Chapter 8

Composite fermions

Learning goals

- We know what a coherent state path integral is.
- We know the concept of a composite fermion.
- We know how to get from composite fermions to a Chern-Simons theory.

- Willett, R. et al., Phys. Rev. Lett. **59**, 1776 (1987)

8.1 Path integrals

8.1.1 Why do we need a path integral

In this section we try to argue why we need a path integral representation of the partition sum

$$Z = \int D[\bar{\phi}\phi]e^{-S[\bar{\phi},\phi]}. \tag{8.1}$$

First of all, we trade non-commuting bosonic *operators* with an integral over all "field" configurations, i.e.,

$$[.,.] \to D[\bar{\phi},\phi]. \tag{8.2}$$

Moreover, we replace complicated anti-commutations for fermions we a simple tool called Grassmann numbers. Before we are going to explain what we exactly mean with expression (8.1), we list a few nice properties that we will gain from a path integral formalism.

1. We can use Gaussian integrals

$$\int D[\bar{\phi}\phi]e^{-\bar{\phi}^{\mathrm{T}}A\phi} = \frac{1}{\det A}. \tag{8.3}$$

2. We can complete the square

$$\int D[\bar{\phi}\phi]e^{-\bar{\phi}^{\mathrm{T}}A\phi+\vartheta^{\mathrm{T}}\bar{\phi}+\bar{\vartheta}^{\mathrm{T}}\phi} = \int D[\bar{\phi}\phi]e^{-(\bar{\phi}-A^{-1}\bar{\vartheta})^{\mathrm{T}}A(\phi-A^{-1}\vartheta)+\bar{\vartheta}^{\mathrm{T}}A^{-1}\vartheta} = \frac{e^{\bar{\vartheta}^{\mathrm{T}}A^{-1}\vartheta}}{\det A}. \tag{8.4}$$

This completing of the square in turn has three important applications:

(a) Greens functions (or more generally, two-point correlators) in a quadratic theory can be calculated by coupling sources ϑ

$$\langle\bar{\phi}_i\phi_j\rangle = \frac{\int D[\bar{\phi}\phi]\bar{\phi}_i\phi_j e^{-S[\bar{\phi},\phi]}}{\int D[\bar{\phi}\phi]e^{-S[\bar{\phi},\phi]}} = \frac{\delta^2}{\delta\vartheta_i\delta\bar{\vartheta}_j}\bigg|_{\vartheta=\bar{\vartheta}=0} e^{\bar{\vartheta}^{\mathrm{T}}A^{-1}\vartheta} = [A^{-1}]_{ij}. \tag{8.5}$$

(b) "Integrating out" linearly coupled quadratic degrees of freedom

$$\int D[\bar{\phi}\phi]D[\bar{\vartheta}\vartheta]e^{-S[\bar{\phi},\phi]+\phi^{\mathrm{T}}\bar{\vartheta}+\bar{\phi}^{\mathrm{T}}\vartheta-\bar{\vartheta}^{\mathrm{T}}B\vartheta} = \int D[\bar{\phi}\phi]e^{-S[\bar{\phi},\phi]+\bar{\phi}^{\mathrm{T}}B^{-1}\phi} = \int D[\bar{\phi}\phi]e^{-S_{\mathrm{eff}}[\bar{\phi},\phi]}. \quad (8.6)$$

(c) Or the reverse of it, called *Hubbard Stratonovich transformation*

$$\int D[\bar{\phi}\phi]e^{-\bar{\phi}^{\mathrm{T}}A\phi+\bar{\phi}^{\mathrm{T}}\phi\bar{\phi}^{\mathrm{T}}\phi} = \int D[\bar{\phi}\phi]D[\vartheta]e^{-(\bar{\vartheta}-\bar{\phi}^{\mathrm{T}}\phi)(\vartheta-\bar{\phi}^{\mathrm{T}}\phi)-\phi^{\mathrm{T}}A\phi+\bar{\phi}^{\mathrm{T}}\phi\bar{\phi}^{\mathrm{T}}\phi} \quad (8.7)$$

$$= \int D[\bar{\phi}\phi]D[\vartheta]e^{-\bar{\vartheta}\vartheta+2\vartheta\bar{\phi}^{\mathrm{T}}\phi-\bar{\phi}^{\mathrm{T}}\phi\bar{\phi}^{\mathrm{T}}\phi-\bar{\phi}^{\mathrm{T}}A\phi+\bar{\phi}^{\mathrm{T}}\phi\bar{\phi}^{\mathrm{T}}\phi} \quad (8.8)$$

$$= \int D[\bar{\phi}\phi]D[\vartheta]e^{-\bar{\vartheta}\vartheta-\bar{\phi}^{\mathrm{T}}(A+2\vartheta)\phi} \quad (8.9)$$

$$= \int D[\vartheta]e^{-\bar{\vartheta}\vartheta-\mathrm{tr}\log[A+2\vartheta]}. \quad (8.10)$$

This is still not a quadratic theory, but the logarithm can be expanded step by step to get an effective theory.

3. We can do mean-field calculations

$$\frac{\delta S[\bar{\phi},\phi]}{\delta\bar{\phi}} = 0 \qquad \Rightarrow \qquad \phi_{\mathrm{MF}}. \quad (8.11)$$

After all these expected profits, let us start introducing such a path integral representation of the partition sum.

8.1.2 Coherent state path integral

Given a quantum mechanical problem defined by a Hamiltonian H, we want to express the partition sum

$$Z = \mathrm{tr}e^{-\beta H} = \sum_n \langle m|e^{-\beta H}|m\rangle, \quad (8.12)$$

as a path integral. For this we use coherent states

$$|\phi\rangle = e^{\eta\sum_i \phi_i c_i^\dagger}|\mathrm{vac}\rangle \quad \Rightarrow c_i|\phi\rangle = \phi_i|\phi\rangle, \quad (8.13)$$

and we used $\eta = \pm 1$ for bosons (fermions), respectively. Remember that they are not orthogonal

$$\langle\phi|\vartheta\rangle = e^{\bar{\phi}^{\mathrm{T}}\vartheta}. \quad (8.14)$$

For bosons, $\phi_i \in \mathbb{C}$. For fermions we need to take care of anti-commutations. This can be achieved by requiring ϕ_i to be Grassmann numbers.
Grassmann numbers are defined by

$$\phi_i\phi_j = -\phi_j\phi_i; \quad \partial_{\phi_i}\phi_j = 0; \quad \int d\phi_i = 0; \quad \int d\phi_i\phi_i = 1. \quad (8.15)$$

From this follows immediately

$$\int d\bar{\phi}_i d\phi_i\, e^{-\bar{\phi}_i a\phi_i} = \int d\bar{\phi}_i d\phi_i\, [1-\phi_i\phi_i a] = a. \quad (8.16)$$

Which immediately leads to

$$\int d(\bar{\phi}\phi)\, e^{-\bar{\phi}^{\mathrm{T}}A\phi} = \prod_n \int d\bar{\phi}_n d\phi_n\, e^{-\sum_{rs}\bar{\phi}_r A_{rs}\phi_s} = \det A. \quad (8.17)$$

Note that this is similar to the bosonic case, however $[\det A]^{-1}$ is replaced with $\det A$.
With the help of the coherent states $|\phi\rangle$ we can now write a complicated but tremendously useful resolution of the unity

$$\mathbb{1} = \int d(\bar{\phi}\phi)\, e^{-\bar{\phi}^{\mathrm{T}}\phi} |\phi\rangle\langle\phi|. \tag{8.18}$$

To proof this identity, we have to show that c_i and $c_i^\dagger$ commute with the right-hand side:

$$c_i \int d(\bar{\phi}\phi)\, e^{-\bar{\phi}^{\mathrm{T}}\phi} |\phi\rangle\langle\phi| = \int d(\bar{\phi}\phi)\, e^{-\bar{\phi}^{\mathrm{T}}\phi} c_i |\phi\rangle\langle\phi| = \int d(\bar{\phi}\phi)\, e^{-\bar{\phi}^{\mathrm{T}}\phi} \phi_i |\phi\rangle\langle\phi| \tag{8.19}$$

$$= -\int d(\bar{\phi}\phi)\, [\partial_{\bar{\phi}_i} e^{-\bar{\phi}^{\mathrm{T}}\phi}] |\phi\rangle\langle\phi| \tag{8.20}$$

$$\overset{\text{P.I.}}{=} \int d(\bar{\phi}\phi)\, e^{-\bar{\phi}^{\mathrm{T}}\phi} [\underbrace{(\partial_{\bar{\phi}_i} |\phi\rangle)}_{=0} \langle\phi| + |\phi\rangle (\partial_{\bar{\phi}_i} \langle\phi|)] \tag{8.21}$$

$$= \int d(\bar{\phi}\phi)\, e^{-\bar{\phi}^{\mathrm{T}}\phi} |\phi\rangle\langle\phi| c_i. \tag{8.22}$$

In the last line we used

$$c_i^\dagger |\phi\rangle = \partial_{\phi_i} |\phi\rangle \qquad \Rightarrow \qquad \partial_{\phi_i} \langle\phi| = \langle\phi_i| a_i. \tag{8.23}$$

With this we showed that c_i indeed commutes with the alleged unity. For $c_i^\dagger$ one starts from the other end end and goes through the same manipulations (show!). As all operators in the Fock space can be written as products (and sums) of the creation and annihilation operators, we have shown that indeed

$$\int d(\bar{\phi}\phi)\, e^{-\bar{\phi}^{\mathrm{T}}\phi} |\phi\rangle\langle\phi| \propto \mathbb{1}. \tag{8.24}$$

Let us check for the proportionality factor

$$\langle \mathrm{vac}|\mathbb{1}|\mathrm{vac}\rangle = 1 = \int d(\bar{\phi}\phi)\, e^{-\bar{\phi}^{\mathrm{T}}\phi} \langle \mathrm{vac}|\phi\rangle\langle\phi|\mathrm{vac}\rangle. \tag{8.25}$$

Let us now rewrite the trace in the partition sum

$$Z = \sum_n \langle n| e^{-\beta H} |n\rangle = \int d(\bar{\phi}\phi)\, \sum_n \langle n|\phi\rangle\langle\phi| e^{-\beta H} |n\rangle e^{-\bar{\phi}^{\mathrm{T}}\phi} \tag{8.26}$$

$$= \int d(\bar{\phi}\phi)\, e^{-\bar{\phi}^{\mathrm{T}}\phi} \sum_n \langle \eta\phi|n\rangle\langle n| e^{-\beta H} |\phi\rangle = \int d(\bar{\phi}\phi)\, e^{-\bar{\phi}^{\mathrm{T}}\phi} \langle \eta\phi| e^{-\beta H} |\phi\rangle. \tag{8.27}$$

Now we need to fix an important property. In order for our path integral approach to go through, we need to normal order our Hamiltonian. This means, we arrange all operators in H such that all $c_i^\dagger$ stand to the left of all c_i. As the fields ϕ_i are just complex numbers (for bosons, at least), this will be the last time we can take care of the operator nature of second quantized quantum mechanics. We write for the normal ordered Hamiltonian explicitly

$$Z = \int d(\bar{\phi}\phi)\, e^{-\bar{\phi}^{\mathrm{T}}\phi} \langle \eta\phi| e^{-\beta H(c^\dagger, c)} |\phi\rangle. \tag{8.28}$$

Next, we re-write

$$\beta H(c^\dagger, c) = \frac{\beta}{N} \sum_{i=1}^{N} H(c^\dagger, c) \tag{8.29}$$

and we insert a unity in between all resulting factors

$$Z = \int_{\phi^1 = \eta\phi^N,\, \bar{\phi}^1 = \eta\bar{\phi}^N} \prod_{i=1}^{N} d(\bar{\phi}^i \phi^i) e^{\frac{\beta}{N} \sum_{i=1}^{N} \frac{(\bar{\phi}^i - \bar{\phi}^{i+1})\phi^i}{\beta/N} + H(\bar{\phi}^i, \phi^i)}. \tag{8.30}$$

Note that the superscript i labels the i'th insertion of the unity. One often calls β the "imaginary time" in relation to the real time propagator $\exp(itH)$. Within this interpretation, i corresponds to the i'th time slice. If we now take the limit $N \to \infty$, we are taking a continuum limit in imaginary time where

$$\phi^i \to \phi(\tau) \qquad \text{and} \qquad \frac{\beta}{N}\sum_i \to \int_0^\beta d\tau. \tag{8.31}$$

We can now write down our sought path integral

$$Z = \int D[\bar{\phi}\phi] e^{-S[\bar{\phi},\phi]}, \tag{8.32}$$

$$S[\bar{\phi},\phi] = \int_0^\beta d\tau\, \bar{\phi}^{\mathrm{T}} \partial_\tau \phi + H(\bar{\phi},\phi), \tag{8.33}$$

$$D[\bar{\phi},\phi] = \lim_{N\to\infty} \prod_{i=1}^N d(\bar{\phi}^i \phi^i); \quad \bar{\phi}(0) = \eta\bar{\phi}(\beta), \ \ \phi(0) = \eta\phi(\beta). \tag{8.34}$$

8.1.3 Kubo formula

We already got acquainted with the Kubo formula in Chap. 3. We want to revisit here in the language of our newly introduced coherent state path integral. Imagine a "force" $F(\mathbf{r},\omega)$ coupled to the "coordinate"

$$\hat{X} = \sum_{\alpha\beta} c^\dagger_\alpha X_{\alpha\beta} c_\beta. \tag{8.35}$$

We then ask for the linear response coefficient

$$X(\mathbf{r},\omega) = \int dr' \chi(\mathbf{r}-\mathbf{r}',\omega) F(\mathbf{r}',\omega). \tag{8.36}$$

In path integral formalism the expetation value on the right hand side is expressed as

$$X(\tau) = \sum_{\alpha\beta} \langle \bar{\phi}_\alpha(\tau) X_{\alpha\beta} \phi_\beta(\tau)\rangle_F, \tag{8.37}$$

where the subscript F indicates that we have to evaluate this expression in the presence of the force F

$$\delta S_F = \int_0^\beta d\tau\, F(\tau)\bar{\phi}_\alpha(\tau) X_{\alpha\beta}\phi_\beta(\tau). \tag{8.38}$$

To generate the expectation value (8.37) we can add another ficticious force F' to the action

$$\delta S_{F'} = \int_0^\beta d\tau\, F'(\tau)\bar{\phi}_\alpha(\tau) X'_{\alpha\beta}\phi_\beta(\tau). \tag{8.39}$$

With this addition, one can write

$$X(\tau) = -\frac{\delta}{\delta F'(\tau)}\Big|_{F'=0} \log(Z[F,F']). \tag{8.40}$$

For the sake of linear response, we imagine F to be small. Therefore, we can apply a Taylor expansion

$$X(\tau) = \int d\tau' \left[\frac{\delta^2}{\delta F'(\tau)\delta F(\tau')}\Big|_{F=F'=0} \log(Z[F,F'])\right] F(\tau') \tag{8.41}$$

With this expression we can immediately indentify the linear response coeffiecient. If we assume at $X(\tau) = 0$ in the absence of the for

$$\chi(\tau,\tau') = -\frac{1}{Z}\frac{\delta^2}{\delta F'(\tau)\delta F(\tau')}\Big|_{F=F'=0} Z[F,F']. \tag{8.42}$$

Electromagnetic response

We consider a system subject to an electromagnetic field $A^\mu = (i\varphi, \mathbf{A})$. The system might react via a redistribution of charge ρ or via an onset of a current $\mathbf{j}$. We write $j_\mu = (i\rho, \mathbf{j})$ and look for

$$j_\mu(x) = \int_{t'<t} dx' K_{\mu\nu}(x - x') A^\nu(x'), \tag{8.43}$$

where x describes the four-coordinate $(it, \mathbf{x})$. We remember that we coupled the A^μ-field as $j_\mu A^\mu$ to the Hamiltonian. Therefore,

$$j_\mu = \frac{\delta S}{\delta A^\mu} \qquad \Rightarrow \qquad F = F' = A^\mu. \tag{8.44}$$

With this we find

$$K_{\mu\nu}(x - x') = -\frac{1}{Z} \frac{\delta^2}{\delta A^\mu(x) \delta A^\nu(x')} Z[A^\mu]. \tag{8.45}$$

Effective theories

If we have a system of charged particles, $H(c^\dagger, c)$, and we are intersted in its electro-magnetic response, all we need to know is $K_{\mu\nu}$. In a path integral language, we say we *integrate out the fermions* to obtain an *effective action* in terms of th A^μ-field alone. The peculiar structure of $K_{\mu\nu}$ will fully describe our system in terms of its electro-mangetic system

$$S_{\text{eff}}[A^\mu] = \int_0^\beta d\tau \int dx dx' \, A^\mu(x) K_{\mu\nu}(x - x') A^\nu(x'). \tag{8.46}$$

8.2 Composite fermions

8.2.1 From a wave functions to a field theory

In the last chapter we got to know the Laughlin wave function for filling fractions $\nu = \frac{1}{2p+1}$ with $p \in \mathbb{N}$

$$\psi(\{z_i\}) = \prod_{i<j} (z_i - z_j)^{\frac{1}{\nu}} e^{-\frac{1}{4} \sum_i |z_i|^2}. \tag{8.47}$$

These wave functions are manifestly in the lowest Landau level and in addition to the $(z_i - z_j)^1$ term needed for the Pauli principle three are two (for $\nu = 1/3$) more zeros attached to the conincidence of two particles. This observation is identical to attaching $1/\nu - 1$ fluxes of 2π to each particle[1]

In the last chapter, we only considered the Laughlin wave function and analyzed its properties. Here, we follow a more ambitious goal. Building on the insight gain throught the Laughlin wave function, we want to construct an effective theory for the fractional quantum Hall effect including the Hamiltonian! However, we want to assume that the important players are not electrons, but the "bound states of electrons with statistical fluxes" that were at the heart of the Laughlin wave function. In other words, we want to go from an electron wave function (theory), to one of *composite fermions* by

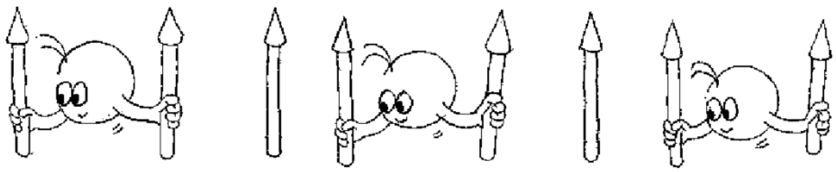
$$\psi(\{\mathbf{x}_i\}) \mapsto \psi(\{\mathbf{x}_i\}) e^{2is \sum_{i<j} \arg(\mathbf{x}_i - \mathbf{x}_j)} \quad \text{with} \quad s \in \mathbb{Z}. \tag{8.49}$$

[1] Up to the fact that a pure flux attachement would require a factor

$$e^{-2i \sum_{i<j} \arg(z_i - z_j)} = \prod_{i<j} \frac{(z_i - z_j)^2}{|z_i - z_j|^2}. \tag{8.48}$$

The absence of the factor $1/|z_i - z_j|^2$ in the Laughlin wave function can be seen as the effect of the projection to the lowest Landau level.

This amounts to attaching $2s$ phase vortices to each electron[2]



Our task is now to find a many-body theory formulated in terms of this new degrees of freedom. In a second quantized version, Eq (8.49) looks like

$$c^\dagger(\mathbf{x}) \quad \mapsto \quad c^\dagger(\mathbf{x}) \exp\left[-2is\int d\mathbf{x}' \arg(\mathbf{x}-\mathbf{x}')\rho(\mathbf{x}')\right]. \tag{8.50}$$

Substituted into the Hamiltonian this leads to

$$H \quad \mapsto \quad \int d\mathbf{x}\, c^\dagger(\mathbf{x}) \left[\frac{1}{2m}\left(-\partial_{\mathbf{x}} + \hat{\mathbf{A}}(\mathbf{x})\right)^2 + V(\mathbf{x})\right] c(\mathbf{x}) + H_{\text{int}}[\rho], \tag{8.51}$$

where

$$\hat{\mathbf{A}}(\mathbf{x}) = \mathbf{A}_{\text{ext}}(\mathbf{x}) + \hat{\mathbf{a}}(\mathbf{x}) \quad \text{with} \quad \hat{\mathbf{a}}(\mathbf{x}) = -2s\int d\mathbf{x}' \, \frac{(x_1 - x_1')\hat{\mathbf{x}}_1 + (x_2 - x_2')\hat{\mathbf{x}}_2}{|\mathbf{x}-\mathbf{x}'|^2}\rho(\mathbf{x}'). \tag{8.52}$$

This is very annoying! The kinetic energy operator became highly non-local and depends on six! operators. Let us fix this. We can relocate the condition (8.52) to another place in the action. Two observations are needed for this:

(i) Eq. (8.52) us only giving rise to the transversal part of $\mathbf{A}$: $\hat{\mathbf{a}} = \hat{\mathbf{a}}_\perp$ as $\sum_i \partial_i \hat{a}_i = 0$.

(ii) $b = \epsilon^{ij}\partial_i a_{\perp,j}$ fulfills $b = -4\pi s\rho(\mathbf{x})$.

Using these two observations we can write

$$Z = \int D[\bar{\psi}\psi] D[a_\perp] D[\phi] e^{iS_{\text{CF}}[\bar{\psi},\psi,a_\perp,\phi] + i\frac{\Theta}{2}S'_{\text{CS}}[a_\perp,\phi]}, \tag{8.53}$$

where $\Theta = 1/2\pi s$. Furthermore,

$$S_{\text{CF}}[\bar{\psi},\psi,a_\perp,\phi] = \int d\mathbf{x} \int dt\, \bar{\psi}\left[i\partial_t + \mu - \phi + \frac{1}{2m}\left(-i\partial_{\mathbf{x}} + \hat{\mathbf{A}}\right)^2 - V\right]\psi + S_{\text{int}}[\bar{\psi},\psi]. \tag{8.54}$$

$$S'_{\text{CS}}[a_\perp,\phi] = -\int d\mathbf{x}\int dt\, \phi \underbrace{\epsilon_{ij}\partial_i a_{\perp,j}}_{b}. \tag{8.55}$$

$\hat{\mathbf{A}}$ is still given by $\mathbf{A}_{\text{ext}} + \hat{\mathbf{a}}$, but the constraint (8.52) is replaced by teh functional δ-fucntion

$$\int D[\phi] e^{i\int d\mathbf{x}\int dt\, \phi\left(\frac{b}{4\pi s}+\rho\right)}. \tag{8.56}$$

With this we are almost done. We see that $a_\perp = (\phi, \mathbf{a}_\perp)$ enters Z like a gauge field. However, S'_{CS} is not gauge invariant. Hence, we propose to use

$$S_{\text{CS}}[a] = -\int dx^\mu \epsilon_{\mu\nu\sigma} a^\mu \partial_\nu a^\sigma. \tag{8.57}$$

with $x^\mu = (x^0, x^1, x^2)$; $\partial_\mu = (-\partial_0, \partial_1, \partial_2)$ wich is gauge invariant. The old S'_{CS} is nothing but S_{CS} evaluated in the Coulomb gauge $\partial_\mu a^\mu = 0$. Therefore, our full effective theory is now given by

$$Z = \int D[\bar{\psi}\psi] D[a] \exp\left\{iS_{\text{CF}}[\bar{\psi},\psi,a] + i\frac{\Theta}{4}S_{\text{CS}}[a]\right\}, \tag{8.58}$$

with

$$S_{\text{CF}}[\bar{\psi},\psi,a] = \int d\mathbf{x}\int dt\, \bar{\psi}\left[i\partial_t + \mu - \phi + \frac{1}{2m}\left(-i\partial_{\mathbf{x}} + \mathbf{A}_{\text{ext}} - \mathbf{a}\right)^2 - V\right]\psi + S_{\text{int}}[\bar{\psi},\psi]. \tag{8.59}$$

[2]Cartoon due Kwon Park.

8.2.2 Analyzing the composite fermion Chern-Simons theory

Before we emabark on the analysis of the above effective theory, let us make a hand-waving mean-field analysis. We see that for $s = 1$, each electron binds two flux quanta. If we *assume* the density to be homogeneous (recall the plasma analogy for the Laughlin wave function), and if we neglect flucutations, then the electrons see *on average* a flux corresponding to $\mathbf{A}_{\text{ext}} - \langle \mathbf{a} \rangle$. In other words, the composite fermions see a smaller **B**-field! Several scenarios are possbile

(i) $\mathbf{A}_{\text{ext}} = \langle \mathbf{a} \rangle \Rightarrow$ no magnetic field. This happens at $\nu = 1/2$. The fact that the the composite fermion prediction at $\nu = 1/2$ looks like a Fermi liquid is one of the great successes of the composite fermion construction [1].

(ii) Maybe, for some filling fraction ν, the effective **B**-field corrsponding to $\mathbf{A}_{\text{ext}} - \langle \mathbf{a} \rangle$ leads to an effective new filling fraction $\nu^* \in \mathbb{Z}$, i.e., the fractional quantum Hall effect for electrons would be mapped to an integer quantum Hall effect for composite fermions.

We are now trying to analyze the composite-fermion Chern-Simons (CF-CS) theory in mean-field. The only term which gives a real headache is the interaction term $S_{\text{int}}[\bar{\psi}, \psi]$. We re-write it using a Hubbard-Stratnovich transformation

$$e^{iS_{\text{int}}} = \int D[\sigma] \exp\left\{ \frac{i}{2} \int dx^3 dx'^3\, \sigma(x)[V^{-1}](x,x')\delta(x_0 - x_0')\sigma(x') + i \int dx^3 (\rho(x) - \rho_0)\sigma(x) \right\}. \tag{8.60}$$

For the interpretation of the σ-field it helps to note that when completing the square, it appears as next to $\bar{\psi}\psi$, hence it describes a (rescaled) density field.[3] Now ψ and $\bar{\psi}$ (and $\rho = \bar{\psi}\psi$) only appear quadratically (linearly) in the action and we can integrate out $\bar{\psi}, \psi$. With this we obtain an effective theory

$$S_{\text{eff}}[\sigma, a] = \underbrace{-i \operatorname{tr} \log \left[i\partial_0 + \mu - a_0 - \sigma + \frac{1}{2m}(-i\nabla + A)^2 \right]}_{S_\psi[a,A]} \tag{8.61}$$

$$- \rho_0 \int dx^3\, \sigma(x) + \frac{1}{2} \int dx^3 dx'^3\, \sigma(x)[V^{-1}](x,x')\delta(x_0 - x_0')\sigma(x') \tag{8.62}$$

$$+ \frac{\Theta}{4} S_{\text{CS}}[a], \tag{8.63}$$

where $A = A_{\text{ext}} + a$. The first line arises from integrating out the fermions ψ. On this effective theory we want to apply a mean-field, or saddle-point, approximation. As there are no ψ-fields present anymore, it can be difficult to interpret the different terms in the theory. To provide remedy to this problem, we note that the local density of fermions is given by taking the derivative of the original fermionic action with respect to $a_0(x)$. This property obviously survives the elimination of the ψ field. Therefore, we can get an "effective" expression for the density by

$$\frac{\delta S_\psi}{\delta a_0} = \rho[a, \sigma]. \tag{8.64}$$

Therefore,

$$\rho[a, \sigma] = \left[i\partial_0 + \mu - a_0 - \sigma + \frac{1}{2m}(-i\nabla + A)^2 \right]^{-1} (x, x). \tag{8.65}$$

Next, let us write down the saddle-point (Euler-Lagrange) equations. We start with

$$\left. \frac{\delta S_{\text{eff}}}{\delta a_o} \right|_{\bar{\sigma}, \bar{a}} = 0 : \qquad \rho[\bar{a}, \bar{\sigma}] = \frac{1}{4\pi s} \bar{b}. \tag{8.66}$$

[3] We also say that we decouple the action in the density-density channel.

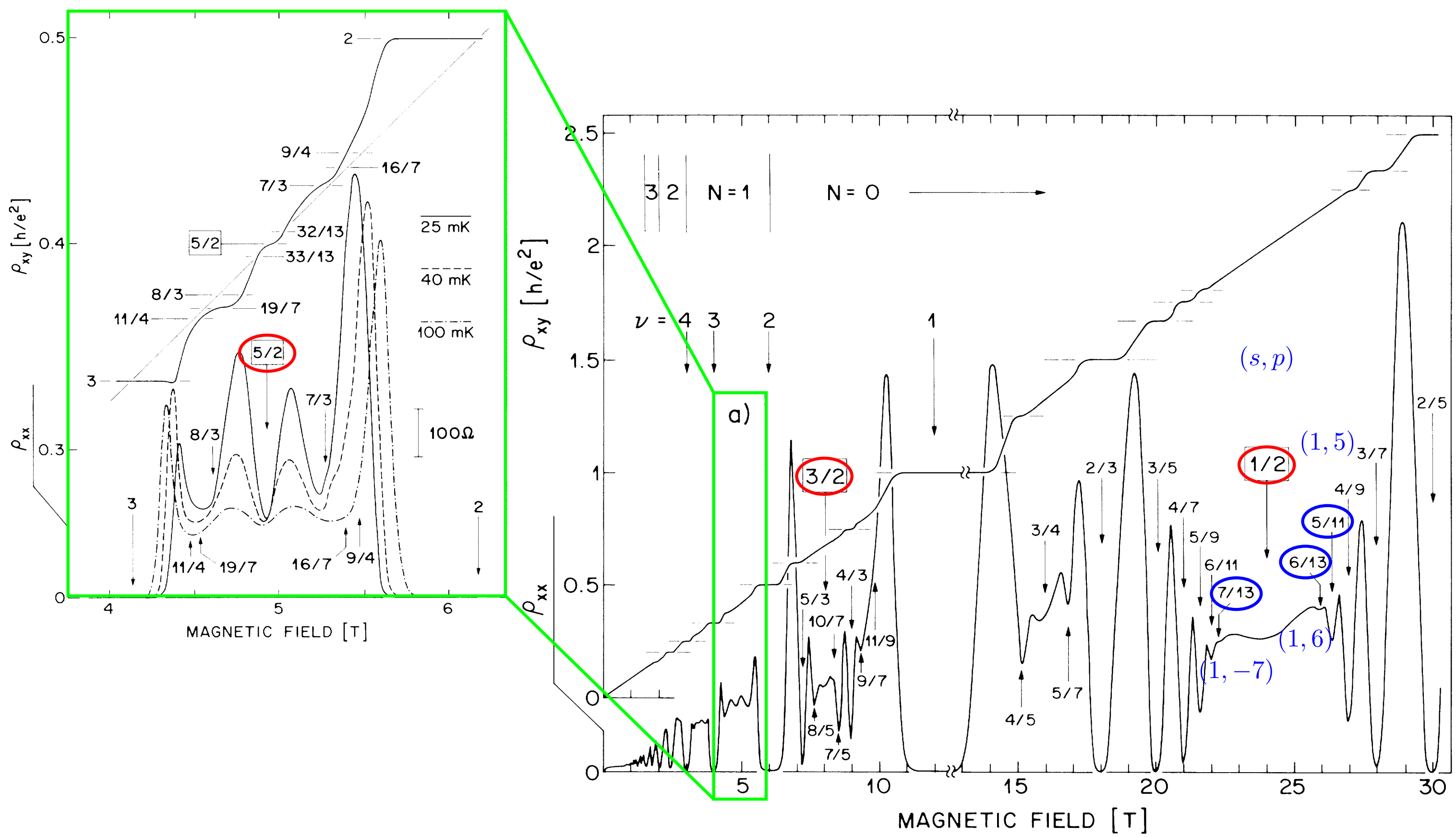


Figure 8.1: Overview of diagonal resistivity ρ_{xx} and Hall resistance ρ_{xy}. The blue numbers denote the fractions which are well explained by an integer quantum Hall plateau of composite fermions. The inset shows the details around $\nu = 5/2$. Figure adapted from Ref. [2] (Copyright (1987) by The American Physical Society).

This is nothing but the expected relation between the $\bar{b}$ field and the density.[4]
Next, we also need to minimize the action with respect to the field σ

$$\frac{\delta S_{\text{eff}}}{\delta\sigma}\Big|_{\bar{\sigma},\bar{a}} = 0 \qquad \Rightarrow \qquad \rho(x) - \rho_0 = -\int dx'^3 [V^{-1}](x,x')\sigma(x')\delta(x_0 - x_0'), \tag{8.67}$$

or

$$\sigma(x) = -\int dx'^3 V(x-x')[\rho(x') - \rho_0]\big|_{x_0'=x_0}. \tag{8.68}$$

Here we recognize that deviations of $\rho(x)$ from its mean value give rise to an "interaction potential" $\sigma(x)$. We can solve the mean-field equations by

$$\rho[\bar{a}, 0] = \rho_0 \tag{8.69}$$

$$\bar{\sigma} = \bar{a} = 0 \tag{8.70}$$

$$\bar{b} = 4\pi s\rho_0 \quad \Rightarrow \mathbf{a} = 2s\nu\mathbf{A}_{\text{ext}}. \tag{8.71}$$

When can we expect this mean-field calculation to be reliable? Certainly, if the resulting ground-state is gapped, we can hope that fluctuations around the mean-field solutions will not do too much harm. One way to ensure a gapped mean-field solution is by asking for the effective $A - A_{\text{ext}} - a$ to give rise to a *filled effective Landau level*. Therefore we ask

$$\nu_{\text{eff}} = p \quad \text{or} \quad \Phi_{\text{eff}} = \frac{2\pi N}{p} \quad \text{with} \quad \Phi_{\text{eff}} = (B_{\text{ext}} - \bar{b})L^2. \tag{8.72}$$

Inserting $b = 4\pi sN/L^2$ we immediately obtain

$$\nu = \frac{2\pi N}{B_{\text{ext}}L^2} = \frac{2\pi N}{\frac{2\pi N}{p} + 4\pi sN} \qquad \Rightarrow \qquad \nu = \frac{p}{1+2sp}. \tag{8.73}$$

We can summarize the mean-field discussion with the following list and Fig. 8.1

(i) We can explain many fractions which are symmetrically distributed around $1/2s$ by an integer quantum Hall effect for composite fermions. Note, however, that the gap is entirely due to interactions!

(ii) For $\nu = p/2s$, CF-CS predicts a Fermi-liquid theory in $B_{\text{eff}} = 0$

   (a) This seems to describe $\nu = 1/2$ well [1].
   (b) For $3/2 = 1/2 + 1$ and $5/2 = 1/2 + 2$ one could have expected the same Fermi-liquid as they are nothing but the $1/2$ plateaus in higher (real) Landau levels. This is however not the case. One can imagine that in these cases, residual interactions beyond the mean-field descriptions lead to an instability of the Fermi surface.

8.2.3 Fluctuations around the mean-field solution

We want to take a step beyond the mean-field considerations. For this, let us expand $S[a, A]$ to second order in a.[5]
We could take the CF-CS action and expand to leading order around $\bar{a}$. However, we can do a much simpler thing. Let us just say that

$$S^{(2)}[a,A] = \frac{1}{2}\int dx^3 dx'^3\,(A+a)^{\mu}(x)K_{\mu\nu}(x-x')(A+a)^{\nu}(x') + \frac{\Theta}{4}S_{\text{CS}}[a]. \tag{8.74}$$

Without actually calculating $K_{\mu\nu}$, we try to constrain it from general considerations

[4] Check that the minimization of the action with respect to a_1 and a_2 only provides the continuity equation of the density and does not give any further constraints in the mean-field value of a.
[5] Why not in σ?

- $K_{\mu\nu}$ has to be gauge invariant.
- $K_{\mu\nu}(q)$ can be expanded in q.
- Via the Kubo formula (8.46), we know that $K_{\mu\nu}$ encodes the electromagnetic response.

We know that $\sigma_{11} = 0$ due to the gap for composite fermions. The transverse response, σ_{12}, however, can be non-zero. Recall, that

$$\sigma_{12} = -i \lim_{\mathbf{q}\to 0} \frac{1}{\omega} K_{12}(\omega, \mathbf{q}). \tag{8.75}$$

From this we conclude that we have

$$K_{\mu\nu} = -i\sigma_{12}^{(0)} \epsilon_{\mu\sigma\nu} q_\sigma. \tag{8.76}$$

Here $\sigma_{12}^{(0)}$ denotes the composite fermion mean-field value for the transverse response. Inserted into the expression for $S^{(2)}[a, A]$ we find

$$S^{(2)}[a, A] = \frac{\sigma_{12}^{(0)}}{2} S_{\text{CS}}[a + A] + \frac{\Theta}{4} S_{\text{CS}}[a]. \tag{8.77}$$

This effective action is clearly (i) gauge invariant, (ii) the lowest order expansion in q, and (iii) provides $K_{\mu\nu}$ that reproduces the electromagnetic of the effective theory. Actually, we would expect that

$$\frac{\delta^2 Z}{\delta A_\mu \delta A_\nu} \tag{8.78}$$

provides us with the desired response function. However, this is only true after we integrated out the fluctuations in a! What we need in the following is the formula valid for quadratic actions (Show!)

$$\int D[a]\, e^{c_1 S[a+b] + c_2 S[b]} = e^{\frac{1}{\frac{1}{c_1} + \frac{1}{c_2}} S[b]}. \tag{8.79}$$

Using this formula we obtain after integrating over the field a

$$S_{\text{eff}}[A] = \frac{1}{\frac{1}{\sigma_{12}^{0}} + \frac{2}{\Theta}} S_{\text{CS}}[A]. \tag{8.80}$$

And hence,

$$\sigma_{12} = \frac{e^2}{h} \frac{p}{1 + 2sp} \qquad s, p \in \mathbb{Z}. \tag{8.81}$$

References

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