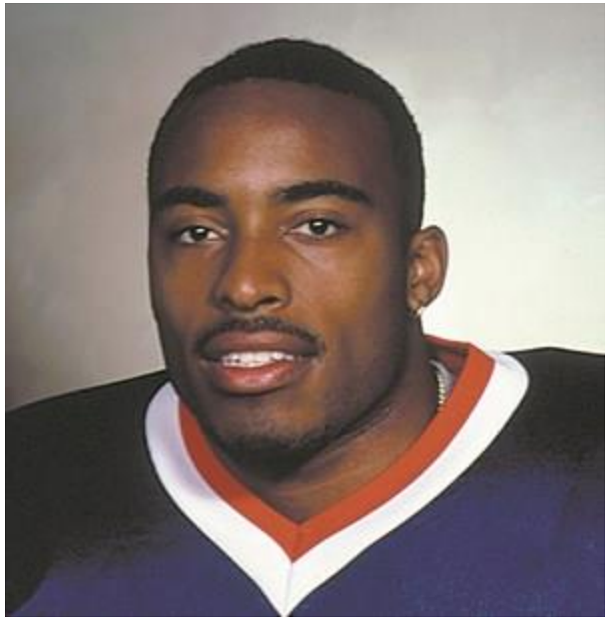
CS3102 Theory of Computation

www.cs.virginia.edu/~njb2b/cs3102

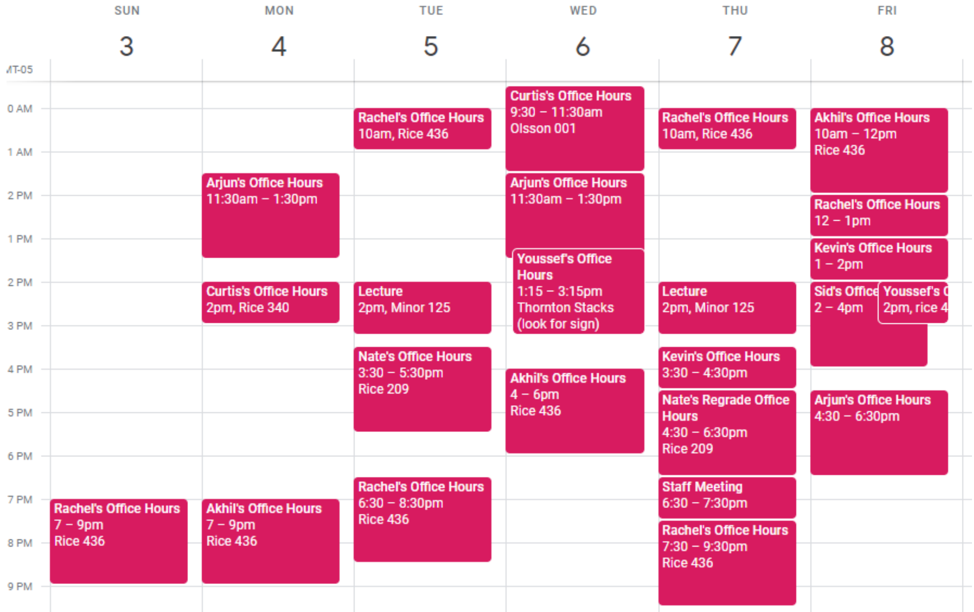


Charlottesville has one barber. This barber cuts everyone's hair, except for those people who cut their own hair. Who cuts the barber's hair?

Russell's Paradox

- Sets can contain other sets
- Maybe a set can contain itself?
- Let S be the set of all sets that do not contain themselves
  - $S = \{T | T \notin T\}$
- Is $S \in S$?

HW1



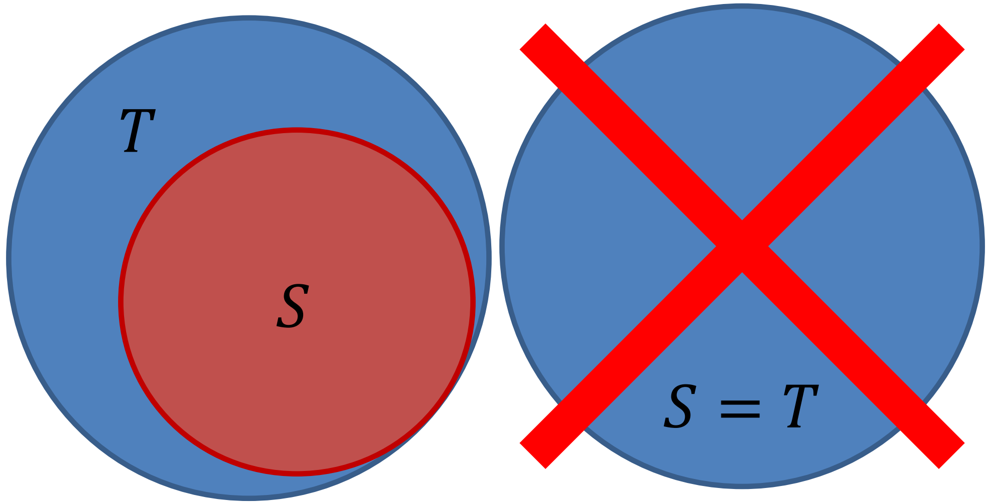
- Due Monday (2/4)
- Programming portion:
  - Write Java code for deciders
  - Use those deciders to produce the sets of strings they compute
- Written portion:
  - Proofs
  - ~~Sets and functions (next time)~~

Interested in a BACS Major?

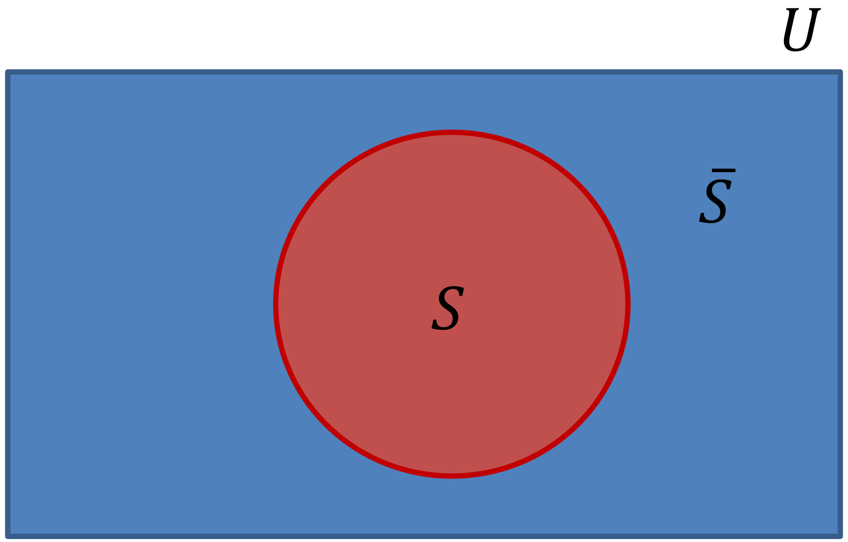
- Information session tonight
  - 5-6pm
  - Rice 130
- Application deadline:
  - Monday, February 18, 9:00am
- http://bit.ly/apply-bacs-s19

Set Operations

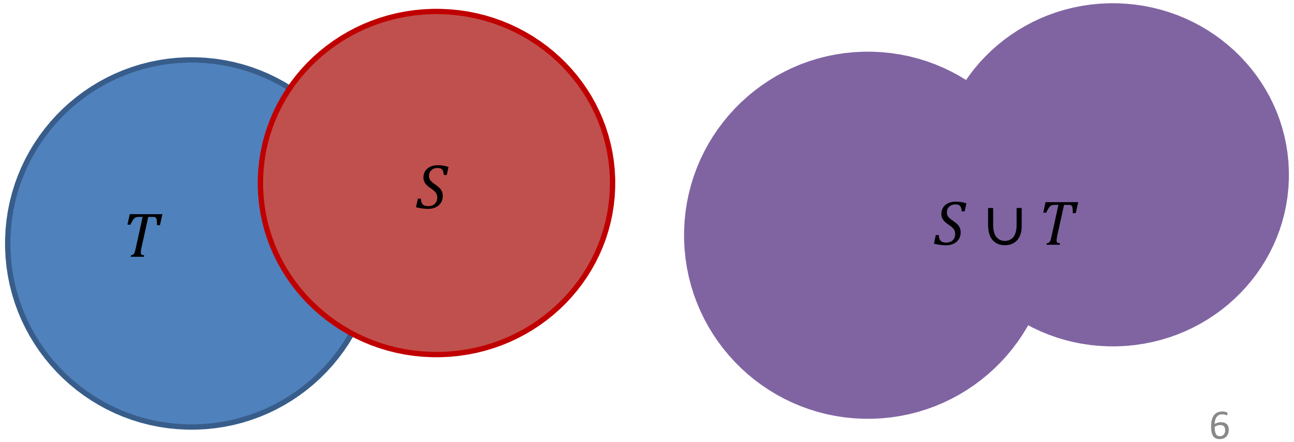
- $S \subset T$
  - S is a proper subset of T
  - Everything in S is also in T, and there's at least one thing in T missing from S
  - $|\{x|x \text{ is a real number such that } x^2 = 4\}| \subset \mathbb{Z}$
  - $\emptyset \subset T$ for any set T except for $\emptyset$
  - $S \subset T \rightarrow |S| \leq |T|$ (why?)
  - \subset in Latex
- 2^S
  - Powerset of S
  - The set of all subsets of S
  - $2^{\{0,1\}} = \{\emptyset, \{0\}, \{1\}, \{0,1\}\}$
  - $|2^S| = 2^{|S|}$ when S is finite
    - Complicated when infinite



Set Operations Cont.

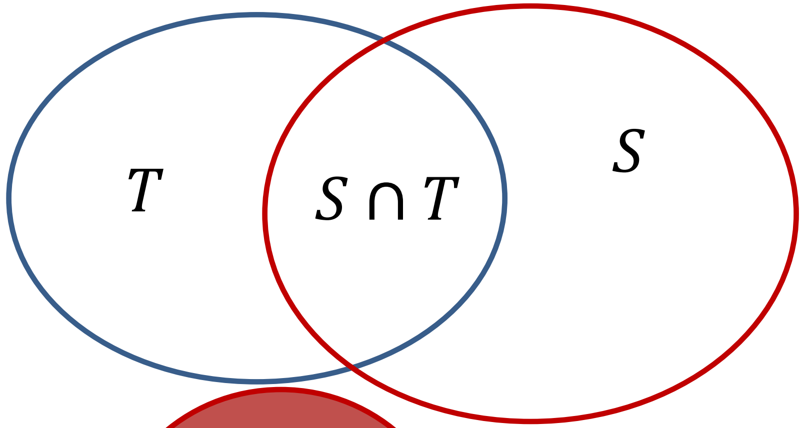


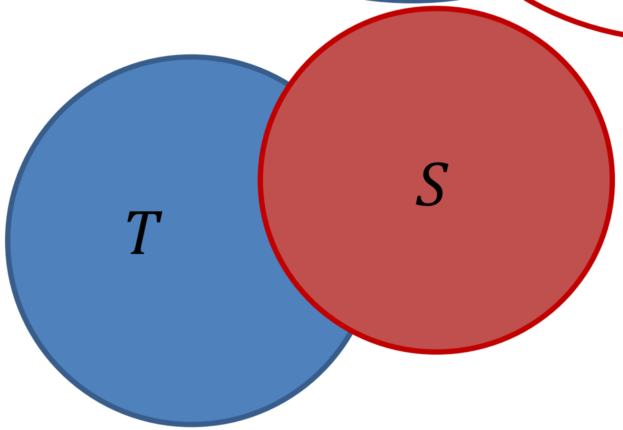
- S^c or $\bar{S}$
  - Complement of S
  - Everything in the "Universe" U that's not in S
    - For languages, the Universe is all strings over a given alphabet
  - What is $\{x|x \text{ is prime}\}^c$?
  - $|\bar{S}| = |U| - |S|$
- $S \cup T$
  - S union T
  - Everything that's in S or T (inclusive)
  - $\{00,01\} \cup \{00,11\} = \{00,01,11\}$
  - $|S \cup T| \leq |S| + |T|$
  - \cup in Latex

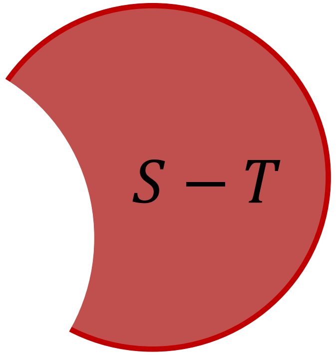


Set Operations Cont.

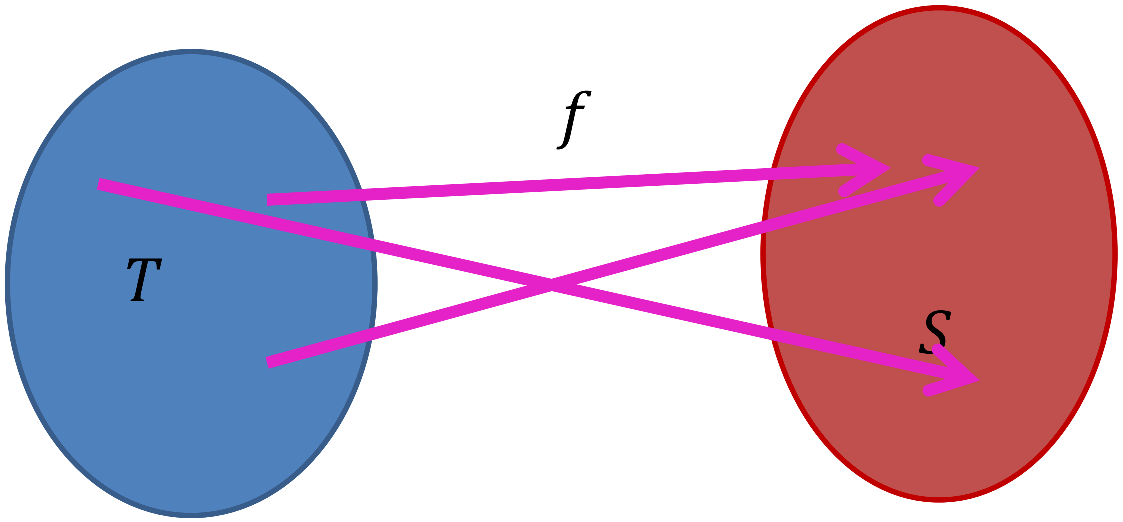
- $S \cap T$
  - S intersect T
  - Everything that's in both of S and T
  - $\{00,01\} \cap \{00,11\} = \{00\}$
  - $|S \cap T| = |S| + |T| - |S \cup T|$
  - \cap in Latex
- $S - T$
  - S minus T
  - Everying in S that's not in T
  - $S - T = S \cap \bar{T}$
- $S \times T$
  - S cross product T
  - Ordered pairs of something from S with something from T
  - $\{a, b\} \times \{a, c\} = \{(a, a), (a, c), (b, a), (b, c)\}$
  - $|S \times T| = |S| \cdot |T|$
  - \times in Latex







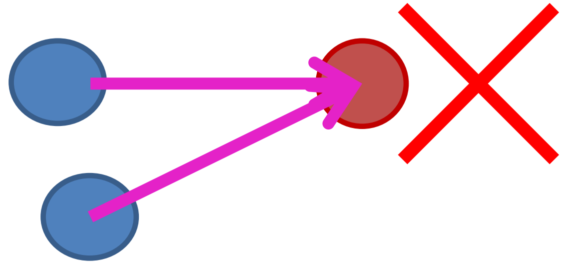
Functions



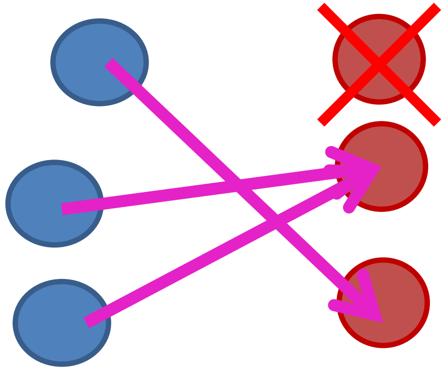
- Denoted $f: S \to T$
  - $s \in S, f(s) \in T$
  - S is the domain of f
  - T is the co-domain of f
  - Everything in S "maps to" exactly one thing in T
- Partial: Some things from S don't map to anything

Properties of Functions

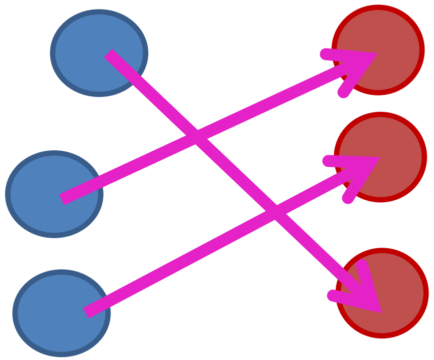
- $f: D \to C$
- One-to-one (1-1), Injective:
  - Every element in the codomain is mapped to by at most one element in the domain
  - Nothing has 2 incoming arrows
  - $\forall x, y \in D, x \neq y \to f(x) \neq f(y)$



- Onto, Surjective:
  - Every element in the codomain is mapped to by at least one element in the domain
  - Everything in Codomain has at least 1 incoming arrow
  - $\forall y \in C, \exists x \in D, f(x) = y$



- 1-1 Correspondence, Bijective:
  - Both injective and surjective
  - Every element in the codomain pairs with exactly one element in the domain
  - Everything in the domain has one outgoing arrow, everything in codomain has one incoming arrow



Functions and Set Cardinalities

- Consider sets S and T and function $f: S \to T$
- If f is 1-1, then $|S| \leq |T|$
  - Everything in T has at most one incoming arrow, but some may have none
- If f is onto, then $|S| \geq |T|$
  - Everything in T has at lease one incoming arrow, but some may have multiple
- If f is bijective, then $|S| = |T|$
  - Everything in T has exactly one incoming arrow, and everything in S has exactly on outgoing arrow
  - s and $f(s)$ are "monogamously paired"

Pigeonhole Principle

- If $|S| > |T|$ then $f: S \to T$ cannot be 1-1
  - S is a set of pigeons
  - T is a set of holes
  - There must be at least one hole shared by at least two pigeons

Set Cardinalities

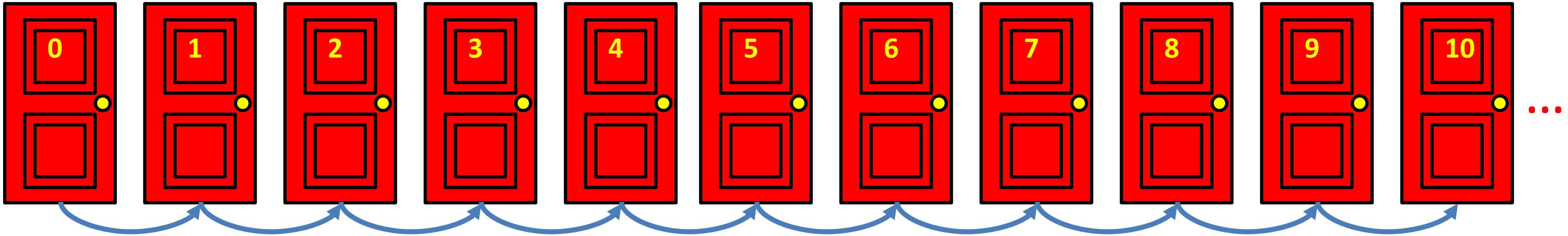
- How can I show that two sets are the same size?

- How Can I show that two sets are different sizes?

- What if the sets are infinite?

Hilbert’s Hotel

Conclusion: $|\mathbb{N}| = |\mathbb{Z}_+|$



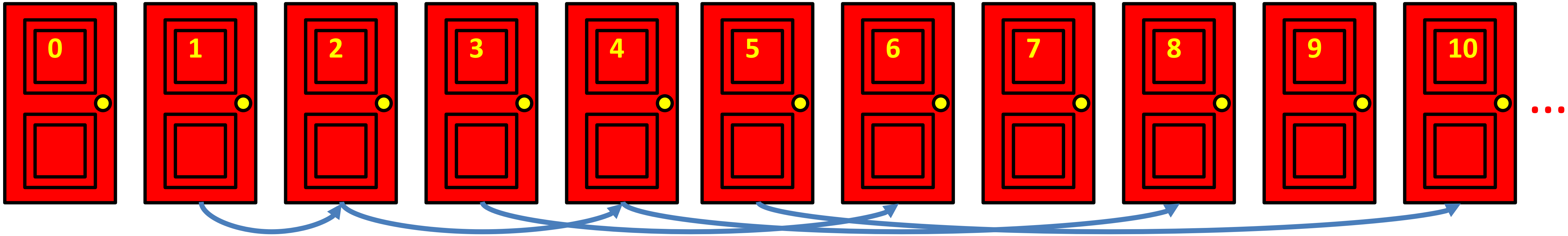
There are an infinite number of rooms,
each room is occupied.
A new guest arrives,
is there space?

As a Bijection

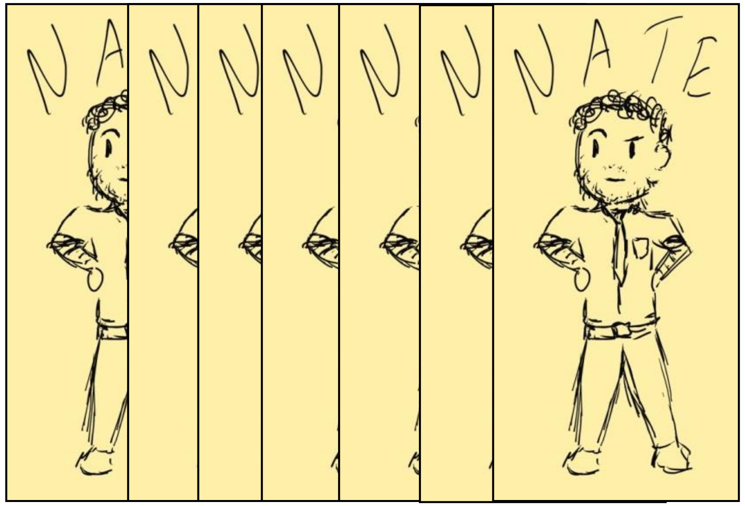
$$f: \mathbb{N} \leftrightarrow \mathbb{Z}_+$$
$$f(x) = x + 1$$

Hilbert's Hotel

Conclusion: $|\mathbb{N}| = |\{x \in \mathbb{N} | x \text{ is even}\}|$



There are an infinite number of rooms,
each room is occupied.
An infinite number of new guest arrive
is there space?



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As a Bijection

$$f: \mathbb{N} \leftrightarrow \{x \in \mathbb{N} | x \text{ is even}\}$$
$$f(x) = 2x$$

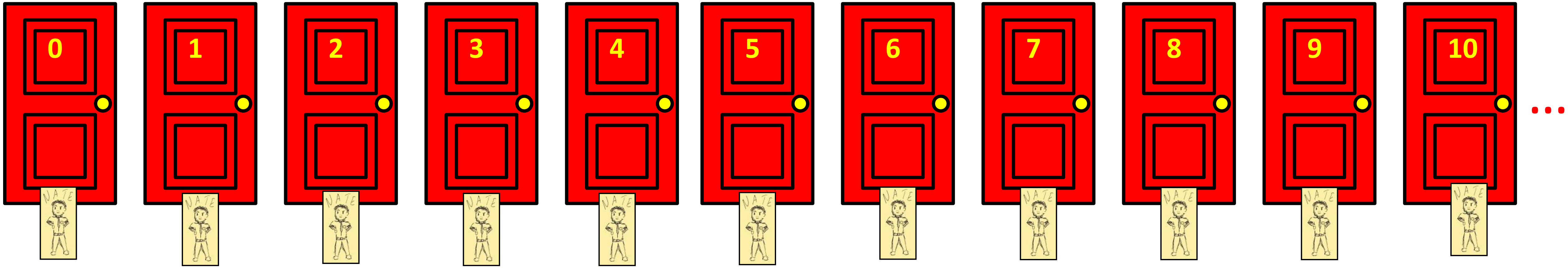
A Similar Bijection

$$f: \mathbb{Z} \leftrightarrow \mathbb{N}$$

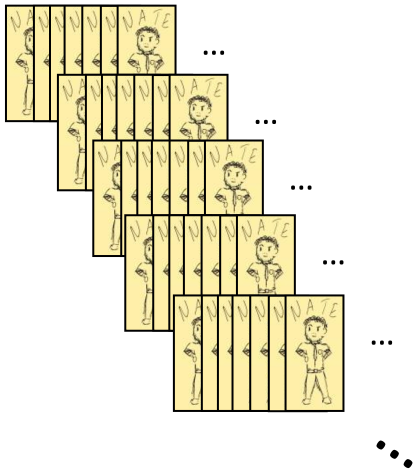
Positive values (and 0) are people already in the rooms
Negative values are the people who just arrived

$$f(x) = \begin{cases} 2x \text{ if } x \geq 0 \\ 2|x| + 1 \text{ otherwise} \end{cases}$$

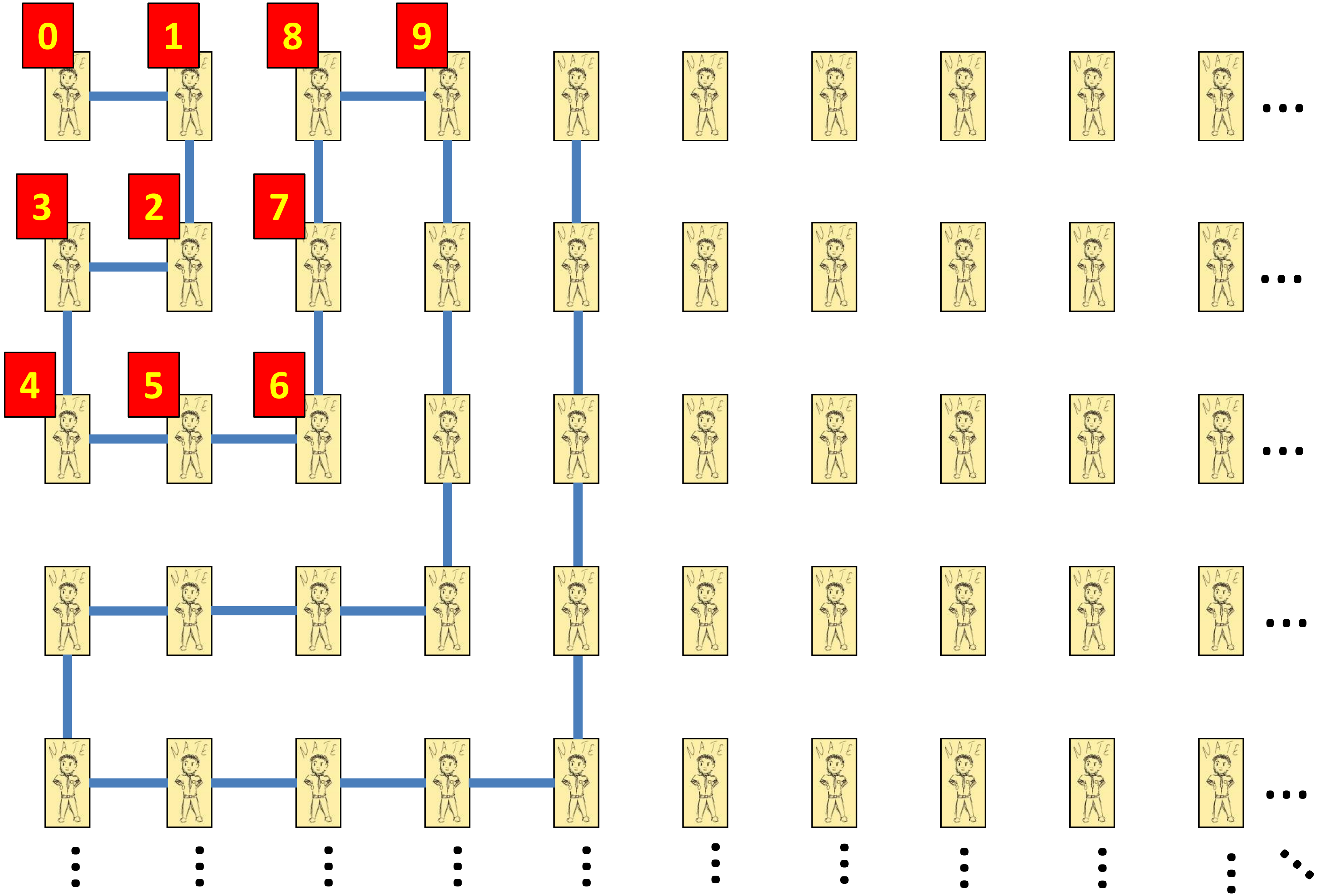
Hilbert’s Hotel



There are an infinite number of rooms,
each room is occupied.
An infinite number of infinitely-sized
families arrive
is there space?

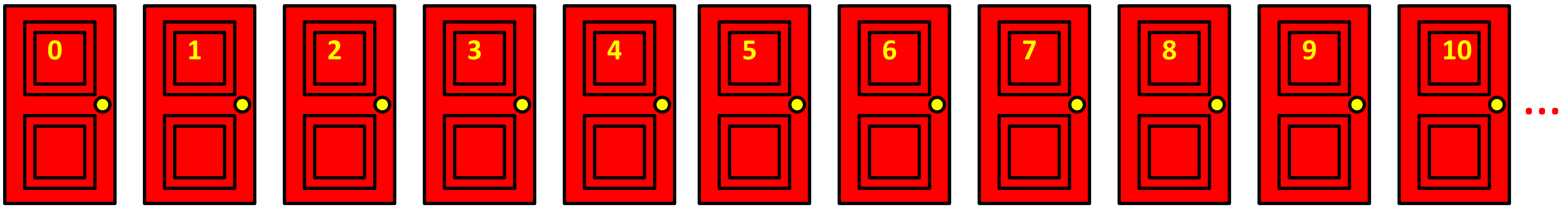


Dovetailing

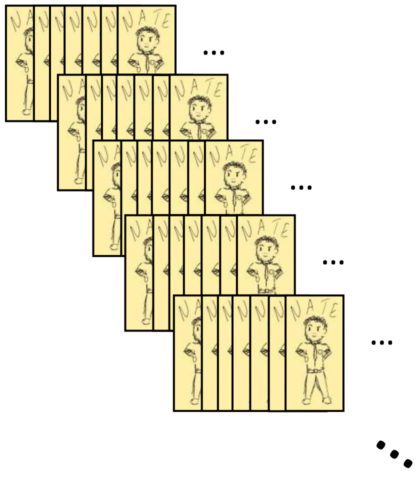


Hilbert's Hotel

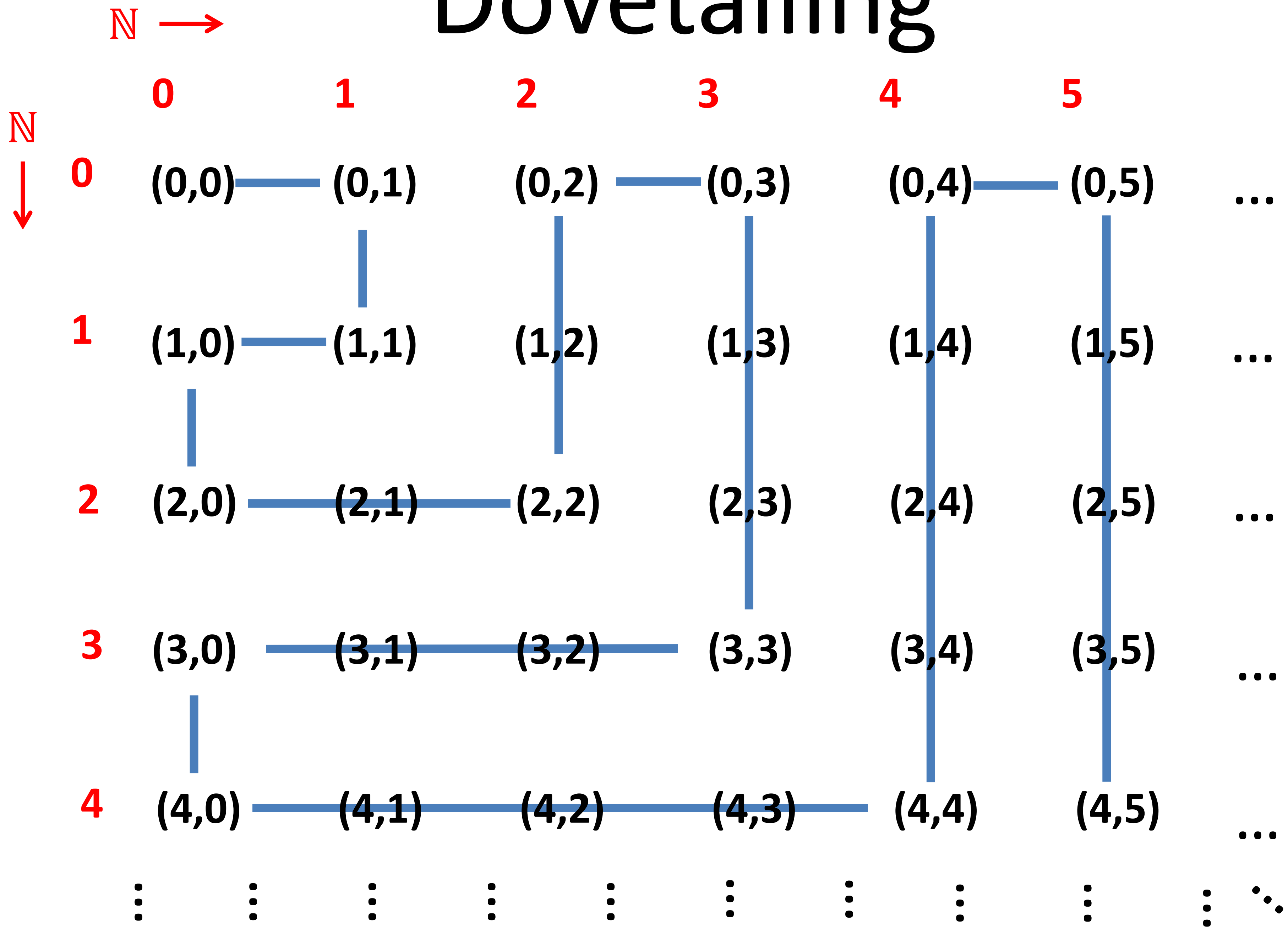
Conclusion: $|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$



There are an infinite number of rooms,
each room is occupied.
An infinite number of infinitely-sized families arrive
is there space?



Dovetailing



$|\mathbb{Q}| = |\mathbb{N}|$

$\mathbb{N} \downarrow$ / $\mathbb{Z}_+ \rightarrow$	1	2	3	4	5	6	
0	$\frac{0}{1}$	$\frac{0}{2}$	$\frac{0}{3}$	$\frac{0}{4}$	$\frac{0}{5}$	$\frac{0}{6}$	$\cdots$
1	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\cdots$
2	$\frac{2}{1}$	$\frac{2}{2}$	$\frac{2}{3}$	$\frac{2}{4}$	$\frac{2}{5}$	$\frac{2}{6}$	$\cdots$
3	$\frac{3}{1}$	$\frac{3}{2}$	$\frac{3}{3}$	$\frac{3}{4}$	$\frac{3}{5}$	$\frac{3}{6}$	$\cdots$
4	$\frac{4}{1}$	$\frac{4}{2}$	$\frac{4}{3}$	$\frac{4}{4}$	$\frac{4}{5}$	$\frac{4}{6}$	$\cdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Countability

- As set is countable if:
  - It is finite
  - It has a bijection with the natural numbers
    - (countably infinite)
- Notation
  - $|\mathbb{N}| = \aleph_0$
    - Aleph-naught
- Is the set of all strings over alphabet $\{a, b\}$ countable?
  - What about other alphabets?

The set of all strings is countable

Length 0	ε 0			**1 string**					
Length 1	a 1	b 2			**2 strings**				
Length 2	aa 3	ab 4	ba 5	bb 6			**4 strings**		
Length 3	aaa 7	aab 8	aba 9	abb 10	baa 11	bab 12	bba 13	bbb 14	**8 strings**
Length 4	$aaaa$	$aaab$	$aaba$	$aabb$	$abaa$	$abab$	$abba$	$abbb$ …	**16 strings**
Length 5	$aaaaa$	$aaaab$	$aaaba$	$aaabb$ …					**32 strings**
⋮									⋮

Important: A countable union of countable sets is countable

Some more countable things

- Any language ever!
- Number of possible Java Programs
- The empty set
- The number of words in the English language
- Number of possible novels