that it is an artifact of the fact we are working in two dimensions that the resistance R is equal to the resistivity ρ . In arbitrary dimension $n \neq 0$ we have

$$\begin{aligned} V_H &= E_2 \cdot d \\ &= \rho_H j_1 \cdot d \\ &= \rho_H \frac{I}{S} d \end{aligned}$$

where S is the cross-section and thus $R_H = \rho_H \frac{d}{S}$. It is merely in $n = 2$ that $S = d$ and we get the seemingly odd expression $R_H = \rho_H$.

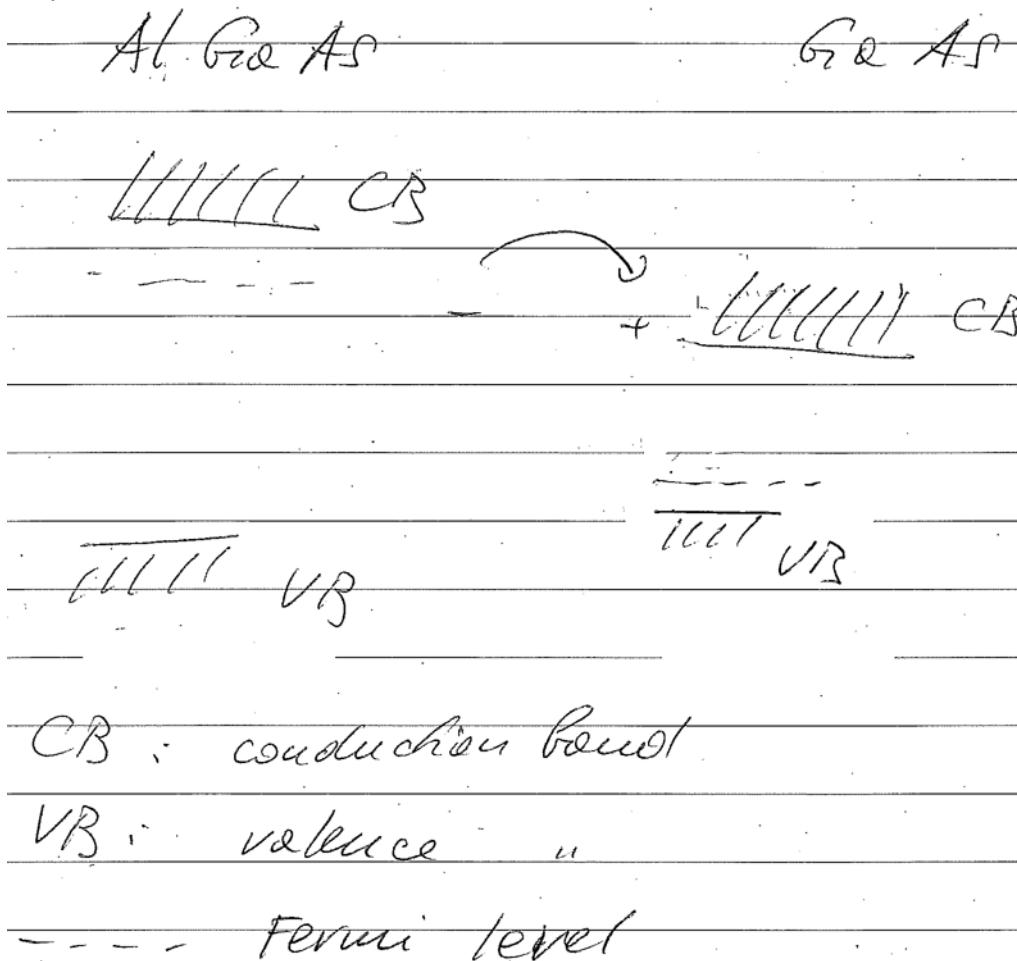
2. THE INTEGER QUANTUM HALL EFFECT

2.1. Experimental Setup.



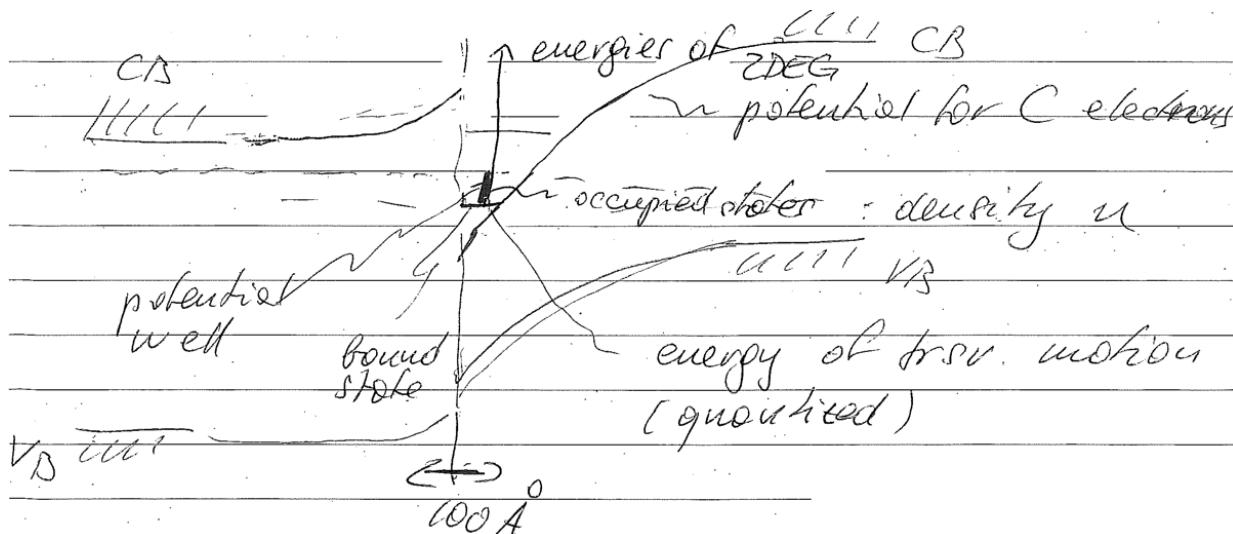
We put a junction of AlGaAs (with ratio 4 : 1 for the Aluminium) and GaAs together. The electrons are bound to the interface so we obtain a two-dimensional electron gas (2DEG). The gas is bound on the interface but free along it.

To understand the interface better, we first picture the two materials separately.



The materials are set up in such a way that the Fermi energy of AlGaAs is higher than that of GaAs. Note that it is possible to depict the band structure in real space due to the fact the scale of the picture is about 100, so that the Heisenberg uncertainty principle doesn't disturb. Also note that the notion of Fermi energy in non-zero temperature makes sense in different heights of the gap (or alternatively explained via donor levels).

When we bring the two media into contact, electrons (assumed with positive charge) spill over from donor sites of AlGaAs, which creates a dipole layer and potential difference.



The potential difference and thus the density of particles can be tuned via a gate voltage. As a result we get bound states in the transverse direction. The electrons in the well have that energy just as the transverse energy, we'd still need to add energy in the longitudinal direction. We obtain a longitudinal 2DEG.

2.2. The Filling Factor.

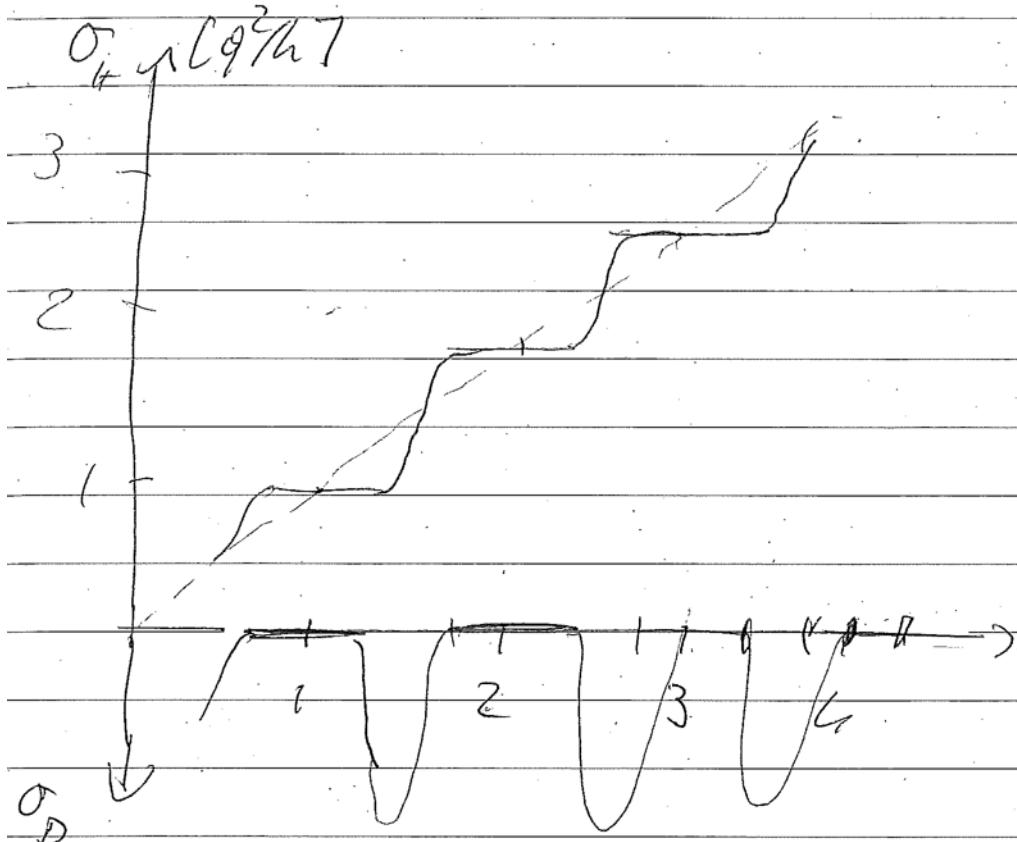
2.2.1. Definition. Define the filling factor $\nu := \frac{n}{|q| \frac{B}{c\hbar}}$ where \hbar is Planck's constant (not divided by 2π).

Note that ν is dimensionless. Indeed, $[qB] = N$ (Newtons), $[h] = N \cdot m \cdot s$, $[c] = \frac{m}{s}$ so that $\left[\frac{qB}{ch} \right] = \frac{1}{m^2}$ and of course as a number density in two dimensions, $[n] = \frac{1}{m^2}$.

Then classically, we can express σ_H using the filling factor to get

$$\begin{aligned}\sigma_H &= \frac{qc}{B} n \\ &= \text{sgn}(q) \frac{q^2}{h} \nu\end{aligned}$$

2.3. Experimental Results. The integer quantum Hall effect (IQHE) is the experimental observation (published in [26]) that σ_H takes on integer values. Indeed, in a typical experiment one obtains the following:



The experiment exhibits the following features:

- We see plateaus of σ_H , which is a quantization. Thus σ_H takes its classically predicted value only for integer values of ν .
- For integer ν or near integer values of it, $\sigma_D = 0$.

- The width of the plateaus increases with disorder. In fact the disorder will turn out to be crucial for the quantization effect, though too much disorder will destroy it.
- The accuracy of the quantization is $\sim 10^{-8}$. Indeed, the value of $\frac{h}{q^2}$ has been measured and found to be

$$\frac{h}{q^2} = 25812.807572\Omega$$

which defines a new unit of resistance, the von Klitzing.

- There is also a fractional quantum Hall effect, which we shall not describe here, as this requires a many-body analysis, whereas the IQHE already appears in the one-particle approximation.

3. THE LANDAU HAMILTONIAN AND ITS LEVELS

To give a theoretical explanation to the quantization of σ_H described in 2.3 we shall consider a system with only a magnetic field and no external electric field, and add the electric field via perturbation theory later, which is justified because in typical experiments the magnetic field is the dominant one.

Thus our system is of electrons in two dimensions (the plane spanned by $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$), subject to a magnetic field (out of the two dimensional plane, so along $\hat{\mathbf{e}}_3$), otherwise free, and spinless. So we work in the one-electron approximation, so the one-particle Hilbert space \mathcal{H} is $L^2(\mathbb{R}^2; \mathbb{C})$. The Hamiltonian (due to Landau) is given by

$$H = \frac{1}{2m} \left(\mathbf{p} - \frac{q}{c} \mathbf{A} \right)^2 \quad (1)$$

where $\mathbf{p} \equiv -i\hbar\nabla$, and we choose a gauge (symmetric gauge) such that

$$\mathbf{A} = \frac{1}{2} \mathbf{B} \times \mathbf{x} \quad (2)$$

Then

$$\begin{aligned} \text{curl } (\mathbf{A})_i &\equiv \epsilon_{ijk} \partial_j A_k \\ &= \epsilon_{ijk} \partial_j \left(\frac{1}{2} \epsilon_{klm} B_l x_m \right) \\ &= \frac{1}{2} \epsilon_{ijk} \epsilon_{klm} B_l \delta_{j,m} \\ &= \frac{1}{2} \underbrace{\epsilon_{ijk} \epsilon_{ljk}}_{2\delta_{i,l}} B_l \\ &= B_i \end{aligned}$$

as necessary. Using the canonical angular momentum,

$$\mathbf{L} \equiv \mathbf{x} \times \mathbf{p}$$

to re-express (1) we have:

$$\begin{aligned} H &\equiv \frac{1}{2m} \left(\mathbf{p} - \frac{q}{c} \mathbf{A} \right)^2 \\ &= \frac{1}{2m} \mathbf{p}^2 + \frac{q^2}{2mc^2} \mathbf{A}^2 - \frac{q}{mc} \mathbf{p} \cdot \mathbf{A} \\ &= \frac{1}{2m} \mathbf{p}^2 + \frac{q^2}{2mc^2} \frac{1}{4} \mathbf{B}^2 \mathbf{x}^2 - \frac{q^2}{2mc^2} \frac{1}{4} \left(\underbrace{\mathbf{B} \cdot \mathbf{x}}_0 \right)^2 - \frac{q}{mc} \mathbf{p} \cdot \frac{1}{2} \mathbf{B} \times \mathbf{x} \\ &= \frac{1}{2m} \mathbf{p}^2 + \frac{1}{2m} \left(\frac{qB}{2c} \right)^2 \mathbf{x}^2 - \frac{1}{2m} \frac{q}{c} \mathbf{B} \cdot \mathbf{L} \\ &= \frac{1}{2m} \left(\mathbf{p}^2 + \left(\frac{qB}{2c} \right)^2 \mathbf{x}^2 - \frac{qB}{c} L_3 \right) \end{aligned} \quad (3)$$

where we have used the fact that \mathbf{x} and \mathbf{p} only have components along $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$, so that $\mathbf{L} \parallel \mathbf{B}$, and $\mathbf{B} \cdot \mathbf{L} = BL_3$.

3.0.1. Claim. $[H, L_3] = 0$.

Proof. Using the fact that $[ab, c] = a[b, c] + [a, c]b$ we have

$$\begin{aligned} [p_i^2, L_3] &= [p_i^2, \epsilon_{3jk} x_j p_k] \\ &= \epsilon_{3jk} p_i [p_i, x_j p_k] + \epsilon_{3jk} [p_i, x_j p_k] p_i \\ &= \epsilon_{3jk} p_i [p_i, x_j] p_k + \epsilon_{3jk} [p_i, x_j] p_k p_i \\ &= \epsilon_{3jk} p_i i \delta_{i,j} p_k + \epsilon_{3jk} i \delta_{i,j} p_k p_i \\ &= 2\epsilon_{3ik} p_i i p_k \\ &= 0 \end{aligned}$$

because partial derivatives commute, and

$$\begin{aligned} [x_i^2, L_3] &= [x_i^2, \epsilon_{3jk} x_j p_k] \\ &= \epsilon_{3jk} x_i [x_i, x_j p_k] + \epsilon_{3jk} [x_i, x_j p_k] x_i \\ &= \epsilon_{3jk} x_i x_j [x_i, p_k] + \epsilon_{3jk} x_j [x_i, p_k] x_i \\ &= 2\epsilon_{3jk} x_k x_j \\ &= 0 \end{aligned}$$

□

3.0.2. *Remark.* Using (3.0.1) one could replace the operator L_3 in the last row of (3) with its eigenvalues $\hbar m_3$ to obtain

$$H = \frac{1}{2m} \left(p^2 + \left(\frac{qB}{2c} \right)^2 x^2 - \frac{qB}{c} \hbar m_3 \right)$$

which is a two-dimensional Harmonic oscillator shifted by a constant term. Then one would have to employ the constraint that the angular momentum values of the states obtained match a given m_3 . We will use an alternative approach below.

3.0.3. **Claim.** (1) The spectrum of H consists of discrete eigenvalues, called the Landau levels, given by

$$\sigma(H) = \left\{ \hbar \frac{qB}{mc} \left(k + \frac{1}{2} \right) \mid k \in \mathbb{N}_{\geq 0} \right\} \quad (4)$$

(2) Each Landau level is infinitely degenerate with

$$\frac{qB}{hc} \quad (5)$$

eigenstates per unit area.

Proof. For convenience we work in units in which $m = \hbar = \frac{qB}{2c} = 1$. Then the last row of (3) is equal to

$$H = \frac{1}{2} (p^2 + x^2) - L_3$$

where $L_3 \equiv -i(x_1 \partial_2 - x_2 \partial_1)$. Define $z := x_1 + ix_2$, $\partial_z := \frac{1}{2}(\partial_1 - i\partial_2)$, $\partial_{\bar{z}} := \frac{1}{2}(\partial_1 + i\partial_2)$. Note that z is not self-adjoint:

$$\begin{aligned} z^* &= x_1^* - ix_2^* \\ &= x_1 - ix_2 \end{aligned}$$

and that

$$(\partial_z)^* = -\partial_{\bar{z}}$$

because using partial integration we have

$$\begin{aligned} \langle f, \partial_z g \rangle &\equiv \int_{\mathbb{R}^2} \bar{f}(x) \partial_z g(x) dx \\ &= \int_{\mathbb{R}^2} \bar{f}(x) \frac{1}{2} (\partial_1 - i\partial_2) g(x) dx \\ &= \int_{\mathbb{R}^2} \left[\frac{1}{2} (-\partial_1 + i\partial_2) \bar{f}(x) \right] g(x) dx \\ &= \int_{\mathbb{R}^2} \overline{\left[\frac{1}{2} (-\partial_1 - i\partial_2) f \right]}(x) g(x) dx \\ &= \langle -\partial_{\bar{z}} f, g \rangle \end{aligned}$$

Then we have

$$\begin{aligned} L_3 &\equiv -i(x_1 \partial_2 - x_2 \partial_1) \\ &= -i \left(\frac{1}{2}(z + \bar{z}) \frac{1}{i} (\partial_{\bar{z}} - \partial_z) - \frac{1}{2i}(z - \bar{z})(\partial_{\bar{z}} + \partial_z) \right) \\ &= -\frac{1}{2}(z + \bar{z})(\partial_{\bar{z}} - \partial_z) + \frac{1}{2}(z - \bar{z})(\partial_{\bar{z}} + \partial_z) \\ &= z\partial_z - \bar{z}\partial_{\bar{z}} \end{aligned}$$

$$\begin{aligned} x^2 &\equiv x_1^2 + x_2^2 \\ &= z\bar{z} \end{aligned}$$

$$\begin{aligned} p^2 &\equiv -\partial_1^2 - \partial_2^2 \\ &= -4\partial_z \partial_{\bar{z}} \end{aligned}$$

So that

$$\begin{aligned} H &= \frac{1}{2}(-4\partial_z \partial_{\bar{z}} + z\bar{z}) - (z\partial_z - \bar{z}\partial_{\bar{z}}) \\ &= 2 \left(\frac{1}{2}\bar{z} - \partial_z \right) \left(\frac{1}{2}z + \partial_{\bar{z}} \right) + 1 \end{aligned}$$

Using the fact that $\partial_z z = 1 + z\partial_z$ as operators on \mathcal{H} . So we define ladder operator $a := \frac{1}{2}z + \partial_{\bar{z}}$ so that $a^* = \frac{1}{2}\bar{z} - \partial_z$ and so

$$H = 2a^*a + 1$$

We remark that

$$\begin{aligned}
 a &\equiv \frac{1}{2}z + \partial_{\bar{z}} \\
 &= \partial_{\bar{z}} \frac{1}{2}z\bar{z} + \partial_{\bar{z}} \\
 &= \exp\left(-\frac{1}{2}z\bar{z}\right) \left\{ \left[\exp\left(\frac{1}{2}z\bar{z}\right) \partial_{\bar{z}} \frac{1}{2}z\bar{z} \right] + \exp\left(\frac{1}{2}z\bar{z}\right) \partial_{\bar{z}} \right\} \\
 &= \exp\left(-\frac{1}{2}z\bar{z}\right) \left\{ \left[\partial_{\bar{z}} \exp\left(\frac{1}{2}z\bar{z}\right) \right] + \exp\left(\frac{1}{2}z\bar{z}\right) \partial_{\bar{z}} \right\} \\
 &= \exp\left(-\frac{1}{2}z\bar{z}\right) \partial_{\bar{z}} \exp\left(\frac{1}{2}z\bar{z}\right)
 \end{aligned}$$

and similarly $a^* = -\exp\left(\frac{1}{2}z\bar{z}\right) \partial_z \exp\left(-\frac{1}{2}z\bar{z}\right)$. Next, note that

$$\begin{aligned}
 [a^*, a] &= \left[\frac{1}{2}\bar{z} - \partial_z, \frac{1}{2}z + \partial_{\bar{z}} \right] \\
 &= \frac{1}{4} \underbrace{[\bar{z}, z]}_0 + \frac{1}{2} \underbrace{[\bar{z}, \partial_{\bar{z}}]}_{-1} - \frac{1}{2} \underbrace{[\partial_z, z]}_1 - \underbrace{[\partial_z, \partial_{\bar{z}}]}_0 \\
 &= -1
 \end{aligned}$$

As a result, we know immediately the spectrum of H , because we know the eigenvalues of a^*a , which are the same as for the harmonic oscillator: $\sigma(a^*a) = \mathbb{N}_{\geq 0}$ so that $\sigma(H) = 2\mathbb{N}_{\geq 0} + 1$. So we have shown the first part of the claim, namely (4). This happens after restoring the units:

$$\begin{aligned}
 \hbar \frac{qB}{mc} &= \underbrace{\hbar}_{\equiv 1} \times \underbrace{\frac{qB}{2c}}_{\equiv 1} \times \underbrace{\frac{1}{m}}_{\equiv 1} \times 2 \\
 &= 2
 \end{aligned}$$

Let ψ_k be an eigenstate of H corresponding to eigenvalue $2k+1$, $k \in \mathbb{N}_{\geq 0}$. That means for $k=0$, $H\psi_0 = \psi_0$, so that $a\psi_0 = 0$. From this we get a differential equation:

$$\begin{aligned}
 \left(\frac{1}{2}z + \partial_{\bar{z}} \right) \psi_0(z) &= 0 \\
 \uparrow \\
 \partial_{\bar{z}} \psi_0(z) &= -\frac{1}{2}z\psi_0(z)
 \end{aligned}$$

so that $\psi_0(z) = \tilde{\psi}_0(z) e^{-\frac{1}{2}z\bar{z}}$ for some $\tilde{\psi}_0$ such that

$$\partial_{\bar{z}} \tilde{\psi}_0(z) = 0 \quad (6)$$

Note that this constraint exactly means that $\tilde{\psi}_0$ is an analytic function because (6) is equivalent to the Cauchy-Riemann equations. However, polynomials are dense in the set of analytic functions so that it suffices to assume that $\tilde{\psi}_0$ is any polynomial. As a result we obtain the fact that the first Landau level with $k=0$ is infinitely degenerate because there are infinitely many polynomials. We may span this dense subspace of eigenstates with $k=0$ by the orthogonal set $\{\psi_{0,m}\}_{m \in \mathbb{N}_{\geq 0}}$ where

$$\psi_{0,m}(z) := (\pi m!)^{-\frac{1}{2}} z^m e^{-\frac{1}{2}z\bar{z}}$$

to see the orthogonality, write $z = re^{i\varphi}$, so that

$$\begin{aligned}
 \langle \psi_{0,m}, \psi_{0,m'} \rangle &\equiv \int_{\mathbb{C}} \overline{\psi_{0,m}(z)} \psi_{0,m'}(z) dz \\
 &= \int_{\mathbb{C}} \overline{(\pi m!)^{-\frac{1}{2}} z^m e^{-\frac{1}{2}z\bar{z}}} (\pi m'!)^{-\frac{1}{2}} z^{m'} e^{-\frac{1}{2}z\bar{z}} dz \\
 &= \int_{\varphi=0}^{2\pi} \int_{r=0}^{\infty} (\pi m!)^{-\frac{1}{2}} r^m e^{-im\varphi} e^{-\frac{1}{2}r^2} (\pi m'!)^{-\frac{1}{2}} r^{m'} e^{im'\varphi} e^{-\frac{1}{2}r^2} r dr d\varphi \\
 &= (\pi m!)^{-\frac{1}{2}} (\pi m'!)^{-\frac{1}{2}} \underbrace{\left(\int_{\varphi=0}^{2\pi} e^{-i(m-m')\varphi} d\varphi \right)}_{2\pi \delta_{m,m'}} \underbrace{\int_{r=0}^{\infty} r^{m+m'+1} e^{-r^2} dr}_{\frac{1}{2} \Gamma\left(\frac{1}{2}(m+m')+1\right)} \\
 &= \frac{2\delta_{m,m'}}{m!} \frac{1}{2} m!
 \end{aligned}$$

The following Landau levels are given by applying the creation operator, as is well known from the harmonic oscillator:

$$\psi_k = (k!)^{-\frac{1}{2}} (a^*)^k \psi_0$$

because

$$\begin{aligned}
 H\psi_k &= (2a^*a + 1)(k!)^{-\frac{1}{2}}(a^*)^k\psi_0 \\
 &= (k!)^{-\frac{1}{2}}(a^*)^k\psi_0 + 2(k!)^{-\frac{1}{2}}a^*\underbrace{a(a^*)^k}_{k(a^*)^{k-1}+(a^*)^k a}\psi_0 \\
 &= (k!)^{-\frac{1}{2}}(a^*)^k\psi_0 + 2(k!)^{-\frac{1}{2}}a^*k(a^*)^k\psi_0 \\
 &= (2k+1)(k!)^{-\frac{1}{2}}(a^*)^k\psi_0 \\
 &= (2k+1)\psi_k
 \end{aligned}$$

As a result, for each energy $k \in \mathbb{N}_{\geq 0}$ we have an infinite number of eigenstates again labelled by $m \in \mathbb{N}_{\geq 0}$: $\{\psi_{k,m}\}_{m \in \mathbb{N}_{\geq 0}}$ spans the eigenspace of $2k+1$.

Next, the density of states is given by an expression of the form

$$\frac{dn(E)}{dE} = \sum_{k \in \mathbb{N}_{\geq 0}} \sum_{m=0}^{\infty} |\psi_{k,m}(z)|^2 \delta\left(E - \underbrace{E_k}_{=(2k+1)}\right)$$

So that we would like to compute $\sum_{m=0}^{\infty} |\psi_{k,m}(z)|^2$ which will give the number of states at energy E_k per unit area (at the point $z \in \mathbb{C}$). Using translational invariance we may assume $z = 0$ so that this quantity is independent of z . For the case when $k = 0$ we have:

$$\begin{aligned}
 \sum_{m=0}^{\infty} |\psi_{0,m}(0)|^2 &= \sum_{m=0}^{\infty} |(\pi m!)^{-\frac{1}{2}} 0^m|^2 \\
 &= \pi^{-1}
 \end{aligned}$$

The case $k \neq 0$ proceeds as follows: We first note that

$$\begin{aligned}
 L_3\psi_{0,m}(z) &= (z\partial_z - \bar{z}\partial_{\bar{z}})(\pi m!)^{-\frac{1}{2}} z^m e^{-\frac{1}{2}z\bar{z}} \\
 &= \left(m - \frac{1}{2}\bar{z}z + \frac{1}{2}\bar{z}z\right)(\pi m!)^{-\frac{1}{2}} z^m e^{-\frac{1}{2}z\bar{z}} \\
 &= m\psi_{0,m}(z)
 \end{aligned}$$

so that using the fact $[L_3, a^*] = -a^*$ we have

$$\begin{aligned}
 L_3\psi_{k,m}(z) &= L_3(k!)^{-\frac{1}{2}}(a^*)^k\psi_{0,k}(z) \\
 &= (k!)^{-\frac{1}{2}}\left((a^*)^k L_3 - k(a^*)^{k-1} a^*\right)\psi_{0,k}(z) \\
 &= (m-k)\psi_{k,m}(z)
 \end{aligned}$$

so that $\psi_{k,m}(0)$ is non-zero only when $m = k$, because $L_3\psi_{k,m}(0) \stackrel{!}{=} 0$ by rotational invariance. Then we can calculate that

$$\begin{aligned}
 \psi_{k,k}(z) &\equiv (k!)^{-\frac{1}{2}}(a^*)^k\psi_{0,k}(z) \\
 &= (k!)^{-\frac{1}{2}}\left[-\exp\left(\frac{1}{2}z\bar{z}\right)\partial_z \exp\left(-\frac{1}{2}z\bar{z}\right)\right]^k (\pi k!)^{-\frac{1}{2}} z^k e^{-\frac{1}{2}z\bar{z}} \\
 &= (k!)^{-1} \pi^{-\frac{1}{2}} \exp\left(\frac{1}{2}z\bar{z}\right) (-\partial_z)^k \exp\left(-\frac{1}{2}z\bar{z}\right) z^k e^{-\frac{1}{2}z\bar{z}} \\
 &= \pi^{-\frac{1}{2}} k!^{-1} \exp\left(\frac{1}{2}z\bar{z}\right) (-\partial_z)^k z^k \exp(-z\bar{z})
 \end{aligned}$$

so that

$$\psi_{k,k}(0) = \pi^{-\frac{1}{2}} (-1)^k$$

and

$$\begin{aligned}
 \sum_{m=0}^{\infty} |\psi_{k,m}(z)|^2 &= |\psi_{k,k}(0)|^2 \\
 &= \pi^{-1}
 \end{aligned}$$

Now after restoring the units we would find that

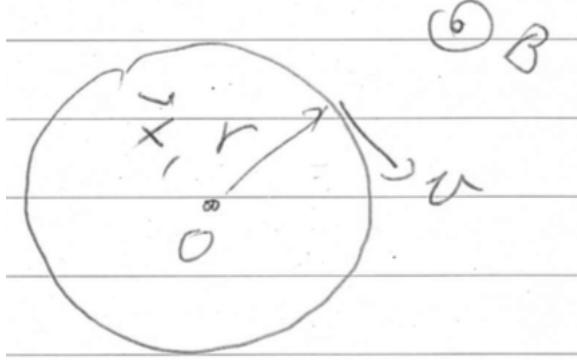
$$\begin{aligned}
 \frac{qB}{ch} &= \frac{qB}{c2\pi\hbar} \\
 &= \underbrace{\frac{qB}{2c}}_{\equiv 1} \times \underbrace{\frac{1}{\hbar}}_{\equiv 1} \times \frac{1}{\pi}
 \end{aligned}$$

so that (5) follows. □

3.0.4. *Remark.* The following should be noted:

- (1) Going back to the many-body system, we use the Pauli exclusion principle to fill up the levels of the system up to the Fermi energy. The density of states dictates how many electrons per unit area may be placed in each level: as many as there are degenerate states per unit area.
- (2) Recall from 2.2.1 that the filling factor is $\nu \equiv \frac{n}{(\frac{qB}{hc})}$ so that (5) shows that ν actually tells how many of the Landau levels are filled.
- (3) If $\nu \in \mathbb{N}_{\geq 0}$ then no Landau level is *partially* filled.
- (4) We have already seen that σ_H is in linear relation to ν so that integer values of ν lead to integer values of $\sigma_H \times \frac{h}{q^2}$.
- (5) However, this does not explain the plateaus, namely, why *near* integer values of ν , $\sigma_H \times \frac{h}{q^2}$ is still integer valued.

Next, we provide a heuristic semi-classical explanation for 3.0.3. In the presence of a magnetic field (and no electric field), an electron undergoes circular motion.



Equating the radial centrifugal force with the Lorentz force we get:

$$m \frac{v^2}{r} = qB \frac{v}{c}$$

so that $mv = qBr\frac{1}{c}$.

Next we employ Bohr's quantization which stipulates that angular momentum is quantized in units of \hbar :

$$L \stackrel{!}{=} \hbar k$$

for some $k \in \mathbb{N}_{\geq 0}$. Because the electron is under the influence of a magnetic field, the momentum p that enters in the computation of the angular momentum should be the *canonical* momentum (that quantity which is conserved, $\frac{\partial L}{\partial x}$, rather than mv). Thus²:

$$\begin{aligned} L &= \mathbf{x} \times \left(mv + \frac{q}{c} \mathbf{A} \right) \\ &= \mathbf{x} \times \left(mv + \frac{q}{c} \frac{\mathbf{B} \times \mathbf{x}}{2} \right) \\ &= -mv r \hat{\mathbf{e}}_3 + \frac{qBr^2}{2c} \hat{\mathbf{e}}_3 \\ &= -qBr \frac{1}{c} r \hat{\mathbf{e}}_3 + \frac{qBr^2}{2c} \hat{\mathbf{e}}_3 \\ &= -\frac{qBr^2}{2c} \hat{\mathbf{e}}_3 \end{aligned}$$

So that $\frac{qBr^2}{2c} \stackrel{!}{=} \hbar k$ and so the radius is quantized

$$r_k^2 = k \hbar \frac{2c}{qB}$$

and so is the energy:

$$\begin{aligned} E_k &= \frac{1}{2} m v_k^2 \\ &= \frac{1}{2m} (mv_k)^2 \\ &= \frac{1}{2m} \left(qBr_k \frac{1}{c} \right)^2 \\ &= \frac{1}{2m} \left(\frac{qB}{c} \right)^2 k \hbar \frac{2c}{qB} \\ &= \frac{qB}{mc} k \hbar \end{aligned}$$

which differs from 4 by $k \mapsto k + \frac{1}{2}$.

²Recall the Lagrangian is $\frac{1}{2} m \mathbf{v}^2 + \frac{q}{c} \mathbf{v} \cdot \mathbf{A}$

Next, note that for a fixed center, the state labeled by k occupies an area of $\pi(r_{k+1}^2 - r_k^2)$. This turns out to be equal to:

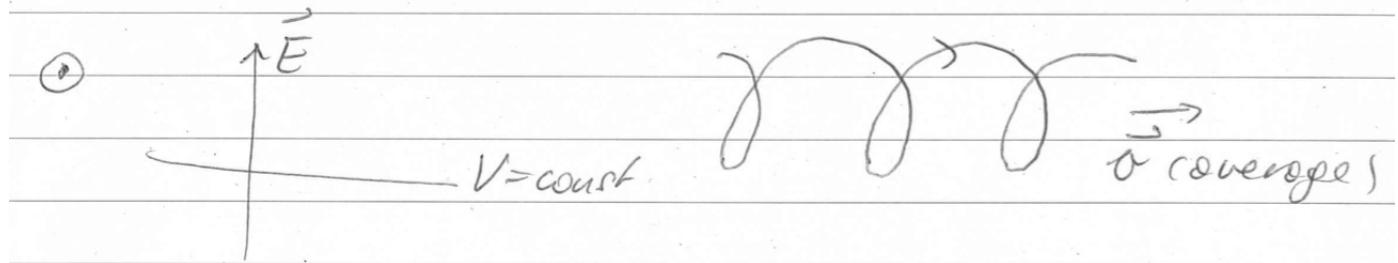
$$\begin{aligned}\pi(r_{k+1}^2 - r_k^2) &= \pi\hbar\frac{2c}{qB} \\ &= \frac{hc}{qB}\end{aligned}$$

Thus the density of states per unit area should be $\frac{qB}{hc}$ so as to not double count states (since the choice of the origin is arbitrary). This leads to (5).

4. STABILITY OF σ_H : EXISTENCE OF PLATEAUS, AND THE ROLE OF DISORDER

4.1. Heuristic Semi-Classical Explanation. Particles in a fixed Landau level correspond classically to trajectories which are circles with fixed radius, with centers which are uniformly distributed in the plane. Each Landau level contributes to this density $\frac{qB}{hc}$ electrons per unit area.

In the presence of an electric field, as already mentioned above, the paths are prolate trochoidal:



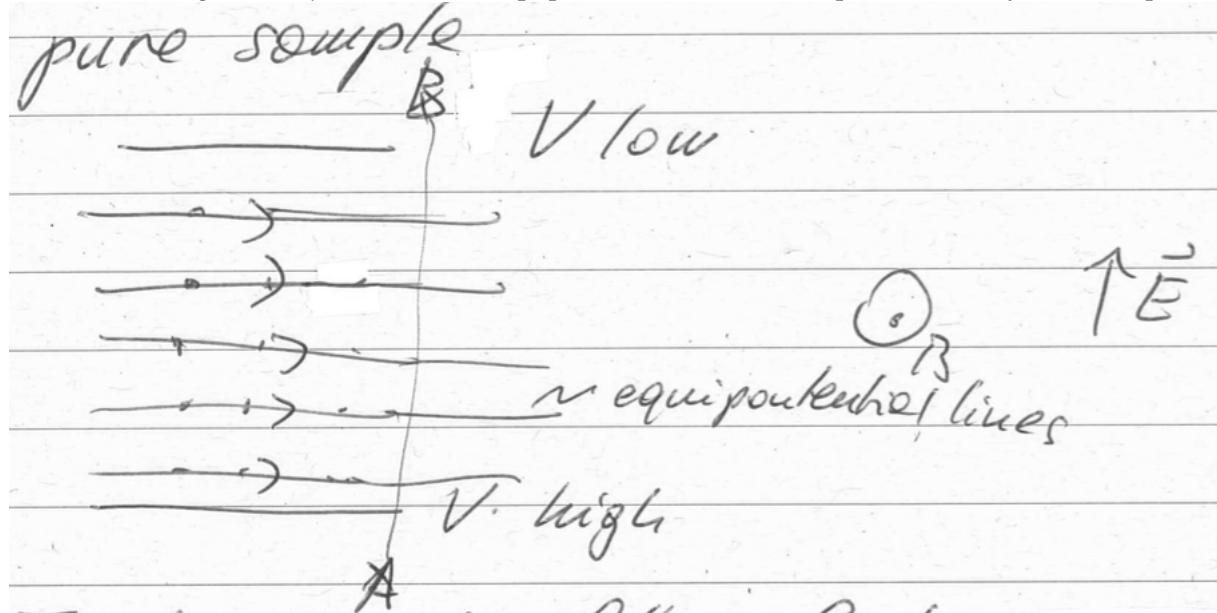
We write $\mathbf{E} = -\nabla V$ where $V(x)$ is the applied external potential. As in the first section, for an equilibrium state we have

$$\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} = 0$$

If we cross this with $\frac{\mathbf{B}}{B^2}$ we get:

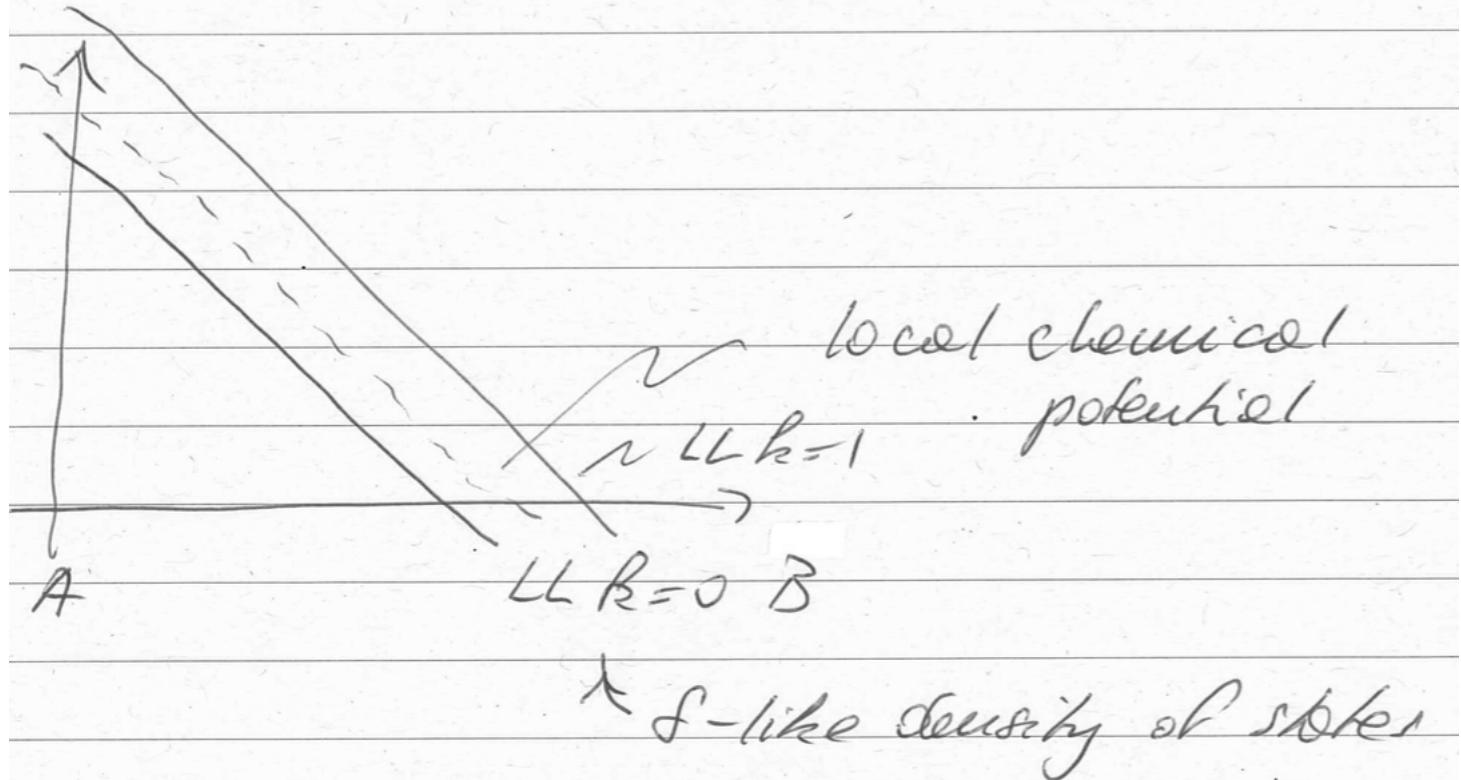
$$\begin{aligned} \mathbf{E} \times \frac{\mathbf{B}}{B^2} + \left(\frac{\mathbf{v}}{c} \times \mathbf{B} \right) \times \frac{\mathbf{B}}{B^2} &= 0 \\ \downarrow \\ -\nabla V \times \frac{\mathbf{B}}{B^2} - \frac{\mathbf{v}}{c} &= 0 \\ \downarrow \\ \frac{\mathbf{v}}{c} &= -\nabla V \times \frac{\mathbf{B}}{B^2} \end{aligned}$$

so that on average, the trajectories follow equipotential lines of V . A top view of the system for a pure sample:

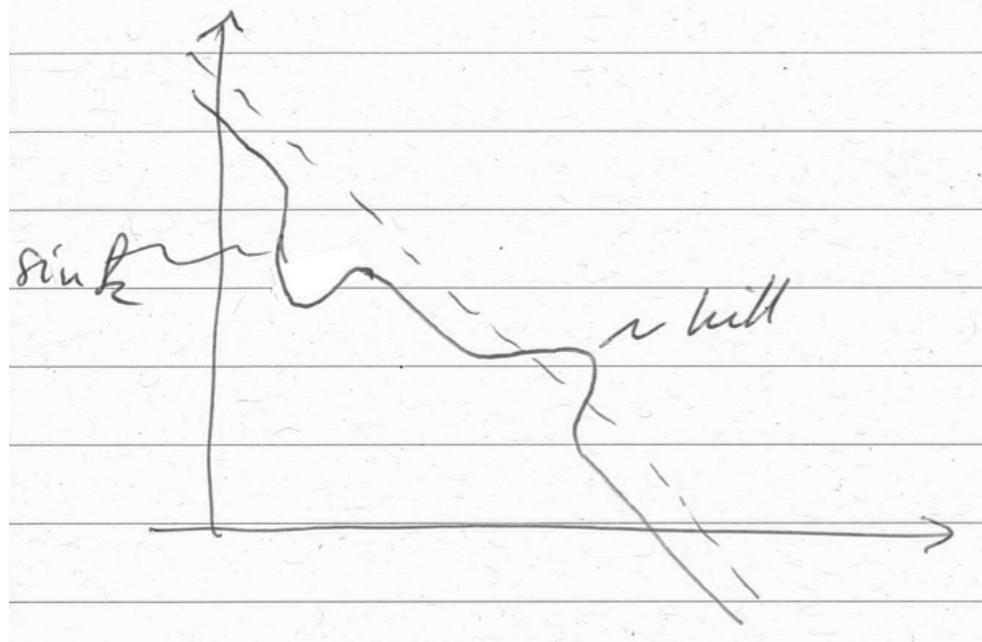


Trying to understand the plateaus of σ_H in this scenario, we imagine lowering v . The result of that is to empty some of the trajectories, so that the current lowers proportionally, and there are no plateaus. Thus an entirely pure sample cannot give rise to plateaus. Going back to the Landau energy levels, before turning on the electric field each level was degenerate. However, with the electric field there is a difference between the lower end of the sample and the upper end of the sample (on the $\hat{\mathbf{e}}_2$ axis), which gives rise to the drift motion. A cross-section view of the energy levels versus the $\hat{\mathbf{e}}_2$ axis along the

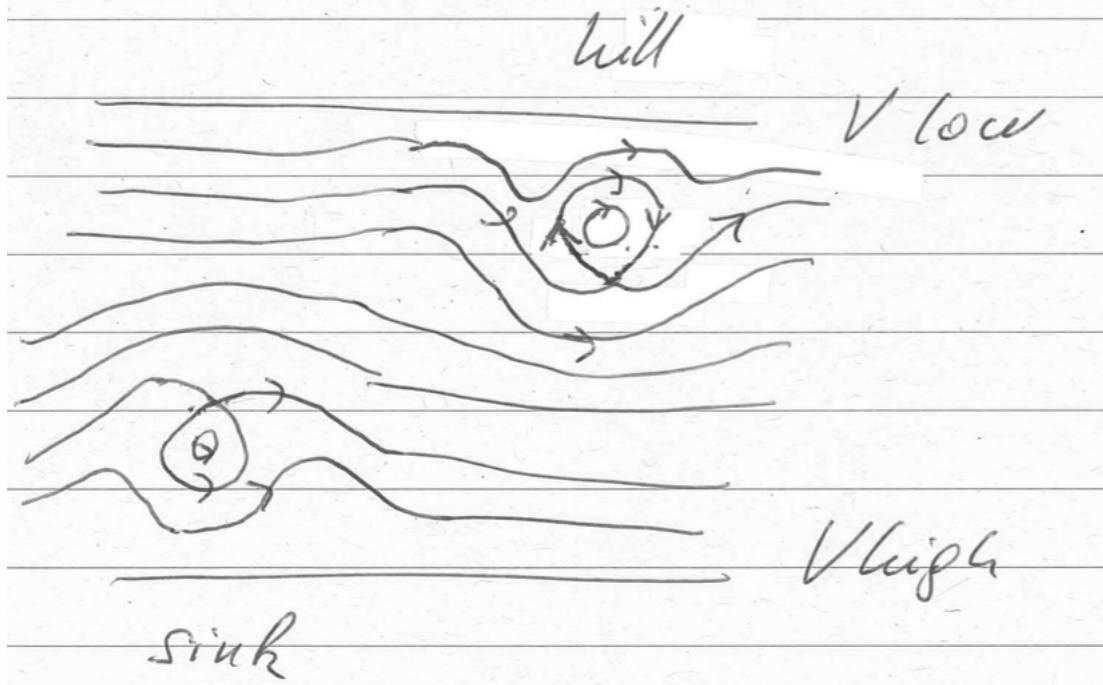
line A-B in the above picture is as follows:



For a disordered sample, there are impurities which give rise to potential hills and sinks:



and the equipotential lines look as follows from top view:



Since the trajectories follow equipotential lines, there are closed loops around some of the hills and sinks, which means bound (localized) states which do not contribute to the conduction. However, these states do contribute to v . Thus, we can lower v a bit to empty those bound states near the hills and the sinks and have the current unaffected. This *does* give rise to plateaus of σ_H .

Note that this is consistent with uniform distribution of ρ of centers in space and time:

$$\begin{aligned}\nabla \cdot \frac{\mathbf{v}}{c} &= \nabla \cdot \left(\frac{\mathbf{B}}{B^2} \times \nabla V \right) \\ &= \partial_i \left(\frac{\mathbf{B}}{B^2} \times \nabla V \right)_i \\ &= \frac{1}{B} \partial_i \epsilon_{i3j} \partial_j V \\ &= 0\end{aligned}$$

and by the continuity equation

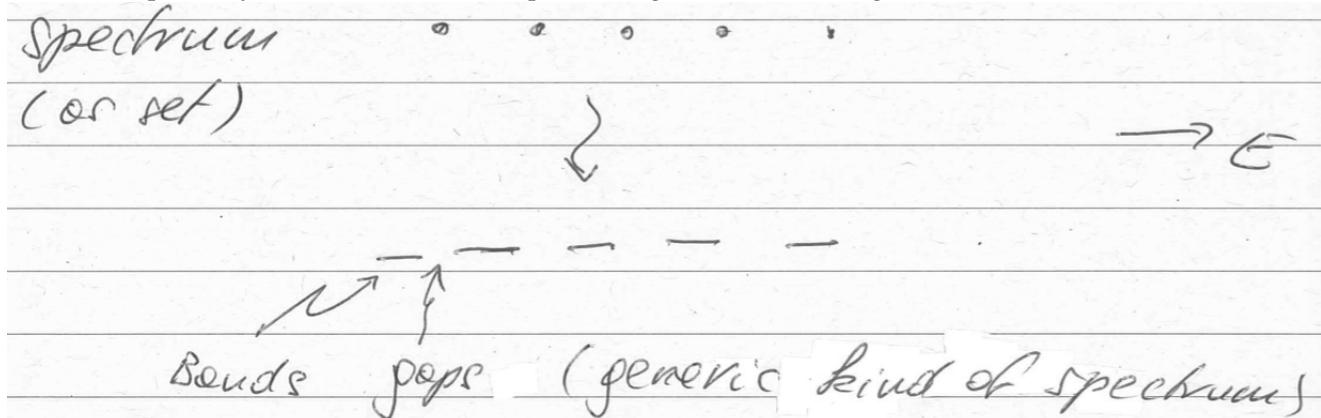
$$\dot{\rho} + \nabla(\rho v) = 0$$

we have

$$\begin{aligned}\dot{\rho} &= -\nabla(\rho v) \\ &= -\rho \nabla v \\ &= 0\end{aligned}$$

assuming that ρ is spatially uniform.

4.2. Quantum Viewpoint. The effect of disorder is on the Landau levels is to broaden them. This can be understood by considering each impurity as generating a small electric (positive or negative) potential which shifts the Landau levels up or down respectively. The effect of all the impurities together is a smearing:



In order to understand this better we must describe the spectral decomposition of self adjoint operators on Hilbert spaces.

Spectral Decomposition. We recall a few notions of functional analysis. The definitions and claims are presented mainly to align notation and conventions. Unfamiliar readers should consult [40] and [38].

4.2.1. Definition. A Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ is a tuple where \mathcal{H} is a \mathbb{C} -vector space and $\langle \cdot, \cdot \rangle : \mathcal{H}^2 \rightarrow \mathbb{C}$ is an inner-product on \mathcal{H} (it is \mathbb{C} -linear in its right slot, conjugate-symmetric ($\langle v_1, v_2 \rangle = \overline{\langle v_2, v_1 \rangle}$), and positive-definite) such that the norm which $\langle \cdot, \cdot \rangle$ induces ($\|v\| \equiv \sqrt{\langle v, v \rangle}$) makes \mathcal{H} into a *complete* metric space.

4.2.2. Remark. It is necessary to employ the completeness constraint only when considering infinite dimensional vector spaces, since up to linear homeomorphism \mathbb{C}^n is the only finite dimensional space we could consider.

4.2.3. Claim. Given two normed spaces \mathcal{H}_1 and \mathcal{H}_2 , there is a norm on the space of linear maps $\mathcal{H}_1 \rightarrow \mathcal{H}_2$ (which is a \mathbb{C} -vector space) given by

$$\|\mathbf{T}\|_{\text{op}} := \sup \left(\left\{ \|\mathbf{T}(v)\|_{\mathcal{H}_2} \mid \|v\|_{\mathcal{H}_1} \leq 1 \right\} \right)$$

Proof. See [40] theorem 4.1. □

4.2.4. Definition. For normed linear spaces \mathcal{H}_1 and \mathcal{H}_2 , define the set of bounded linear operators as

$$\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2) := \left\{ T : \mathcal{H}_1 \rightarrow \mathcal{H}_2 \mid T \text{ is } \mathbb{C}\text{-linear and } \|T\|_{\text{op}} < \infty \right\}$$

and we also define $\mathcal{B}(\mathcal{H}) := \mathcal{B}(\mathcal{H}, \mathcal{H})$.

4.2.5. Remark. In what follows we assume \mathcal{D} is a dense linear subspace of a Hilbert space \mathcal{H} and that $H : \mathcal{D} \rightarrow \mathcal{H}$ is a linear map (not necessarily bounded).

4.2.6. Definition. Define a new subset of \mathbb{C} , called the eigenvalues of H :

$$\mathcal{E}(H) := \{ \lambda \in \mathbb{C} \mid \exists v_\lambda \in \mathcal{H} \setminus \{0\} : H(v_\lambda) = \lambda v_\lambda \}$$

4.2.7. Remark. The eigenvalues of a linear map on finite dimensional vector spaces (a matrix) is a familiar concept. We generalize $\mathcal{E}(H)$ to infinite dimensional Hilbert spaces below:

4.2.8. Definition. Define a new subset of \mathbb{C} , called the spectrum of H , via its complement, the resolvent set $\rho(H)$:

$$\sigma(H) := \mathbb{C} \setminus \underbrace{\left(\left\{ \lambda \in \mathbb{C} \mid \exists (H - \lambda \mathbb{1}_{\mathcal{H}})^{-1} \in \mathcal{B}(\mathcal{H}) \right\} \right)}_{\rho(H)}$$

4.2.9. Remark. If the inverse exists at all then it is linear, so one never has to verify the linearity of $(H - \lambda \mathbb{1}_{\mathcal{H}})^{-1}$; For fixed $\lambda \in \mathbb{C}$, there are three ways $H - \lambda \mathbb{1}_{\mathcal{H}}$ could have no bounded inverse map $\mathcal{H} \rightarrow \mathcal{D}$:

(1) If $(H - \lambda \mathbb{1}_{\mathcal{H}})$ is not injective, then $\ker(H - \lambda \mathbb{1}_{\mathcal{H}}) \neq 0$, so that $\exists v \in \mathcal{H} \setminus \{0\}$ such that $H(v) = \lambda(v)$. That is, λ is an eigenvalue of H : $\lambda \in \mathcal{E}(H)$. This is the only possibility if $\dim(\mathcal{H}) < \infty$, because an injective linear map is necessarily surjective on a finite dimensional vector space by the rank nullity relation. Also, any linear map on finite dimensional spaces is bounded. Thus we have:

$$\dim(\mathcal{H}) < \infty \implies \sigma(H) = \mathcal{E}(H)$$

The converse is false.

(2) $(H - \lambda \mathbb{1}_{\mathcal{H}})$ could fail to be surjective. For example, if $\mathcal{H} := l^2(\mathbb{N}; \mathbb{C})$ and $R : \mathcal{H} \rightarrow \mathcal{H}$ is defined via

$$(x_1, x_2, x_3, \dots) \mapsto (0, x_1, x_2, \dots)$$

then clearly R is not surjective because $(1, 0, 0, \dots) \in \mathcal{H}$ is not in its range. It is however injective because if $Rv = \lambda v$ then it follows that $v = 0$. It is also clearly bounded because

$$\begin{aligned} \|Rv\|^2 &= \sum_{i=2}^{\infty} |v_i|^2 \\ &= \|v\|^2 - |v_1|^2 \\ &\leq \|v\|^2 \end{aligned}$$

Then we say that $0 \in \sigma(R)$ even though 0 is not an eigenvalue of R :

$$\underbrace{\mathcal{E}(R)}_{\emptyset} \subsetneq \sigma(R)$$

(3) It could be that $\exists (H - \lambda \mathbb{1}_{\mathcal{H}})^{-1} : \mathcal{H} \rightarrow \mathcal{D}$ (that is, $H - \lambda \mathbb{1}_{\mathcal{H}}$ is bijective), but that $(H - \lambda \mathbb{1}_{\mathcal{H}})^{-1}$ is none the less not bounded. This case is not possible if $(H - \lambda \mathbb{1}_{\mathcal{H}})$ is a closed operator (and a self-adjoint operator is always closed).

4.2.10. Claim. There exists a unique linear map $H^* : \tilde{\mathcal{D}} \rightarrow \mathcal{H}$, called the adjoint of H , where

$$\tilde{\mathcal{D}} := \{ v \in \mathcal{H} \mid (v' \mapsto \langle v, H(v') \rangle) \forall v' \in \mathcal{D} \in \mathcal{B}(\mathcal{D}, \mathbb{C}) \}$$

is a linear subspace of \mathcal{H} and such that

$$\langle v_2, H(v_1) \rangle = \langle H^*(v_2), v_1 \rangle$$

for all $(v_1, v_2) \in \mathcal{D} \times \tilde{\mathcal{D}}$. If $H \in \mathcal{B}(\mathcal{H})$ then $\|H\|_{\text{op}} = \|H^*\|_{\text{op}}$.

Proof. See the text after definition 13.1 in [40]. □

4.2.11. Definition. H is called *self-adjoint* iff $H = H^*$.

4.2.12. Claim. If H is self-adjoint then $\sigma(H) \subseteq \mathbb{R}$.

Proof. Theorem 11.28 (a) in [40] for the case that $H \in \mathcal{B}(\mathcal{H})$. See also [45] theorem 2.18. □

4.2.13. Claim. If H is self-adjoint, then

$$\sigma(H) = \{\lambda \in \mathbb{C} \mid \forall \varepsilon > 0 \exists v_\varepsilon \in \mathcal{H} \setminus \{0\} : \|H(v_\varepsilon) - \lambda v_\varepsilon\| < \varepsilon \|v_\varepsilon\|\}$$

that is, the spectrum of a self-adjoint H consists of eigenvalues as well as “approximate” eigenvalues. The sequence $(v_\varepsilon)_{\varepsilon > 0}$ is called a Weyl sequence.

Proof. We divide the proof into the two directions:

Case 1. ⊥ ←→ ⊥ Let $\lambda \in \mathbb{C}$ be given and assume the converse. Then $\exists \varepsilon > 0$ such that $\forall v \in \mathcal{H} \setminus \{0\}$ we have $\|H(v) - \lambda v\| \geq \varepsilon \|v\|$. Then $(H - \lambda \mathbb{1})$ is injective (otherwise $\ker(H - \lambda \mathbb{1}) \neq \{0\}$ so we have some $u \in \mathcal{H} \setminus \{0\}$ such that $H(u) - \lambda u = 0$ which contradicts the assumption). As a result, we may define $(H - \lambda \mathbb{1})^{-1} : (H - \lambda \mathbb{1})(\mathcal{H}) \rightarrow \mathcal{D}$. Now we show that $(H - \lambda \mathbb{1})^{-1} : (H - \lambda \mathbb{1})(\mathcal{H}) \rightarrow \mathcal{D}$ is bounded. Let $v \in (H - \lambda \mathbb{1})(\mathcal{H})$ be given. Then $\exists u \in \mathcal{D}$ such that $(H - \lambda \mathbb{1})u = v$. Then

$$\begin{aligned} \|(H - \lambda \mathbb{1})^{-1} v\| &= \|(H - \lambda \mathbb{1})^{-1}(H - \lambda \mathbb{1})u\| \\ &= \|u\| \\ &\leq \frac{1}{\varepsilon} \|(H - \lambda \mathbb{1})u\| \\ &= \frac{1}{\varepsilon} \|(H - \lambda \mathbb{1})(H - \lambda \mathbb{1})^{-1} v\| \\ &= \frac{1}{\varepsilon} \|v\| \end{aligned}$$

so that $(H - \lambda \mathbb{1})^{-1}$ is bounded on its domain. Now we show that $(H - \lambda \mathbb{1})(\mathcal{H})$ is dense in \mathcal{H} : Suppose that $y \perp (H - \lambda \mathbb{1})(\mathcal{H})$. Then $x \mapsto \underbrace{\langle y, (H - \lambda \mathbb{1})x \rangle}_0$ is continuous in \mathcal{D} . Hence $y \in \tilde{\mathcal{D}}$ (note that $\tilde{\mathcal{D}} = \mathcal{D}$ for self-adjoint operators) so that

$$\begin{aligned} \underbrace{\langle y, (H - \lambda \mathbb{1})x \rangle}_0 &= \langle (H - \lambda \mathbb{1})^* y, x \rangle \\ &= \langle (H - \lambda \mathbb{1})y, x \rangle \\ &= 0 \end{aligned}$$

for all $x \in \mathcal{D}$. Hence $(H - \lambda \mathbb{1})y = 0$ so $y \in \ker(H - \lambda \mathbb{1})$ so that by the fact $H - \lambda \mathbb{1}$ is injective we have that $y = 0$. Thus $(H - \lambda \mathbb{1})(\mathcal{H})$ is dense in \mathcal{H} . Then we can define a continuous extension from $(H - \lambda \mathbb{1})(\mathcal{H})$ to the whole of \mathcal{H} of which agrees with $(H - \lambda \mathbb{1})^{-1}$. Then we have found that $(H - \lambda \mathbb{1})$ has a bijective bounded inverse in $\mathcal{B}(\mathcal{H})$ so that λ is *not* in the spectrum of H .

Case 2. ←→ Let $\lambda \in \mathbb{C}$ be given such that $\forall \varepsilon > 0 \exists v_\varepsilon \in \mathcal{H} \setminus \{0\} : \|H(v_\varepsilon) - \lambda v_\varepsilon\| < \varepsilon \|v_\varepsilon\|$. Then we can find a sequence $(v_n)_{n \in \mathbb{N}} \subseteq \mathcal{H} \setminus \{0\}$ such that $\forall n \in \mathbb{N}$ we have $\|(H - \lambda \mathbb{1})v_n\| < \frac{1}{n} \|v_n\|$. Assume that $\exists (H - \lambda \mathbb{1})^{-1} \in \mathcal{B}(\mathcal{H})$. Then

$$\frac{\|(H - \lambda \mathbb{1})^{-1}(H - \lambda \mathbb{1})v_n\|}{\|(H - \lambda \mathbb{1})v_n\|} > n$$

But $(H - \lambda \mathbb{1})v_n$ are included in the set of the vectors that enter in the computation of $\|(H - \lambda \mathbb{1})^{-1}\|_{\text{op}}$ so that $(H - \lambda \mathbb{1})^{-1}$ is *not* bounded. Hence we have a contradiction. □

4.2.14. Exercise. Let $\mathcal{H} := L^2((0, \infty), \mathbb{C})$ and $H : \mathcal{H} \rightarrow \mathcal{H}$ be given by

$$(x \mapsto f(x)) \stackrel{H}{\mapsto} (x \mapsto xf(x))$$

Then show that:

- (1) H is not bounded.
- (2) H is self-adjoint.
- (3) $\sigma(H) = (0, \infty)$.
- (4) $\mathcal{E}(H) = \emptyset$.
- (5) Find a sequence of approximate eigenstates for a given approximate eigenvalue $x \in (0, \infty)$.

4.2.15. Claim. $\sigma(H) \in \text{Closed}(\mathbb{C})$.

| *Proof.* Theorem VIII.2 in [38]. □

4.2.16. Definition. Define the point spectrum of H as

$$\sigma_p(H) := \text{cl}_C(\mathcal{E}(H))$$

where cl_C is the closure in C .

4.2.17. Remark. Using 4.2.15 we have that $\sigma_p(H) \subseteq \sigma(H)$. It should also be noted that some authors (including [40]) use the phrase “point spectrum” to refer to what we defined as $\mathcal{E}(H)$ and the term “pure point spectrum” to refer to what we defined as $\sigma_p(H)$.

4.2.18. Definition. A projection-valued-measure (also called resolution of the identity) is a map from measurable subsets of \mathbb{R} to orthogonal projections on \mathcal{H}

$$\mathbb{R} \supseteq \underbrace{M}_{\text{measurable}} \mapsto P(M) \in \mathcal{B}(\mathcal{H})$$

such that

- (1) As already stated, each $P(M)$ is an orthogonal projection: $P(M)^2 = P(M) = P(M)^*$.
- (2) $P(\emptyset) = 0$ and $P(\mathbb{R}) = 1_{\mathcal{H}}$.
- (3) For pairwise disjoint measurable sets $(M_i)_{i \in \mathbb{N}}$ we have

$$P\left(\bigcup_{i \in \mathbb{N}} M_i\right) = \lim_{n \rightarrow \infty} \sum_{i=1}^n P(M_i)$$

where the limit is in the topology induced by $\|\cdot\|_{op}$.

- (4) For intersections we have

$$P(M_1 \cap M_2) = P(M_1)P(M_2)$$

4.2.19. Claim. (*The Spectral Theorem*) For self-adjoint operators H there is a one-to-one correspondence with projection-valued-measures (as defined above) such that

$$H = \int_{\mathbb{R}} \lambda dP_{\lambda}$$

where $P_{\lambda} := P((-\infty, \lambda])$ and furthermore

$$\sigma(H) = \text{supp}(P)$$

| *Proof.* See [38] theorem VIII.6. □

4.2.20. Remark. The physical interpretation is that P_M projects onto all states (vectors in \mathcal{H}) which have energy in the set $M \subseteq \mathbb{R}$. Also note that

$$\text{supp}(P) = \mathbb{R} \setminus \left(\bigcup_{M \in \{\mathcal{M}' \in \text{Open}(\mathbb{R}) \mid P(\mathcal{M}') = 0\}} M \right)$$

4.2.21. Definition. Define the point of P (the one which is associated with a self-adjoint H via the spectral theorem) as

$$P_p(M) := \sum_{\lambda \in M} P(\{\lambda\})$$

Note that $P(\{\lambda\})$ is non-zero only if $\lambda \in \mathcal{E}(H)$. If \mathcal{H} is separable, $\mathcal{E}(H)$ is countable and thus the sum is a countable sum, which is well-defined.

4.2.22. Corollary. Then we have

$$\begin{aligned} \sigma_p(H) &\equiv \text{cl}_C(\mathcal{E}(H)) \\ &= \text{supp}(P_p) \end{aligned}$$

4.2.23. Remark. $\sigma_p(H)$ may not consist solely of isolated points. In particular, the eigenvalues may be dense on an interval of \mathbb{R} and then the point spectrum will be that interval.

4.2.24. Definition. We define

$$P_c(M) := P(M) - P_p(M)$$

thus $P_c(\{\lambda\}) = 0$ for any $\lambda \in \mathbb{R}$. Then the continuous spectrum is defined as

$$\sigma_c(H) := \text{supp}(P_c)$$

4.2.25. Remark. We obtain a spectral decomposition of a self-adjoint operator H as:

$$\sigma(H) = \sigma_p(H) \cup \sigma_c(H)$$

which is possibly *not* disjoint.

4.2.26. Claim. (*RAGE theorem*) $P_p(M)$ projects onto localized states.

Proof. If we pick an eigenvalue in the spectrum, then there is an eigenvector corresponding to it:

$$Hv = \lambda v$$

and so with the unitary evolution we have

$$e^{-iHt}v = e^{-i\lambda t}v$$

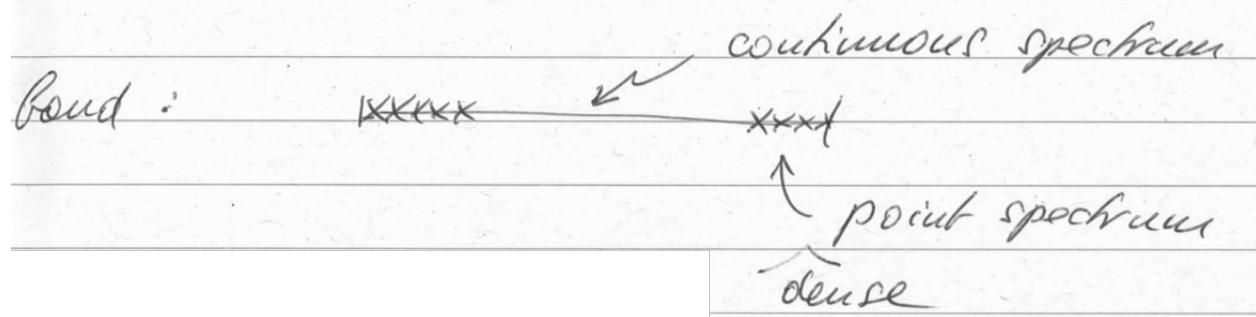
so that as time goes by, v obtains a phase, but otherwise does not change. In particular, $\|v\|^2$ remains unchanged. For the same statement on $P_p(\mathbb{R})$ see theorem 5.7 in [45], which roughly says that a vector ψ is in the point eigenspace iff

$$\lim_{R \rightarrow \infty} \sup \left(\left\{ \|\chi_{\|x\| > R} e^{-iHt}\psi\| \mid t \in \mathbb{R} \right\} \right) = 0$$

where χ is the characteristic function. \square

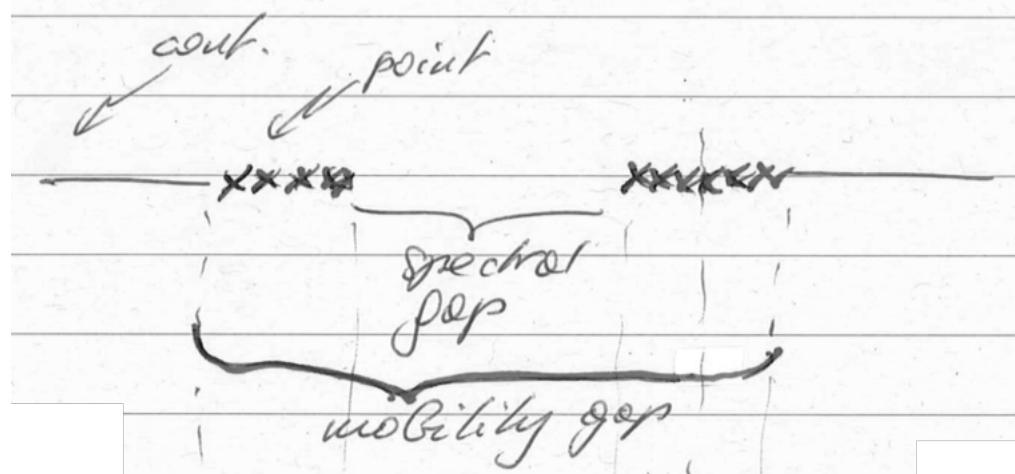
4.3. Anderson Localization. In 1958 Anderson published in [3] an article (for which he won the Nobel prize in 1977) which describes the following

4.3.1. Fact. *If H is the Hamiltonian of a particle in a disordered potential then H has a dense point spectrum (at least) at band edges.*



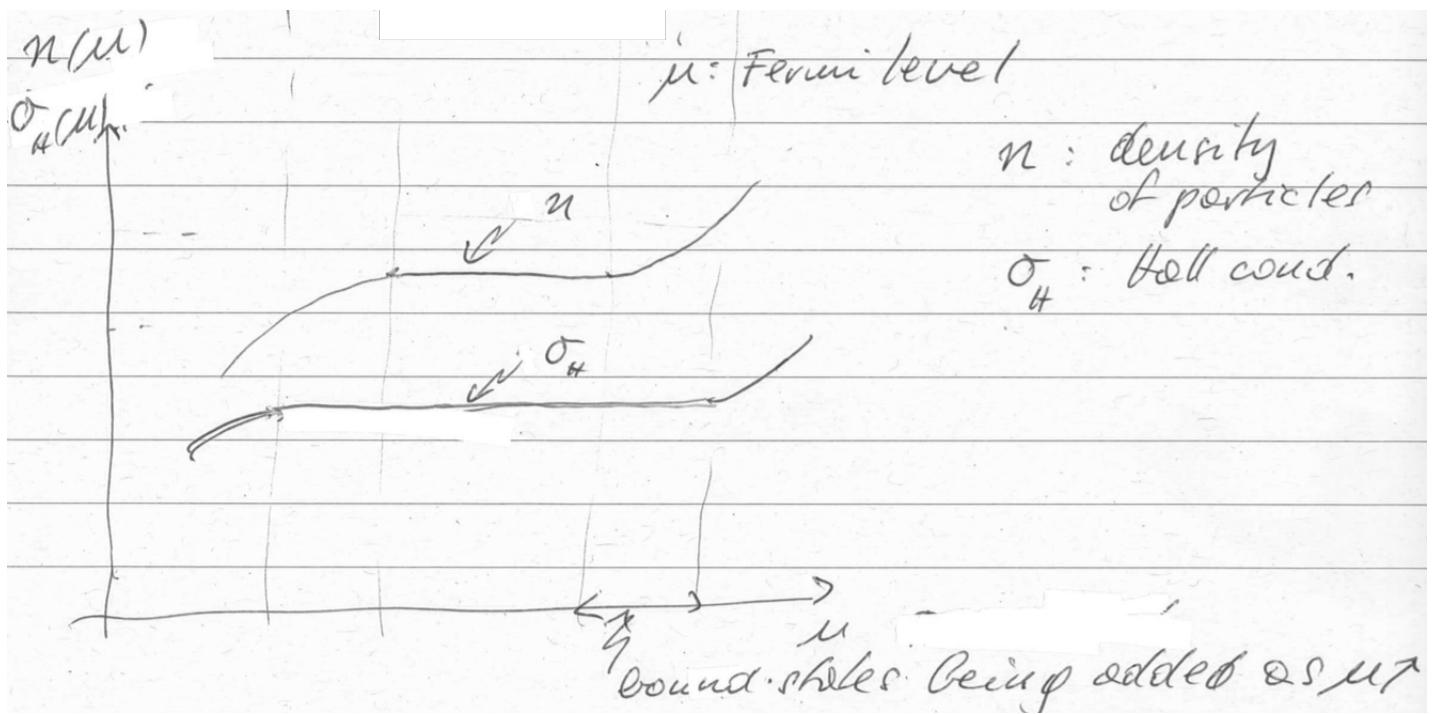
Note that Anderson also showed that \nexists tunneling between the localized states so that they *really* do not contribute to the conduction. For a more mathematically rigorous approach to this problem see [16].

4.4. Application to IQHE. In quantum Hall effect, using 4.3.1, the spectrum of the Landau levels, which is smeared from isolated dots into bands by disorder, is now divided into point and continuous parts:

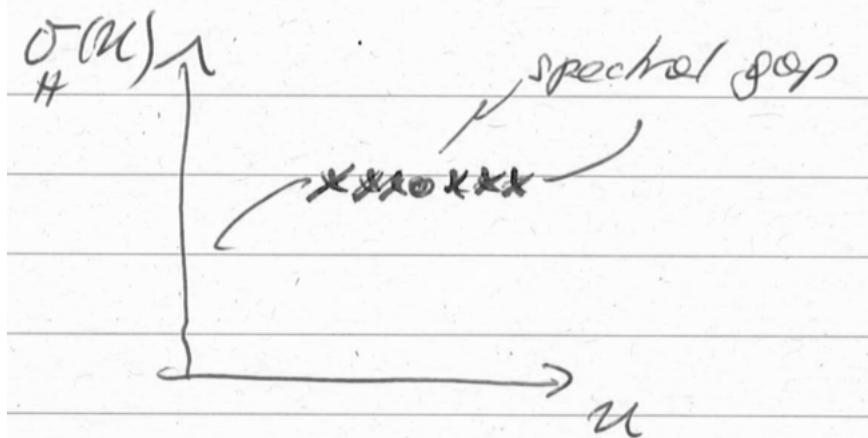


and a new concept is defined: *the mobility gap*, which is the interval of \mathbb{R} including the spectral gap *and* the point spectrum. The crucial fact about the point spectrum is given in 4.2.26, and using this we can finally explain the existence of plateaus of σ_H :

Suppose we start with the Fermi energy μ somewhere in the middle of the gap and start raising it. Then the electron density n does not change: there are no states to occupy inside the spectral gap. However, when μ enters into the point spectrum, there *are* states to occupy. However, these states cannot contribute to conduction because the pure spectrum corresponds to localized states (this is the content of 4.2.26). As a result, we see that n can be changed while keeping σ_H constant: this is exactly the existence of plateaus. Once μ is changed sufficiently that it goes out of the mobility gap, states that *can* contribute to conduction get occupied and σ_H changes.



Observe how the spectral gap only contributes to one point on the plateaus:



So far we have explained the existence of plateaus, but not why they occur at integer values.

5. MORE HEURISTIC EXPLANATIONS OF THE QUANTIZATION OF σ_H

5.1. The IQHE as a Charge Pump.

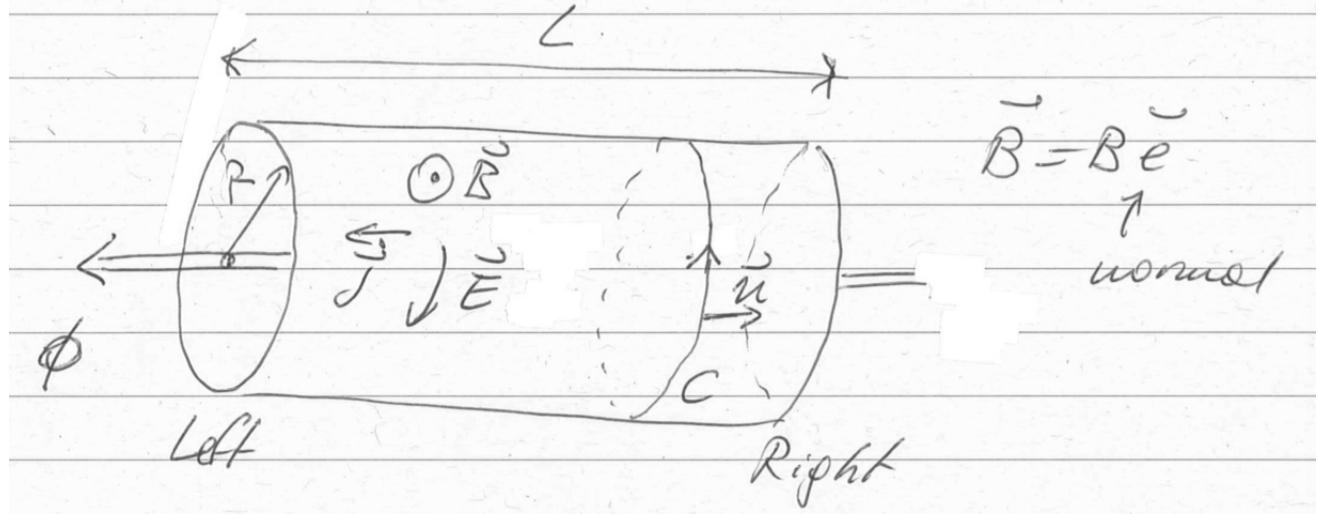
5.1.1. *Remark.* (Streda's Formula 1982) Before we present Laughlin's argument, we should mention that there is a similar idea due to Streda ([44]). Consider a two-dimensional planar system. Then a change in the magnetic field $\Delta \mathbf{B}(\mathbf{x})$ which is out of plane, induces a change $\Delta\rho(\mathbf{x})$ in charge density:

$$\Delta\rho(\mathbf{x}) = -\sigma_H \frac{\Delta\mathbf{B}(\mathbf{x})}{c}$$

Indeed, using the fact that $\mathbf{j} = -\sigma_H \hat{\mathbf{B}} \times \mathbf{E}$, the continuity equation and Faraday's law we have

$$\begin{aligned}\partial_t \rho &= -\operatorname{div}(\mathbf{j}) \\ &= -\operatorname{div}(-\sigma_H \hat{\mathbf{B}} \times \mathbf{E}) \\ &= -\sum_i \partial_i (-\sigma_H \hat{\mathbf{B}} \times \mathbf{E})_i \\ &= \sigma_H \sum_i \partial_i \sum_{j,k} \epsilon_{ijk} (\hat{\mathbf{B}})_j (\mathbf{E})_k \\ &= \sigma_H \hat{\mathbf{B}} \cdot \operatorname{curl}(\mathbf{E}) \\ &= -\sigma_H \hat{\mathbf{B}} \cdot \frac{1}{c} \partial_t \mathbf{B}\end{aligned}$$

Alternatively, one could imagine rolling the two dimensional plane of conducting material on itself into a cylinder. This was the idea Laughlin presented in [29]:



In addition, imagine applying magnetic flux ϕ in the middle of the cylinder (along its axis) which may have a certain time dependence. Note that this is in addition, and independently of the already present background homogeneous magnetic field \mathbf{B} .

5.1.2. **Claim.** As ϕ increases from ϕ to $\phi + \Delta\phi$, a charge $\Delta Q = -\sigma_H \frac{\Delta\phi}{c}$ will be transported (or "pumped") from the left edge to the right edge.

Proof. We give macroscopic considerations:

We use cylindrical coordinates so that $\hat{\mathbf{e}}_z$ is the axis of the cylinder, $\hat{\mathbf{e}}_\rho$ is the radial direction and $\hat{\mathbf{e}}_\varphi$ is the azimuthal direction.

Make ϕ time dependent. Then by Faraday's law we know there will be an electric field \mathbf{E} induced along $\hat{\mathbf{e}}_\varphi$ such that

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{1}{c} \frac{d\phi}{dt}$$

where C is a ring around the cylinder as denoted in the picture above. As a result there will be a Hall current

$$\mathbf{j} = -\sigma_H \hat{\mathbf{e}}_\rho \times \mathbf{E}$$

(assuming there is no dissipative conductivity, that is, $\sigma_D = 0$). Then the current across the fiducial line C is

$$\begin{aligned}
 \frac{dQ}{dt} &= I \\
 &= \oint_C \mathbf{j} \cdot \hat{\mathbf{e}}_z dl \\
 &= \oint_C (-\sigma_H \hat{\mathbf{e}}_\rho \times \mathbf{E}) \cdot \hat{\mathbf{e}}_z dl \\
 &= -\sigma_H \oint_C \underbrace{(\hat{\mathbf{e}}_\rho \times \hat{\mathbf{e}}_z) \cdot \mathbf{E} dl}_{-\hat{\mathbf{e}}_\varphi} \\
 &= \sigma_H \oint_C \mathbf{E} \cdot dl \\
 &= \sigma_H \left(-\frac{1}{c} \frac{d\phi}{dt} \right) \\
 &= -\sigma_H \frac{1}{c} \frac{d\phi}{dt}
 \end{aligned}$$

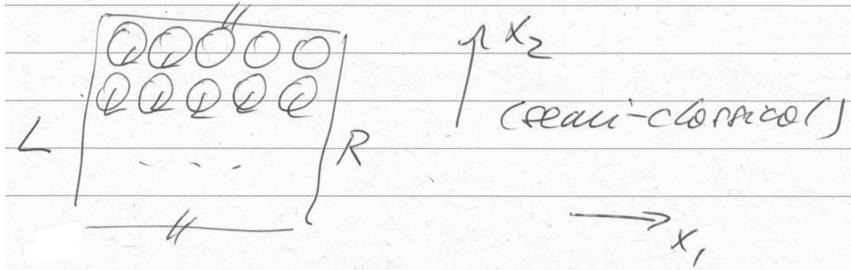
□

5.1.3. Corollary. If we choose to change ϕ by $\Delta\phi = \frac{hc}{q}$, which is called "the flux quantum", then we will have

$$\frac{\Delta Q}{q} = -\sigma_H \frac{h}{q^2} \quad (7)$$

5.1.4. Claim. There is a quantization of σ_H iff there is a quantization of charge transport.

Proof. We employ microscopic considerations:



We introduce gauge potentials in the Landau gauge (different from the symmetric gauge of (2)). We have two independent magnetic fields, \mathbf{B} (called the background field, denoted by B) and the one corresponding to ϕ (called the flux field, denoted by F).

$$\mathbf{A}_B := \begin{bmatrix} 0 \\ Bx_1 \end{bmatrix}$$

so that

$$\begin{aligned}
 \text{curl}(\mathbf{A}_B) &= \partial_{x_1} (\mathbf{A}_B)_2 - \partial_{x_2} (\mathbf{A}_B)_1 \\
 &= B
 \end{aligned}$$

and

$$\mathbf{A}_F := \begin{bmatrix} 0 \\ \frac{\phi}{2\pi R} \end{bmatrix}$$

so that

$$\text{curl}(\mathbf{A}_F) = 0$$

(as it should since there is no magnetic flux coming from ϕ on the cylinder) yet

$$\begin{aligned}
 \phi &\equiv \int_{\text{surface enclosed by } C} \mathbf{B}_F \cdot d\mathbf{s} \\
 &\stackrel{\text{Stokes}}{=} \oint_C \mathbf{A}_F \cdot dl \\
 &= \frac{\phi}{2\pi R} 2\pi R
 \end{aligned}$$

and

$$\mathbf{A} = \mathbf{A}_B + \mathbf{A}_F$$

The Hamiltonian is given by

$$\begin{aligned} H &= \frac{1}{2m} \left(\mathbf{p} - \frac{q}{c} \mathbf{A} \right)^2 \\ &= \frac{1}{2m} \left(\mathbf{p} - \frac{q}{c} \left(Bx_1 \hat{\mathbf{e}}_2 + \frac{\phi}{2\pi R} \hat{\mathbf{e}}_2 \right) \right)^2 \\ &= \frac{1}{2m} \left\{ p_1^2 + \left[p_2 - \frac{q}{c} \left(Bx_1 + \frac{\phi}{2\pi R} \right) \right]^2 \right\} \end{aligned}$$

with the first line as in (1), and the difference due to the difference in gauge and the addition of a flux gauge potential.

Note that $[H, p_2] = 0$ because the only thing that could prevent that is the existence of a term containing x_2 in H , but there is no such term. As a result, p_2 is conserved and also quantized due to the periodic boundary conditions:

$$\begin{aligned} \exp(ip_2 2\pi R) &= \exp(ip_2 \cdot 0) \\ &\Downarrow \\ p_2 2\pi R &= 2\pi \hbar n \end{aligned}$$

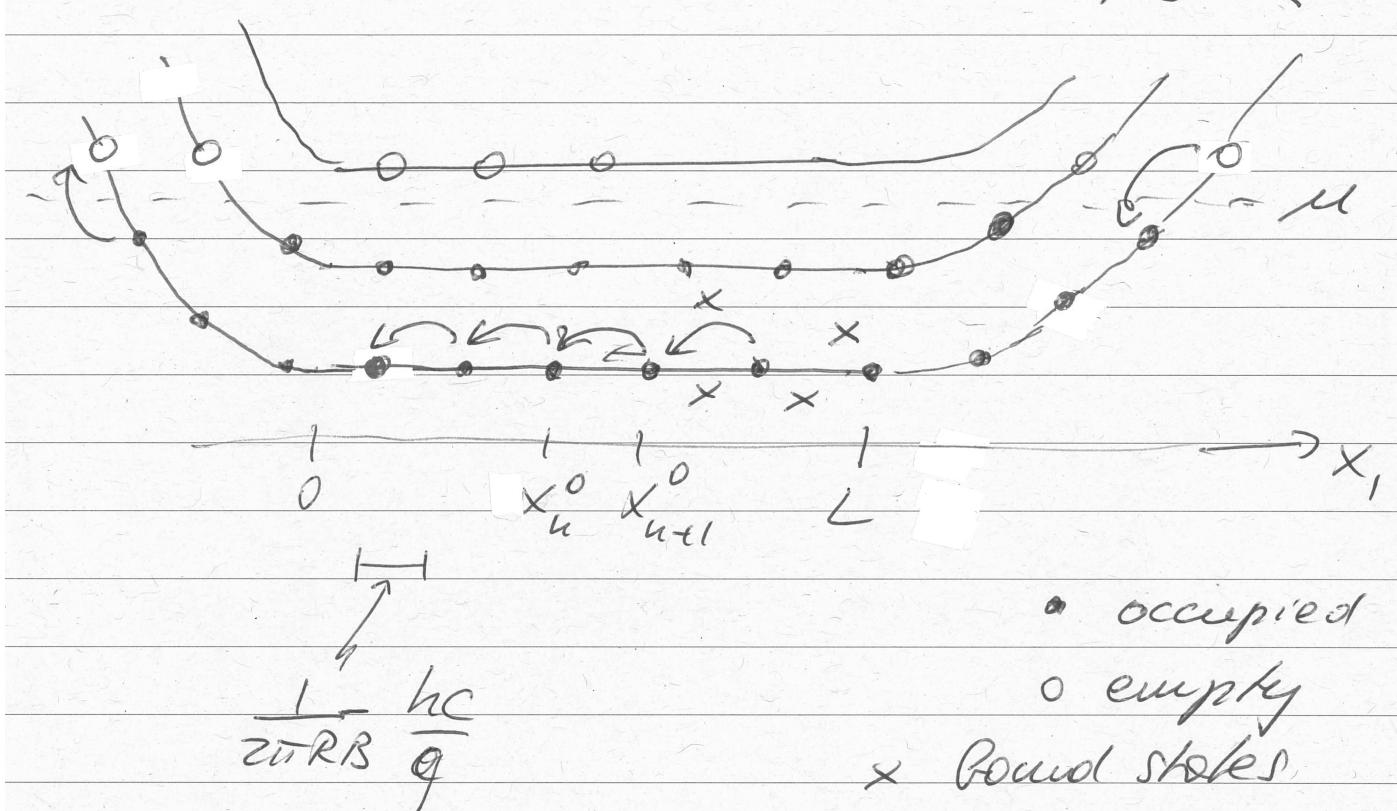
for some $n \in \mathbb{Z}$. So we may replace p_2 with $\frac{\hbar}{R}n$ in the Hamiltonian:

$$\begin{aligned} H &= \frac{1}{2m} \left\{ p_1^2 + \left[\frac{\hbar}{2\pi R} n - \frac{q}{c} \left(Bx_1 + \frac{\phi}{2\pi R} \right) \right]^2 \right\} \\ &= \frac{1}{2m} \left\{ p_1^2 + \left(\frac{qB}{c} \right)^2 \left[\frac{c}{qB} \frac{\hbar}{2\pi R} n - \left(x_1 + \frac{1}{B} \frac{\phi}{2\pi R} \right) \right]^2 \right\} \\ &= \frac{1}{2m} \left\{ p_1^2 + \left(\frac{qB}{c} \right)^2 \left[x_1 - \underbrace{\frac{1}{2\pi RB} \left(n \frac{\hbar c}{q} - \phi \right)}_{x_1^0(n)} \right]^2 \right\} \\ &= \frac{1}{2m} \left[p_1^2 + \left(\frac{qB}{c} \right)^2 (x_1 - x_1^0(n))^2 \right] \end{aligned} \tag{8}$$

so that we get a simple harmonic oscillator with its origin position shifted by $x_1^0(n)$ for each choice of $n \in \mathbb{Z}$.

Now we employ the fact that x_1 is not unconstrained but rather must obey $x_1 \in [0, L]$ (as in the picture above) so that we don't have infinitely many choices for $x_1^0(n)$. At any rate each such choice gives rise to the familiar discrete spectrum of the harmonic oscillator and near the edges the levels must bend to reflect the fact that the material ends. So we get the following schematic landscape of energy versus the x_1 coordinate:

Landau levels



We use the adiabatic principle to assert that as we change ϕ slowly and continuously, a time evolved eigenstate of the instantaneous Hamiltonian remains an eigenstate of the later instantaneous Hamiltonian. Occupation is inherited. Thus, as ϕ increases,

$$x_1^0(n) \equiv \frac{1}{2\pi RB} \left(n \frac{hc}{q} - \phi \right)$$

decreases, that is, the centers move leftwards. Note that a change of $\Delta\phi = \frac{hc}{q}$ for a particular n results in

$$\begin{aligned} x_1^0(n)_{\text{new}} &= \frac{1}{2\pi RB} \left(n \frac{hc}{q} - \left(\phi + \frac{hc}{q} \right) \right) \\ &= \frac{1}{2\pi RB} \left((n-1) \frac{hc}{q} - \phi \right) \\ &= x_1^0(n-1)_{\text{old}} \end{aligned}$$

so that to change the flux by $\Delta\phi = \frac{hc}{q}$ is equivalent to merely shifting or relabeling the whole picture $n \mapsto n-1$: Thus only what happens at the edges $x_1 = 0$ and $x_1 = L$ will matter, using the fact that occupation is inherited. Thus we see that at the right edge some of the empty states that used to be above the Fermi line will now move below it: charge is “lost” on the right edge. On the left edge, some of the occupied states that used to be below the Fermi line will now move above it: charge is “gained” on the left edge. To compute the total transfer of charge from left to right, we merely have to count the number of occupied Landau levels (as the unoccupied ones don’t participate in this analysis):

$$\Delta Q = -q \times (\# \text{ of occupied Landau levels})$$

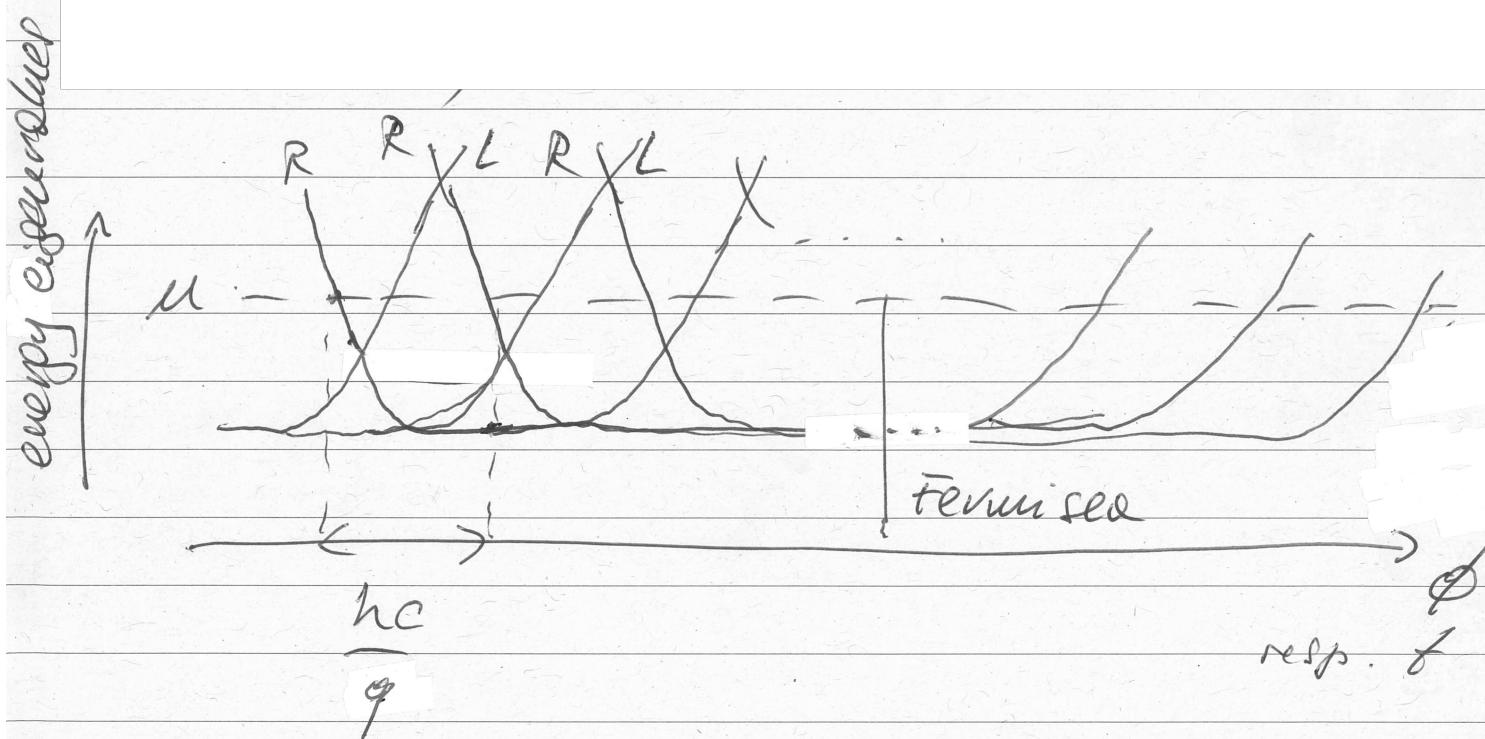
which is a quantized number. Thus using (7) we have:

$$\boxed{\sigma_H = (\# \text{ of occupied Landau levels}) \times \frac{h}{q^2}} \quad (9)$$

Note that the number of occupied levels is a deterministic integer, not a random one. □

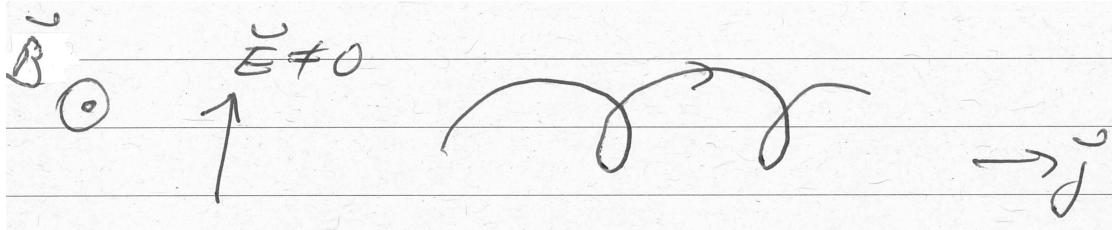
5.1.5. *Remark.* This result is robust even in the presence of disorder. To see this, note that $\text{curl } (\mathbf{A}_F) = 0$ (indeed, there is no field on the cylinder), but that does *not* mean that we may write $\mathbf{A}_F = \text{grad } (\chi)$ for some global scalar function χ . This is not possible due to the fact that the domain is not simply connected. However, locally (there is some open subset such that) we *may* write $\mathbf{A}_F = \text{grad } (\chi)$. Disorder adds bound states to the system, states whose wave-functions have compact support (in fact support that does not extend as a loop around the cylinder). Thus, for such states, we may write $\mathbf{A}_F = \text{grad } (\chi)$ within the support of their wave functions. By doing so, we may perform a gauge transformation in order to entirely eliminate \mathbf{A}_F . As a result, bound states are not affected by changes in ϕ and are entirely exempt from the analysis of 5.1.4. Note that the harmonic oscillator states are bound in x_1 but extend over the whole of x_2 , so that in the support of the wave function of such states as in 5.1.4 we may *not* write $\mathbf{A}_F = \text{grad } (\chi)$.

5.1.6. Remark. One could also see the quantization as a transport of spectral flow:



In this picture we consider only one Landau level (for simplicity) and ignore disorder. Then within one width $\Delta\phi = \frac{hc}{q}$, one state falls into the Fermi sea of R and one emerges out of the Fermi sea of L. Note that this argument does not work when disorder is included unless the system is infinite.

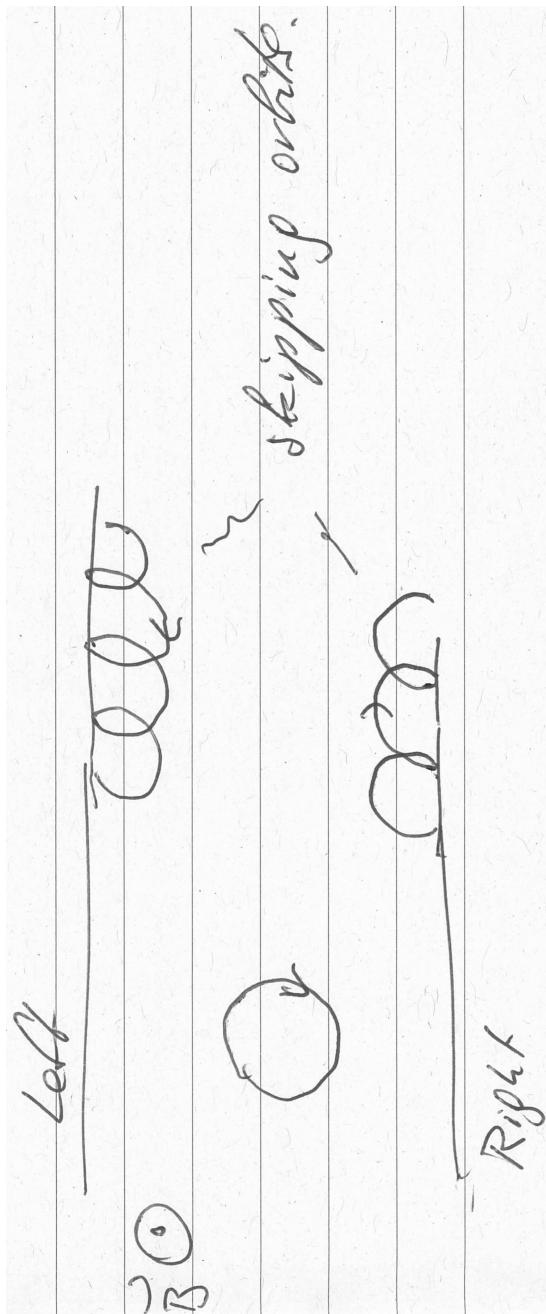
5.2. IQHE as an Edge Effect. First consider the classical picture of 1. There we considered a system which extends infinitely in both axes, and concluded that in the presence of perpendicular magnetic and electric fields there would be drift cyclotron motion:



$$\mathbf{j} = -\sigma_B \hat{\mathbf{B}} \times \mathbf{E}$$

We call this a drift motion a “bulk” effect because in this analysis we completely ignore the finiteness (the edges) of the system. In order to “obtain” it we didn’t need any edges.

Now instead we follow an idea due to Halperin ([22]). Consider a sample of finite size which *does* have edges along the ϵ_2 axis: left and right edges:

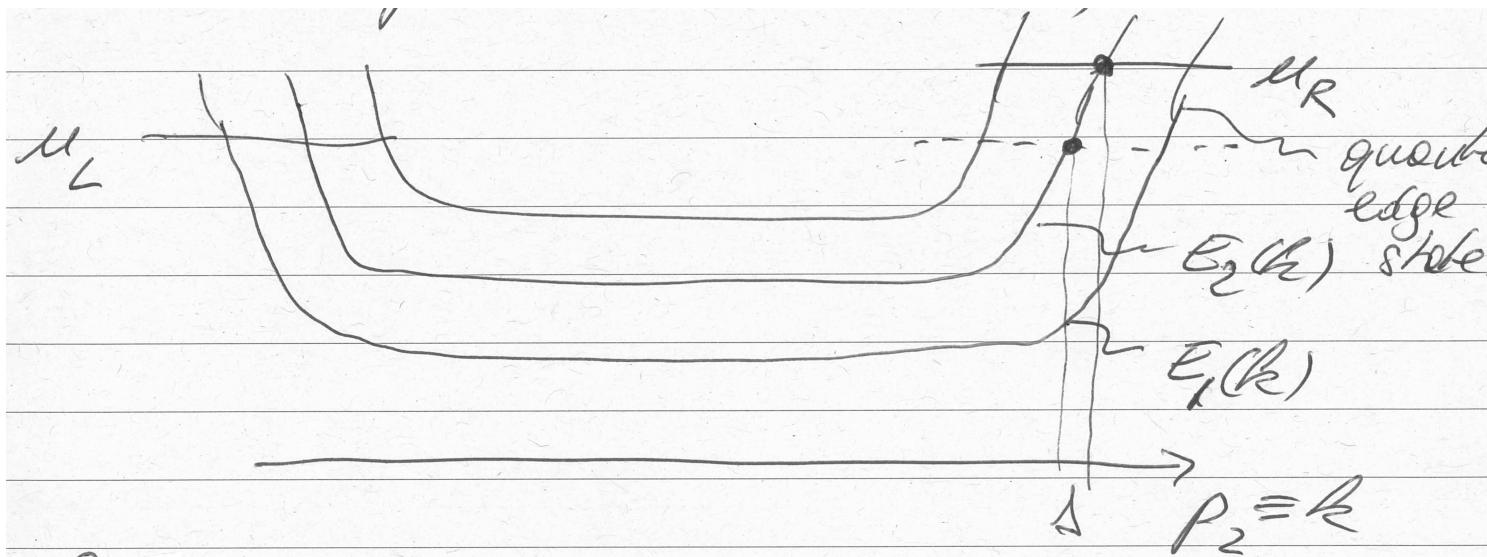


In the absence of an electric field the electrons follow circular motion in the bulk, just as before, and this leads to no overall drift motion. However, at the edges, the electrons "bump" into the end of the sample and as a result are driven along the edge, in opposite directions in each edge as in the picture. Now as in 1 we have a potential drop from the right edge to the left edge. But now interpret this not as generating a field E but rather as the difference in the Fermi level of the skipping orbits. We obtain

$$qV = \mu_R - \mu_L$$

Hence the current is carried by the skipping orbits and results from their different occupation.

Now we analyze the system quantum mechanically. Here the Hamiltonian is the same as 8, except that $\phi = 0$ now and p_2 is not quantized because x_2 doesn't have periodic boundary conditions (but p_2 is still conserved). As a result, now $p_2 = \hbar k$ where $k \in \mathbb{R}$. Thus we get the following diagram of Landau levels versus p_2 :



Now observe that the group velocity of each Landau level is

$$\frac{d\omega_n}{dk} = \frac{1}{\hbar} \frac{dE_n}{dk}$$

The current from each level n is canceled out between left and right edges and the only part that contributes is the difference that results from $\mu_L \neq \mu_R$. So the current is given by:

$$\begin{aligned} I_n &= q \int_{\Delta} \underbrace{\frac{1}{2\pi}}_{\text{density of states in } k\text{-space}} \frac{d\omega_n}{dk} dk \\ &= q \int_{\Delta} \frac{1}{2\pi} \frac{1}{\hbar} \frac{dE_n}{dk} dk \\ &= \frac{q}{\hbar} \int_{\mu_L}^{\mu_R} dE_n \\ &= \frac{q}{\hbar} (\mu_R - \mu_L) \\ &= \frac{q^2}{\hbar} V \end{aligned}$$

so that the total current is given by

$$I = (\# \text{ of occupied Landau levels below } \mu_L) \times \frac{q^2}{\hbar} V$$

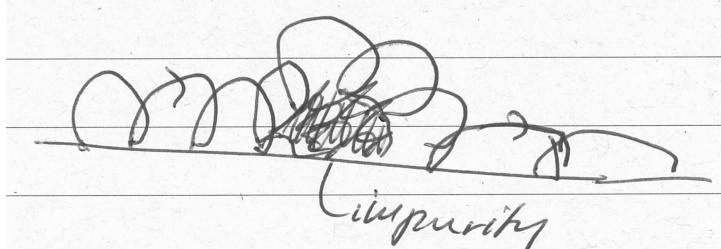
Since $I = \sigma V$, we find that in the edge-picture, σ_E is equal to

$$\sigma_E = (\# \text{ of occupied Landau levels below } \mu_L) \times \frac{q^2}{\hbar}$$

The number of occupied Landau levels below μ_L is called "edge channels". We find that the quantization of σ_H is equivalent to an integer number of edge channels.

5.2.1. Remark. We find that the Hall conductivity is $\sigma_E = \sigma_B$ between the Halperin picture and the Laughlin picture because the number of edge channels is the number of filled Landau levels.

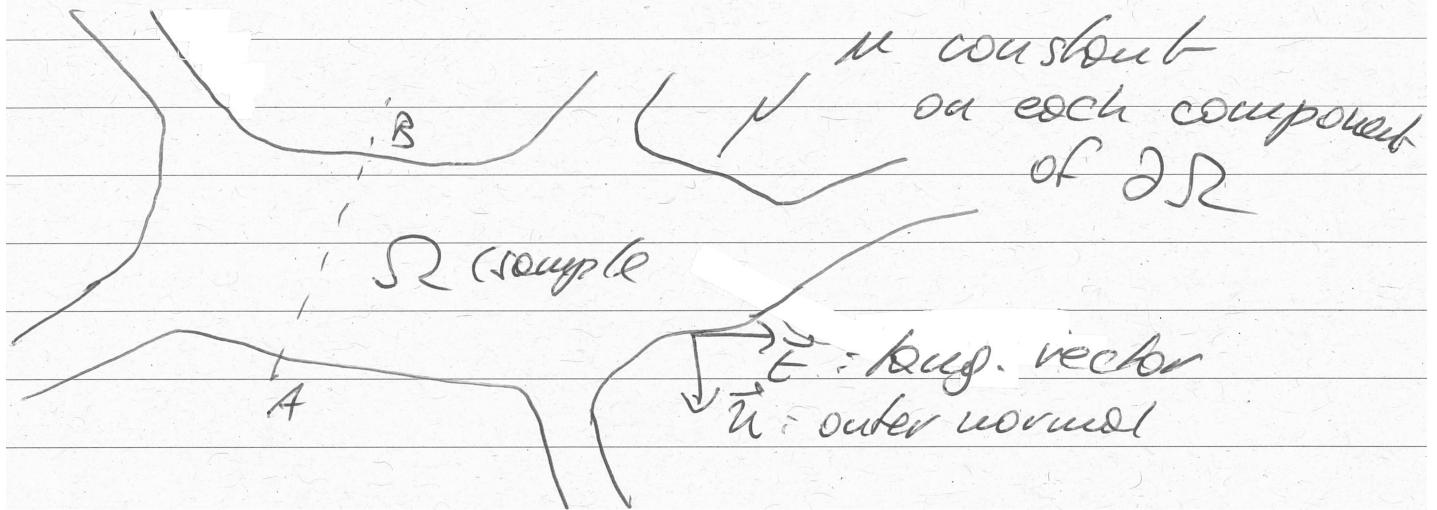
5.2.2. Remark. Disorder does *not* affect this result. If there is an impurity on one of the edges as the following picture depicts:



then there is no backscattering possible: orbits coming from very far away below cannot be trapped by the impurity and so must travel around it.

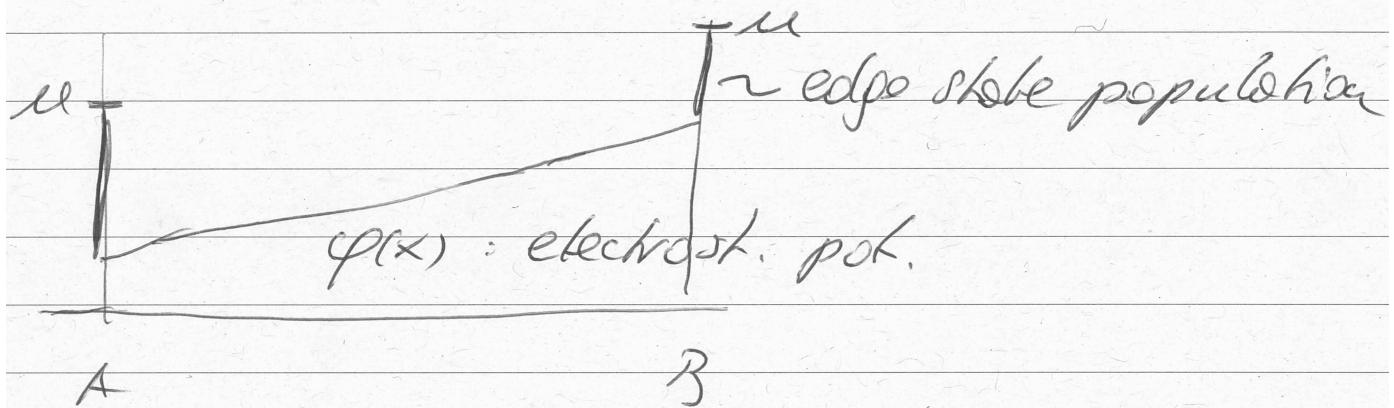
5.2.3. Remark. In a real world samples usually both bulk *and* edge currents are observed.

5.3. Bulk and Edge Equality from a Phenomenological Perspective. In this section we give phenomenological arguments for the equality of σ_B and σ_E . Note that this argument is not rigorous because the whole premise of Ohm's law is classical. Consider a sample of various connected-edge-components:



We denote the sample as a subset Ω of \mathbb{R}^2 and its edge as $\partial\Omega$. We assume that μ is constant on each connected component of $\partial\Omega$. We denote by t the tangential vector along $\partial\Omega$ and by n the out normal vector to $\partial\Omega$.

Consider a cross section along the sample Ω between the points A and B:



If we define a 2×2 ninety-degree-rotation matrix ε by

$$\varepsilon := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

then the bulk current density in Ω is given by

$$\mathbf{j}_B := -\sigma_B \varepsilon \mathbf{E} \quad (10)$$

whereas the edge current at $\partial\Omega$ is given by

$$I_E = \sigma_E (\mu - \varphi(\mathbf{x})) + \text{const}$$

where φ is the potential at $\partial\Omega$. Note that $\mathbf{E} = -\text{grad}(\varphi)$ and $\mathbf{t} = \varepsilon \mathbf{n}$. Let $\chi_S : \mathbb{R}^2 \rightarrow \{0, 1\}$ be the characteristic function of a set $S \subseteq \mathbb{R}^2$:

$$\chi_S(\mathbf{x}) := \begin{cases} 0 & \mathbf{x} \notin S \\ 1 & \mathbf{x} \in S \end{cases}$$

then

$$\text{grad}(\chi_\Omega) = -\mathbf{n} \delta_{\partial\Omega}$$

where $\delta_{\partial\Omega}$ is the delta-distribution supported on $\partial\Omega$.

Now actually we should write (10) with χ_Ω so that outside of Ω the current density would be zero. Then we get:

$$\begin{aligned} \mathbf{j}_B &= -\chi_\Omega \sigma_B \varepsilon \mathbf{E} \\ &= \chi_\Omega \sigma_B \varepsilon \text{grad}(\varphi) \end{aligned}$$

whereas the edge current density is given by

$$\begin{aligned} \mathbf{j}_E &= I_E \delta_{\partial\Omega} \mathbf{t} \\ &= \sigma_E (\mu - \varphi(\mathbf{x})) \delta_{\partial\Omega} \varepsilon \mathbf{n} \\ &= \sigma_E (\mu - \varphi(\mathbf{x})) \varepsilon \delta_{\partial\Omega} \mathbf{n} \\ &= -\sigma_E (\mu - \varphi(\mathbf{x})) \varepsilon \text{grad}(\chi_\Omega) \end{aligned}$$

Next, we know that in the stationary regime, $\operatorname{div}(\mathbf{j}) = 0$ and $\mathbf{j} \equiv \mathbf{j}_B + \mathbf{j}_E$ so that

$$\begin{aligned}
 \operatorname{div}(\mathbf{j}_B) &= -\operatorname{div}(\mathbf{j}_E) \\
 &\uparrow \\
 \operatorname{div}(\chi_\Omega \sigma_B \varepsilon \operatorname{grad}(\varphi)) &= -\operatorname{div}(-\sigma_E (\mu - \varphi(\mathbf{x})) \varepsilon \operatorname{grad}(\chi_\Omega)) \\
 &\uparrow \\
 \sigma_B \operatorname{grad}(\chi_\Omega) \cdot \varepsilon \operatorname{grad}(\varphi) + \sigma_B \chi_B \underbrace{\operatorname{div}(\varepsilon \operatorname{grad}(\varphi))}_{0 \text{ in } 2D} &= \sigma_E (\mu - \varphi(\mathbf{x})) \underbrace{\operatorname{div}(\varepsilon \operatorname{grad}(\chi_\Omega))}_0 + \sigma_E \underbrace{\operatorname{grad}((\mu - \varphi(\mathbf{x}))) \cdot \varepsilon \operatorname{grad}(\chi_\Omega)}_{-\operatorname{grad}(\varphi)} \\
 &\uparrow \\
 \sigma_B \operatorname{grad}(\chi_\Omega) \cdot \varepsilon \operatorname{grad}(\varphi) &= \sigma_E (-\operatorname{grad}(\varphi) \cdot \varepsilon \operatorname{grad}(\chi_\Omega)) \\
 &\uparrow \quad (\varepsilon^T = -\varepsilon) \\
 \sigma_B \operatorname{grad}(\chi_\Omega) \cdot \varepsilon \operatorname{grad}(\varphi) &= \sigma_E \varepsilon \operatorname{grad}(\varphi) \cdot \operatorname{grad}(\chi_\Omega) \\
 &\downarrow \\
 \sigma_B &= \sigma_E
 \end{aligned}$$

6. THE KUBO FORMULA

Above we have computed the Hall conductivity for the Landau Hamiltonian. In a quest to generalize the computation to arbitrary Hamiltonians we will use perturbation theory, where the perturbation is an electric field.

The Kubo formula is a particular formulation of perturbation theory in quantum mechanics³ which turned out to be extremely useful in explaining the quantum Hall effect as first shown in [46].

6.1. General Formulation of the Kubo Formula. Before presenting the Kubo formula in the context of the quantum Hall effect, we give a more general presentation which applies to general scenarios. The source for this material is at [28].

Assume there is a quantum system with Hamiltonian H and equilibrium (possibly mixed) state $\rho_0 : \mathcal{H} \rightarrow \mathcal{H}$. Note that for us a state now is given by a density matrix which is a weighted sum of rank-one projectors $\rho_0 = \sum_i w_i |\psi_i\rangle\langle\psi_i|$, where the weights w_i sum to one (see [1] for details). The assumption of equilibrium implies:

$$[H, \rho_0] = 0$$

Now apply a time-dependent self-adjoint perturbation of the form

$$-f(t)\lambda A$$

where $\lambda > 0$ is fixed, A is a self-adjoint operator and $f : \mathbb{R} \rightarrow \mathbb{R}$ is some function such that

$$\lim_{t \rightarrow -\infty} f(t) = 0$$

and

$$f(0) \stackrel{!}{=} 1$$

For concreteness, we will take $f(t) = \exp(\varepsilon t)$ where $\varepsilon > 0$ is fixed.

The response of the system is measured by probing it with a self-adjoint observable B at time 0 and expressing this to first order in λ .

The expectation value of an observable in state ρ is given by

$$\langle O \rangle_\rho \equiv \text{Tr}[\rho O]$$

So the expectation of B at time zero is:

$$\langle B \rangle_{\rho(0)} = \langle B \rangle_{\rho_0} + \lambda \chi_{BA} + \mathcal{O}(\lambda^2)$$

where the state $\rho(t)$ is the perturbed state. We would like to compute $\lim_{\varepsilon \rightarrow 0} \chi_{BA}$, the limit of no time-dependence.

Note that there is a certain ambiguity in the procedure, in the sense that we arbitrarily decided that at time $-\infty$ there is no perturbation and then it would be turned on adiabatically, and only at the end of the calculation we change ε so that it is as if the perturbation was always present. We would indeed get a different result if we were to take the other limit, where we start with a system which at time 0 is perturbed and at time $+\infty$ slowly goes on to be unperturbed. Our choice of boundary condition thus corresponds to specifying the causality.

Also note that some criticism has been raised about taking the limit $\varepsilon \rightarrow 0$ in the end, which is unjustified. More rigorous attempts have been presented in [15].

6.1.1. Claim. (Kubo formula)

$$\chi_{BA} = i \int_0^\infty \text{Tr} [\exp(itH) B \exp(-itH) [A, \rho_0]] dt$$

Proof. We follow [27]. The state of the system at time t is denoted by $\rho(t)$ and it obeys the Liouville equation

$$i\dot{\rho}(t) = [H - f(t)\lambda A, \rho(t)]$$

with initial condition $\rho(-\infty) = \rho_0$. Expand $\rho(t)$ as

$$\rho(t) = \rho_0 + \Delta\rho(t)$$

and obtain

$$\begin{aligned} i\dot{\Delta}\rho(t) &= [H - f(t)\lambda A, \rho_0 + \Delta\rho(t)] \\ &= \underbrace{[H, \rho_0]}_0 + [-f(t)\lambda A, \rho_0] + [H, \Delta\rho(t)] + \underbrace{[-f(t)\lambda A, \Delta\rho(t)]}_{\propto \lambda^2} \\ &= -f(t)\lambda [A, \rho_0] + [H, \Delta\rho(t)] + \mathcal{O}(\lambda^2) \\ &= -f(t)\lambda A^\times \rho_0 + H^\times \Delta\rho(t) + \mathcal{O}(\lambda^2) \end{aligned} \tag{11}$$

where we used the notation $O^\times(\cdot) \equiv [O, \cdot]$ (sometimes also denoted by the adjoint notation ad_O for O)

³Note that we cannot employ the usual time independent Rayleigh–Schrödinger perturbation theory (as presented in [41] pp. 303) since one of its assumptions is that the spectrum is discrete, something which is not true in general for the perturbed Hamiltonian. For instance, perturbing from the Landau Hamiltonian we get

$$H = (\mathbf{p} - \mathbf{A})^2 - E_2 x_2$$

for an electric-field perturbing in the $\hat{\mathbf{e}}_2$ -direction. If we now use the gauge $\mathbf{A} = e_B^B x_2 \hat{\mathbf{e}}_1$ we get see that the Hamiltonian is not dependent on x_1 so that p_1 may be replaced by $\hbar k_1$ (not quantized) and so since the spectrum will (eventually) depend on k_1 , it is not discrete.

For this reason the Kubo formula is used as a sort of trick, in which time-dependence is only removed at the very end.

Claim. $e^{a^\times} b = e^a b e^{-a}$

Proof. One can proceed either in a pedestrian way by computing the explicit expression for $(a^\times)^n$ (make guess and proof by induction) or by defining

$$F(t) := e^{ta} b e^{-ta} \quad \forall t \in \mathbb{R}$$

and

$$G(t) := e^{ta^\times} b \quad \forall t \in \mathbb{R}$$

Next note that F and G both solve the differential equation

$$\tilde{F}'(t) = a^\times \tilde{F}(t)$$

with initial condition $\tilde{F}(0) = b$. Since the solution to a first order ordinary differential equation is unique, $F = G$ and in particular $F(1) = G(1)$. \square

Claim. The solution of (11) is given by:

$$\begin{aligned} \Delta\rho(t) &= i \int_{-\infty}^t \exp(-i(t-t')H^\times) \lambda A^\times \rho_0 f(t') dt' \\ &\equiv i \int_{-\infty}^t \exp(-i(t-t')H) \lambda [A, \rho_0] \exp(i(t-t')H) f(t') dt' \end{aligned}$$

Proof. Using the fact that

$$\frac{d}{dx} \int_{a(x)}^{b(x)} f(x, y) dy = f(x, b(x)) b'(x) - f(x, a(x)) a'(x) + \int_{a(x)}^{b(x)} [\partial_x f(x, y)] dy$$

the left hand side of (11) would be

$$\begin{aligned} i\dot{\Delta}\rho(t) &= i \frac{d}{dt} i \int_{-\infty}^t \exp(-i(t-t')H^\times) \lambda A^\times \rho_0 f(t') dt' \\ &= -\exp(-i(t-t')H^\times) \lambda A^\times \rho_0 f(t) - \int_{-\infty}^t \frac{d}{dt} [\exp(-i(t-t')H^\times) \lambda A^\times \rho_0 f(t')] dt' \\ &= -\lambda A^\times \rho_0 f(t) - \int_{-\infty}^t \exp(-i(t-t')H^\times) (-iH^\times) \lambda A^\times \rho_0 f(t') dt' \\ &= -\lambda A^\times \rho_0 f(t) + H^\times i \int_{-\infty}^t \exp(-i(t-t')H^\times) \lambda A^\times \rho_0 f(t') dt' \\ &= -\lambda A^\times \rho_0 f(t) + H^\times \Delta\rho(t) \end{aligned}$$

where we have used the fact that

$$[\exp(-i(t-t')H^\times), H^\times] = 0$$

and also note that the initial value is obeyed: $\Delta\rho(-\infty) = 0$. \square

Then we have

$$\begin{aligned} \langle B \rangle_{\rho(0)} &\equiv \text{Tr}[\rho(0) B] \\ &= \text{Tr}[(\rho_0 + \Delta\rho(0) + \mathcal{O}(\lambda^2)) B] \\ &= \underbrace{\text{Tr}[\rho_0 B]}_{\langle B \rangle_{\rho_0}} + \underbrace{\text{Tr}[\Delta\rho(0) B]}_{\lambda \chi_{BA}} + \mathcal{O}(\lambda^2) \end{aligned}$$

so that

$$\begin{aligned}
 \chi_{BA} &= \frac{1}{\lambda} \text{Tr} [\Delta\rho(0) B] \\
 &= \frac{1}{\lambda} \text{Tr} \left[i \int_{-\infty}^0 \exp(-i(0-t') H^\times) \lambda A^\times \rho_0 f(t') dt' B \right] \\
 &= i \int_{-\infty}^0 \text{Tr} \{ [\exp(itH^\times) (A^\times \rho_0)] B \} f(t) dt \\
 &\equiv i \int_{-\infty}^0 \text{Tr} [\exp(itH) [A, \rho_0] \exp(-itH) B] f(t) dt \\
 &= i \int_{-\infty}^0 \text{Tr} [\exp(-itH) B \exp(itH) [A, \rho_0]] f(t) dt \\
 &= i \int_0^\infty \text{Tr} [\exp(itH) B \exp(-itH) [A, \rho_0]] f(-t) dt
 \end{aligned}$$

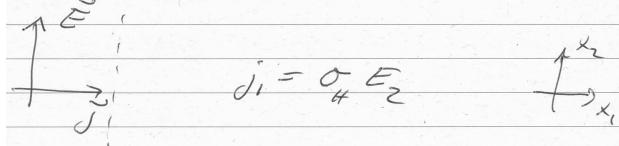
We now take care of the limit:

$$\lim_{\varepsilon \rightarrow 0} \chi_{BA} = \lim_{\varepsilon \rightarrow 0} i \int_0^\infty \text{Tr} [\exp(itH) B \exp(-itH) [A, \rho_0]] \exp(-\varepsilon t) dt$$

We now use Lebesgue's dominated convergence theorem ([39] pp. 26) with the dominating function being $t \mapsto |\text{Tr} [\exp(itH) B \exp(-itH) [A, \rho_0]]|$ (need to show it is L^1) to take the limit $\varepsilon \rightarrow 0$ into the integrand and obtain our result. \square

6.2. Kubo Formula for the Integer Quantum Hall Effect. We now specialize to the IQHE in order to compute the Hall conductivity (which plays the role of the measurable B in the previous section). For more rigorous treatment see [15].

If we define the axes as follows:



Then we could write $j_1 = \sigma_H E_2$. This relation holds locally (but it is valid only on macroscopic scales). So we allow that E_2 is not homogeneous. The current across the fiducial (dashed) line is:

$$\begin{aligned}
 I &= \int dx_2 j_1 \\
 &= \sigma_H \underbrace{\int dx_2 E_2}_V
 \end{aligned}$$

where V is the potential difference between where E starts and where it ends.

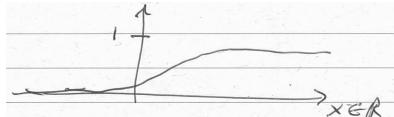
We make the approximation that the electrons do not interact so that we may use the single-particle Hamiltonian H to describe them. Therefore H corresponds to the energy of a single particle *before* the application of the electric field. Correspondingly the density matrix will be the single particle density matrix.

6.2.1. Definition. (Switch Function) A switch function is a C^∞ map $\Lambda : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\lim_{x \rightarrow -\infty} \Lambda(x) = 0$$

and

$$\lim_{x \rightarrow \infty} \Lambda(x) = 1$$



Using a switch function Λ , we could write the perturbation which the electric field introduces as

$$-V_0 \Lambda(x_2)$$

where the magnitude of the potential is the constant V_0 which is assumed to be small. Indeed, since

$$\begin{aligned}
 \mathbf{E} &\equiv -\nabla V \\
 &= V_0 \partial_{x_2} \Lambda(x_2)
 \end{aligned}$$

then the electric field will be non-zero in a compact extent of x_2 , the region where Λ' is non-zero.

Let $\varepsilon > 0$ be given. Define

$$f(t) := e^{\varepsilon t} \quad \forall t \in \mathbb{R}$$

so that we have $f(0) = 1$ and

$$\lim_{t \rightarrow -\infty} f(t) = 0$$

In the end of the computation we will take $\varepsilon \rightarrow 0$. Using that we could write the perturbed Hamiltonian as

$$H'(t) = H - V_0 \Lambda(x_2) f(t)$$

observe that x_2 is an operator on Hilbert space (the multiplication operator) which in the basis $\{|x\rangle\}_{x \in \mathbb{R}^2}$ acts as by $x_2 |x'\rangle \equiv x'_2 |x'\rangle$ and the meaning of $\Lambda(x_2)$ as an operator on Hilbert space is then obtained is via the symbolic calculus (see [40] definition 10.26 via contour integration).

The observable we probe is the current (from which we can compute the Hall conductivity). Current is rate of change of charge, so that ultimately we need to compute the rate of change of charge to the right of the fiducial line. In the single particle picture this is merely given by whether the particle is on the right or not, which we encode by the observable:

$$\Lambda(x_1)$$

(if x_1 is far on the right, it will be equal 1, if x_1 is far on the left it will be 0) so that the current is the rate of change of this observable, given by the Heisenberg equation of motion:

$$\begin{aligned} \partial_t \Lambda(x_1) &\equiv i[H'(t), \Lambda(x_1)] \\ &= i[H, \Lambda(x_1)] \end{aligned} \quad (12)$$

where the second equality holds only up to zeroth order in V_0 . Note that we shouldn't take the first order in V_0 here because later on there will be another contribution of a power of V_0 coming from ρ .

The initial state of the system (the one particle density matrix) is the Fermi projection P_μ :

$$P_\mu = \chi_{(-\infty, \mu]}(H)$$

where χ is the characteristic function and μ is the chemical potential. We write this as the ground state because we neglect the interactions.

6.2.2. Claim. The Hall conductivity is

$$\sigma_H = i \text{Tr}[P_\mu [[\Lambda(x_1), P_\mu], [\Lambda(x_2), P_\mu]]] \quad (13)$$

Proof. The equation of motion is given by

$$\partial_t \rho(t) = -i[H'(t), \rho(t)] \quad (14)$$

where $\rho(t)$ is the density matrix. Note the sign difference between (12) and (14).

The initial condition for (14)

$$\lim_{t \rightarrow -\infty} \rho(t) = P_\mu$$

is too naive since the limit $\lim_{t \rightarrow -\infty} \rho(t)$ does not necessarily exist. What we would rather impose is that

$$\|\rho(t) - e^{-iHt} P_\mu e^{iHt}\| \xrightarrow{t \rightarrow -\infty} 0$$

which implies

$$\left\| \underbrace{e^{iHt} \rho(t) e^{-iHt}}_{\equiv \rho_I(t)} - P_\mu \right\| \xrightarrow{t \rightarrow -\infty} 0 \quad (15)$$

Since e^{iHt} is unitary.

In the interaction picture, we define

$$\begin{aligned} \Delta H_I(t) &\equiv e^{iHt} (-V_0 \Lambda(x_2) f(t)) e^{-iHt} \\ \rho_I(t) &\equiv e^{iHt} \rho(t) e^{-iHt} \end{aligned}$$

and so from (14) it follows that:

$$\partial_t \rho_I(t) = -i[\Delta H_I(t), \rho_I(t)]$$

with boundary condition obtained from (15):

$$\lim_{t \rightarrow -\infty} \rho_I(t) = P_\mu$$

with solution to first order in the perturbation given by:

$$\rho_I(t) = P_\mu + i \int_{-\infty}^t e^{\varepsilon t'} e^{iHt'} [V_0 \Lambda(x_2), P_\mu] e^{-iHt'} dt'$$

indeed,

$$\begin{aligned} \partial_t \rho_I(t) &= i e^{\varepsilon t} e^{iHt} [V_0 \Lambda(x_2), P_\mu] e^{-iHt} \\ &\stackrel{[H, P_\mu]=0}{=} -i [f(t) e^{iHt} V_0 \Lambda(x_2) e^{-iHt}, P_\mu] \\ &= -i [\Delta H_I(t), P_\mu] \\ &= -i [\Delta H_I(t), \rho_I(t)] + \mathcal{O}(V_0^2) \end{aligned}$$

but in zeroth order in V_0 , $P_\mu = \rho_I(t)$.

Then we have that if $I = \sigma_h V$ then

$$\begin{aligned}\sigma_H &= \frac{1}{V_0} \lim_{\varepsilon \rightarrow 0^+} \text{Tr} [\rho_I(0) I] \\ \text{Tr}[P_\mu I] &= 0 \quad \frac{1}{V_0} \lim_{\varepsilon \rightarrow 0^+} \text{Tr} [(\rho_I(0) - P_\mu) I] \\ &= \frac{1}{V_0} \lim_{\varepsilon \rightarrow 0^+} \text{Tr} \left[\left(i \int_{-\infty}^0 e^{\varepsilon t'} e^{iHt'} [V_0 \Lambda(x_2), P_\mu] e^{-iHt'} dt' \right) (i[H, \Lambda(x_1)]) \right] \\ &= \lim_{\varepsilon \rightarrow 0^+} \text{Tr} \left[\left(\int_{-\infty}^0 e^{-iHt} i[H, \Lambda(x_1)] e^{iHt} e^{\varepsilon t} i[\Lambda(x_2), P_\mu] dt \right) \right]\end{aligned}$$

next note that

$$\begin{aligned}e^{-iHt} i[H, \Lambda(x_1)] e^{iHt} &= ie^{-iHt} (H \Lambda(x_1) - \Lambda(x_1) H) e^{iHt} \\ &= - \left(e^{-iHt} (-iH) \Lambda(x_1) e^{iHt} + e^{-iHt} \Lambda(x_1) e^{iHt} (iH) \right) \\ &= - \frac{d}{dt} \left(e^{-iHt} \Lambda(x_1) e^{iHt} - \Lambda(x_1) \right)\end{aligned}$$

so that

$$\begin{aligned}\sigma_H &= \lim_{\varepsilon \rightarrow 0^+} \text{Tr} \left[\int_{-\infty}^0 \left(- \frac{d}{dt} \left(e^{-iHt} \Lambda(x_1) e^{iHt} - \Lambda(x_1) \right) \right) e^{\varepsilon t} i[\Lambda(x_2), P_\mu] dt \right] \\ &= \lim_{\varepsilon \rightarrow 0^+} \text{Tr} \left[\underbrace{\left(-e^{-iHt} \Lambda(x_1) e^{iHt} - \Lambda(x_1) \right) e^{\varepsilon t} i[\Lambda(x_2), P_\mu] \Big|_{-\infty}^0}_{0} + \right. \\ &\quad \left. + \int_{-\infty}^0 (e^{-iHt} \Lambda(x_1) e^{iHt} - \Lambda(x_1)) \frac{d}{dt} (e^{\varepsilon t} i[\Lambda(x_2), P_\mu]) dt \right] \\ &= \lim_{\varepsilon \rightarrow 0^+} i\varepsilon \text{Tr} \left[\int_{-\infty}^0 e^{\varepsilon t} \left(e^{-iHt} \Lambda(x_1) e^{iHt} - \Lambda(x_1) \right) [\Lambda(x_2), P_\mu] dt \right]\end{aligned}$$

also note that

$$\begin{aligned}[\Lambda(x_2), P_\mu] &= \Lambda(x_2) P_\mu - P_\mu \Lambda(x_2) \\ &= \Lambda(x_2) P_\mu - P_\mu \Lambda(x_2) P_\mu - P_\mu \Lambda(x_2) + P_\mu \Lambda(x_2) P_\mu \\ &= (\mathbb{1} - P_\mu) \Lambda(x_2) P_\mu - P_\mu \Lambda(x_2) (\mathbb{1} - P_\mu) \\ &= (\mathbb{1} - P_\mu) \Lambda(x_2) \underbrace{P_\mu^2}_P - \underbrace{(\mathbb{1} - P_\mu) P_\mu}_0 \Lambda(x_2) P_\mu + P_\mu \Lambda(x_2) \underbrace{P_\mu (\mathbb{1} - P_\mu)}_0 - \underbrace{P_\mu^2}_P \Lambda(x_2) (\mathbb{1} - P_\mu) \\ &= (\mathbb{1} - P_\mu) [\Lambda(x_2), P_\mu] P_\mu + P_\mu [\Lambda(x_2), P_\mu] (\mathbb{1} - P_\mu)\end{aligned}$$

so that

$$\begin{aligned}\sigma_H &= \lim_{\varepsilon \rightarrow 0^+} i\varepsilon \text{Tr} \left[\int_{-\infty}^0 e^{\varepsilon t} \left(e^{-iHt} \Lambda(x_1) e^{iHt} - \Lambda(x_1) \right) ((\mathbb{1} - P_\mu) [\Lambda(x_2), P_\mu] P_\mu + P_\mu [\Lambda(x_2), P_\mu] (\mathbb{1} - P_\mu)) dt \right] \\ &= \lim_{\varepsilon \rightarrow 0^+} i\varepsilon \text{Tr} \left[\int_{-\infty}^0 e^{\varepsilon t} \left(P_\mu \left(e^{-iHt} \Lambda(x_1) e^{iHt} - \Lambda(x_1) \right) (\mathbb{1} - P_\mu) + (\mathbb{1} - P_\mu) \left(e^{-iHt} \Lambda(x_1) e^{iHt} - \Lambda(x_1) \right) P_\mu \right) [\Lambda(x_2), P_\mu] dt \right] \\ &= \lim_{\varepsilon \rightarrow 0^+} i\varepsilon \text{Tr} \left[\int_{-\infty}^0 e^{\varepsilon t} \left(e^{-iH^\times t} (P_\mu \Lambda(x_1) (\mathbb{1} - P_\mu) + (\mathbb{1} - P_\mu) \Lambda(x_1) P_\mu) \right) [\Lambda(x_2), P_\mu] dt \right] - \\ &\quad - \lim_{\varepsilon \rightarrow 0^+} i\varepsilon \text{Tr} \left[\int_{-\infty}^0 e^{\varepsilon t} ((P_\mu \Lambda(x_1) (\mathbb{1} - P_\mu) + (\mathbb{1} - P_\mu) \Lambda(x_1) P_\mu)) [\Lambda(x_2), P_\mu] dt \right]\end{aligned}$$

Claim. $\lim_{\varepsilon \rightarrow 0^+} i\varepsilon \text{Tr} \left[\int_{-\infty}^0 e^{\varepsilon t} \left(e^{-iH^\times t} (P_\mu \Lambda(x_1) (\mathbb{1} - P_\mu) + (\mathbb{1} - P_\mu) \Lambda(x_1) P_\mu) \right) [\Lambda(x_2), P_\mu] dt \right] = 0$

Proof. Start with only one term (the other is complementary):

$$\lim_{\varepsilon \rightarrow 0^+} i\varepsilon \text{Tr} \left[\int_{-\infty}^0 e^{\varepsilon t} \left(e^{-iHt} (\mathbb{1} - P_\mu) \Lambda(x_1) P_\mu e^{iHt} \right) [\Lambda(x_2), P_\mu] dt \right]$$

By the spectral theorem ([40] theorem 13.33) we may write schematically

$$\begin{aligned} P_\mu &= \int_{-\infty}^\mu dP_{\lambda_-} \\ P_\mu e^{iHt} &= \int_{-\infty}^\mu dP_{\lambda_-} e^{i\lambda_- t} \end{aligned}$$

and

$$\begin{aligned} \mathbb{1} - P_\mu &= \int_\mu^\infty dP_{\lambda_+} \\ e^{-iHt} (\mathbb{1} - P_\mu) &= \int_\mu^\infty dP_{\lambda_+} e^{-i\lambda_+ t} \end{aligned}$$

so that our expression becomes

$$\begin{aligned} &= \lim_{\varepsilon \rightarrow 0^+} i\varepsilon \text{Tr} \left[\int_{\lambda_+ = \mu}^\infty \int_{\lambda_- = -\infty}^0 \int_{-\infty}^0 e^{\varepsilon t} \left(e^{-i\lambda_+ t} dP_{\lambda_+} \Lambda(x_1) dP_{\lambda_-} e^{i\lambda_- t} \right) [\Lambda(x_2), P_\mu] dt \right] \\ &= \lim_{\varepsilon \rightarrow 0^+} \text{Tr} \left[\int_{\lambda_+ = \mu}^\infty \int_{\lambda_- = -\infty}^0 dP_{\lambda_+} \Lambda(x_1) dP_{\lambda_-} [\Lambda(x_2), P_\mu] \right] i\varepsilon \int_{-\infty}^0 e^{\varepsilon t - i(\lambda_+ - \lambda_-)t} dt \end{aligned}$$

the integral over time is

$$\begin{aligned} i\varepsilon \int_{-\infty}^0 e^{\varepsilon t - i(\lambda_+ - \lambda_-)t} dt &= i\varepsilon \frac{e^{\varepsilon t - i(\lambda_+ - \lambda_-)t}}{\varepsilon - i(\lambda_+ - \lambda_-)} \Big|_{-\infty}^0 \\ &\stackrel{\varepsilon \geq 0}{=} \frac{i\varepsilon}{\varepsilon - i(\lambda_+ - \lambda_-)} \\ &= \frac{-\varepsilon}{\lambda_+ - \lambda_- + i\varepsilon} \end{aligned}$$

Note that IF $\lambda_+ \neq \lambda_-$ then in the limit $\varepsilon \rightarrow 0^+$, this expression becomes zero. This situation happens when there is a spectral gap, so that $\lambda_- < \lambda_+$ for both integrations. If there is no spectral gap this argument fails, since we could still get some contributions when $\lambda_+ = \lambda_-$. It is also possible to generalize this for when there is only a mobility gap but no spectral gap, but we refrain from this at the moment.

The second term proceeds analogously.

see in graph/sep

Jop: $\lambda_+ - \lambda_- \geq \delta > 0$.

□

Next note that

$$\begin{aligned} P_\mu \Lambda(x_1) (\mathbb{1} - P_\mu) + (\mathbb{1} - P_\mu) \Lambda(x_1) P_\mu &= \Lambda(x_1) P_\mu - 2P_\mu \Lambda(x_1) P_\mu + P_\mu \Lambda(x_1) \\ &= \Lambda(x_1) P_\mu^2 - P_\mu \Lambda(x_1) P_\mu - P_\mu \Lambda(x_1) P_\mu + P_\mu^2 \Lambda(x_1) \\ &= [\Lambda(x_1), P_\mu] P_\mu - P_\mu [\Lambda(x_1), P_\mu] \\ &= [[\Lambda(x_1), P_\mu], P_\mu] \end{aligned}$$

so that

$$\begin{aligned} \sigma_H &= -\lim_{\varepsilon \rightarrow 0^+} i\varepsilon \text{Tr} \left[\int_{-\infty}^0 e^{\varepsilon t} ([[[\Lambda(x_1), P_\mu], P_\mu], P_\mu]] [\Lambda(x_2), P_\mu] dt \right] \\ &= -i\text{Tr} [[[[\Lambda(x_1), P_\mu], P_\mu], P_\mu]] [\Lambda(x_2), P_\mu]] \underbrace{\lim_{\varepsilon \rightarrow 0^+} \varepsilon \int_{-\infty}^0 e^{\varepsilon t} dt}_0 \\ &= -i\text{Tr} [[[[\Lambda(x_1), P_\mu], P_\mu], P_\mu]] [\Lambda(x_2), P_\mu]] \end{aligned}$$

and using the fact that

$$\begin{aligned} \text{Tr} [[A, B] C] &= \text{Tr} [ABC - BAC] \\ &= \text{Tr} [BCA - BAC] \\ &= -\text{Tr} [B[A, C]] \end{aligned}$$

we obtain the desired result. \square

6.2.3. Remark. The commutator in (13) reflects the anti-symmetry of exchange $x_1 \leftrightarrow x_2$; this is the anti-symmetry of the conductivity matrix.

6.2.4. Remark. As noted above, the proof fails if there is no spectral gap, but can be generalized to the case of a mobility gap. For details see [19].

6.3. Discussion of the Kubo Formula.

Traces. In (13) there is a trace of an operator on Hilbert space. The question arises as to when this trace is well-defined and finite. The main textbook for this topic is [42] or [38] volume 1 page 206.

6.3.1. Definition. For an element A in a Banach algebra with involution $*$ the statement

$$A \geq 0$$

is equivalent to the statement that $A = A^*$ and $\sigma(A) \subseteq [0, \infty)$. Note that the Banach algebra we normally consider is $\mathcal{B}(\mathcal{H})$, the bounded linear maps $\mathcal{H} \rightarrow \mathcal{H}$.

6.3.2. Claim. If $A \geq 0$ for some $A \in \mathcal{B}(\mathcal{H})$ and $\{\varphi_n\}_{n \in \mathbb{N}}$ is an orthonormal basis for the separable Hilbert space \mathcal{H} then the (possibly infinite) quantity

$$\sum_{n \in \mathbb{N}} \langle \varphi_n, A \varphi_n \rangle$$

does not depend on the choice of the basis $\{\varphi_n\}_{n \in \mathbb{N}}$.

Proof. Since $A \geq 0$, according to theorem 11.26 in [40], $\sqrt{A} \in \mathcal{B}(\mathcal{H})$ exists as an operator such that $\sqrt{A}^2 = A$ and \sqrt{A} is also self-adjoint. Then if $\{\psi_n\}_{n \in \mathbb{N}}$ is another orthonormal basis for \mathcal{H} , then

$$\begin{aligned} \sum_{n \in \mathbb{N}} \langle \varphi_n, A \varphi_n \rangle &= \sum_{n \in \mathbb{N}} \langle \varphi_n, \sqrt{A}^2 \varphi_n \rangle \\ &= \sum_{n \in \mathbb{N}} \langle \sqrt{A}^* \varphi_n, \sqrt{A} \varphi_n \rangle \\ &= \sum_{n \in \mathbb{N}} \langle \sqrt{A} \varphi_n, \sqrt{A} \varphi_n \rangle \\ &= \sum_{n \in \mathbb{N}} \|\sqrt{A} \varphi_n\|^2 \\ &= \sum_{n \in \mathbb{N}} \left\| \sum_{m \in \mathbb{N}} \langle \psi_m, \sqrt{A} \varphi_n \rangle \psi_m \right\|^2 \\ &= \sum_{n \in \mathbb{N}} \left(\sum_{m \in \mathbb{N}} |\langle \psi_m, \sqrt{A} \varphi_n \rangle|^2 \right) \\ &\stackrel{*}{=} \sum_{m \in \mathbb{N}} \left(\sum_{n \in \mathbb{N}} |\langle \psi_m, \sqrt{A} \varphi_n \rangle|^2 \right) \\ &= \sum_{m \in \mathbb{N}} \|\sqrt{A} \psi_m\|^2 \\ &\equiv \sum_{m \in \mathbb{N}} \langle \sqrt{A} \psi_m, \sqrt{A} \psi_m \rangle \\ &= \sum_{m \in \mathbb{N}} \langle \psi_m, A \psi_m \rangle \end{aligned}$$

where $*$ was valid because all terms are positive so rearrangements are possible. \square

6.3.3. Definition. If $A \geq 0$ for some $A \in \mathcal{B}(\mathcal{H})$ then define

$$\text{Tr}(A) := \sum_{n \in \mathbb{N}} \langle \varphi_n, A \varphi_n \rangle$$

where $\{\varphi_n\}_{n \in \mathbb{N}}$ is any orthonormal basis of \mathcal{H} .

6.3.4. Claim. For any $A \in \mathcal{B}(\mathcal{H})$, $A^* A \geq 0$.

Proof. The self-adjoint condition is easy:

$$\begin{aligned} (A^* A)^* &\equiv A^* (A^*)^* \\ &= A^* A \end{aligned}$$

because a separable Hilbert space is reflexive ($(A^*)^* = A$). Next, if $f : \mathbb{C} \rightarrow \mathbb{C}$ is the entire map $z \mapsto \bar{z}z$ then $A^*A = f(A)$ as in definition 10.26 of [40]. By theorem 10.28 of [40], since f is entire, we have $\sigma(f(A)) = f(\sigma(A))$. That is,

$$\begin{aligned}\sigma(f(A)) &= \sigma(A^*A) \\ &= f(\sigma(A)) \\ &\equiv \{f(z) \mid z \in \sigma(z)\} \\ &= \{|z|^2 \mid z \in \sigma(z)\}\end{aligned}$$

so that $\sigma(A^*A) \subseteq [0, \infty)$ as necessary and $A^*A \geq 0$. \square

6.3.5. Corollary. For any $A \in \mathcal{B}(\mathcal{H})$, $\sqrt{A^*A}$ is defined.

6.3.6. Definition. For any $A \in \mathcal{B}(\mathcal{H})$, define $|A| := \sqrt{A^*A}$. Note that $|A| \geq 0$ by theorem 10.28 of [40]. (also note that $|A|$ is self-adjoint)

6.3.7. Definition. $A \in \mathcal{B}(\mathcal{H})$ is trace class, written $A \in \mathcal{J}_1(\mathcal{H})$, iff

$$\text{Tr}(|A|) < \infty$$

6.3.8. Claim. $\mathcal{J}_1(\mathcal{H})$ is a Banach algebra together with the norm $\|\cdot\|_1 \equiv \text{Tr}(|\cdot|)$. Moreover, we have

$$\mathcal{J}_1(\mathcal{H}) \subseteq \mathcal{K}(\mathcal{H}) \subseteq \mathcal{B}(\mathcal{H})$$

where $\mathcal{K}(\mathcal{H})$ are the compact operators, and $\mathcal{J}_1(\mathcal{H})$ is a two-sided ideal of $\mathcal{B}(\mathcal{H})$.

Proof. This is the content of theorem VI.19 in [38] volume 1 page 207. \square

6.3.9. Claim. Finite rank operators are trace-class and they are dense in $\mathcal{J}_1(\mathcal{H})$ with respect to $\|\cdot\|_1$.

Proof. This is the corollary of Theorem VI.21 in [38]. \square

6.3.10. Remark. Note that finite rank operators are also compact, and that they are dense in the space of compact operators on \mathcal{H} with respect to the usual operator norm $\|\cdot\|$.

6.3.11. Claim. For any $A \in \mathcal{J}_1(\mathcal{H})$, the following expression is absolutely convergent

$$\sum_{n \in \mathbb{N}} \langle \varphi_n, A\varphi_n \rangle$$

where $\{\varphi_n\}_{n \in \mathbb{N}}$ is any orthonormal basis of \mathcal{H} , and the expression is independent of the choice of basis $\{\varphi_n\}_{n \in \mathbb{N}}$.

Proof. We follow theorem VI.24 in [38]. According to [38] pp. 197 theorem VI.10, we may write $A = U|A|$ where U is a unique partial isometry determined by $\ker(A) = \ker(U)$. Then we may write $A = U\sqrt{|A|}\sqrt{|A|}$ as $|A| \geq 0$. Then

$$\begin{aligned} |\langle \varphi_n, A\varphi_n \rangle| &= \left| \langle \varphi_n, U\sqrt{|A|}\sqrt{|A|}\varphi_n \rangle \right| \\ &= \left| \langle \sqrt{|A|}U^*\varphi_n, \sqrt{|A|}\varphi_n \rangle \right| \\ \text{Cauchy-Schwarz} &\leq \left\| \sqrt{|A|}U^*\varphi_n \right\| \left\| \sqrt{|A|}\varphi_n \right\| \end{aligned}$$

so that

$$\begin{aligned} \sum_{n \in \mathbb{N}} |\langle \varphi_n, A\varphi_n \rangle| &\leq \sum_{n \in \mathbb{N}} \left\| \sqrt{|A|}U^*\varphi_n \right\| \left\| \sqrt{|A|}\varphi_n \right\| \\ &\stackrel{\text{C.S.}}{\leq} \sqrt{\sum_{n \in \mathbb{N}} \left(\left\| \sqrt{|A|}U^*\varphi_n \right\|^2 \right)} \sqrt{\sum_{n \in \mathbb{N}} \left(\left\| \sqrt{|A|}\varphi_n \right\|^2 \right)} \end{aligned}$$

Now it is possible to show that since $\text{tr}(|A|) < \infty$ these two sums converge. Hence, $\sum_{n \in \mathbb{N}} \langle \varphi_n, A\varphi_n \rangle$ is absolutely convergent. The fact that this expression is independent of the choice of φ_n is left as an exercise to the reader. \square

As a result, it makes sense to make the following

6.3.12. Definition. If $A \in \mathcal{J}_1(\mathcal{H})$, define

$$\boxed{\text{Tr}(A) := \sum_{n \in \mathbb{N}} \langle \varphi_n, A\varphi_n \rangle}$$

where $\{\varphi_n\}_{n \in \mathbb{N}}$ is any orthonormal basis of \mathcal{H} .

Note that if $A \in \mathcal{J}_1(\mathcal{H})$, then we could also write

$$\text{Tr}(A) = \sum_{n \in \mathbb{N}} \lambda_n$$

where $\{\lambda_n\}_{n \in \mathbb{N}}$ are the eigenvalues of A (the only content of the spectrum as $A \in \mathcal{K}(\mathcal{H})$).

6.3.13. **Claim.** If $A \in \mathcal{J}_1(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{H})$ then

$$\text{Tr}(AB) = \text{Tr}(BA)$$

Proof. Note that $(AB, BA) \in \mathcal{J}_1(\mathcal{H})^2$ because $\mathcal{J}_1(\mathcal{H})$ is a two-sided ideal in $\mathcal{B}(\mathcal{H})$.

First consider the case where B is unitary. Then

$$\begin{aligned}\text{Tr}(AB) &= \sum_{n \in \mathbb{N}} \langle \varphi_n, AB\varphi_n \rangle \\ &= \sum_{n \in \mathbb{N}} \langle B^*B\varphi_n, AB\varphi_n \rangle \\ &= \sum_{n \in \mathbb{N}} \langle B\varphi_n, BAB\varphi_n \rangle \\ &= \text{Tr}(BA)\end{aligned}$$

where the last equality follows from the fact that $\{B\varphi_n\}_{n \in \mathbb{N}}$ is also an orthonormal basis of \mathcal{H} . Since Tr is linear and any bounded linear operator may be written as the sum of four unitary maps: For an arbitrary $B \in \mathcal{B}(\mathcal{H})$, write

$$B = \frac{1}{2}(B + B^*) - \frac{1}{2}(i(B - B^*))$$

Indeed,

$$\begin{aligned}\left\langle \varphi, \left(\frac{1}{2}(B + B^*) - \frac{1}{2}(i(B - B^*)) \right) \varphi \right\rangle &= \frac{1}{2} \langle \varphi, B\varphi \rangle + \frac{1}{2} \langle \varphi, B^*\varphi \rangle + \frac{1}{2i} \langle \varphi, B\varphi \rangle - \frac{1}{2i} \langle \varphi, B^*\varphi \rangle \\ &= \frac{1}{2} \langle \varphi, B\varphi \rangle + \frac{1}{2} \langle B\varphi, \varphi \rangle + \frac{1}{2i} \langle \varphi, B\varphi \rangle - \frac{1}{2i} \langle B\varphi, \varphi \rangle \\ &= \frac{1}{2} (\langle \varphi, B\varphi \rangle + \overline{\langle \varphi, B\varphi \rangle}) + \frac{1}{2i} (\langle \varphi, B\varphi \rangle - \overline{\langle \varphi, B\varphi \rangle}) \\ &= \langle \varphi, B\varphi \rangle\end{aligned}$$

so that using [40] the corollary after theorem 12.7 we have the equality. For any $A \in \mathcal{B}(\mathcal{H})$ which is self-adjoint and has $\|A\| \leq 1$, note that

$$A \pm i\sqrt{1 - A^2}$$

are both unitary, where the square root is well defined because $A^2 \leq 1$. Indeed:

$$\begin{aligned}(A + i\sqrt{1 - A^2})(A + i\sqrt{1 - A^2})^* &= (A + i\sqrt{1 - A^2})(A - i\sqrt{1 - A^2}) \\ &= A^2 \underbrace{-iA\sqrt{1 - A^2} + i\sqrt{1 - A^2}A}_{0} + 1 - A^2 \\ &= 1\end{aligned}$$

and similarly for the other order. Also note $A = \frac{1}{2}(A + i\sqrt{1 - A^2}) + \frac{1}{2}(A - i\sqrt{1 - A^2})$ so that A is a sum of two unitaries.

If $\|A\| > 1$, Write $\frac{A}{2\|A\|}$ as a sum of two unitaries. \square

6.3.14. **Example.** Let $\mathcal{H} = l^2(\mathbb{Z}; \mathbb{C}) \equiv \left\{ \psi : \mathbb{Z} \rightarrow \mathbb{C} \mid \sum_{n \in \mathbb{Z}} |\psi(n)|^2 < \infty \right\}$. \mathcal{H} has an orthonormal basis given by $\{ \delta_m \}_{m \in \mathbb{Z}}$ where $\delta_m : \mathbb{Z} \rightarrow \mathbb{C}$ is given by $\delta_m(n) = \delta_{m,n}$, the Kronecker delta. A shift operator to the right $A : \mathcal{H} \rightarrow \mathcal{H}$ is given by its action on $\{ \delta_m \}_{m \in \mathbb{Z}}$ as $A\delta_m := \delta_{m+1}$.

Claim. $A \notin \mathcal{J}_1(\mathcal{H})$.

Proof. Note that A is unitary: A^* is the left shift operator given by $A^*\delta_m = \delta_{m-1}$. This can be seen via

$$\begin{aligned}\langle \varphi, A^*\varphi \rangle &= \left\langle \sum_{n \in \mathbb{Z}} \varphi(n) \delta_n, A^* \sum_{m \in \mathbb{Z}} \varphi(m) \delta_m \right\rangle \\ &= \left\langle A \sum_{n \in \mathbb{Z}} \varphi(n) \delta_n, \sum_{m \in \mathbb{Z}} \varphi(m) \delta_m \right\rangle \\ &= \left\langle \sum_{n \in \mathbb{Z}} \varphi(n) \delta_{n+1}, \sum_{m \in \mathbb{Z}} \varphi(m) \delta_m \right\rangle \\ &= \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \overline{\varphi(n)} \varphi(m) \delta_{n+1, m} \\ &= \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \overline{\varphi(n)} \varphi(m) \delta_{n, m-1} \\ &= \left\langle \sum_{n \in \mathbb{Z}} \varphi(n) \delta_n, \sum_{m \in \mathbb{Z}} \varphi(m) \delta_{m-1} \right\rangle\end{aligned}$$

the fact that adjoints are unique and the corollary after theorem 12.7 in [40]. As a result, $AA^* = A^*A = \mathbb{1}$ (shift left and the right is doing nothing), so that $|A| = \mathbb{1}$ and so

$$\begin{aligned}\text{Tr}(|A|) &= \sum_{n \in \mathbb{Z}} \langle \delta_n, \mathbb{1} \delta_n \rangle \\ &= \sum_{n \in \mathbb{Z}} 1 \\ &= \infty\end{aligned}$$

□

Contrast this with the fact that

$$\begin{aligned}\langle \delta_n, A \delta_n \rangle &= \langle \delta_n, \delta_{n+1} \rangle \\ &= 0\end{aligned}$$

so that formally

$$\begin{aligned}\sum_{n \in \mathbb{Z}} \langle \delta_n, A \delta_n \rangle &= 0 \\ &< \infty\end{aligned}$$

so that one might have guessed that $\text{Tr}(A) = 0$. The point is that in another orthonormal basis we would've obtained another value for $\sum_{n \in \mathbb{N}} \langle \varphi_n, A \varphi_n \rangle$. The independence of choice of basis only follows when $A \in \mathcal{J}_1(\mathcal{H})$.

6.3.15. Example. Choose the same Hilbert space \mathcal{H} as in the previous example. Now let A be the multiplication operator by $f : \mathbb{Z} \rightarrow \mathbb{C}$, that is,

$$(A\psi)(n) = f(n)\psi(n) \quad \forall n \in \mathbb{Z}$$

Claim. If $f \in l^1(\mathbb{Z}; \mathbb{C})$ then $A \in \mathcal{J}_1(\mathcal{H})$.

Proof. Note that A^* is multiplication by \bar{f} (verify) and so A^*A is multiplication by $|f|^2$ so that $|A|$ is multiplication by $|f|$ (spectral theorem). Then

$$\begin{aligned}\text{Tr}(|A|) &= \sum_{n \in \mathbb{Z}} \langle \delta_n, |A| \delta_n \rangle \\ &= \sum_{n \in \mathbb{Z}} \langle \delta_n, |f(n)| \delta_n \rangle \\ &= \sum_{n \in \mathbb{Z}} |f(n)| \\ &< \infty\end{aligned}$$

by assumption. □

Tight Binding Models. Now that we have a concept of traces in separable Hilbert spaces in general, we turn to a more concrete description of the Hilbert spaces which will be used in the applications of (13).

As an approximation, we use \mathbb{Z}^2 instead of \mathbb{R}^2 : this is the tight-binding approximation. Correspondingly the Hilbert space is $\mathcal{H} = l^2(\mathbb{Z}^2; \mathbb{C})$. This space is spanned by an orthonormal basis $\{\delta_n\}_{n \in \mathbb{Z}^2}$ where $\delta_n : \mathbb{Z}^2 \rightarrow \mathbb{C}$ is given by

$$\begin{aligned}\delta_n(\mathbf{m}) &= \delta_{n, \mathbf{m}} \\ &\equiv \delta_{n_1, m_1} \delta_{n_2, m_2}\end{aligned}$$

with $\delta_{n, \mathbf{m}} \equiv \begin{cases} 0 & n = \mathbf{m} \\ 1 & n \neq \mathbf{m} \end{cases}$. We will use $\|\mathbf{n}\| \equiv |n_1| + |n_2|$.

6.3.16. Fact. The Hamiltonians we consider are local. That is, $\exists D \geq 1$ such that $\langle \delta_n, H \delta_m \rangle = 0$ if $\|\mathbf{n} - \mathbf{m}\| > D$. Note that if $D = 0$ then H is entirely diagonal, which means it has no hopping terms, that is, the kinetic energy is zero. We thus exclude that possibility.

6.3.17. Example. The value $D = 1$ corresponds to the nearest neighbor approximation.

6.3.18. Example. Define $T : \mathcal{H} \rightarrow \mathcal{H}$ by

$$\langle \delta_n, T \delta_m \rangle = \begin{cases} 1 & \|\mathbf{n} - \mathbf{m}\| = 1 \\ 0 & \text{otherwise} \end{cases}$$

and $V : \mathcal{H} \rightarrow \mathcal{H}$ by

$$\langle \delta_n, V \delta_m \rangle = \begin{cases} \tilde{V}(\mathbf{n}) & \|\mathbf{n} - \mathbf{m}\| = 0 \\ 0 & \text{otherwise} \end{cases}$$

where $\tilde{V} : \mathbb{Z}^2 \rightarrow \mathbb{R}$ is some map. Then we have for $H = T + V$ that $D = 1$, and T is called the discrete Laplacian.

6.3.19. **Claim.** If P_μ is the Fermi projection of H :

$$P_\mu \equiv \chi_{(-\infty, \mu]}(H)$$

and if μ lies in a spectral gap of H then P_μ is “almost local”, that is,

$$|\langle \delta_n, P_\mu \delta_m \rangle| \leq C \exp(-c \|n - m\|)$$

for some C and c positive constants. If μ is not in the spectral gap then the decay is merely polynomial. This is almost true even for μ in a mobility gap.

Proof. The contents of this proof can be found in [2]. □

The Kubo Formula is Well-Defined. The goal in this section is to show that (13) is indeed well defined. That is,

6.3.20. **Claim.** $P_\mu [[\Lambda(x_1), P_\mu], [\Lambda(x_2), P_\mu]] \in \mathcal{J}_1(\mathcal{H})$

Proof. Our first goal is to determine the range of the operator $[\Lambda(x_1), P_\mu]$. For that matter, pick any $m \in \mathbb{Z}^2$, to which there corresponds a state δ_m . Then

$$\begin{aligned} [\Lambda(x_1), P_\mu] \delta_m &= \Lambda(x_1) P_\mu \delta_m - P_\mu \Lambda(x_1) \delta_m \\ &= \Lambda(x_1) P_\mu \delta_m - P_\mu \Lambda(m_1) \delta_m \\ &= [\Lambda(x_1) - \Lambda(m_1)] P_\mu \delta_m \end{aligned}$$

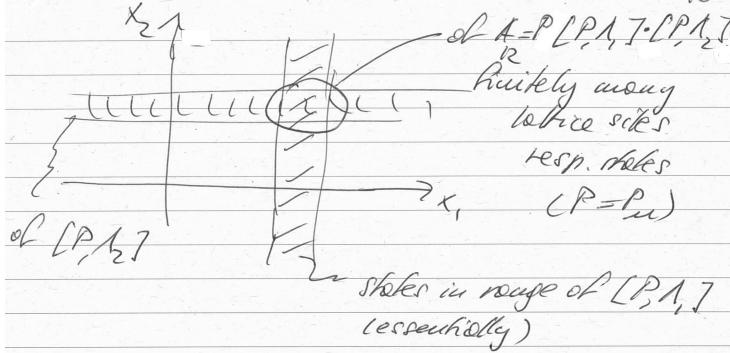
Now using the fact that 6.3.19 we may infer that $P_\mu \delta_m$ is non-zero mostly around m . In particular, we may assume that $|\langle P_\mu \delta_m, x_1 P_\mu \delta_m \rangle - m_1|$ is not a large number.

As such, if m_1 is very large positive number, so that the switch function $\Lambda(m_1)$ is 1, then $\langle P_\mu \delta_m, x_1 P_\mu \delta_m \rangle$ should also be quite large, in fact large enough so that $\Lambda(x_1)$ acting on $P_\mu \delta_m$ will give 1 as well, so that all together $[\Lambda(x_1), P_\mu] \delta_m$ for very large m_1 will be zero.

If on the other hand m_1 is a very large negative number, then $\Lambda(m_1) = 0$ and $\langle P_\mu \delta_m, x_1 P_\mu \delta_m \rangle$ should also be large and negative, so that $\Lambda(x_1)$ acting on $P_\mu \delta_m$ should give 0. Again we get zero for $[\Lambda(x_1), P_\mu] \delta_m$.

The conclusion is that $[\Lambda(x_1), P_\mu] \delta_m$ is non-zero only if m is such that m_1 is in $\Lambda'^{-1}(\mathbb{R} \setminus \{0\})$, which, by assumption on Λ , is a bounded region in space. Note that this argument is heuristic, in reality, since P_μ has exponential decay, more precise estimates must be dealt with.

Exactly the same argument can be made for $[\Lambda(x_2), P_\mu] \delta_m$ so that the following picture describes the area in \mathbb{Z}^2 where both are supported:



The upshot is that only finitely many lattice sites (that is, finitely many states δ_m) are in the range of $[\Lambda(x_1), P_\mu] [\Lambda(x_2), P_\mu]$ or $[\Lambda(x_2), P_\mu] [\Lambda(x_1), P_\mu]$ and as such in the range of $P_\mu [[\Lambda(x_1), P_\mu], [\Lambda(x_2), P_\mu]]$. Since finite rank operators are trace class, we arrive at our result. Strictly speaking, if we were to take the precise estimates mentioned before then we wouldn't say the operators are finite-rank, but merely trace-class.

If one defines $A_{ij} := P_\mu [\Lambda(x_i), P_\mu] [\Lambda(x_j), P_\mu]$ for $(i, j) \in \{(1, 2), (2, 1)\}$ then we have actually shown that A_{12} and A_{21} are separately trace class, and we have the additional formula

$$\begin{aligned} \sigma_H &= i \text{Tr}(A_{12} - A_{21}) \\ &= i (\text{Tr}(A_{12}) - \text{Tr}(A_{21})) \end{aligned}$$

6.3.21. **Claim.** We have

$$A_{ij} = -P_\mu \Lambda(x_i) P_\mu^\perp \Lambda(x_j) P_\mu$$

Proof. Note that

$$\begin{aligned} P_\mu [\Lambda(x_1), P_\mu] &\equiv P_\mu \Lambda(x_1) P_\mu - P_\mu P_\mu \Lambda(x_1) \\ &= P_\mu \Lambda(x_1) \underbrace{(P_\mu - 1)}_{-P_\mu^\perp} \\ &= -P_\mu \Lambda(x_1) P_\mu^\perp \end{aligned}$$

and

$$\begin{aligned} P_\mu^\perp [\Lambda(x_2), P_\mu] &= P_\mu^\perp \Lambda(x_2) P_\mu - \underbrace{P_\mu^\perp P_\mu}_{0} \Lambda(x_2) \\ &= P_\mu^\perp \Lambda(x_2) P_\mu \end{aligned}$$

and so as a result

$$\begin{aligned} A_{12} &= P_\mu [\Lambda(x_1), P_\mu] [\Lambda(x_2), P_\mu] \\ &= (-P_\mu \Lambda(x_1) P_\mu^\perp) [\Lambda(x_2), P_\mu] \\ &= -P_\mu \Lambda(x_1) P_\mu^\perp \Lambda(x_2) P_\mu \end{aligned}$$

□

6.3.22. Corollary. We can now write

$$\begin{aligned} \sigma_H &= -i\text{Tr} (P_\mu \Lambda(x_1) P_\mu^\perp \Lambda(x_2) P_\mu) + i\text{Tr} (P_\mu \Lambda(x_2) P_\mu^\perp \Lambda(x_1) P_\mu) \\ &= -i\text{Tr} (P_\mu^\perp \Lambda(x_2) P_\mu \Lambda(x_1) P_\mu^\perp) + i\text{Tr} (P_\mu^\perp \Lambda(x_1) P_\mu \Lambda(x_2) P_\mu^\perp) \end{aligned} \quad (16)$$

where in the second line we have used the fact that $P_\mu \Lambda(x_i) P_\mu^\perp \in \mathcal{J}_1(\mathcal{H})$ separately.

6.3.23. Claim. σ_H does not depend on the choice of the switch function Λ .

Proof. Actually in our formulation so far, the switch function was identical when it was used with the argument with x_1 or x_2 . But that did not have to be the case. Here we show what happens if we change merely the switch function that is used to compute the current, that is, the term $\Lambda(x_1)$.

So let $\tilde{\Lambda}$ be another switch function. Define $\Delta\Lambda(x_1) := \Lambda(x_1) - \tilde{\Lambda}(x_1)$. Note that $\Delta\Lambda$ is compactly supported in x_1 , because sufficiently far to the left or right, both Λ and $\tilde{\Lambda}$ are either both 0 or both 1, so that their difference, $\Delta\Lambda$ is 0 sufficiently far to the left or right.

Also note that $P_\mu^\perp \Lambda(x_2) P_\mu$ has compact support in x_2 .

Now if we want to see the difference in σ_H , we examine the expression in (16):

$$P_\mu \Delta\Lambda(x_1) P_\mu^\perp \Lambda(x_2) P_\mu$$

which will give us the difference in σ_H between computing it with Λ or with $\tilde{\Lambda}$ for the x_1 argument. Then because $\Delta\Lambda(x_1)$ has compact support in x_1 and $P_\mu^\perp \Lambda(x_2) P_\mu$ has compact support in x_2 , all together $\Delta\Lambda(x_1) P_\mu^\perp \Lambda(x_2) P_\mu \in \mathcal{J}_1(\mathcal{H})$. Note that this is now stronger than what we used in (16) because now we may use the cyclicity to move P alone (P is bounded, $\Delta\Lambda(x_1) P_\mu^\perp \Lambda(x_2) P_\mu$ is trace class, so we may employ cyclicity):

$$\begin{aligned} \text{Tr}(\Delta A_{12}) &= -\text{Tr}(P_\mu \Delta\Lambda(x_1) P_\mu^\perp \Lambda(x_2) P_\mu) \\ &= -\text{Tr}(\Delta\Lambda(x_1) P_\mu^\perp \Lambda(x_2) P_\mu^2) \\ &= -\text{Tr}(\Delta\Lambda(x_1) P_\mu^\perp \Lambda(x_2) P_\mu) \end{aligned}$$

Similarly we have

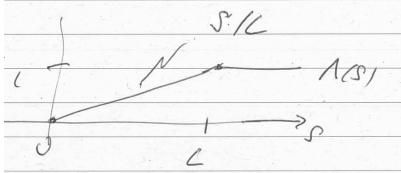
$$\begin{aligned} \text{Tr}(\Delta A_{21}) &= \text{Tr}(P_\mu^\perp \Delta\Lambda(x_1) P_\mu \Lambda(x_2) P_\mu^\perp) \\ &= \text{Tr}(\Delta\Lambda(x_1) P_\mu \Lambda(x_2) P_\mu^\perp) \end{aligned}$$

so that

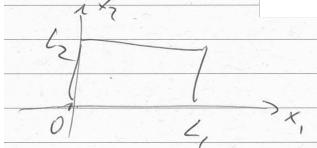
$$\begin{aligned}
 \Delta\sigma_H &= i\text{Tr}(\Delta A_{12} - \Delta A_{21}) \\
 &= i\text{Tr}\left(-\Delta\Lambda(x_1)P_\mu\Lambda(x_2)P_\mu^\perp + \Delta\Lambda(x_1)P_\mu\Lambda(x_2)P_\mu^\perp\right) \\
 &= i\text{Tr}\left(\Delta\Lambda(x_1)\left(-P_\mu\Lambda(x_2)P_\mu^\perp + P_\mu\Lambda(x_2)P_\mu^\perp\right)\right) \\
 &= i\text{Tr}(\Delta\Lambda(x_1)[\Lambda(x_2), P_\mu]) \\
 &= i\sum_{n \in \mathbb{Z}^2} \langle \delta_n, \Delta\Lambda(x_1)[\Lambda(x_2), P_\mu] \delta_n \rangle \\
 &= i\sum_{n \in \mathbb{Z}^2} \langle \delta_n, \Delta\Lambda(x_1)(\Lambda(x_2)P_\mu - P_\mu\Lambda(x_2)) \delta_n \rangle \\
 &= i\sum_{n \in \mathbb{Z}^2} \langle \delta_n, \Delta\Lambda(n_1)(\Lambda(n_2)P_\mu - P_\mu\Lambda(n_2)) \delta_n \rangle \\
 &= 0
 \end{aligned}$$

□

6.4. The Kubo-Thouless Formula in the Infinite Volume Limit. Next we want to consider the limit when $\text{supp}(\Lambda')$ becomes infinite.



We let $\chi_L(x)$ be the characteristic function of the rectangle here:



where $L \equiv (L_1, L_2)$.

Then except for small errors in the boundaries of the rectangle, we have

$$\sigma_H = \lim_{\|L\| \rightarrow \infty} \frac{1}{L_1 L_2} \text{Tr}(\chi_L P_\mu [[x_1, P_\mu], [x_2, P_\mu]]) \quad (17)$$

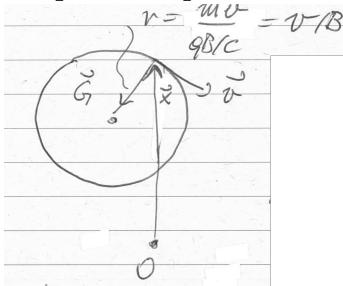
This works because where the functions Λ change, they are linear of the form $\Lambda(x_i) = \frac{x_i}{L_i}$ and otherwise they are zero far enough to the left and unsupported by the whole operator far enough to the right. Thus the characteristic function captures most of this information. If the limit exists, we may write

$$\sigma_H = i\text{Tr}'(P_\mu [[x_1, P_\mu], [x_2, P_\mu]]) \quad (18)$$

where Tr' is trace per unit area. Similarly to (16) we can also write

$$\sigma_H = -i\text{Tr}'\left(P_\mu x_1 P_\mu^\perp x_2 P_\mu - P_\mu x_2 P_\mu^\perp x_1 P_\mu\right) \quad (19)$$

6.5. Explicit Computation for the Landau Hamiltonian. As usual we start with a classical picture:



Our model is that of independent electrons moving freely under the influence of a magnetic field (and the electric field is later added as a perturbation). The classical unperturbed orbits are circular motion. We denote by \mathbf{G} the vector from the particle's position \mathbf{x} to the center of the circular motion. The vector from the origin, the center of the circular motion, called "the guiding center", is given by $\mathbf{G} + \mathbf{x}$.

We know that the radius is

$$r = \frac{mv}{q\frac{B}{c}}$$

or in appropriate units ($m = \frac{q}{c} = \hbar = 1$) $r = \frac{v}{B}$ so that

$$\mathbf{G} = \frac{-\hat{\mathbf{e}}_3 \times \mathbf{v}}{B}$$

Classically, $\mathbf{x} + \mathbf{G}$ is a constant of motion (obviously from the picture the centers of the circular motion are constant).

6.5.1. Exercise. In fact also quantum mechanically we have

$$[\mathcal{H}, x_i + G_i] = 0$$

Recall also that for the Landau levels we had degeneracy of $\frac{B}{2\pi}$ states per unit area (see 3.0.3). As a result,

$$\text{Tr}'(P_{\text{one Landau Level}}) = \frac{B}{2\pi}$$

where $P_{\text{one Landau Level}}$ is the projector onto one of the Landau levels. Thus

$$\text{Tr}'(P_\mu) = \frac{B}{2\pi} n$$

where n is the number of Landau levels with energy below μ .

Since $[\mathcal{H}, x_i + G_i] = 0$, we have $[P_\mu, x_i + G_i] = 0$. But that means $P_\mu(x_i + G_i)P_\mu^\perp = 0$ whence $P_\mu x_i P_\mu^\perp = -P_\mu G_i P_\mu^\perp$.

Note also that x_i is an “extensive” quantity in the sense that it grows as we take $L_i \rightarrow \infty$ and G_i is an “intensive” quantity, it does not change with L , since it is essentially the velocity.

Thus we start from (19) and make the replacement $x_i \rightarrow G_i$ to get:

$$\begin{aligned} \sigma_H &= -i\text{Tr}'(P_\mu G_1 P_\mu^\perp G_2 P_\mu - P_\mu G_2 P_\mu^\perp G_1 P_\mu) \\ &= -i\text{Tr}'(P_\mu G_1 (\mathbb{1} - P_\mu) G_2 P_\mu - P_\mu G_2 (\mathbb{1} - P_\mu) G_1 P_\mu) \\ &= -i\text{Tr}'(P_\mu G_1 G_2 P_\mu - P_\mu G_2 G_1 P_\mu - P_\mu G_1 P_\mu G_2 P_\mu + P_\mu G_2 P_\mu G_1 P_\mu) \\ &\stackrel{*}{=} -i\text{Tr}'\left(P_\mu [G_1, G_2] P_\mu - \underbrace{P_\mu G_1 P_\mu G_2 P_\mu + P_\mu G_2 P_\mu G_1 P_\mu}_{\text{o by cyclicity}}\right) \\ &= -i\text{Tr}'(P_\mu [G_1, G_2] P_\mu) \end{aligned}$$

we evaluate the commutator

$$\begin{aligned} [G_1, G_2] &= \frac{1}{B^2} [v_2, -v_1] \\ &= \frac{1}{B^2} [v_1, v_2] \\ &= \frac{1}{B^2} [\mathbf{p}_1 - \mathbf{A}_1, \mathbf{p}_2 - \mathbf{A}_2] \\ &= \frac{1}{B^2} i(\partial_1 \mathbf{A}_2 - \partial_2 \mathbf{A}_1) \\ &= \frac{i}{B} \end{aligned}$$

As a result we find

$$\begin{aligned} \sigma_H &= -i\text{Tr}'(P_\mu [G_1, G_2] P_\mu) \\ &= \frac{1}{B}\text{Tr}'(P_\mu P_\mu) \\ &= \frac{n}{2\pi} \end{aligned}$$

which is what we obtained previously in (9). Note that we can get different signs for σ_H by working with holes rather than electrons (thus changing the sign of q).

6.5.2. Remark. There is one transition in * where we used the cyclicity in order to argue that one term in the trace is zero. Actually this term should be

$$\begin{aligned} \lim_{L \rightarrow \infty} \frac{1}{L_1 L_2} \text{Tr}(X_L P_\mu G_1 P_\mu G_2 P_\mu) &= \lim_{L \rightarrow \infty} \frac{1}{L_1 L_2} \text{Tr}(P_\mu G_2 P_\mu X_L P_\mu G_1 P_\mu) \\ &= \lim_{L \rightarrow \infty} \frac{1}{L_1 L_2} \text{Tr}(X_L P_\mu G_2 P_\mu G_1 P_\mu) + \lim_{L \rightarrow \infty} \frac{1}{L_1 L_2} \text{Tr} \left(\underbrace{[P_\mu G_2 P_\mu, X_L]}_{\text{supported at bdry of rectangle}} \quad P_\mu G_1 P_\mu \right) \end{aligned}$$

and we may neglect the second surface term as it $L_1 L_2$ is much larger than it. The support is at the boundary because X_L only changes on the boundary and G is proportional to the velocity. Note that it was crucial to change from x_i to G_i because $x_i \propto L_i$ at the boundary and so we couldn't have neglected the surface term with x_i .

7. LAUGHLIN'S PUMP REVISITED

7.1. The Index of a Pair of Projections. There is another way to compute the Hall conductivity which is closely related to the argument of Laughlin's pump. In order to describe this, we need to introduce the mathematical concept of an index of a pair of projections, which was first developed in [6] and in [8].

In what follows, P and Q are two orthogonal (meaning self-adjoint) projections on a Hilbert space \mathcal{H} . Note we do *not* assume $Q = P^\perp$.

7.1.1. Definition. If P and Q are both of finite rank, define

$$\text{Ind}(P, Q) := \dim(\text{im}(P)) - \dim(\text{im}(Q))$$

which measures the difference in the “size” (i.e. rank) of the projections.

Our main goal is to extend this definition for the case when P and Q are not necessarily of finite rank. To this end, consider the following

7.1.2. Example. (Hilbert's Grand Hotel) Let $\mathcal{H} = l^2(\mathbb{N}_{\geq 1})$ and define two self-adjoint projections

$$P := \mathbb{1}$$

and

$$Q := \mathbb{1} - \delta_1 \langle \delta_1, \cdot \rangle$$

Then $\text{im}(P) = \mathcal{H}$ and $\text{im}(Q) \cong l^2(\mathbb{N}_{\geq 2})$. Even though strictly speaking 7.1.1 does not apply here because $\dim(\text{im}(P)) = \dim(\text{im}(Q)) = \infty$, intuitively it seems like the difference between the two projections, spanned by the vector δ_1 , should result in an index of 1. Perhaps

$$\begin{aligned} \text{Ind}(P, Q) &\stackrel{?}{=} \text{Tr}(P - Q) \\ &= 1 \end{aligned}$$

Indeed when 7.1.1 does apply then

$$\begin{aligned} \text{Tr}(P - Q) &= \sum_{n \in \{ n \in \mathbb{N} \mid \varphi_n \in \text{im}(P) \cup \text{im}(Q) \}} \langle \varphi_n, (P - Q) \varphi_n \rangle \\ &= \sum_{n \in \{ n \in \mathbb{N} \mid \varphi_n \in \text{im}(P) \cup \text{im}(Q) \}} \langle \varphi_n, P \varphi_n \rangle - \sum_{n \in \{ n \in \mathbb{N} \mid \varphi_n \in \text{im}(P) \cup \text{im}(Q) \}} \langle \varphi_n, Q \varphi_n \rangle \\ &= \dim(\text{im}(P)) - \dim(\text{im}(Q)) \end{aligned}$$

More generally we make the

7.1.3. Definition. Whenever the right hand side of the following equation is finite, we define

$$\boxed{\text{Ind}(P, Q) := \dim(\text{im}(P) \cap \ker(Q)) - \dim(\ker(P) \cap \text{im}(Q))} \quad (20)$$

7.1.4. Claim. 7.1.3 agrees with 7.1.1 when both are defined.

Proof. Since P and Q are finite rank, we may use the rank-nullity theorem, which says that if $T : V \rightarrow W$ is a linear map between two finite vector spaces then

$$\dim(\ker(T)) = \dim(V) - \dim(\text{im}(T))$$

Apply this on the map $\tilde{Q} : \text{im}(P) \rightarrow \text{im}(Q)$ given by $\psi \mapsto Q(\psi)$. Then

$$\begin{aligned} \ker(\tilde{Q}) &\equiv \{ P\psi \in \mathcal{H} \mid QP\psi = 0 \} \\ &= \{ \psi \in \mathcal{H} \mid \psi \in \text{im}(P) \cap \ker(Q) \} \\ &= \text{im}(P) \cap \ker(Q) \end{aligned}$$

and the rank-nullity theorem on \tilde{Q} gives

$$\begin{aligned} \dim(\text{im}(P) \cap \ker(Q)) &\stackrel{\text{R.N.T.}}{=} \dim(\text{im}(P)) - \dim(\text{im}(\tilde{Q})) \\ &= \dim(\text{im}(P)) - \dim(\text{im}(QP)) \end{aligned}$$

Next, define $\tilde{P} : \text{im}(QP) \rightarrow \text{im}(P)$ by $\psi \mapsto P(\psi)$.

Note that \tilde{P} is injective:

$$\begin{aligned}
 \psi &\in \ker(\tilde{P}) \\
 &\Downarrow \\
 P Q P \varphi &= 0 \text{ for some } \varphi \text{ such that } Q P \varphi = \psi \\
 &\Downarrow \\
 \langle \varphi, P Q P \varphi \rangle &= 0 \\
 &\Downarrow \\
 \langle \varphi, P Q Q P \varphi \rangle &= 0 \\
 &\Downarrow \\
 \langle Q P \varphi, Q P \varphi \rangle &= 0 \\
 &\Downarrow \\
 \psi &= 0
 \end{aligned}$$

then we have

$$\underbrace{\dim(\ker(\tilde{P}))}_{0} = \dim(\text{im}(Q P)) - \dim\left(\underbrace{\text{im}(\tilde{P})}_{\text{im}(P Q P)}\right)$$

so that

$$\begin{aligned}
 \dim(\text{im}(Q P)) &= \dim(\text{im}(P Q P)) \\
 &\leq \dim(\text{im}(P Q))
 \end{aligned}$$

and similarly for the opposite direction so that $\dim(\text{im}(P Q)) = \dim(\text{im}(Q P))$. Thus we have

$$\begin{aligned}
 \dim(\text{im}(P) \cap \ker(Q)) - \dim(\text{im}(Q) \cap \ker(P)) &= \dim(\text{im}(P)) - \dim(\text{im}(Q P)) - \dim(\text{im}(Q)) - \dim(\text{im}(P Q)) \\
 &= \dim(\text{im}(P)) - \dim(\text{im}(Q))
 \end{aligned}$$

□

7.1.5. Claim. $\text{im}(P) \cap \ker(Q) = \ker(P - Q - \mathbb{1})$.

Proof. [≤] If $\psi \in \text{im}(P) \cap \ker(Q)$ then $P\psi = \psi$ and $Q\psi = 0$. Then

$$\begin{aligned}
 (P - Q - \mathbb{1})\psi &= \psi - \psi \\
 &= 0
 \end{aligned}$$

as desired.

[⊇] Assume $(P - Q - \mathbb{1})\psi = 0$. Then

$$\begin{aligned}
 \langle \psi, (P - Q - \mathbb{1})\psi \rangle &= 0 \\
 -\langle \psi, Q\psi \rangle + \langle \psi, (\mathbb{1} - P)\psi \rangle &= 0 \\
 \langle \psi, Q\psi \rangle + \langle \psi, P^\perp\psi \rangle &= 0 \\
 \langle Q\psi, Q\psi \rangle + \langle P^\perp\psi, P^\perp\psi \rangle &= 0 \\
 \|Q\psi\|^2 + \|P^\perp\psi\|^2 &= 0
 \end{aligned}$$

since the last two terms are both non-negative, they must be separately equal to zero and the result follows. □

7.1.6. Corollary. We can now rewrite the index as

$$\begin{aligned}
 \text{Ind}(P, Q) &= \dim(\ker(P - Q - \mathbb{1})) - \dim(\ker(Q - P - \mathbb{1})) \\
 &= \dim(\ker(P - Q - \mathbb{1})) - \dim(\ker(P - Q + \mathbb{1}))
 \end{aligned}$$

In words, the index is the difference in the multiplicity of the eigenvalue 1 of $P - Q$ with the multiplicity of the eigenvalue -1 of $P - Q$.

7.1.7. Claim. The index is stable: If $\|P - Q\| < 1$ then

$$\text{Ind}(P, Q) = 0$$

Proof. Let $\psi \in \ker(P - Q - \mathbb{1})$. Then $\psi = (P - Q)\psi$. So

$$\begin{aligned}
 \|\psi\| &= \|(P - Q)\psi\| \\
 &\leq \|P - Q\|\|\psi\| \\
 &< 1 \cdot \|\psi\| \\
 &< \|\psi\|
 \end{aligned}$$

and we must conclude that $\psi = 0$, so that $\ker(P - Q - \mathbb{1}) = 0$. Similarly $\ker(P - Q + \mathbb{1}) = 0$. \square

7.1.8. Claim. *The index is unitary invariant: If U is a unitary map then*

$$\text{Ind}(P, Q) = \text{Ind}(UPU^*, UQU^*)$$

Proof. Let U be a unitary transformation. Define $A := P - Q$.

$$\text{Ind}(P, Q) = \dim(\ker(A - \mathbb{1})) - \dim(\ker(A + \mathbb{1}))$$

and

$$\begin{aligned} \text{Ind}(UPU^*, UQU^*) &= \dim(\ker(UPU^* - UQU^* - \mathbb{1})) - \dim(\ker(UPU^* - UQU^* + \mathbb{1})) \\ &= \dim(\ker(U(A - \mathbb{1})U^*)) - \dim(\ker(U(A + \mathbb{1})U^*)) \end{aligned}$$

But

$$\begin{array}{ccc} \psi & \in & \ker(U(A - \mathbb{1})U^*) \\ \downarrow & & \downarrow \\ U(A - \mathbb{1})U^*\psi & = & 0 \\ \downarrow & & \downarrow \\ (A - \mathbb{1})U^*\psi & = & 0 \\ \downarrow & & \downarrow \\ U^*\psi & \in & \ker(A - \mathbb{1}) \end{array}$$

where we have used the fact that U is an isomorphism. Since it is, the dimension of the kernel may be evaluated in the transformed space $U^*\mathcal{H}$ instead of \mathcal{H} to yield the same number. The same goes for $A + \mathbb{1}$. \square

7.1.9. Claim. *If P and Q are as above then*

$$\begin{aligned} (P - Q)^2 P &= P(P - Q)^2 \\ &= P - PQP \end{aligned}$$

so that

$$[(P - Q)^2, P] = 0$$

and similarly

$$[(P - Q)^2, Q] = 0$$

Proof. We have

$$\begin{aligned} [(P - Q)^2, P] &\equiv (P - Q)^2 P - P(P - Q)^2 \\ &= (P + Q - PQ - QP)P - P(P + Q - PQ - QP) \\ &= P + QP - PQP - QP - P - PQ + PQ + PQP \\ &= P - PQP - P + PQP \\ &= 0 \end{aligned}$$

by symmetry the same holds for Q . \square

7.1.10. Claim. *If $(P - Q)^{2n_0+1} \in \mathcal{J}_1(\mathcal{H})$ for some $n_0 \in \mathbb{N}_{\geq 0}$ then*

$$(P - Q)^{2n+1} \in \mathcal{J}_1(\mathcal{H})$$

and

$$\text{Tr}((P - Q)^{2n+1}) = \text{Tr}((P - Q)^{2n_0+1})$$

for all $n \geq n_0$.

Proof. From 7.1.9 we have

$$\begin{cases} (P - Q)^{2n_0+2} P = (P - Q)^{2n_0} (P - PQP) \\ (P - Q)^{2n_0+2} Q = (P - Q)^{2n_0} (Q - QPQ) \end{cases}$$

subtracting the two we obtain

$$(P - Q)^{2n_0+3} = (P - Q)^{2n_0+1} - (P - Q)^{2n_0} [PQ, QP] \quad (21)$$

Next note that

$$\begin{aligned}
 [PQ, QP] &= PQP - QPQ \\
 &= PQQP - QPPP - PQPQ + PQPQ \\
 &= [PQ, [Q, P]] \\
 &= [PQ, QP - Q^2 - PQ + Q^2] \\
 &= [PQ, [Q, P - Q]]
 \end{aligned}$$

so that

$$\begin{aligned}
 (P - Q)^{2n_0} [PQ, QP] &= (P - Q)^{2n_0} [PQ, [Q, P - Q]] \\
 &\stackrel{*}{=} [PQ, [Q, (P - Q)^{2n_0+1}]]
 \end{aligned}$$

where in $*$ we have used 7.1.9.

Now, $\text{Tr}([PQ, [Q, (P - Q)^{2n_0+1}]]) = 0$ because:

- (1) $(P - Q)^{2n_0+1} \in J_1(\mathcal{H})$.
- (2) $\|Q\| = 1$ so that Q is bounded.
- (3) $[Q, (P - Q)^{2n_0+1}] \in J_1(\mathcal{H})$ as $J_1(\mathcal{H})$ is a two-sided ideal in $\mathcal{B}(\mathcal{H})$.
- (4) PQ is bounded as $\|PQ\| \leq \|P\|\|Q\| = 1$.
- (5) Hence $PQ [Q, (P - Q)^{2n_0+1}]$ and $[Q, (P - Q)^{2n_0+1}] PQ$ are trace-class, again as $J_1(\mathcal{H})$ is a two-sided ideal in $\mathcal{B}(\mathcal{H})$.
- (6) Thus we obtain

$$\begin{aligned}
 \text{Tr}([PQ, [Q, (P - Q)^{2n_0+1}]]) &= \text{Tr}(PQ [Q, (P - Q)^{2n_0+1}] - [Q, (P - Q)^{2n_0+1}] PQ) \\
 &= \text{Tr}(PQ [Q, (P - Q)^{2n_0+1}]) - \text{Tr}([Q, (P - Q)^{2n_0+1}] PQ) \\
 &\stackrel{\text{**}}{=} \text{Tr}(PQ [Q, (P - Q)^{2n_0+1}]) - \text{Tr}(PQ [Q, (P - Q)^{2n_0+1}]) \\
 &= 0
 \end{aligned}$$

where in $**$ we have used 6.3.13. □

7.1.11. Claim. (Avron, Seiler, Simon in [8]) If $\exists n_0 \in \mathbb{N}_{\geq 0}$ such that $(P - Q)^{2n_0+1} \in J_1(\mathcal{H})$ then

$$\text{Ind}(P, Q) = \text{Tr}((P - Q)^{2n_0+1})$$

Proof. Writing the trace in the eigenbasis of $P - Q$ (which exists because $P - Q$ is compact by assumption) we get

$$\text{Tr}((P - Q)^{2n+1}) = \sum_{\lambda} \lambda^{2n+1} m_{\lambda}$$

where λ is an eigenvalue of $P - Q$, and m_{λ} is its multiplicity. Next note that $\|P - Q\| \leq 1$ so that $\sigma(P - Q) \subseteq [-1, 1]$ and so in the limit $n \rightarrow \infty$, all eigenvalues which are strictly smaller than 1 in their absolute value converge to zero, and we are left with

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \text{Tr}((P - Q)^{2n+1}) &= (+1)m_{+1} + (-1)m_{-1} \\
 &= m_1 - m_{-1} \\
 &\equiv \text{Ind}(P, Q)
 \end{aligned}$$

but on the other hand, 7.1.10 has showed us that $\text{Tr}((P - Q)^{2n+1}) = \text{Tr}((P - Q)^{2n_0+1})$ for any $n \geq n_0$ so that the left hand side becomes

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \text{Tr}((P - Q)^{2n+1}) &= \lim_{n \rightarrow \infty} \text{Tr}((P - Q)^{2n_0+1}) \\
 &= \text{Tr}((P - Q)^{2n_0+1})
 \end{aligned}$$

as desired.

The proof is now complete. For the record, we also recount here the other proof which was presented during the lecture.

Start by defining:

$$A := P - Q$$

and

$$\begin{aligned}
 B &:= (\mathbb{1} - P) - Q \\
 &= \mathbb{1} - (P + Q)
 \end{aligned}$$

Claim. $A^2 + B^2 = \mathbb{1}$

Proof. First compute

$$\begin{aligned} A^2 &= P^2 + Q^2 - PQ - QP \\ &= P + Q - \{P, Q\} \end{aligned}$$

and then

$$\begin{aligned} B^2 &= (\mathbb{1} - P)^2 + Q^2 - (\mathbb{1} - P)Q - Q(\mathbb{1} - P) \\ &= (\mathbb{1} - P) + Q - \{\mathbb{1} - P, Q\} \\ &= (\mathbb{1} - P) + Q - 2Q + \{P, Q\} \\ &= \mathbb{1} - P - Q + \{P, Q\} \end{aligned}$$

□

Claim. $\{A, B\} = 0$

Proof. We start with

$$\begin{aligned} \{A, \mathbb{1}\} &= \{P - Q, \mathbb{1}\} \\ &= 2(P - Q) \end{aligned}$$

and

$$\begin{aligned} \{A, P + Q\} &= \{P - Q, P + Q\} \\ &= (P - Q)(P + Q) + (P + Q)(P - Q) \\ &= 2P - 2Q \end{aligned}$$

so that

$$\begin{aligned} \{A, B\} &= \{A, \mathbb{1} - (P + Q)\} \\ &= \{A, \mathbb{1}\} - \{A, P + Q\} \\ &= 0 \end{aligned}$$

as desired.

□

Next define the multiplicity of the eigenvalue λ of A (possibly zero):

$$m_\lambda := \dim(\ker(A - \lambda)) \quad \forall \lambda \in \mathbb{C}$$

Claim. $m_\lambda = m_{-\lambda}$ for all $\lambda \notin \{\pm 1\}$.

Proof. Let $\lambda \notin \{\pm 1\}$ be given. Define $\tilde{B} : \ker(A - \lambda) \rightarrow \ker(A + \lambda)$ by $\tilde{B}\psi := B\psi$. We want to show \tilde{B} is a bijection. So let $\psi \in \ker(A - \lambda)$ be given.

Using the fact that $\{A, B\} = 0$ we have

$$\begin{aligned} (A + \lambda)B\psi &= -B(A - \lambda)\psi \\ &= 0 \end{aligned}$$

so that $B\psi \in \ker(A + \lambda)$ and \tilde{B} is well-defined.

Assume $\ker(\tilde{B}) \neq \{0\}$. Then $\exists \psi \in \ker(A - \lambda) \setminus \{0\}$ such that $B\psi = 0$. Hence $B^2\psi = 0$. But $B^2 = \mathbb{1} - A^2$ so that

$$\begin{aligned} B^2\psi &= (\mathbb{1} - A^2)\psi \\ &= (1 - \lambda^2)\psi \end{aligned}$$

as $\psi \in \ker(A - \lambda)$. Since $\lambda \notin \{\pm 1\}$ then $\psi = 0$, which is a contradiction. We must conclude that $\ker(\tilde{B}) = \{0\}$.

Finally let $\psi \in \ker(A + \lambda)$ be given. Then from the above $B\psi \in \ker(A - \lambda)$ so that $\frac{B\psi}{1 - \lambda^2} \in \ker(A - \lambda)$, which is well-defined as $\lambda \notin \{\pm 1\}$. Then

$$\begin{aligned} \tilde{B}\left(\frac{B\psi}{1 - \lambda^2}\right) &= \frac{B^2\psi}{1 - \lambda^2} \\ &= \frac{(1 - A^2)\psi}{1 - \lambda^2} \\ &= \psi \end{aligned}$$

so that \tilde{B} is surjective.

□

Finally we can compute $\text{Tr}((P - Q)^{2n_0+1})$. Let $\{\varphi_l\}_{l \in \mathbb{N}}$ be an eigenbasis of $P - Q$ with eigenvalues $\{\lambda_l\}_{l \in \mathbb{N}}$. Then we have

$$\begin{aligned}\text{Tr}((P - Q)^{2n_0+1}) &= \text{Tr}(A^{2n_0+1}) \\ &= \sum_{l \in \mathbb{N}} \langle \varphi_l, A^{2n_0+1} \varphi_l \rangle \\ &= \sum_{l \in \mathbb{N}} \langle \varphi_l, \lambda_l^{2n_0+1} \varphi_l \rangle \\ &= \sum_{l \in \mathbb{N}} \lambda_l^{2n_0+1} \\ &= \sum_{\lambda^{2n_0+1} \in \sigma(A^{2n_0+1})} \lambda^{2n_0+1} m_\lambda \\ &= \sum_{\lambda^{2n_0+1} \in \sigma(A^{2n_0+1}) \wedge \lambda^{2n_0+1} > 0} \lambda^{2n_0+1} (m_\lambda + m_{-\lambda}) \\ &= m_1 - m_{-1} \\ &\equiv \text{Ind}(P, Q)\end{aligned}$$

□

7.1.12. Remark. We were able to do all the above manipulations with an eigenbasis of $P - Q$ because $P - Q$ was compact, which follows from the fact that $(P - Q)^{2n_0+1} \in \mathcal{J}_1(\mathcal{H})$. However, it should be noted that it is possible that $\text{Ind}(P, Q)$ is defined via 7.1.3 even when $(P - Q)^{2n+1} \notin \mathcal{J}_1(\mathcal{H})$ for any $n \in \mathbb{N}_{\geq 0}$ (but such cases are exotic).

7.1.13. Claim. If P and Q are two orthogonal projections such that $\text{Ind}(P, Q)$ is defined, then

$$\text{Ind}(P, Q) = \text{Ind}(C_{QP})$$

where $C_{QP} : \text{im}(P) \rightarrow \text{im}(Q)$ is given by $\psi \mapsto QP\psi$ for all $\psi \in \text{im}(P)$ and the right hand side of the equation is the Fredholm index of the Fredholm operator C_{QP} . Thus we obtain an expression for the index of a pair of projections as the Fredholm index of some operator.

Proof. This is from [6] (proposition 3.1):

Note the definition (where we use $\text{coker}(F) \simeq \ker(F^*)$)

$$\text{Ind}(C_{QP}) \equiv \dim(\ker(C_{QP})) - \dim(\ker(C_{QP}^*))$$

But

$$\begin{aligned}\ker(C_{QP}) &= \{\psi \in \text{im}(P) : QP\psi = 0\} \\ &= \{\psi \in \text{im}(P) : Q\psi = 0\} \\ &= \ker(Q) \cap \text{im}(P) \\ &= \ker(P - Q - 1\mathbb{I})\end{aligned}$$

where in the last step we have used 7.1.5 and

$$\begin{aligned}\ker(C_{QP}^*) &= \{\psi \in \text{im}(Q)^* : (QP)^* \psi = 0\} \\ &= \{\psi \in \text{im}(Q) : PQ\psi = 0\} \\ &= \{\psi \in \text{im}(Q) : P\psi = 0\} \\ &= \ker(P) \cap \text{im}(Q) \\ &= \ker(P - Q + 1\mathbb{I})\end{aligned}$$

where we used the fact Q and P are self-adjoint.

□

7.1.14. Corollary. If U is a unitary map then

$$\text{Ind}(P, UPU^*) = \text{Ind}_{\text{im}(P) \rightarrow \text{im}(P)}(PU^*P)$$

Proof. From 7.1.13 we have

$$\text{Ind}(P, UPU^*) = \text{Ind}_{\text{im}(P) \rightarrow \text{im}(UPU^*)}(UPU^*P)$$

But U is a bijection, and is thus Fredholm with index 0. So $\text{im}(UPU^*) \simeq \text{im}(P)$ and $\text{Ind}(U) = 0$. In addition, the Fredholm index is additive under composition of maps so that we get the result. □

7.1.15. Claim. *The index is additive: For any self-adjoint projections P , Q and R we have*

$$\text{Ind}(P, R) = \text{Ind}(P, Q) + \text{Ind}(Q, R)$$

assuming either $P - Q$ or $Q - R$ are compact.

Proof. Using 7.1.13 we have to prove is:

$$\text{Ind}(C_{RP} : \text{im}(P) \rightarrow \text{im}(R)) = \text{Ind}(C_{QP} : \text{im}(P) \rightarrow \text{im}(Q)) + \text{Ind}(C_{RQ} : \text{im}(Q) \rightarrow \text{im}(R))$$

we know that the Fredholm index is additive under composition (using the “snake lemma” for instance, see [11] pp. 6) so that

$$\text{Ind}(C_{QP} : \text{im}(P) \rightarrow \text{im}(Q)) + \text{Ind}(C_{RQ} : \text{im}(Q) \rightarrow \text{im}(R)) = \text{Ind}(C_{RQ} \circ C_{QP} : \text{im}(P) \rightarrow \text{im}(R))$$

but $C_{RP} : \text{im}(P) \rightarrow \text{im}(R)$ is given by

$$\text{im}(P) \ni \psi \mapsto RP\psi$$

and $C_{RQ} \circ C_{QP} : \text{im}(P) \rightarrow \text{im}(R)$ is given by

$$\text{im}(P) \ni \psi \mapsto RQP\psi$$

so that their difference map is $(C_{RP} - C_{RQ} \circ C_{QP}) : \text{im}(P) \rightarrow \text{im}(R)$ is given by

$$\begin{aligned} \text{im}(P) \ni \psi &\mapsto RP\psi - RQP\psi \\ &= R(1 - Q)P\psi \\ &= R(P - Q)P\psi \end{aligned}$$

but we have assumed $P - Q$ to be compact, and the product of compact and bounded operators is again compact, which means the difference map $(C_{RP} - C_{RQ} \circ C_{QP}) : \text{im}(P) \rightarrow \text{im}(R)$ is compact. Now we can use a well-known result ([11] pp. 37 Exercise 7) which says that if two maps differ by a compact map then their Fredholm index is the same. \square

7.1.16. Claim. *If U is unitary such that $U - 1$ is compact then*

$$\text{Ind}(P, UPU^*) = 0$$

Proof. We use the representation via the Fredholm index to get

$$\begin{aligned} \text{Ind}(P, UPU^*) &= -\text{Ind}(PUP) \\ &= -\text{Ind}\left(PUP - \underbrace{P(U-1P)}_{\text{cpt.}}\right) \\ &= \text{Ind}(P) \\ &= \dim(\ker(P)) - \dim(\ker(P^*)) \\ &= 0 \end{aligned}$$

\square

7.1.17. Claim. *If U_1 and U_2 are two unitaries such that $U_1 - U_2$ is compact then*

$$\text{Ind}(U_1PU_1^*, U_2PU_2^*) = 0$$

Proof. Using the additivity we have

$$\text{Ind}(U_1PU_1^*, U_2PU_2^*) = \text{Ind}(U_1PU_1^*, P) + \text{Ind}(P, U_2PU_2^*)$$

\square

7.1.18. Claim. *If U is unitary such that $\|U - 1\| < \frac{1}{2}$ then*

$$\text{Ind}(P, UPU^*) = 0$$

Proof. We can write

$$\begin{aligned}
 \|P - UPU^*\| &= \|(\mathbb{1} - U)P + UP(\mathbb{1} - U^*)\| \\
 &\leq \|UP(\mathbb{1} - U^*)\| + \|(\mathbb{1} - U)P\| \\
 &\leq \underbrace{\|U\|}_{\leq 1} \underbrace{\|P\|}_{\leq 1} \|\mathbb{1} - U^*\| + \|\mathbb{1} - U\| \underbrace{\|P\|}_{\leq 1} \\
 &= \|\mathbb{1} - U^*\| + \|\mathbb{1} - U\| \\
 &< 1
 \end{aligned}$$

and now use 7.1.7. □

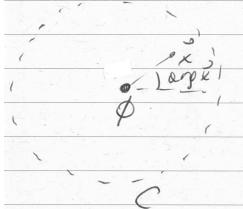
7.1.19. Claim. If U is unitary and has an eigenbasis, and $(P - UPU^*) \in \mathcal{J}_1(\mathcal{H})$ then $\text{Ind}(P, UPU^*) = 0$.

Proof. Since $(P - UPU^*) \in \mathcal{J}_1(\mathcal{H})$ we can use 7.1.11 with $n_0 = 0$ in this case. So if $\{\varphi_n\}_{n \in \mathbb{N}}$ is the eigenbasis of U with eigenvalues $\{\lambda_n\}_{n \in \mathbb{N}}$ then

$$\begin{aligned}
 \text{Ind}(P, UPU^*) &= \text{Tr}(P - UPU^*) \\
 &= \sum_{n \in \mathbb{N}} \langle \varphi_n, (P - UPU^*) \varphi_n \rangle \\
 &= \sum_{n \in \text{some smaller set}} \left\langle \varphi_n, \left(1 - |\lambda_n|^2\right) \varphi_n \right\rangle \\
 |\lambda_n| = 1 &\quad 0
 \end{aligned}$$
□

7.2. The Hall Conductivity via Laughlin's Pump and the index of a Pair of Projections. We now want to apply the “machinery” of the previous section to calculate the Hall conductivity for a generic system. This exercise is thus parallel to the one where we have used the Kubo formula, and later on we shall prove the two formulations are in fact equal. The material in this section may also be found in [19].

We will use again the same idea of the Laughlin pump, but in a slightly different geometry:



We now have a sample which is an infinite two-dimensional plane, with magnetic field which is perpendicular to it, and the electric field will now be *radial* into the origin. To produce an electric field, we imagine there is a magnetic flux which is increased from 0 to ϕ which passes through the origin perpendicular to the plane. This change in flux produces an electric field which is radial inwards into the origin. Then the charge Q traversing the fiducial dashed-line C inwards is given by

$$Q = \sigma_H \phi$$

where we use units of $c = \hbar = 1$.

The flux quantum is now $\phi = 2\pi$.

The flux is generated by a gauge potential \mathbf{A} such that

$$\oint_C \mathbf{A} \cdot d\mathbf{s} = \phi$$

One possible choice is $\mathbf{A} = \nabla \left(\frac{\phi}{2\pi} \arg(\mathbf{x}) \right) = \nabla \chi$ where $\chi \equiv \frac{\phi}{2\pi} \arg(\mathbf{x})$. Now observe that if χ were single valued, then a gauge of $\mathbf{A} = 0$ is equivalent to $\mathbf{A} = \nabla \chi$ and the Hamiltonian H is equivalent to UHU^* with U a unitary map given by $U = \exp(i\chi)$. Note that even though χ is not single valued, U is single valued when $\phi = 2\pi$.

7.2.1. Claim. If C is the circle at infinity then the Hall conductivity is given by

$$\sigma_H = \frac{1}{2\pi} \text{Ind}(P, UPU^*)$$

(22)

where U is as above with $\phi = 2\pi$ (so $U = e^{i\arg(\mathbf{x})} = \frac{x_1 + ix_2}{|x_1 + ix_2|}$) and P is the Fermi projection of the system with $\phi = 0$. Note that in our convention, $\text{Tr}(P)$ is the number of particles, so that $\text{Tr}(P)$ is infinite.

Proof. Note that because H and UHU^* are the Hamiltonians of the systems with and without magnetic flux, UPU^* is the Fermi projection for the system with $\phi = 2\pi$.

Now assume that ϕ is time dependent and increases from time $t = 0$ to time $t = t_0$ such that $\phi(0) = 0$ and $\phi(t_0) = 2\pi$. Let $\tilde{U}(t, 0)$ be the propagator (time evolution operator) of a state from time 0 to time t for some $t \in [0, t_0]$ of the system described by H .

Then because the index of a pair of projections “somehow” measures the difference between the two of them (the difference in the number of states they project onto, their rank), it follows that

$$\text{Ind}(\tilde{U}(t_0, 0)P\tilde{U}(t_0, 0)^*, UPU^*)$$

measures how many states (that is, how much charge) moved from the circle C at infinity as t went from 0 to t_0 , $\tilde{U}(t_0, 0)P\tilde{U}(t_0, 0)^*$ being the time-evolved Fermi sea and UPU^* being the Fermi sea which had no time evolution and with flux of $\phi = 2\pi$ to begin with. So we write

$$\begin{aligned} Q &= \text{Ind}(\tilde{U}(t_0, 0)P\tilde{U}(t_0, 0)^*, UPU^*) \\ \text{additivity} &= \text{Ind}(\tilde{U}(t_0, 0)P\tilde{U}(t_0, 0)^*, P) + \text{Ind}(P, UPU^*) \\ &= \text{Ind}(\tilde{U}(t, 0)P\tilde{U}(t, 0)^*, P)|_{t=t_0} + \text{Ind}(P, UPU^*) \end{aligned}$$

Since \tilde{U} is continuous in t , $f(t) := \text{Ind}(\tilde{U}(t, 0)P\tilde{U}(t, 0)^*, P)$ is a continuous function of t which is integer valued and 0 for $t = 0$. Thus it must be zero at all other times. Thus

$$Q = \text{Ind}(P, UPU^*)$$

so that

$$\sigma_H = \frac{1}{2\pi} \text{Ind}(P, UPU^*)$$

as desired. \square

7.2.2. Remark. Note that our way to associate the charge that moved inward into the circle at infinity with

$$\text{Ind}(\tilde{U}(t_0, 0)P\tilde{U}(t_0, 0)^*, UPU^*)$$

is not entirely well-founded, because there is no concrete physical reason to associate the index of a pair of projections with the number of states differing between them. All we have done in the discussion about an index of pair of projections is merely show that when both notions are defined (i.e. in the finite case) they agree. However in the infinite case it is not completely clear what the index of pair of projections exactly measures. Thus the more cautious way to show the claim would be to work with a *finite* circle C , get some expression for the charge that passed into the circle, and only in the end take the limit of the radius of C going to infinity. In the end one would get that this limit should converge to

$$\text{Ind}(P, UPU^*)$$

7.2.3. Remark. One might wonder why was it necessary to assume that the electric field is small (so that we could use perturbation theory) whereas in the above proof there was no mentioning of how big or small the electric field (that is, the rate of change of the flux ϕ) has to be for the derivation to be valid. It appears that if one were to really follow through with the purist derivation which is suggested in 7.2.2, then for the calculations to go through with a *finite* radius for C , it must be necessary for $\dot{\phi}$ to be small for the adiabatic approximation to hold.

7.2.4. Claim. $(P - UPU^*)^3 \in \mathcal{J}_1(\mathcal{H})$

Proof. First note that $(P - UPU^*) \notin \mathcal{J}_1(\mathcal{H})$. This can be substantiated from the following heuristic argument (this is not a proof yet):

$$\begin{aligned} \langle \delta_n, (P - UPU^*) \delta_m \rangle &= \langle \delta_n, P \delta_m \rangle - \langle \delta_n, UPU^* \delta_m \rangle \\ &= \langle \delta_n, P \delta_m \rangle - \left\langle e^{-i \arg(n)} \delta_n, P e^{-i \arg(m)} \delta_m \right\rangle \\ &= \langle \delta_n, P \delta_m \rangle \left(1 - e^{i(\arg(n) - \arg(m))} \right) \\ &= \langle \delta_n, P \delta_m \rangle \left(1 - e^{i\angle(m, 0, n)} \right) \end{aligned}$$

since we are inquiring about the question if this object is trace class or not, we consider the case when $\|n\|$ is large. Recall that $\langle \delta_n, P \delta_m \rangle$ decays exponentially with $\|m - n\|$. So that we get

$$\langle \delta_n, (P - UPU^*) \delta_m \rangle \xrightarrow{\|n\| \text{ large}, \|m - n\| \text{ small}} \langle \delta_n, P \delta_m \rangle (-i\angle(m, 0, n))$$

when $\|m - n\|$ is held fixed, $\angle(m, 0, n) \propto \mathcal{O}\left(\frac{1}{\|n\|}\right)$. However, this is not summable in two dimensions. Similarly one would find for the third power that the matrix element is $\propto \mathcal{O}\left(\frac{1}{\|n\|^3}\right)$, which is summable.

The reason this is not a proof is that, as in 6.3.7, in order to compute the trace, one first has to ascertain that

$$\text{Tr}(|P - UPU^*|) < \infty$$

which we have *not* done, so it is illegitimate to compute $\sum_{n \in \mathbb{Z}^2} \langle \delta_n, (P - UPU^*) \delta_n \rangle$ which happens to be equal to zero (but this sum is *not* the trace, because, again, the trace is not defined). This is very similar to the situation in 6.3.14.

In order to properly prove that $(P - UPU^*) \notin \mathcal{J}_1(\mathcal{H})$, one would have to show that $\sum_{n \in \mathbb{Z}^2} \langle \delta_n, |P - UPU^*| \delta_n \rangle$ does not converge. One way to see that is to explicitly compute $|P - UPU^*|$. Another way is to compute the Hilbert-Schmidt norm

of $P - UPU^*$. If this is infinite, then the trace-class norm is also infinite, as it is known that the Hilbert-Schmidt norm is smaller than the trace class norm:

$$\begin{aligned}
\|P - UPU^*\|_{HS}^2 &\equiv \text{Tr}((P - UPU^*)(P - UPU^*)^*) \\
&= \text{Tr}((P - UPU^*)^2) \\
&= \sum_{n \in \mathbb{Z}^2} \langle \delta_n, (P - UPU^*)^2 \delta_n \rangle \\
&= \sum_{n \in \mathbb{Z}^2} \sum_{m \in \mathbb{Z}^2} \langle \delta_n, (P - UPU^*) \delta_m \rangle \langle \delta_m, (P - UPU^*) \delta_n \rangle \\
&= \sum_{n \in \mathbb{Z}^2} \sum_{m \in \mathbb{Z}^2} \langle \delta_n, (P - UPU^*) \delta_m \rangle \langle (P - UPU^*) \delta_m, \delta_n \rangle \\
&= \sum_{n \in \mathbb{Z}^2} \sum_{m \in \mathbb{Z}^2} |\langle \delta_n, (P - UPU^*) \delta_m \rangle|^2 \\
&= \sum_{n \in \mathbb{Z}^2} \sum_{m \in \mathbb{Z}^2} |\langle \delta_n, P \delta_m \rangle (1 - e^{i(\arg(n) - \arg(m))})|^2 \\
&= \sum_{n \in \mathbb{Z}^2} \sum_{m \in \mathbb{Z}^2} |\langle \delta_n, P \delta_m \rangle|^2 4 \left[\sin\left(\frac{1}{2}\angle(\mathbf{m}, \mathbf{0}, \mathbf{n})\right) \right]^2
\end{aligned}$$

and now we obtain something which is proportional to $\mathcal{O}\left(\frac{1}{\|\mathbf{n}\|^2}\right)$, again, not summable in two-dimensions. \square

7.2.5. Claim. *If H is invariant under the reversal of time, then $\sigma_H = 0$.*

Proof. Time-reversal is implemented on the Hilbert space as a map $\theta : \mathcal{H} \rightarrow \mathcal{H}$ such that:

- (1) θ is an anti- \mathbb{C} -linear, anti-unitary map.
- (2) $\theta^2 = -1$ (because we deal with Fermions).

And the invariance under time-reversal implies $[H, \theta] = 0$. Thus the Fermi projection also obeys $[P, \theta] = 0$, that is, $P = \theta P \theta^{-1}$. Then we have

$$\begin{aligned}
\sigma_H &= \frac{1}{2\pi} \text{Ind}(P, UPU^*) \\
&= \frac{1}{2\pi} \text{Ind}(\theta P \theta^{-1}, U \theta P \theta^{-1} U^*) \\
&\stackrel{*}{=} \frac{1}{2\pi} \text{Ind}(\theta P \theta^{-1}, \theta U^* P U \theta^{-1}) \\
&= \frac{1}{2\pi} \text{Ind}(P, U^* P U) \\
&= \frac{1}{2\pi} \text{Ind}(UPU^*, P) \\
&= -\sigma_H
\end{aligned}$$

where in * we have used the fact that

$$\begin{aligned}
\theta U &= \theta e^{i \arg(x)} \\
&= e^{-i \arg(x)} \theta \\
&= U^* \theta
\end{aligned}$$

which follows from the anti- \mathbb{C} -linearity of θ . \square

7.2.6. Claim. *Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a bounded odd function such that*

$$g(\alpha) = \alpha + \mathcal{O}(\alpha^3) \quad (23)$$

near $\alpha = 0$, $\mathbf{u}^{(i)} \in \mathbb{Z}^2$ be given for all $i \in \{1, 2, 3\}$. Then

$$\sum_{\mathbf{p} \in \mathbb{Z}^{2*}} \left(g\left(\angle(\mathbf{u}^{(2)}, \mathbf{p}, \mathbf{u}^{(3)})\right) + g\left(\angle(\mathbf{u}^{(3)}, \mathbf{p}, \mathbf{u}^{(1)})\right) + g\left(\angle(\mathbf{u}^{(1)}, \mathbf{p}, \mathbf{u}^{(2)})\right) \right) = 2\pi \text{Area}(\mathbf{u}^{(1)}, \mathbf{u}^{(2)}, \mathbf{u}^{(3)}) \quad (24)$$

where we assume $\angle(\mathbf{u}^{(i)}, \mathbf{p}, \mathbf{u}^{(j)}) \in (-\pi, \pi)$ is the angle of viewed from $\mathbf{p} \in \mathbb{R}^2$ of $\mathbf{u}^{(j)}$ relative to $\mathbf{u}^{(i)}$ and $\mathbb{Z}^{2} := \mathbb{Z}^2 + \left[\frac{1}{2}\right]$.*

Proof. First note that if $g = 1$, then the value of the expression

$$\left(g\left(\angle(\mathbf{u}^{(2)}, \mathbf{p}, \mathbf{u}^{(3)})\right) + g\left(\angle(\mathbf{u}^{(3)}, \mathbf{p}, \mathbf{u}^{(1)})\right) + g\left(\angle(\mathbf{u}^{(1)}, \mathbf{p}, \mathbf{u}^{(2)})\right) \right) \quad (25)$$

is quite simple and given by

$$2\pi \begin{cases} 1 & \mathbf{p} \in \Delta(\mathbf{u}^{(1)}, \mathbf{u}^{(2)}, \mathbf{u}^{(3)}) \\ \frac{1}{2} & \mathbf{p} \in \partial\Delta(\mathbf{u}^{(1)}, \mathbf{u}^{(2)}, \mathbf{u}^{(3)}) \\ 0 & \mathbf{p} \notin \Delta(\mathbf{u}^{(1)}, \mathbf{u}^{(2)}, \mathbf{u}^{(3)}) \cap \partial\Delta(\mathbf{u}^{(1)}, \mathbf{u}^{(2)}, \mathbf{u}^{(3)}) \end{cases}$$

the way to see this is to draw a triangle $\Delta(\mathbf{u}^{(1)}, \mathbf{u}^{(2)}, \mathbf{u}^{(3)})$ three times and consider the possible cases.

As a result, (25) measures whether \mathbf{p} is inside the triangle, on its boundary or out of it, with certain weight, and so the sum over the whole of \mathbb{Z}^{2*} gives us the number of lattice points inside the triangle, that is, its area times 2π .

Thus we seek to prove that for $f := g - 1$ we have

$$\sum_{\mathbf{p} \in \mathbb{Z}^{2*}} (f(\angle(\mathbf{u}^{(2)}, \mathbf{p}, \mathbf{u}^{(3)})) + f(\angle(\mathbf{u}^{(3)}, \mathbf{p}, \mathbf{u}^{(1)})) + f(\angle(\mathbf{u}^{(1)}, \mathbf{p}, \mathbf{u}^{(2)}))) = 0 \quad (26)$$

Note that due to our assumption (23) we have $f(\alpha) \propto \mathcal{O}(\alpha^3)$ so that $f(\angle(\mathbf{u}^{(i)}, \mathbf{p}, \mathbf{u}^{(j)})) \propto \frac{1}{\|\mathbf{p}\|^3}$ for large $\|\mathbf{p}\|$. As a result, even though each of the three summands in the left hand side of (24) is *not* summable because $g(\angle(\mathbf{u}^{(i)}, \mathbf{p}, \mathbf{u}^{(j)})) \propto \frac{1}{\|\mathbf{p}\|}$, now each of the three terms in the left hand side of (26) is summable and so we need to prove:

$$\sum_{\mathbf{p} \in \mathbb{Z}^{2*}} f(\angle(\mathbf{u}^{(2)}, \mathbf{p}, \mathbf{u}^{(3)})) + \sum_{\mathbf{p} \in \mathbb{Z}^{2*}} f(\angle(\mathbf{u}^{(3)}, \mathbf{p}, \mathbf{u}^{(1)})) + \sum_{\mathbf{p} \in \mathbb{Z}^{2*}} f(\angle(\mathbf{u}^{(1)}, \mathbf{p}, \mathbf{u}^{(2)})) = 0 \quad (27)$$

But for any $(i, j) \in \{1, 2, 3\}^2$,

$$\sum_{\mathbf{p} \in \mathbb{Z}^{2*}} f(\angle(\mathbf{u}^{(i)}, \mathbf{p}, \mathbf{u}^{(j)})) = 0$$

This is because we may split this sum from a sum on the whole lattice \mathbb{Z}^{2*} into two sums on two half-lattices, where the line dividing the two is given by the vector $\mathbf{u}^{(j)} - \mathbf{u}^{(i)}$. The point is that for each point on the first half-lattice, there is a symmetric point on the other half lattice. The two contribute exactly the same magnitude with opposite signs, because if \mathbf{p}^* is the reflected point corresponding to \mathbf{p} then

$$\begin{aligned} \angle(\mathbf{u}^{(i)}, \mathbf{p}^*, \mathbf{u}^{(j)}) &= \angle(\mathbf{u}^{(j)}, \mathbf{p}, \mathbf{u}^{(i)}) \\ &= -\angle(\mathbf{u}^{(i)}, \mathbf{p}, \mathbf{u}^{(j)}) \end{aligned}$$

so that the whole sum is zero. \square

7.2.7. Claim. The area of a triangle is given by

$$\sum_{\mathbf{p} \in \mathbb{Z}^{2*}} [(\Lambda(m_1 - p_1) - \Lambda(n_1 - p_1)) (\Lambda(n_2 - p_2) - \Lambda(l_2 - p_2)) - (1 \leftrightarrow 2)] = 2\text{Area}(\mathbf{m}, \mathbf{n}, \mathbf{l})$$

for any triangle with the vertices $(\mathbf{m}, \mathbf{n}, \mathbf{l})$.

Proof. Note that

$$\sum_{\mathbf{p}_i \in \mathbb{Z}^*} [\Lambda(m_i - p_i) - \Lambda(n_i - p_i)]$$

roughly gives the number of lattice points between m_i and n_i , that is, it gives the number $m_i - n_i$ where “roughly” means if Λ were a sharp step function which is 0 for negative values and 1 for positive values. Since we are working on a lattice, the Kubo formula goes through when we exchange the smooth switch function Λ with a step function. Thus we assume that is what is being done throughout and so for the rest of the proof Λ is a step function (Note that we could have also instead integrated over a continuous \mathbf{p} over \mathbb{R}^2).

Then we have

$$\begin{aligned} \sum_{\mathbf{p} \in \mathbb{Z}^{2*}} [(\Lambda(m_1 - p_1) - \Lambda(n_1 - p_1)) (\Lambda(n_2 - p_2) - \Lambda(l_2 - p_2)) - (1 \leftrightarrow 2)] &= (m_1 - n_1)(n_2 - l_2) - (m_2 - n_2)(n_1 - l_1) \\ &= \|(\mathbf{m} - \mathbf{n}) \times (\mathbf{n} - \mathbf{l})\| \hat{\mathbf{e}} \\ &= 2|\text{Area}(\mathbf{m}, \mathbf{n}, \mathbf{l})| \hat{\mathbf{e}} \end{aligned}$$

where the vector $\hat{\mathbf{e}}$ gives the orientation of the triangle and so all together we obtain the result. \square

7.2.8. Claim. (Bellissard et al. or in the present form [8]) The expression for σ_H via the Kubo formula given in (13) is equal to the expression for σ_H via an index of pair of projections given in (22), provided that the Fermi energy μ is in a mobility gap. That is,

$$\frac{1}{2\pi} \text{Tr}((P - UPU^*)^3) = i\text{Tr}(P[[\Lambda(x_1), P], [\Lambda(x_2), P]])$$

In particular, this proves that the Kubo formula results in an integer number for the Hall conductivity.

Proof. The proof may be found in [2] and in [19] pp. 9.

We start by computing the matrix elements in the position basis:

$$\begin{aligned}
 \langle \delta_{\mathbf{n}_1}, (\mathbf{P} - \mathbf{U}\mathbf{P}\mathbf{U}^*)^3 \delta_{\mathbf{n}_2} \rangle &= \langle \delta_{\mathbf{n}_1}, (\mathbf{P} - \mathbf{U}\mathbf{P}\mathbf{U}^*) \mathbb{1} (\mathbf{P} - \mathbf{U}\mathbf{P}\mathbf{U}^*) \mathbb{1} (\mathbf{P} - \mathbf{U}\mathbf{P}\mathbf{U}^*) \delta_{\mathbf{n}_2} \rangle \\
 &= \sum_{(\mathbf{n}_3, \mathbf{n}_4) \in \mathbb{Z}^4} \langle \delta_{\mathbf{n}_1}, (\mathbf{P} - \mathbf{U}\mathbf{P}\mathbf{U}^*) \delta_{\mathbf{n}_3} \rangle \langle \delta_{\mathbf{n}_3}, (\mathbf{P} - \mathbf{U}\mathbf{P}\mathbf{U}^*) \delta_{\mathbf{n}_4} \rangle \langle \delta_{\mathbf{n}_4}, (\mathbf{P} - \mathbf{U}\mathbf{P}\mathbf{U}^*) \delta_{\mathbf{n}_2} \rangle \\
 &= \sum_{(\mathbf{n}_3, \mathbf{n}_4) \in \mathbb{Z}^4} \langle \delta_{\mathbf{n}_1}, (\mathbf{P} - \mathbf{U}\mathbf{P}\mathbf{U}^*) \delta_{\mathbf{n}_3} \rangle \langle \delta_{\mathbf{n}_3}, \mathbf{P} \delta_{\mathbf{n}_4} \rangle \langle \delta_{\mathbf{n}_4}, (\mathbf{P} - \mathbf{U}\mathbf{P}\mathbf{U}^*) \delta_{\mathbf{n}_2} \rangle \\
 &= \sum_{(\mathbf{n}_3, \mathbf{n}_4) \in \mathbb{Z}^4} \langle \delta_{\mathbf{n}_1}, \mathbf{P} \delta_{\mathbf{n}_3} \rangle \left(1 - e^{i\angle(\mathbf{n}_3, 0, \mathbf{n}_1)}\right) \langle \delta_{\mathbf{n}_3}, \mathbf{P} \delta_{\mathbf{n}_4} \rangle \left(1 - e^{i\angle(\mathbf{n}_4, 0, \mathbf{n}_3)}\right) \langle \delta_{\mathbf{n}_4}, \mathbf{P} \delta_{\mathbf{n}_2} \rangle \left(1 - e^{i\angle(\mathbf{n}_2, 0, \mathbf{n}_4)}\right) \\
 \text{Tr}((\mathbf{P}, \mathbf{U}\mathbf{P}\mathbf{U}^*)^3) &= \sum_{\mathbf{n}_1 \in \mathbb{Z}^2} \langle \delta_{\mathbf{n}_1}, (\mathbf{P} - \mathbf{U}\mathbf{P}\mathbf{U}^*)^3 \delta_{\mathbf{n}_1} \rangle \\
 &= \sum_{(\mathbf{n}_1, \mathbf{n}_3, \mathbf{n}_4) \in \mathbb{Z}^6} \langle \delta_{\mathbf{n}_1}, \mathbf{P} \delta_{\mathbf{n}_3} \rangle \left(1 - e^{i\angle(\mathbf{n}_3, 0, \mathbf{n}_1)}\right) \langle \delta_{\mathbf{n}_3}, \mathbf{P} \delta_{\mathbf{n}_4} \rangle \left(1 - e^{i\angle(\mathbf{n}_4, 0, \mathbf{n}_3)}\right) \langle \delta_{\mathbf{n}_4}, \mathbf{P} \delta_{\mathbf{n}_1} \rangle \left(1 - e^{i\angle(\mathbf{n}_1, 0, \mathbf{n}_4)}\right) \\
 &= \sum_{(\mathbf{n}, \mathbf{l}, \mathbf{m}) \in \mathbb{Z}^6} \langle \delta_{\mathbf{n}}, \mathbf{P} \delta_{\mathbf{l}} \rangle \langle \delta_{\mathbf{l}}, \mathbf{P} \delta_{\mathbf{m}} \rangle \langle \delta_{\mathbf{m}}, \mathbf{P} \delta_{\mathbf{n}} \rangle \left(1 - e^{i\angle(\mathbf{l}, 0, \mathbf{n})}\right) \left(1 - e^{i\angle(\mathbf{m}, 0, \mathbf{l})}\right) \left(1 - e^{i\angle(\mathbf{n}, 0, \mathbf{m})}\right)
 \end{aligned}$$

next we simplify

$$\begin{aligned}
 \left(1 - e^{i\angle(\mathbf{l}, 0, \mathbf{n})}\right) \left(1 - e^{i\angle(\mathbf{m}, 0, \mathbf{l})}\right) \left(1 - e^{i\angle(\mathbf{n}, 0, \mathbf{m})}\right) &= 1 - e^{i\angle(\mathbf{l}, 0, \mathbf{n})} - e^{i\angle(\mathbf{m}, 0, \mathbf{l})} - e^{i\angle(\mathbf{n}, 0, \mathbf{m})} - \underbrace{e^{i\angle(\mathbf{l}, 0, \mathbf{n})} e^{i\angle(\mathbf{m}, 0, \mathbf{l})} e^{i\angle(\mathbf{n}, 0, \mathbf{m})}}_1 \\
 &= + \underbrace{e^{i\angle(\mathbf{l}, 0, \mathbf{n})} e^{i\angle(\mathbf{m}, 0, \mathbf{l})}}_{e^{i\angle(\mathbf{m}, 0, \mathbf{n})}} + e^{i\angle(\mathbf{m}, 0, \mathbf{l})} e^{i\angle(\mathbf{n}, 0, \mathbf{m})} + e^{i\angle(\mathbf{l}, 0, \mathbf{n})} e^{i\angle(\mathbf{n}, 0, \mathbf{m})} \\
 &= -e^{i\angle(\mathbf{l}, 0, \mathbf{n})} - e^{i\angle(\mathbf{m}, 0, \mathbf{l})} - e^{i\angle(\mathbf{n}, 0, \mathbf{m})} + \\
 &\quad + e^{i\angle(\mathbf{m}, 0, \mathbf{n})} + e^{i\angle(\mathbf{n}, 0, \mathbf{l})} + e^{i\angle(\mathbf{l}, 0, \mathbf{m})} \\
 &= 2i(\sin(\angle(\mathbf{m}, 0, \mathbf{n})) + \sin(\angle(\mathbf{n}, 0, \mathbf{l})) + \sin(\angle(\mathbf{l}, 0, \mathbf{m})))
 \end{aligned}$$

Next note that the expression for $\text{Tr}((\mathbf{P}, \mathbf{U}\mathbf{P}\mathbf{U}^*)^3)$ cannot depend on the choice of the origin as we could have picked any translate $\mathbf{U}(\mathbf{x}) = \exp(i \arg(\mathbf{x} + \mathbf{p}))$ to do the job. Thus:

$$\begin{aligned}
 \text{Tr}((\mathbf{P}, \mathbf{U}\mathbf{P}\mathbf{U}^*)^3) &= \sum_{(\mathbf{n}, \mathbf{l}, \mathbf{m}) \in \mathbb{Z}^6} \langle \delta_{\mathbf{n}}, \mathbf{P} \delta_{\mathbf{l}} \rangle \langle \delta_{\mathbf{l}}, \mathbf{P} \delta_{\mathbf{m}} \rangle \langle \delta_{\mathbf{m}}, \mathbf{P} \delta_{\mathbf{n}} \rangle 2i(\sin(\angle(\mathbf{m}, 0, \mathbf{n})) + \sin(\angle(\mathbf{n}, 0, \mathbf{l})) + \sin(\angle(\mathbf{l}, 0, \mathbf{m}))) \\
 &= \sum_{(\mathbf{n}, \mathbf{l}, \mathbf{m}) \in \mathbb{Z}^6} \langle \delta_{\mathbf{n}}, \mathbf{P} \delta_{\mathbf{l}} \rangle \langle \delta_{\mathbf{l}}, \mathbf{P} \delta_{\mathbf{m}} \rangle \langle \delta_{\mathbf{m}}, \mathbf{P} \delta_{\mathbf{n}} \rangle 2i(\sin(\angle(\mathbf{m}, \mathbf{p}, \mathbf{n})) + \sin(\angle(\mathbf{n}, \mathbf{p}, \mathbf{l})) + \sin(\angle(\mathbf{l}, \mathbf{p}, \mathbf{m}))) \quad \forall \mathbf{p} \in \mathbb{R}^2 \\
 &\stackrel{*}{=} \frac{1}{L^2} \sum_{\mathbf{p} \in \Lambda_L^*} \sum_{(\mathbf{n}, \mathbf{l}, \mathbf{m}) \in \mathbb{Z}^6} \langle \delta_{\mathbf{n}}, \mathbf{P} \delta_{\mathbf{l}} \rangle \langle \delta_{\mathbf{l}}, \mathbf{P} \delta_{\mathbf{m}} \rangle \langle \delta_{\mathbf{m}}, \mathbf{P} \delta_{\mathbf{n}} \rangle \times \\
 &\quad \times 2i(\sin(\angle(\mathbf{m}, \mathbf{p}, \mathbf{n})) + \sin(\angle(\mathbf{n}, \mathbf{p}, \mathbf{l})) + \sin(\angle(\mathbf{l}, \mathbf{p}, \mathbf{m}))) \quad \forall L > 0 \\
 &= \lim_{L \rightarrow \infty} \frac{1}{L^2} \sum_{\mathbf{p} \in \Lambda_L^*} \sum_{(\mathbf{n}, \mathbf{l}, \mathbf{m}) \in \mathbb{Z}^6} \langle \delta_{\mathbf{n}}, \mathbf{P} \delta_{\mathbf{l}} \rangle \langle \delta_{\mathbf{l}}, \mathbf{P} \delta_{\mathbf{m}} \rangle \langle \delta_{\mathbf{m}}, \mathbf{P} \delta_{\mathbf{n}} \rangle \times \\
 &\quad \times 2i(\sin(\angle(\mathbf{m}, \mathbf{p}, \mathbf{n})) + \sin(\angle(\mathbf{n}, \mathbf{p}, \mathbf{l})) + \sin(\angle(\mathbf{l}, \mathbf{p}, \mathbf{m})))
 \end{aligned}$$

where in $*$ we used the fact that since the expression does not depend on \mathbf{p} , we could also take the average instead and we have denoted $\Lambda_L^* := \{\mathbf{p} \in \mathbb{Z}^{2*} \mid \|\mathbf{p}\| \leq L\}$.

Because the projectors are localized, we could replace the ranges of the two sums without a big error (see [19] pp. 10)

$$\begin{aligned}
\text{Tr} \left((P, UPU^*)^3 \right) &= \lim_{L \rightarrow \infty} \frac{1}{L^2} \sum_{p \in \mathbb{Z}^{2*}} \sum_{(\mathbf{n}, \mathbf{l}, \mathbf{m}) \in \Lambda_L^3} \langle \delta_{\mathbf{n}}, P\delta_{\mathbf{l}} \rangle \langle \delta_{\mathbf{l}}, P\delta_{\mathbf{m}} \rangle \langle \delta_{\mathbf{m}}, P\delta_{\mathbf{n}} \rangle \times \\
&\quad \times 2i (\sin(\angle(\mathbf{m}, \mathbf{p}, \mathbf{n})) + \sin(\angle(\mathbf{n}, \mathbf{p}, \mathbf{l})) + \sin(\angle(\mathbf{l}, \mathbf{p}, \mathbf{m}))) \\
&\stackrel{*}{=} \lim_{L \rightarrow \infty} \frac{1}{L^2} \sum_{(\mathbf{n}, \mathbf{l}, \mathbf{m}) \in \Lambda_L^3} \langle \delta_{\mathbf{n}}, P\delta_{\mathbf{l}} \rangle \langle \delta_{\mathbf{l}}, P\delta_{\mathbf{m}} \rangle \langle \delta_{\mathbf{m}}, P\delta_{\mathbf{n}} \rangle 2i \times 2\pi \text{Area}(\mathbf{m}, \mathbf{n}, \mathbf{l}) \\
&\stackrel{**}{=} \lim_{L \rightarrow \infty} \frac{1}{L^2} \sum_{(\mathbf{n}, \mathbf{l}, \mathbf{m}) \in \Lambda_L^3} \langle \delta_{\mathbf{n}}, P\delta_{\mathbf{l}} \rangle \langle \delta_{\mathbf{l}}, P\delta_{\mathbf{m}} \rangle \langle \delta_{\mathbf{m}}, P\delta_{\mathbf{n}} \rangle i \\
&\quad \times 2\pi \sum_{p \in \mathbb{Z}^{2*}} [(\Lambda(m_1 - p_1) - \Lambda(n_1 - p_1)) (\Lambda(n_2 - p_2) - \Lambda(l_2 - p_2)) - (1 \leftrightarrow 2)] \\
&= \lim_{L \rightarrow \infty} \frac{1}{L^2} \sum_{p \in \Lambda_L^*} \sum_{(\mathbf{n}, \mathbf{l}, \mathbf{m}) \in \mathbb{Z}^{2*}} \langle \delta_{\mathbf{n}}, P\delta_{\mathbf{l}} \rangle \langle \delta_{\mathbf{l}}, P\delta_{\mathbf{m}} \rangle \langle \delta_{\mathbf{m}}, P\delta_{\mathbf{n}} \rangle i \\
&\quad \times 2\pi [(\Lambda(m_1 - p_1) - \Lambda(n_1 - p_1)) (\Lambda(n_2 - p_2) - \Lambda(l_2 - p_2)) - (1 \leftrightarrow 2)] \\
&\stackrel{\star}{=} 2\pi i \sum_{(\mathbf{n}, \mathbf{l}, \mathbf{m}) \in \mathbb{Z}^{2*}} \langle \delta_{\mathbf{n}}, P\delta_{\mathbf{l}} \rangle \langle \delta_{\mathbf{l}}, P\delta_{\mathbf{m}} \rangle \langle \delta_{\mathbf{m}}, P\delta_{\mathbf{n}} \rangle [(\Lambda(m_1) - \Lambda(n_1)) (\Lambda(n_2) - \Lambda(l_2)) - (1 \leftrightarrow 2)] \\
&= 2\pi i \sum_{(\mathbf{l}, \mathbf{m}, \mathbf{n}) \in \mathbb{Z}^6} \langle \delta_{\mathbf{l}}, P\delta_{\mathbf{m}} \rangle \langle \delta_{\mathbf{m}}, P\delta_{\mathbf{n}} \rangle \langle \delta_{\mathbf{n}}, P\delta_{\mathbf{l}} \rangle [(\Lambda_1(\mathbf{m}) - \Lambda_1(\mathbf{n})) (\Lambda_2(\mathbf{n}) - \Lambda_2(\mathbf{l})) - (1 \leftrightarrow 2)] \\
&= 2\pi i \sum_{\mathbf{l} \in \mathbb{Z}^2} \langle \delta_{\mathbf{l}}, P((\Lambda_1 P - P\Lambda_1)(\Lambda_2 P - P\Lambda_2) - (1 \leftrightarrow 2)) \delta_{\mathbf{l}} \rangle \\
&= 2\pi i \text{Tr}(P[[\Lambda_1, P], [\Lambda_2, P]])
\end{aligned}$$

where we used in \star 7.2.6 since \sin satisfies the conditions required by the claim, and in $**$ we used 7.2.7, and in \star we again used the fact that the summand does not depend on the choice of origin, so it is equal to its average. \square

7.2.9. *Remark.* 7.2.8 can be thought of schematically as an analogy to the Gauss-Bonnet theorem ([30] pp. 167), which roughly says that if \mathcal{M} is a compact smooth orientable manifold of dimension two and g is its genus then

$$1 - g = \frac{1}{4\pi} \int_{\mathcal{M}} R dA \quad (28)$$

where R is the scalar curvature of \mathcal{M} and dA is the element of area on the surface. The analogy with 7.2.8 is based on the fact that

- (1) The index of a pair of projections, being a difference of the dimensions of two spaces, is obviously an integer. In addition we have seen that it is stable under smooth deformations. So is the left hand side of (28): the genus is stable under smooth deformations.
- (2) The Kubo formula is a sort of curvature (in the sense that will become clear below in the periodic sample case, or using non-commutative geometry): One can make the following analogies:

$$\begin{aligned}
R(X, Y) &= \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]} \\
\nabla_X &\sim [\Lambda, \cdot] \\
\int_{\mathcal{M}} \cdot dA &\sim \text{Tr}(\cdot)
\end{aligned}$$

8. THE PERIODIC CASE

Even though we already have two expressions for the Hall conductivity, namely the one that arises as a linear response via the Kubo formula, (13), and the one that arises from the Laughlin argument as a charge pump, (22), we will now develop (13) further for the case when the sample is periodic. Even though this is not justified physically (as a completely periodic sample will have no disorder, thus no mobility gap and thus no plateaus, thus giving an incomplete description of the quantum Hall effect), it was historically the first explanation for the fact that the conductivity is an integer value at integer values of the filling factor (see [46]) and further, the geometric and topological nature of the effect is seen via this description. To maintain this geometric perspective for the non-periodic case one could follow [10], though in our treatment we do not venture so far.

8.1. Vector Bundles. To understand the periodic case we need the concept of “vector bundles”, which we present below. A good reference for this section is the first part of [5] (math), or [35] (physics). Intuitively, one should think of a vector bundle as a family of vector spaces parametrized by an arbitrary topological space, such that locally the whole space looks like a Cartesian product of the parameter space times the vector space. Thus the “point” of vector bundles is the global structure, analogously to manifolds which locally “look” like Euclidean space.

8.1.1. Definition. Let $X \in \text{Obj}(\text{Top})$ be given, which is called the base space. A *family of vector spaces over X* is the following set of data and conditions:

- (1) $E \in \text{Obj}(\text{Top})$, called *the total space*.
- (2) $p \in \text{Mor}_{\text{Top}}(E, X)$, called *the projection map*.
- (3) Local continuous vector space structure: for each $x \in X$, we define

$$E_x := p^{-1}(\{x\})$$

together with the subspace topology, and call it *the fiber over x* . The data is then a local vector addition map

$$a_x \in \text{Mor}_{\text{Top}}(E_x \times E_x, E_x)$$

and a local scalar multiplication map

$$m_x \in \text{Mor}_{\text{Top}}(\mathbb{C} \times E_x, E_x)$$

such that these maps and E_x form a topological vector space:

$$(E_x, a_x, m_x) \in \text{Obj}(\text{TVS})$$

8.1.2. Definition. A *section* of a family $p : E \rightarrow X$ is some continuous map $s \in \text{Mor}_{\text{Top}}(X, E)$ such that:

$$p \circ s = \mathbb{1}_X$$

8.1.3. Definition. A *homomorphism* from one family $p : E \rightarrow X$ to another $\tilde{p} : \tilde{E} \rightarrow X$ (or a “morphism of families of vector spaces over X ”) is a continuous map $\varphi \in \text{Mor}_{\text{Top}}(E, \tilde{E})$ such that:

- (1) φ “respects” the base-space:

$$\tilde{p} \circ \varphi = p$$

Equivalently, the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{\varphi} & \tilde{E} \\ p \downarrow & \circlearrowright & \searrow \tilde{p} \\ X & & \end{array}$$

- (2) For each $x \in X$, the induced map $\varphi_x : E_x \rightarrow \tilde{E}_x$ defined via $\varphi_x := \varphi|_{E_x}$, is also \mathbb{C} -linear, that is,

$$\varphi|_{E_x} \in \text{Mor}_{\text{Vect}_{\mathbb{C}}}(E_x, \tilde{E}_x)$$

8.1.4. Definition. An *isomorphism* from one family $p : E \rightarrow X$ to another $\tilde{p} : \tilde{E} \rightarrow X$ is a morphism of families of vector spaces over X ,

$$\varphi : E \rightarrow \tilde{E}$$

which is bijective and such that φ^{-1} is *also* continuous.

8.1.5. Remark. Note that then, automatically, we have

$$\tilde{p} = p \circ \varphi^{-1}$$

and

$$\varphi_x^{-1} \in \text{Mor}_{\text{Vect}_{\mathbb{C}}}(\tilde{E}_x, E_x)$$

8.1.6. Example. (*The product family*) Let $n \in \mathbb{N}_{>0}$ be given and $X \in \text{Obj}(\text{Top})$. Define $E := X \times \mathbb{C}^n$ with \mathbb{C}^n having the standard topology and E having the product topology and $p : E \rightarrow X$ as the projection onto the first factor:

$$(x, v) \xrightarrow{p} x$$

By definition of the product topology, p is continuous. If $x \in X$, then

$$\begin{aligned} E_x &\equiv p^{-1}(\{x\}) \\ &= \{x\} \times \mathbb{C}^n \\ &\cong \mathbb{C}^n \end{aligned}$$

where the last isomorphism is in $\text{Mor}_{\text{Vect}_{\mathbb{C}}^n}$. $\underbrace{E}_{X \times \mathbb{C}^n}$ is called *the product family over X with fiber \mathbb{C}^n* . If \tilde{E} is any other family of vector spaces over X which is isomorphic (as a family of vector spaces over X) to some product family, then \tilde{E} is called *a trivial family*.

8.1.7. Claim. (*The restriction of the basespace*) Let $Y \subseteq X$ and E be a family of vector spaces over X with projection p . Then $\tilde{p} : p^{-1}(Y) \rightarrow Y$ defined by

$$\tilde{p}(e) = p(e) \quad \forall e \in p^{-1}(Y)$$

is a family over Y .

It is called the *restriction of E to Y* and is denoted by

$$\tilde{p} : E|_Y \rightarrow Y$$

Proof. Let $i : Y \hookrightarrow X$ be the inclusion map and $i^* : E|_Y \hookrightarrow E$ be also the inclusion map. Then $\tilde{p} = p \circ i^*$ so that \tilde{p} is also continuous.

The vector space operations are also continuous with respect to the subspace topology. Indeed, consider vector addition in $E|_Y$. Let $y \in Y$ be given, and denote the fiber at y in $E|_Y$ by \tilde{E}_y and the fiber at y in E by E_y as usual. Then vector addition at y in $E|_Y$, $\tilde{a}_y : \tilde{E}_y^2 \rightarrow \tilde{E}_y$ is defined via vector addition in E , $a_y : E_y^2 \rightarrow E_y$, by

$$\tilde{a}_y(e_1, e_2) \equiv a_y(i^*(e_1), i^*(e_2)) \quad \forall (e_1, e_2) \in \tilde{E}_y^2$$

and because $a_y \circ (i^* \times i^*)$ is continuous, \tilde{a}_y is continuous. Similarly for scalar multiplication. \square

8.1.8. Definition. A family E of vector spaces over X is said to *locally trivial* iff for every $x \in X$ there exists $U \in \text{Nbhd}_X(x)$ such that $E|_U$ (as in 8.1.7) is a trivial family (as in 8.1.6). A family E of vector spaces over X which is locally trivial is a *vector bundle* (unlike a family of vector over X , which is not required to be locally trivial). A trivial family is a *trivial bundle*.

8.1.9. Definition. Let $p : E \rightarrow X$ be a vector bundle. A *sub-bundle* F is a subset of E such that $p \circ i : F \rightarrow X$ is a vector bundle over X , where $i : F \hookrightarrow E$ is the inclusion map, and F is considered with the subspace topology.

8.1.10. Claim. For $p : E \rightarrow X$ a vector bundle, the fiber is locally fixed. As a result it is fixed on each connected component of X .

Proof. Because $p : E \rightarrow X$ is a vector bundle, for each $x \in X$ there is some neighborhoud $U \in \text{Nbhd}_X(x)$ such that $\varphi : E|_U \cong U \times V$ for some $V \in \text{obj}(\text{Vect}_{\mathbb{C}})$, where φ is a isomorphism of family of vector spaces over X . In particular φ is \mathbb{C} -linear when restricted to a particular point, $\varphi|_x : E_x \cong \underbrace{\{x\} \times V}_{(U \times V)_x}$ for all $x \in U$. Thus, in U , all the fibers E_x are really

isomorphic to V as vector spaces.

Now for this given V , define

$$S := \{x' \in X \mid \exists U' \in \text{Nbhd}_X(x') \wedge \exists \text{ isomorphism of families } \varphi' : E|_{U'} \rightarrow U' \times V\}$$

Then $S \in \text{Open}(X)$ because such a U' as in the above condition actually has $U' \subseteq S$ (with the same U' for all its points). But $S \in \text{Closed}(X)$ as well. Thus S must be a connected component of X , and on it, V is “fixed”. \square

8.1.11. Corollary. For connected X and finite fiber, $X \ni x \mapsto \dim(E_x) \in \mathbb{N}$ is constant. It is called the *rank*, or *dimension*, of the bundle.

8.1.12. Definition. (*Transition Maps*) Let $p : E \rightarrow X$ be a vector bundle, and let $x \in X$ be given. Let U_1 and U_2 be two neighborhouds of x in X such that there are family isomorphisms $E|_{U_1} \xrightarrow{\varphi_1} U_1 \times V$ and $E|_{U_2} \xrightarrow{\varphi_2} U_2 \times V$. Then we have the restricted maps which are also isomorphisms

$$\begin{aligned} \varphi_1|_{U_1 \cap U_2} : E|_{U_1 \cap U_2} &\rightarrow U_1 \cap U_2 \times V \\ \varphi_2|_{U_1 \cap U_2} : E|_{U_1 \cap U_2} &\rightarrow U_1 \cap U_2 \times V \end{aligned}$$

so that the following map is defined and is an isomorphism of $U_1 \cap U_2 \times V$ to itself:

$$\varphi_{12} : U_1 \cap U_2 \times V \rightarrow U_1 \cap U_2 \times V$$

given by

$$\varphi_{12} := \varphi_1|_{U_1 \cap U_2} \circ (\varphi_2|_{U_1 \cap U_2})^{-1}$$

Since the range of φ_{12} is $U_1 \cap U_2 \times V$, we can compose it with a projection onto the second factor to get a map with range V :

$$U_1 \cap U_2 \times V \rightarrow V$$

or better yet we get the map

$$t_{12} : U_1 \cap U_2 \rightarrow \text{Aut}(V)$$

That is,

$$\varphi_{12}(x, v) = (x, t_{12}(x)v) \in U_1 \cap U_2 \times V$$

When $\dim(V) < \infty$, we call the transition maps $t_{\alpha\beta}(x, \cdot)$ transition matrices.

8.1.13. Claim. *For the transition maps we have the following properties:*

- (1) $t_{\alpha\alpha}(x) = 1_V$.
- (2) $t_{\alpha\beta}(x) = t_{\beta\alpha}(x)^{-1}$.
- (3) $t_{\alpha\beta}(x) \circ t_{\beta\gamma}(x) = t_{\alpha\gamma}(x)$.

Proof. Let α, β and γ label three different trivializations, whose intersection contains some given point $x \in X$. That is, $E|_{U_\kappa} \stackrel{\varphi_\kappa}{\cong} U_\kappa \times V$ for all $\kappa \in \{\alpha, \beta, \gamma\}$. Then

$$\begin{aligned} \varphi_{\alpha\alpha} &\equiv \varphi_\alpha|_{U_\alpha \cap U_\alpha} \circ (\varphi_\alpha|_{U_\alpha \cap U_\alpha})^{-1} \\ &= 1_{U_\alpha \times V} \end{aligned}$$

so that the corresponding $t_{\alpha\alpha}(x, \cdot)$ must be 1_V .

Next,

$$\begin{aligned} (\varphi_{\beta\alpha})^{-1} &\equiv \left(\varphi_\beta|_{U_\beta \cap U_\alpha} \circ (\varphi_\alpha|_{U_\beta \cap U_\alpha})^{-1} \right)^{-1} \\ &= \varphi_\alpha|_{U_\beta \cap U_\alpha} \circ (\varphi_\beta|_{U_\beta \cap U_\alpha})^{-1} \\ &\equiv \varphi_{\alpha\beta} \end{aligned}$$

so the corresponding transition map obeys the same constraint.

Similarly for the last statement. □

8.1.14. Claim. *Let V and X be two given topological spaces, with an open cover $\{U_\alpha\}_{\alpha \in A}$ for X , and a set of continuous transition functions $t_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \text{Aut}(V)$ on each nonempty intersection, such that $t_{\alpha\beta}$ satisfy the conditions of 8.1.13. Then these data determines uniquely a vector bundle over X (up to vector bundle isomorphisms).*

Proof. Define

$$E := (\coprod_{\alpha \in A} U_\alpha \times V) / \sim$$

where $U_\beta \times V \ni (x, v) \sim (y, u) \in U_\alpha \times V$ iff $x = y$ and $t_{\alpha\beta}(x)v = u$. This is indeed an equivalence relation:

- Reflexive: $(x, v) \sim (x, v)$ as $x = x$ and $t_{\alpha\alpha}(x)v = 1_V v = v$.
- Symmetric: If $(x, v) \sim (y, u)$ then $x = y$ and $t_{\alpha\beta}(x)v = u$, so that $y = x$ and

$$\begin{aligned} v &= (t_{\alpha\beta}(x))^{-1} u \\ &= t_{\beta\alpha}(x) u \\ &= t_{\beta\alpha}(y) u \end{aligned}$$

so that $(y, u) \sim (x, v)$ indeed.

- Transitive: If $(x, v) \sim (y, u)$ and $(y, u) \sim (z, w)$ then $x = z$ and

$$\begin{aligned} t_{\alpha\gamma}(x)v &= t_{\alpha\beta}(x) \circ t_{\beta\gamma}(x)v \\ &= t_{\alpha\beta}(y) \circ t_{\beta\gamma}(x)v \\ &= t_{\alpha\beta}(y) \circ u \\ &= w \end{aligned}$$

The topology on E is given by the following hierarchy:

- (1) Each U_α has its subspace topology inherited from X and V comes with a predetermined topology.
- (2) $U_\alpha \times V$ has the product topology.
- (3) $(\coprod_{\alpha \in A} U_\alpha \times V)$ has the disjoint union topology.
- (4) E has the quotient topology.

We then define the projection $p : E \rightarrow X$ by

$$p([(x, v)]) := x$$

and also define maps for each $\alpha \in A$, $\varphi_\alpha : E|_{U_\alpha} \rightarrow U_\alpha \times V$ by

$$\varphi_\alpha([(x, v)]) \mapsto (x, v)$$

Claim. p is continuous.

Proof. Let $q : \coprod_{\alpha \in A} U_\alpha \times V \rightarrow E$ be the quotient map, which is continuous and open by definition of the quotient topology, and let $i_{\alpha'} : U_{\alpha'} \times V \rightarrow \coprod_{\alpha \in A} U_\alpha \times V$ be the canonical injections, which are continuous and clopen by definition of the disjoint union topology. Let $U \in \text{Open}(X)$. Then

$$\begin{aligned} p^{-1}(U) &\equiv \{[(x, v)] \in E \mid x \in U\} \\ &= \{q((x, v)) \in E \mid x \in U\} \\ &= \{q(i_\alpha(x, v)) \in E \mid x \in U \cap U_\alpha\} \\ &= \bigcup_{\alpha \in A} (q \circ i_\alpha)(U \cap U_\alpha) \\ &\in \text{Open}(E) \end{aligned}$$

as the composition of two open maps is again open. \square

Note that p also respects the base space.

Then if V is a topological vector space then each fiber is

$$E_x \equiv \{[(y, v)] \in E \mid y = x\}$$

also has continuous vector space operations, so that E is indeed a family of vector spaces over X .

Claim. φ_α is well-defined.

Proof. Let $x \in U_\alpha$. If $(x, v) \sim (x, u)$ then we want $\varphi_\alpha([(x, v)]) = \varphi_\alpha([(x, u)])$. But $(x, v) \sim (x, u)$ means $t_{\alpha\alpha}(x)v = u$. But $t_{\alpha\alpha}(x) = 1_V$ so that $v = u$ indeed. \square

Claim. φ_α is a family isomorphism.

Proof. φ_α is injective: If $\varphi_\alpha([x, v]) = \varphi_\alpha([y, u])$ then $(x, v) \sim (y, u)$. But since $t_{\alpha\alpha} = 1$, $(x, v) = (y, u)$. φ_α is surjective.

Note that $\varphi_\alpha^{-1} : U_\alpha \times V \rightarrow E|_{U_\alpha}$ is given by

$$\varphi_\alpha^{-1} = q \circ i_\alpha$$

and since these two maps are both continuous and open, φ_α is continuous and so is φ_α^{-1} .

Lastly, restricted to one point, we have

$$\begin{aligned} (\varphi_\alpha)_x &\equiv \varphi_\alpha|_{E_x} \\ &= \{[(y, v)] \in E \mid y = x\} \rightarrow \{x\} \times V : t_{\alpha\beta}(x) \end{aligned}$$

for some α, β , so that this is indeed a linear map. \square

8.1.15. *Remark.* If we are given two trivializations, (φ, U) and $(\tilde{\varphi}, U)$ (U being the same), then

$$\tilde{\varphi} \circ \varphi^{-1} : U \times V \rightarrow U \times V$$

for some vector space V . Call the map onto the second factor $g : U \times V \rightarrow V$, that is

$$\varphi(x, v) = (x, g(x, v))$$

Then $g(x) \in \text{Aut}(V)$ for any $x \in U$. In fact, any map $U \rightarrow \text{Aut}(V)$ defines another trivialization $\tilde{\varphi}$ from φ .

One may ask how the transition maps t change from via gauge transformations g . Then we have

$$\begin{aligned} \tilde{\varphi}_{\beta\alpha} &\equiv \tilde{\varphi}_\beta \circ \tilde{\varphi}_\alpha^{-1} \\ &= \tilde{\varphi}_\beta \circ 1 \circ 1 \circ \tilde{\varphi}_\alpha^{-1} \\ &= \tilde{\varphi}_\beta \circ \varphi_\beta^{-1} \circ \varphi_\beta \circ \varphi_\alpha^{-1} \circ \varphi_\alpha \circ \tilde{\varphi}_\alpha^{-1} \\ &= (\tilde{\varphi}_\beta \circ \varphi_\beta^{-1}) \circ (\varphi_\beta \circ \varphi_\alpha^{-1}) \circ (\varphi_\alpha \circ \tilde{\varphi}_\alpha^{-1}) \\ &= (\tilde{\varphi}_\beta \circ \varphi_\beta^{-1}) \circ \varphi_{\beta\alpha} \circ (\varphi_\alpha \circ \tilde{\varphi}_\alpha^{-1}) \end{aligned}$$

so that on the level of the vector component

$$\tilde{t}_{\beta\alpha}(x, \cdot) = g_\beta(x, \cdot) \circ t_{\beta\alpha}(x, \cdot) \circ g_\alpha(x, \cdot)^{-1}$$

8.1.16. Example. (The Tangent Bundle of a Manifold) In this example we consider $\text{Vect}_{\mathbb{R}}$ instead of $\text{Vect}_{\mathbb{C}}$. Let X be a differentiable real manifold of dimension n . At each point $x \in X$, a real vector space $T_x M$ is defined as the vector space of all linear and Leibnitz maps from $\mathcal{F}(X) \equiv \{f : X \rightarrow \mathbb{R} \mid f \text{ is smooth}\}$ to \mathbb{R} . Via the charts we get transition maps, which we can use to define a vector bundle as in 8.1.14.

Explicitly, the tangent bundle is defined by

$$TX \equiv \bigcup_{x \in X} \{x\} \times T_x M$$

and a projection is defined by $p : TX \rightarrow X$ by

$$TX \ni (x, V) \rightarrow x \in X$$

Then TX is a vector bundle with fiber \mathbb{R}^n . In fact, let A be an indexing set, and $(O_\alpha, \psi_\alpha)_{\alpha \in A}$ an atlas for X , that is $\psi_\alpha : O_\alpha \rightarrow U_\alpha$ is a homeomorphism for some $U_\alpha \in \text{Open}(\mathbb{R}^n)$, where $O_\alpha \in \text{Open}(X)$. Then for each $x \in X$ there is some $\alpha_x \in A$ such that $O_{\alpha_x} \ni x$. In this case,

$$\left\{ \mathcal{F}(X) \ni f \mapsto \left(\partial_i (f \circ \psi_{\alpha_x}^{-1}) \right) (\psi_{\alpha_x}(x)) \in \mathbb{R} \right\}_{i=1}^n$$

is a basis for $T_x X$, and so we obtain a local trivialization of $p^{-1}(O_{\alpha_x})$

$$\varphi_{\alpha_x} : p^{-1}(O_{\alpha_x}) \rightarrow O_{\alpha_x} \times \mathbb{R}^n$$

given by

$$T_{x'} X \ni (x', V) \xrightarrow{\varphi_{\alpha_x}} (x', V(\psi_{\alpha_x}^1), \dots, V(\psi_{\alpha_x}^n))$$

(recall that $V(\psi_{\alpha_x}^i)$ is the i th component of the vector $V \in T_{x'} X$ when $T_{x'} X$ is spanned in the basis above).

Then we define the topology on TX by

$$\text{Open}(TX) = \left\{ W \subseteq TX \mid \varphi_\alpha \left(W \cap p^{-1}(O_\alpha) \right) \in O_\alpha \times \mathbb{R}^n \forall \alpha \in A \right\}$$

For instance, note that TS^1 is a trivial vector bundle, that is

$$TS^1 \cong S^1 \times \mathbb{R}$$

whereas TS^2 is not

$$TS^2 \not\cong S^2 \times \mathbb{R}^2$$

8.1.17. Example. (The tautological line bundle) Let $V \in \text{Obj}(\text{Vect}_{\mathbb{C}})$ (not necessarily finite dimensional) and let

$$\begin{aligned} X &:= \mathbb{P}(V) \\ &\equiv \text{Gr}_1(V) \\ &\equiv \{\mathbb{C}v \subseteq V \mid v \in V \setminus \{0\}\} \end{aligned}$$

Thus every point in the base space X is really a line $\mathbb{C}v$ (a one-dimensional subspace) in V . Define

$$E := \{(\mathbb{C}v, \tilde{v}) \in X \times V \mid \tilde{v} \in \mathbb{C}v\}$$

with the subspace topology on it, with $p : E \rightarrow X$ given by $(\mathbb{C}v, \tilde{v}) \mapsto \mathbb{C}v$.

Then E is a vector bundle over X .

Proof. First, as restriction of a projection map, p is continuous. Next, let $\mathbb{C}v_0 \in X$ be given. Then

$$\begin{aligned} E_{\mathbb{C}v_0} &\equiv p^{-1}(\{\mathbb{C}v_0\}) \\ &= \{(\mathbb{C}v, \tilde{v}) \in E \mid p((\mathbb{C}v, \tilde{v})) \in \{\mathbb{C}v_0\}\} \\ &= \{(\mathbb{C}v, \tilde{v}) \in E \mid \mathbb{C}v = \mathbb{C}v_0\} \\ &= \{(\mathbb{C}v_0, \tilde{v}) \in X \times V \mid \tilde{v} \in \mathbb{C}v_0\} \\ &= \{\mathbb{C}v_0\} \times \mathbb{C}v_0 \\ &\simeq \mathbb{C} \end{aligned}$$

where the last equivalence is a morphism in $\text{Mor}_{\text{Vect}_{\mathbb{C}}}$. The vector space operations are defined as $a_{\mathbb{C}v_0} : E_{\mathbb{C}v_0}^2 \rightarrow E_{\mathbb{C}v_0}$ given by

$$((\mathbb{C}v_0, \lambda_1 v_0), (\mathbb{C}v_0, \lambda_2 v_0)) \mapsto (\mathbb{C}v_0, (\lambda_1 + \lambda_2) v_0)$$

This is continuous because it is the restriction of $a : V^2 \rightarrow V$ to $\mathbb{C}v_0^2$, that is, the composition of two continuous maps, (with the inclusion map which is by definition continuous) and thus a continuous map. Similarly, $m_{\mathbb{C}v_0} : \mathbb{C} \times E_{\mathbb{C}v_0} \rightarrow E_{\mathbb{C}v_0}$ given by

$$(\alpha, (\mathbb{C}v_0, \lambda v_0)) \mapsto (\mathbb{C}v_0, \alpha \lambda v_0)$$

is continuous because it is the restriction of $m : \mathbb{C} \times V \rightarrow \mathbb{C}$ to $\mathbb{C} \times \mathbb{C}v_0$.

Then E is a family of vector spaces over X indeed.

Next, we want to show E is locally trivial so that it is also a vector bundle. For simplicity assume that V has the structure of a Hilbert space (this is not strictly necessary but will make notation easier). Then define $U_{v_0} \in \text{Nbhd}_X(\mathbb{C}v_0)$

as

$$U_{v_0} := \{ Cv \mid \langle v, v_0 \rangle \neq 0 \}$$

which is open because

$$\{ v \in V \mid \langle v, v_0 \rangle = 0 \} \in \text{Closed}(V)$$

and the the quotients map $V \rightarrow X$ is open. Then we should have

$$\underbrace{E|_{U_{v_0}}}_{\cong p^{-1}(U_{v_0})} \cong U_{v_0} \times Cv_0$$

Indeed,

$$\begin{aligned} E|_{U_{v_0}} &\equiv p^{-1}(U_{v_0}) \\ &= \{ (Cv, \tilde{v}) \in E \mid p((Cv, \tilde{v})) \in U_{v_0} \} \\ &= \{ (Cv, \tilde{v}) \in E \mid Cv \in U_{v_0} \} \\ &= \{ (Cv, \tilde{v}) \in E \mid \langle v, v_0 \rangle \neq 0 \} \end{aligned}$$

this is homeomorphic to $U_{v_0} \times Cv_0$ via the map $\varphi : (Cv, \tilde{v}) \mapsto (Cv, \frac{\langle \tilde{v}, v_0 \rangle}{\langle v_0, v_0 \rangle} v_0)$, because if $\langle v, v_0 \rangle \neq 0$ and $\tilde{v} \in Cv$ then $Cv \cong Cv_0$ via the orthogonal projection. This map is continuous, with continuous inverse, and bijective. \square

8.1.18. Example. Let \mathcal{H} be a Hilbert space over \mathbb{C} , X be a manifold and let $E_x \subseteq \mathcal{H}$ be a subspace of \mathcal{H} which depends smoothly on $x \in X$. Then we define

$$E := \{ (x, u) \in X \times \mathcal{H} \mid u \in E_x \} \subseteq X \times \mathcal{H}$$

with the subset topology, and $p : E \rightarrow X$ by $(x, u) \mapsto x$. Then $p : E \rightarrow X$ is a complex vector bundle, which is a subbundle (to be defined later) of the trivial bundle $X \times \mathcal{H}$. Even though $X \times \mathcal{H}$ is trivial, E may be non-trivial. For instance, one way to obtain the subspaces E_x is to assume we have an operator $H : \mathcal{H} \rightarrow \mathcal{H}$ which depends on a parameter $x \in X$, and which has an isolated eigenvalue $\lambda(x) \in \mathbb{C}$ with corresponding eigenspace E_x .

8.1.19. Remark. Let $s : X \rightarrow E$ be a section of a vector bundle. For any $x \in X$, we have some $U \in \text{Nbhd}_X(x)$ such that $E|_U \stackrel{\varphi}{\cong} U \times V$ for some $V \in \text{Obj}(\text{Vect}_{\mathbb{C}})$. Then

$$\varphi(s(x')) \in U \times V \quad \forall x' \in U$$

and if $\pi_2 : U \times V \rightarrow V$ is the projection onto the second slot, then

$$\pi_2(\varphi(s(x'))) \in V \quad \forall x' \in U$$

so that we have a continuous vector valued function

$$\pi_2 \circ \varphi \circ s : U \rightarrow V$$

8.1.20. Claim. Let a vector bundle $p : E \rightarrow X$ be given such that $x \mapsto \dim(E_x)$ is constant, say, with value $n \in \mathbb{N}_{>0}$. Then E is trivial iff there are n sections $\{s_i\}_{i=1}^n$ such that for each $x \in X$,

$$\text{span}(s_i(x) \mid i \in \{1, \dots, n\}) = E_x$$

Proof. First assume that we have such a set of sections $\{s_i\}_{i=1}^n$ with

$$\text{span}(s_i(x) \mid i \in \{1, \dots, n\}) = E_x$$

for all $x \in X$. Let $e \in E$. Then $p(e) \in X$. Then there is some $U \in \text{Nbhd}_{p(e)}(X)$ such that $\varphi : E|_U \rightarrow U \times V$ is a linear isomorphism for $V \cong \mathbb{C}^n$. Then let $(\varphi_1, \dots, \varphi_n)$ be the expansion coefficients of $\pi_2(\varphi(e)) \in V$ in the basis $\{s_i(p(e))\}_{i=1}^n$, that is,

$$\varphi_i := \langle s_i(p(e)), \pi_2(\varphi(e)) \rangle$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $V \cong \mathbb{C}^n$. Now we can define

$$\psi : E \rightarrow X \times \mathbb{C}^n$$

by

$$\varphi(e) := (p(e), \varphi_1, \dots, \varphi_n)$$

and must verify that $\psi \in \text{Mor}_{\text{Vect}_{\mathbb{C}}^n(X)}$. By construction we have that ψ respects the base space X . Next let $x \in X$ be given. Then $\psi_x : E_x \rightarrow \{x\} \times \mathbb{C}^n$ is a \mathbb{C} -linear, because $e \mapsto \langle s_i(p(e)), \pi_2(\varphi(e)) \rangle$ is linear inside the same fiber ($p(e_1) = p(e_2)$):

$$\begin{aligned} \alpha e_1 + e_2 &\mapsto \langle s_i(p(e)), \pi_2(\varphi(\alpha e_1 + e_2)) \rangle \\ &= \langle s_i(p(e)), \alpha \pi_2(\varphi(e_1)) + \pi_2(\varphi(e_2)) \rangle \\ &= \alpha \langle s_i(p(e)), \pi_2(\varphi(e_1)) \rangle + \langle s_i(p(e)), \pi_2(\varphi(e_2)) \rangle \end{aligned}$$

Routine verifications show that ψ is a continuous bijection with continuous inverse.

Conversely, if E is trivial, then we have some $\psi : E \rightarrow X \times \mathbb{C}^n$ which is a vector bundle isomorphism. Then we may define the sections as

$$s_i(x) := \psi^{-1}(e_i)$$

where e_i is the i th basis vector of \mathbb{C}^n . □

8.1.21. Example. Consider $X = S^1$ (the circle) and $V = \mathbb{R}$. Then there exactly two possibilities up to isomorphisms. Using 8.1.14 we can work with transition maps. To trivialize $S^1 \equiv \{ e^{i\varphi} \in \mathbb{C} \mid \varphi \in \mathbb{R} \}$, we choose a covering of two open sets: $U_1 := S^1 \setminus \{-1\}$ and $U_2 := S^1 \setminus \{+1\}$. Then $U_1 \cap U_2 = S^1 \setminus \{\pm 1\}$. Then to specify E we only need to specify a map

$$t_{12} : U_1 \cap U_2 \times \mathbb{R} \rightarrow \mathbb{R}$$

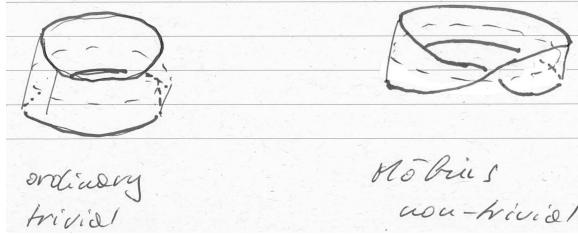
Then because $t_{12} \in \text{Aut}(\mathbb{R})$ (that is, t_{12} cannot be zero and thus cannot change sign along $U_1 \cap U_2$) we obtain that the two possibilities, up to isomorphisms, are

$$t_{12}(e^{i\varphi}, v) := v \quad \forall e^{i\varphi} \in S^1, v \in \mathbb{R}$$

and

$$t_{12}(e^{i\varphi}, v) := \begin{cases} v & \Im\{e^{i\varphi}\} > 0 \\ -v & \Im\{e^{i\varphi}\} < 0 \end{cases} \quad \forall e^{i\varphi} \in S^1, v \in \mathbb{R}$$

The first option corresponds to the trivial bundle $S^1 \times \mathbb{R}$ whereas the second one corresponds to the Moebius band.



Then there are exactly two possibilities up to vector bundle isomorphisms.

8.2. Bloch Decomposition.

In this section we will be working in two space dimensions.

8.2.1. Definition. An *affine space* is a set together with a group homomorphism $t : V \rightarrow \text{Sym}(A)$ where V is a some vector space considered as a group under vector addition, and $\text{Sym}(A)$ is the group of bijections $A \rightarrow A$, such that for any $a \in A$, the map $\varphi_a : V \rightarrow A$ given by $v \mapsto t(v)a$ is bijective.

8.2.2. Remark. Thus, an affine space is a vector space without an origin, and the maps φ_a make this identification: $\varphi_a^{-1}(a)$ maps to $0 \in \mathbb{R}^2$.

We allow for two possibilities on the physical space:

Case 1. (Continuous) Physical space X is the affine space \mathbb{E}^2 (the affine space associated with \mathbb{R}^2).

Case 2. (Discrete) Physical space X is a lattice $L \subseteq \mathbb{E}^2$, that is, after choosing an origin, we would get the set $\{ n_1 a_1 + n_2 a_2 \in \mathbb{R}^2 \mid (n_1, n_2) \in \mathbb{Z}^2 \}$ with a_1 and a_2 two linearly independent vectors of \mathbb{R}^2 which determine L .

8.2.3. Definition. (*Translation Symmetry Group*) The translation symmetry group, \mathcal{L} , is a group isomorphic to \mathbb{Z}^2 , which acts on X by

$$x \mapsto x + n_1 a_1 + n_2 a_2 =: T_n x \tag{29}$$

for any $n \in \mathbb{Z}^2$. a_1 and a_2 are two linearly independent vectors in \mathbb{R}^2 .

8.2.4. Definition. (*Unit Cell*) The unit cell is defined as $C := X/\mathcal{L}$. In the continuous case, we have

$$\begin{aligned} C &\equiv \mathbb{E}^2/\mathcal{L} \\ &\cong \mathbb{T}^2 \end{aligned}$$

where \mathbb{T}^2 is the 2-torus, but the isomorphism is not canonical. In the discrete case we have

$$\begin{aligned} C &\equiv L/\mathcal{L} \\ &\cong \{1, \dots, N\} \end{aligned} \tag{30}$$

that is, a set of finitely many points.

8.2.5. Definition. (*Characters*) If G is a locally compact Abelian topological group, a character of G is a continuous group homomorphism $G \rightarrow S^1$.

8.2.6. Definition. (*Dual Group*) The dual group G^* is the group of characters of G , with the group operation given by pointwise multiplication of characters, and inverse being the complex conjugate of a character, and topology being the compact-open topology as a subset of all continuous maps $G \rightarrow S^1$.

8.2.7. Definition. (*Exponential of classes in the 2-torus*) Let $k \in \mathbb{T}^2$ and $n \in \mathbb{Z}^2$. Recall that \mathbb{T}^2 is the 2-torus defined by $\mathbb{T}^2 \equiv (\mathbb{R}/2\pi\mathbb{Z})^2$, so that $k = (\kappa_1 + 2\pi\mathbb{Z}) \times (\kappa_2 + 2\pi\mathbb{Z})$ (a product of two sets) for some $\kappa \in \mathbb{R}^2$ (the choice of κ is not unique). Define

$$\exp(i\kappa \cdot n) := \exp(i\kappa \cdot n)$$

This is well defined. Indeed, if $\kappa' \in \mathbb{R}^2$ is another choice such that $k = (\kappa'_1 + 2\pi\mathbb{Z}) \times (\kappa'_2 + 2\pi\mathbb{Z})$ as well then $(\kappa_i - \kappa'_i) \in 2\pi\mathbb{Z}$ so that

$$\begin{aligned} \exp(i\kappa' \cdot n) &= \exp\left(i \sum_{i=1}^2 \kappa'_i n_i\right) \\ &= 1 \times \exp\left(i \sum_{i=1}^2 \kappa'_i n_i\right) \\ &= \exp\left(i \sum_{i=1}^2 (\kappa_i - \kappa'_i)\right) \exp\left(i \sum_{i=1}^2 \kappa'_i n_i\right) \\ &= \exp\left(i \sum_{i=1}^2 (\kappa'_i n_i + (\kappa_i - \kappa'_i) n_i)\right) \\ &= \exp\left(i \sum_{i=1}^2 (\kappa_i n_i)\right) \end{aligned}$$

Note that this was possible because $n \in \mathbb{Z}^2$. If n were in \mathbb{R}^2 this would have failed, and so

$$\exp(i\kappa \cdot x)$$

is *not* defined for arbitrary $x \in \mathbb{R}^2$.

8.2.8. Definition. *The Brillouin zone* is the dual group of $\mathcal{L}, \mathcal{L}^*$. Explicitly, it is the set of all maps

$$\mathbb{Z}^2 \ni n \mapsto e^{ik \cdot n} \in S^1$$

where the maps are indexed by $k \in \mathbb{T}^2$.

The wave functions are then maps $\psi : X \rightarrow \mathbb{C}$ belonging to a Hilbert space

$$\begin{aligned} \mathcal{H} &\equiv L^2(X) \\ &= \begin{cases} L^2(\mathbb{E}^2) \\ L^2(L) \end{cases} \end{aligned}$$

which is the space of all square integrable or summable maps.

\mathcal{L} is represented on \mathcal{H} by a group homomorphism $U : \mathcal{L} \rightarrow U(\mathcal{H})$ into the group unitary maps on \mathcal{H} . We actually also allow that $U(n) : \mathcal{H} \rightarrow \mathcal{H}$ act on arbitrary maps $X \rightarrow \mathbb{C}$ and not just on \mathcal{H} . The fact that U is a group homomorphism implies that

$$U(n) \circ U(m) = U(n+m)$$

8.2.9. Example. One example for a choice of U is simply ordinary translations, that is,

$$((U(n))\psi)(x) := \psi(T_{-n}(x))$$

where T_n was defined in (29). Then

$$\begin{aligned} (U(n)U(m)\psi)(x) &= U(n)\psi(T_{-m}(x)) \\ &= \psi(T_{-m}(T_{-n}(x))) \end{aligned}$$

but T is also a group homomorphism so that

$$T_{-m}T_{-n} = T_{-(n+m)}$$

as desired.

The following material about direct integrals may be found in [37] section XIII.16 for the case of constant fibers or in [31]. Its essence is the generalization of complete reducibility to the case of unitary representations on infinite dimensional Hilbert spaces.

8.2.10. Definition. (*Direct Integrals of Hilbert Spaces*) Let (X, μ) be a measure space. Assume that for each $x \in X$, $\tilde{\mathcal{H}}(x)$ is a Hilbert space. Define

$$\int_X^{\oplus} \tilde{\mathcal{H}}(x) d\mu(x) := \left\{ \left[\psi : X \rightarrow \bigcup_{x \in X} \tilde{\mathcal{H}}(x) \right] \mid \psi(x) \in \tilde{\mathcal{H}}(x) \forall x \in X \wedge \left(x \mapsto \|\psi(x)\|_{\tilde{\mathcal{H}}(x)} \in L^2(X, \mu) \right) \right\}$$

where $\psi \sim \varphi$ iff they agree μ -almost-everywhere. As usual we will drop the class notation $[\psi]$ in favor of the lighter ψ . Define vector addition and scalar multiplication on $\int_X^{\oplus} \tilde{\mathcal{H}}(x) d\mu(x)$ pointwise in x . An inner product on $\int_X^{\oplus} \tilde{\mathcal{H}}(x) d\mu(x)$ is defined via

$$\langle \psi, \varphi \rangle_{\int_X^{\oplus} \tilde{\mathcal{H}}(x) d\mu(x)} := \int_X \langle \psi(x), \varphi(x) \rangle_{\tilde{\mathcal{H}}(x)} d\mu(x) \quad (31)$$

8.2.11. Remark. If μ is a finite sum of delta measures then we recover the usual notion of direct sum of Hilbert spaces. Thus, we obtain a sort of “continuous direct sum of vector spaces”.

8.2.12. Claim. $\int_X^{\oplus} \tilde{\mathcal{H}}(x) d\mu(x)$ is a Hilbert space, separable if (X, μ) is separable.

8.2.13. Claim. (*Complete Reducibility*) For a given separable Hilbert space \mathcal{H} we have

$$\mathcal{H} = \int_{\mathbb{T}^2}^{\oplus} \tilde{\mathcal{H}}(k) dk$$

where

$$\tilde{\mathcal{H}}(k) := \left\{ \psi \in \mathcal{H} \mid U_n \psi = e^{-ik \cdot n} \psi \quad \forall n \in \mathcal{L} \right\} \quad \forall k \in \mathbb{T}^2$$

and any given $\psi \in \mathcal{H}$ may be written as

$$\psi = \int_{\mathbb{T}^2} \tilde{\psi}(k) dk$$

where $(\tilde{\psi} : \mathbb{T}^2 \rightarrow \bigcup_{k \in \mathbb{T}^2} \tilde{\mathcal{H}}(k)) \in \int_{\mathbb{T}^2}^{\oplus} \tilde{\mathcal{H}}(k) dk$ (an integral of vector-valued functions, defined for instance in [40] pp. 77)

Proof. One has to show an isomorphism between

$$\mathcal{H} \rightarrow \int_{\mathbb{T}^2}^{\oplus} \tilde{\mathcal{H}}(k) dk$$

Let $\psi \in \mathcal{H}$ be given. Define the map $\tilde{\psi} : \mathbb{T}^2 \rightarrow \mathcal{H}$ by

$$k \xrightarrow{\tilde{\psi}} \frac{1}{(2\pi)^2} \sum_{n \in \mathbb{Z}^2} e^{ik \cdot n} U_n \psi \quad (32)$$

Then the claim is that

$$\mathcal{H} \ni \psi \mapsto \tilde{\psi} \in \mathcal{H}^{\mathbb{T}^2}$$

is a linear isometric bijection. First we show $\tilde{\psi}(k) \in \tilde{\mathcal{H}}(k)$. If $m \in \mathcal{L}$,

$$\begin{aligned} U_m \tilde{\psi}(k) &= \frac{1}{(2\pi)^2} \sum_{n \in \mathbb{Z}^2} e^{ik \cdot n} U_m U_n \psi \\ &= \frac{1}{(2\pi)^2} \sum_{n \in \mathbb{Z}^2} e^{ik \cdot n} U_{n+m} \psi \\ &\stackrel{n \mapsto n+m}{=} \frac{1}{(2\pi)^2} \sum_{n \in \mathbb{Z}^2} e^{ik \cdot (n-m)} U_n \psi \\ &= e^{-ik \cdot m} \tilde{\psi}(k) \end{aligned}$$

Next,

$$\begin{aligned} \int_{\mathbb{T}^2} \tilde{\psi}(k) dk &= \int_{\mathbb{T}^2} \left(\frac{1}{(2\pi)^2} \sum_{n \in \mathbb{Z}^2} e^{ik \cdot n} U_n \psi \right) dk \\ &= \sum_{n \in \mathbb{Z}^2} \underbrace{\frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} e^{ik \cdot n} dk}_{\delta_{n,0}} U_n \psi \\ &= U_0 \psi \\ &= \psi \end{aligned}$$

□

8.2.14. Claim. A vector bundle is defined via

$$E := \left\{ (k, \psi) \in \mathbb{T}^2 \times \mathcal{H} \mid \psi \in \tilde{\mathcal{H}}(k) \right\}$$

where E is given the subspace topology, and the projection map is defined by $p : E \rightarrow \mathbb{T}^2$ by $(k, \psi) \mapsto k$.

Proof.

□

8.2.15. Remark. Using the above construction of 8.2.13, for any $\psi, \tilde{\psi} : \mathbb{T}^2 \rightarrow E$ is a section.

8.2.16. Claim. For each $k \in \mathbb{T}^2$, $\tilde{\mathcal{H}}(k) \cong L^2(\mathcal{C})$ where \mathcal{C} is from 8.2.4.

Proof. Let $k \in \mathbb{T}^2$. Note that $\mathcal{C} \equiv X/\mathcal{L} \equiv \{[x] \mid x \in X\}$ and $[x] \equiv \{y \in X \mid \exists n \in \mathcal{L} : T_n x = y\}$. Make an arbitrary choice of a primitive cell

$$\mathcal{P} \subseteq X \tag{33}$$

which is bijective with \mathcal{C} , and simply connected. Now define a map $\eta : \tilde{\mathcal{H}}(k) \rightarrow L^2(\mathcal{P})$ on any $\psi \in \tilde{\mathcal{H}}(k)$ by:

$$\eta(\psi) := \psi|_{\mathcal{P}}$$

For any $x \in X$, there is a $n_{(x_0, x)} \in \mathcal{L}$ unique such that $(x - n_{(x_0, x)}) \in \mathcal{P}$.

Then the inverse of η on some $f \in L^2(\mathcal{P})$ is given by

$$(\eta^{-1}(f))(x) = e^{ik \cdot n_{(x_0, x)}} f(x - n_{(x_0, x)}) \quad \forall x \in X$$

Then using the fact that $n_{(x_0, x+n)} = n_{(x_0, x)} + n$ for all $n \in \mathcal{L}$ we have

$$\begin{aligned} (U_{-n}(\eta^{-1}(f)))(x) &= (\eta^{-1}(f))(T_n x) \\ &= (\eta^{-1}(f))(x + n) \\ &= e^{ik \cdot n_{(x_0, x+n)}} f(x + n - n_{(x_0, x+n)}) \\ &= e^{ik \cdot n_{(x_0, x)} + n} f(x + n - n_{(x_0, x)} - n) \\ &= e^{ik \cdot n} e^{ik \cdot n_{(x_0, x)}} f(x - n_{(x_0, x)}) \\ &= e^{ik \cdot n} (\eta^{-1}(f))(x) \end{aligned}$$

so that $\eta^{-1}(f) \in \tilde{\mathcal{H}}(k)$ indeed. It is the inverse because $n_{(x_0, x)}|_{x \in \mathcal{P}} = 0$ and

$$\begin{aligned} (\eta^{-1}\eta\psi)(x) &= e^{ik \cdot n_{(x_0, x)}} \psi(x - n_{(x_0, x)}) \\ &= e^{ik \cdot n_{(x_0, x)}} U_{n_{(x_0, x)}} \psi(x) \\ &= \psi(x) \end{aligned}$$

Next use the fact that $\mathcal{C} \cong \mathcal{P}$ to conclude $L^2(\mathcal{C}) \cong L^2(\mathcal{P})$.

□

8.2.17. Corollary. $p : E \rightarrow \mathbb{T}^2$ is a trivial vector bundle.

Proof. Define a map $\zeta : E \rightarrow \mathbb{T}^2 \times L^2(\mathcal{C})$ by

$$\zeta((k, \psi)) := (k, \eta(\psi))$$

ζ is bijective, continuous and with a continuous inverse, and restricted to each k , it is η , which is linear.

□

8.2.18. Claim. E also has a local trivialization using “Bloch waves”.

Proof. We will show that the fiber can be chosen as $\tilde{\mathcal{H}}(0)$. Let $k_0 \in \mathbb{T}^2$ be given. As \mathbb{T}^2 is a manifold, $\exists U \in \text{Nbhd}_{\mathbb{T}^2}(k_0)$ such that

$$\eta : U \rightarrow \eta(U) \in \text{Open}(\mathbb{R}^2)$$

is a homeomorphism. Let $x_0 \in X$ be given, and declare it to be the origin of X , so that $\varphi_{x_0}^{-1}(x) \in \mathbb{R}^2$ or $\varphi_{x_0}^{-1}(x) \in \mathbb{Z}^2$ for all $x \in X$ (recall the map φ_{x_0} from 8.2.1 which converts an affine space into a linear space). For notational simplicity, we will write simply k instead of $\eta(k)$ and x instead of $\varphi_{x_0}^{-1}(x)$.

Now pick any

$$(k, \psi) \in E|_U$$

By definition it follows that $k \in U$ and $\psi \in \tilde{\mathcal{H}}(k)$. If we define a function

$$u_k(x) := e^{-ik \cdot x} \psi(x) \quad \forall x \in X$$

Then we can write the trivialization map as

$$E|_U \rightarrow U \times \tilde{\mathcal{H}}(0)$$

$$(k, \psi) \mapsto (k, u_k)$$

Let us show that this map is well-defined. So we should show that $u_k \in \tilde{\mathcal{H}}(0)$, that is, that

$$U_n u_k = u_k \forall n \in \mathcal{L}$$

Indeed,

$$\begin{aligned} (U_n u_k)(x) &= u_k(x - n) \\ &= e^{-ik \cdot (x - n)} \psi(x - n) \\ &= e^{-ik \cdot (x - n)} U_n \psi(x) \\ \stackrel{\psi \in \tilde{\mathcal{H}}(k)}{=} & e^{-ik \cdot (x - n)} e^{-ik \cdot n} \psi(x) \\ &= e^{-ik \cdot x} \psi(x) \\ &= u_k(x) \end{aligned}$$

The map u_k is usually called a “Bloch wave”. Unlike $\psi \in \tilde{\mathcal{H}}(k)$, u_k is \mathcal{L} -periodic. It is also customary to write

$$\psi(x) = e^{ik \cdot x} u_k(x) \in \tilde{\mathcal{H}}(k)$$

for all $k \in U$ (but not all $k \in \mathbb{T}^2$), $x \in X$. □

8.2.19. *Remark.* 8.2.18 relies on the isomorphism $\varphi_{x_0}^{-1} : X \rightarrow \mathbb{R}^2$ or $\varphi_{x_0}^{-1} : X \rightarrow \mathbb{Z}^2$ and more importantly on the chart

$$\eta : \mathbb{T}^2 \supseteq U \rightarrow \eta(U) \subseteq \mathbb{R}^2$$

This is something that is swept under the rug in physics textbooks such as [4] (pp. 133), where equation (8.3) does not mention that the k values for which it holds are restricted to a chart of \mathbb{T}^2 , and that this equation cannot hold globally on \mathbb{T}^2 ! In this sense, [4]’s equation (8.6) is much more accurate because it does not rely on a chart for \mathbb{T}^2 , but rather only on 8.2.7.

8.2.20. *Remark.* Note that even though for the trivialization of 8.2.18 requires more than one chart on the base space \mathbb{T}^2 , the bundle E is of course still trivial as shown in 8.2.17 (a bundle being trivial is an intrinsic property which does not depend on any one choice of charts). This can also be shown explicitly with the choice of trivialization given in 8.2.18, which is left as an exercise to the reader.

8.2.21. **Fact.** We assume the Hamiltonian $H : \mathcal{H} \rightarrow \mathcal{H}$ is \mathcal{L} -invariant:

$$[U_n, H] = 0 \quad \forall n \in \mathcal{L} \tag{34}$$

8.2.22. **Definition.** (*Direct Integrals of Operators*) Let a function

$$\tilde{H} : X \rightarrow \bigcup_{x \in X} \mathcal{S}(\tilde{\mathcal{H}}(x))$$

be given (where $\mathcal{S}(\tilde{\mathcal{H}}(x))$ is the set of self-adjoint linear operators (not necessarily bounded) $\mathcal{D}(\tilde{H}(x)) \rightarrow \tilde{\mathcal{H}}(x)$ such that $\tilde{H}(x) \in \mathcal{S}(\tilde{\mathcal{H}}(x))$ for all $x \in X$ and $x \mapsto \langle \varphi(x), (\tilde{H}(x) + i)^{-1} \psi(x) \rangle_{\tilde{\mathcal{H}}(x)}$ is measurable for all $(\varphi, \psi) \in \left(\int_X^\oplus \tilde{\mathcal{H}}(x) d\mu(x)\right)^2$). We define a new operator

$$\int_X^\oplus \tilde{H}(x) d\mu(x) : \mathcal{D}\left(\int_X^\oplus \tilde{H}(x) d\mu(x)\right) \rightarrow \int_X^\oplus \tilde{\mathcal{H}}(x) d\mu(x)$$

by

$$\left(\left(\int_X^\oplus \tilde{H}(x') d\mu(x') \right) (\psi) \right) (x) := \tilde{H}(x) \psi(x) \quad \forall x \in X$$

where

$$\mathcal{D}\left(\int_X^\oplus \tilde{H}(x) d\mu(x)\right) := \left\{ \psi \in \int_X^\oplus \tilde{\mathcal{H}}(x) d\mu(x) \mid \psi(x) \in \mathcal{D}(\tilde{H}(x)) \text{ a.e.} \wedge \int_X \|\tilde{H}(x) \psi(x)\|_{\tilde{\mathcal{H}}(x)}^2 d\mu(x) < \infty \right\}$$

8.2.23. **Claim.** $\int_X^\oplus \tilde{H}(x) d\mu(x)$ as defined above is self-adjoint.

Proof. Theorem XIII.85 in [37]. Note that we have a slightly different form where the fibers are not constant. □

8.2.24. Claim. As a result of (34), we have

$$H = \int_{\mathbb{T}^2}^{\oplus} \tilde{H}(k) dk \quad (35)$$

where for each $k \in \mathbb{T}^2$, $\tilde{H}(k) : \mathcal{D}(\tilde{H}(k)) \rightarrow \tilde{\mathcal{H}}(k)$ is some self-adjoint operator. Furthermore, $\tilde{H}(k)$ has discrete spectrum (as $\tilde{H}(k) \cong L^2(\mathbb{C})$ and \mathbb{C} is compact) with eigenvalues which we label $(\varepsilon_n(k))_{n \in \mathbb{N}}$.

Proof. Define $\tilde{H}(k) : \mathcal{D}(\tilde{H}(k)) \rightarrow \tilde{\mathcal{H}}(k)$ by

$$\tilde{H}(k) := H|_{\tilde{\mathcal{H}}(k)}$$

with

$$\mathcal{D}(\tilde{H}(k)) := \mathcal{D}(H) \cap \tilde{\mathcal{H}}(k)$$

For $\tilde{H}(k)$ to be well defined, we need that $H\tilde{\psi}(k) \in \tilde{\mathcal{H}}(k)$ for all $\tilde{\psi}(k) \in \tilde{\mathcal{H}}(k)$:

$$\begin{aligned} U_n H \tilde{\psi}(k) &= H U_n \tilde{\psi}(k) \\ &= H e^{-ik \cdot n} \tilde{\psi}(k) \\ &= e^{-ik \cdot n} H \tilde{\psi}(k) \end{aligned}$$

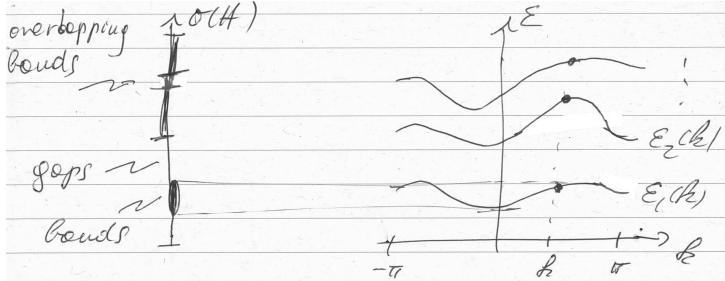
so that is indeed the case.

Then we would like to have $H = \int_{\mathbb{T}^2}^{\oplus} \tilde{H}(k) dk$. So we let $\int_{\mathbb{T}^2}^{\oplus} \tilde{H}(k) dk$ act on some $\psi \in \mathcal{D}(H)$, where for any $\psi \in \mathcal{H}$, $\tilde{\psi} : \mathbb{T}^2 \rightarrow \mathcal{H}$ was defined in (32):

$$\begin{aligned} \left(\int_{\mathbb{T}^2}^{\oplus} \tilde{H}(k) dk \right) \psi &= \left(\int_{\mathbb{T}^2}^{\oplus} \tilde{H}(k) dk \right) \left(\int_{\mathbb{T}^2} \tilde{\psi}(k) dk \right) \\ &\equiv \int_{\mathbb{T}^2} (\tilde{H}(k) \tilde{\psi}(k)) dk \\ &\equiv \int_{\mathbb{T}^2} \left(H \left(\frac{1}{(2\pi)^2} \sum_{n \in \mathbb{Z}^2} e^{ik \cdot n} U_n \psi \right) \right) dk \\ &= \sum_{n \in \mathbb{Z}^2} \underbrace{\frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} e^{ik \cdot n} dk}_{\delta_{n,0}} U_n H \psi \\ &= H \psi \end{aligned}$$

□

A schematic view of the spectrum would be as follows:



And we would have

$$\underbrace{\sigma(H)}_{\text{continuous}} = \bigcup_{k \in \mathbb{T}^2} \underbrace{\sigma(\tilde{H}(k))}_{\text{discrete}}$$

8.2.25. Claim. Let P be a projection associated to an isolated part of $\sigma(H)$ (one or more bands). (34) then implies that

$$[P, U_n] = 0 \forall n \in \mathcal{L}$$

As a result, we may also decompose P as

$$P = \int_{\mathbb{T}^2}^{\oplus} \tilde{P}(k) dk$$

Proof. We may write

$$P = \frac{i}{2\pi} \int_{\Gamma} (H - z\mathbb{1})^{-1} dz$$

where Γ is a closed contour around the isolated part of the spectrum. Then we may write

$$(H - z\mathbb{1})^{-1} = -\frac{1}{z} \sum_{j=0}^{\infty} \left(\frac{1}{z} H \right)^j$$

which converges if $\|H\| < |z|$. Then, we have

$$\begin{aligned} [P, U_n] &= \left[\frac{i}{2\pi} \int_{\Gamma} (H - z\mathbb{1})^{-1} dz, U_n \right] \\ &= \left[\frac{i}{2\pi} \int_{\Gamma} -\frac{1}{z} \sum_{j=0}^{\infty} \left(\frac{1}{z} H \right)^j dz, U_n \right] \\ &= \frac{i}{2\pi} \int_{\Gamma} -\frac{1}{z} \sum_{j=0}^{\infty} \left(\frac{1}{z} \right)^j [H^j, U_n] dz \\ &= 0 \end{aligned}$$

the rest follows (8.2.24). Explicitly,

$$\tilde{P}(k) := P|_{\tilde{\mathcal{F}}(k)} \quad \forall k \in \mathbb{T}^2$$

□

8.2.26. Corollary. Because P is associated with an isolated part of the spectrum, $\dim(\text{im}(\tilde{P}(k)))$ is constant in k . As a result, a sub-bundle of E is defined via $\text{im}(\tilde{P}(k)) \hookrightarrow E_k$. Denote this sub-bundle by \mathcal{P} .

Proof. Use [5] lemma 1.3.1 with the map $i : \mathcal{P} \hookrightarrow E$ which is a monomorphism because the dimension is constant. □

8.2.27. Remark. \mathcal{P} may be non-trivial, despite E always being trivial (8.2.17).

8.3. Magnetic Translations. The work on magnetic translations was first presented in [47].

For this section, let $x_0 \in X$ be given and then we have the bijection $\varphi_{x_0} : \mathbb{R}^2 \rightarrow X$ so that for all intents and purposes in this section X is \mathbb{R}^2 (continuous case) or a subset of \mathbb{R}^2 (discrete case). Again we have $\mathcal{L} = \mathbb{Z}^2$.

8.3.1. Fact. The system is under the effect of a magnetic field $B : X \rightarrow \mathbb{R}$ such that B is periodic:

$$B(x + n) = B(x) \quad \forall n \in \mathcal{L}$$

8.3.2. Remark. As before, our system is two dimensional on X and we always assume that the magnetic field is perpendicular to it, so that really it is simply a scalar.

8.3.3. Example. A homogeneous field $B(x) = B_0$ for some fixed $B_0 \in \mathbb{R}^2$ is an example of a periodic magnetic field.

8.3.4. Definition. Define the flux through the primitive cell \mathcal{P} ((33)) as $\phi := \int_{\mathcal{P}} B(x) dx$.

8.3.5. Claim. The magnetic vector potential A corresponding to B cannot be chosen to be periodic if $\phi \neq 0$.

Proof. Since B is periodic, we may write a Fourier decomposition of it as

$$B(x) = \sum_{q \in (2\pi\mathbb{Z})^2} \tilde{B}(q) e^{iq \cdot x}$$

with

$$\tilde{B}(q) = \frac{1}{(2\pi)^2} \int_{\mathcal{P}} B(x) e^{-iq \cdot x} dx$$

In particular,

$$\begin{aligned} \phi &\equiv \int_{\mathcal{P}} B(x) dx \\ &= (2\pi)^2 \tilde{B}(0) \end{aligned}$$

If A is chosen to be periodic, then it may also be decomposed as

$$A(x) = \sum_q \tilde{A}(q) e^{iq \cdot x}$$

Now we have the relation (repeating indices are summed over {1, 2, 3})

$$\begin{aligned} B &\equiv (\nabla \times A)_3 \\ &\equiv \epsilon_{3ij} \partial_i A_j \\ &= \epsilon_{3ij} \partial_i \sum_q \tilde{A}_j(q) e^{iq \cdot x} \\ &= \epsilon_{3ij} \sum_q i q_i \tilde{A}_j(q) e^{iq \cdot x} \end{aligned}$$

from which it follows due to the orthogonality of $e^{iq \cdot x}$ that

$$\tilde{B}(q) = i \epsilon_{3ij} q_i \tilde{A}_j(q) \tag{36}$$

Hence

$$\begin{aligned}\phi &= (2\pi)^2 \tilde{B}(0) \\ &= (2\pi)^2 i\varepsilon_{3ij} \cdot 0 \cdot \tilde{A}_j(q) \\ &= 0\end{aligned}$$

Thus we have shown that if A is periodic, $\phi = 0$.

For completeness, the solution of (36) is given by

$$\tilde{A}_i(q) = \begin{cases} \frac{i}{q \cdot q} \tilde{B}(q) \varepsilon_{3ij} q_j & q \neq 0 \\ \text{free} & q = 0 \end{cases}$$

Indeed,

$$\begin{aligned}i\varepsilon_{3ij} q_i \tilde{A}_j(q) &= i\varepsilon_{3ij} q_i \frac{i}{q \cdot q} \tilde{B}(q) \varepsilon_{3lj} q_l \\ &= \frac{\tilde{B}(q)}{q \cdot q} q_i \varepsilon_{3ij} \varepsilon_{3lj} q_l \\ &= \frac{\tilde{B}(q)}{q \cdot q} q_i \delta_{il} q_l \\ &= \tilde{B}(q)\end{aligned}$$

□

8.3.6. Corollary. Since for a general system, $\phi \neq 0$, and the magnetic vector potential A appears in the Hamiltonian H (for example as in (1)), H cannot be chosen to be \mathcal{L} -periodic unless we restrict to the uninteresting case that $\phi = 0$.

8.3.7. Claim. (Magnetic Translations) For every $n \in \mathcal{L}$, there exists a map $\chi_n : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that H is invariant under the magnetic translations, defined as:

$$\tilde{U}_n := e^{i\chi_n} U_n$$

That is, for such χ_n we will have:

$$[H, \tilde{U}_n] = 0 \quad \forall n \in \mathcal{L} \quad (37)$$

Proof. Denote the Hamiltonian, which depends on A , by $H(A)$. Then as stated above, we will not have

$$U_n H(A) = H(A) U_n$$

but rather since A is not periodic we will have:

$$U_n H(A) = H(A_n) U_n$$

where $A_n(x) := A(x - n)$. On the other hand with an arbitrary gauge transformation χ we have

$$e^{i\chi} H(A) = H(A + \nabla\chi) e^{i\chi}$$

Then

$$\begin{aligned}e^{i\chi} U_n H(A) &= e^{i\chi} H(A_n) U_n \\ &= H(A_n + \nabla\chi) e^{i\chi} U_n \\ &\stackrel{!}{=} H(A) e^{i\chi} U_n\end{aligned}$$

and the last equation would be fulfilled if

$$\begin{aligned}A_n + \nabla\chi &\stackrel{!}{=} A \\ \downarrow \\ \nabla\chi(x) &= A(x) - A(x - n)\end{aligned}$$

this last equation has a solution iff $\operatorname{curl}(A(x) - A(x - n)) = 0$, which is equivalent to the fact that B is periodic. Hence there is a solution to the last equation, which we denote by χ_n . □

8.3.8. Remark. The magnetic translations $n \mapsto \tilde{U}_n$ do not form a group homomorphism as a representation $\mathcal{L} \rightarrow \mathcal{B}(\mathcal{H})$:

$$\begin{aligned}\tilde{U}_n \tilde{U}_m &= e^{i\chi_n} U_n e^{i\chi_m} U_m \\ &= e^{i\chi_n} e^{i\chi_m(-n)} U_n U_m \\ &= e^{i\chi_n} e^{i\chi_m(-n)} U_{n+m} \\ &= e^{i(\chi_n + \chi_m(-n) - \chi_{n+m})} e^{i\chi_{n+m}} U_{n+m} \\ &\neq e^{i\chi_{n+m}} U_{n+m} \\ &= \tilde{U}_{n+m}\end{aligned}$$

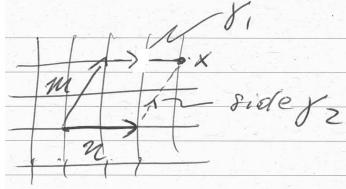
Hence, since

$$\chi_n + \chi_m(-n) - \chi_{n+m} \notin 2\pi\mathbb{Z}$$

in general, $n \mapsto \tilde{U}_n$ forms a *projective representation* rather than a linear representation.

8.3.9. Claim. $\chi_n(x) + \chi_m(x - n) - \chi_{n+m}(x)$ is equal to the magnetic flux through a plaquette spanned by $(n, m) \in \mathcal{L}^2$ at x and is hence independent of x as the magnetic field B is assumed to be periodic.

Proof. Consider a plaquette \mathcal{P} spanned by $(n, m) \in \mathcal{L}^2$ whose top right corner is at some $x \in X$:



The flux through it is given by

$$\begin{aligned}\phi_{nm} &= \int_{\mathcal{P}} B dx \\ \stackrel{\text{Stokes'}}{=} &\int_{\partial \mathcal{P}} A \cdot ds \\ &= \int_{\gamma_1} \underbrace{(A(x' - m) - A(x'))}_{\equiv -\nabla \chi_m(x')} \cdot ds + \int_{\gamma_2} \underbrace{(A(x') - (A' - n))}_{\equiv \nabla \chi_n(x')} \cdot ds \\ &= [\chi_m(x) - \chi_m(x - n)] + [\chi_n(x) - \chi_n(x - m)]\end{aligned}$$

Next note that

$$A(x) - A(x - n - m) = A(x) - A(x - m) + A(x - m) - A(x - n - m)$$

so that curling this last equation we obtain the relation

$$\chi_{n+m}(x) = \chi_m(x) + \chi_n(x - m)$$

from which we get

$$\begin{aligned}\phi_{nm} &= \chi_m(x - n) + \chi_n(x) - \chi_n(x - m) - \chi_m(x) \\ &= \chi_m(x - n) + \chi_n(x) - \chi_{n+m}(x)\end{aligned}$$

as desired. □

8.3.10. Corollary. The magnetic translations obey

$$\tilde{U}_n \tilde{U}_m = e^{i\phi_{n,m}} \tilde{U}_{n+m}$$

for all $(n, m) \in \mathcal{L}^2$. Since the magnetic flux $\phi_{n,m}$ is independent of x , this defines a projective representation.

Furthermore, if

$$\phi_{a_1, a_2} \in 2\pi\mathbb{Z} \tag{38}$$

where $(a_1, a_2) \in \mathbb{R}^2$ are the two vectors which span the unit cell then so that $e^{i\phi_{n,m}} = 1$ for all $(n, m) \in \mathcal{L}^2$ then this defines a linear representation.

8.3.11. Remark. The requirement of (38) can be somewhat alleviated by assuming that

$$\phi_{a_1, a_2} = 2\pi \frac{p}{q}$$

for some $(p, q) \in \mathbb{Z}^2$ which are relatively prime. Then

$$\phi_{n a_1, m a_2} = 2\pi p$$

for suitable $(n, m) \in \mathbb{N}_{\geq 0}^2$, and we are back to the integer case. So one merely has to work with larger unit cells and one recovers *linear* representations rather than projective representations. Thus with slight loss of generality (slight as \mathbb{Q} is dense in \mathbb{R}) we shall assume (38) in what follows and thus we will have

$$\tilde{U}_n \tilde{U}_m = \tilde{U}_{n+m}$$

so that $n \mapsto \tilde{U}_n$ is really a linear representation, under which H is invariant. Again we may perform a Bloch decomposition of \mathcal{H} into invariant subspaces

$$\tilde{\mathcal{H}}(k) := \left\{ \psi \in \mathcal{H} \mid \tilde{U}_n \psi = e^{-ik \cdot n} \psi \forall n \in \mathcal{L} \right\}$$

8.4. Classifying Quantum Hall Systems–The Chern Number. Working in the same setting as in the preceding section (still having chosen an origin for X so that it is a linear rather than affine space).

8.4.1. Claim. Equation (37) implies that

$$[P, \tilde{U}_n] = 0 \quad \forall n \in \mathcal{L}$$

where P is the Fermi projection, projecting onto the subspace associated with an isolated part of the spectrum $\sigma(H)$. As a result, again

$$P = \int_{\mathbb{T}^2} \tilde{P}(k) dk$$

Proof. We follow the same way as in (8.2.25). □

8.4.2. Claim. We have

$$i[P, x_i] = \int_{\mathbb{T}^2}^{\oplus} (\partial_{k_i} \tilde{P}(k)) dk$$

where x_i is the i th-coordinate position operator.

Proof. We start by seeing how the i th-coordinate of the position operator, x_i , behaves on an arbitrary vector $\psi \in \mathcal{H}$, evaluated at some $y \in X$:

$$\begin{aligned} (x_i \psi)(y) &= y_i \psi(y) \\ &= y_i \int_{\mathbb{T}^2} (\tilde{\psi}(k))(y) dk \end{aligned}$$

now we choose a trivialization as in 8.2.18 (the integral on \mathbb{T}^2 will then be divided into charts, but we suppress this now):

$$(\tilde{\psi}(k))(y) = e^{ik \cdot y} u_k(y)$$

so that

$$\begin{aligned} (x_i \psi)(y) &= y_i \int_{\mathbb{T}^2} e^{ik \cdot y} u_k(y) dk \\ &= \int_{\mathbb{T}^2} y_i e^{ik \cdot y} u_k(y) dk \\ &= \int_{\mathbb{T}^2} (-i \partial_{k_i} e^{ik \cdot y}) u_k(y) dk \\ &= -i \int_{\mathbb{T}^2} [(\partial_{k_i} \tilde{\psi}(k))(y) - e^{ik \cdot y} \partial_{k_i} u_k(y)] dk \end{aligned}$$

but the first term is zero. For, WLOG $i = 1$, we have

$$\begin{aligned} \int_{\mathbb{T}^2} \partial_{k_1} \tilde{\psi}(k) dk &= \int_{k_1=0}^{2\pi} \int_{k_2=0}^{2\pi} \partial_{k_1} \tilde{\psi}(k) dk_1 dk_2 \\ &= \int_{k_2=0}^{2\pi} [\tilde{\psi}(2\pi, k_2) - \tilde{\psi}(0, k_2)] dk_2 \\ &= 0 \end{aligned}$$

because $0 = 2\pi$ on S^1 . So we have found that

$$(x_i \psi)(y) = \int_{\mathbb{T}^2} e^{ik \cdot y} i \partial_{k_i} u_k(y) dk$$

and similarly

$$(x_i P \psi)(y) = \int_{\mathbb{T}^2} e^{ik \cdot y} i \partial_{k_i} (\tilde{P}(k) u_k(y)) dk$$

so that

$$\begin{aligned} (i[P, x_i] \psi)(y) &= i(P x_i \psi)(y) - i(x_i P \psi)(y) \\ &= i \int_{\mathbb{T}^2} e^{ik \cdot y} \tilde{P}(k) i \partial_{k_i} u_k(y) dk - i \left(\int_{\mathbb{T}^2} e^{ik \cdot y} i \partial_{k_i} (\tilde{P}(k) u_k(y)) dk \right) \\ &= i \int_{\mathbb{T}^2} e^{ik \cdot y} (\tilde{P}(k) i \partial_{k_i} u_k(y) - i \partial_{k_i} (\tilde{P}(k) u_k(y))) dk \\ &= \int_{\mathbb{T}^2} e^{ik \cdot y} (\partial_{k_i} \tilde{P}(k)) u_k(y) dk \\ &= \int_{\mathbb{T}^2} ((\partial_{k_i} \tilde{P}(k)) \tilde{\psi}(k))(y) dk \end{aligned}$$

□

8.4.3. Claim. If A is \mathcal{L} -invariant, so that we may write

$$A = \int_{\mathbb{T}^2}^{\oplus} \tilde{A}(k) dk$$

then we have

$$\text{tr}'_{\mathcal{H}}(A) = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} \text{tr}_{\tilde{\mathcal{H}}(k)}(A(k)) dk$$

where $\text{tr}'_{\mathcal{H}}$ is the trace per unit volume in the total space \mathcal{H} .

Proof. The trace per unit volume is defined as (recall (17))

$$\text{tr}'_{\mathcal{H}}(A) \equiv \lim_{L \rightarrow \infty} \frac{1}{L^2} \text{tr}_{\mathcal{H}}(\chi_L A)$$

where χ_L is the characteristic function which is zero outside $[-\frac{1}{2}L, \frac{1}{2}L]^2 \subseteq \mathbb{R}^2$. Next note that if $\psi \in \tilde{\mathcal{H}}(k)$ for some $k \in \mathbb{T}^2$, then using (33) we have:

$$\begin{aligned} \langle \psi, \chi_L A \psi \rangle_{\mathcal{H}} &= \int_X \overline{\psi(x)} \chi_L(x) (A\psi)(x) dx \\ &= \sum_{n \in \mathcal{L} \wedge |n| < L} \int_{\mathcal{P}} \overline{\psi(x+n)} (A\psi)(x+n) dx \\ &= \sum_{n \in \mathcal{L} \wedge |n| < L} \int_{\mathcal{P}} \overline{\psi(x) e^{ik \cdot n}} (A\psi)(x) e^{ik \cdot n} dx \\ &= \sum_{n \in \mathcal{L} \wedge |n| < L} \int_{\mathcal{P}} \overline{\psi(x)} (A\psi)(x) dx \\ \stackrel{L^2 \text{ unit cells}}{=} & L^2 \int_{\mathcal{P}} \overline{\psi(x)} (A\psi)(x) dx \\ &= L^2 \langle \psi, \tilde{A}(k) \psi \rangle_{L^2(\mathcal{C})} \end{aligned}$$

Using the fact that A commutes with U and $A\psi \in \tilde{\mathcal{H}}(k)$ as well. Note that if $\{\varphi_{k,n}\}_{n \in \mathbb{N}}$ is some orthonormal basis of $\tilde{\mathcal{H}}(k)$ then $\frac{1}{L} \varphi_{k,n}$ is an orthonormal basis for $L^2([- \frac{1}{2}L, \frac{1}{2}L]^2)$, with periodic boundary conditions which means $k_i \in \frac{2\pi}{L} \mathbb{Z}$. Then the area of any one point is $\Delta k = \frac{(2\pi)^2}{L^2}$ so that

$$\begin{aligned} \frac{1}{L^2} \text{tr}_{\mathcal{H}}(\chi_L A) &= \frac{1}{L^2} \sum_{n \in \mathbb{N}} \sum_k \left\langle \frac{1}{L} \varphi_{k,n}, \chi_L A \frac{1}{L} \varphi_{k,n} \right\rangle_{\mathcal{H}_L} \\ &= \frac{1}{L^4} \sum_{n \in \mathbb{N}} \sum_k \langle \varphi_{k,n}, \chi_L A \varphi_{k,n} \rangle_{\mathcal{P}} \\ &= \frac{1}{L^2} \sum_{n \in \mathbb{N}} \sum_k \langle \varphi_{k,n}, \tilde{A}(k) \varphi_{k,n} \rangle_{\mathcal{P}} \\ &= \frac{1}{L^2} \sum_k \text{tr}_{\tilde{\mathcal{H}}(k)}(\tilde{A}(k)) \\ &= \frac{1}{L^2} \sum_k \text{tr}_{\tilde{\mathcal{H}}(k)}(\tilde{A}(k)) \underbrace{\frac{1}{\Delta k L^2}}_{(2\pi)^2} \\ &= \frac{1}{(2\pi)^2} \underbrace{\sum_k \text{tr}_{\tilde{\mathcal{H}}(k)}(\tilde{A}(k)) \Delta k}_{\text{a Riemann sum}} \\ \stackrel{L \rightarrow \infty}{=} & \frac{1}{(2\pi)^2} \int_{k \in \mathbb{T}^2} \text{tr}_{\tilde{\mathcal{H}}(k)}(\tilde{A}(k)) \end{aligned}$$

□

8.4.4. Corollary. As a result the Kubo formula for the periodic case becomes

$$\sigma_H = \frac{1}{i(2\pi)^2} \int_{\mathbb{T}^2} \text{Tr}_{\tilde{\mathcal{H}}(k)}(\tilde{P}_{\mu}(k) [(\partial_{k_1} \tilde{P}_{\mu}(k)), (\partial_{k_2} \tilde{P}_{\mu}(k))]) dk$$

Proof. Starting from (18) we have

$$\begin{aligned} \sigma_H &= i \text{Tr}'(P_{\mu} [[x_1, P_{\mu}], [x_2, P_{\mu}]]) \\ &= i \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} \text{Tr}_{\tilde{\mathcal{H}}(k)}(\tilde{P}_{\mu}(k) [(\partial_{k_1} \tilde{P}_{\mu}(k)), (\partial_{k_2} \tilde{P}_{\mu}(k))]) dk \\ &= \frac{1}{i(2\pi)^2} \int_{\mathbb{T}^2} \text{Tr}_{\tilde{\mathcal{H}}(k)}(\tilde{P}_{\mu}(k) [(\partial_{k_1} \tilde{P}_{\mu}(k)), (\partial_{k_2} \tilde{P}_{\mu}(k))]) dk \end{aligned}$$

□

8.4.5. Remark. Note that

$$Ch_1(\mathcal{P}) = \frac{1}{2\pi i} \int_{\mathbb{T}^2} \text{Tr}_{\tilde{\mathcal{H}}(k)}(\tilde{P}_{\mu}(k) [(\partial_{k_1} \tilde{P}_{\mu}(k)), (\partial_{k_2} \tilde{P}_{\mu}(k))]) dk \quad (39)$$

is the first Chern number of the bundle \mathcal{P} (defined in the appendix), so that we have found that in the periodic case,

$$\boxed{\sigma_H = \frac{1}{2\pi} Ch_1(\mathcal{P})}$$

8.4.6. Remark. The remainder of this section and the next is devoted to showing explicitly some of the characteristics of the functor $\text{Ch}_1 : \text{Vect}_{\mathbb{C}}(X) \rightarrow \mathbb{Z}$ in the concrete context of quantum mechanics, even though the mathematical derivation of this functor (as it appears in the appendix) would make this whole presentation redundant. For those readers who don't wish to venture into the appendix, (39) may be regarded as a definition of a certain quantity (which turns out to be an integer), $\text{Ch}_1(\mathcal{P})$, associated with the occupied sub-bundle \mathcal{P} of E defined by $k \mapsto \tilde{P}(k)$.

8.4.7. Claim. (Sanity Check) Even though we know $\text{Ch}_1(\mathcal{P}) \propto \sigma_H$ and σ_H is a physical quantity, so that it is real, it's possible to explicitly see that:

$$\text{Ch}_1(\mathcal{P}) \in \mathbb{R}$$

Proof. Using the fact that $\overline{\text{tr}(A)} = \text{tr}(A^*)$ we have

$$\begin{aligned} \overline{\text{Ch}_1(\mathcal{P})} &= \overline{\frac{1}{2\pi i} \int_{\mathbb{T}^2} \text{Tr}_{\tilde{\mathcal{H}}(k)} (\tilde{P}_\mu(k) [(\partial_{k_1} \tilde{P}_\mu(k)), (\partial_{k_2} \tilde{P}_\mu(k))]) dk} \\ &= -\frac{1}{2\pi i} \int_{\mathbb{T}^2} \text{Tr}_{\tilde{\mathcal{H}}(k)} ((\tilde{P}_\mu(k) [(\partial_{k_1} \tilde{P}_\mu(k)), (\partial_{k_2} \tilde{P}_\mu(k))])^*) dk \\ &= -\frac{1}{2\pi i} \int_{\mathbb{T}^2} \text{Tr}_{\tilde{\mathcal{H}}(k)} ([(\partial_{k_2} \tilde{P}_\mu(k)), (\partial_{k_1} \tilde{P}_\mu(k))] \tilde{P}_\mu(k)) dk \\ &= \frac{1}{2\pi i} \int_{\mathbb{T}^2} \text{Tr}_{\tilde{\mathcal{H}}(k)} (\tilde{P}_\mu(k) [(\partial_{k_1} \tilde{P}_\mu(k)), (\partial_{k_2} \tilde{P}_\mu(k))]) dk \\ &= \text{Ch}_1(\mathcal{P}) \end{aligned}$$

where we have used the cyclicity of the trace. \square

8.4.8. Claim. (Time-Reversal Invariance) A time-reversal invariant system (as in 7.2.5) has

$$\text{Ch}_1(\mathcal{P}) = 0$$

Proof. We assume that $[\theta, U_n] = 0$ to be compatible with the existing structure of the system. Then if $\psi \in \tilde{\mathcal{H}}(k)$, we have $U_n \psi = e^{-ik \cdot n} \psi$ and so

$$\begin{aligned} U_n \theta \psi &= \theta U_n \psi \\ &= \theta e^{-ik \cdot n} \psi \\ \theta \text{ is anti-linear} &\equiv e^{ik \cdot n} \theta \psi \end{aligned}$$

so that it turns out that $\theta \psi \in \tilde{\mathcal{H}}(-k)$, and so it turns out that θ maps $\tilde{\mathcal{H}}(k) \rightarrow \tilde{\mathcal{H}}(-k)$. Also we have

$$\theta H = H \theta$$

so that if $\psi \in \tilde{\mathcal{H}}(k)$ then

$$\begin{aligned} \theta H \psi &= H \theta \psi \\ &\downarrow \quad (\tilde{H}(k) = H|_{\tilde{\mathcal{H}}(k)}) \\ \theta \tilde{H}(k) \psi &= \tilde{H}(-k) \theta \psi \end{aligned}$$

so that we obtain the have the relation

$$\theta \tilde{H}(k) \theta^{-1} = \tilde{H}(-k)$$

and similarly because $[P, \theta] = 0$,

$$\theta \tilde{P}(k) \theta^{-1} = \tilde{P}(-k)$$

and

$$\theta \partial_{k_i} \tilde{P}(k) \theta^{-1} = -\partial_{k_i} \tilde{P}(-k)$$

Also, the anti-unitary of θ means that

$$\begin{aligned} \langle \theta \psi, \varphi \rangle &= \overline{\langle \psi, \theta^* \varphi \rangle} \\ \theta^* \equiv \theta^{-1} &\quad \overline{\langle \psi, \theta^{-1} \varphi \rangle} \end{aligned}$$

so that

$$\begin{aligned}
 \text{tr}(A) &= \sum_n \langle \varphi_n, A\varphi_n \rangle \\
 (\{\theta\varphi_n\}_n \text{ is also a basis of } \mathcal{H}) &\quad = \sum_n \langle \theta\varphi_n, A\theta\varphi_n \rangle \\
 &= \sum_n \overline{\langle \varphi_n, \theta^{-1}A\theta\varphi_n \rangle} \\
 &= \overline{\text{tr}(\theta^{-1}A\theta)}
 \end{aligned}$$

so that

$$\begin{aligned}
 \text{Ch}_1(\mathcal{P}) &= \frac{1}{2\pi i} \int_{\mathbb{T}^2} \text{Tr}_{\tilde{\mathcal{H}}(k)} (\tilde{P}_\mu(k) [(\partial_{k_1} \tilde{P}_\mu(k)), (\partial_{k_2} \tilde{P}_\mu(k))]) dk \\
 &= \frac{1}{2\pi i} \int_{\mathbb{T}^2} \overline{\text{Tr}_{\tilde{\mathcal{H}}(k)} (\theta^{-1} \tilde{P}_\mu(k) [(\partial_{k_1} \tilde{P}_\mu(k)), (\partial_{k_2} \tilde{P}_\mu(k))] \theta)} dk \\
 &= \frac{1}{2\pi i} \int_{\mathbb{T}^2} \overline{\text{Tr}_{\tilde{\mathcal{H}}(k)} (\theta^{-1} \tilde{P}_\mu(k) \theta \theta^{-1} [(\partial_{k_1} \tilde{P}_\mu(k)), (\partial_{k_2} \tilde{P}_\mu(k))] \theta)} dk \\
 &= \frac{1}{2\pi i} \int_{\mathbb{T}^2} \overline{\text{Tr}_{\tilde{\mathcal{H}}(k)} (\tilde{P}_\mu(-k) [(-\partial_{k_1} \tilde{P}_\mu(-k)), (-\partial_{k_2} \tilde{P}_\mu(-k))])} dk \\
 &\stackrel{-k \mapsto k}{=} \frac{1}{2\pi i} \int_{\mathbb{T}^2} \overline{\text{Tr}_{\tilde{\mathcal{H}}(k)} (\tilde{P}_\mu(k) [(\partial_{k_1} \tilde{P}_\mu(k)), (\partial_{k_2} \tilde{P}_\mu(k))])} dk \\
 &= -\frac{1}{2\pi i} \int_{\mathbb{T}^2} \text{Tr}_{\tilde{\mathcal{H}}(k)} (\tilde{P}_\mu(k) [(\partial_{k_1} \tilde{P}_\mu(k)), (\partial_{k_2} \tilde{P}_\mu(k))]) dk \\
 &= -\text{Ch}_1(\mathcal{P}) \\
 \text{Ch}_1(\mathcal{P}) \in \mathbb{R} &\stackrel{?}{=} -\text{Ch}_1(\mathcal{P})
 \end{aligned}$$

□

8.4.9. Claim. (*Even though it is an integer in general, for now we present a proof that $\text{Ch}_1(\mathcal{P}) \in \mathbb{Z}$ for the special case of line bundles. 8.4.16 shows the general case.*) If $\text{rank}(\mathcal{P}) = 1$ then $\text{Ch}_1(\mathcal{P}) \in \mathbb{Z}$.

Proof. We have that $\mathcal{P} \rightarrow \mathbb{T}^2$ is a line bundle, and so we may pick a (not-necessarily global) section $\psi(k)$ for all $k \in U$ for some $U \in \text{Open}(\mathbb{T}^2)$. Without loss of generality we pick it such that $\|\psi(k)\|_{\tilde{\mathcal{H}}(k)} = 1$ for all $k \in U$. Then

$$\tilde{P}(k) = \psi(k) \langle \psi(k), \cdot \rangle \quad \forall k \in U$$

Note that:

- (1) Even though $\tilde{P}(k)$ is defined globally for all $k \in \mathbb{T}^2$, $\psi(k)$ is only defined for $k \in U$.
- (2) Of course even though we chose a normalization the section is still not unique: it still has a $U(1)$ gauge-freedom:

$$\psi(k) \mapsto e^{i\alpha(k)}\psi(k) \tag{40}$$

for some continuous $\alpha : U \rightarrow \mathbb{R}$.

Anyway, define a vector field on U by

$$v(k) := \begin{bmatrix} \langle \psi(k), \partial_{k_1} \psi(k) \rangle \\ \langle \psi(k), \partial_{k_2} \psi(k) \rangle \end{bmatrix} \tag{41}$$

Then

Claim. We have

$$\text{Tr}_{\tilde{\mathcal{H}}(k)} (\tilde{P}_\mu(k) [(\partial_{k_1} \tilde{P}_\mu(k)), (\partial_{k_2} \tilde{P}_\mu(k))]) = [\text{curl}(v(k))]_3$$

Proof. Note the two identities

$$\begin{aligned}\partial_{k_i} \tilde{P}(k) &= \partial_{k_i} (\psi(k) \langle \psi(k), \cdot \rangle) \\ &= (\partial_{k_i} \psi(k)) \langle \psi(k), \cdot \rangle + \psi(k) \langle \partial_{k_i} \psi(k), \cdot \rangle\end{aligned}$$

and

$$\begin{aligned}\|\psi(k)\|^2 &= 1 \\ &\downarrow \\ \langle \psi(k), \psi(k) \rangle &= 1 \\ &\downarrow \\ \partial_{k_i} \langle \psi(k), \psi(k) \rangle &= 0 \\ &\downarrow \\ \langle \partial_{k_i} \psi(k), \psi(k) \rangle + \langle \psi(k), \partial_{k_i} \psi(k) \rangle &= 0 \\ &\downarrow \\ \langle \psi(k), \partial_{k_i} \psi(k) \rangle &= -\langle \partial_{k_i} \psi(k), \psi(k) \rangle\end{aligned}$$

and then

$$\begin{aligned}(\partial_{k_i} \tilde{P}(k)) \psi(k) &= ((\partial_{k_i} \psi(k)) \langle \psi(k), \cdot \rangle + \psi(k) \langle \partial_{k_i} \psi(k), \cdot \rangle) \psi(k) \\ &= \partial_{k_i} \psi(k) + \psi(k) \langle \partial_{k_i} \psi(k), \psi(k) \rangle\end{aligned}$$

and

$$\begin{aligned}\langle \psi(k), \cdot \rangle \partial_{k_i} \tilde{P}(k) &= \langle \psi(k), \cdot \rangle ((\partial_{k_i} \psi(k)) \langle \psi(k), \cdot \rangle + \psi(k) \langle \partial_{k_i} \psi(k), \cdot \rangle) \\ &= \langle \psi(k), \partial_{k_i} \psi(k) \rangle \langle \psi(k), \cdot \rangle + \langle \partial_{k_i} \psi(k), \cdot \rangle\end{aligned}$$

and then

$$\begin{aligned}&\text{Tr}_{\tilde{\mathcal{H}}(k)} (\tilde{P}_\mu(k) [(\partial_{k_1} \tilde{P}_\mu(k)), (\partial_{k_2} \tilde{P}_\mu(k))]) \\ &= \sum_{i,j=1}^3 \varepsilon_{3ij} \text{Tr}_{\tilde{\mathcal{H}}(k)} (\tilde{P}_\mu(k) (\partial_{k_i} \tilde{P}_\mu(k)) (\partial_{k_j} \tilde{P}_\mu(k))) \\ &= \varepsilon_{3ij} \text{Tr}_{\tilde{\mathcal{H}}(k)} (\psi(k) \langle \psi(k), \cdot \rangle (\partial_{k_i} \tilde{P}_\mu(k)) (\partial_{k_j} \tilde{P}_\mu(k))) \\ &= \varepsilon_{3ij} \langle \psi(k), (\partial_{k_i} \tilde{P}_\mu(k)) (\partial_{k_j} \tilde{P}_\mu(k)) \psi(k) \rangle \\ &= \varepsilon_{3ij} \langle \psi(k), \cdot \rangle (\partial_{k_i} \tilde{P}_\mu(k)) (\partial_{k_j} \tilde{P}_\mu(k)) \psi(k) \\ &= \varepsilon_{3ij} ((\langle \psi(k), \partial_{k_i} \psi(k) \rangle \langle \psi(k), \cdot \rangle + \langle \partial_{k_i} \psi(k), \cdot \rangle) (\partial_{k_j} \psi(k) + \psi(k) \langle \partial_{k_j} \psi(k), \psi(k) \rangle)) \\ &= \varepsilon_{3ij} \langle \psi(k), \partial_{k_i} \psi(k) \rangle \langle \psi(k), \partial_{k_j} \psi(k) \rangle + \varepsilon_{3ij} \langle \psi(k), \partial_{k_i} \psi(k) \rangle \langle \partial_{k_j} \psi(k), \psi(k) \rangle + \\ &\quad + \varepsilon_{3ij} \langle \partial_{k_i} \psi(k), \partial_{k_j} \psi(k) \rangle + \varepsilon_{3ij} \langle \partial_{k_i} \psi(k), \psi(k) \rangle \langle \partial_{k_j} \psi(k), \psi(k) \rangle \\ &\stackrel{\varepsilon_{3ij} = -\varepsilon_{3ji}}{=} \varepsilon_{3ij} \langle \partial_{k_i} \psi(k), \partial_{k_j} \psi(k) \rangle \\ &= \varepsilon_{3ij} \left(\partial_{k_i} \langle \psi(k), \partial_{k_j} \psi(k) \rangle - \langle \psi(k), \partial_{k_i} \partial_{k_j} \psi(k) \rangle \right) \\ &\stackrel{\varepsilon_{3ij} = -\varepsilon_{3ji}}{=} \varepsilon_{3ij} \partial_{k_i} \langle \psi(k), \partial_{k_j} \psi(k) \rangle \\ &= \varepsilon_{3ij} \partial_{k_i} v(k) \\ &\equiv [\text{curl}(v(k))]_3\end{aligned}$$

□

Remark. We have found

$$\text{Ch}_1(\mathcal{P}) = \frac{1}{2\pi i} \int_{\mathbb{T}^2} [\text{curl}(v(k))]_3 dk$$

and one might be tempted to use Stokes' theorem to conclude that

$$\begin{aligned}\text{Ch}_1(\mathcal{P}) &= \frac{1}{2\pi i} \int_{\partial \mathbb{T}^2} v(k) \cdot dk \\ &\stackrel{\partial \mathbb{T}^2 = \emptyset}{=} 0\end{aligned}$$

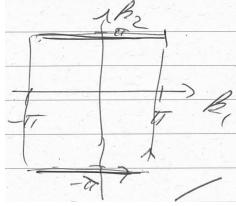
However, this would be a false, since $v(k)$ is only defined for $k \in U$ and not for all $k \in \mathbb{T}^2$! Thus when performing the integral one would have to stitch between different charts, in each of which v might be different.

It follows from 16.4.16 that if X is contractible, then

$$\begin{aligned}\text{Vect}_n(X) &= [X \rightarrow G_n(\mathbb{C}^\infty)] \\ &= [\text{point} \rightarrow G_n(\mathbb{C}^\infty)]\end{aligned}$$

so that all vector-bundles over X are isomorphic, and in particular, all are isomorphic to the trivial one $X \times \mathbb{C}^n$. Now as a result of 8.1.20, if X is contractible, there is a global section on every vector bundle over X .

Thus our strategy is to cut the torus \mathbb{T}^2 so that the resulting base manifold is contractible, define a global section on this new base manifold, and express $\text{Ch}_1(\mathcal{P})$ in terms of transition functions of the global sections between different boundary lines of this new base manifold.



If we make two cuts on the torus, \mathbb{T}^2 , we obtain the square, which is contractible.

$$\mathbb{T}^2 := [-\pi, \pi]^2 \subseteq \mathbb{R}^2$$

Hence there is a global section on $E|_{\mathbb{T}^2} \cong \mathbb{T}^2 \times \mathbb{C}$, $k \mapsto \psi(k)$, such that $\|\psi(k)\| = 1$ for all $k \in \mathbb{T}^2$. Choose two continuous maps $\theta_i : [-\pi, \pi] \rightarrow \mathbb{R}$ by the relations

$$\psi(k_1, \pi) \stackrel{!}{=} e^{i\theta_1(k_1)} \psi(k_1, -\pi)$$

and

$$\psi(\pi, k_2) \stackrel{!}{=} e^{i\theta_2(k_2)} \psi(-\pi, k_2)$$

which can indeed be chosen because ψ never vanishes and is always normalized. Then, in particular, at the opposite corners we have

$$\begin{aligned}\psi(\pi, \pi) &= e^{i\theta_1(\pi)} \psi(\pi, -\pi) \\ &= e^{i\theta_1(\pi)} e^{i\theta_2(-\pi)} \psi(-\pi, -\pi)\end{aligned}$$

going first via θ_1 and then via θ_2 . However, we could also go the other way around:

$$\begin{aligned}\psi(\pi, \pi) &= e^{i\theta_2(\pi)} \psi(-\pi, \pi) \\ &= e^{i\theta_2(\pi)} e^{i\theta_1(-\pi)} \psi(-\pi, -\pi)\end{aligned}$$

Hence we obtain

$$e^{i\theta_1(\pi)} e^{i\theta_2(-\pi)} \psi(-\pi, -\pi) = e^{i\theta_2(\pi)} e^{i\theta_1(-\pi)} \psi(-\pi, -\pi)$$

which implies

$$\boxed{\theta_1(\pi) + \theta_2(-\pi) - \theta_2(\pi) - \theta_1(-\pi) \in 2\pi\mathbb{Z}} \quad (42)$$

Now we compute

$$\begin{aligned}\partial_{k_1} \psi(k_1, \pi) &= \partial_{k_1} [e^{i\theta_1(k_1)} \psi(k_1, -\pi)] \\ &= e^{i\theta_1(k_1)} [i\theta'_1(k_1) \psi(k_1, -\pi) + \partial_{k_1} \psi(k_1, -\pi)]\end{aligned}$$

hence

$$\begin{aligned}v_1(k_1, \pi) &\equiv \langle \psi(k_1, \pi), \partial_{k_1} \psi(k_1, \pi) \rangle \\ &= \left\langle \psi(k_1, \pi), e^{i\theta_1(k_1)} [i\theta'_1(k_1) \psi(k_1, -\pi) + \partial_{k_1} \psi(k_1, -\pi)] \right\rangle \\ &= i\theta'_1(k_1) \underbrace{\left\langle \psi(k_1, \pi), e^{i\theta_1(k_1)} \psi(k_1, -\pi) \right\rangle}_{\psi(k_1, \pi)} + \underbrace{\left\langle e^{-i\theta_1(k_1)} \psi(k_1, \pi), \partial_{k_1} \psi(k_1, -\pi) \right\rangle}_{\psi(k_1, -\pi)} \\ &= i\theta'_1(k_1) + v(k_1, -\pi)\end{aligned}$$

for v_2 we have

$$\begin{aligned}\partial_{k_2} \psi(\pi, k_2) &= \partial_{k_2} [e^{i\theta_2(k_2)} \psi(-\pi, k_2)] \\ &= e^{i\theta_2(k_2)} [i\theta'_2(k_2) \psi(-\pi, k_2) + \partial_{k_2} \psi(-\pi, k_2)]\end{aligned}$$

$$\begin{aligned}
v_2(\pi, k_2) &\equiv \langle \psi(\pi, k_2), \partial_{k_2} \psi(\pi, k_2) \rangle \\
&= \left\langle \psi(\pi, k_2), e^{i\theta_2(k_2)} [i\theta'_2(k_2) \psi(-\pi, k_2) + \partial_{k_2} \psi(-\pi, k_2)] \right\rangle \\
&= i\theta'_2(k_2) \underbrace{\left\langle \psi(\pi, k_2), e^{i\theta_2(k_2)} \psi(-\pi, k_2) \right\rangle}_1 + \underbrace{\left\langle e^{-i\theta_2(k_2)} \psi(\pi, k_2), \partial_{k_2} \psi(-\pi, k_2) \right\rangle}_{\psi(-\pi, k_2)} \\
&= i\theta'_2(k_2) + v_2(-\pi, k_2)
\end{aligned}$$

all together we obtain

$$\begin{aligned}
\text{Ch}_1(\mathcal{P}) &= \frac{1}{2\pi i} \int_{\mathbb{T}^2} [\text{curl}(v(k))]_3 dk \\
\text{up to a set of measure zero} &\stackrel{=}{=} \frac{1}{2\pi i} \int_{\mathbb{T}^2} [\text{curl}(v(k))]_3 dk \\
\text{Stokes} &\stackrel{=}{=} \frac{1}{2\pi i} \int_{\partial \mathbb{T}^2} v(k) \cdot dk \\
&= \frac{1}{2\pi i} \int_{k_1=-\pi}^{k_1=\pi} v_1(k_1, -\pi) dk_1 + \frac{1}{2\pi i} \int_{k_2=-\pi}^{k_2=\pi} v_2(\pi, k_2) dk_2 + \\
&\quad + \frac{1}{2\pi i} \int_{k_1=\pi}^{k_1=-\pi} v_1(k_1, +\pi) dk_1 + \frac{1}{2\pi i} \int_{k_2=\pi}^{k_2=-\pi} v_2(-\pi, k_2) dk_2 \\
&= \frac{1}{2\pi i} \int_{k_1=-\pi}^{k_1=\pi} [v_1(k_1, -\pi) - v_1(k_1, +\pi)] dk_1 + \\
&\quad + \frac{1}{2\pi i} \int_{k_2=-\pi}^{k_2=\pi} [v_2(\pi, k_2) - v_2(-\pi, k_2)] dk_2 \\
&= \frac{1}{2\pi i} \int_{k_1=-\pi}^{k_1=\pi} [-i\theta'_1(k_1)] dk_1 + \\
&\quad + \frac{1}{2\pi i} \int_{k_2=-\pi}^{k_2=\pi} [i\theta'_2(k_2)] dk_2 \\
&= \frac{1}{2\pi} (\theta_1(-\pi) - \theta_1(\pi) + \theta_2(\pi) - \theta_2(-\pi)) \\
&\in \mathbb{Z}
\end{aligned}$$

where the last line follows by (42). \square

8.4.10. Remark. From 8.4.9 it is clear that if \exists a global section on \mathcal{P} then by definition of θ_i , $\theta_i = 0$ and so $\text{Ch}_1(\mathcal{P}) = 0$. Conversely, if $\text{Ch}_1(\mathcal{P}) = 0$ then the bundle is trivial (see the appendix, or 8.4.11 for an explicit construction), so that by 8.1.20, there is a global section on it.

8.4.11. Claim. If $\text{rank}(\mathcal{P}) = 1$ and $\text{Ch}_1(\mathcal{P}) = 0$ then there is a global section on \mathcal{P} which is normalized to 1.

Proof. As we remarked in (40), there is an additional gauge transformation after choosing the normalization

$$\psi(k) \mapsto e^{i\alpha(k)} \psi(k) \quad \forall k \in \mathbb{T}^2$$

for some continuous $\alpha : \mathbb{T}^2 \rightarrow \mathbb{R}$. This gauge transformation gives rise to new transition functions $\tilde{\theta}_i$ which are related to the old ones via:

$$\begin{aligned}
\tilde{\psi}(k) &:= e^{i\alpha(k)} \psi(k) \\
\psi(k_1, \pi) &= e^{i\theta_1(k_1)} \psi(k_1, -\pi) \\
\psi(\pi, k_2) &= e^{i\theta_2(k_2)} \psi(-\pi, k_2) \\
\tilde{\psi}(k_1, \pi) &= e^{i\tilde{\theta}_1(k_1)} \tilde{\psi}(k_1, -\pi) \\
\tilde{\psi}(\pi, k_2) &= e^{i\tilde{\theta}_2(k_2)} \tilde{\psi}(-\pi, k_2) \\
\tilde{\psi}(k_1, \pi) &= e^{i\alpha(k_1, \pi)} \psi(k_1, \pi) \\
&= e^{i\alpha(k_1, \pi)} e^{i\theta_1(k_1)} \psi(k_1, -\pi) \\
&= e^{i\alpha(k_1, \pi)} e^{i\theta_1(k_1)} e^{-i\alpha(k_1, -\pi)} \tilde{\psi}(k_1, -\pi)
\end{aligned}$$

so that

$$\tilde{\theta}_1(k_1) = \alpha(k_1, \pi) + \theta_1(k_1) - \alpha(k_1, -\pi)$$

for $\tilde{\theta}_2$:

$$\begin{aligned}\tilde{\psi}(\pi, k_2) &= e^{i\alpha(\pi, k_2)}\psi(\pi, k_2) \\ &= e^{i\alpha(\pi, k_2)}e^{i\theta_2(k_2)}\psi(-\pi, k_2) \\ &= e^{i\alpha(\pi, k_2)}e^{i\theta_2(k_2)}e^{-i\alpha(-\pi, k_2)}\tilde{\psi}(-\pi, k_2)\end{aligned}$$

so that

$$\tilde{\theta}_2(k_2) = \alpha(\pi, k_2) + \theta_2(k_2) - \alpha(-\pi, k_2)$$

Thus with an appropriate choice of α , we might be able to eliminate θ_1 or θ_2 (but it will turn out that we cannot in general eliminate both). Indeed, if we demand

$$\begin{array}{ccc}\tilde{\theta}_1(k_1) & \stackrel{!}{\in} & 2\pi\mathbb{Z} & \forall k_1 \in [-\pi, \pi] \\ \downarrow & & & \\ \alpha(k_1, \pi) + \theta_1(k_1) - \alpha(k_1, -\pi) & \in & 2\pi\mathbb{Z} & \forall k_1 \in [-\pi, \pi] \\ \downarrow & & & \\ \alpha(k_1, k_2)|_{k_2=-\pi}^{k_2=\pi} + \theta_1(k_1) & \in & 2\pi\mathbb{Z} & \forall k_1 \in [-\pi, \pi] \\ \downarrow & & & \\ \alpha(k_1, k_2)|_{k_2=-\pi}^{k_2=\pi} & \in & 2\pi\mathbb{Z} - \theta_1(k_1) & \forall k_1 \in [-\pi, \pi]\end{array}$$

so that if we pick

$$\alpha(k_1, k_2) := \frac{2\pi n - \theta_1(k_1)}{2\pi} k_2$$

(any choice of $n \in \mathbb{Z}$ will work) the relation is obeyed. As a result, we were able to obtain $\tilde{\theta}_1(k_1) \in 2\pi\mathbb{Z}$ so that $e^{i\tilde{\theta}_1(k_1)} = 1$ and we may forget the cut along $k_2 = \pi$. Thus instead of a square

$$\ddot{\mathbb{T}}^2 \equiv [-\pi, \pi]^2$$

we obtain a cylinder

$$\dot{\mathbb{T}}^2 := [-\pi, \pi] \times S^1$$

And we have a global section on the cylinder $\dot{\mathbb{T}}^2$ (which is by the way *not* contractible). Any additional gauge transformation must respect

$$\alpha(k_1, k_2)|_{k_2=-\pi}^{k_2=\pi} \stackrel{!}{\in} 2\pi\mathbb{Z} \quad \forall k_1 \in [-\pi, \pi] \tag{43}$$

otherwise it will re-introduce a θ_1 transition function. After eliminating θ_1 , we find

$$\boxed{\text{Ch}_1(\mathcal{P}) = \frac{1}{2\pi} [\theta_2(\pi) - \theta_2(-\pi)]}$$

Thus, $\text{Ch}_1(\mathcal{P})$ is the winding of the map $S^1 \rightarrow S^1$ given by:

$$S^1 \ni k_2 \mapsto \exp(i\theta_2(k_2)) \in S^1$$

Now if we wanted to also eliminate θ_2 as well we would have to require that

$$\begin{array}{ccc}\alpha(\pi, k_2) + \theta_2(k_2) - \alpha(-\pi, k_2) & \in & 2\pi\mathbb{Z} & \forall k_2 \in S^1 \\ \downarrow & & & \\ \alpha(k_1, k_2)|_{k_1=-\pi}^{k_1=\pi} & \in & 2\pi\mathbb{Z} - \theta_2(k_2) & \forall k_2 \in S^1\end{array}$$

and again we pick

$$\alpha(k_1, k_2) := \frac{2\pi n - \theta_2(k_2)}{2\pi} k_1$$

(any choice of $n \in \mathbb{Z}$ will do) and now employ the constraint (43) to get:

$$\begin{array}{ccc}\left[\frac{2\pi n - \theta_2(\pi)}{2\pi} k_1 - \frac{2\pi n - \theta_2(-\pi)}{2\pi} k_1 \right] & \stackrel{!}{\in} & 2\pi\mathbb{Z} & \forall k_1 \in [-\pi, \pi] \\ \downarrow & & & \\ \frac{-\theta_2(\pi) + \theta_2(-\pi)}{2\pi} k_1 & \stackrel{!}{\in} & 2\pi\mathbb{Z} & \forall k_1 \in [-\pi, \pi] \\ \downarrow & & & \\ -\text{Ch}_1(\mathcal{P}) k_1 & \stackrel{!}{\in} & 2\pi\mathbb{Z} & \forall k_1 \in [-\pi, \pi]\end{array}$$

which can only happen for all k_1 when $\text{Ch}_1(\mathcal{P}) = 0$. But the possibility of setting $e^{i\theta_1(k_1)} = e^{i\theta_2(k_2)} = 1$ is equivalent to the existence of a global section. Thus we see that, indeed, when $\text{Ch}_1(\mathcal{P}) = 0$ we explicitly constructed a global section. \square

8.4.12. Corollary. *As a result we obtain the fact that $\text{Ch}_1(\mathcal{P}) \neq 0$ is an obstruction to choosing a global section on \mathcal{P} , and if $\text{Ch}_1(\mathcal{P}) = 0$ there is a global section.*

8.4.13. Remark. The following sequence of statements about the first Chern number of bundles of rank higher than 1 appeared in the lecture without proof. Their proofs are a very slight generalization of the ones above. In keeping with the order of presentation of the lecture, a redundancy is introduced into this section. It would have been more efficient, in fact, to immediately present the proof for bundles of arbitrary finite rank.

8.4.14. Claim. If $\text{rank}(\mathcal{P}) = N$, then one can pick N -linearly-independent global sections (a section of the “frame bundle”) on $\tilde{\mathbb{T}}^2 \subseteq \mathbb{T}^2$.

Proof. As in 8.4.9 we know that on the cut torus, $\tilde{\mathbb{T}}^2$, which is contractible, there is a global section of the frame bundle. We denote it this section by

$$\{\psi_i(k)\}_{i=1}^N \quad \forall k \in \tilde{\mathbb{T}}^2$$

where without loss of generality we assume that at each k , $\{\psi_i(k)\}_{i=1}^N$ are not merely linearly independent, but even form an orthonormal basis of \mathcal{P}_k . Again, as before, we may introduce transition matrices (rather than functions) between the boundary lines: Pick two functions $T_i : [-\pi, \pi] \rightarrow U(\mathbb{C}^N)$ for all $i \in \{1, 2\}$ such that

$$\begin{aligned} \psi_i(k_1, \pi) &\stackrel{!}{=} \sum_{j=1}^N \psi_j(k_1, -\pi) [T_1(k_1)]_{ji} \quad \forall i \in \{1, \dots, N\} \\ \psi_i(\pi, k_2) &\stackrel{!}{=} \sum_{j=1}^N \psi_j(-\pi, k_2) [T_2(k_2)]_{ji} \quad \forall i \in \{1, \dots, N\} \end{aligned}$$

Note that on the left hand side of these equations we have vectors, not components of vectors.

We have an additional gauge transformation as in (40) in the form of a map $A : \tilde{\mathbb{T}}^2 \rightarrow U(\mathbb{C}^N)$ such that the transformed frame is

$$\tilde{\psi}_i(k) = \sum_{j=1}^N \psi_j(k) [A(k)]_{ji}$$

So that

$$\begin{aligned} \tilde{\psi}_i(k_1, \pi) &= \sum_{j=1}^N \psi_j(k_1, \pi) [A(k_1, \pi)]_{ji} \\ &= \sum_{j=1}^N \sum_{l=1}^N \psi_l(k_1, -\pi) [T_1(k_1)]_{lj} [A(k_1, \pi)]_{ji} \\ &= \sum_{j=1}^N \psi_j(k_1, -\pi) [T_1(k_1) A(k_1, \pi)]_{ji} \\ &= \sum_{j=1}^N \tilde{\psi}_j(k_1, -\pi) \left[A(k_1, -\pi)^{-1} T_1(k_1) A(k_1, \pi) \right]_{ji} \end{aligned}$$

and we find that

$$\tilde{T}_1(k_1) = A(k_1, -\pi)^{-1} T_1(k_1) A(k_1, \pi)$$

and similarly

$$\tilde{T}_2(k_2) = A(-\pi, k_2)^{-1} T_2(k_2) A(\pi, k_2)$$

Is there a choice of A such that \tilde{T}_1 or \tilde{T}_1 will be $\mathbb{1}_{N \times N}$?

$$\begin{aligned} \tilde{T}_1(k_1) &\stackrel{!}{=} \mathbb{1}_{N \times N} \quad \forall k_1 \in [-\pi, \pi] \\ A(k_1, -\pi)^{-1} T_1(k_1) A(k_1, \pi) &\stackrel{!}{=} \mathbb{1}_{N \times N} \quad \forall k_1 \in [-\pi, \pi] \\ A(k_1, \pi) A(k_1, -\pi)^{-1} &\stackrel{!}{=} T_1(k_1)^{-1} \quad \forall k_1 \in [-\pi, \pi] \end{aligned}$$

so that if we pick

$$A(k_1, k_2) := \left(T_1(k_1)^{-1} \right)^{\frac{k_2}{2\pi}}$$

the relation is obeyed (Note that we can raise $T_1(k_1)^{-1}$ to arbitrary powers since it is non-singular). As a result, we obtain that

$$\tilde{T}_1(k_1) \stackrel{!}{=} \mathbb{1}_{N \times N} \quad \forall k_1 \in [-\pi, \pi]$$

and we may forget the cut along $k_2 = \pi$. Thus instead of a square

$$\tilde{\mathbb{T}}^2 \equiv [-\pi, \pi]^2$$

we obtain a cylinder

$$\dot{\mathbb{T}}^2 := [-\pi, \pi] \times S^1$$

And we have a global section on the cylinder $\dot{\mathbb{T}}^2$ which is *not* contractible. Any additional gauge transformation must respect

$$A(k_1, \pi) A(k_1, -\pi)^{-1} \stackrel{!}{=} \mathbb{1}_{N \times N} \quad \forall k_1 \in [-\pi, \pi]$$

otherwise it will re-introduce a T_1 transition matrix function. \square

8.4.15. Corollary. *If $\text{rank } (\mathcal{P}) = N$, then on cylinder $\dot{\mathbb{T}}^2 \subseteq \mathbb{T}^2$, we have a global section of the frame-bundle*

$$\{\psi_i(k)\}_{i=1}^N \quad \forall k \in \dot{\mathbb{T}}^2$$

(orthonormal basis of \mathcal{P}_k for each $k \in \dot{\mathbb{T}}^2$) with a transition matrix-valued function $T_2 : S^1 \rightarrow U(\mathbb{C}^N)$ such that

$$\psi_i(\pi, k_2) \stackrel{!}{=} \sum_{j=1}^N \psi_j(-\pi, k_2) [T_2(k_2)]_{ji} \quad \forall i \in \{1, \dots, N\}$$

8.4.16. Claim. *For a bundle \mathcal{P} of any finite rank,*

$$\text{Ch}_1(\mathcal{P}) = \text{winding number of the map } S^1 \ni k_2 \mapsto \det(T_2(k_2)) \in S^1$$

where T_2 is as in 8.4.15.

Proof. Since $\{\psi_i(k)\}_{i=1}^N$ spans \mathcal{P}_k we have

$$\tilde{P}(k) = \sum_{i=1}^n \psi_i(k) \langle \psi_i(k), \cdot \rangle$$

We define similarly to (41) N -vector-fields

$$v_i(k) := \begin{bmatrix} \langle \psi_i(k), \partial_{k_1} \psi_i(k) \rangle \\ \langle \psi_i(k), \partial_{k_2} \psi_i(k) \rangle \end{bmatrix} \quad \forall i \in \{1, \dots, N\}, k \in \dot{\mathbb{T}}^2$$

And like before the integrand can be expressed in terms of this vector field. It is a miracle that it splits completely:

Claim. We have the following relation

$$\text{Tr}_{\mathcal{H}(k)} (\tilde{P}_\mu(k) [(\partial_{k_1} \tilde{P}_\mu(k)), (\partial_{k_2} \tilde{P}_\mu(k))]) = \sum_{i=1}^N [\text{curl}(v_i(k))]_3$$

Proof. Note the two identities

$$\begin{aligned}\partial_{k_i} \tilde{P}(k) &= \partial_{k_i} \sum_{l=1}^N \psi_l(k) \langle \psi_l(k), \cdot \rangle \\ &= \sum_{l=1}^N [(\partial_{k_i} \psi_l(k)) \langle \psi_l(k), \cdot \rangle + \psi_l(k) \langle \partial_{k_i} \psi_l(k), \cdot \rangle]\end{aligned}$$

and

$$\begin{aligned}\langle \psi_i(k), \psi_j(k) \rangle &= \delta_{ij} \\ &\downarrow \\ \langle \partial_{k_l} \psi_i(k), \psi_j(k) \rangle &= -\langle \psi_i(k), \partial_{k_l} \psi_j(k) \rangle\end{aligned}$$

Then we have

$$\begin{aligned}(\partial_{k_j} \tilde{P}_\mu(k)) \psi_l(k) &= \sum_{m=1}^N [(\partial_{k_j} \psi_m(k)) \langle \psi_m(k), \cdot \rangle + \psi_m(k) \langle \partial_{k_j} \psi_m(k), \cdot \rangle] \psi_l(k) \\ &= \sum_{m=1}^N \left[(\partial_{k_j} \psi_m(k)) \underbrace{\langle \psi_m(k), \psi_l(k) \rangle}_{\delta_{m,l}} + \psi_m(k) \langle \partial_{k_j} \psi_m(k), \psi_l(k) \rangle \right] \\ &= \partial_{k_j} \psi_l(k) + \sum_{m=1}^N \psi_m(k) \langle \partial_{k_j} \psi_m(k), \psi_l(k) \rangle\end{aligned}$$

and

$$\begin{aligned}\langle \psi_l(k), \cdot \rangle \partial_{k_i} \tilde{P}_\mu(k) &= \langle \psi_l(k), \cdot \rangle \sum_{r=1}^N [(\partial_{k_i} \psi_r(k)) \langle \psi_r(k), \cdot \rangle + \psi_r(k) \langle \partial_{k_i} \psi_r(k), \cdot \rangle] \\ &= \sum_{r=1}^N \left[\langle \psi_l(k), \partial_{k_i} \psi_r(k) \rangle \langle \psi_r(k), \cdot \rangle + \underbrace{\langle \psi_l(k), \psi_r(k) \rangle}_{\delta_{l,r}} \langle \partial_{k_i} \psi_r(k), \cdot \rangle \right] \\ &= \sum_{r=1}^N \langle \psi_l(k), \partial_{k_i} \psi_r(k) \rangle \langle \psi_r(k), \cdot \rangle + \langle \partial_{k_i} \psi_l(k), \cdot \rangle\end{aligned}$$

so that

$$\begin{aligned}&\text{Tr}_{\mathcal{H}(k)} (\tilde{P}_\mu(k) [(\partial_{k_1} \tilde{P}_\mu(k)), (\partial_{k_2} \tilde{P}_\mu(k))]) \\ &= \sum_{i,j=1}^3 \varepsilon_{3ij} \text{Tr}_{\mathcal{H}(k)} (\tilde{P}_\mu(k) (\partial_{k_i} \tilde{P}_\mu(k)) (\partial_{k_j} \tilde{P}_\mu(k))) \\ &= \varepsilon_{3ij} \text{Tr}_{\mathcal{H}(k)} \left(\left(\sum_{l=1}^N \psi_l(k) \langle \psi_l(k), \cdot \rangle \right) (\partial_{k_i} \tilde{P}_\mu(k)) (\partial_{k_j} \tilde{P}_\mu(k)) \right) \\ &= \varepsilon_{3ij} \sum_{l=1}^N \langle \psi_l(k), (\partial_{k_i} \tilde{P}_\mu(k)) (\partial_{k_j} \tilde{P}_\mu(k)) \psi_l(k) \rangle \\ &= \varepsilon_{3ij} \sum_{l=1}^N \langle \psi_l(k), \cdot \rangle (\partial_{k_i} \tilde{P}_\mu(k)) (\partial_{k_j} \tilde{P}_\mu(k)) \psi_l(k) \\ &= \varepsilon_{3ij} \sum_{l=1}^N \left(\sum_{r=1}^N \langle \psi_l(k), \partial_{k_i} \psi_r(k) \rangle \langle \psi_r(k), \cdot \rangle + \langle \partial_{k_i} \psi_l(k), \cdot \rangle \right) \left(\partial_{k_j} \psi_l(k) + \sum_{m=1}^N \psi_m(k) \langle \partial_{k_j} \psi_m(k), \psi_l(k) \rangle \right) \\ &= \varepsilon_{3ij} \underbrace{\sum_{l=1}^N \sum_{r=1}^N \langle \psi_l(k), \partial_{k_i} \psi_r(k) \rangle \langle \psi_r(k), \partial_{k_j} \psi_l(k) \rangle}_{0} + \\ &\quad + \varepsilon_{3ij} \underbrace{\sum_{l=1}^N \sum_{r=1}^N \langle \psi_l(k), \partial_{k_i} \psi_r(k) \rangle \langle \partial_{k_j} \psi_r(k), \psi_l(k) \rangle}_{0} + \\ &\quad + \varepsilon_{3ij} \sum_{l=1}^N \langle \partial_{k_i} \psi_l(k), \partial_{k_j} \psi_l(k) \rangle + \\ &\quad + \varepsilon_{3ij} \underbrace{\sum_{l=1}^N \sum_{m=1}^N \langle \partial_{k_i} \psi_l(k), \psi_m(k) \rangle \langle \partial_{k_j} \psi_m(k), \psi_l(k) \rangle}_{0}\end{aligned}$$

These terms are zero because the sum of entries of an anti-symmetric matrix is zero. The result then follows as in the line bundle case. \square

Next to compute the first Chern number we need

$$\begin{aligned}
 \partial_{k_2} \psi_i(\pi, k_2) &= \partial_{k_2} \sum_{j=1}^N \psi_j(-\pi, k_2) [T_2(k_2)]_{ji} \\
 &= \sum_{j=1}^N (\partial_{k_2} \psi_j(-\pi, k_2)) [T_2(k_2)]_{ji} + \sum_{j=1}^N \psi_j(-\pi, k_2) [\partial_{k_2} T_2(k_2)]_{ji} \\
 \sum_{i=1}^N [v_i(\pi, k_2)]_2 &\equiv \sum_{i=1}^N \langle \psi_i(\pi, k_2), \partial_{k_2} \psi_i(\pi, k_2) \rangle \\
 &= \sum_{i=1}^N \left\langle \psi_i(\pi, k_2), \sum_{j=1}^N (\partial_{k_2} \psi_j(-\pi, k_2)) [T_2(k_2)]_{ji} + \sum_{i=1}^N \sum_{j=1}^N \psi_j(-\pi, k_2) [\partial_{k_2} T_2(k_2)]_{ji} \right\rangle \\
 &= \sum_{i=1}^N \sum_{j=1}^N \left\langle \psi_i(\pi, k_2) \overline{[T_2(k_2)^T]_{ij}}, (\partial_{k_2} \psi_j(-\pi, k_2)) \right\rangle + \\
 &\quad + \sum_{i=1}^N \sum_{j=1}^N \left\langle \sum_{r=1}^N \psi_r(-\pi, k_2) [T_2(k_2)]_{ri}, \psi_j(-\pi, k_2) \right\rangle [\partial_{k_2} T_2(k_2)]_{ji} \\
 T_2(k_2) \text{ unitary} &= \sum_{j=1}^N \langle \psi_j(-\pi, k_2), (\partial_{k_2} \psi_j(-\pi, k_2)) \rangle + \sum_{i=1}^N \sum_{j=1}^N [T_2(k_2)^{-1}]_{ij} [\partial_{k_2} T_2(k_2)]_{ji} \\
 &= \sum_{i=1}^N \langle \psi_i(-\pi, k_2), (\partial_{k_2} \psi_i(-\pi, k_2)) \rangle + \text{Tr} (T_2(k_2)^{-1} \partial_{k_2} T_2(k_2)) \\
 &= \sum_{i=1}^N [v_i(-\pi, k_2)]_2 + \text{Tr} (T_2(k_2)^{-1} \partial_{k_2} T_2(k_2))
 \end{aligned}$$

As a result we have

$$\begin{aligned}
 \text{Ch}_1(\mathcal{P}) &= \frac{1}{2\pi i} \int_{\mathbb{T}^2} \sum_{i=1}^N [\text{curl}(v_i(k))]_3 dk \\
 \text{up to a set of measure zero} &= \frac{1}{2\pi i} \int_{\mathbb{T}^2} \sum_{i=1}^N [\text{curl}(v_i(k))]_3 dk \\
 \stackrel{\text{Stokes}}{=} & \sum_{i=1}^N \frac{1}{2\pi i} \int_{\partial \mathbb{T}^2} v_i(k) \cdot dk \\
 &= \sum_{i=1}^N \frac{1}{2\pi i} \left(\int_{k_2=-\pi}^{k_2=\pi} [v_i(\pi, k_2)]_2 dk_2 + \int_{k_2=\pi}^{k_2=-\pi} [v_i(-\pi, k_2)]_2 dk_2 \right) \\
 &= \sum_{i=1}^N \frac{1}{2\pi i} \int_{k_2=-\pi}^{k_2=\pi} [v_i(\pi, k_2) - v_i(-\pi, k_2)]_2 dk_2 \\
 &= \frac{1}{2\pi i} \int_{k_2=-\pi}^{k_2=\pi} \text{Tr} (T_2(k_2)^{-1} \partial_{k_2} T_2(k_2)) dk_2 \\
 &= \frac{1}{2\pi i} \int_{k_2 \in S^1} \text{Tr} (T_2(k_2)^{-1} \partial_{k_2} T_2(k_2)) dk_2 \\
 \stackrel{\text{Jacobi's formula}}{=} & \frac{1}{2\pi i} \int_{k_2 \in S^1} \frac{\partial_{k_2} \det(T_2(k_2))}{\det(T_2(k_2))} dk_2 \\
 &\equiv \text{winding number of map } k_2 \mapsto \det(T_2(k_2))
 \end{aligned}$$

□

8.4.17. Corollary.

[8.4.16](#) shows that

$$\text{Ch}_1(\mathcal{P}) \in \mathbb{Z}$$

also for bundles of rank higher than 1.

8.4.18. Remark. This discussion could have also been phrased using the concept of the fundamental group of a space. Recall (or learn from [\[34\]](#) page 321) that π_1 is a functor from the category of pointed spaces into Grp which is defined as follows. If $X \in \text{Obj}(\text{Top})$ and $x_0 \in X$ then:

$$\pi_1(X, x_0) \equiv \text{loops based at } x_0 / \sim$$

where \sim is identification up to continuous deformations, the group composition law is concatenation of loops, and the inverse of a loop is a loop that runs in the opposite direction.

Next note that

$$\pi_1(X, x_0) \cong \pi_1(X, x_1)$$

for all x_0, x_1 in the same path-connected component of X . In particular if X is path-connected we may forget about the base point and simply write

$$\pi_1(X)$$

It is well known that $U(\mathbb{C}^N)$ is path-connected, and also that

$$\pi_1(U(\mathbb{C}^N)) \cong \mathbb{Z}$$

To see this, one has to prove that

$$\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$$

and note that

$$U(\mathbb{C}^N) \cong SU(\mathbb{C}^N) \rtimes U(1)$$

and that $\pi_1(SU(\mathbb{C}^N)) = \{0\}$ (as the determinant is always 1) and of course

$$\pi_1(U(1)) \cong \pi_1(S^1) \cong \mathbb{Z}$$

is just the plain old winding number of a map $S^1 \rightarrow S^1$.

We find that indeed $Ch_1(\mathcal{P})$ computes the corresponding element of $\pi_1(U(\mathbb{C}^N))$ to which the loop

$$S^1 \ni k_2 \mapsto T_2(k_2) \in U(\mathbb{C}^N)$$

belongs to.

Finally note that if $T_2(k_2) \in U(\mathbb{C}^N)$ then the winding number of the map

$$k_2 \mapsto \det(T_2(k_2))$$

is equal to the sum of winding numbers of the maps

$$k_2 \mapsto \lambda_i(k_2)$$

where $\lambda_i(k_2)$ is the i th eigenvalue of the matrix $T_2(k_2)$.

8.4.19. Example. Take for example a tight-binding model on \mathbb{Z}^2 in which each unit cell has N sites. Then

$$\mathcal{H}(k) \cong L^2(\mathcal{C}) \cong \mathbb{C}^N$$

as was established in the combination of (30) and 8.2.16. In this example, let us choose

$$N := 2$$

Then we assume exactly one state is occupied and that there is always a gap between the eigenvalues, with the Fermi energy in between:

$$\varepsilon_-(k) < \mu < \varepsilon_+(k) \quad (44)$$

and we have $\text{rank}(\mathcal{P}) = 1$. Then the Hamiltonian in each fiber is given by

$$\tilde{H}(k) \in \text{End}(\mathbb{C}^2)$$

And there are some restrictions which must be imposed to make it into a Hamiltonian:

- (1) It should be self-adjoint.
- (2) It should always have a gap to satisfy (44)

If we denote $\sigma_0 := \mathbb{1}_{2 \times 2}$ and $\{\sigma_i\}_{i=1}^3$ as the three Pauli matrices then it is known that $\{\sigma_i\}_{i=0}^3$ spans $\text{Mat}_{2 \times 2}(\mathbb{C})$. To satisfy the two conditions, the coefficients of this linear span must be real (as the (four) Pauli matrices themselves are self-adjoint) and the two eigenvalues must never be equal. We write

$$\tilde{H}(k) := \sum_{i=0}^3 h_i(k) \sigma_i$$

for some $h_i : \mathbb{T}^2 \rightarrow \mathbb{R}^4$ and the eigenvalues are given by (straight forward computation of 2×2 linear algebra):

$$\varepsilon_{\pm}(k) = h_0(k) \pm \sqrt{\sum_{i=1}^3 (h_i(k))^2}$$

so that for the eigenvalues to never be equal we need

$$\sum_{i=1}^3 (h_i(k))^2 \neq 0 \quad \forall k \in \mathbb{T}^2$$

so that if we denote $\mathbf{h}(k) := \begin{bmatrix} h_1(k) \\ h_2(k) \\ h_3(k) \end{bmatrix}$ then we have a map defined

$$\mathbb{T}^2 \ni k \mapsto \mathbf{h}(k) \in \mathbb{R}^3 \setminus \{0\}$$

which in turn induces a map

$$\mathbb{T}^2 \ni k \mapsto \underbrace{\frac{\mathbf{h}(k)}{\|\mathbf{h}(k)\|_{\mathbb{R}^3}}}_{-\mathbf{e}(k)} \in S^2$$

This map has a degree, and its degree is precisely $\text{Ch}_1(\mathcal{P})$. Let us see this explicitly. The projector onto the occupied state is given by

$$\begin{aligned}\tilde{P}(k) &\equiv \frac{\varepsilon_+(k) \mathbb{1}_{2 \times 2} - \tilde{H}(k)}{\varepsilon_+(k) - \varepsilon_-(k)} \\ &= \frac{1}{2} (\mathbb{1}_{2 \times 2} + \mathbf{e}(k) \cdot \vec{\sigma})\end{aligned}$$

with $\vec{\sigma} \equiv \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix}$, and so

$$\partial_{k_i} \tilde{P}(k) = \frac{1}{2} \partial_{k_i} \mathbf{e}(k) \cdot \vec{\sigma}$$

and

$$\begin{aligned}(\mathbf{a} \cdot \vec{\sigma})(\mathbf{b} \cdot \vec{\sigma}) &= \sum_{i=1}^3 \sum_{j=1}^3 a_i \sigma_i b_j \sigma_j \\ &= \sum_{i=1}^3 \sum_{j=1}^3 a_i b_j \left(i \sum_{l=1}^3 \varepsilon_{lij} \sigma_l + \delta_{ij} \right) \\ &= \mathbf{a} \cdot \mathbf{b} \mathbb{1}_{2 \times 2} + i(\mathbf{a} \times \mathbf{b}) \cdot \vec{\sigma}\end{aligned}$$

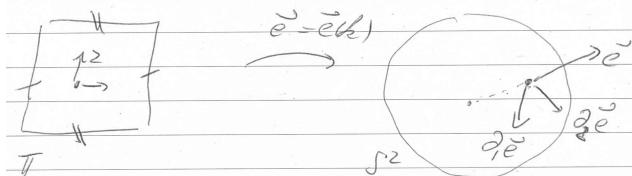
$$\text{tr}((\mathbf{a} \cdot \vec{\sigma})(\mathbf{b} \cdot \vec{\sigma})) = 2\mathbf{a} \cdot \mathbf{b}$$

and

$$\begin{aligned}(\mathbf{a} \cdot \vec{\sigma})(\mathbf{b} \cdot \vec{\sigma})(\mathbf{c} \cdot \vec{\sigma}) &= (\mathbf{a} \cdot \mathbf{b} \mathbb{1}_{2 \times 2} + i(\mathbf{a} \times \mathbf{b}) \cdot \vec{\sigma})(\mathbf{c} \cdot \vec{\sigma}) \\ &= (\mathbf{a} \cdot \mathbf{b})(\mathbf{c} \cdot \vec{\sigma}) + i((\mathbf{a} \times \mathbf{b}) \cdot \vec{\sigma})(\mathbf{c} \cdot \vec{\sigma}) \\ &= (\mathbf{a} \cdot \mathbf{b})(\mathbf{c} \cdot \vec{\sigma}) + i(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} \mathbb{1}_{2 \times 2} - ((\mathbf{a} \times \mathbf{b}) \times \mathbf{c}) \cdot \vec{\sigma} \\ \text{tr}((\mathbf{a} \cdot \vec{\sigma})(\mathbf{b} \cdot \vec{\sigma})(\mathbf{c} \cdot \vec{\sigma})) &= 2i(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}\end{aligned}$$

$$\begin{aligned}\text{Ch}_1(\mathcal{P}) &= \frac{1}{2\pi i} \int_{\mathbb{T}^2} \text{Tr}_{\mathcal{F}(k)} (\tilde{P}(k) [(\partial_{k_1} \tilde{P}(k)), (\partial_{k_2} \tilde{P}(k))]) dk \\ &= \frac{1}{2\pi i} \int_{\mathbb{T}^2} \sum_{i,j=1}^3 \varepsilon_{3ij} \text{Tr}_{\mathcal{F}(k)} (\tilde{P}(k) (\partial_{k_i} \tilde{P}(k)) (\partial_{k_j} \tilde{P}(k))) dk \\ &= \frac{1}{2\pi i} \int_{\mathbb{T}^2} \sum_{i,j=1}^3 \varepsilon_{3ij} \text{Tr}_{\mathcal{F}(k)} \left(\frac{1}{2} (\mathbb{1}_{2 \times 2} + \mathbf{e}(k) \cdot \vec{\sigma}) \left(\frac{1}{2} \partial_{k_i} \mathbf{e}(k) \cdot \vec{\sigma} \right) \left(\frac{1}{2} \partial_{k_j} \mathbf{e}(k) \cdot \vec{\sigma} \right) \right) dk \\ &= \frac{1}{2\pi i} \int_{\mathbb{T}^2} \sum_{i,j=1}^3 \varepsilon_{3ij} \text{Tr}_{\mathcal{F}(k)} \left(\frac{1}{8} \left((\partial_{k_i} \mathbf{e}(k) \cdot \vec{\sigma}) (\partial_{k_j} \mathbf{e}(k) \cdot \vec{\sigma}) + (\mathbf{e}(k) \cdot \vec{\sigma}) (\partial_{k_i} \mathbf{e}(k) \cdot \vec{\sigma}) (\partial_{k_j} \mathbf{e}(k) \cdot \vec{\sigma}) \right) \right) dk \\ &= \frac{1}{2\pi i} \int_{\mathbb{T}^2} \frac{1}{8} \left(2 \underbrace{\sum_{i,j=1}^3 \varepsilon_{3ij} (\partial_{k_i} \mathbf{e}(k)) \cdot (\partial_{k_j} \mathbf{e}(k))}_{0} + 2i \mathbf{e}(k) \cdot \sum_{i,j=1}^3 \varepsilon_{3ij} (\partial_{k_i} \mathbf{e}(k) \times \partial_{k_j} \mathbf{e}(k)) \right) dk \\ &= \frac{1}{2\pi} \int_{\mathbb{T}^2} \frac{1}{4} \mathbf{e}(k) \cdot 2 (\partial_{k_1} \mathbf{e}(k) \times \partial_{k_2} \mathbf{e}(k)) dk \\ &= \frac{1}{4\pi} \int_{\mathbb{T}^2} \mathbf{e}(k) \cdot (\partial_{k_1} \mathbf{e}(k) \times \partial_{k_2} \mathbf{e}(k)) dk\end{aligned}\tag{45}$$

Schematically we have a map $\mathbb{T}^2 \rightarrow S^2$:



Compare this formula with the area of a parametrized surface in \mathbb{R}^3 :

$$\mathbb{R}^2 \ni k \mapsto \mathbf{f}(k) \in \mathbb{R}^3$$

$$\text{Area} = \int_k \|\partial_{k_1} \mathbf{f}(k) \times \partial_{k_2} \mathbf{f}(k)\|_{\mathbb{R}^3} dk$$

Since $\partial_{k_1} \mathbf{e}(k) \times \partial_{k_2} \mathbf{e}(k)$ is always parallel or anti-parallel to $\mathbf{e}(k)$, we get that

$$\mathbf{e}(k) \cdot (\partial_{k_1} \mathbf{e}(k) \times \partial_{k_2} \mathbf{e}(k)) = \pm \|\partial_{k_1} \mathbf{e}(k) \times \partial_{k_2} \mathbf{e}(k)\|_{\mathbb{R}^3}$$

So that in this case, $\text{Ch}_1(\mathcal{P})$ gives us the (signed) number of times the map

$$\mathbf{e} : \mathbb{T}^2 \rightarrow S^2$$

wraps on S^2 .

Let us elaborate a bit further on these concepts in a more formal way.

8.4.20. Definition. (*Regular Value*) Let $f : M \rightarrow N$ be differentiable, where M and N are differentiable manifolds. The point $y \in N$ is called *regular* iff for all $x \in f^{-1}(\{y\})$, the tangent map

$$f_* (x_i) : T_x M \rightarrow T_y N$$

given by

$$X \mapsto X(\cdot \circ f)$$

is surjective, where we consider the tangent vector X as a derivation (a linear map from functions $M \rightarrow \mathbb{R}$ into \mathbb{R} which is Leibniz).

8.4.21. Claim. (*Sard's Theorem*) *The set of values of a map $f : M \rightarrow N$ which are not regular has measure zero.*

Proof. For example see [13] page 80. □

8.4.22. Definition. (*Degree of Map at Regular Value*) Let $f : M \rightarrow N$ be smooth where M and N are compact, oriented smooth manifolds. The degree of f at the regular value $y \in N$ is defined as

$$\deg(f, y) := n_+(f, y) - n_-(f, y)$$

where

$$n_{\pm}(f, y) := \left| \left\{ x \in f^{-1}(\{y\}) \mid f_*(x) \text{ is orientation preserving (+) or reversing (-)} \right\} \right|$$

8.4.23. Claim. $\deg(f, y) = \deg(f, \tilde{y})$ for any two regular values y and \tilde{y} .

Proof. One way to see this is to prove the equivalence of this definition with the one given in 8.4.25. For a direct proof see [14] page 103 theorem 13.1.2. □

8.4.24. Definition. (*Degree of Map*) We thus define

$$\deg(f) := \deg(f, y)$$

for any y a regular value of f .

8.4.25. Remark. In algebraic topology there is also a definition of the degree of a map as follows: For closed connected orientable manifolds of the same dimension n , the top homology group is (not canonically) isomorphic to \mathbb{Z} , and the map f induces a morphism in Grp

$$f_* : H_n(M) \rightarrow H_n(N)$$

which sends a generator of $H_n(M)$, $[M]$ (equivalent to a choice of orientation for M) to a generator of $H_n(N)$, $[N]$ (equivalent to a choice of orientation for N) via

$$[M] \xrightarrow{f_*} d[N]$$

for some $d \in \mathbb{Z}$. Then we define the degree of the map f as

$$\deg(f) := d$$

For more details see [33]. The two definitions are of course equivalent.

8.4.26. Example. If $f : S^1 \rightarrow S^1$ then $\deg(f)$ is the winding number.

8.4.27. Example. Let $\mathbf{e} : \mathbb{T}^2 \rightarrow S^2$. The formula we have found in (45):

$$\text{Ch}_1(\mathcal{P}) = \frac{1}{4\pi} \int_{\mathbb{T}^2} \mathbf{e}(k) \cdot (\partial_{k_1} \mathbf{e}(k) \times \partial_{k_2} \mathbf{e}(k)) dk$$

actually computes the degree of the map \mathbf{e} , so that we have that in the special case of a two-level system,

$$\text{Ch}_1(\mathcal{P}) = \deg(\mathbf{e})$$

Proof. We want to prove that

$$\deg(\mathbf{e}) = \frac{1}{4\pi} \int_{\mathbb{T}^2} \mathbf{e}(\mathbf{k}) \cdot (\partial_{k_1} \mathbf{e}(\mathbf{k}) \times \partial_{k_2} \mathbf{e}(\mathbf{k})) d\mathbf{k}$$

Write \mathbb{T}^2 as

$$\mathbb{T}^2 = \coprod_i U_i$$

(disjoint union, which holds true up to a null set) so that $\mathbf{e} : U_i \rightarrow \mathbf{e}(U_i)$ is a diffeomorphism $\forall i$, and then

$$\int_{U_i} \mathbf{e}(\mathbf{k}) \cdot (\partial_{k_1} \mathbf{e}(\mathbf{k}) \times \partial_{k_2} \mathbf{e}(\mathbf{k})) d\mathbf{k} = \pm \int_{\mathbf{e}(U_i)} dy$$

where the sign is determined by whether the diffeomorphism is orientation preserving or reversing for that particular i , so that the integral on the whole is

$$\begin{aligned} \int_{\mathbb{T}^2} \mathbf{e}(\mathbf{k}) \cdot (\partial_{k_1} \mathbf{e}(\mathbf{k}) \times \partial_{k_2} \mathbf{e}(\mathbf{k})) d\mathbf{k} &= \int_{S^2} (n_+(\mathbf{e}, y) - n_-(\mathbf{e}, y)) dy \\ &\equiv \deg(\mathbf{e}) 4\pi \end{aligned}$$

□

8.4.28. Example. For concreteness, consider the following Hamiltonian, with $M \in \mathbb{R}$ a parameter of the model:

$$\tilde{H}(\mathbf{k}) = \sin(k_1)\sigma_1 + \sin(k_2)\sigma_2 + (M + \cos(k_1) + \cos(k_2))\sigma_3$$

We want to compute the first Chern number of this model.

Picking up from (45) we have

$$\mathbf{h}(\mathbf{k}) = \begin{bmatrix} \sin(k_1) \\ \sin(k_2) \\ M + \cos(k_1) + \cos(k_2) \end{bmatrix}$$

so that

$$\begin{aligned} \|\mathbf{h}(\mathbf{k})\|_{\mathbb{R}^3} &= \sqrt{\sin(k_1)^2 + \sin(k_2)^2 + [M + \cos(k_1) + \cos(k_2)]^2} \\ &= \sqrt{2 + M^2 + 2M\cos(k_1) + 2M\cos(k_2) + 2\cos(k_1)\cos(k_2)} \end{aligned}$$

Then

Claim. $\|\mathbf{h}(\mathbf{k})\|_{\mathbb{R}^3} \neq 0$ for all $\mathbf{k} \in \mathbb{T}^2$ iff $M \notin \{0, 2, -2\}$.

Proof. We have

$$\begin{aligned} \|\mathbf{h}(\mathbf{k})\|_{\mathbb{R}^3} &= 0 \\ \downarrow \\ \begin{bmatrix} \sin(k_1) \\ \sin(k_2) \\ M + \cos(k_1) + \cos(k_2) \end{bmatrix} &= 0 \\ \downarrow \\ \begin{cases} k_1 & \in \pi\mathbb{Z} \\ k_2 & \in \pi\mathbb{Z} \\ M + \cos(k_1) + \cos(k_2) & = 0 \end{cases} \end{aligned}$$

From this last expression it is clear that if $M \notin \{0, 2, -2\}$ then the last line will never be zero, and conversely, if $M \in \{0, 2, -2\}$, then there are certain points $\mathbf{k} \in \mathbb{T}^2$ for which the last line will be zero. In particular:

Case 1. If $M = 0$, then $\cos(k_1) \stackrel{!}{=} \pm 1$ and $\cos(k_2) \stackrel{!}{=} \mp 1$ so that $\mathbf{k} = (0, \pi)$ and $\mathbf{k} = (\pi, 0)$ both do the job.

Case 2. If $M = \pm 2$, then $\cos(k_1) \stackrel{!}{=} \mp 1$ and $\cos(k_2) \stackrel{!}{=} \mp 1$ so that $\mathbf{k} = (\pi, \pi)$ does the job for $M = +2$ and $\mathbf{k} = (0, 0)$ does the job for $M = -2$.

□

Claim. The north and south pole, $\begin{bmatrix} 0 \\ 0 \\ \pm 1 \end{bmatrix} \in S^2$, are regular values of \mathbf{e} .

Proof. If $(k_1, k_2) \in (\pi\mathbb{Z})^2$ then $h_1(\mathbf{k}) = 0$ and $h_2(\mathbf{k}) = 0$ so that $\mathbf{e}(\mathbf{k}) = \pm 1$ necessarily (assuming $M \notin \{0, \pm 2\}$). For this scenario, there are only four possible points on \mathbb{T}^2 :

$$\mathbf{k} \in \{(0, 0), (0, \pi), (\pi, 0), (\pi, \pi)\} \quad (46)$$

In order to avoid using charts we consider the torus as a subset of \mathbb{R}^3 using the parametrization of $\mathbb{T}^2 \equiv (\mathbb{R}/2\pi\mathbb{Z})^2$ as a donut in \mathbb{R}^3 :

$$\psi : \mathbb{T}^2 \rightarrow \begin{bmatrix} [R + r \cos(k_2)] \cos(k_1) \\ [R + r \cos(k_2)] \sin(k_1) \\ r \sin(k_2) \end{bmatrix}$$

for some $(R, r) \in (0, \infty)^2$ with $R > r$. We have

$$\begin{aligned} \cos(k_1) &= \frac{\psi_x}{\sqrt{\psi_x^2 + \psi_y^2}} \\ \sin(k_1) &= \frac{\psi_y}{\sqrt{\psi_x^2 + \psi_y^2}} \\ \cos(k_2) &= \sqrt{\psi_x^2 + \psi_y^2} - R \\ \sin(k_2) &= \frac{1}{r} \psi_z \end{aligned}$$

so that $h : \mathbb{T}^2 \rightarrow \mathbb{R}^3$ induces a map $\hat{h} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ explicitly given by

$$\begin{aligned} \hat{h} \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) &:= h \circ \psi^{-1} \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) \\ &= \begin{bmatrix} \frac{y}{\sqrt{x^2+y^2}} \\ \frac{1}{r} z \\ M + \frac{x}{\sqrt{x^2+y^2}} + \sqrt{x^2+y^2} - R \end{bmatrix} \end{aligned}$$

which gives a map $\tilde{e} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ with range on $S^2 \subseteq \mathbb{R}^3$ given by:

$$\tilde{e} \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) \equiv \frac{1}{\sqrt{\left(\frac{y}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{1}{r} z\right)^2 + \left(M + \frac{x}{\sqrt{x^2+y^2}} + \sqrt{x^2+y^2} - R\right)^2}} \begin{bmatrix} \frac{y}{\sqrt{x^2+y^2}} \\ \frac{1}{r} z \\ M + \frac{x}{\sqrt{x^2+y^2}} + \sqrt{x^2+y^2} - R \end{bmatrix}$$

The four points we are interested in,

$$(k_1, k_2) \in \{(0, 0), (0, \pi), (\pi, 0), (\pi, \pi)\}$$

correspond to $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \left\{ \begin{bmatrix} \pm R \pm r \\ 0 \\ 0 \end{bmatrix} \right\}$ and so the tangent map, evaluated at these points, is

$$\begin{aligned} D\tilde{e} \left(\begin{bmatrix} R+r \\ 0 \\ 0 \end{bmatrix} \right) &\equiv \begin{bmatrix} \partial_x \tilde{e}_1 & \partial_y \tilde{e}_1 & \partial_z \tilde{e}_1 \\ \partial_x \tilde{e}_2 & \partial_y \tilde{e}_2 & \partial_z \tilde{e}_2 \\ \partial_x \tilde{e}_3 & \partial_y \tilde{e}_3 & \partial_z \tilde{e}_3 \end{bmatrix} \left(\begin{bmatrix} R+r \\ 0 \\ 0 \end{bmatrix} \right) \\ &= \begin{bmatrix} 0 & \frac{1}{(R+r)(M+1+r)} & 0 \\ 0 & 0 & \frac{1}{r(M+1+r)} \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

which is indeed surjective onto the xy -plane in \mathbb{R}^3 , which is isomorphic to the tangent spaces to S^2 at the south and north poles. The other three points follow similarly. \square

Then we can immediately compute the degree of e via its characterization in 8.4.22:

The four points in (46) correspond respectively to values of $e_3(k)$ given by:

$$\begin{aligned} e_3((0, 0)) &= -\text{sgn}(M+2) \\ e_3((0, \pi)) &= -\text{sgn}(M) \\ e_3((\pi, 0)) &= -\text{sgn}(M) \\ e_3((\pi, \pi)) &= -\text{sgn}(M-2) \end{aligned}$$

so that we have

Value of M	# of pre-images of south pole	# of pre-images of north pole
$M < -2$	$ \emptyset = 0$	$ \{(0, 0), (0, \pi), (\pi, 0), (\pi, \pi)\} = 4$
$-2 < M < 0$	$ \{(0, 0)\} = 1$	$ \{(0, \pi), (\pi, 0), (\pi, \pi)\} = 3$
$0 < M < 2$	$ \{(0, 0), (0, \pi), (\pi, 0)\} = 3$	$ \{(\pi, \pi)\} = 1$
$M > 2$	$ \{(0, 0), (0, \pi), (\pi, 0), (\pi, \pi)\} = 4$	$ \emptyset = 0$

so that using the definition 8.4.22 we have:

Value of M	$\deg(\mathbf{e})$
$M < -2$	$\deg(\mathbf{e}) = \deg\left(\mathbf{e}, \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}\right) = 0$
$-2 < M < 0$	$\deg(\mathbf{e}) = \deg\left(\mathbf{e}, \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}\right) = \pm 1$
$0 < M < 2$	$\deg(\mathbf{e}) = \deg\left(\mathbf{e}, \begin{bmatrix} 0 \\ 0 \\ +1 \end{bmatrix}\right) = \pm 1$
$M > 2$	$\deg(\mathbf{e}) = \deg\left(\mathbf{e}, \begin{bmatrix} 0 \\ 0 \\ +1 \end{bmatrix}\right) = 0$

and all that is left is to determine the sign for the two cases. Near $(k_1, k_2) = (0, 0)$,

$$\mathbf{h}(k_1, k_2) \sim \begin{bmatrix} k_1 \\ k_2 \\ M+2 \end{bmatrix}$$

so that \mathbf{e}_* near that point is $\mathbf{e}_*((0, 0)) \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and so is orientation preserving. The other case has opposite orientation.

Thus we conclude

Value of M	$\deg(\mathbf{e})$
$-2 < M < 0$	$\deg(\mathbf{e}) = +1$
$0 < M < 2$	$\deg(\mathbf{e}) = -1$

9. CONNECTIONS AND CURVATURE ON VECTOR BUNDLES

In an attempt to explain the classification of vector bundles in a more systematic way (rather than the enigmatic definition we presented; see 8.4.6) we now turn to what is referred to as the Chern-Weil description of characteristic classes. This section follows Appendix C in [33] or [35] chapter 10. Also see the appendix for yet another way to define characteristic classes.

Our goal is to define a connection and a curvature on a vector bundle (as a generalization of the connection and curvature which one would encounter in general relativity) and see that the first Chern number is really an integral on this curvature, which would close the circle with 7.2.9.

9.1. Preliminary Notions about Vector Bundles.

9.1.1. Definition. (*Cartesian Product of Vector Bundles*) Let $p_1 : E_1 \rightarrow M_1$ and $p_2 : E_2 \rightarrow M_2$ be two vector bundles. Then we define the product bundle

$$E_1 \times E_2 \rightarrow M_1 \times M_2$$

as follows:

- (1) The total space is the Cartesian product of the two total spaces. The product topology is used for the total space.
- (2) The projection map $p : E_1 \times E_2 \rightarrow M_1 \times M_2$ is defined by

$$p(e_1, e_2) := (p_1(e_1), p_2(e_2))$$

which is continuous by definition of the product topology.

- (3) There is a natural vector space structure on $(E_1 \times E_2)_{(m_1, m_2)}$ defined as

$$(E_1 \times E_2)_{(m_1, m_2)} := (E_1)_{m_1} \oplus (E_2)_{m_2}$$

9.1.2. Definition. (*Whitney Sum of Vector Bundles*) The Whitney sum is defined via 16.2.2 using

$$\oplus : \text{Vect}_{\mathbb{C}} \times \text{Vect}_{\mathbb{C}} \rightarrow \text{Vect}_{\mathbb{C}}$$

as a continuous functor. Explicitly, Let E_1 and E_2 both be vector bundles over the same base space M . We define the *Whitney sum* of E_1 and E_2 , denoted as

$$E_1 \oplus E_2$$

as the vector bundle whose fiber at each $m \in M$ is given by

$$(E_1 \oplus E_2)_m := (E_1)_m \oplus (E_2)_m$$

Note that

$$i : M \rightarrow M \times M$$

$$m \mapsto (m, m)$$

defines an induced vector bundle $i^*(E_1 \times E_2)$ (as in 16.2.4) which is isomorphic to the restricted bundle $(E_1 \times E_2)|_{i(M)}$. This is how the topology of the Whitney sum is defined.

9.1.3. Definition. (*Tensor Product of Bundles*) Simiarly the tensor product bundle is defined via 16.2.2 using

$$\otimes : \text{Vect}_{\mathbb{C}} \times \text{Vect}_{\mathbb{C}} \rightarrow \text{Vect}_{\mathbb{C}}$$

as a continuous functor.

9.2. The Ehresmann Connection.

9.2.1. Definition. (*The Vertical Subspace*) If $\pi : E \rightarrow M$ is a vector bundle and M is a smooth manifold, then E is a smooth manifold as well, and so, it also has a tangent bundle over E which we denote by

$$TE \rightarrow E$$

Its typical fiber at $u \in E$, $T_u E$, is comprised of maps $C^\infty(E, \mathbb{C}) \rightarrow \mathbb{C}$ which are \mathbb{C} -linear and Leibniz at u and its projection is defined as

$$q : \mathcal{X} \mapsto u$$

where $\mathcal{X} : C^\infty(E, \mathbb{C}) \rightarrow \mathbb{C}$ is such a tangent vector to u .

But the initial vector bundle projection map

$$\pi : E \rightarrow M$$

can also be considered as a map between manifolds; for any $u \in E$ we have the tangent map

$$(\pi_*)_u : T_u E \rightarrow T_{\pi(u)} M$$

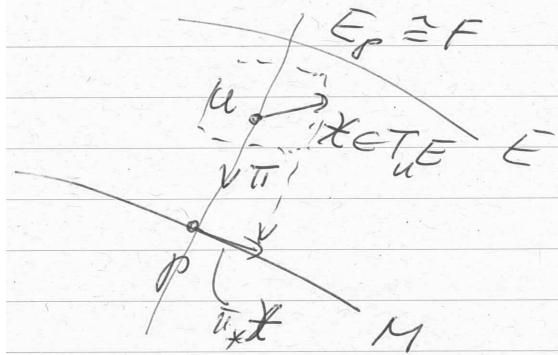
$$\mathcal{X} \mapsto \mathcal{X}(\cdot \circ \pi)$$

π_* is the pushforward differential, also denoted as $D\pi$ or $d\pi$. Actually via π_* one could also consider TE as a vector bundle over TM . So we have two base spaces for TE :

$$\begin{aligned} TE &\xrightarrow{q} E \\ TE &\xrightarrow{\pi_*} TM \end{aligned}$$

Note that π and q are not linear (M and E are not even vector spaces), but $(\pi_*)_u$ is linear by definition (as a map between two vector spaces—the tangent spaces). As such, its kernel $\ker((\pi_*)_u)$ defines a vector subspace of its domain $T_u E$, which we call *the vertical subspace* and denote by V_u :

$$V_u := \ker((\pi_*)_u)$$



9.2.2. Claim. Let $\pi : E \rightarrow M$ be a vector bundle over a manifold M , $u \in E$ be given. We know that $E_{\pi(u)}$ is the fiber at $\pi(u)$, which is a finite dimensional vector space over \mathbb{C} , and thus also a smooth manifold (the simplest example of such). So it also has a tangent space, $TE_{\pi(u)}$; The fiber of $TE_{\pi(u)}$ at u is denoted by $T_u(E_{\pi(u)})$. We claim that

$$V_u = T_u(E_{\pi(u)})$$

Proof. We now characterize $T_u E$ via curves in E and their derivatives: Let $\gamma : [0, 1] \rightarrow E$ be a curve in E such that $\gamma(0) = u$. Then this curve defines a tangent vector

$$\mathcal{X} \equiv \partial_t (\cdot \circ \gamma)|_{t=0}$$

◻ Assume that $\mathcal{X} \in T_u(E_{\pi(u)})$. Then $\gamma(t) \in E_{\pi(u)}$ for all $t \in [0, 1]$. That is, $\pi(\gamma(t)) = \pi(u)$ for all $t \in [0, 1]$. As a result,

$$\begin{aligned} ((\pi_*)_u)(\mathcal{X}) &\equiv \mathcal{X}(\cdot \circ \pi) \\ &= \partial_t \left(\underbrace{\cdot \circ \pi \circ \gamma}_{\pi(u)} \right) \Big|_{t=0} \\ &= \partial_t \left(\underbrace{\cdot \circ \pi(u)}_{\text{does not depend on } t} \right) \Big|_{t=0} \\ &= 0 \end{aligned}$$

so that we find

$$\begin{aligned} \mathcal{X} &\in \ker((\pi_*)_u) \\ &\equiv V_u \end{aligned}$$

◻ Assume that $\mathcal{X} \in V_u$. Thus $((\pi_*)_u)(\mathcal{X}) = 0$. So

$$\begin{aligned} \mathcal{X}(\cdot \circ \pi) &= 0 \\ \uparrow \\ \partial_t (\cdot \circ \pi \circ \gamma)|_{t=0} &= 0 \\ \uparrow \\ \pi \circ \gamma \text{ does not depend on } t \end{aligned}$$

so that $\pi \circ \gamma : [0, 1] \rightarrow M$ must be a constant, equal to u . Hence the result follows. ◻

9.2.3. Definition. (Ehresmann Connection) We would like to have a canonical assignment of a complement of V_u in $T_u E$. An Ehresmann connection on E is a smooth assignment $u \mapsto H_u \subseteq T_u E$ such that

- (1) $T_u E = H_u \oplus V_u$ for each $u \in E$, that is $TE = H \oplus V$ where H and V are the bundles over E with fibers H_u and V_u at each $u \in E$ respectively.

(2) H is a vector sub-bundle of T_E .

Note that the splitting in the first requirement is equivalent to the specification of a projection

$$v : T_u E \rightarrow V_u \quad (47)$$

such that $v \circ v = v$ via the fact that $\ker(v) \equiv H_u$ (Recall that a projection is always defined by two subspaces if there is no inner product, which we do not necessarily have).

9.3. The Covariant Derivative.

9.3.1. Remark. (*Definition of Ehresmann Connection via Covariant Derivative*) An equivalent definition of the Ehresmann connection is as follows. Let $s : M \rightarrow E$ be a section on E . That is,

$$\pi \circ s = \mathbb{1}_M$$

We denote by $\Gamma(E)$ the space of all sections on E and by $\mathcal{F}(M)$ the space of all smooth maps $M \rightarrow \mathbb{C}$ together with pointwise multiplication and addition. Then $\Gamma(E)$ is an $\mathcal{F}(M)$ -module: for all $f \in \mathcal{F}(M)$ and $s \in \Gamma(E)$ we have

$$(fs)(p) = f(p)s(p)$$

$$(s_1 + s_2)(p) = s_1(p) + s_2(p)$$

There is also a tangent map induced by a section: $s : M \rightarrow E$ induces

$$(s_*)_p : T_p M \rightarrow T_{s(p)} E$$

which is given by

$$X \mapsto X(\cdot \circ s) \quad \forall X \in T_p M$$

9.3.2. Remark. Note that (contrary to what was written in earlier versions of this document) if $f \in \mathcal{F}(M)$, $g \in \mathcal{F}(E)$ and $X \in T_p M$ then in general

$$(fs)_*(X) \neq fs_*(X) + X(f)s$$

Example. Define $M := \mathbb{R}$, $E := \mathbb{R}^2$, $p \in M$. There is only one tangent vector on \mathbb{R} up to proportionality, $\partial_1|_p \in T_p \mathbb{R}$. A section $s : \mathbb{R} \rightarrow \mathbb{R}^2$ is a map

$$x_1 \mapsto (x_1, \tilde{s}(x_1)) \in \mathbb{R}^2$$

for some $\tilde{s} : \mathbb{R} \rightarrow \mathbb{R}$. A scalar on \mathbb{R}^2 is a map $g : \mathbb{R}^2 \rightarrow \mathbb{R}$. Then

$$\begin{aligned} X(g \circ s) &\equiv \partial_1|_p(g \circ s) \\ &= \partial_1|_p(g(\mathbb{1}_{\mathbb{R}}, \tilde{s})) \\ &= \partial_1|_{(p, \tilde{s}(p))} g \underbrace{\partial_1|_p \mathbb{1}_{\mathbb{R}}}_1 + (\partial_2|_{(p, \tilde{s}(p))} g) \partial_1|_p \tilde{s} \\ &= \partial_1|_{(p, \tilde{s}(p))} g + (\partial_2|_{(p, \tilde{s}(p))} g) \partial_1|_p \tilde{s} \end{aligned}$$

Pick $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ as $g := \pi_1$. Then $\partial_1|_{(p, \tilde{s}(p))} g = 1$ and $\partial_2|_{(p, \tilde{s}(p))} g = 0$ so that

$$X(g \circ s) = 1$$

Note that this result is independent of the choice of \tilde{s} . In particular, for any constant scalar map $f : \mathbb{R} \rightarrow \mathbb{R}$ which is not equal to 1, we have

$$X(g \circ fs) = 1$$

Yet

$$\underbrace{X(f)}_0 g \circ s + f(p) \underbrace{X(g \circ s)}_1 = f(p) \neq 1$$

Such an $\mathcal{F}(M)$ -linear relation only holds for the covariant derivative (as will be seen below), but not for the differential d .

9.3.3. Definition. Using the projection $v : T_u E \rightarrow V_u$ in (47) we define, for any given $s \in \Gamma(E)$, a map $\nabla(s)$:

$$\nabla(s) : T_{\pi(u)} M \rightarrow V_u$$

by

$$X \mapsto \underbrace{v(s_*(X))}_{=: \nabla_X(s)}$$

9.3.4. Claim. $\nabla_X(s)$ is tensorial in $s \in \Gamma(E)$ and $\mathcal{F}(M)$ -linear in $X \in TM$.

Proof. Let $f \in \mathcal{F}(M)$, $(X_1, X_2) \in TM^2$, $(s_1, s_2) \in \Gamma(E)^2$ and $g \in \mathcal{F}(E)$. Then:

$$\begin{aligned}
\nabla_X(s_1 + s_2) &\equiv v((s_1 + s_2)_*(X)) \\
&= v(X(\cdot \circ (s_1 + s_2))) \\
&\stackrel{(X \text{ is linear})}{=} v(X(\cdot \circ s_1) + X(\cdot \circ s_2)) \\
&\stackrel{(v \text{ is linear})}{=} v(X(\cdot \circ s_1)) + v(X(\cdot \circ s_2)) \\
&\equiv \nabla_X(s_1) + \nabla_X(s_2) \\
\\
\nabla_{fX_1}(s_1) &= v(fX_1(\cdot \circ s_1)) \\
&\stackrel{(v \text{ is linear})}{=} fv(X_1(\cdot \circ s_1)) \\
&= f\nabla_{X_1}(s_1) \\
\\
\nabla_{X_1+X_2}(s_1) &\equiv v(s_1_*(X_1 + X_2)) \\
&\stackrel{(\text{differential is linear})}{=} v(X_1(\cdot \circ s_1) + X_2(\cdot \circ s_2)) \\
&\stackrel{(v \text{ is linear})}{=} \nabla_{X_1}(s_1) + \nabla_{X_2}(s_2)
\end{aligned}$$

For brevity define $s := s_1$ and $X := X_1$. Define $m := \dim_{\mathbb{C}}(M)$, let $p \in M$, define $n := \dim_{\mathbb{C}}(E_p)$, pick some $U \in \text{Nbhd}_M(p)$ such that there is a chart $\psi : U \rightarrow \psi(U) \in \text{Open}(\mathbb{C}^m)$ at p which is also a trivialization of E , that is, there is a family-of-vector-spaces-isomorphism η :

$$\eta : E|_U \rightarrow U \times \mathbb{C}^n$$

Define

$$\Phi := (\psi, \mathbb{1}_{\mathbb{C}^n}) \circ \eta$$

so that

$$\Phi : E|_U \rightarrow \psi(U) \times \mathbb{C}^n \in \text{Open}(\mathbb{C}^m \times \mathbb{C}^n)$$

and we define $\Phi_m := \pi_m \circ \Phi$ and $\Phi_n := \pi_n \circ \Phi$ as the projections to the two components. Then

$$\begin{aligned}
g \circ \Phi^{-1} : \psi(U) \times \mathbb{C}^n &\rightarrow \mathbb{C} \\
f \circ \psi^{-1} : \psi(U) &\rightarrow \mathbb{C}
\end{aligned}$$

and

$$\Phi \circ s \circ \psi^{-1} : \psi(U) \rightarrow \psi(U) \times \mathbb{C}^n$$

such that

$$(\Phi \circ s \circ \psi^{-1})(x) = (x, \tilde{s}(x)) \quad \forall x \in \psi(U)$$

for some map $\tilde{s} : \psi(U) \rightarrow \mathbb{C}^n$ (this is to preserve basepoints). Thus we write

$$\Phi \circ s \circ \psi^{-1} = (\mathbb{1}_{\psi(U)}, \tilde{s})$$

Now we have ($\nabla_X (fs)$ acts on g , but we write \cdot instead for brevity)

$$\begin{aligned}
 \nabla_X (fs) &\equiv v((fs)_*(X)) \\
 &= v(X(\cdot \circ fs)) \\
 &= v\left(\sum_{i=1}^m \underbrace{X(\pi_i \circ \psi)}_{=: X_i^\psi} \partial_i|_{\psi(p)} (\cdot \circ fs \circ \psi^{-1})\right) \\
 &= v\left(\sum_{i=1}^m X_i^\psi \partial_i|_{\psi(p)} (\cdot \circ \Phi^{-1} \circ \Phi \circ fs \circ \psi^{-1})\right) \\
 &= v\left(\sum_{i=1}^m X_i^\psi \sum_{j=1}^{m+n} \partial_j|_{(\Phi \circ fs \circ \psi^{-1})(\psi(p))} (\cdot \circ \Phi^{-1}) \partial_i|_{\psi(p)} (\Phi \circ fs \circ \psi^{-1})_j\right) \\
 &= \sum_{i=1}^m X_i^\psi \sum_{j=m+1}^{m+n} \partial_j|_{(\Phi \circ fs)(p)} (\cdot \circ \Phi^{-1}) \partial_i|_{\psi(p)} (\Phi \circ fs \circ \psi^{-1})_j \\
 &= \sum_{i=1}^m X_i^\psi \sum_{j=m+1}^{m+n} \partial_j|_{(\Phi \circ fs)(p)} (\cdot \circ \Phi^{-1}) \left((\partial_i|_{\psi(p)} f \circ \psi^{-1}) (\Phi \circ s \circ \psi^{-1})_j \right) (\psi(p)) + f(p) \partial_i|_{\psi(p)} (\Phi \circ s \circ \psi^{-1})_j \\
 &= \underbrace{\sum_{i=1}^m X_i^\psi \partial_i|_{\psi(p)} f \circ \psi^{-1}}_{X(f)} \sum_{j=m+1}^{m+n} \partial_j|_{(\Phi \circ fs)(p)} (\cdot \circ \Phi^{-1}) \left((\Phi \circ s \circ \psi^{-1})_j \right) (\psi(p)) + \\
 &\quad + f(p) \sum_{i=1}^m X_i^\psi \sum_{j=n+1}^{m+n} \partial_j|_{(\Phi \circ fs)(p)} (\cdot \circ \Phi^{-1}) \partial_i|_{\psi(p)} (\Phi \circ s \circ \psi^{-1})_j \\
 &= X(f) \sum_{j=m+1}^{m+n} (\Phi_j \circ s)(p) \partial_j|_{(\Phi \circ fs)(p)} (\cdot \circ \Phi^{-1}) + f(p) \sum_{j=n+1}^{m+n} \left(\sum_{i=1}^m X_i^\psi \partial_i|_{\psi(p)} (\Phi \circ s \circ \psi^{-1})_j \right) \partial_j|_{(\Phi \circ fs)(p)} (\cdot \circ \Phi^{-1})
 \end{aligned}$$

Next, note that by 9.2.2,

$$V_{s(p)} = T_{s(p)}(E_p)$$

and v projects onto the vertical subspace, so that

$$\nabla_X (fs) \in T_{s(p)}(E_p)$$

But since E_p is a vector space, it is linearly isomorphic to its tangent space at any point, that is,

$$E_p \xrightarrow{\varphi_{p,e}} T_e(E_p)$$

via the map

$$\varphi_{p,e} \circ \Phi|_{\{\psi(p)\} \times \mathbb{C}^n}^{-1} : \left(\psi(p), \underbrace{u_1, \dots, u_n}_{\in \mathbb{C}^n} \right) \mapsto \sum_{j=m+1}^{m+n} u_j \partial_j|_{\Phi(e)} (\cdot \circ \Phi^{-1})$$

We get that

$$\varphi_{p,f(p)s(p)}^{-1}(\nabla_X(fs)) = X(f)s + f(p)\varphi_{p,s(p)}^{-1}(\nabla_X(s))$$

which is the more precise statement meant by

$$\nabla_X(fs) = X(f)s + f(p)\nabla_X(s)$$

□

9.3.5. *Remark.* Now we want to generalize this from tangent vectors $X \in T_{\pi(u)}M$ to vector fields $X \in \Gamma(TM)$.

So let $X \in \Gamma(TM)$. Then for any $p \in M$, $X|_p \in T_p M$, and a section $s \in \Gamma(E)$, we have

$$M \ni p \mapsto \nabla_{X|_p}(s) \in V_{s(p)}$$

But according to 9.2.2,

$$V_{s(p)} = T_{s(p)}(E_p)$$

and for any vector space, the tangent space is isomorphic to the vector space itself

$$T_{s(p)}(E_p) \cong E_p$$

so that we have an induced map map

$$M \ni p \mapsto \nabla_{X|_p}(s) \in E_p$$

Since this depends on X only through its value at p , we get a map tensorial in X , or $\mathcal{F}(M)$ -linear in X . That is, we have a section, which depends on X . Thus we have a *covariant derivative*: A map from sections to sections of the tensor-product

bundle $T^*M \otimes E$ (1-forms with values in E):

$$\boxed{\nabla : \Gamma(E) \rightarrow \Gamma(T^*M \otimes E)}$$

$$s \xrightarrow{\nabla}$$

$$\underbrace{\nabla(s)}_{\nabla_X(s)}$$

At each $p \in M$, takes a vector $x \in T_p M$ and produces a value $\nabla_X(s) \in E_p$

Note that ∇ is not $\mathcal{F}(M)$ -linear in s : If $f \in \mathcal{F}(M)$, $s \in \Gamma(E)$ and $X \in TM$ then using 9.3.4 we have

$$\nabla_X(fs) = f\nabla_X(s) + X(f)s \quad (48)$$

9.3.6. Definition. (*The Parallel Transport induced by the Ehresmann Connection*) A parallel transport is the assignment to every curve

$$\gamma : [0, 1] \rightarrow M$$

And to every $(s, t) \in [0, 1]^2$ a map

$$\tau(t, s) : E_{\gamma(s)} \rightarrow E_{\gamma(t)}$$

such that:

- (1) $\tau(s, s) = 1_{E_{\gamma(s)}}$
- (2) $\tau(t, s)\tau(s, r) = \tau(t, r)$
- (3) Let $u \in E_{\gamma(0)}$. Then the vector field tangent to the curve

$$[0, 1] \rightarrow E$$

given by

$$t \mapsto (\tau(t, 0))(u)$$

which we denote by X should be horizontal. That is,

$$X|_{(\tau(t, 0))(u)} = \partial_t (\cdot \circ (\tau(t, 0))(u))|_t \in T_{(\tau(t, 0))(u)} E$$

and the requirement is that

$$X|_{(\tau(t, 0))(u)} \stackrel{!}{=} H_{(\tau(t, 0))(u)}$$

which is equivalent to

$$v(X|_{(\tau(t, 0))(u)}) \stackrel{!}{=} 0$$

where v is the vertical projection, which is also equivalent to

$$\begin{aligned} \nabla_{\dot{\gamma}}(\tau(t, 0))(u) &\equiv v\partial_t (\cdot \circ (\tau(t, 0))(u)) \\ &\stackrel{!}{=} 0 \end{aligned}$$

9.4. Gauge Potentials.

9.4.1. Definition. Let $\pi : E \rightarrow M$ be a vector bundle with typical fiber $F \in \text{Obj}(\text{Vect}_{\mathbb{C}}^n)$. Let (φ, U) be a local bundle chart:

$$\varphi : E|_U \rightarrow U \times F$$

Then we have defined a horizontal subspace of $T(E|_U)$ given by

$$T_p U \times \{0\}$$

which is *different* than the Ehresmann connection (the sub-bundle H from above). Now instead of what was called v above (a projector onto the vertical subspace) we need a projector onto

$$T_{s(p)} F \subseteq T_p U \times T_{s(p)} F$$

for some section $s : U \rightarrow F$. Then sections on $E|_U$ are maps

$$s : U \rightarrow F$$

and have pushforwards given by

$$s_* : T_p U \rightarrow T_{s(p)} F$$

and the connection associated to the horizontal subspace $T_p U \times \{0\}$ is denoted by d and is given by

$$d_X(s) \equiv s_*(X) \in T_{s(p)} F$$

(since it is already in $T_{s(p)} F$, the vertical subspace, there is no need to apply a projection).

9.4.2. Claim. $(\nabla_X(s) - d_X(s))(p)$ depends on s only through $s(p)$ and is thus linear in X_p and in $s(p)$. Equivalently stated, it is $\mathcal{F}(M)$ -linear in s .

Proof. Let $f \in \mathcal{F}(M)$. Then

$$\begin{aligned}\nabla_X(fs) - d_X(fs) &= v(fs)_*(X) - (fs)_*(X) \\ &= (v-1) \left(f(p)(s_*)_p(X) + X(f)s_* \right) \\ &= f(p)(\nabla_X(s) - d_X(s)) + \underbrace{(v-1)X(f)s_*}_0\end{aligned}$$

□

9.4.3. Definition. (*Local connection 1-form*) Since $\nabla_X(s) - d_X(s)$ is $\mathcal{F}(M)$ -linear in s , we can define

$$A(X_p)s(p) := \nabla_X(s) - d_X(s)$$

for some $A(X_p) : F \rightarrow F$ linear (matrix), that is,

$$A(X_p) \in \mathfrak{gl}(F) \equiv \text{Lie(GL}(F)\text{)} \equiv \text{End}(F)$$

and for short we may write this relation as

$$A \equiv \nabla - d$$

with $A \in \mathfrak{gl}(F) \otimes T^*U$ which is called the local connection 1-form. If $\{f^\alpha\}_\alpha$ is some basis of F , then let $\omega(X_p)$ be the matrix of that linear map in the basis $\{f^\alpha\}$:

$$(A(X))(f^\alpha) \equiv \sum_\beta \omega(X)^\alpha{}_\beta f^\beta$$

Then ω itself can be thought of as a matrix where each entry is a 1-form (it takes a vector $X \in T_p M$ and produces a complex number, which is just the definition of a one form). These matrix 1-forms $\omega(X)^\alpha{}_\beta$ are also called connection 1-forms. If we define the constant sections as

$$\begin{aligned}\tilde{f}^\alpha : U &\rightarrow F \\ p &\mapsto f^\alpha\end{aligned}$$

then for some $X \in TU$ we have

$$\begin{aligned}d_X(\tilde{f}^\alpha) &\equiv \tilde{f}^\alpha_*(X) \\ &\equiv X \left(\underbrace{\cdot \circ \tilde{f}^\alpha}_{\text{constant map}} \right) \\ \text{tangent vector zero on constant maps} &= 0\end{aligned}$$

so that

$$\begin{aligned}\nabla_X(\tilde{f}^\alpha) &= \nabla_X(\tilde{f}^\alpha) - d_X(\tilde{f}^\alpha) \\ &= A(X)(f^\alpha) \\ &= \sum_\beta \omega^\alpha{}_\beta(X)f^\beta\end{aligned}$$

If $\psi : U \rightarrow \psi(U)$ is a chart on M then

$$\left\{ \partial_i(\cdot \circ \psi^{-1}) \right\}_i$$

is a basis for TU and with respect to it we define the Christoffel symbols:

$$\Gamma_{i\beta}^\alpha := \omega^\alpha{}_\beta(\partial_i(\cdot \circ \psi^{-1}))$$

9.4.4. Claim. (*The Connection 1-forms under Gauge Transformations*) Let $\pi : E \rightarrow M$ be a vector bundle and let $\varphi_i : E|_U \rightarrow U \times F$ for $i \in \{1, 2\}$ be two local bundle trivializations of the same set $U \in \text{Open}(M)$. As we have seen in 8.1.15, the map

$$\varphi_2 \circ \varphi_1^{-1} : U \times F \rightarrow U \times F$$

is of the form

$$(p, f) \mapsto (p, g(p)f)$$

where g is some matrix-valued map $U \rightarrow \text{Aut}(F)$. Then we have the following transformation law of the connection 1-forms:

$$A_1 = g^{-1}A_2g + g^{-1}(dg)$$

Proof. If $s : M \rightarrow E$ is a section, then restricted to U , we may write this section as $s_i : U \rightarrow F$ as follows

$$s_i(p) := \pi_2(\varphi_i(s(p)))$$

where $\pi_2 : U \times F \rightarrow F$ is the projection. Then by definition of g we have

$$s_2(p) = g(p)s_1(p)$$

so that if $X \in TU$ then $\nabla_X(s) : U \rightarrow F$ is also a section so that we have

$$(\nabla_X(s))_2 = g(p)(\nabla_X(s))_1$$

Then plugging in $\nabla = d + A$ we get

$$\begin{aligned} (d_X + A_2(X))(s_2(p)) &= g(p)((d_X + A_1(X))(s_1(p))) \\ &\quad \uparrow \\ (d_X + A_2(X))(g(p)s_1(p)) &= g(p)((d_X + A_1(X))(s_1(p))) \\ &\quad \uparrow \\ \underbrace{g(p)^{-1}d_X(g(p)s_1(p)) - d_X(s_1(p))}_{(g(p)^{-1}(d_X(g(p))))s_1(p)} + g(p)^{-1}A_2(X)g(p)s_1(p) &= A_1(X)(s_1(p)) \end{aligned}$$

so that the result follows.

Note that this transformation law is *not* tensorial. □

9.4.5. Corollary. Note that if $\text{rank}(E) = 1$ then $g(p) \in \text{Aut}(\mathbb{C})$ so that we are not dealing with matrices but rather with numbers, which commute, and so the transformation law becomes

$$A_1 = A_2 + g^{-1}(dg)$$

9.5. Curvature.

9.5.1. Definition. (The curvature corresponding to a connection) Let $\pi : E \rightarrow M$ be a vector bundle and let

$$\nabla : \Gamma(E) \rightarrow \Gamma(T^*M \otimes E)$$

be a connection on E . Then the curvature corresponding to ∇ is the operator

$$R : \Gamma(E) \rightarrow \Gamma\left(\underbrace{\Lambda^2 T^*M}_{2\text{-forms}} \otimes E\right)$$

defined via

$$R(X, Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]}$$

for all vector fields $(X, Y) \in TM$.

9.5.2. Claim. For $s \in \Gamma(E)$, $(R(X, Y))(s)$ is tensorial in X, Y and s .

Proof. Let $f \in \mathcal{F}(M)$. Then using (48) we have

$$\begin{aligned} (R(X, Y))(fs) &= \nabla_X(\nabla_Y(fs)) - \nabla_Y(\nabla_X(fs)) - \nabla_{[X, Y]}(fs) \\ &= \nabla_X(f\nabla_Y(s) + Y(f)s) - \nabla_Y(f\nabla_X(s) + X(f)s) - f\nabla_{[X, Y]}(s) - [X, Y](f)s \\ &= f\nabla_X(\nabla_Y s) + X(f)\nabla_Y(s) + Y(f)\nabla_X(s) + X(Y(f))s \\ &\quad - f\nabla_Y(\nabla_X s) - Y(f)\nabla_X(s) - X(f)\nabla_Y(s) - Y(X(f))s \\ &\quad - f\nabla_{[X, Y]}(s) - [X, Y](f)s \\ &= f\left(\nabla_X(\nabla_Y s) - \nabla_Y(\nabla_X s) - \nabla_{[X, Y]}(s)\right) \end{aligned}$$

And using 9.3.4 we have

$$\begin{aligned} (R(X, fY))(s) &= \nabla_X \nabla_{fY}s - \nabla_{fY} \nabla_X s - \nabla_{[X, fY]}s \\ [X, fY] = X(f)Y + f[X, Y] &\equiv f\nabla_X(\nabla_Y s) + X(f)\nabla_Y(s) - f\nabla_Y \nabla_X s - X(f)\nabla_Y s - f\nabla_{[X, Y]}s \end{aligned}$$
□

9.5.3. Corollary. As a result of this tensorial property, R determines not just a map

$$R : \Gamma(E) \rightarrow \Gamma(\Lambda^2 T^*M \otimes E)$$

but actually a $R \in \Gamma(\Lambda^2 T^*M \otimes \text{End}(E))$ where

$$\text{End}(E) \cong E^* \otimes E$$

and has fibers $(\text{End}(E))_p \equiv \text{End}(E_p) \equiv \text{gl}(E_p)$. Then in a local chart, R is given by $F \in \text{gl}(F) \otimes \Lambda^2 T^*U$ where F is called the curvature 2-form. It is a matrix whose entries are 2-forms on $U \in \text{Open}(M)$.

9.5.4. Claim. We have

$$F = dA + A \wedge A$$

Proof. The definition of the exterior derivative and the wedge product are as follows:

$$(dA)(X, Y) \equiv X(A(Y)) - Y(A(X)) - (A([X, Y]))(s)$$

(the fact that $A(Y)$ is really a matrix of one-forms and not a bona-fide one-form doesn't matter as X acts linearly) and

$$((A \wedge A)(X, Y)) \equiv A(X)A(Y) - A(Y)A(X)$$

Then using the same basis as in 9.3.4 we have

$$\begin{aligned} F(X, Y) &\equiv (d_X + A(X))(d_Y + A(Y)) - (X \leftrightarrow Y) - (d_{[X, Y]} + A([X, Y])) \\ &= d_X d_Y + \underbrace{d_X A(Y)}_{X(A(Y)) + A(Y)d_X} + A(X) d_Y + A(X) A(Y) + \dots \end{aligned}$$

which follows as

$$\begin{aligned} d_X A(Y)s &= X(\cdot \circ A(Y)s) \\ &= X(A(Y))s + A(Y)X(\cdot \circ s) \\ &= X(A(Y))s + A(Y)d_X s \end{aligned}$$

Now use the fact that

$$\begin{aligned} (d_X d_Y - d_Y d_X)(s) &= X(Y(\cdot \circ s)) - Y(X(\cdot \circ s)) \\ &\equiv d_{[X, Y]}s \end{aligned}$$

□

9.5.5. Claim. Under the same transformation as in 9.4.4 we have that the curvature 2-form transforms as

$$F_1 = g^{-1} F_2 g$$

Proof. Using the above formula, the fact that $d^2 g = 0$ and

$$dg^{-1} = -g^{-1}(dg)g^{-1}$$

we have

$$\begin{aligned} F_1 &= dA_1 + A_1 \wedge A_1 \\ &= d(g^{-1}A_2 g + g^{-1}(dg)) + (g^{-1}A_2 g + g^{-1}(dg)) \wedge (g^{-1}A_2 g + g^{-1}(dg)) \\ &= dg^{-1}A_2 g + dg^{-1}(dg) + g^{-1}A_2 g \wedge g^{-1}A_2 g + g^{-1}A_2 g \wedge g^{-1}(dg) + g^{-1}(dg) \wedge g^{-1}A_2 g + g^{-1}(dg) \wedge g^{-1}(dg) \\ &= (dg^{-1}) \wedge A_2 g + g^{-1}(dA_2)g + g^{-1}A_2 dg + (dg^{-1}) \wedge (dg) \\ &\quad + g^{-1}A_2 \wedge A_2 g + g^{-1}A_2 \wedge (dg) + g^{-1}(dg) \wedge g^{-1}A_2 g + g^{-1}(dg) \wedge g^{-1}(dg) \\ &= -g^{-1}(dg)g^{-1} \wedge A_2 g + g^{-1}(dA_2)g + g^{-1}A_2 \wedge dg - g^{-1}(dg)g^{-1} \wedge dg \\ &\quad + g^{-1}A_2 \wedge A_2 g + g^{-1}A_2 \wedge dg + g^{-1}(dg) \wedge g^{-1}A_2 g + g^{-1}(dg) \wedge g^{-1}(dg) \\ &= g^{-1}A_2 \wedge dg + g^{-1}A_2 \wedge dg + g^{-1}(dA_2)g + g^{-1}A_2 \wedge A_2 g \\ &= g^{-1}(dA_2)g + g^{-1}A_2 \wedge A_2 g \end{aligned}$$

□

9.5.6. Remark. The above transformation rule is tensorial. Note that even though F is not gauge invariant, its trace is.

$$\text{tr}(F_1) = \text{tr}(F_2)$$

9.5.7. Claim. In a local basis $\{f^\alpha\}_\alpha$ of F , we have

$$F(X, Y) f^\alpha = \Omega^\alpha{}_\beta(X, Y) f^\beta$$

for some matrix $\Omega(X, Y) \in \mathfrak{gl}(F)$ and then

$$\Omega^\alpha{}_\beta = d\omega^\alpha{}_\beta + \omega^\alpha{}_\gamma \wedge \omega^\gamma{}_\beta \quad (49)$$

Proof. Use the identity $F = dA + A \wedge A$.

□

9.5.8. Claim. (The Bianchi Identity) If $[A, F] \equiv A \wedge F - F \wedge A$ then

$$dF + [A, F] = 0$$

Proof. A straight-forward computation shows

$$\begin{aligned} dF + [A, F] &= d(dA + A \wedge A) + A \wedge (dA + A \wedge A) - (dA + A \wedge A) \wedge A \\ &= dA \wedge A - A \wedge dA + A \wedge dA + A \wedge A \wedge A - dA \wedge A - A \wedge A \wedge A \\ &= 0 \end{aligned}$$

□

9.5.9. Example. (Electrodynamics) Let M be the spacetime manifold. Let E be a vector bundle with typical fiber \mathbb{R} (so $\mathfrak{gl}(\mathbb{R}) = \mathbb{R}$)

(Note that in physics the Lie algebra of $U(1)$, $u(1)$, is defined as $i\mathbb{R}$ and in math it is \mathbb{R})

Then

$$\begin{aligned}\mathfrak{gl}(\mathbb{R}) \otimes T^*U &= \mathbb{R} \otimes T^*U \\ &= T^*U\end{aligned}$$

so that

$$A = A_\mu dx^\mu$$

is the vector potential, $F = dA$ is the electromagnetic field tensor (since $A \wedge A = 0$ because the typical fiber is of rank 1). The Bianchi identity implies

$$\underbrace{dF}_{d^2A=0} + [F, A] = 0$$

↑

$$[F, A] = 0$$

9.5.10. Example. (Differential Geometry on Manifolds) Let

$$E := TM$$

and

$$F := \mathbb{R}^4$$

with ∇ being the Levi-Civita connection. Then

$$\omega^\alpha{}_\beta(\cdot)$$

are the connection 1-forms. Then (49) is the second Cartan structure identity, and the Bianchi identity says

$$\begin{aligned}dF + [F, A] &= 0 \\ \downarrow \\ d\Omega^\alpha{}_\beta + [\Omega^\alpha{}_\beta, \omega^\alpha{}_\beta] &= 0\end{aligned}$$

which is precisely the second Bianchi identity.

9.6. The Berry Connection.

9.6.1. Definition. (The Berry Connection) Let $\pi : M \times F \rightarrow M$ be the trivial bundle with typical fiber F , such that F is equipped with an inner product. Let \mathcal{P} be a (possibly non-trivial) sub-bundle of $M \times F$ with fibers $(\mathcal{P})_k$ determined by orthogonal projections

$$\tilde{P}(k) : F \rightarrow F$$

$$(\mathcal{P})_k \equiv \text{im}(\tilde{P}(k))$$

Then a connection on \mathcal{P} is defined via

$$\nabla := \tilde{P}(k) d$$

that is,

$$\begin{aligned}\nabla_X(s) &\equiv \tilde{P}(k) d_X(s) \\ &= \tilde{P}(k) s_*(X) \\ &= \tilde{P}(k) X(\cdot \circ s)\end{aligned}$$

9.6.2. Remark. (Analogy with Levi-Civita Connection) Let N be a Riemannian manifold with a Levi-Civita connection ∇_N and $M \subseteq N$ be a sub-manifold. Then the Levi-Civita connection on M is

$$\nabla_M = P \nabla_N$$

with $P : T_p N \rightarrow T_p M$ the orthogonal projection. The analogy is complete for $N = \mathbb{R}^n$, $\nabla_N = d$ and a sub-bundle $TM \subseteq TN|_M = M \times \mathbb{R}^n$.

9.6.3. Remark. Let $s \in \Gamma(\mathcal{P})$. Then

$$\begin{aligned}ds &= d\tilde{P}s \\ (d_X \tilde{P}s = X(\tilde{P})s + \tilde{P}s_*(X)) &\equiv (d\tilde{P})s + \tilde{P}ds \\ &\equiv \nabla \\ &= ((d\tilde{P})\tilde{P} + \nabla)s\end{aligned}$$

so that

$$\nabla = d - (d\tilde{P})\tilde{P} \quad (50)$$

However,

$$A \neq -(d\tilde{P})\tilde{P}$$

since ∇ is not expressed in a local bundle chart!

9.6.4. Claim. *The Berry curvature is given by*

$$R(X, Y) = [X\tilde{P}, Y\tilde{P}] \tilde{P}$$

Proof. We start from

$$R(X, Y) \equiv \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]}$$

and use (50) in the form

$$\nabla_X = d_X - (X\tilde{P})\tilde{P}$$

so that

$$\begin{aligned} \nabla_X \nabla_Y &= (d_X - (X\tilde{P})\tilde{P})(d_Y - (Y\tilde{P})\tilde{P}) \\ &= d_X d_Y - d_X (Y\tilde{P})\tilde{P} - (X\tilde{P})\tilde{P} d_Y + (X\tilde{P})\tilde{P} (Y\tilde{P})\tilde{P} \\ &= d_X d_Y - X(Y\tilde{P})\tilde{P} - (Y\tilde{P})(X\tilde{P}) - (Y\tilde{P})\tilde{P} d_X - (X\tilde{P})\tilde{P} d_Y + (X\tilde{P})\tilde{P} (Y\tilde{P})\tilde{P} \end{aligned}$$

But

$$(X\tilde{P})\tilde{P} (Y\tilde{P})\tilde{P} - (Y\tilde{P})\tilde{P} (X\tilde{P})\tilde{P} = 0$$

so that the result follows. \square

9.6.5. Corollary. *If $\dim(M) = 2$, we have*

$$\begin{aligned} \frac{1}{2\pi i} \int_M \underbrace{\text{tr}(R)}_{2-form \text{ on } M} &= \frac{1}{2\pi i} \int_M \text{tr}(R(\partial_{k_1}, \partial_{k_2})) dk_1 dk_2 \\ &= \frac{1}{2\pi i} \int_M \text{tr}([\partial_{k_1}\tilde{P}(k), \partial_{k_2}\tilde{P}(k)]\tilde{P}(k)) dk_1 dk_2 \\ &\equiv \text{Ch}_1(\mathcal{P}) \end{aligned}$$

and if $\dim(M) = 4$ we have

$$\frac{1}{2} \frac{1}{(2\pi i)^2} \int_M \underbrace{\text{tr}(R \wedge R)}_{4-form \text{ on } M} =: \text{Ch}_2(\mathcal{P})$$

which is the second Chern number, and is also an integer.

9.6.6. Remark. There is a way to show that the Chern numbers are independent of the particular connection chosen (see appendix).

10. THE BULK-EDGE CORRESPONDENCE IN THE PERIODIC CASE

Recall from 5.2 that one can consider the integer quantum Hall effect as an edge effect. In fact, we even presented a phenomenological proof that the Hall conductivity in the two perspectives is equal (5.3). In this section we will present a rigorous proof of this fact for the case (in the periodic setting) using Levinson's theorem, a proof which was published in [20].

10.1. The System. We consider a two-dimensional sample in a tight-binding model on

$$X = \mathbb{Z}^2$$

(for the bulk) or

$$X^\sharp = \mathbb{Z} \times \mathbb{N}$$

(for the edge) where the Hilbert space is $\ell^2(X; \mathbb{C})$ or $\ell^2(X^\sharp; \mathbb{C})$ and such that there is an operator

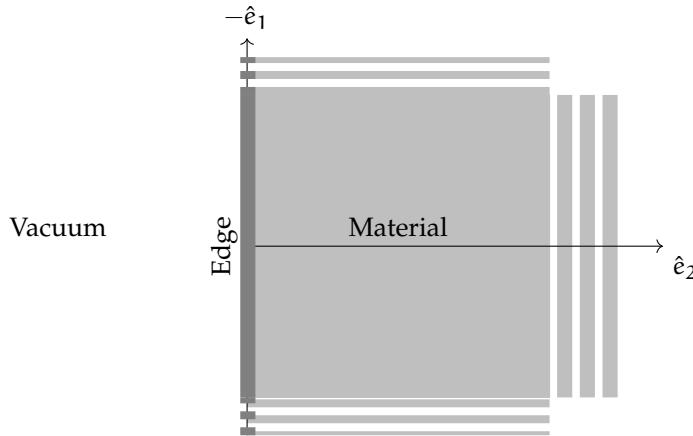
$$H \in \mathcal{B}\left(\ell^2(X; \mathbb{C})\right)$$

and an operator

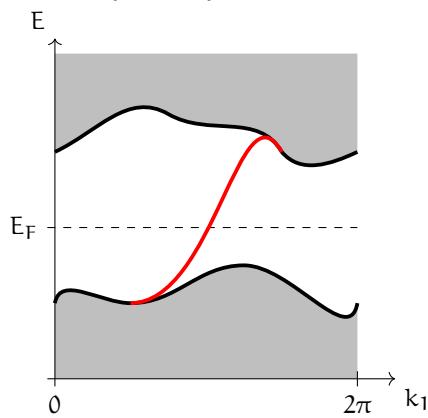
$$H^\sharp \in \mathcal{B}\left(\ell^2(X^\sharp; \mathbb{C})\right)$$

which obey the following relation: If $(H)_{(n, m)} \in \mathbb{C}$ for $(n, m) \in X^2$ is the matrix element of H with respect to the position basis, then

$$(H^\sharp)_{(n, m)} \stackrel{!}{=} (H)_{(n, m)} \quad \forall (n, m) \in (X^\sharp)^2$$



10.2. The Edge. The edge Hamiltonian now only has translation symmetry along the first axis and the second axis has no translation symmetry. As a result, the edge unit cell is *not* compact and so has continuous spectrum.



10.2.1. Definition. Define the signed number of times that the Fermi energy crosses the discrete spectrum of the edge (after having done Bloch decomposition on the axis on which it is possible) by

$$\text{Index}(H^\sharp) := \text{signed } \# \text{ of crossings of } E_F \text{ with the discrete spectrum}$$

for example in the above picture we have $\text{Index}(H^\sharp) = +1$.

10.2.2. *Remark.* Index (H^\sharp) is an integer. Due to the stability of the spectrum it is also clear that if we perturb H^\sharp compactly this integer remains constant. This index is also sometimes called “spectral flow” or the Maslov index of $k_1 \mapsto H^\sharp(k_1)$.

10.2.3. **Claim.** *The Hall conductivity in the edge system is given by*

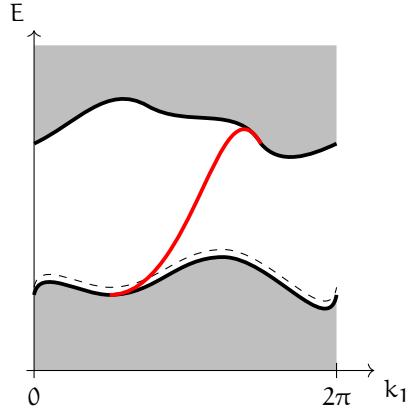
$$\sigma_H = 2\pi \text{Index}(H^\sharp)$$

Proof. We assume the chemical potential on one edge is μ_+ and μ_- on the other edge, where $\mu_+ \neq \mu_-$ (otherwise the current on one edge cancels out the current on the other edge as they flow in opposite directions). Using the formula $j = \rho v$ where ρ is the density of carriers and v is the velocity of the carriers, we have

$$\begin{aligned} I &= \frac{1}{2\pi} \sum_j \int_{k_-^j}^{k_+^j} v(k) dk \\ &= \frac{1}{2\pi} \sum_j \int_{k_-^j}^{k_+^j} \frac{1}{h} \frac{\partial E}{\partial k} dk \\ &= \frac{1}{h} \frac{1}{2\pi} \sum_j [E(k_+^j) - E(k_-^j)] \\ &= \frac{1}{h} \frac{1}{2\pi} \sum_j [\mu_+ - \mu_-] \\ &= \frac{1}{h} \frac{1}{2\pi} \sum_j V \end{aligned}$$

where the sum is on intersection points of either μ_+ or μ_- with the gapless edge states, v is the velocity, and V is the potential between the two edges. Thus we obtain that for each ascending crossing of the gapless edge mode with either μ_+ or μ_- we must count +1 for the conductance (given by $\sigma = \frac{I}{V}$) and -1 for a descending crossing. \square

10.3. The Equality



10.3.1. **Claim.** *The Hall conductivity as computed in the edge system is equal to its analog in the bulk system. That is, we have*

$$\text{Index}(H^\sharp) = \text{Ch}_1(\mathcal{P})$$

where \mathcal{P} is the occupied sub-bundle of E , the Bloch-bundle corresponding to the bulk system.

Proof. We will not give the whole proof, but rather just establish the context. As the picture above shows, instead of having the Fermi energy in the middle of the gap, we can just as well have it be infinitesimally close to the upper occupied band edge and count the incipience or disappearing of edge states from that band edge.

Assume that $\text{rank}(\mathcal{P}) = N$ and concentrate on just one band in this bundle, that is, some rank-1 projection $\tilde{P}_j(k)$ with $j \in \{1, \dots, N\}$.

Assume that for fixed $k_1 \in S^1$, $k_2 \mapsto \varepsilon_j(k)$ has two extremum points, namely one maximum and one minimum, which we denote by $k_2^{\max}(k_1)$ and $k_2^{\min}(k_1)$. Then these two curves

$$k_1 \mapsto k_2^{\max}(k_1)$$

and

$$k_1 \mapsto k_2^{\min}(k_1)$$

cut \mathbb{T}^2 into two open domains in which $k_2 \mapsto \varepsilon_j(k)$ is either increasing or decreasing with respect to the orientation of S^1 . In these two subsets of \mathbb{T}^2 , as we have seen in 8.4.11, one can find global sections for the line-bundle, and the first Chern number is given by the winding number of the transition matrix along the cut. Since we have two cuts, we need the difference of the two windings (requires more explanation) of two transition matrices. One goes from the region of

decreasing to the region of increasing, which we call $T_1^{-+}(k_1)$, and traces the path

$$k_1 \mapsto k_2^{\min}(k_1)$$

in \mathbb{T}^2 and the other goes from the region of increasing to the region of decreasing which we call $T_1^{+-}(k_1)$ and traces the path

$$k_1 \mapsto k_2^{\max}(k_1)$$

in \mathbb{T}^2 .

$$\text{Ch}_1(\mathcal{P}_j) = \text{Winding Number}(T_1^{-+}) - \text{Winding Number}(T_1^{+-})$$

However, these transition matrices also have a physical meaning: they can be thought of as scattering S-matrices, where a particle scatters with the edge and bounces back in the other direction. Indeed, since for fixed k_1 , $k_2 \mapsto \varepsilon_j(k)$ is either increasing or decreasing, and $v_i \sim \partial_{k_i} \varepsilon_j(k)$ we can think of these curves as points of scattering, as the sign of the speed is reversed.

The point now is to use Levinson's theorem

$$\lim_{\delta \rightarrow 0} \arg(T^{+-}(k_1)) \Big|_{k_1^{(1)}}^{k_1^{(2)}} = 2\pi N$$

where N is the signed number of discrete eigenvalues of $H^\sharp(k_1)$ emerging (counted as minus) or disappearing (counted as plus) at the upper band edge $\varepsilon_j(k_1, k_2^{\max}(k_1))$ as k_1 runs from $k_1^{(1)}$ to $k_1^{(2)}$. For the full details see [20] page 19 and the actual proof on page 41. \square

Part 2. Time-Reversal Invariant Topological Insulators

As we have seen in 7.2.5, the Hall conductivity of a system with time-reversal invariance is zero. This makes perfect sense, because a magnetic field breaks time-reversal invariance, and of course without magnetic field there is no Hall effect.

Despite this, there is still some interesting topology associated with such systems. Because in this section we restrict ourselves to the periodic case (even though we don't have to. See [24] for an analog of the index of pair of projections in the time-reversal invariant disordered setting) mathematically one could say that the case of no time reversal invariance corresponds to *complex* vector bundles whereas the addition of extra symmetries changes the field of the vector space of the typical fiber of the vector bundle. For instance, time reversal invariance on Fermions corresponds to quaternionic vector bundles. For Bosons, one could say it corresponds to real vector bundles. Then there is a rich theory for the classification of vector bundles over other (than \mathbb{C}) fields, for instance, Pontryagin classes for quaternionic bundles and Stiefel-Whitney classes for real bundles (see [33]).

Our goal here is to describe the discovery of [23].

11. TIME-REVERSAL INVARIANT SYSTEMS

11.1. The Time-Reversal Symmetry Operation. Let \mathcal{H} be a single particle Hilbert space over \mathbb{C} , and assume that we have some symmetry operation $\Theta : \mathcal{H} \rightarrow \mathcal{H}$ which reverses the direction of time.

By Wigner's theorem, we know that any symmetry must be implemented as a \mathbb{C} -linear (unitary) or anti- \mathbb{C} -linear (anti-unitary) map $\mathcal{H} \rightarrow \mathcal{H}$ (see [9]).

If U_t is the one-parameter group of time-translations (typically $U_t = \exp(iHt)$ with $H \in \mathcal{B}(\mathcal{H})$ the Hamiltonian), we expect time reversal Θ to be a map $\mathcal{H} \rightarrow \mathcal{H}$ such that

$$\Theta U_t \psi = U_{-t} \Theta \psi \quad \forall \psi \in \mathcal{H}$$

so that

$$\Theta iH = -iH\Theta$$

and if Θ is anti- \mathbb{C} -linear we have

$$[\Theta, H] = 0$$

whereas if Θ is \mathbb{C} -linear we have

$$\{\Theta, H\} = 0$$

The anti-commutation relation with the Hamiltonian implies that $\sigma(H)$ is symmetric about zero. Thus, if $\sigma(H)$ is bounded below and unbounded above Θ has to be anti- \mathbb{C} -linear. So it is usually assumed that Θ is indeed anti- \mathbb{C} -linear, which we also assume in what follows. However, there are some condensed matter physics systems that have symmetric spectrum so that it makes sense to allow also for the other possibility.

11.1.1. Definition. (*Time-Reversal*) Time reversal is an anti- \mathbb{C} -linear anti-unitary map $\Theta : \mathcal{H} \rightarrow \mathcal{H}$.

11.1.2. Claim. $\Theta^2 = \pm 1$ and the choice is determined by the system under consideration rather than by our choice of how to implement time reversal.

Proof. Since Θ is an involution (reversing time twice gives the same direction of time), applying it twice must give at "worst" a phase, which we denote by $c \in \mathbb{C}$ (with $|c| = 1$):

$$\Theta^2 = c1$$

Then

$$\begin{aligned} \Theta^3 &= \Theta^2 \Theta \\ &= c\Theta \end{aligned}$$

and

$$\begin{aligned} \Theta^3 &= \Theta \Theta^2 \\ &= \Theta c \\ &= \bar{c}\Theta \end{aligned}$$

so that $c = \bar{c}$ and so $c = \pm 1$.

Next assume that $\tilde{\Theta}$ is some other choice of a time-reversal symmetry operator. Then $\tilde{\Theta} = \tilde{c}\Theta$ for some $\tilde{c} \in \mathbb{C}$ and $|\tilde{c}| = 1$. Thus

$$\begin{aligned} \tilde{\Theta}^2 &= (\tilde{c}\Theta)^2 \\ &= \tilde{c}\Theta\tilde{c}\Theta \\ &= \tilde{c}\bar{\tilde{c}}\Theta^2 \\ &= |\tilde{c}|^2\Theta^2 \\ &= \Theta^2 \end{aligned}$$

11.1.3. Remark. For half-integer-spin systems we choose $\Theta^2 = -1$ and for integer-spin systems we choose $\Theta^2 = +1$. □

11.1.4. *Remark.* Note that an anti- \mathbb{C} -linear map $\Theta : \mathcal{H} \rightarrow \mathcal{H}$ on a complex vector space \mathcal{H} with $\Theta^2 = +1$ defines a real-structure on \mathcal{H} . Θ can be thought of as complex conjugation on \mathcal{H} , so that the “real” vectors in \mathcal{H} are those for which $\Theta\psi = \psi$. If we define the projectors

$$P_{\pm} \equiv \frac{1}{2}(\mathbb{1} \pm \Theta)$$

then $P_{\pm}\mathcal{H}$ is an \mathbb{R} -vector space with

$$\dim_{\mathbb{R}}(P_+\mathcal{H}) = \dim_{\mathbb{C}}(\mathcal{H})$$

and

$$\mathcal{H}^{\mathbb{R}} = (P_+\mathcal{H}) \oplus (P_-\mathcal{H})$$

where $\mathcal{H}^{\mathbb{R}}$ is the \mathbb{R} -vector-space obtained from \mathcal{H} by forgetting how to multiply vectors by i . Note that \mathbb{R} -vector bundles (which arise in the translation invariant case) are classified by the Stiefel-Whitney classes.

11.1.5. *Remark.* An anti- \mathbb{C} -linear map $\Theta : \mathcal{H} \rightarrow \mathcal{H}$ on a complex vector space \mathcal{H} with $\Theta^2 = -1$ defines a quaternionic structure on \mathcal{H} . Θ, i and $i\Theta$ are the three generators of quaternionic algebra. \mathbb{H} -vector bundles are classified by the Pontryagin classes.

11.1.6. **Example.** (*Electron with spin- $\frac{1}{2}$*) Let

$$\mathcal{H} = L^2(X) \otimes \mathbb{C}^2$$

where X is either \mathbb{R}^d or \mathbb{Z}^d . Then define

$$\Theta := \mathbb{1}_{L^2(X)} \otimes \theta$$

with

$$\begin{aligned} \theta : \mathbb{C}^2 &\rightarrow \mathbb{C}^2 \\ v &\mapsto \sigma_2 \bar{v} \quad \forall v \in \mathbb{C}^2 \end{aligned}$$

and σ_2 the second Pauli spin matrix. If we denote complex conjugation by $\mathcal{C} : \mathbb{C}^N \rightarrow \mathbb{C}^N$ then we can write $\theta = \sigma_2 \mathcal{C}$.

(1) Squares to minus one:

$$\begin{aligned} \Theta^2 &= (\mathbb{1}_{L^2(X)} \otimes \theta)^2 \\ &= \mathbb{1}_{L^2(X)} \otimes \theta^2 \end{aligned}$$

and

$$\begin{aligned} \theta^2 &= \sigma_2 \mathcal{C} \sigma_2 \mathcal{C} \\ &= \sigma_2 (-\sigma_2) \underbrace{\mathcal{C}^2}_{\mathbb{1}} \\ &= -\underbrace{(\sigma_2)^2}_{\mathbb{1}} \\ &= -\mathbb{1} \end{aligned}$$

(2) Anti-commutes with Pauli-matrices:

$$\Theta (\mathbb{1} \otimes \sigma_i) \Theta^{-1} = \mathbb{1} \otimes \theta \sigma_i \theta^{-1}$$

and

$$\begin{aligned} \theta \sigma_i \theta^{-1} &= \sigma_2 \mathcal{C} \sigma_i \mathcal{C}^{-1} \sigma_2^{-1} \\ &= \sigma_2 \mathcal{C} \sigma_i \mathcal{C} \sigma_2 \\ &= \sigma_2 (-\delta_{i,2} \sigma_2 + (1 - \delta_{i,2}) \sigma_i) \sigma_2 \\ &= -\delta_{i,2} \sigma_2 + (1 - \delta_{i,2}) \underbrace{\sigma_2 \sigma_i}_{\substack{i\varepsilon_{12i} \sigma_1 \\ i\varepsilon_{12i} i\varepsilon_{12j} \sigma_j}} \sigma_2 \\ &\quad \underbrace{-\delta_{i,j} \sigma_j}_{-\delta_{i,j} \sigma_j} \\ &= -\delta_{i,2} \sigma_2 - (1 - \delta_{i,2}) \sigma_i \\ &= -\sigma_i \end{aligned}$$

Then the spin vector $\mathbf{S} = \frac{1}{2}\hbar\vec{\sigma}$ is odd under θ , just like angular momentum $\mathbf{L} \equiv \mathbf{x} \times \mathbf{p}$.

11.2. Kramers' Theorem.

11.2.1. **Claim.** *If the Hamiltonian of a system H is time-reversal invariant, then any eigenvalue λ of H has a degenerate subspace of dimension at least 2.*

Proof. Time reversal invariance of H is equivalent to

$$[H, \Theta] = 0$$

Now assume that $\lambda \in \mathbb{R}$ is an eigenvalue of H , with some eigenvector $\psi_\lambda \in \mathcal{H}$. Then

$$H\psi_\lambda = \lambda\psi_\lambda$$

so that

$$\begin{aligned} H\Theta\psi_\lambda &= \Theta H\psi_\lambda \\ &= \Theta\lambda\psi_\lambda \\ &= \bar{\lambda}\Theta\psi_\lambda \\ &\stackrel{\lambda \in \mathbb{R}}{=} \lambda\Theta\psi_\lambda \end{aligned}$$

so that $\Theta\psi_\lambda$ is also an eigenvector of H with the same eigenvalue λ . However, for degeneracy we still need that ψ_λ and $\Theta\psi_\lambda$ are linearly independent. Assume otherwise. Then $\Theta\psi_\lambda \sim \psi_\lambda$, that is $\Theta\psi_\lambda = \alpha\psi_\lambda$ for some $\alpha \in \mathbb{C} \setminus \{0\}$. But then

$$\Theta^2\psi_\lambda = -\psi_\lambda$$

and

$$\begin{aligned} \Theta^2\psi_\lambda &= \Theta\Theta\psi_\lambda \\ &= \Theta\alpha\psi_\lambda \\ &= \bar{\alpha}\Theta\psi_\lambda \\ &= \bar{\alpha}\alpha\psi_\lambda \\ &= |\alpha|^2\psi_\lambda \end{aligned}$$

and since $\psi_\lambda \neq 0$ we have $|\alpha|^2 = -1$ which is a contradiction. \square

11.2.2. Claim. *If the Hamiltonian of a system H is time-reversal invariant, then any eigenvalue λ of H has a degenerate subspace of even dimension.*

Proof. Let λ be an eigenvalue of H and assume that the eigenspace of λ is odd-dimensional. That is, assume there are vectors $\{\psi_i\}_{i=1}^{2n+1}$ for some $n \in \mathbb{N}_{\geq 0}$ where ψ_i are linearly independent and

$$H\psi_i = \lambda\psi_i$$

and without loss of generality $\langle \psi_i, \psi_j \rangle = \delta_{i,j}$. We proceed by induction on n to show that the set of vectors in the eigenspace must be of the form $\{\psi_1, \Theta\psi_1, \psi_2, \Theta\psi_2, \dots\}$. For the case $n = 0$ we have the claim above. Now assume that for some n we have

$$\{\psi_1, \Theta\psi_1, \psi_2, \Theta\psi_2, \dots, \psi_{n-1}, \Theta\psi_{n-1}, \psi_n\}$$

as the eigenspace where all vectors above are linearly independent, and that $\Theta\psi_n$ is in the span of the above set. Then

$$\Theta\psi_n = \alpha_n\psi_n + \sum_{i=1}^{n-1} (\alpha_i\psi_i + \beta_i\Theta\psi_i) \quad (51)$$

for some $(\alpha_i, \beta_i) \in \mathbb{C}^2$ not all zero. Then

$$\Theta^2\psi_n = -\psi_n$$

and yet

$$\begin{aligned} \Theta^2\psi_n &= \Theta\Theta\psi_n \\ &= \Theta \left[\alpha_n\psi_n + \sum_{i=1}^{n-1} (\alpha_i\psi_i + \beta_i\Theta\psi_i) \right] \\ &= \bar{\alpha}_n\Theta\psi_n + \sum_{i=1}^{n-1} (\bar{\alpha}_i\Theta\psi_i - \bar{\beta}_i\psi_i) \\ &= \bar{\alpha}_n \left(\alpha_n\psi_n + \sum_{i=1}^{n-1} (\alpha_i\psi_i + \beta_i\Theta\psi_i) \right) + \sum_{i=1}^{n-1} (\bar{\alpha}_i\Theta\psi_i - \bar{\beta}_i\psi_i) \\ &= |\alpha_n|^2\psi_n + \sum_{i=1}^{n-1} [(\bar{\alpha}_n\alpha_i - \bar{\beta}_i)\psi_i + (\bar{\alpha}_n\beta_i + \bar{\alpha}_i)\Theta\psi_i] \end{aligned}$$

so that $\alpha_n = 0$ and so

$$\sum_{i=1}^{n-1} [-\bar{\beta}_i\psi_i + \bar{\alpha}_i\Theta\psi_i] = 0$$

and $\psi_i, \Theta\psi_i$ are all linearly independent, so that $\alpha_i = \beta_i = 0$ as well. Thus (51) is a contradiction. \square

12. TRANSLATION INVARIANT SYSTEMS

12.0.1. Fact. *Assume that lattice translations (as in 8.2.9) U_n commute with Θ :*

$$[U_n, \Theta] = 0 \quad \forall n \in \mathcal{L}$$

12.0.2. **Claim.** *On the Bloch-decomposed Hilbert space*

$$\mathcal{H} = \int_{\mathbb{T}^2}^{\oplus} \tilde{\mathcal{H}}(\mathbf{k}) d\mathbf{k}$$

the map $\Theta : \mathcal{H} \rightarrow \mathcal{H}$ obeys: if $\mathbf{k} \in \mathbb{T}^2$ is given then $\Theta\psi \in \tilde{\mathcal{H}}(-\mathbf{k})$ for all $\psi \in \tilde{\mathcal{H}}(\mathbf{k})$.

Proof. Let $\psi \in \tilde{\mathcal{H}}(\mathbf{k})$. Then $U_n\psi = e^{-i\mathbf{k}\cdot\mathbf{n}}\psi$. Then

$$\begin{aligned} U_n\Theta\psi &= \Theta U_n\psi \\ &= \Theta e^{-i\mathbf{k}\cdot\mathbf{n}}\psi \\ &= e^{i\mathbf{k}\cdot\mathbf{n}}\Theta\psi \end{aligned}$$

□

12.0.3. **Example.** We have $[\Theta, U_n] = 0$ when Θ is local, that is, if $\mathcal{H} = L^2(X) \otimes \mathbb{C}^N$ with

$$U_n = \tilde{U}_n \otimes \mathbb{1}_{\mathbb{C}^N}$$

and

$$\Theta = \mathbb{1}_{L^2(X)} \otimes \theta$$

for some θ on \mathbb{C}^N .

12.0.4. **Corollary.** *As a result, we have as in the proof of 8.4.8*

$$\Theta \tilde{H}(\mathbf{k}) = \tilde{H}(-\mathbf{k}) \Theta \quad \forall \mathbf{k} \in \mathbb{T}^2$$

in particular,

$$\sigma(\tilde{H}(\mathbf{k})) = \sigma(\tilde{H}(-\mathbf{k}))$$

and likewise for the Fermi projection

$$\Theta \tilde{P}(\mathbf{k}) = \tilde{P}(-\mathbf{k}) \Theta$$

and, the conclusion in 8.4.8 of course still holds:

$$\text{Ch}_1(\mathcal{P}) = 0$$

12.0.5. **Definition.** Define the time-reversal-invariant-momenta (henceforth denoted by TRIM) as the following subset of \mathbb{T}^2 :

$$\begin{aligned} \text{TRIM} &:= \left\{ \mathbf{k} \in \mathbb{T}^2 \mid \mathbf{k} = -\mathbf{k} \right\} \\ &= \{(0, 0), (0, \pi), (\pi, 0), (\pi, \pi)\} \end{aligned}$$

12.0.6. **Remark.** Note that for one dimensional systems on $\mathbb{T} \equiv S^1$ we would have

$$\text{TRIM} = \{0, \pi\}$$

12.0.7. **Corollary.** *Note that*

$$[\Theta, \tilde{H}(\mathbf{k})] = 0 \quad \forall \mathbf{k} \in \text{TRIM}$$

so that the discrete eigenvalues of the edge system $\varepsilon_j(\mathbf{k})$ are Kramers degenerate at $\mathbf{k} \in \text{TRIM}$. This also shows (as the rank(\mathcal{P}) must be constant in $\mathbf{k} \in \mathbb{T}^2$) that $\text{rank}(\mathcal{P}) \in 2\mathbb{N}$.

12.0.8. **Remark.** The general mathematical structure of such a system is that of an equivariant vector bundle (or a complex vector bundle with a real structure). We have the bundle $\pi : E \rightarrow M$ with a map $\tau : M \rightarrow M$ such that $\tau^2 = \mathbb{1}_M$, $\mathcal{T} : E \rightarrow E$ with $\mathcal{T}^2 = -\mathbb{1}_E$ such that the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{\mathcal{T}} & E \\ \pi \downarrow & & \downarrow \pi \\ M & \xrightarrow{\tau} & M \end{array}$$

Note that, in particular, we have $\mathcal{T}e \in E_{\tau(p)}$ for all $e \in E_p$ and $p \in M$. If $M = \mathbb{T}^2$ then we usually define $\tau\mathbf{k} := -\mathbf{k}$.

Note that we now use the symbol \mathcal{T} for the time-reversal operation in the level of bundles (whereas Θ was the time-reversal operation in the level of the single particle Hilbert space).

13. THE FU-KANE-MELE INVARIANT

This work was first published in [17] (which is a more correct perspective of the slightly earlier work [23]).

13.1. The Pfaffian. For a general discussion of Pfaffians see [18].

13.1.1. Definition. $\text{Pf}(A)$ is the Pfaffian of an anti-symmetric $2n \times 2n$ matrix, given by

$$\text{Pf}(A) = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \text{sgn}(\sigma) \prod_{i=1}^n (A)_{\sigma(2i-1), \sigma(2i)} \quad (52)$$

where S_{2n} is the symmetric group of all permutations of the set $\{1, \dots, 2n\}$ and sgn is the signature of a permutation (being -1 iff the permutation is composed of an odd number of transpositions and $+1$ otherwise).

13.1.2. Example. $\text{Pf} \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} = a$ and $\det \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} = a^2$.

Proof. We have $n = 1$ for $\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} =: A$ so that:

$$\begin{aligned} \text{Pf} \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} &= \frac{1}{2^1 \times 1!} \sum_{\sigma \in S_2} \text{sgn}(\sigma) \underbrace{\prod_{i=1}^1 (A)_{\sigma(2i-1), \sigma(2i)}}_{A_{\sigma(1), \sigma(2)}} \\ &= \frac{1}{2} (A_{1,2} - A_{2,1}) \\ &= \frac{1}{2} (a - (-a)) \\ &= a \end{aligned}$$

□

13.1.3. Claim. $\text{Pf}(A \oplus B) = \text{Pf}(A) \text{Pf}(B)$ where A is a $2n \times 2n$ anti-symmetric matrix and B is a $2m \times 2m$ anti-symmetric matrix.

Proof. First note that

$$(A \oplus B)_{i,j} \equiv \begin{cases} A_{i,j} & i \leq 2n \wedge j \leq 2n \\ B_{i-2n, j-2n} & i > 2n \wedge j > 2n \\ 0 & i > 2n \vee j > 2n \end{cases}$$

and indeed $A \oplus B$ is anti-symmetric iff both A and B are. Next, let us examine the expression for the Pfaffian of $A \oplus B$:

$$\begin{aligned} \text{Pf}(A \oplus B) &= \frac{1}{2^{n+m} (n+m)!} \sum_{\sigma \in S_{2(n+m)}} \text{sgn}(\sigma) \prod_{i=1}^{n+m} (A \oplus B)_{\sigma(2i-1), \sigma(2i)} \\ &= \frac{1}{2^{n+m} (n+m)!} \sum_{\sigma \in S_{2n+2m}} \text{sgn}(\sigma) (A \oplus B)_{\sigma(1), \sigma(2)} (A \oplus B)_{\sigma(3), \sigma(4)} \cdots (A \oplus B)_{\sigma(2n+2m-1), \sigma(2n+2m)} \end{aligned}$$

From this expression it is clear that there are many permutations in the sum for which the summand is zero, indeed, all σ such that there is some $i \in \{1, \dots, n+m\}$ with $\sigma(2i-1) > 2n$ and $\sigma(2i) \leq 2n$ or alternatively $\sigma(2i-1) \leq 2n$ and $\sigma(2i) > 2n$. Thus the sum $\sum_{\sigma \in S_{2n+2m}}$ reduces to a sum only on the internal blocks $2n$ and $2m$: $\sum_{\sigma \in S_{2n}} \sum_{\pi \in S_{2m}}$. The sign of the composition of two permutations is the product of the two signs. One only has to take care that the sum $\sum_{\sigma \in S_{2n+2m}}$ contains a redundancy of $\binom{n+m}{n}$ combinations (the answer to the question where to place the n pairs within the string of $n+m$ pairs, for instance, the permutation 2134 and 3421 in S_4 are identical for our purposes if $n=1$ and $m=1$) so that all together we have

$$\begin{aligned} \text{Pf}(A \oplus B) &= \frac{1}{2^{n+m} (n+m)!} \underbrace{\binom{n+m}{n}}_{=\frac{(n+m)!}{n!m!}} \sum_{\sigma \in S_{2n}} \sum_{\pi \in S_{2m}} \text{sgn}(\sigma) \text{sgn}(\pi) \prod_{i=1}^n A_{\sigma(2i-1), \sigma(2i)} \prod_{j=1}^m B_{\pi(2j-1), \pi(2j)} \\ &= \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \text{sgn}(\sigma) \prod_{i=1}^n A_{\sigma(2i-1), \sigma(2i)} \frac{1}{2^m m!} \sum_{\pi \in S_{2m}} \text{sgn}(\pi) \prod_{j=1}^m B_{\pi(2j-1), \pi(2j)} \\ &\equiv \text{Pf}(A) \text{Pf}(B) \end{aligned}$$

□

13.1.4. Claim. $\det(A) = [\text{Pf}(A)]^2$ if A is anti-symmetric.

Proof. This is [35] Proposition 1.3.

Here is a recipe for an alternative proof:

(1) Show that in the Leibniz formula for the determinant

$$\det(A) = \sum_{\sigma \in S_{2n}} \operatorname{sgn}(\sigma) \prod_{i=1}^{2n} A_{i, \sigma(i)}$$

the sum over *all* permutations is actually not necessary: due to the fact that A is anti-symmetric, all terms corresponding to permutations that contain cycles of odd length will be zero. Define S_{2n}^{even} as the set of all permutations which contain only even length cycles. Then one will have shown:

$$\det(A) = \sum_{\sigma \in S_{2n}^{\text{even}}} \operatorname{sgn}(\sigma) \prod_{i=1}^{2n} A_{i, \sigma(i)}$$

(2) Define any two permutations $(\sigma_1, \sigma_2) \in S_{2n}$ ² to be “equivalent” iff σ_2 can be obtained from σ_1 by flipping pairs and then permuting them. Explicitly, there should be a set $S \subseteq \{1, \dots, n\}$ and a permutation $\alpha \in S_n$ (note we use S_n here and *not* S_{2n}) such that for all $i \in \{1, \dots, n\}$

$$\begin{aligned} \sigma_2(2i-1) &= \begin{cases} \sigma_1(2\alpha(i)) & i \in S \\ \sigma_1(2\alpha(i)-1) & i \in \{1, \dots, n\} \setminus S \end{cases} \\ \sigma_2(2i) &= \begin{cases} \sigma_1(2\alpha(i)-1) & i \in S \\ \sigma_1(2\alpha(i)) & i \in \{1, \dots, n\} \setminus S \end{cases} \end{aligned}$$

Then we write $\sigma_1 \sim \sigma_2$. \sim defines an equivalence relation on S_{2n} , the class corresponding to σ is denoted by $[\sigma]$ and the set of all classes by S_{2n}^{pairs} .

(3) Note that $|[\sigma]| = 2^n n!$ for any $[\sigma] \in S_{2n}^{\text{pairs}}$ and that if $[\sigma_1] = [\sigma_2]$ then the corresponding summands in (52) are equal:

$$\operatorname{sgn}(\sigma_1) \prod_{i=1}^n (A)_{\sigma_1(2i-1), \sigma_1(2i)} = \operatorname{sgn}(\sigma_2) \prod_{i=1}^n (A)_{\sigma_2(2i-1), \sigma_2(2i)}$$

As a result, (52) becomes:

$$\operatorname{Pf}(A) = \sum_{[\sigma] \in S_{2n}^{\text{pairs}}} \operatorname{sgn}(\sigma) \prod_{i=1}^n (A)_{\sigma(2i-1), \sigma(2i)}$$

(4) Prove that there is a bijection

$$\varphi : S_{2n}^{\text{pairs}} \times S_{2n}^{\text{pairs}} \rightarrow S_{2n}^{\text{even}}$$

and that

$$\operatorname{sgn}(\varphi([\sigma_1], [\sigma_2])) = \operatorname{sgn}(\sigma_1) \operatorname{sgn}(\sigma_2)$$

for all $([\sigma_1], [\sigma_2]) \in (S_{2n}^{\text{pairs}})^2$.

□

13.1.5. **Claim.** $\operatorname{Pf}(BAB^T) = \det(B) \operatorname{Pf}(A)$ where B is any $2n \times 2n$ matrix and A is an anti-symmetric $2n \times 2n$ matrix.

Proof. First note that if A is anti-symmetric then BAB^T will be anti-symmetric as well:

$$\begin{aligned} (BAB^T)_{j,i} &= \sum_{l,m=1}^{2n} B_{j,l} A_{l,m} B^T_{m,i} \\ &= \sum_{l,m=1}^{2n} B_{j,l} A_{l,m} B_{i,m} \\ (\stackrel{A=-A^T}{=}) &= - \sum_{l,m=1}^{2n} B_{j,l} A_{m,l} B_{i,m} \\ &= - \sum_{l,m=1}^{2n} B_{i,m} A_{m,l} B_{j,l} \\ &= - \sum_{l,m=1}^{2n} B_{i,m} A_{m,l} B^T_{l,j} \\ &= - (BAB^T)_{i,j} \end{aligned}$$

Next we have using 13.1.4:

$$\begin{aligned} \left[\text{Pf} \left(B A B^T \right) \right]^2 &= \det \left(B A B^T \right) \\ &= [\det(B)]^2 \det(A) \\ &= [\det(B)]^2 [\text{Pf}(A)]^2 \end{aligned}$$

so that

$$\text{Pf} \left(B A B^T \right) = \pm \det(B) \text{Pf}(A)$$

Now the sign *must* be + by using the special case $B = \mathbb{1}_{2n \times 2n}$. □

13.2. The W Overlaps Matrix.

13.2.1. **Definition.** (*The W Matrix*) Let $\pi : E \rightarrow \mathbb{T}^2$ be a \mathcal{T} -equivariant bundle with fibers E_k equipped with an inner product. By 8.4.8 we know that

$$\text{Ch}_1(E) = 0$$

so that by 8.4.11 there is an orthonormal frame (which is a basis for E_k) with respect to which \mathcal{T} is antiunitary:

$$\{\psi_i(k)\}_{k=1}^{2M}$$

for some $M \in \mathbb{N}_{\geq 1}$. Define a matrix $k \mapsto W(k)$ with entries in $\mathcal{F}(\mathbb{T}^2)$ by the following equation:

$$W(k)_{ij} := \langle \psi_i(k), \mathcal{T}\psi_j(-k) \rangle$$

which makes sense as $\mathcal{T}\psi_j(-k) \in E_k$ so that both elements are in E_k and the inner product makes sense.

13.2.2. **Claim.** $W(k)^* W(k) = \mathbb{1}$ for all $k \in \mathbb{T}^2$.

Proof. We have

$$\begin{aligned} [W(k)^* W(k)]_{ij} &= \sum_l [W(k)^*]_{il} [W(k)]_{lj} \\ &= \sum_l \overline{[W(k)]_{li}} [W(k)]_{lj} \\ &= \sum_l \overline{\langle \psi_l(k), \mathcal{T}\psi_i(-k) \rangle} \langle \psi_l(k), \mathcal{T}\psi_j(-k) \rangle \\ &= \sum_l \langle \mathcal{T}\psi_i(-k), \psi_l(k) \rangle \langle \psi_l(k), \mathcal{T}\psi_j(-k) \rangle \\ &\stackrel{(\sum_l \psi_l(k) \langle \psi_l(k), \cdot \rangle = \mathbb{1})}{=} \langle \mathcal{T}\psi_i(-k), \mathcal{T}\psi_j(-k) \rangle \\ &\stackrel{(\mathcal{T} \text{ is antiunitary})}{=} \overline{\langle \psi_i(-k), \psi_j(-k) \rangle} \\ &\stackrel{(\{\psi_i(k)\} \text{ is an OBN})}{=} \langle \psi_j(-k), \psi_i(-k) \rangle \\ &= \delta_{i,j} \\ &= [\mathbb{1}]_{i,j} \end{aligned}$$

□

13.2.3. **Corollary.** $\det(W(k)) \neq 0$ for all $k \in \mathbb{T}^2$.

Proof. Taking the determinant of the equation above we have

$$\begin{aligned} \det(W(k)^* W(k)) &= \det(\mathbb{1}) \\ &\uparrow \\ \overline{\det(W(k))} \det(W(k)) &= 1 \\ &\uparrow \\ |\det(W(k))|^2 &= 1 \end{aligned}$$

□

13.2.4. **Claim.** $W(k)^T = -W(-k)$.

Proof. We have

$$\begin{aligned}
 [W(k)^T]_{ij} &= [W(k)]_{ji} \\
 &= \langle \psi_j(k), \mathcal{T}\psi_i(-k) \rangle \\
 (\mathcal{T}^2 = -1) &\quad \langle -\mathcal{T}^2\psi_j(k), \mathcal{T}\psi_i(-k) \rangle \\
 &= -\langle \mathcal{T}\psi_j(k), \mathcal{T}\psi_i(-k) \rangle \\
 (\mathcal{T} \text{ is antiunitary}) &\quad -\overline{\langle \mathcal{T}\psi_j(k), \psi_i(-k) \rangle} \\
 &= -\langle \psi_i(-k), \mathcal{T}\psi_j(k) \rangle \\
 &\equiv -[W(-k)]_{ij}
 \end{aligned}$$

□

13.2.5. **Corollary.** $\det(W(k)) = \det(W(-k))$.

Proof. Taking the determinant of the equation in the preceding claim we obtain

$$\begin{aligned}
 \det(W(k)^T) &= \det(-W(-k)) \\
 &\Downarrow \\
 \det(W(k)) &= (-1)^2 \det(W(-k)) \\
 &\Downarrow \\
 \det(W(k)) &= \det(W(-k))
 \end{aligned}$$

□

13.2.6. **Corollary.** $W(k)$ is an anti-symmetric matrix for all $k \in \text{TRIM}$ and thus its Pfaffian is defined on TRIM.

13.2.7. **Corollary.** Let $\gamma : S^1 \rightarrow \mathbb{T}^2$ be the loop defined by

$$\underbrace{[0, 2\pi]}_{S^1} \ni \varphi \mapsto (0, \varphi) \in \mathbb{T}^2$$

or

$$\underbrace{[0, 2\pi]}_{S^1} \ni \varphi \mapsto (\varphi, 0) \in \mathbb{T}^2$$

or

$$\underbrace{[0, 2\pi]}_{S^1} \ni \varphi \mapsto (\pi, \varphi) \in \mathbb{T}^2$$

Then

$$\det(W(\gamma(\cdot))) : S^1 \rightarrow S^1$$

has winding number zero.

Proof. First it is clear from the proof of 13.2.3 that the range of the map $\det(W(\gamma(\cdot)))$ is indeed $S^1 \subseteq \mathbb{C}$. Next, because of 13.2.5, any winding that the path does until its midpoint must be undone on the way back to the end which travels through the negative part of \mathbb{T}^2 (the path is constructed in such a way that half of it is in k and the other half is in $-k$), so that all together any winding motion must be undone by the time we get back to the base point. □

13.2.8. **Corollary.** As a result, a consistent choice (between the two possible choices) of a sign for the square root of $\det(W(k))$ can be made for all $k \in \text{TRIM}$ by defining the sign arbitrarily at one point and then determining the sign by a continuous path to the other points.

13.3. The Fu-Kane Index.

13.3.1. **Definition.** (The Fu-Kane Index) Define

$$\text{Index}(E) := \prod_{k \in \text{TRIM}} \frac{\text{Pf}(W(k))}{\sqrt{\det(W(k))}} \tag{53}$$

where the sign of the square root of the determinant is chosen consistently on TRIM.

13.3.2. *Remark.* From the discussion about it is clear that this quantity is well-defined and that

$$\text{Index}(E) \in \{1, -1\}$$

because of 13.1.4:

$$\text{Index}(E) = \prod_{k \in \text{TRIM}} \text{sign}(\text{Pf}(W(k))) \quad (54)$$

For this reason one often speaks of the Fu-Kane index as a \mathbb{Z}_2 index, although strictly speaking

$$\begin{aligned} \mathbb{Z}_2 &\equiv \mathbb{Z}/2\mathbb{Z} \\ &= \{0 + 2\mathbb{Z}, 1 + 2\mathbb{Z}\} \end{aligned}$$

is an additive group and $\{1, -1\}$ is meant in the sense of a multiplicative group. There is a slight misfortune with the notation, as the symbol 1 for the multiplicative group means trivial group element whereas it is a generator for the additive group.

13.3.3. *Remark.* It can be shown that $\text{Index}(E) = +1$ corresponds to the \mathcal{T} -isomorphism class of a trivial \mathcal{T} -equivariant vector bundles and $\text{Index}(E) = -1$ is the other possible \mathcal{T} -isomorphism class.

13.3.4. *Remark.* The Fu-Kane index does *not* correspond to a response of the system to a perturbation, as the first Chern number did (with relation to the Kubo formula which was a response to perturbing the system by an electric field).

13.3.5. **Claim.** $\text{Index}(E)$ does not depend on the choice of the orthonormal basis $\{\psi_i\}_{i=1}^{2M}$.

Proof. Actually from 13.2.7 it is clear that $\prod_{k \in S} \frac{\text{Pf}(W(k))}{\sqrt{\det(W(k))}}$ has an unambiguous sign where

$$S \in \{\{(0, 0), (0, \pi)\}, \{(\pi, 0), (\pi, \pi)\}, \{(0, 0), (\pi, 0)\}, \{(\pi, 0), (\pi, \pi)\}\}$$

so that one could naively propose *two* indices being associated with E . However, consider how the index transforms if we work with

$$\tilde{\psi}_i(k) := \sum_j T_{ij}(k) \psi_j(k)$$

for some $T : \mathbb{T}^2 \rightarrow U(\mathbb{C}^{2M})$. Then

$$\begin{aligned} [\tilde{W}(k)]_{ij} &\equiv \langle \tilde{\psi}_i(k), \mathcal{T} \tilde{\psi}_j(-k) \rangle \\ &= \left\langle \sum_l T_{il}(k) \psi_l(k), \mathcal{T} \sum_m T_{jm}(-k) \psi_m(-k) \right\rangle \\ (\mathcal{T} \text{ is anti-linear}) &= \sum_{l, m} \overline{T_{il}(k) T_{jm}(-k)} \langle \psi_l(k), \mathcal{T} \psi_m(-k) \rangle \\ &= \sum_{l, m} [\overline{T(k)}]_{il} [W(k)]_{lm} [\overline{T(-k)}]_{jm} \\ &= \sum_{l, m} [\overline{T(k)}]_{il} [W(k)]_{lm} [T(-k)^T]_{mj} \\ &= [T(k) W(k) \overline{T(-k)^T}]_{ij} \end{aligned}$$

so that

$$\det(\tilde{W}(k)) = \det(\overline{T(k)}) \det(\overline{T(-k)}) \det(W(k))$$

and at TRIM we have by 13.1.5

$$\text{Pf}(\tilde{W}(k)) = \det(\overline{T(k)}) \text{Pf}(W(k))$$

so that

$$\begin{aligned}
 \prod_{k \in S} \frac{\text{Pf}(\tilde{W}(k))}{\sqrt{\det(\tilde{W}(k))}} &= \prod_{k \in S} \frac{\det(\overline{T(k)}) \text{Pf}(W(k))}{\sqrt{\det(\overline{T(k)}) \det(\overline{T(-k)}) \det(W(k))}} \\
 &= \prod_{k \in S} \frac{\det(\overline{T(k)}) \text{Pf}(W(k))}{\sqrt{\det(\overline{T(k)})^2 \det(W(k))}} \\
 &= \prod_{k \in S} \text{sign}(\det(\overline{T(k)})) \text{sign}(\text{Pf}(W(k))) \\
 &= (-1)^{n_S} \prod_{k \in S} \text{sign}(\text{Pf}(W(k)))
 \end{aligned}$$

where n_S is the winding of $\overline{T(k)}$ along the line defined by S . By continuity it must be independent of which path we take along \mathbb{T}^2 and so even though

$$\prod_{k \in S} \frac{\text{Pf}(\tilde{W}(k))}{\sqrt{\det(\tilde{W}(k))}}$$

might change sign upon changing basis $\psi \mapsto \tilde{\psi}$, for any choice of S ,

$$\prod_{k \in \text{TRIM}} \frac{\text{Pf}(\tilde{W}(k))}{\sqrt{\det(\tilde{W}(k))}} = \underbrace{((-1)^{n_S})^2}_{\text{any } S} \prod_{k \in \text{TRIM}} \frac{\text{Pf}(W(k))}{\sqrt{\det(W(k))}}$$

so that the result follows. \square

13.3.6. Claim.

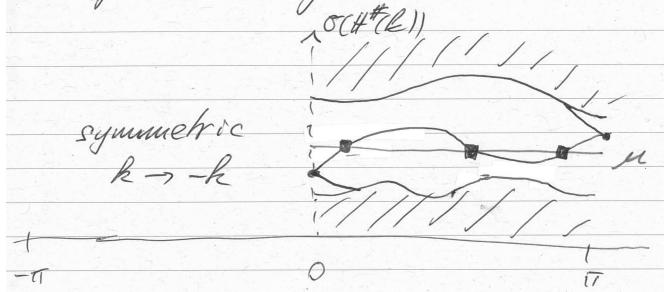
$\text{Index}(E_1 \oplus E_2) = \text{Index}(E_1) \text{Index}(E_2)$.

Proof. The \mathcal{T} -bundle $E_1 \oplus E_2$ is composed of fibers that are the direct sums of the fibers of E_1 and E_2 . As a result, the W -overlap matrix corresponding to $E_1 \oplus E_2$ will be a direct sum of the W_1 -overlap matrix with the W_2 -overlap matrix. Then one can use 13.1.3 in (54). \square

14. THE EDGE INDEX FOR TWO-DIMENSIONAL SYSTEMS

14.1. The Spectrum of Time-Reversal-Invariant Edge System. In this section we assume that we have exactly the same setting as in 10.1, with the additional condition that

$$\Theta \tilde{H}^\#(k_1) = \tilde{H}^\#(-k_1) \Theta$$



14.1.1. Claim.

$\sigma_{pp}(\tilde{H}^\#(k_1)) = \sigma_{pp}(\tilde{H}^\#(-k_1))$.

Proof. Let $\lambda \in \sigma_{pp}(\tilde{H}^\#(k_1))$. Then $\tilde{H}^\#(k_1)\psi = \lambda\psi$ for some ψ . Then

$$\begin{aligned}
 \tilde{H}^\#(-k_1)\Theta\psi &= \Theta\tilde{H}^\#(k_1)\psi \\
 &= \Theta\lambda\psi \\
 &= \bar{\lambda}\Theta\psi \\
 &\stackrel{(\lambda \in \mathbb{R})}{=} \lambda\Theta\psi
 \end{aligned}$$

\square

14.1.2. Remark. Note that at $k_1 \in \{0, \pi\}$ (the Time-Reversal-Invariant-Momenta in one dimension), we can also invoke Kramers' theorem to obtain that there is even degeneracy at those points.

14.2. The Edge Index.

14.2.1. Definition.

Define

$$\text{Index}(H^\#) := (-1)^N$$

where N is the number of eigenvalue crossings of Fermi line at *half* the values of k_1 , that is, $k_1 \in [0, \pi]$.

14.2.2. *Remark.* Note that if we took N to be the number of crossings for $k_1 \in [-\pi, \pi]$, then we would *always* get an even number by 14.1.1.

14.2.3. *Remark.* Note that we would get the same result if instead we counted the *signed* number of crossings, analogously to the quantum Hall half-periodic edge index. However, it is easy to see that in the case of time-reversal invariant systems it is not necessary to discern the signs.

14.2.4. *Remark.* If we perturb H^\sharp in a compact fashion, $\text{Index}(H^\sharp)$ remains constant. Additionally, if we move the Fermi level (within the gap) $\text{Index}(H^\sharp)$ also remains constant.

14.2.5. **Claim.** We have

$$\text{Index}(H^\sharp) = \text{Index}(E)$$

Proof. For a proof see [20]. □

15. THE RELATION BETWEEN THE FIRST CHERN NUMBER AND THE FU-KANE INDEX

It should be noted that in general there is *no* relation between the two quantities defined in (39) and (53). It is only in a special case of taking the direct sum of two quantum Hall systems that there is a direct relation, which we describe below.

Let H_0 be the Hamiltonian (not necessarily time-reversal invariant) on a Hilbert space \mathcal{H} with corresponding Fermi projection P_0 . Define

$$\hat{H}_0 := \Theta_0 H_0 \Theta_0^*$$

where $\Theta_0 : \mathcal{H} \rightarrow \mathcal{H}$ is time reversal on \mathcal{H} . Define, on $\mathcal{H} \oplus \mathcal{H}$ the following two operators:

$$H := \begin{bmatrix} H_0 & 0 \\ 0 & \hat{H}_0 \end{bmatrix}$$

and

$$\Theta := \begin{bmatrix} 0 & \Theta_0 \\ \Theta_0 & 0 \end{bmatrix}$$

15.0.1. **Claim.** $[H, \Theta] = 0$.

Proof. We have

$$\begin{aligned} \begin{bmatrix} H_0 & 0 \\ 0 & \hat{H}_0 \end{bmatrix} \begin{bmatrix} 0 & \Theta_0 \\ \Theta_0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & \Theta_0 \\ \Theta_0 & 0 \end{bmatrix} \begin{bmatrix} H_0 & 0 \\ 0 & \hat{H}_0 \end{bmatrix} &= \begin{bmatrix} 0 & H_0 \Theta_0 \\ \hat{H}_0 \Theta_0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & \Theta_0 \hat{H}_0 \\ \Theta_0 H_0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & H_0 \Theta_0 - \Theta_0 \hat{H}_0 \\ \hat{H}_0 \Theta_0 - \Theta_0 H_0 & 0 \end{bmatrix} \end{aligned}$$

but

$$\begin{aligned} \hat{H}_0 \Theta_0 - \Theta_0 \hat{H}_0 &= \Theta_0 H_0 \underbrace{\Theta_0^* \Theta_0}_{-1} - \Theta_0 H_0 \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} H_0 \Theta_0 - \Theta_0 \hat{H}_0 &= H_0 \Theta_0 - \Theta_0 \Theta_0 H_0 \Theta_0^* \\ &= H_0 \Theta_0^* \underbrace{\Theta_0 \Theta_0}_{-1} - \underbrace{\Theta_0 \Theta_0}_{-1} H_0 \Theta_0^* \\ &= -H_0 \Theta_0^* + H_0 \Theta_0^* \\ &= 0 \end{aligned}$$
□

15.0.2. *Remark.* The Fermi projection of H is

$$P := \begin{bmatrix} P_0 & 0 \\ 0 & \hat{P}_0 \end{bmatrix}$$

15.0.3. **Example.** Let H_0 be the Hamiltonian for a spinless electron with $\mathcal{H} := L^2(X)$ and then H *does* include spin:

$$\mathcal{H} \oplus \mathcal{H} = L^2(X) \otimes \mathbb{C}^2$$

Then we have

$$\text{Index}(\mathcal{P}) = (-1)^{\text{Ch}_1(\mathcal{P}_0)} \tag{55}$$

where \mathcal{P} is the occupied sub-bundle corresponding to P and \mathcal{P}_0 is the occupied sub-bundle corresponding to P_0 .

Proof. We will use the bulk-edge correspondence to show (55) in the edge perspective.

Claim. $\hat{H}_0^\#(k_1) = H_0^\#(-k_1)$.

Proof. The proof is identical to the beginning of 8.4.8. \square

Then $\text{Ch}_1(\mathcal{P}_0)$ is equal to the signed number of crossings of eigenvalues of $H_0^\#(k_1)$ on $k_1 \in [-\pi, \pi]$ by the bulk-edge correspondence for the quantum Hall effect.

Claim. The signed number of crossings of eigenvalues of $H_0^\#(k_1)$ on $k_1 \in [-\pi, \pi]$ is equal to the signed number of crossings of eigenvalues of $H^\#(k_1)$ on $k_1 \in [0, \pi]$.

Proof. If $\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$ is a crossing of $H^\#$ at $k_1 \in [0, \pi]$ then

$$\begin{aligned} H^\#(k_1) \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} &= E_F \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \\ \Updownarrow \\ \begin{bmatrix} H_0^\#(k_1) & 0 \\ 0 & \hat{H}_0^\#(k_1) \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} &= E_F \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \\ \Updownarrow \\ \begin{cases} H_0^\#(k_1)\psi_1 = E_F\psi_1 \\ \hat{H}_0^\#(k_1)\psi_2 = E_F\psi_2 \end{cases} & \\ \Updownarrow \\ \begin{cases} H_0^\#(k_1)\psi_1 = E_F\psi_1 \\ H_0^\#(-k_1)\psi_2 = E_F\psi_2 \end{cases} & \end{aligned}$$

Now by the bulk edge correspondence for time-reversal-invariant topological insulators, this latter quantity is equal to $\text{Index}(\mathcal{P})$. \square

15.0.4. *Remark.* Note that if we now perturb H compactly so as to lose the block-diagonal form but preserve time-reversal invariance, the index remains constant. Thus, one approach to compute the index of a time-reversal invariant system using the first Chern number is to perturb it compactly (if that is possible) to a time-reversal-invariant system so that it has the block-form and then compute the first Chern number of only one block. The parity of the first Chern number will be the Fu-Kane index.

Appendix

The following material did not appear in the lectures.

16. MORE ABOUT VECTOR BUNDLES

The material in this section is taken from [5].

16.1. Basic Properties.

16.1.1. Claim. (Slight generalization of (8.1.12)) Let $\varphi : E \rightarrow F$ be a vector bundle morphism between two vector bundles over X . Assume the fibers of E and F are constant, and are given respectively by V and W . Then φ determines a continuous map $\Phi : X \rightarrow \text{Mor}_{\text{Vect}_C}(V, W)$, and conversely, any continuous map $\Phi : X \rightarrow \text{Mor}_{\text{Vect}_C}(V, W)$ determines a vector bundle morphism $\varphi : E \rightarrow F$.

Proof. Let $x_0 \in X$ be given. Since E is a vector bundle, $\exists U \in \text{Nbhd}_X(x_0)$ such that there is an isomorphism $\psi : E|_U \rightarrow U \times V$ and another $\tilde{U} \in \text{Nbhd}_X(x_0)$ such that there is an isomorphism $\tilde{\psi} : F|_{\tilde{U}} \rightarrow \tilde{U} \times W$. Then $U \cap \tilde{U} \in \text{Nbhd}_X(x_0)$ and we have the restricted isomorphisms

$$\psi : E|_{U \cap \tilde{U}} \rightarrow (U \cap \tilde{U}) \times V$$

and

$$\tilde{\psi} : F|_{U \cap \tilde{U}} \rightarrow (U \cap \tilde{U}) \times W$$

Then for all $x \in U \cap \tilde{U}$ we define a map $\Phi(x) : V \rightarrow W$ by

$$(\Phi(x))(v) := \pi_2 \circ \tilde{\psi} \circ \varphi \circ \psi^{-1}(x, v) \quad \forall v \in V$$

If V and W are finite dimensional, then $\text{Mor}_{\text{Vect}_C}(V, W)$ is also a finite dimensional vector space ($\cong V^* \otimes W$) which has the standard topology, in which Φ is continuous. If V and W are infinite dimensional one has to choose with which operator topology one wants to work with.

Conversely, any such map $\Phi : X \rightarrow \text{Mor}_{\text{Vect}_C}(V, W)$ can be used to define a morphism $\varphi \in \text{Mor}_{\text{Vect}_C}(E, F)$ by

$$(\varphi|_{U \cap \tilde{U}})(e) := (x, \Phi \circ \pi_2 \circ \psi(e))$$

□

16.1.2. Remark. Note that since the group of invertible elements $G(\mathcal{B}(V, W))$ in $\mathcal{B}(V, W)$ is open in norm, $\Phi^{-1}(G(\mathcal{B}(V, W))) \in \text{Open}(X)$, where $\Phi^{-1}(G(\mathcal{B}(V, W)))$ is the set of all points $x \in X$ for which $\Phi(x)$ is an isomorphism.

16.2. New Bundles out of Old Ones.

16.2.1. Definition. (Continuous Functor) Let T be a functor of n covariant arguments

$$T : (\text{Vect}_C)^n \rightarrow \text{Vect}_C$$

where $n \in \mathbb{N}_{>0}$. $((\text{Vect}_C)^n)$, the category of n products of Vect_C , is not to be confused with Vect_C^n , the category of n -dimensional vector spaces over \mathbb{C} .

Examples for T are the direct sum functor

$$(V, W) \mapsto V \oplus W$$

the tensor product functor

$$(V, W) \mapsto V \otimes W$$

the homomorphism functor

$$(V, W) \mapsto \underbrace{\text{Mor}_{\text{Vect}_C}(V, W)}_{\cong V^* \otimes W}$$

and so on (Recall that any contravariant functor on \mathcal{C} may be regarded as a covariant functor on the opposite category \mathcal{C}^{op} so there is no need to consider here contravariant functors at all). Then T is called a *continuous functor* iff for all

$$((V_1, \dots, V_n), (W_1, \dots, W_n)) \in (\text{Obj } ((\text{Vect}_C)^n))^2$$

the map

$$T : \text{Mor}_{(\text{Vect}_C)^n}((V_1, \dots, V_n), (W_1, \dots, W_n)) \rightarrow \text{Mor}_{\text{Vect}_C}(T(V_1, \dots, V_n), T(W_1, \dots, W_n))$$

is continuous, where we use the standard topology on

$$\text{Mor}_{(\text{Vect}_C)^n}((V_1, \dots, V_n), (W_1, \dots, W_n))$$

and

$$\text{Mor}_{\text{Vect}_C}(T(V_1, \dots, V_n), T(W_1, \dots, W_n))$$

This topology is defined for instance when we use instead of Vect_C the category of finite dimensional vector spaces or of Hilbert spaces and use the operator norm.

16.2.2. Claim. (*Induced Bundle from a Continuous Functor*) Let $p_1 : E_1 \rightarrow X, \dots, p_n : E_n \rightarrow X$ be n given vector bundles over X and let T be a continuous functor of n covariant arguments

$$T : (\text{Vect}_{\mathbb{C}})^n \rightarrow \text{Vect}_{\mathbb{C}}$$

Then there is defined a vector bundle $T(E_1, \dots, E_n)$ over X , such that

$$(T(E_1, \dots, E_n))_x = T((E_1)_x, \dots, (E_n)_x)$$

for all $x \in X$. For example, with the direct sum functor,

$$\left(\begin{array}{c} E_1 \oplus E_2 \\ \text{direct sum on bundles, which will be defined below} \end{array} \right)_x = \underbrace{(E_1)_x \oplus (E_2)_x}_{\text{direct sum on vector spaces, which is already defined}}$$

Proof. We divide into cases.

Case 1: E_i are all product spaces: $E_i = X \times V_i$ for some vector spaces V_i . Then define

$$T(E_1, \dots, E_n) := X \times T(V_1, \dots, V_n)$$

with the product topology. The projection is defined naturally.

Case 2: E_i are all trivial. Then let the isomorphisms

$$\alpha_i : E_i \rightarrow X \times V_i$$

be given for some vector spaces V_i . Now define

$$T(E_1, \dots, E_n) := \coprod_{x \in X} T((E_1)_x, \dots, (E_n)_x) \quad (56)$$

as a set, with the projection $p(x, T(e_1, \dots, e_n)) := x$. Note that we also have a bijection $T(\alpha)$ naturally defined:

$$T(\alpha) : T(E_1, \dots, E_n) \rightarrow X \times T(V_1, \dots, V_n)$$

by

$$(x, T(e_1, \dots, e_n)) \xrightarrow{T(\alpha)} (x, T(\pi_2 \circ \alpha_1(e_1), \dots, \pi_2 \circ \alpha_n(e_n)))$$

We define $\text{Open}(T(E_1, \dots, E_n))$ as the unique topology on $T(E_1, \dots, E_n)$ such that $T(\alpha)$ is a homeomorphism. Note that since $T(\alpha)$ is bijective, this is the initial topology on $T(E_1, \dots, E_n)$ generated by $T(\alpha)$. If

$$\tilde{\alpha}_i : E_i \rightarrow X \times \tilde{V}_i$$

is another set of isomorphisms, then $T(\alpha)$ and $T(\tilde{\alpha})$ induce the same topology, since $T(\tilde{\alpha} \circ \alpha^{-1}) = T(\tilde{\alpha})T(\alpha)^{-1}$ is a homeomorphism.

Case 3: E_i are not all trivial. Then we define $T(E_1, \dots, E_n)$ again as in (56) as a set, and its topology is defined as follows: $W \subseteq T(E_1, \dots, E_n)$ is defined to be open iff $W \cap (T(E_1|_U, \dots, E_n|_U)) \in \text{Open}(T(E_1|_U, \dots, E_n|_U))$ for all $U \in \text{Open}(X)$ such that $E_i|_U$ are all trivial, where $\text{Open}(T(E_1|_U, \dots, E_n|_U))$ has been defined in Case 2 already. \square

16.2.3. Claim. Let $\varphi : F \rightarrow E$ be a bundle monomorphism (two bundles over X). Then $\varphi(F)$ is a sub-bundle of E and $\varphi : F \rightarrow \varphi(F)$ is an isomorphism.

Proof. Let $x_0 \in X$. Then $\exists U \in \text{Nbhd}_X(x_0), (V_1, V_2) \in \text{Obj}(\text{Vect}_{\mathbb{C}})^2$, and two isomorphisms

$$\psi_i : E|_U \rightarrow U \times V_i$$

Now choose some $W_{x_0} \subseteq V_1$ as a subspace which is complementary to $\pi_2 \circ \psi_1 \circ \varphi(F|_{x_0})$. Then $\psi_1^{-1}(U \times W_{x_0})$ is a sub-bundle of $E|_U$. Define

$$\theta : F|_U \oplus \psi_1^{-1}(U \times W_{x_0}) \rightarrow E|_U$$

by

$$f \oplus \psi_1^{-1}(x, w) \mapsto \psi_1^{-1}(\psi_1(\varphi(f)) + w)$$

Then $\theta|_x$ is an isomorphism for any $x \in U$ so that $\exists Q \in \text{Nbhd}_X(x)$ such that $\theta|_Q$ is an isomorphism. Since $F|_U$ is a sub-bundle of $F|_U \oplus \psi_1^{-1}(U \times W_{x_0})$, $\underbrace{\theta(F|_U)}_{\varphi|_U(F)}$ is a sub-bundle of $\underbrace{\theta(F|_U \oplus \psi_1^{-1}(U \times W_{x_0}))}_E$ on Q . Thus, locally, $\varphi(F)$ is a direct summand of E . \square

16.2.4. Claim. (*The Pullback Family of Vector Spaces*) Let $f \in \text{Mor}_{\text{Top}}(Y, X)$, and $p : E \rightarrow X$ be a family of vector spaces over X . Then the following definition gives a new family of vector spaces over Y : Define the induced family $f^*(p) : f^*(E) \rightarrow Y$ as follows:

$$f^*(E) := \{(y, e) \in Y \times E \mid f(y) = p(e)\}$$

where $f^*(p) : f^*(E) \rightarrow Y$ is defined as the projection onto the first coordinate. The topology on $f^*(E)$ is defined as the subspace topology from $Y \times E$.

Proof. $f^*(p)$ is continuous, because it is the composition of the inclusion map $f^*(E) \hookrightarrow Y \times E$ (which is continuous) with the projection map $Y \times E \rightarrow Y$ (which is also continuous).

Then a typical fiber is

$$\begin{aligned} f^*(E)_{y_0} &\equiv f^*(p)^{-1}(\{y_0\}) \\ &\equiv \{(y, e) \in f^*(E) \mid (f^*(p))(y, e) = y_0\} \\ &= \{(y_0, e) \in Y \times E \mid p(e) = f(y_0)\} \\ &= \{(y_0, e) \in \{y_0\} \times E \mid p(e) = f(y_0)\} \\ &= \{y_0\} \times \{e \in E \mid p(e) = f(y_0)\} \\ &= \{y_0\} \times p^{-1}(\{f(y_0)\}) \\ &\equiv \{y_0\} \times E_{f(y_0)} \end{aligned}$$

so that we can define the continuous vector space multiplication on $f^*(E)_{y_0}$ as $f^*(m)_{y_0} : f^*(E)_{y_0}^2 \rightarrow f^*(E)_{y_0}$ by

$$f^*(m)_{y_0}((y_0, e), (y_0, \tilde{e})) := (y_0, m_{f(y_0)}(e, \tilde{e}))$$

and addition $f^*(a)_{y_0} : \mathbb{C} \times f^*(E)_{y_0} \rightarrow f^*(E)_{y_0}$ by

$$f^*(a)_{y_0}(\lambda, (y_0, e)) := (y_0, a_{f(y_0)}(\lambda, e))$$

These two maps are indeed continuous since $m_{f(y_0)}$ and $a_{f(y_0)}$ are continuous, and the map

$$\begin{aligned} f^*(E)_{y_0}^2 &\rightarrow \{y_0\} \times E_{f(y_0)}^2 \\ ((y_0, e), (y_0, \tilde{e})) &\mapsto (y_0, (e, \tilde{e})) \end{aligned}$$

and

$$\begin{aligned} \mathbb{C} \times f^*(E)_{y_0} &\rightarrow \{y_0\} \times \mathbb{C} \times E_{f(y_0)} \\ (\lambda, (y_0, e)) &\mapsto (y_0, (\lambda, e)) \end{aligned}$$

are continuous. □

16.2.5. Claim. If $g : Z \rightarrow Y$ then there is a natural isomorphism of vector families over Z ,

$$g^*(f^*(E)) \cong (f \circ g)^*(E)$$

Proof. Define the map $\psi : (f \circ g)^*(E) \rightarrow g^*(f^*(E))$ by

$$(z, e) \mapsto (z, (g(z), e)) \quad \forall (z, e) \in (f \circ g)^*(E)$$

We claim that ψ is an isomorphism of families of vector spaces over Z . To do that we must show that:

(1) ψ is continuous: $(f \circ g)^*(E) \subseteq Z \times E$ and $g^*(f^*(E)) \subseteq \underbrace{Z \times (Y \times E)}_{Z \times Z \times E}$. Define the map

$$\varphi := \mathbb{1}_Z \times g \times \mathbb{1}_E$$

which is continuous on $Z \times Z \times E$ as the Cartesian product of continuous functions is again continuous. Also define the map $\eta : Z \times E \rightarrow Z \times Z \times E$ given by $\eta : (z, e) \mapsto (z, z, e)$. Note that ψ is the composition of restrictions of these two maps to the corresponding bundles. Thus we have to show that η is continuous. Suffice to work with a basis for $\text{Open}(Z \times Z \times E)$, so that we assume $U_1 \times U_2 \times U_3 \in \text{Open}(Z) \times \text{Open}(Z) \times \text{Open}(E)$. We want to show that $\eta^{-1}(U_1 \times U_2 \times U_3) \in \text{Open}(Z \times E)$.

$$\begin{aligned} \eta^{-1}(U_1 \times U_2 \times U_3) &\equiv \{(z, e) \in Z \times E \mid (z, z, e) \in U_1 \times U_2 \times U_3\} \\ &= \underbrace{(U_1 \cap U_2)}_{\in \text{Open}(Z)} \times \underbrace{U_3}_{\in \text{Open}(E)} \\ &\in \text{Open}(Z) \times \text{Open}(E) \end{aligned}$$

As a result ψ is continuous, as a composition of two continuous maps.

(2) The projection onto Z is respected by ψ by construction, so that projecting before or after performing ψ leads to the same point in ψ .

(3) Let $z_0 \in Z$ be given. Then we have the two fibers, $(f \circ g)^*(E)_{z_0}$ and $g^*(f^*(E))_{z_0}$, and $\psi|_{(f \circ g)^*(E)_{z_0}}$ should be a vector space morphism between them. We found out above that $(f \circ g)^*(E)_{z_0} = \{z_0\} \times E_{f(g(z_0))}$. Next,

$$\begin{aligned} g^*(f^*(E))_{z_0} &\equiv \left\{ (z_0, (y, e)) \in Z \times Y \times E \mid g(z_0) = \underbrace{p_1(y, e)}_y \wedge f(y) = p(e) \right\} \\ &= \{(z_0, (g(z_0), e)) \in Z \times Y \times E \mid f(g(z_0)) = p(e)\} \\ &= \{z_0\} \times \{g(z_0)\} \times E_{f(g(z_0))} \end{aligned}$$

so that $\psi|_{(f \circ g)^*(E)_{z_0}} : (z_0, e) \mapsto (z_0, g(z_0), e)$, being the identity on the factor $E_{f(g(z_0))}$, is a vector space morphism.

(4) It is also clear that ψ is injective, and it is surjective because

$$\begin{aligned} g^*(f^*(E)) &\equiv \{(z, (y, e)) \in Z \times f^*(E) \mid g(z) = y\} \\ &= \{(z, (g(z), e)) \in Z \times f^*(E) \mid \text{true}\} \\ &= \psi((f \circ g)^*(E)) \end{aligned}$$

(5) Lastly, the inverse map $\psi^{-1} : (z, (g(z), e)) \mapsto (z, e)$ is continuous because it is a restriction of a projection map $Z \times Y \times E \rightarrow Z \times E$.

□

16.2.6. Claim. (*The Pullback Bundle*) If $p : E \rightarrow X$ is a vector bundle and $f \in \text{Mor}_{\text{Top}}(Y, X)$ then $f^*(p) : f^*(E) \rightarrow Y$ is also a vector bundle (rather than merely a family of vector spaces over Y).

Proof. Let $y_0 \in Y$ be given. Then $\exists U \in \text{Nbhd}_X(f(y_0))$ such that we have a family of vector spaces over U isomorphism:

$$\varphi : E|_U \rightarrow U \times V$$

for some vector space V . Since f is continuous, $f^{-1}(U) \in \text{Open}(Y)$. Furthermore, $y_0 \in f^{-1}(U)$ since $f(y_0) \in U$. Thus $f^{-1}(U) \in \text{Nbhd}_Y(y_0)$. Thus we have an induced vector bundle $f^*(E|_U)$ over $f^{-1}(U)$. We want to define an isomorphism of families over $f^{-1}(U)$

$$\psi : f^*(E)|_{f^{-1}(U)} \rightarrow f^{-1}(U) \times V$$

Define

$$\psi(y, e) := (y, \pi_2 \circ \varphi(e)) \quad \forall (y, e) \in f^*(E)|_{f^{-1}(U)}$$

where $\pi_2 : U \times V \rightarrow V$ is the projection to the second factor. Thus ψ is continuous (as the composition of continuous maps), with inverse given by

$$f^{-1}(U) \times V \ni (y, v) \xrightarrow{\psi^{-1}} (y, \varphi^{-1}(f(y), v)) \in f^*(E)|_{f^{-1}(U)}$$

which is also continuous. ψ is linear on each point because $\pi_2 \circ \varphi$ is.

□

16.3. Pullbacks of Homotopic Maps are Isomorphic.

16.3.1. Claim. (*Bundle form of the Tietze extension*) Let $X \in \text{Obj}(\text{Top})$ be compact and Hausdorff, $Y \in \text{Closed}(X)$ and E be a vector bundle over X . Then any section $s : Y \rightarrow E|_Y$ can be extended to X .

Proof. Recall the Tietze extension theorem ([34] pp. 219): If X' is a topological space normal space, $Y' \in \text{Closed}(X')$ and $V' \in \text{Obj}(\text{Vect}_{\mathbb{R}})$ and $f \in \text{Mor}_{\text{Top}}(Y', V')$, then there exists an extension $g \in \text{Mor}_{\text{Top}}(X', V')$ of f : $g|_{Y'} = f$. The statement of the theorem is equivalent to the normality of X' .

Let $s : Y \rightarrow E|_Y$ be a given section. Let $x_0 \in X$ be given. Then $\exists U \in \text{Nbhd}_X(x)$ such that $\varphi : E|_U \rightarrow U \times V$ is a family isomorphism for some vector space V . Then as in 8.1.19,

$$\pi_2 \circ \varphi \circ s : U \rightarrow V$$

is a vector valued function. Note that $U \cap Y \in \text{Closed}(U)$ so that we apply the Tietze extension theorem on $\pi_2 \circ \varphi \circ s|_{U \cap Y}$ to get a new map $t : U \rightarrow V$ such that $t|_{U \cap Y} = \pi_2 \circ \varphi \circ s|_{U \cap Y}$.

Since X is compact, there is a finite open cover $\{U_\alpha\}_\alpha$ of such sets, in each of which there is such an extension, call it $t_\alpha : U_\alpha \rightarrow V$. Let $\{p_\alpha : X \rightarrow \mathbb{R}\}_\alpha$ be a partition of unity with $\text{supp}(p_\alpha) \subseteq U_\alpha$.

Then define, $\forall \alpha$, a section $s_\alpha : X \rightarrow E$ by

$$s_\alpha(x) := \begin{cases} p_\alpha(x) t_\alpha(x) & x \in U_\alpha \\ 0 & x \notin U_\alpha \end{cases}$$

and then $\sum_\alpha s_\alpha$ is a section whose restriction to Y is s .

□

16.3.2. Claim. (*Extending Bundle Isomorphism*) Let $Y \in \text{Closed}(X)$ where X is compact Hausdorff, and let E and F be two bundles over X . If $f : E|_Y \rightarrow F|_Y$ is a family of vector spaces isomorphism, then there exists some $U \in \text{Open}(X)$ such that $U \supseteq Y$ and some $g : E|_U \rightarrow F|_U$ which extends f and which is also a family of vector spaces isomorphism.

Proof. Note that the space of morphisms $E|_Y \rightarrow F|_Y$, $\text{Mor}_{\text{Vect}(Y)}(E|_Y, F|_Y)$, is also a vector bundle over Y , in which f is a section. This is a sub-bundle of the space of morphisms $E \rightarrow F$, $\text{Mor}_{\text{Vect}(X)}(E, F)$. Thus we use 16.3.1 on f to obtain a section of $\text{Mor}_{\text{Vect}(X)}(E, F)$, $g : E \rightarrow F$. Define U to be the set of x for which g is an isomorphism. Since the subset of isomorphisms is open in the set of all morphisms, and g is continuous, we obtain that U is open. \square

16.3.3. Claim. (*Homotopy Invariance of Pullbacks*) Let $(f, g) \in \text{Mor}_{\text{Top}}(Y, X)^2$ such that $[f] = [g]$ in $\text{Mor}_{\text{hTop}}(Y, X)$ (that is, f and g are homotopic). Let $p : E \rightarrow X$ be a vector bundle over X . If Y is compact, then the two vector bundles $f^*(E)$ and $g^*(E)$ as defined in 16.2.6 are isomorphic as families of vector spaces over X .

Proof. Let $I = [0, 1]$, $F : Y \times I \rightarrow X$ be the homotopy between f and g (so that $F(\cdot, 0) = f$ and $F(\cdot, 1) = g$), $\pi_1 : Y \times I \rightarrow Y$, and define maps, $G_t : Y \rightarrow X$, $\forall t \in I$, by

$$G_t(y) := F(y, t) \quad \forall y \in Y$$

Let $t_0 \in I$ be given. Observe that $Y \times \{t_0\} \in \text{Closed}(Y \times I)$, and that $Y \times I$ is compact and Hausdorff, as Y is. Next observe that the following isomorphism of restricted bundles over $Y \times \{t_0\}$:

$$F^*(E)|_{Y \times \{t_0\}} \cong \pi_1^*(G_t^*(E))|_{Y \times \{t_0\}} \quad (57)$$

Indeed, using 16.2.5 with

$$Y \times \{t_0\} \xrightarrow{\pi_1} Y \xrightarrow{G_t} X$$

and

$$Y \times \{t_0\} \xrightarrow{F} X$$

we have $F = G_t \circ \pi_1$ on $Y \times \{t_0\}$, resulting in (57). Call the isomorphism (57) s . Now employ (16.3.2) on

$$s : F^*(E)|_{Y \times \{t_0\}} \rightarrow \pi_1^*(G_t^*(E))|_{Y \times \{t_0\}}$$

to obtain an extension,

$$\tilde{s} : F^*(E)|_{Y \times U} \rightarrow \pi_1^*(G_t^*(E))|_{Y \times U}$$

which is also an isomorphism, where $U \in \text{Nbhd}_I(t_0)$. Hence $G_t^*(E)$ is locally isomorphic as a function of $t \in I$. But I is connected so that $F^*(E)|_{Y \times \{t\}}$ are all isomorphic, for any pair of values of t on I , so and the result follows by using $t = 0$ and $t = 1$. \square

16.4. Homotopic Characterization of Vector Bundles.

16.4.1. Definition. (*Grassmannian*) Let V be a complex vector space of some finite dimension and $n \in \mathbb{N}_{>0}$ be given such that $n < \dim(V)$.

$$G_n(V) := \left\{ W \subseteq \mathcal{H} \mid W \text{ is a vector subspace of } V \wedge \underbrace{\text{codim}(W)}_{\dim(V/W)} = n \right\}$$

Each element $W \in G_n(\mathcal{H})$ determines a projection map $V \rightarrow V$ via the following construction. Let $\{e_i\}_{i=1}^{\dim(V)}$ be an orthonormal basis for V , such that there is some subset $N \subseteq \{1, \dots, \dim(V)\}$ such that $\{e_i\}_{i \in N}$ is a basis for W . Then the projection is defined as

$$v \mapsto \sum_{i \in N} \langle e_i, v \rangle e_i \quad \forall v \in V$$

In this sense we obtain a map

$$\psi : G_n(V) \rightarrow \text{End}(V)$$

Now, $\text{End}(V) \cong V^* \otimes V$ so that it is also finite dimensional and has a topology induced by the standard topology of $\mathbb{C}^{\dim(V)^2}$ after an isomorphism $\text{End}(V) \cong \mathbb{C}^{\dim(V)^2}$. We thus give $G_n(V)$ the initial topology with respect to the map ψ : the smallest topology such that ψ is continuous. One can then prove that this topology is independent of the choice of bases.

Note that $G_n(V)$ can be generalized to the case that V is an infinite dimensional separable Hilbert space. In both the finite and infinite case, one could define a differentiable structure on $G_n(V)$ thereby making it into a smooth manifold. In fact it is possible to show that

$$G_n(\mathbb{C}^{n+m}) \cong U(n+m) / (U(n) \times U(m))$$

as a smooth manifold isomorphism.

Also note that for reasons which will become clear below, $G_n(V)$ is known as the classifying space of the group $\text{GL}(\mathbb{C}^n)$, which is the structure group of any complex vector bundle considered as a fiber bundle (defined for instance in [43] pp. 89). Thus, $G_n(V)$ is also denoted $\text{BGL}(\mathbb{C}^n)$.

16.4.2. Claim. $G_n(\mathbb{C}^{n+m}) \cong G_m(\mathbb{C}^{n+m})$ for all $(n, m) \in (\mathbb{N}_{>0})^2$.

16.4.3. Example. One can prove that $G_1(\mathbb{C}^2)$ is diffeomorphic to the 2-sphere S^2 in \mathbb{R}^3 . In fact, this map is closely related to the Hopf fibration $S^3 \rightarrow S^2$. A point $W \in G_1(\mathbb{C}^2)$ is determined by a non-zero vector $v \in \mathbb{C}^2 \cong \mathbb{R}^4$. Any non-zero vector in \mathbb{R}^4 determines a point on S^3 , at which point one may use the Hopf-fibration map to reach S^2 . Explicitly, one could also

think of quantum mechanics: A point in $G_1(\mathbb{C}^2)$ is the vector subset $\mathbb{C}v$ determined by some $v \in \mathbb{C}^2 \setminus \{0\}$. This defines the projection

$$\frac{1}{\|v\|_{\mathbb{C}^2}^2} \begin{bmatrix} |v_1|^2 & v_1 \bar{v}_2 \\ \bar{v}_1 v_2 & |v_2|^2 \end{bmatrix} \in \text{End}(\mathbb{C}^2)$$

where $v_i \in \mathbb{C}$ is the i th component of $v \in \mathbb{C}^2$. Now physicists will recognize that the most general self-adjoint projection in \mathbb{C}^2 can be written as

$$\frac{1}{2} (\mathbb{1}_{2 \times 2} - h_1 \sigma_1 - h_2 \sigma_2 - h_3 \sigma_3)$$

for some $\begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix} \in S^2 \subseteq \mathbb{R}^3$ where σ_i are the Pauli matrices. This means that we can map

$$G_1(\mathbb{C}^2) \ni \mathbb{C} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \mapsto \frac{1}{\|v\|_{\mathbb{C}^2}^2} \begin{bmatrix} -2\Re\{v_1 \bar{v}_2\} \\ 2\Im\{v_1 \bar{v}_2\} \\ |v_2|^2 - |v_1|^2 \end{bmatrix} \in S^2$$

using the fact that $\frac{1}{2} \text{tr}(\sigma_i \sigma_j) = \delta_{i,j}$. This is precisely the Hopf fibration map when $\|v\|_{\mathbb{C}^2}^2 = 1$. One can show that this map is indeed a diffeomorphism.

16.4.4. Definition. (*The Tautological Bundle*, slight generalization of (8.1.17)) Let $G_n(V)$ be a Grassmannian manifold as above. Define the tautological vector-bundle, a vector bundle over the manifold $G_n(V)$, as the total space

$$F_{G_n(V)} := \{(W, v) \in G_n(V) \times V \mid v \in W\}$$

together with the subspace topology from $G_n(V) \times V$ and the natural projection. Then the fiber above a point $W \in G_n(V)$ is exactly $W \subseteq V$.

16.4.5. Definition. (*The Classifying Bundle*) Let $G_n(V)$ and $F_{G_n(V)}$ be as above. Define the *quotient bundle*

$$E_{G_n(V)} := (G_n(V) \times V) / F_{G_n(V)} \quad (58)$$

where $G_n(V) \times V$ is the trivial vector bundle over $G_n(V)$. Then $E_{G_n(V)}$ is called the classifying bundle over $G_n(V)$.

16.4.6. Definition. (*The Induced Map*) Let E be a vector bundle over a connected space X (so $\dim(E_x) =: n$ is constant), V a finite dimensional vector space, $X \times V$ the trivial bundle over X , and $\varphi : X \times V \rightarrow E$ be a bundle epimorphism. Then we have a map

$$f_\varphi : X \rightarrow G_n(V)$$

induced by φ . It is given by

$$f_\varphi(x) := \ker(\varphi_x)$$

where φ_x is the restriction of the map φ to the fiber $\{x\} \times V$ of $X \times V$.

16.4.7. Claim. f_φ is continuous.

Proof. Without loss of generality assume that $V = \mathbb{C}^m$ for some $m \in \mathbb{N}_{\geq 0}$. (If $m < n$ then φ cannot be an epimorphism). Let $x_0 \in X$ be given. We need to show that f_φ is continuous at x_0 . Since E is a vector bundle, there is some $U \in \text{Open}(X)$ such that there is a bundle isomorphism $\tilde{\varphi} : E|_U \rightarrow U \times W$ for some vector space W . Then φ restricts to a bundle morphism

$$\varphi|_U : U \times V \rightarrow E|_U$$

so that $\tilde{\varphi} \circ \varphi|_U$ is an epimorphism

$$\tilde{\varphi} \circ \varphi|_U : U \times V \rightarrow U \times W$$

which is $\mathbb{1}_U$ on its first factor, and

$$\pi_2 \circ \tilde{\varphi} \circ \varphi|_U : U \times V \rightarrow W$$

is also surjective. Thus, $\pi_2 \circ \tilde{\varphi} \circ \varphi|_U$ is some matrix of $\dim(V)$ columns and $\dim(W)$ rows with entries continuously dependent on $x \in U$. As such, the projection onto its kernel is also a matrix whose entries are continuous in x . \square

16.4.8. Claim. If $p : E \rightarrow X$ is a manifold over X and $\varphi : X \times V \rightarrow E$ is a bundle epimorphism for some V and $f_\varphi : X \rightarrow G_n(V)$ is as above, then

$$E \cong f_\varphi^*(E_{G_n(V)})$$

where $f_\varphi^*(E_{G_n(V)})$ is the pull-back of the classifying bundle $E_{G_n(V)}$ defined in (58).

Proof. Before we start we must describe more concretely the quotient bundle. Since $E_{G_n(V)}$ is defined such that its fiber above a point $W \in G_n(V)$, denoted by E_W , is the quotient vector space

$$\begin{aligned} E_W &\equiv \underbrace{\{W\} \times V}_{\cong V} / \underbrace{\{W\} \times W}_{\cong W} \\ &\cong \{W\} \times (V/W) \end{aligned}$$

then a generic point in this fiber is given by

$$(W, v + W)$$

with $v \in V$. Recall from 16.2.4 that

$$\begin{aligned} f_\varphi^*(E_{G_n(V)}) &\equiv \left\{ (x, (W, v + W)) \in X \times E_{G_n(V)} \mid f_\varphi(x) = W \wedge v \in V \right\} \\ &= \left\{ (x, (\ker(\varphi_x), v + \ker(\varphi_x))) \in X \times E_{G_n(V)} \mid v \in V \right\} \end{aligned}$$

Thus we define the map $i : f_\varphi^*(E_{G_n(V)}) \rightarrow E$ by

$$i(x, (\ker(\varphi_x), v + \ker(\varphi_x))) := \varphi(x, v)$$

We show that i is well defined: If $\tilde{v} + \ker(\varphi_x) = v + \ker(\varphi_x)$ then

$$\begin{aligned} \varphi(x, \tilde{v}) &= \varphi(x, v + \tilde{v} - v) \\ &\stackrel{\text{linearity}}{=} \varphi(x, v) + \varphi \underbrace{\left(x, \underbrace{(\tilde{v} - v)}_{\in \ker(\varphi_x)} \right)}_0 \\ &= \varphi(x, v) \end{aligned}$$

and that i is injective: its inverse $i^{-1} : E \rightarrow f_\varphi^*(E_{G_n(V)})$ is given by

$$i^{-1}(e) := (p(e), (f_\varphi(p(e)), \chi_{p(e)}^{-1}(e)))$$

where $\chi_x : V/\ker(\varphi_x) \rightarrow E_x$ are bijections which are naturally defined by $\varphi_x : \{x\} \times V \rightarrow E_x$. Then

$$\begin{aligned} (i^{-1} \circ i)(x, (\ker(\varphi_x), v + \ker(\varphi_x))) &= i^{-1}(\varphi(x, v)) \\ &= (p(\varphi(x, v)), (f_\varphi(p(\varphi(x, v))), \chi_{p(\varphi(x, v))}^{-1}(\varphi(x, v)))) \\ &= (x, (f_\varphi(x), \chi_x^{-1}(\varphi(x, v)))) \\ &= \left(x, \left(\ker(\varphi_x), \underbrace{\chi_x^{-1}(\varphi(x, v))}_{\equiv \ker(\varphi_x) + v} \right) \right) \\ &= (x, (\ker(\varphi_x), \ker(\varphi_x) + v)) \end{aligned}$$

indeed. Next, if $e \in E$ then $\chi_{p(e)}^{-1}(e) = v + \ker(\varphi_{p(e)})$ for some $v \in V$ such that $\varphi(p(e), v) = e$ so that and

$$\begin{aligned} (i \circ i^{-1})(e) &= i(p(e), (f_\varphi(p(e)), \chi_{p(e)}^{-1}(e))) \\ &= \varphi(p(e), v) \\ &= e \end{aligned}$$

as necessary. i is linear on fibers because φ is a bundle morphism and is hence itself linear on fibers. One still has to show that i is a homeomorphism. \square

16.4.9. **Claim.** Let $n \in \mathbb{N}_{>0}$ and $m \in \mathbb{N}_{>n}$. Then there is a continuous injective map

$$\iota_{m-1 \rightarrow m} : G_n(\mathbb{C}^{m-1}) \rightarrow G_n(\mathbb{C}^m)$$

Proof. Let $W \in G_n(\mathbb{C}^{m-1})$ be given. Then $\dim(W) = m-1-n$. Then $\dim(W^\perp) = n$. Thus $W^\perp \times \{0\}$ is an n -dimensional subspace of \mathbb{C}^m . So map

$$W \mapsto (W^\perp \times \{0\})^\perp$$

This map is continuous because the corresponding map on the projections

$$\text{End}(\mathbb{C}^{m-1}) \rightarrow \text{End}(\mathbb{C}^m)$$

is continuous. \square

16.4.10. **Definition.** If $(l, m) \in \mathbb{N}_{\geq n}^2$ and $l \geq m$, we define

$$\iota_{m \rightarrow l} : G_n(\mathbb{C}^m) \rightarrow G_n(\mathbb{C}^l)$$

as

$$\iota_{m \rightarrow l} := \iota_{l-1 \rightarrow l} \circ \cdots \circ \iota_{m+1 \rightarrow m+2} \circ \iota_{m \rightarrow m+1}$$

16.4.11. **Claim.** Let $n \in \mathbb{N}_{>0}$ and $m \in \mathbb{N}_{>n}$. Then

$$\iota_{m-1}^*(E_{G_n(\mathbb{C}^m)}) \cong E_{G_n(\mathbb{C}^{m-1})}$$

where $E_{G_n(\mathbb{C}^m)}$ is defined in 16.4.5.

Proof. We have

$$\begin{aligned} \iota_{m-1}^*(E_{G_n(\mathbb{C}^m)}) &\equiv \left\{ (W, e) \times \in G_n(\mathbb{C}^{m-1}) \times E_{G_n(\mathbb{C}^m)} \mid \iota_{m-1}(W) = p_{E_{G_n(\mathbb{C}^m)}}(e) \right\} \\ &= \left\{ (W, (\tilde{W}, \tilde{W} + v)) \times \in G_n(\mathbb{C}^{m-1}) \times E_{G_n(\mathbb{C}^m)} \mid \iota_{m-1}(W) = \tilde{W} \wedge v \in \mathbb{C}^m \right\} \\ &= \left\{ (W, (\iota_{m-1}(W), \iota_{m-1}(W) + v)) \times \in G_n(\mathbb{C}^{m-1}) \times E_{G_n(\mathbb{C}^m)} \mid v \in \mathbb{C}^m \right\} \end{aligned}$$

So use the map $i : E_{G_n(\mathbb{C}^{m-1})} \rightarrow \iota_{m-1}^*(E_{G_n(\mathbb{C}^m)})$ given by

$$\left(W, W + \underbrace{v}_{\in \mathbb{C}^{m-1}} \right) \mapsto \left(W, \left(\iota_{m-1}(W), \iota_{m-1}(W) + \underbrace{(v, 0)}_{\in \mathbb{C}^m} \right) \right)$$

and show that it is a bundle isomorphism. \square

16.4.12. **Claim.** Let E be a vector bundle over X with $\dim(E) = n \in \mathbb{N}_{>0}$. Then for sufficiently large $m \in \mathbb{N}_{>n}$, there is a unique homotopy-class of maps $[f : X \rightarrow G_n(\mathbb{C}^m)]$ such that $f^*(E_{G_n(\mathbb{C}^m)}) \cong E$ for any representative of the class.

Proof. Using [5] 1.4.13 for any vector bundle $p : E \rightarrow X$ there exists some $m \in \mathbb{N}$ such that there is a bundle epimorphism $\varphi : X \times \mathbb{C}^m \rightarrow E$. Then we know that

$$E \cong f_\varphi^*(E_{G_n(\mathbb{C}^m)})$$

by 16.4.8. We would like to define the homotopy class as the class of the induced map $[f_\varphi]$, where f_φ is as above. However, one must show that this class does not depend on the choice of φ .

So let two possible choices be given: $\varphi_i : X \times \mathbb{C}^{m_i} \rightarrow E$ with $i \in \{0, 1\}$. Our goal is to show that f_{φ_0} is homotopic to f_{φ_1} if both of their domains are placed into a large enough space.

Define for all $t \in [0, 1]$,

$$\psi_t : X \times \mathbb{C}^{m_0} \times \mathbb{C}^{m_1} \rightarrow E$$

by

$$\psi_t(x, v_0, v_1) := \underbrace{(1-t)\varphi_0(x, v_0) + t\varphi_1(x, v_1)}_{\text{addition in a fiber of } E}$$

Note that this is an epimorphism, since φ_i are epimorphisms. Note that the map induced by ψ_t (in the sense of 16.4.6) is

$$f_{\psi_t} : X \rightarrow G_n \left(\underbrace{\mathbb{C}^{m_0} \times \mathbb{C}^{m_1}}_{\mathbb{C}^{m_0+m_1}} \right)$$

Then note that

$$f_{\psi_0} = \iota_{m_0 \rightarrow m_1+m_2} \circ f_{\varphi_0}$$

by construction. However,

$$f_{\psi_1} \neq \iota_{m_1 \rightarrow m_1+m_2} \circ f_{\varphi_1}$$

because f_{φ_1} sends x to the kernel of φ_1 , whereas f_{ψ_1} sends x to the kernel of ψ_1 . But the kernel of ψ_1 will project to all the coordinates corresponding to \mathbb{C}^{m_0} , whereas $\iota_{m_1 \rightarrow m_1+m_2}$ will inject from the first coordinate of $\mathbb{C}^{m_0+m_1}$. Thus we define an isomorphism

$$T : G_n(\mathbb{C}^{m_0+m_1}) \rightarrow G_n(\mathbb{C}^{m_0+m_1})$$

which takes a subspace $W \in G_n(\mathbb{C}^{m_0+m_1})$ and permutes its coordinates so that the first m_0 coordinates are always zero. Then we have

$$f_{\Psi_1} = T \circ \iota_{m_1 \rightarrow m_1+m_2} \circ f_{\varphi_1}$$

However, note that T is homotopic to the identity, so that

$$\begin{aligned} [\iota_{m_1 \rightarrow m_1+m_2} \circ f_{\varphi_1}] &= [1_{G_n(\mathbb{C}^{m_0+m_1})} \circ \iota_{m_1 \rightarrow m_1+m_2} \circ f_{\varphi_1}] \\ &= [T \circ \iota_{m_1 \rightarrow m_1+m_2} \circ f_{\varphi_1}] \\ &= [f_{\Psi_1}] \\ \underset{\forall t \in [0, 1]}{\equiv} & [f_{\Psi_t}] \\ &= [f_{\Psi_0}] \\ &= [\iota_{m_0 \rightarrow m_1+m_2} \circ f_{\varphi_0}] \end{aligned}$$

□

16.4.13. Remark. Note that for fixed $n \in \mathbb{N}_{>0}$ we have a natural monomorphism

$$[X \rightarrow G_n(\mathbb{C}^m)] \hookrightarrow [X \rightarrow G_n(\mathbb{C}^{m+1})]$$

given by

$$[f] \mapsto [\iota_m \circ f]$$

With this inclusion we get a direct system

$$\{[X \rightarrow G_n(\mathbb{C}^m)]\}_{m=n}^\infty$$

so that we may define the direct limit

$$\lim_{\rightarrow m} [X \rightarrow G_n(\mathbb{C}^m)] := \left(\bigsqcup_{m=n}^{\infty} [X \rightarrow G_n(\mathbb{C}^m)] \right) / \sim$$

where $\{m_1\} \times [f] \sim \{m_2\} \times [g]$ iff $\exists M \in \mathbb{N}$ such that

$$[\iota_{m_1 \rightarrow M} \circ f] = [\iota_{m_2 \rightarrow M} \circ g]$$

16.4.14. Definition. (*Stable Homotopy*) An equivalence class in $\lim_{\rightarrow m} [X \rightarrow G_n(\mathbb{C}^m)]$ is called a class of *stable homotopies*.

16.4.15. Claim. If X is compact then

$$\lim_{\rightarrow m} [X \rightarrow G_n(\mathbb{C}^m)] = [X \rightarrow G_n(\mathbb{C}^\infty)]$$

where

$$G_n(\mathbb{C}^\infty) := \varinjlim_m G_n(\mathbb{C}^m)$$

with the direct limit topology. $G_n(\mathbb{C}^\infty)$ is called the infinite Grassmannian. It gives us a more convenient way to classify maps up to regular homotopy rather than stable homotopy, at the price of working with a direct space. Alternatively, one could define

$$G_n(\mathbb{C}^\infty) := G_n(\mathcal{H})$$

where \mathcal{H} is a complex-infinite-dimensional separable Hilbert space with the same topology as in 16.4.1.

16.4.16. Corollary. As a result we obtain that there is a bijection between the set of isomorphism classes of vector bundles of rank n over X , denoted by $\text{Vect}_n(X)$, and the set of homotopy classes of maps $X \rightarrow G_n(\mathbb{C}^\infty)$:

$$[X \rightarrow G_n(\mathbb{C}^\infty)] = \text{Vect}_n(X)$$

so that we obtain a complete homotopic characterization of vector bundles. This is the culmination of the first part of [5].

16.5. Classification of Vector Bundles and the Chern Number.

16.5.1. Example. If $X = S^m$, the m -sphere, which is compact, then we obtain the following result:

$$\begin{aligned} \text{Vect}_n(S^m) &= [S^m \rightarrow G_n(\mathbb{C}^\infty)] \\ &\equiv \pi_m(G_n(\mathbb{C}^\infty)) \end{aligned}$$

where $\pi_m(\cdot)$ is the m -th homotopy group functor. The computation of $\pi_m(G_n(\mathbb{C}^\infty))$ follows from Bott's periodicity [12] to give, for n sufficiently large:

$$\pi_m(G_n(\mathbb{C}^\infty)) = \begin{cases} \mathbb{Z} & m \in 2\mathbb{Z} \\ \{0\} & m \notin 2\mathbb{Z} \end{cases} \quad (59)$$

In fact a similar result may be proven when the sphere S^m is replaced with the m -torus \mathbb{T}^m , which produces exactly the "A" row of the Kitaev table [25].

The remaining rows of the Kitaev table may be computed by the notion of G -bundles (see [5] section 1.6) for some group G (the group would be the group of symmetries of a particular symmetry class in the Kitaev table).

16.5.2. *Remark.* We give a sketch of the approach outlined in the book [33]. The main goal is to give an explicit formula for the computation of the integer in (59) when $X = \mathbb{T}^2$, as given in [7] equation (3). This number is usually called the first Chern number.

Using 16.4.16 we come up with a general strategy to distinguish vector bundles which are not isomorphic:

- (1) Let $p : E \rightarrow X$ and $q : F \rightarrow X$ be two given vector bundles over X . We want to ascertain whether $E \cong F$ (bundle-isomorphism).
- (2) Then by 16.4.12, there are two maps,

$$f_E : X \rightarrow G_n(\mathbb{C}^m)$$

$$f_F : X \rightarrow G_n(\mathbb{C}^l)$$

and $E \cong F$ iff

$$[j_{m \rightarrow N} \circ f_E] = [j_{l \rightarrow N} \circ f_F] \quad (60)$$

for some $N \in \mathbb{N}$ where $j_{R \rightarrow S} : G_n(\mathbb{C}^R) \rightarrow G_n(\mathbb{C}^S)$ is the natural inclusion map for $S \geq R$.

- (3) In practice it is, however, extremely difficult to determine the condition (60). What we can do is pass on to algebraic categories, which are easier to compute. This is the point of algebraic topology.
- (4) One such algebraic category which is often used in mathematics is the group (or ring) category using the functor $\mathcal{F} := H^r(\cdot; \mathbb{Z})$ of singular cohomology with integer coefficients. Then if $\mathcal{F}([f_E]) \neq \mathcal{F}([f_F])$ we will know that $E \not\cong F$. This is not going to be a full classification since it is known that there are spaces with completely trivial cohomology groups which are non-trivial.
- (5) Thus we obtain a map

$$f_E^* : H^{2r}(G_n(\mathbb{C}^m); \mathbb{Z}) \rightarrow H^{2r}(X; \mathbb{Z})$$

which is induced by $[f_E]$, and for a certain choice of an element c_r in the group $H^{2r}(G_n(\mathbb{C}^m); \mathbb{Z})$ (these choices are a matter of mathematical conventions which someone felt was natural) we obtain $f_E^*(c_r)$, which is called the r th Chern class of E .

- (6) Note that we have taken only even homology classes because the odd ones are zero for the Grassmannians. In addition, it is known that if the dimension of a smooth manifold is D then all singular cohomology classes above D are zero. Thus, there is a notion of a “top” Chern class, which corresponds to the dimension of X . Then c_r is chosen such that the “top” Chern class is the Euler class of X .
- (7) When $X = \mathbb{T}^2$, there is only one Chern class, the first Chern class, because $\dim(X) = 2$.
- (8) The first Chern number for a vector bundle over a base space which is a smooth manifold of dimension two is defined to be the top Chern cohomology class acting on the fundamental homology class (the generator of the top homology group which is also isomorphic to \mathbb{Z}).
- (9) In the appendix of [33] it is proven how to show that this definition above is equal to the one given in (39).

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