CS3102 Theory of Computation

www.cs.virginia.edu/~njb2b/cs3102

When your professor tries to meme

How to prove $p \rightarrow q$

- Direct Proof $(p \rightarrow q)$
  - Start with premise
  - Repeatedly apply definitions, equivalences, and inferences
  - End with conclusion
- Indirect Proof $(\neg q \rightarrow \neg p)$ AKA, proof by contrapositive
  - Start with negation of the conclusion
  - Repeatedly apply definitions, equivalences, and inferences
  - End with negation of the premise
- Proof by Contradiction $\left(\neg(p \wedge \neg q)\right)$
  - Start with $p \wedge \neg q$
  - Repeatedly apply definitions, equivalences, and inferences
  - End with False

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Proofs Techniques Cont.

- Construction
  - Shows: $\exists x \in S, P(x)$
  - Give/Build an example which works
- Proof by Cases
  - Shows: $\forall x \in S, P(x)$
  - Show $\left(\forall x \in S_1, P(x)\right) \wedge \left(\forall x \in S_2, P(x)\right) \wedge (S_1 \cup S_2 = S)$
- Induction
  - Shows: $\forall x \in \mathbb{N}, P(x)$
  - Show $P(0)$, then show $P(k) \rightarrow P(k+1)$ for $k \geq 0$

Proof: n^2 is even $\leftrightarrow n$ is even

- How would we prove this?
- Recall: $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
  - Suffice to show both of:
    - $p \rightarrow q$
    - $q \rightarrow p$

Proof: if n^2 is even $\leftarrow n$ is even

- **Direct Proof :**
  - Start with premise
  - Repeatedly apply definitions, equivalences, and inferences
  - End with conclusion

<u>Show</u> that $\boldsymbol{n = 2k}$ implies $\boldsymbol{n^2 = 2k'}$

<u>Start with</u> $n = 2k$

1. $n = 2k$
2. $n^2 = (2k)(2k)$
3. $n^2 = 4k^2$
4. $n^2 = 2(2k^2)$
5. Let $k' = 2k^2$, n^2 is even

Proof: if n^2 is even $\rightarrow n$ is even

- **Indirect Proof (proof by contraposive):**
  - Start with negation of the conclusion
  - Repeatedly apply definitions, equivalences, and inferences
  - End with negation of the premise

<u>Show</u> that $\boldsymbol{n = 2k + 1}$ implies $\boldsymbol{n^2 = 2k' + 1}$

<u>Start with</u> $n = 2k + 1$

1. $n = 2k + 1$

2. $n^2 = (2k + 1)(2k + 1)$

3. $n^2 = 4k^2 + 4k + 1$

4. $n^2 = 2(2k^2 + 2k) + 1$

5. Let $k' = 2k^2 + 2k$, n^2 is odd

Proof: √2 is not rational

- **Proof by contradiction**
  - Start with $p \wedge \neg q$
  - Repeatedly apply definitions, equivalences, and inferences
  - End with False

<u>Show</u> that $\boldsymbol{\left(\frac{a}{b}\right)^2 = 2}$ and $\boldsymbol{a, b \in \mathbb{N}}$ is impossible

<u>Start with</u>$\left(\frac{a}{b}\right)^2 = 2 \wedge a, b \in \mathbb{N}n = 2k + 1$

1. Assume toward reaching a contradiction that $\left(\frac{a}{b}\right)^2 = 2$ and a, b are integers, and $\frac{a}{b}$ is in simplest terms (i.e. $\gcd(a, b) = 1$)
2. Since a^2 is even, it must be that a is even, so a^2 is divisible by 4
3. For $\left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2} = 2$, it must then be that b^2 is even, meaning b is also even
4. Since a, b are both even, $\gcd(a, b) \geq 2$, which is a contradiction

Proof: For any integer x, there is a power of 3 larger than x

- **Proof by construction:**
  - Give/Build an example which works

<u>Show</u>: I can use $\boldsymbol{x}$ to build a power of 3 larger than $\boldsymbol{x}$

1. Note that $3^{\lceil \log_3 x \rceil} \geq x$
2. So $3 \cdot 3^{\lceil \log_3 x \rceil} > x$
3. $3 \cdot 3^{\lceil \log_3 x \rceil}$ is an integer because $\lceil y \rceil$ is always an integer (by definition of ceiling).

Proof: $n^4 - 4n^2$ is divisible by 3

- **Proof by Cases:**
  1. Enumerate all possible circumstances for the given
  2. Show that each circumstance results in the conclusion

1. $n^4 - 4n^2$
2. $n^2(n^2 - 4)$
3. $n \cdot n(n-2)(n+2)$
4. **Cases:** $\boldsymbol{n \equiv 0 \text{ mod } 3, n \equiv 1\ mod\ 3, n \equiv 2\ mod\ 3}$
   1. $n \equiv 0 \text{ mod } 3$: $3|n$, thus $3|(n^4 - 4n^2)$
   2. $n \equiv 1 \text{ mod } 3$: $3|(n+2)$, thus $3|(n^4 - 4n^2)$
   3. $n \equiv 2 \text{ mod } 3$: $3|(n-2)$, thus $3|(n^4 - 4n^2)$

When writing a proof

1. Mention method of proof you’re using
2. List everything given (all assumptions you’re making)
3. State formally what you are going to prove
4. State your proof as clearly and concisely as you can
   – providing accompanying intuition is often helpful)


Twice an odd number is sometimes odd
CHANGE MY MIND


Some even numbers
are the products of
two odds
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A number can
be both even
and odd
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There is a
largest number
CHANGE MY MIND


$\min(x,y)\cdot\max(x,y)$
$\neq x\cdot y$
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Mathematical Induction

- We what to show $\forall x \in \mathbb{N}, P(x)$ for some proposition $P(x)$
  - E.g. $\forall x \in \mathbb{N}$, Domino x will fall
- Base Case: First show $P(0)$
  - Show that Domino 0 (the first domino) will fall
- Inductive Hypothesis: Assume $P(k)$ for an arbitrary $k \geq 0$.
  - Assume arbitrary domino k will fall
- Inductive Step: Show $P(k) \rightarrow P(k+1)$
  - Show that when arbitrary domino k falls, then the next domino $k+1$ will fall.

Mathematical Induction

- Base Case: The first domino will fall
- Inductive step: If any domino k falls, then domino $k+1$ will fall

Falls because of base Case

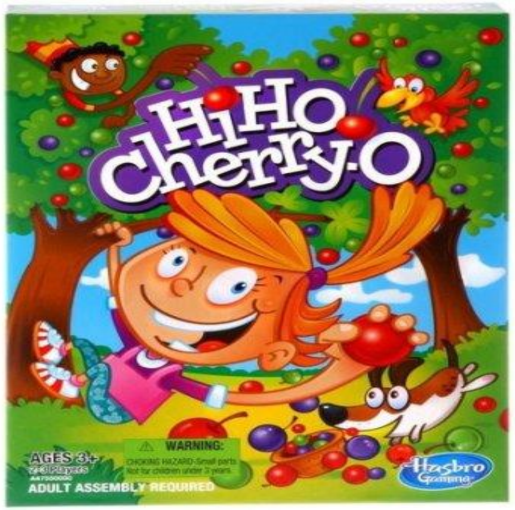
Falls because of Inductive Step

Falls because of Inductive Step

Falls because of Inductive Step

All dominos fall!!

Hi-Ho-Cherry-O



- Each Player takes turns removing 1, 2, 3, or 4 cherries from play
- First player unable to pick a cherry loses


It's better to
go first
CHANGE MY MIND

Player 2 always wins

- If number of cherries is a multiple of 5, then player 2 always wins.
  - $\forall x \in \mathbb{N}$, Player 2 wins for $5x$ cherries
- Base Case: when $x = 0$, player 2 wins
  - Proof: when there are 0 cherries, player 1 has none to take, so player 2 wins
- Inductive Hypothesis: Assume player 2 wins when there are $5k$ cherries
- Inductive Step: Show that if player 2 wins with $5k$ cherries, then player 2 wins with $5(k+1)$ cherries
  - Proof: By construction: If player 1 takes n cherries, where $1 \leq n \leq 4$, then player 2 can take $5 - n$ cherries. If we had $5(k+1)$ cherries, then we now have $5(k+1) - n - (5 - n) = 5k$ cherries. Therefore player 2 wins by the inductive hypothesis

Problems with induction

- Useless for helping you to find the answer
- You have to know the answer first
- Does not provide insights into why something is true
- Does not give any clues on how to correct if you’re wrong