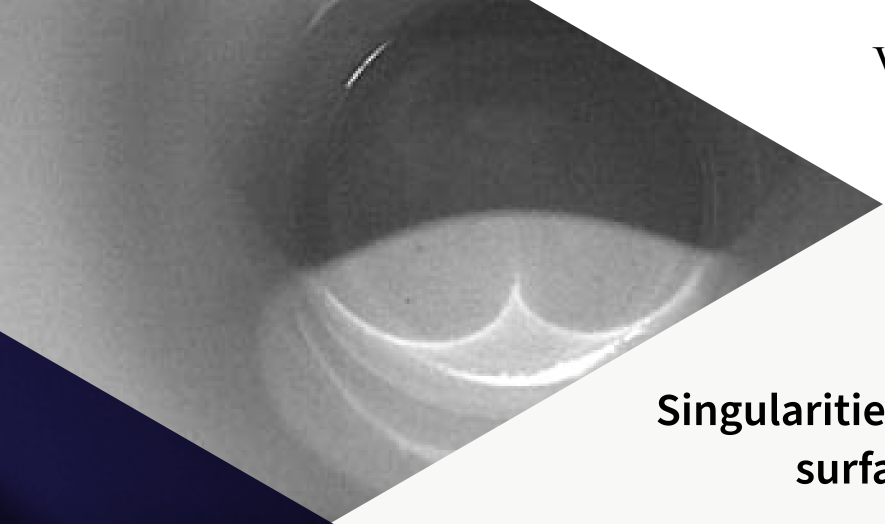

Universitat de València

Singularities of frontal surfaces

C. Muñoz-Cabello

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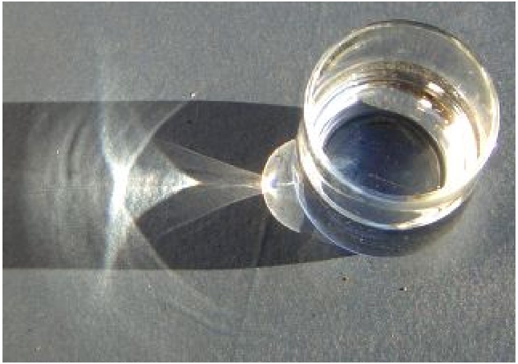
Overview

1. Introduction
2. Frontal equivalence
3. Stabilizing frontal maps
4. Exploring frontal surfaces

Introduction

Introduction
What is a frontal hypersurface?



(Image by Heiner Otterstedt, 2006)

- Light cardioids

Introduction
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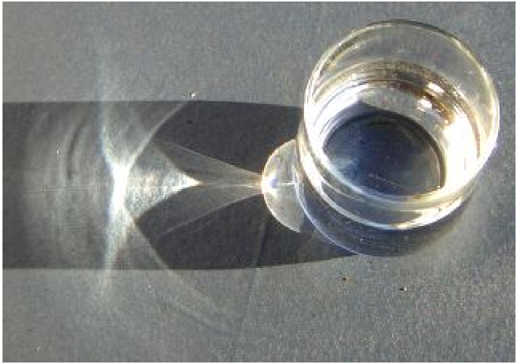

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- Control systems (*degrees of freedom*)

Introduction

What is a frontal hypersurface?

(Image by Patrick Dirden, 2010)

- Light cardioids
- Control systems (*degrees of freedom*)
- Sound barrier

Introduction
What is a frontal hypersurface?

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- Light cardioids
- Control systems (*degrees of freedom*)
- Sound barrier
- Differential geometry (geodesic flows)

Introduction

Ishikawa’s theory of infinitesimal Legendre equivalence

- Corank 1 stable frontals come from a certain family of maps, called *open Whitney umbrellas*

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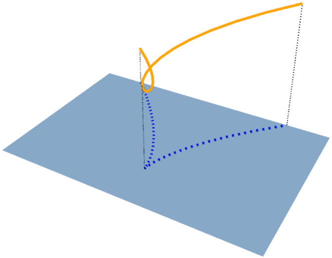
Ishikawa's theory of infinitesimal Legendre equivalence

- Corank 1 stable frontals come from a certain family of maps, called *open Whitney umbrellas*
- Stability is codified in terms of a certain $\mathbb{C}$-algebra Q
- Legendrian codimension measures how far a frontal is from being stable

Frontal equivalence

Frontal equivalence

All we need to know is downstairs



We say $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ is **frontal** if there exists a $\nu\colon (\mathbb{C}^{n+1}, 0) \to T^*\mathbb{C}^{n+1}$ such that $0 \notin \nu[f(S)]$ and

$$\nu(df \circ \xi) = 0 \quad \forall \xi \in \theta_n$$

Frontal equivalence

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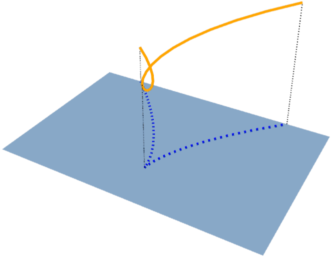

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Proposition

Let $f, g\colon (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ be $\mathscr{A}$-equivalent germs. If f is frontal, g is frontal.

Frontal equivalence
Frontal stability

- An unfolding F of f is **frontal** if it has a frontal representative. The map f is $\mathscr{F}$**-stable** if every frontal unfolding is trivial.

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- Given a frontal f,

$$T\mathscr{A}_e f \subseteq \mathscr{F}(f) = \left\{ \left.\frac{df_t}{dt}\right|_{t=0} : (f_t, t) \text{ frontal }, f_0 = f \right\}$$

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- We say f is $\mathscr{F}$-finite if

$$\operatorname{codim}_{\mathscr{F}} f = \dim_{\mathbb{C}} \frac{\mathscr{F}(f)}{T\mathscr{A}_e f} < \infty;$$

Frontal equivalence
Characterising frontal stability

Theorem

A corank 1 frontal is stable as a frontal if and only if it has frontal codimension 0.

Frontal equivalence

Characterising frontal stability

Theorem

A corank 1 frontal is stable as a frontal if and only if it has frontal codimension 0.

Theorem (Mather-Gaffney criterion for frontals)

A corank 1 frontal $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ with codim $V(p_y, \mu_y) > 1$ is $\mathscr{F}$-finite if and only if there is a finite representative $f\colon X \to Y$ such that $f\colon X\backslash f^{-1}(0) \to Y\backslash\{0\}$ is locally $\mathscr{F}$-stable.

Frontal equivalence

Relation to Legendrian stability

Lemma

Given a corank 1 frontal f with Nash lift $\tilde{f}$, an unfolding $F = (f_u, u)$ of f is frontal if and only if $\widetilde{f_u}$ is an integral deformation of $\tilde{f}$.

Frontal equivalence

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Lemma

Given a corank 1 frontal f with Nash lift $\tilde{f}$, an unfolding $F = (f_u, u)$ of f is frontal if and only if $\widetilde{f_u}$ is an integral deformation of $\tilde{f}$.

Theorem

The frontal codimension of f coincides with the Legendrian codimension of $(\pi, \tilde{f})$ (incl. ∞).

Frontal equivalence

A faithful reflection of the situation above

Lemma

Two frontal germs f, g are $\mathscr{A}$-equivalent if and only if their Nash lifts are Legendrian equivalent.

Frontal equivalence

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Theorem

A frontal unfolding $F = (f_u, u)$ of f is stable (resp. versal) if and only if $\widetilde{f_u}$ is stable (resp. versal).

Frontal equivalence

Mond's classification of simple fold surfaces in $\mathbb{C}^3$

A parametrized surface germ $f\colon (\mathbb{C}^2, S) \to (\mathbb{C}^3, 0)$ is a fold if its Thom-Boardman symbol is $\Sigma^{1,0}$.

Frontal equivalence

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Mond, 1983

Parametrized surface germs admit a representative in the form $f(x, y) = (x, y^2, yp(x, y^2))$ with $p \in \mathbb{C}\{x, y\}$.

Frontal equivalence

Mond's classification of simple fold surfaces in $\mathbb{C}^3$

Mond's classification		Codimension	Notes
S_k	$y^3 + x^{k+1}y$	k	
B_k	$x^2y + y^{2k+1}$	k	$k \geq 2$
C_k	$xy^3 + x^ky$	k	$k \geq 3$
F_4	$y^5 + x^3y$	4	

Frontal equivalence

Mond's classification of simple fold surfaces in $\mathbb{C}^3$

Mond's classification		Frontalised surface		Codimension	Notes
S_k	$y^3 + x^{k+1}y$	$\check{S}_k$	$y^5 + x^{k+1}y^3$	k	
B_k	$x^2y + y^{2k+1}$	$\check{B}_k$	$x^2y^3 + y^{2k+3}$	k	$k \geq 2$
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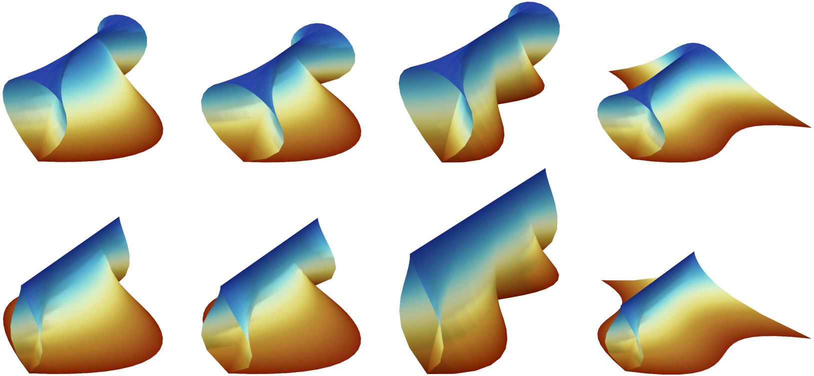
F_4	$y^5 + x^3y$	$\check{F}_4$	$y^7 + x^3y^3$	4	

Frontal equivalence

Mond’s fold surfaces and their frontalisations



Frontal equivalence
Properties of frontal surfaces

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  - no singular points other than cuspidal edges near the origin;
  - the singular set has finitely many irreducible components.

Stabilizing frontal maps

Stabilizing frontal maps
Frontal reduction

Theorem

Given an analytic plane curve γ with miniversal unfolding Γ, there is an immersion h such that

$$\Gamma_{\mathscr{F}}(u,t) := (h^*\Gamma)(u,t) = (\gamma_{h(u)}(t), u)$$

is a frontal miniversal unfolding of γ. We call this pull-back the **frontal reduction** of Γ.

Stabilizing frontal maps

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Corollary

If γ has isolated instability,

$$\operatorname{codim}_{\mathscr{F}_e} \gamma = \operatorname{codim}_{\mathscr{A}_e} \gamma - \operatorname{mult} \gamma + 1$$

Stabilizing frontal maps

Every corank 1 frontal admits a stable unfolding

Corollary

Let f be a corank 1 frontal with generic slice γ, and Γ be a versal unfolding of γ. There is a frontal stable unfolding F of f.

Stabilizing frontal maps

Constructing a stable unfolding

Setting $\gamma(t) = (p(t), q(t))$ with $\mu = q'/p'$ and

$$f(u,y) = (u, P(u,y), Q(u,y)); \qquad M = Q_y/P_y;$$
$$\Gamma_{\mathscr{F}}(v,y) = (v, P'(v,y), Q'(v,y)); \qquad M' = Q'_y/P'_y,$$

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we obtain a stable unfolding of f by taking

$$F(u,v,y) = (u, v, P''(u,v,y), Q''(u,v,y));$$
$$P''(u,v,y) = P(u,y) + P'(v,y) - p(y);$$
$$Q''(u,v,y) = \int_0^y P''_s(u,v,s)(M(u,s) + M'(v,s) - \mu(s))\,ds$$

Exploring frontal surfaces

Exploring frontal surfaces

Double points split into two branches

If $f(x,y) = (x, p(x,y), q(x,y))$, the **double point space** of f is given by

$$D^2(f) = \left\{ (x,y,y') \in \mathbb{C}^3 \colon \frac{p(x,y) - p(x,y')}{y - y'} = \frac{q(x,y) - q(x,y')}{y - y'} = 0 \right\}$$

Exploring frontal surfaces

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If $\pi(x,y,y') = (x,y)$, we set the complex space $D(f) = \pi(D^2(f))$ with the complex space structure induced by π.

Exploring frontal surfaces

The six $\mathscr{F}$-stable frontal surfaces

Theorem

If $p_y | q_y$, the generating function for $D(f)$ has the form $\lambda(x, y) = p_y^2(x, y)\tau(x, y)$.

Exploring frontal surfaces

The six $\mathscr{F}$-stable frontal surfaces

Theorem

If $p_y|q_y$, the generating function for $D(f)$ has the form $\lambda(x,y) = p_y^2(x,y)\tau(x,y)$.

Corollary

Let $\lambda \in \mathscr{O}_2$ be the generating function for $D(f)$, and $\mu = q_y/p_y$:

1. if λ/p_y is regular, f is either a cuspidal edge or a curve of transverse double points;
2. if $V(p_y, \mu_y) = \{0\}$, f is $\mathscr{F}$-finite if and only if λ/p_y has an isolated singularity at 0.

Exploring frontal surfaces

Frontal disentanglement

A smooth family (f_t) is an $\mathscr{F}$-**stabilisation** of a frontal $f\colon (\mathbb{C}^2, S) \to (\mathbb{C}^3, 0)$ if the 1-parameter unfolding $F = (f_t, t)$ is frontal and f_t is frontal stable for $t \neq 0$.

Exploring frontal surfaces

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Lemma-Definition

If f has an isolated instability, then it admits an $\mathscr{F}$-stabilisation (f_t) and we call $\Delta_{\mathscr{F}}(f) = \operatorname{Im} f_t$ ($t \neq 0$) the **frontal disentanglement** of f.

Exploring frontal surfaces
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Lê, 1987

If f has an isolated instability, $\Delta_{\mathscr{F}}(f)$ has the homotopy type of a wedge of spheres.

Exploring frontal surfaces

Frontal Marar-Mond formulas

Theorem (frontal Marar-Mond formulas)

Given a corank 1 frontal $f\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ with isolated instability,

$$\mu(f(C), 0) = 2S + \mu(C, 0);$$
$$2\mu(f(D_+), 0) = 2K + 2T + \mu(D_+, 0) - W - S + 1$$

Exploring frontal surfaces

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- C: Cuspidal edge curve
- S: Swallowtails
- K: Cuspidal double points
- D_+: Transverse double point curve
- T: Triple points
- W: Folded Whitney umbrellas

Exploring frontal surfaces

Counting the invariants

Given a corank 1 frontal $f\colon (\mathbb{C}^2,0)\to(\mathbb{C}^3,0)$ with isolated instability,

$$\dim_{\mathbb{C}} \frac{\mathscr{O}_2}{(p_y, p_{yy})} = S; \qquad \dim_{\mathbb{C}} \frac{\mathscr{O}_2}{(p_y, \tau)} = 2S + K + W;$$

$$\dim_{\mathbb{C}} \frac{\mathscr{O}_3}{(p_y, \alpha, \alpha')} = 2S + K; \qquad \dim_{\mathbb{C}} \frac{\mathscr{O}_3}{\mathscr{F}_2(f)} = T + S + K.$$

These identities have been compiled into a Singular library, `frontals.lib`, which can be found at `cuspidalcoffee.github.io`.

Exploring frontal surfaces

Frontal Milnor number

Proposition

Given an analytic plane curve $\gamma\colon (\mathbb{C},0)\to(\mathbb{C}^2,0)$ with $\kappa=|\Sigma(\gamma)|<\infty$,

$$\mu_{\mathscr{F}}(\gamma)=\mu_I(\gamma)-\kappa$$

Exploring frontal surfaces

Frontal Milnor number

Proposition

Given an analytic plane curve $\gamma\colon (\mathbb{C},0) \to (\mathbb{C}^2,0)$ with $\kappa = |\Sigma(\gamma)| < \infty$,

$$\mu_{\mathscr{F}}(\gamma) = \mu_I(\gamma) - \kappa$$

Theorem

Given a corank 1 frontal $f\colon (\mathbb{C}^2,0) \to (\mathbb{C}^3,0)$ with isolated instability,

$$\mu_{\mathscr{F}}(f) = \mu(f(D_+),0) - S - W + T + 1$$

Exploring frontal surfaces

Mond's conjecture

Recall that Mond defines the **image Milnor number** μ_I of $f\colon (\mathbb{C}^2, S) \to (\mathbb{C}^3, 0)$ as the number of spheres in a disentanglement.

Mond's conjecture

If f is $\mathscr{A}$-finite, $\mu_I(f) \geq \operatorname{codim}_{\mathscr{A}_e}(f)$, with equality if and only if f is quasihomogeneous.

Exploring frontal surfaces

Mond's conjecture

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Proposed frontal conjecture

Let $f\colon (\mathbb{C}^2, S) \to (\mathbb{C}^3, 0)$ be an $\mathscr{F}$-finite frontal map. Then $\mu_{\mathscr{F}}(f) \geq \operatorname{codim}_{\mathscr{F}_e}(f)$, with equality if and only if f is quasihomogeneous.