Derivation of the Normal Cycles Metric

Derivation of the Current Shape Norm

Let $\mathbb{S} \subset \mathbb{R}^n$ be some closed, oriented surface in $\mathbb{R}^n$ and $\boldsymbol{v}(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ be some vector field (we consider $n = 2$ or $n = 3$). We define the *"current"* representation of the shape $\mathbb{S}$ as the linear functional $\mu_{\mathbb{S}}\big(\boldsymbol{v}(\cdot)\big)$ which we specify as the surface integral of the vector field

$$\mu_{\mathbb{S}}\big(\boldsymbol{v}(\cdot)\big) := \int_{\mathbb{S}} \langle \boldsymbol{v}(\boldsymbol{x}), \hat{\boldsymbol{n}}(\boldsymbol{x}) \rangle \, \mathrm{ds}(\boldsymbol{x}) \in \mathbb{R} \,, \tag{1}$$

where $\hat{\boldsymbol{n}}(\boldsymbol{x})$ denotes the unit normal vector and $\mathrm{ds}(\boldsymbol{x})$ the elemental area of the surface, both at $\boldsymbol{x} \in \mathbb{S}$, and $\langle \cdot, \cdot \rangle$ is the standard Euclidean inner (dot) product. Importantly, this representation is sensitive to the shape $\mathbb{S}$ through the domain of the integration (i.e. we integrate over a particular surface).

We restrict the vector field to come from some space of *smooth* vector fields $\boldsymbol{v}(\cdot) \in \mathbb{V}$. If we specify $\mathbb{V}$ as a Reproducing Kernel Hilbert Space (RKHS), with some kernel $\kappa(\boldsymbol{x}, \boldsymbol{x}')$, then we may place a norm $\|\boldsymbol{v}(\cdot)\|_{\mathbb{V}}$ on the vector field $\boldsymbol{v}(\cdot) = \int \kappa(\boldsymbol{x}, \cdot) \, \mathrm{d}\mu_{\mathbb{S}}(\boldsymbol{x}) \in \mathbb{V}$ defined by the Hilbert space inner product

$$\|\boldsymbol{v}(\cdot)\|^2_{\mathbb{V}} = \langle \boldsymbol{v}(\cdot), \boldsymbol{v}(\cdot) \rangle_{\mathbb{V}} := \iint \kappa(\boldsymbol{x}, \boldsymbol{x}') \; \mathrm{d}\mu_{\mathbb{S}}(\boldsymbol{x}) \; \mathrm{d}\mu_{\mathbb{S}}(\boldsymbol{x}') \,. \tag{2}$$

From this, we may consider a dual norm, defined over the space of shapes, for the metric as

$$\|\, \mu(\mathbb{S}) \,\|_{\mathbb{V}^\star} := \sup_{\|\boldsymbol{v}(\cdot)\|_{\mathbb{V}} \leq 1} \Big| \, \mu_{\mathbb{S}}\big(\boldsymbol{v}(\cdot)\big) \, \Big| \,, \tag{3}$$

where we consider the largest value of the current representation $\mu_{\mathbb{S}}(\cdot)$ of the shape $\mathbb{S}$ when the velocity field cannot exceed unit magnitude under its norm. For example, Figure 1 shows a random currents field whereas Figure 2 shows the field that maximises $\mu_{\mathbb{S}}(\cdot)$ under the unit norm constraint; intuitively, the velocity field should align with the surface normals (shown as green arrows) as much as possible.

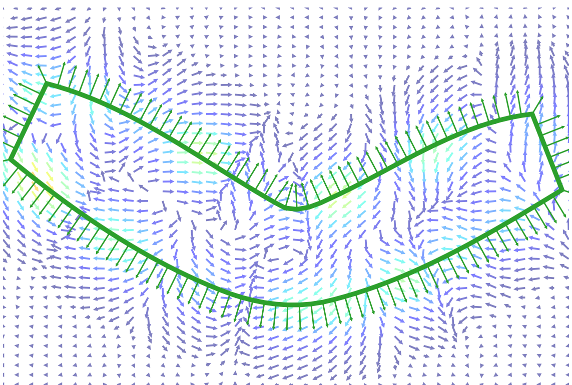

Figure 1: A random currents field.

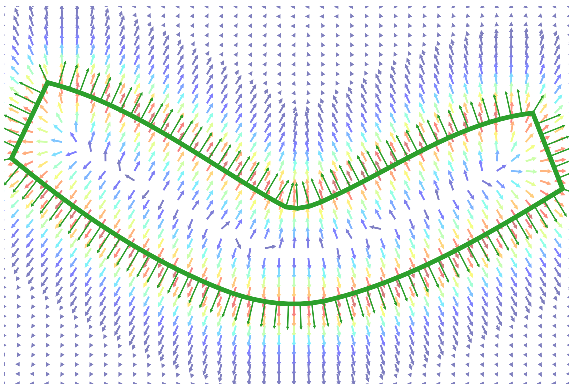

Figure 2: Field that maximises $\mu_{\mathbb{S}}$.

We note that this is a dual norm over the shape space in the sense that the shape $\mathbb{S}$ is defining a space of continuous linear functionals, through the (linear) integral in the metric, that map the original (vector field) space to $\mathbb{R}$, i.e. $\mu_{\mathbb{S}}\big(\boldsymbol{v}(\cdot)\big) : \mathbb{V} \to \mathbb{R}$.

Considering a Lagrange multiplier β, we can rewrite the dual norm as

$$\sup_{\boldsymbol{v}(\cdot) \in \mathbb{V}} \min_{\beta \geq 0} \int_{\mathbb{S}} \langle \boldsymbol{v}(\boldsymbol{x}), \hat{\boldsymbol{n}}(\boldsymbol{x}) \rangle \, \mathrm{ds}(\boldsymbol{x}) + \beta\Big(1 - \|\boldsymbol{v}(\cdot)\|^2_{\mathbb{V}}\Big) \,. \tag{4}$$

We now define the (Lagrangian dual) function $g(\beta)$ as

$$g(\beta) := \sup_{\boldsymbol{v}(\cdot)\in\mathbb{V}} \int_{\mathbb{S}} \langle \boldsymbol{v}(\boldsymbol{x}), \hat{\boldsymbol{n}}(\boldsymbol{x})\rangle \, \mathrm{ds}(\boldsymbol{x}) - \beta \left\| \boldsymbol{v}(\cdot) \right\|_{\mathbb{V}}^2 + \beta \ . \tag{5}$$

Following the representer theorem, we consider the subspace of $\mathbb{V}$ spanned by linear combinations of the kernel representers located over the surface $\mathbb{S}$ as

$$\mathbb{V}_{\mathbb{S}} := \operatorname{span}\Big\{ \kappa(\cdot, \boldsymbol{x}) \, \boldsymbol{a}(\boldsymbol{x}) : \boldsymbol{x} \in \mathbb{S} \ , \ \boldsymbol{a}(\boldsymbol{x}) : \mathbb{R}^n \to \mathbb{R}^n \Big\} \ . \tag{6}$$

Projecting an arbitrary vector field $\boldsymbol{v}(\cdot)$ from the RKHS $\mathbb{V}$ into this subspace $\mathbb{V}_{\mathbb{S}}$ will leave a component in the remaining orthogonal space $\mathbb{V}_{\perp}$; we denote this decomposition

$$\boldsymbol{v}(\cdot) = \boldsymbol{v}_{\mathbb{S}}(\cdot) + \boldsymbol{v}_{\perp}(\cdot) \ . \tag{7}$$

Considering the supremum objective within $g(\beta)$, we have terms to evaluate the field over the shape and the norm term. For the shape term, we have

$$\boldsymbol{v}(\boldsymbol{x}) = \big\langle \boldsymbol{v}_{\mathbb{S}}(\cdot) + \boldsymbol{v}_{\perp}(\cdot), \kappa_{\boldsymbol{x}}(\cdot) \big\rangle_{\mathbb{V}} = \big\langle \boldsymbol{v}_{\mathbb{S}}(\cdot), \kappa_{\boldsymbol{x}}(\cdot) \big\rangle_{\mathbb{V}} + \cancel{\big\langle \boldsymbol{v}_{\perp}(\cdot), \kappa_{\boldsymbol{x}}(\cdot) \big\rangle_{\mathbb{V}}} = \big\langle \boldsymbol{v}_{\mathbb{S}}(\cdot), \kappa_{\boldsymbol{x}}(\cdot) \big\rangle_{\mathbb{V}} \ , \tag{8}$$

using the reproducing property and orthogonality; decomposing the norm term, we have

$$\left\| \boldsymbol{v}(\cdot) \right\|_{\mathbb{V}}^2 = \left\| \boldsymbol{v}_{\mathbb{S}}(\cdot) \right\|_{\mathbb{V}}^2 + \left\| \boldsymbol{v}_{\perp}(\cdot) \right\|_{\mathbb{V}}^2 \leq \left\| \boldsymbol{v}_{\mathbb{S}}(\cdot) \right\|_{\mathbb{V}}^2 \ . \tag{9}$$

We can conclude, therefore, that the optimal vector field $\boldsymbol{v}^\star(\cdot)$ for $g(\beta)$ will lie in $\mathbb{V}_{\mathbb{S}}$ and take the form

$$\boldsymbol{v}^\star(\cdot) = \int_{\mathbb{S}} \kappa(\boldsymbol{x}, \cdot) \, \boldsymbol{a}(\boldsymbol{x}) \, \mathrm{ds}(\boldsymbol{x}) \ , \ \text{ for some } \ \boldsymbol{a}(\cdot) : \mathbb{R}^n \to \mathbb{R}^n \ . \tag{10}$$

Thus the norm term will take the form

$$\left\| \boldsymbol{v}^\star(\cdot) \right\|_{\mathbb{V}}^2 = \int_{\mathbb{S}} \int_{\mathbb{S}} \kappa(\boldsymbol{x}, \boldsymbol{x}') \, \langle \boldsymbol{a}(\boldsymbol{x}), \boldsymbol{a}(\boldsymbol{x}') \rangle \, \mathrm{ds}(\boldsymbol{x}) \, \mathrm{ds}(\boldsymbol{x}') \ , \tag{11}$$

and $g(\beta)$ can be defined in terms of $\boldsymbol{a}(\cdot)$ as

$$g(\beta) = \sup_{\boldsymbol{a}(\cdot)} \int_{\mathbb{S}} \int_{\mathbb{S}} \kappa(\boldsymbol{x}, \boldsymbol{x}') \, \langle \boldsymbol{a}(\boldsymbol{x}), \hat{\boldsymbol{n}}(\boldsymbol{x}') \rangle - \beta \, \kappa(\boldsymbol{x}, \boldsymbol{x}') \, \langle \boldsymbol{a}(\boldsymbol{x}), \boldsymbol{a}(\boldsymbol{x}') \rangle \, \mathrm{ds}(\boldsymbol{x}) \, \mathrm{ds}(\boldsymbol{x}') + \beta \ . \tag{12}$$

The quadratic form indicates that the optimal $\boldsymbol{a}^\star(\cdot)$ will be found at

$$\boldsymbol{a}^\star(\boldsymbol{x}) = \frac{1}{2\beta} \, \hat{\boldsymbol{n}}(\boldsymbol{x}) \ . \tag{13}$$

We can now write $g(\beta)$ as

$$\begin{aligned} g(\beta) &= \int_{\mathbb{S}} \int_{\mathbb{S}} \kappa(\boldsymbol{x}, \boldsymbol{x}') \left\langle \frac{1}{2\beta} \, \hat{\boldsymbol{n}}(\boldsymbol{x}'), \hat{\boldsymbol{n}}(\boldsymbol{x}) \right\rangle - \beta \, \kappa(\boldsymbol{x}, \boldsymbol{x}') \left\langle \frac{1}{2\beta} \, \hat{\boldsymbol{n}}(\boldsymbol{x}), \frac{1}{2\beta} \, \hat{\boldsymbol{n}}(\boldsymbol{x}') \right\rangle \mathrm{ds}(\boldsymbol{x}) \, \mathrm{ds}(\boldsymbol{x}') + \beta \\ &= \frac{1}{4\beta} \int_{\mathbb{S}} \int_{\mathbb{S}} \kappa(\boldsymbol{x}, \boldsymbol{x}') \, \langle \hat{\boldsymbol{n}}(\boldsymbol{x}), \hat{\boldsymbol{n}}(\boldsymbol{x}') \rangle \, \mathrm{ds}(\boldsymbol{x}) \, \mathrm{ds}(\boldsymbol{x}') + \beta \ . \end{aligned} \tag{14}$$

To finish the dual norm, we now optimise for β as

$$\min_{\beta \geq 0} g(\beta) \ , \quad \Longrightarrow \frac{\partial}{\partial \beta} \left(\frac{1}{4\beta} \alpha + \beta \right) = -\frac{\alpha}{4\beta^2} + 1 = 0 \ , \quad \Longrightarrow \beta^\star = \frac{\sqrt{\alpha}}{2} \geq 0 \ , \tag{15}$$

$$\text{where} \quad \alpha := \int_{\mathbb{S}} \int_{\mathbb{S}} \kappa(\boldsymbol{x}, \boldsymbol{x}') \left\langle \hat{\boldsymbol{n}}(\boldsymbol{x}), \hat{\boldsymbol{n}}(\boldsymbol{x}') \right\rangle \mathrm{d}s(\boldsymbol{x})\, \mathrm{d}s(\boldsymbol{x}') \ . \tag{15}$$

Therefore, the optimal $g(\beta^\star)$ is given by

$$g(\beta^\star) = \frac{1}{2\sqrt{\alpha}}\,\alpha + \frac{\sqrt{\alpha}}{2} = \alpha \ , \tag{16}$$

and consequently the dual norm, the *"currents metric"*, is found to be

$$\|\, \mu(\mathbb{S}) \,\|_{\mathbb{V}^\star} := \sup_{\|\boldsymbol{v}(\cdot)\|_{\mathbb{V}} \leq 1} |\, \mu_{\mathbb{S}}\big(\boldsymbol{v}(\cdot)\big) \,| = \int_{\mathbb{S}} \int_{\mathbb{S}} \kappa(\boldsymbol{x}, \boldsymbol{x}') \left\langle \hat{\boldsymbol{n}}(\boldsymbol{x}), \hat{\boldsymbol{n}}(\boldsymbol{x}') \right\rangle \mathrm{d}s(\boldsymbol{x})\, \mathrm{d}s(\boldsymbol{x}') \ . \tag{17}$$

Shape Matching with Currents

We can use the dual norm to derive a discrepancy measure between two shapes. The norm is associated with an inner product

$$\|\, \mathbb{S} \,\|^2_{\mathbb{V}^\star} = \langle \mathbb{S}, \mathbb{S} \rangle_{\mathbb{V}^\star} \ , \tag{18}$$

where we drop the μ for clarity under the implicit assumption of the use of currents metric. Consequently, we may invoke the linearity of the operator to show that for two shapes, $\mathbb{S}$ and $\mathbb{S}'$, we have

$$\begin{aligned} \|\, \mathbb{S} - \mathbb{S}' \,\|^2_{\mathbb{V}^\star} &= \langle \mathbb{S} - \mathbb{S}', \mathbb{S} - \mathbb{S}' \rangle_{\mathbb{V}^\star} \\ &= \langle \mathbb{S}, \mathbb{S} \rangle_{\mathbb{V}^\star} - 2 \,\langle \mathbb{S}, \mathbb{S}' \rangle_{\mathbb{V}^\star} + \langle \mathbb{S}', \mathbb{S}' \rangle_{\mathbb{V}^\star} \ , \end{aligned} \tag{19}$$

where we have

$$\langle \mathbb{S}, \mathbb{S}' \rangle_{\mathbb{V}^\star} := \int_{\mathbb{S}} \int_{\mathbb{S}'} \kappa(\boldsymbol{x}, \boldsymbol{x}') \left\langle \hat{\boldsymbol{n}}(\boldsymbol{x}), \hat{\boldsymbol{n}}'(\boldsymbol{x}') \right\rangle \mathrm{d}s(\boldsymbol{x})\, \mathrm{d}s'(\boldsymbol{x}') \ . \tag{20}$$

Intuitively, we can consider the resulting vector field from the cross-term $\langle \mathbb{S}, \mathbb{S}' \rangle_{\mathbb{V}^\star}$ of (19). When the two shapes are different, Figure 3, it is not possible for the velocity field to match both sets of normals; when the shapes are aligned, Figure 4, the field has no conflicts and alignment with the normals is possible.

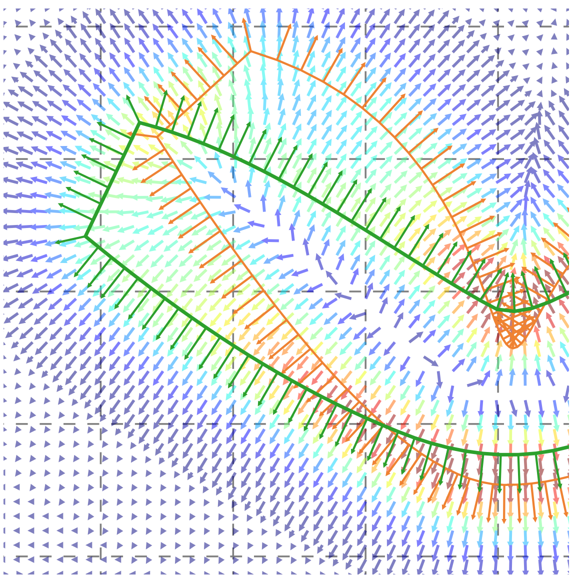

Figure 3: Visualisation of the implicit field for the $\langle \mathbb{S}, \mathbb{S}' \rangle_{\mathbb{V}^\star}$ cross term when shapes are not aligned.

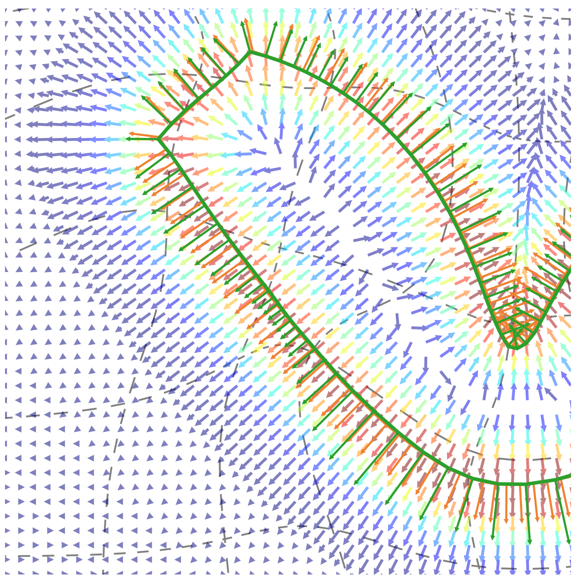

Figure 4: Visualisation of the implicit field for the $\langle \mathbb{S}, \mathbb{S}' \rangle_{\mathbb{V}^\star}$ cross term when shapes are aligned.

Generalisation of Currents

We take the surface as some smooth manifold $\mathbb{S} \in \mathbb{R}^n$ with inherent dimension m; that is a surface in 3D would have $n = 3$ and $m = 2$. We allow $\omega(\boldsymbol{x}) \in \Omega_0^m(\mathbb{R}^n)$ (the space of continuous differential m-forms) to be some differential m-form and then we can generalise the current formulation to

$$\mu_{\mathbb{S}}(\omega(\cdot)) := \int_{\mathbb{S}} \langle \omega(\boldsymbol{x}), \tau_{\mathbb{S}}(\boldsymbol{x}) \rangle \, \mathrm{d}\mathcal{H}^m(\boldsymbol{x}) \ , \tag{21}$$

where $\tau_{\mathbb{S}}(\boldsymbol{x}) := \boldsymbol{e}_1(\boldsymbol{x}) \wedge \ldots \wedge \boldsymbol{e}_{m(\boldsymbol{x})}$ is a wedge product of orthonormal basis vectors of the tangent space to surface $\mathbb{S}$ at $\boldsymbol{x}$ and we integrate with respect to the m-dimensional Hausdorff measure $\mathcal{H}^m(\boldsymbol{x})$, a suitable generalisation of area for reachable sets. When $\mathbb{S}$ is a smooth, m-dimensional manifold, this is the standard integration of an m-form over a smooth manifold (an application of *differential forms*).

If we parameterise the manifold $\mathbb{S}$ via $\varphi(\boldsymbol{u}) : \mathbb{D} \to \mathbb{R}^n$, where $\mathbb{D} \subset \mathbb{R}^m$, then we have

$$\mu_{\mathbb{S}}(\omega(\cdot)) = \int_{\mathbb{D}} \omega(\varphi(\boldsymbol{u})) \left| \frac{\partial \varphi}{\partial \boldsymbol{u}} \right| \mathrm{d}u_1 \ldots \mathrm{d}u_m \ , \tag{22}$$

using the determinant of the Jacobian of the manifold.

Normal Cycles

The currents metric requires the identification of a spatial lengthscale ℓ for the RKHS kernel, e.g.

$$\kappa(\boldsymbol{x}, \boldsymbol{x}') := \exp\left(-\frac{\|\boldsymbol{x} - \boldsymbol{x}'\|^2}{2\,\ell^2} \right) . \tag{23}$$

This lengthscale is fixed across the entire shape and should be commensurate with the variation (curvature) of the shape. For shapes with locally varying curvature, a global lengthscale is inappropriate and limits the fidelity of surface matching using currents.

To make progress, we can consider a higher order object, the *Normal Cycle*, that is sensitive to higher order differential changes in the surface (i.e. sensitive to curvature). Instead of defining a current over the surface alone, we define a current over the **Normal Bundle** of the surface.

The unit Normal Bundle $\mathcal{N}_{\mathbb{S}}$ for shape $\mathbb{S}$ is defined as

$$\mathcal{N}_{\mathbb{S}} := \left\{ (\boldsymbol{x}, \hat{\boldsymbol{n}}(\boldsymbol{x})) \,\middle|\, \boldsymbol{x} \in \mathbb{S},\, \hat{\boldsymbol{n}}(\boldsymbol{x}) \in \left(\mathrm{T}_{\boldsymbol{x}}[\mathbb{S}]\right)^{\perp} \right\} , \tag{24}$$

that is the tuple of the surface points and their unit normals. We use $\mathrm{T}_{\boldsymbol{x}}[\mathbb{S}]$ to denote the tangent space to $\mathbb{S}$ at $\boldsymbol{x}$ and $\left(\mathrm{T}_{\boldsymbol{x}}[\mathbb{S}]\right)^{\perp}$ to denote elements orthogonal to this space as the normal.

We denote this product space as $\boldsymbol{w} := (\boldsymbol{x}, \hat{\boldsymbol{n}}(\boldsymbol{x})) \in \mathbb{R}^n \times \mathbb{E}^{n-1}$ where $\mathbb{E}^{n-1} := \{\boldsymbol{a} \mid \boldsymbol{a} \in \mathbb{R}^n, \|\boldsymbol{a}\| = 1\}$ denotes the unit sphere in $\mathbb{R}^n$ (of inherent dimension $n - 1$).

Curvature Sensitivity Example: a Curve in $\mathbb{R}^2$

If we take a smooth curve in $\mathbb{R}^2$ as an example then we could parameterise the manifold of $\mathcal{N}_{\mathbb{S}}$ as

$$\boldsymbol{w} := \begin{pmatrix} x \\ y \\ n_x \\ n_y \end{pmatrix} \in \mathbb{R}^2 \times \mathbb{E}^1 \ , \quad \mathbb{S} := \{\varphi(u), u \in \mathbb{D}\} \ , \quad \mathbb{D} := [0, 1] \ , \tag{25}$$

and consequently the tangent space is the 1-form space $\tau_{\mathbb{S}} = \left\langle \mathrm{dx}, \mathrm{dy}, \mathrm{dn_x}, \mathrm{dn_y} \right\rangle$. Thus the Normal Cycle current is evaluated as

$$\begin{aligned}
\mu_{\mathbb{S}}(\omega) &:= \int_{\mathcal{N}_{\mathbb{S}}} \langle \omega(\boldsymbol{w}), \tau_{\mathbb{S}}(\boldsymbol{w}) \rangle \, \mathrm{d}\mathcal{H}^m(\boldsymbol{w}) \\
&= \int_{\mathbb{D}} \omega(\boldsymbol{\varphi}(u)) \left| \frac{\partial \boldsymbol{\varphi}}{\partial u} \right| \mathrm{d}u \\
&= \int_{\mathbb{D}} a(\varphi_x(u)) \left| \frac{\partial \varphi_x}{\partial u} \right| + b\big(\varphi_y(u)\big) \left| \frac{\partial \varphi_y}{\partial u} \right| + c\big(\varphi_{n_x}(u)\big) \left| \frac{\partial \varphi_{n_x}}{\partial u} \right| + d\Big(\varphi_{n_y}(u)\Big) \left| \frac{\partial \varphi_{n_y}}{\partial u} \right| \mathrm{d}u \; ,
\end{aligned} \tag{26}$$

where $\omega(\boldsymbol{w}) := a(\boldsymbol{w})\,\mathrm{dx} + b(\boldsymbol{w})\,\mathrm{dy} + c(\boldsymbol{w})\,\mathrm{dn_x} + d(\boldsymbol{w})\,\mathrm{dn_y}$.

If the curve is given by $\mathbb{S} := \left\{ (x(u), y(u)) \,\middle|\, u \in \mathbb{D} \right\}$ then the unit normal is given by

$$\hat{n}(u) := \frac{1}{\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}u}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}u}\right)^2}} \left(-\frac{\mathrm{d}y}{\mathrm{d}u}, \frac{\mathrm{d}x}{\mathrm{d}u} \right) \; , \tag{27}$$

and we may now specify the manifold as

$$\boldsymbol{\varphi}(u) := \left(x(u), y(u), -\frac{y_u}{\sqrt{x_u^2 + y_u^2}}, \frac{x_u}{\sqrt{x_u^2 + y_u^2}} \right) \; , \tag{28}$$

where $x_u := \frac{\mathrm{d}x(u)}{\mathrm{d}u}$ and $y_u := \frac{\mathrm{d}y(u)}{\mathrm{d}u}$.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}u}\left[\frac{x_u}{\sqrt{x_u^2 + y_u^2}} \right] &= \frac{x_{uu}\left(\sqrt{x_u^2 + y_u^2}\right) - x_u \frac{\mathrm{d}}{\mathrm{d}u}\left[\sqrt{x_u^2 + y_u^2}\right]}{x_u^2 + y_u^2} \\
&= \frac{x_{uu}\left(\sqrt{x_u^2 + y_u^2}\right) - x_u \left[\frac{1}{\sqrt{x_u^2 + y_u^2}}(x_u\, x_{uu} + y_u\, y_{uu})\right]}{x_u^2 + y_u^2} \\
&= \frac{x_{uu}\left(x_u^2 + y_u^2\right) - \left(x_u^2\, x_{uu} + x_u\, y_u\, y_{uu}\right)}{\left(x_u^2 + y_u^2\right)^{\frac{3}{2}}} \\
&= \frac{y_u^2\, x_{uu} - x_u\, y_u\, y_{uu}}{\left(x_u^2 + y_u^2\right)^{\frac{3}{2}}} = \frac{y_u(y_u\, x_{uu} - x_u\, y_{uu})}{\left(x_u^2 + y_u^2\right)^{\frac{3}{2}}}
\end{aligned} \tag{29}$$

Thus we have

$$\frac{\mathrm{d}\boldsymbol{\varphi}}{\mathrm{d}u} = \left(x_u, y_u, -\frac{x_u(x_u\, y_{uu} - y_u\, x_{uu})}{\left(x_u^2 + y_u^2\right)^{\frac{3}{2}}}, \frac{y_u(y_u\, x_{uu} - x_u\, y_{uu})}{\left(x_u^2 + y_u^2\right)^{\frac{3}{2}}} \right) \tag{30}$$

Importantly, these second order derivatives are the source of the curvature sensitivity of the normal cycles measure; we recall the equation of curvature for a planar curve $c(u) := \frac{|x_u\, y_{uu} - x_{uu}\, y_u|}{\left(x_u^2 + y_u^2\right)^{\frac{3}{2}}}$ for comparison.

Kernel Formulation for Normal Cycles

If we embed $\omega(\boldsymbol{w})$ in some RKHS space $\mathbb{W}$ with kernel $\kappa_{\mathbb{W}}(\boldsymbol{w}, \boldsymbol{w}')$ then the representer theorem will ensure that the dual norm will correspond to the inner product

$$\langle \mathrm{N}[\mathbb{S}], \mathrm{N}[\mathbb{S}'] \rangle_{\mathbb{W}^\star} := \int_{\mathcal{N}_\mathbb{S}} \int_{\mathcal{N}_{\mathbb{S}'}} \kappa_\mathbb{W}(\boldsymbol{w}, \boldsymbol{w}') \, \langle \tau_\mathbb{S}(\boldsymbol{w}), \tau_{\mathbb{S}'}(\boldsymbol{w}') \rangle \, \mathrm{d}\mathcal{H}^m(\boldsymbol{w}) \, \mathrm{d}\mathcal{H}^m(\boldsymbol{w}') \,, \tag{31}$$

where we use $\mathrm{N}[\mathbb{S}]$ to denote the Normal Cycle; the derivation follows equivalently from the process for currents.

From the inner product, we define the dual metric as the Normal Cycle Metric, denoted by

$$\| \mathrm{N}[\mathbb{S}] \|^2_{\mathbb{W}^\star} := \langle \mathcal{N}_\mathbb{S}, \mathcal{N}_{\mathbb{S}'} \rangle_{\mathbb{W}^\star} \,. \tag{32}$$

Discrete Approximation of the Normal Cycle Metric

The formal computation of the Normal Cycle requires integrals over continuous surfaces. For arbitrary surfaces, we have a discrete approximation of the surface, for example a mesh, rather than a closed form expression.

To approximate the metric, we can evaluate the normal cycle for each discrete element (i.e. a line segment or a triangular mesh) and then use a combination rule to take the union of all the discrete elements as the set; the union is specifed as

$$\mathrm{N}[\mathbb{S}_A \cup \mathbb{S}_B] = \mathrm{N}[\mathbb{S}_A] + \mathrm{N}[\mathbb{S}_B] - \mathrm{N}[\mathbb{S}_A \cap \mathbb{S}_B] \,. \tag{33}$$

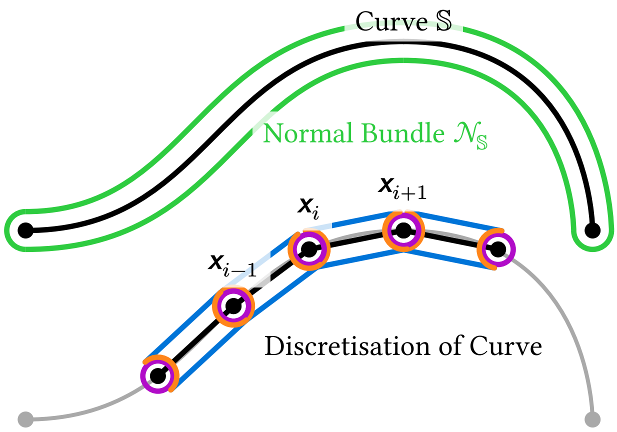

Figure 5: Illustration of discretisation of a curve in $\mathbb{R}^2$. Curve is decomposed into segments each with cylindrical components, spherical components, and half-spherical components.

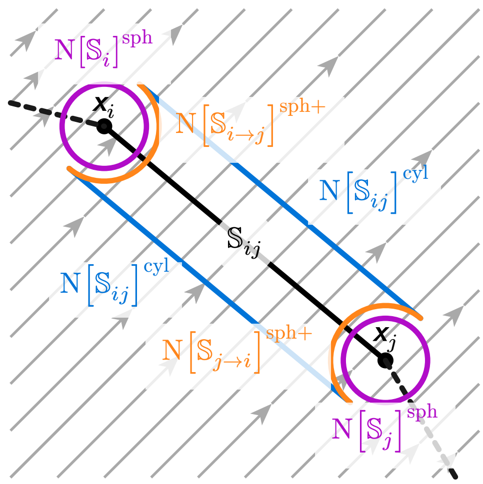

Figure 6: Breakdown of the components of the normal cycle for a single segment of a curve in $\mathbb{R}^2$.

Illustration for Curve in $\mathbb{R}^2$

Figure 5 illustrates the decomposition for an (open) curve in $\mathbb{R}^2$. If we discretise our curve into a series of segments $\mathbb{S} = \left\{\mathbb{S}_{ij}\right\}$ with line segment $\mathbb{S}_{ij}$ connecting $\boldsymbol{x}_i$ to $\boldsymbol{x}_j$ then we can decompose the relevant consituent parts of the normal cycle as in Figure 6. The normal bundle for each segment can be decomposed into two parts, a "cylindrical" part along the line segment (not including the end points) and "spherical" component, comprising two half spheres, at the end points. To make computation easier, given the union rule (33), we decompose the computation two components:

- The segment component connecting $\boldsymbol{x}_i$ to $\boldsymbol{x}_j$, $\mathrm{N}\left[\mathbb{S}_{i\leftrightarrow j}\right]^{\mathrm{seg}}$; comprising the cylindrical component, $\mathrm{N}\left[\mathbb{S}_{ij}\right]^{\mathrm{cyl}}$ minus the two inward facing half-spheres, $\mathrm{N}\left[\mathbb{S}_{i\to j}\right]^{\mathrm{sph}+}$ and $\mathrm{N}\left[\mathbb{S}_{j\to i}\right]^{\mathrm{sph}+}$.

- The vertex components, $\mathrm{N}[\mathbb{S}_i]^{\mathrm{vert}}$ and $\mathrm{N}\left[\mathbb{S}_j\right]^{\mathrm{vert}}$; comprising the complete spheres, $\mathrm{N}[\mathbb{S}_i]^{\mathrm{sph}}$ and $\mathrm{N}\left[\mathbb{S}_j\right]^{\mathrm{sph}}$, at the vertices with centres $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$.

This allows the entire computation to be decomposed into

$$\mathrm{N}[\mathbb{S}] = \sum_{\mathbb{S}_i \in \mathcal{F}[\mathbb{S}]} \mathrm{N}[\mathbb{S}_i]^{\mathrm{vert}} + \sum_{\mathbb{S}_i, \mathbb{S}_j \in \mathcal{E}[\mathbb{S}]} \mathrm{N}\left[\mathbb{S}_{i \leftrightarrow j}\right]^{\mathrm{seg}}, \tag{34}$$

where $\mathcal{F}[\mathbb{S}]$ are the faces (triangles or line segments) and $\mathcal{E}[\mathbb{S}]$ the edges of the discrete surface.

For each segment $\mathbb{S}_{ij}$, we have

$$\begin{aligned} \mathrm{N}\left[\mathbb{S}_{ij}\right] &= \mathrm{N}\left[\mathbb{S}_{ij}\right]^{\mathrm{seg}} + \mathrm{N}[\mathbb{S}_i]^{\mathrm{vert}} + \mathrm{N}\left[\mathbb{S}_j\right]^{\mathrm{vert}}, \\ \text{with} \quad \mathrm{N}\left[\mathbb{S}_{ij}\right]^{\mathrm{seg}} &:= \mathrm{N}\left[\mathbb{S}_{ij}\right]^{\mathrm{cyl}} - \mathrm{N}\left[\mathbb{S}_{i \to j}\right]^{\mathrm{sph+}} - \mathrm{N}\left[\mathbb{S}_{j \to i}\right]^{\mathrm{sph+}}, \\ \text{and} \quad \mathrm{N}[\mathbb{S}_i]^{\mathrm{vert}} &:= \mathrm{N}[\mathbb{S}_i]^{\mathrm{sph}}, \ \mathrm{N}\left[\mathbb{S}_j\right]^{\mathrm{vert}} := \mathrm{N}\left[\mathbb{S}_j\right]^{\mathrm{sph}}. \end{aligned} \tag{35}$$

To illustrate the simplest case, consider a separable kernel $\kappa_{\mathbb{W}}(\boldsymbol{w}, \boldsymbol{w}') := \kappa_{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{x}')\, \kappa_{\boldsymbol{n}}(\hat{\boldsymbol{n}}, \hat{\boldsymbol{n}}')$ and then make the normal component of the kernel constant, that is $\kappa_{\boldsymbol{n}}(\hat{\boldsymbol{n}}, \hat{\boldsymbol{n}}') := 1$.

We assume that the field $\omega(\boldsymbol{x})$ is effectively constant over the small segment (this is a reasonable assumption as we dictate the lengthscale of the spatial kernel and can ensure that this is the case). In this setting, i.e. constant normal kernel and constant field, we can see that the cylindrical components will cancel as the opposite sides of the normal bundle are oriented in opposite directions so we have $\mathrm{N}\left[\mathbb{S}_{ij}\right]^{\mathrm{cyl}} = 0$. Similarly, the full spheres of the vertex components will cancel as all orientations are equal and opposite; so we have $\mathrm{N}[\mathbb{S}_i]^{\mathrm{vert}} = \mathrm{N}\left[\mathbb{S}_j\right]^{\mathrm{vert}} = 0$. We are thus left with the two half-spheres at the end points, which are oriented inwards along the segment.

We use $\mathbb{E}_+^{n-1}[\vec{\boldsymbol{a}}] := \left\{\boldsymbol{x} \,\middle|\, \boldsymbol{x} \in \mathbb{E}^{n-1}, \boldsymbol{x} \cdot \boldsymbol{a} > 0, \|\boldsymbol{a}\| = 1\right\}$ to denote the half-sphere oriented in the direction of $\boldsymbol{a}$. Thus we have

$$\begin{aligned} \left\langle \mathrm{N}\left[\mathbb{S}_{i\to j}\right]^{\mathrm{sph+}}, \mathrm{N}\left[\mathbb{S}'_{i'\to j'}\right]^{\mathrm{sph+}} \right\rangle_{\mathbb{W}^\star} &= \int_{\delta(\boldsymbol{x}_i)\times\mathbb{E}_+^1\left[\overrightarrow{\boldsymbol{x}_{ij}}\right]} \int_{\delta\left(\boldsymbol{x}'_{i'}\right)\times\mathbb{E}_+^1\left[\overrightarrow{\boldsymbol{x}'_{i'j'}}\right]} \kappa_{\mathbb{W}}(\boldsymbol{w}, \boldsymbol{w}')\, \langle \tau_{\mathbb{S}}(\boldsymbol{w}), \tau_{\mathbb{S}'}(\boldsymbol{w}')\rangle \,\mathrm{d}\mathcal{H}^m(\boldsymbol{w})\,\mathrm{d}\mathcal{H}^m(\boldsymbol{w}') \\ &= \kappa_{\boldsymbol{x}}(\boldsymbol{x}_i, \boldsymbol{x}'_{i'}) \int_{\mathbb{E}_+^1\left[\overrightarrow{\boldsymbol{x}_{ij}}\right]} \int_{\mathbb{E}_+^1\left[\overrightarrow{\boldsymbol{x}'_{i'j'}}\right]} \langle \hat{\boldsymbol{n}}, \hat{\boldsymbol{n}}'\rangle \,\mathrm{d}\mathcal{H}^m(\hat{\boldsymbol{n}})\,\mathrm{d}\mathcal{H}^m(\hat{\boldsymbol{n}}') \\ &= \kappa_{\boldsymbol{x}}(\boldsymbol{x}_i, \boldsymbol{x}'_{i'}) \left\langle \int_{\mathbb{E}_+^1\left[\overrightarrow{\boldsymbol{x}_{ij}}\right]} \hat{\boldsymbol{n}}\,\mathrm{d}\mathcal{H}^m(\hat{\boldsymbol{n}}), \int_{\mathbb{E}_+^1\left[\overrightarrow{\boldsymbol{x}'_{i'j'}}\right]} \hat{\boldsymbol{n}}'\,\mathrm{d}\mathcal{H}^m(\hat{\boldsymbol{n}}') \right\rangle \\ &= \kappa_{\boldsymbol{x}}(\boldsymbol{x}_i, \boldsymbol{x}'_{i'}) \left\langle \frac{\pi}{2}\frac{\boldsymbol{x}_j - \boldsymbol{x}_i}{\|\boldsymbol{x}_j - \boldsymbol{x}_i\|}, \frac{\pi}{2}\frac{\boldsymbol{x}'_{j'} - \boldsymbol{x}'_{i'}}{\|\boldsymbol{x}'_{j'} - \boldsymbol{x}'_{i'}\|} \right\rangle. \end{aligned} \tag{36}$$

So, in total, for two discrete curves $\mathbb{S}$ and $\mathbb{S}'$, we have

$$\langle \mathrm{N}[\mathbb{S}], \mathrm{N}[\mathbb{S}']\rangle_{\mathbb{W}^\star} = \frac{\pi^2}{4}\sum_i \sum_{i'} \kappa_{\boldsymbol{x}}(\boldsymbol{x}_i, \boldsymbol{x}'_{i'}) \left\langle \sum_{j\in\{i-1,i+1\}} \overrightarrow{\boldsymbol{x}_{ij}}, \sum_{j'\in\{i'-1,i'+1\}} \overrightarrow{\boldsymbol{x}'_{i'j'}} \right\rangle, \tag{37}$$

where $\overrightarrow{\boldsymbol{x}_{ij}} := \frac{\boldsymbol{x}_j - \boldsymbol{x}_i}{\|\boldsymbol{x}_j - \boldsymbol{x}_i\|}$ is the unit vector along the segment from $\boldsymbol{x}_i$ to $\boldsymbol{x}_j$ and the summation is over all incident edges for each vertex.