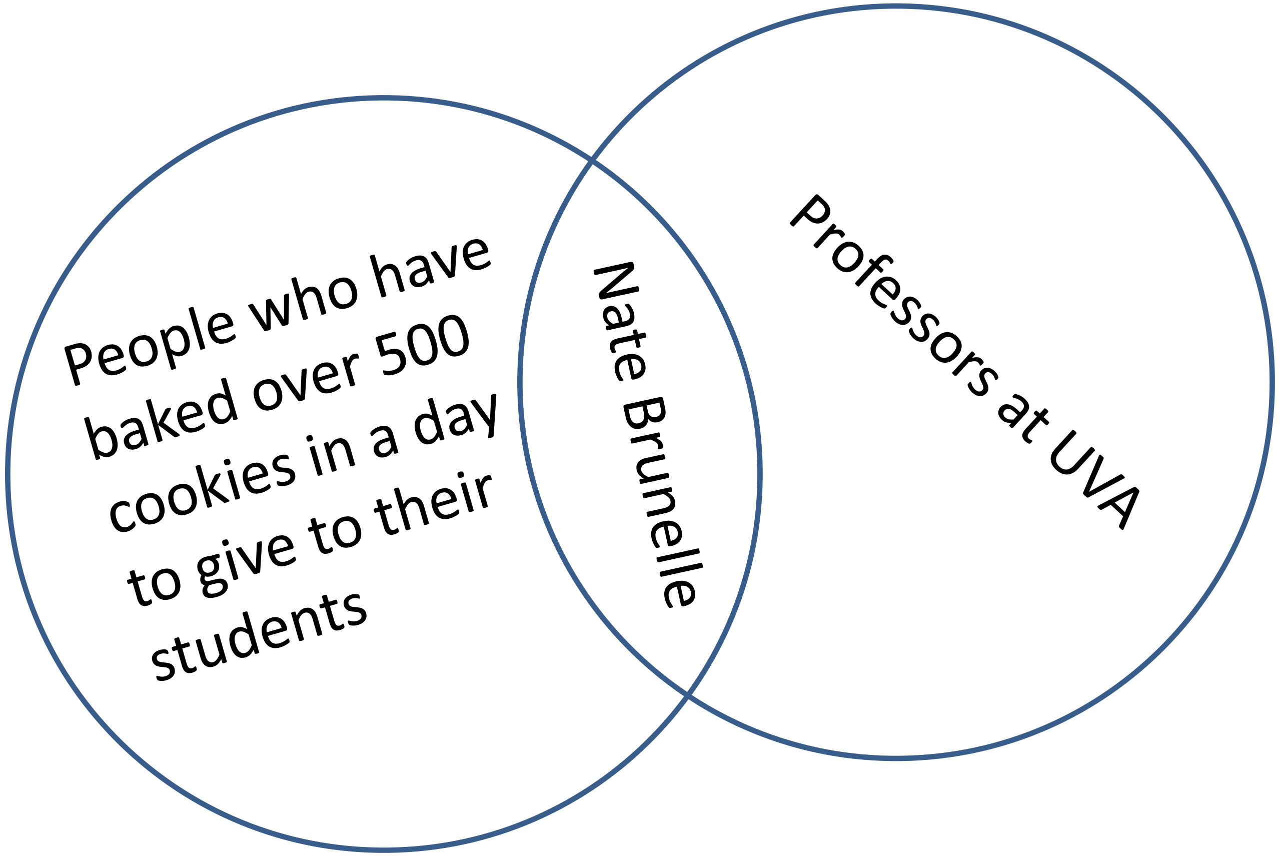
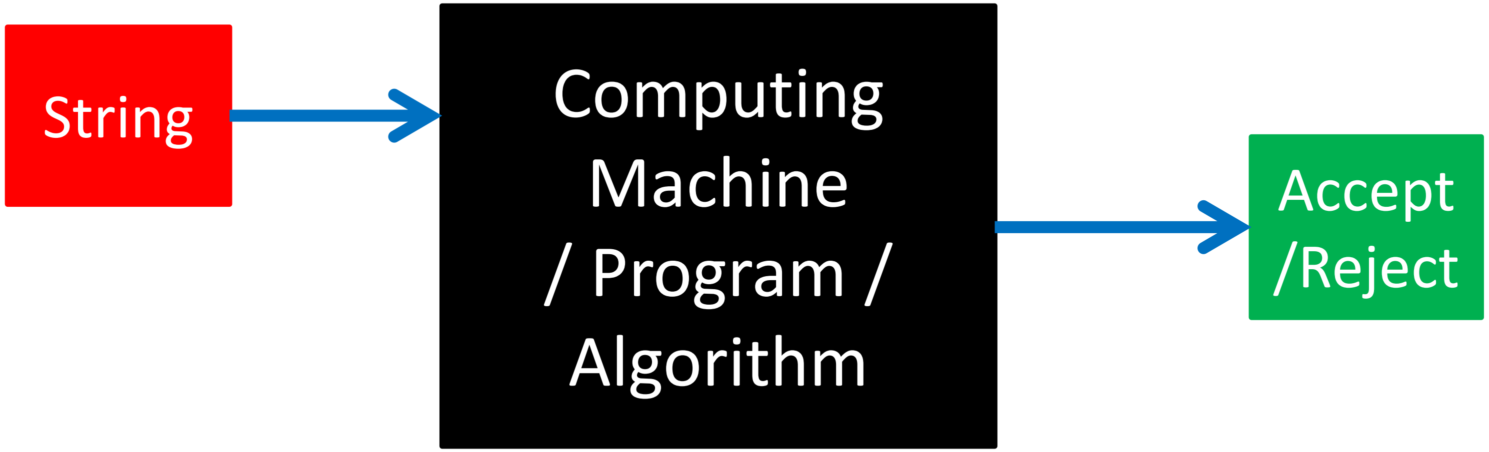
CS3102 Theory of Computation

www.cs.virginia.edu/~njb2b/cs3102



Computability

- A Decider's "behavior" is the set of strings it accepts
- A set of strings is **computable** by Java if we can find a program which accepts exactly those strings



What goes in here?
For now, Java.

Computations as Sets of Strings

- Things a computer can do:
  - Plays a movie
    - All strings representing color, pixel, frame triples for the movie.
  - Searches the web
    - All strings representing url, score pairs for the query
  - Removes an appendix
    - All strings representing safe places to cut with the laser
  - Finds digits of pi
    - All strings representing a prefix of the decimal expansion of π

∞

Languages

- Natural
  - English
  - Vocabulary and grammar
- Programming
  - Java
  - Commands and Syntax
- Formal
  - A set of strings
  - Strings only, no rules for assembly

Example languages

- Language of names of members of The Beatles
  - $\{John, Paul, George, Ringo\}$
- Language of all bit strings representing odd integers
  - $\{1,01,11,001,011,101,111, \ldots\}$
- Language of all prefixes of π
  - $\{3, 3.1, 3.14, 3.141, 3.1415,3.14159,3.141592, \ldots\}$

Strings

- A language is a set of strings
  - May be infinite
- What is a string?
  - A finite sequence of characters
- Alphabet:
  - The set of characters available
  - Must be finite

Alphabet examples

- $\{a, b, c, d, e, f, g, h, \ldots\}$
  - Sesame Street alphabet
- $\{0,1\}$
  - Binary Alphabet
- $\{0,1, \ldots, 9\}$
  - Decimal alphabet
- Most alphabets we will talk about in this class are small
  - $\{0,1\}, \{a, b\}, \{a\}, \{a, b, c\}$

Operations on Strings

- Length
  - $|s|$= Number of characters in the string s
  - $|Ringo| = 5$
- Concatenation
  - $s \cdot t = st =$ string which has all of the characters from s followed by all of the characters from t
  - $John \cdot Paul = JohnPaul$
  - $|s \cdot t| = |s| + |t|$
- Exponentiation
  - s^k =The string created by concatenation s with itself k times
  - $(George)^5 = GeorgeGeorge\ George\ George\ George$
  - $\left|s^k\right| = |s| \cdot k$

Empty String (“”)

- Notation for this class: ε
  - \varepsilon in Latex
- $|\varepsilon| = 0$
- $s \cdot \varepsilon = s$
- $\varepsilon^k = \varepsilon$

Operations on Languages

- Any operation on Sets
  - Union
  - Intersection
  - Complementation
- Concatenation

Set review

- Listing out the elements of a set works well for sets that are small and finite. What about larger sets?
- Set Builder Notation!

$$S = \{x \in A | x \text{ is blue}\}$$

The set of all x in A

Vertical Bar is read "such that"

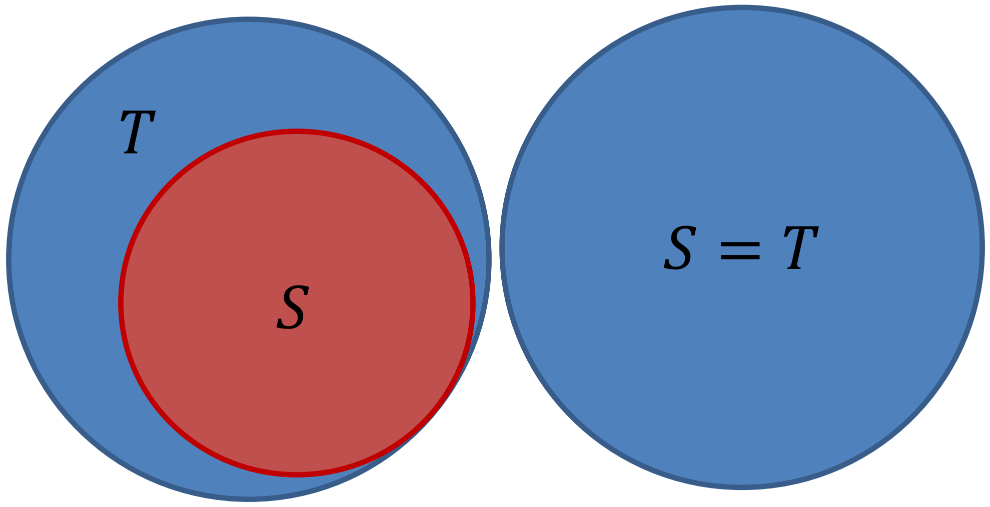
Property (or properties) of x that must be met in order to be an element of S

Examples

- **List the members of these sets**
  - $\{x|x \text{ is a real number such that } x^2 = 4\}$
  - $\{x|x \text{ is the square of an integer and } x < 100\}$
  - $\{x|x \text{ is the integer such that } x^2 = 2\}$
- **Use set builder notation to give a description of each of these sets**
  - $\{0, 3, 6, 9, 12, \ldots\}$
  - $\{2, 3, 5, 7, 11, 13, \ldots\}$

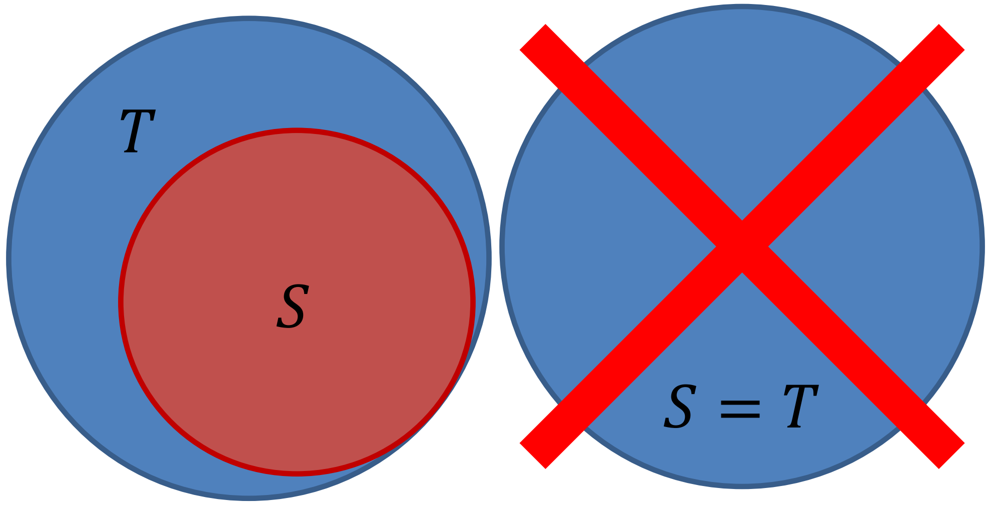
Set Operations

- $|S|$
  - Cardinality (size) of S
  - May be finite
  - $|\{x|x \text{ is a real number such that } x^2 = 4\}| = 2$
  - $|\{x|x \text{ is prime}\}| = \infty$ (more on this later)
  - $|\emptyset| = 0$
- $S \subseteq T$
  - S is a subset of T
  - Everything in S is also in T
  - $|\{x|x \text{ is a real number such that } x^2 = 4\}| \subseteq \mathbb{Z}$
  - $\emptyset \subseteq T$ for any set T
  - $S \subseteq T \rightarrow |S| \leq |T|$
  - \subseteq in Latex



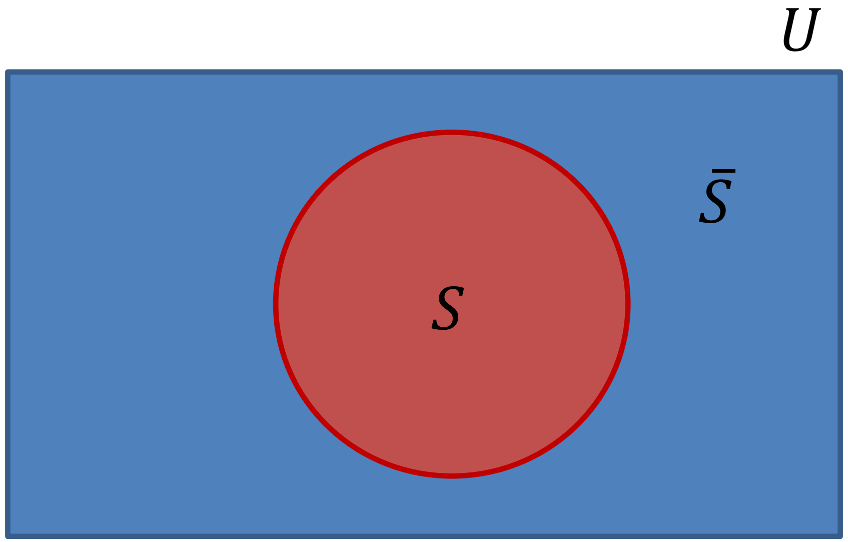
Set Operations

- $S \subset T$
  - S is a proper subset of T
  - Everything in S is also in T, and there's at least one thing in T missing from S
  - $|\{x|x \text{ is a real number such that } x^2 = 4\}| \subset \mathbb{Z}$
  - $\emptyset \subset T$ for any set T except for $\emptyset$
  - $S \subset T \rightarrow |S| \leq |T|$ (why?)
  - \subset in Latex
- 2^S
  - Powerset of S
  - The set of all subsets of S
  - $2^{\{0,1\}} = \{\emptyset, \{0\}, \{1\}, \{0,1\}\}$
  - $|2^S| = 2^{|S|}$ when S is finite
    - Complicated when infinite

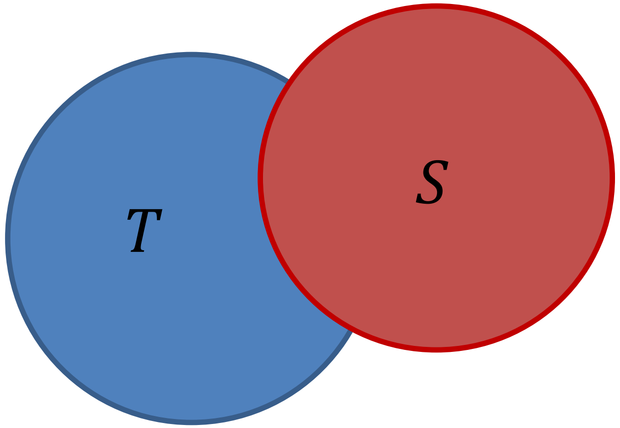


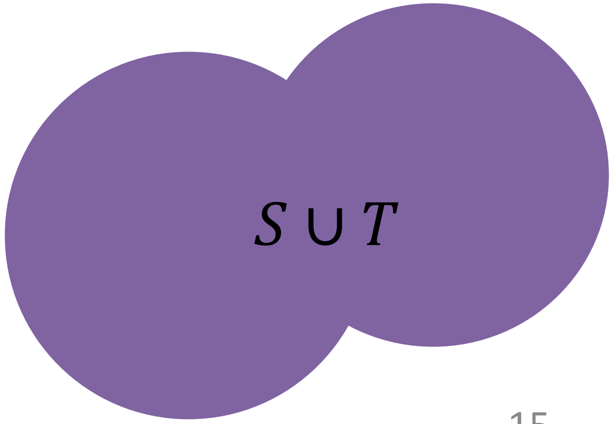
Set Operations Cont.

- S^c or $\bar{S}$
  - Complement of S
  - Everything in the “Universe” U that’s not in S
    - For languages, the Universe is all strings over a given alphabet
  - What is $\{x|x \text{ is prime}\}^c$?
  - $|\bar{S}| = |U| - S$



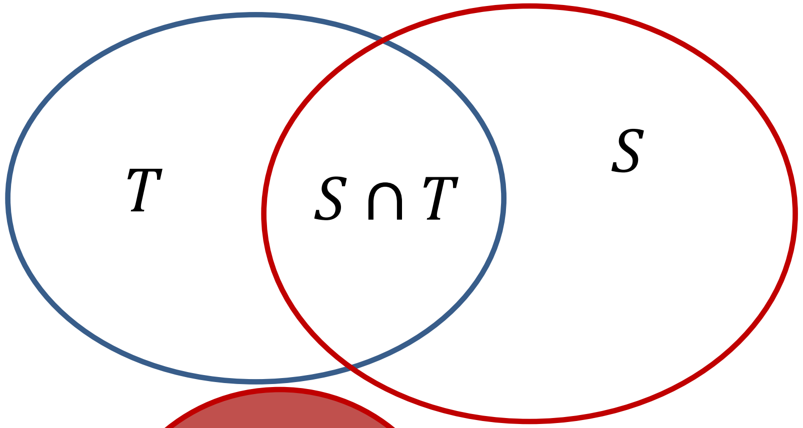
- $S \cup T$
  - S union T
  - Everything that’s in S or T (inclusive)
  - $\{00{,}01\} \cup \{00{,}11\} = \{00{,}01{,}11\}$
  - $|S \cup T| \leq |S| + |T|$
  - \cup in Latex

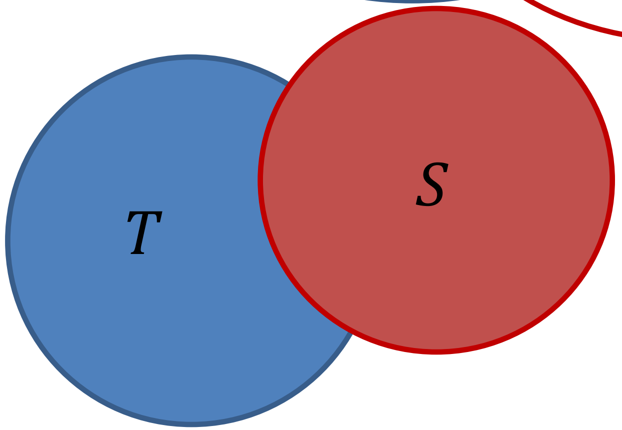


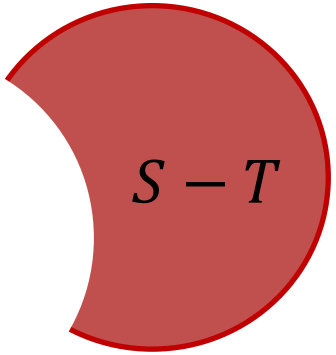


Set Operations Cont.

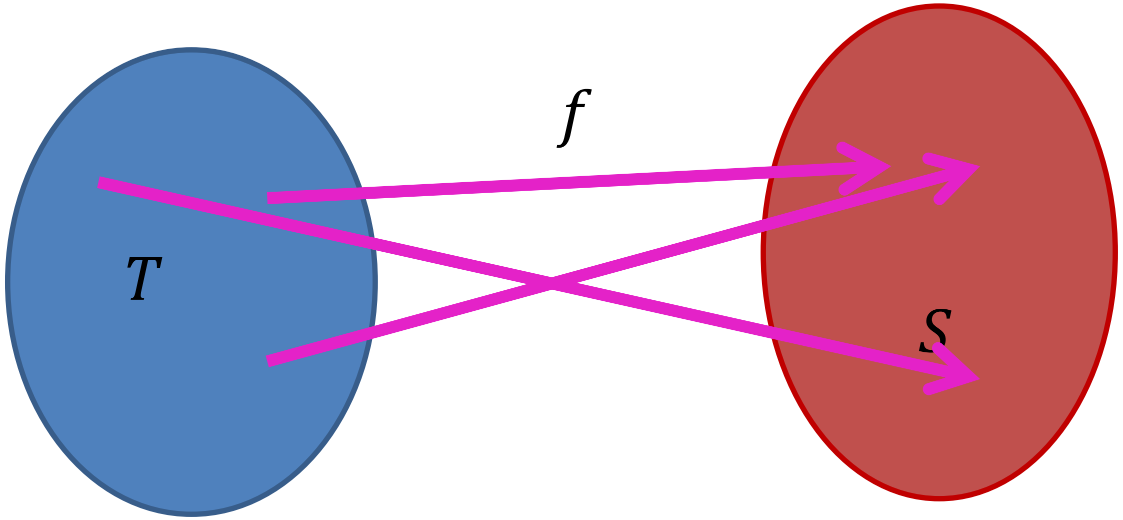
- $S \cap T$
  - S intersect T
  - Everything that's in both of S and T
  - $\{00,01\} \cap \{00,11\} = \{00\}$
  - $|S \cap T| = |S| + |T| - |S \cup T|$
  - \cap in Latex
- $S - T$
  - S minus T
  - Everying in S that's not in T
  - $S - T = S \cap \bar{T}$
- $S \times T$
  - S cross product T
  - Ordered pairs of something from S with something from T
  - $\{a, b\} \times \{a, c\} = \{(a, a), (a, c), (b, a), (b, c)\}$
  - $|S \times T| = |S| \cdot |T|$
  - \times in Latex







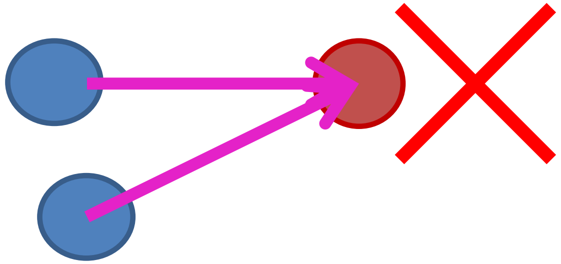
Functions



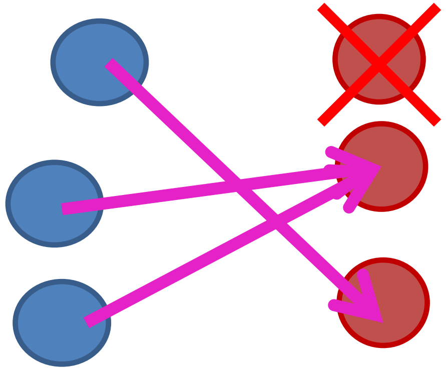
- Denoted $f: S \to T$
  - $s \in S, f(s) \in T$
  - S is the domain of f
  - T is the co-domain of f
  - Everything in S "maps to" exactly one thing in T
- Partial: Some things from S don't map to anything

Properties of Functions

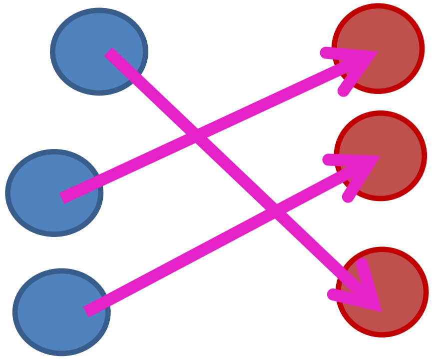
- $f: D \to C$
- One-to-one (1-1), Injective:
  - Every element in the codomain is mapped to by at most one element in the domain
  - Nothing has 2 incoming arrows
  - $\forall x, y \in D, x \neq y \to f(x) \neq f(y)$



- Onto, Surjective:
  - Every element in the codomain is mapped to by at least one element in the domain
  - Everything in Codomain has at least 1 incoming arrow
  - $\forall y \in C, \exists x \in D, f(x) = y$



- 1-1 Correspondence, Bijective:
  - Both injective and surjective
  - Every element in the codomain pairs with exactly one element in the domain
  - Everything in the domain has one outgoing arrow, everything in codomain has one incoming arrow



Functions and Set Cardinalities

- Consider sets S and T and function $f: S \to T$
- If f is 1-1, then $|S| \leq |T|$
  - Everything in T has at most one incoming arrow, but some may have none
- If f is onto, then $|S| \geq |T|$
  - Everything in T has at lease one incoming arrow, but some may have multiple
- If f is bijective, then $|S| = |T|$
  - Everything in T has exactly one incoming arrow, and everything in S has exactly on outgoing arrow
  - s and $f(s)$ are "monogamously paired"

Set Cardinalities

- How can I show that two sets are the same size?
- How Can I show that two sets are different sizes?
- What if the sets are infinite?

Brain Teaser

- Sets can contain other sets
- Maybe a set can contain itself?
- Let S be the set of all sets that do not contain themselves
  - $S = \{T | T \notin T\}$
- Is $S \in S$?

HW1

- Due Monday (2/4)
- Programming portion:
  - Write Java code for deciders
  - Use those deciders to produce the sets of strings they compute
- Written portion:
  - Proofs
  - ~~Sets and functions (next time)~~