Analysis of optimal deterministic protocols for the tetrahedron measurement

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Abstract

This report addresses the challenge of improving the accuracy of noisy quantum measurements, particularly focusing on the tetrahedron measurement as a case study. The report presents a mathematical framework using positive operator valued measures (POVMs) and contrasts them with ideal projection operators. A protocol involving an ancillary qubit is introduced to enhance measurement precision and a systematic analysis of this protocol ensues. The optimal deterministic protocol is derived leading to results which showcase substantial improvements in measurement accuracy. The report concludes by discussing potential future directions in this area.

Keywords: quantum, measurement, POVM, tetrahedron

1 Introduction

The act of measurement is one of the fundamental postulates of quantum theory [1]. Quantum measurement allows us to read classical information from quantum states while changing the state in the process, however for practical applications this process is never truly perfect. Noisy quantum measurements which give unreliable results are a consistent problem for quantum technologies, especially for NISQ era quantum computers [2]. Over recent years significant progress has been made towards error correcting circuits, which can account for errors in noisy quantum gates [3]. While these circuits can account for measurement errors, there has only been a small amount of exploration on how to improve these measurement errors in their own right [4–6] and these current efforts are not generalised. In [7] it is demonstrated how we can improve the accuracy of a noisy quantum measurement by using repeated copies of the measurement on an ancillary system of qubits. In this report we build on the previous work by analysing in detail an example using the tetrahedron measurement. The report is organised as follows:

- In section 2 we outline how to model noisy quantum measurements mathematically using positive operator valued measures and compare this to a perfect measurement using projection operators.
- In section 3 we introduce the tetrahedron measurement as an example of a positive operator valued measure.
- In section 4 we show how we can map the tetrahedron measurement of section 3 onto a measurement in the standard basis.
- In section 5 we introduce a figure of merit which quantifies how close an arbitrary measurement is to a perfect measurement. We also define the concept of a protocol involving an ancillary qubit, a pre-measurement unitary and some post-processing, which can improve the accuracy of the tetrahedron measurement.
- In section 6 we demonstrate how to fully optimise [8] a protocol and then maximally optimise the protocol introduced in section 5.
- Finally, in section 7 we discuss the results and the direction of future work.

2 Positive operator valued measures

Much of what is to follow in this section can be found in Nielsen & Chuang [1]. It is often taught that we use projection operators to measure a quantum state where projection operators obey the equations

$$P^\dagger = P, \quad P^2 = P. \tag{1}$$

We can use projection operators to define measurement if we are given a set $\{P_i\}$ such that

$$\sum_i P_i = I \tag{2}$$

where I is the identity operator. Associated with each operator is an outcome state and a classical value i which is measured. The probability of measuring outcome i on a state $|\psi\rangle$ is

$$Prob(i) = \langle\psi| P_i |\psi\rangle \tag{3}$$

and the state after measurement is

$$|\phi\rangle = \frac{P_i |\psi\rangle}{\sqrt{\langle\psi| P_i |\psi\rangle}}. \tag{4}$$

This is how measurement is often introduced however it is not the most general case. Imagine a noisy measurement device which will collapse a quantum state and return a classical value but with a non-zero probability will return the incorrect value, suggesting your system has collapsed to one state when it is in fact in another. This process is modelled by the theory of noisy quantum channels and has associated with it a set of Kraus operators $\{E_k\}$. A set of Kraus operators satisfies the equation

$$\sum_k E_k^\dagger E_k = I. \tag{5}$$

Like with the projection operators, associated with each of these Kraus operators is a classical value k which we measure with a probability

$$Prob(k) = tr(E_k \rho E_k^\dagger) \tag{6}$$

where ρ is a density matrix representing a quantum state. The state post measurement is

$$\rho_k' = \frac{E_k \rho E_k^\dagger}{tr(E_k \rho E_k^\dagger)} \tag{7}$$

which is unlike the projection operator case as the post measurement state could be a mixed state. A mixed post measurement state suggests classical uncertainty in the state which implies uncertainty in the measurement outcome k, hence this models a noisy measurement. If we do not care about the post measurement state and are only interested in the classical outcome k and its associated probability, then by rewriting equation 6 as

$$tr(E_k \rho E_k^\dagger) = tr(\rho E_k^\dagger E_k) = tr(\rho M_k) = \langle\psi| M_k |\psi\rangle \tag{8}$$

we can see that the probability of outcome k is completely characterised by the operator $M_k = E_k^\dagger E_k$. This is a more general definition of measurement and is known as a positive operator valued measure (POVM). Like the projection operators, we have a set $\{M_k\}$ which obeys equations 2 and 3 however POVMs do not have to obey equation 1 and it is <u>not</u> possible to determine the post measurement state like in equations 4 and 7. This is because a given POVM can be described by an infinite set of Kraus operators as we can always apply an arbitrary unitary U after a Kraus operator and end up at the same POVM. For example, take the set of Kraus operators $\{UE_k\}$ then the corresponding POVM is also

$$E_k^\dagger U^\dagger U E_k = E_k^\dagger E_k = M_k. \tag{9}$$

This means with only a POVM $\{M_k\}$ we cannot know what the associated Kraus operators are and therefore cannot know the state post measurement. This makes POVMs a useful mathematical tool as they are more "light-weight" than Kraus operators, however they are not a complete model of quantum measurements as they do not give the post measurement state.

3 Tetrahedron measurement

To illustrate an example we shall discuss the projection operators on the standard basis and compare this to the focus of the report, the tetrahedron POVM. Consider the projection operators $P_0 = |0\rangle\langle 0|$ and $P_1 = |1\rangle\langle 1|$ which satisfy equations 1 and

2 and the arbitrary state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ where $\alpha, \beta \in \mathbb{C}$, $|\alpha|^2 + |\beta|^2 = 1$. Using equation 3 we find the probabilities of each outcome to be

$$\begin{aligned} Prob(0) &= \langle\psi| P_0 |\psi\rangle = |\alpha|^2 \\ Prob(1) &= \langle\psi| P_1 |\psi\rangle = |\beta|^2. \end{aligned} \tag{10}$$

Since we are using projection operators, these are the probabilities you would find if performing a perfect measurement in the standard basis. Now consider the tetrahedron POVM $\{M_k\}$, $k \in \{0, 1, 2, 3\}$ which has elements

$$M_k = \frac{1}{2} |\phi_k\rangle \langle\phi_k| \tag{11}$$

where

$$\begin{aligned} |\phi_0\rangle &= |0\rangle, \\ |\phi_1\rangle &= \frac{1}{\sqrt{3}}(|0\rangle + \sqrt{2}|1\rangle), \\ |\phi_2\rangle &= \frac{1}{\sqrt{3}}(|0\rangle + \sqrt{2}e^{-i\frac{2\pi}{3}}|1\rangle), \\ |\phi_3\rangle &= \frac{1}{\sqrt{3}}(|0\rangle + \sqrt{2}e^{i\frac{2\pi}{3}}|1\rangle). \end{aligned} \tag{12}$$

These operators satisfy equation 2 but not equation 1. The states defining the operators in equation 12 represent the corners of a tetrahedron inside a Bloch sphere and are plotted in figure 1. Using equation 3 we find the probabilities of each outcome to be

$$\begin{aligned} Prob(0) &= \frac{1}{2}|\alpha|^2, \\ Prob(1) &= \frac{1}{6}(1 + |\beta|^2 + \sqrt{2}(\alpha\beta^* + \alpha^*\beta)), \\ Prob(2) &= \frac{1}{6}(1 + |\beta|^2 + \sqrt{2}(e^{i\frac{2\pi}{3}}\alpha\beta^* + e^{-i\frac{2\pi}{3}}\alpha^*\beta)), \\ Prob(3) &= \frac{1}{6}(1 + |\beta|^2 + \sqrt{2}(e^{-i\frac{2\pi}{3}}\alpha\beta^* + e^{i\frac{2\pi}{3}}\alpha^*\beta)). \end{aligned} \tag{13}$$

4 Mapping the tetrahedron measurement onto a measurement in the standard basis

It was shown in [7] that if we are able to reproduce the statistics of a perfect measurement, returning 0 with probability $|\alpha|^2$ and 1 with probability $|\beta|^2$, then for all intents and purposes we have replicated the perfect measurement. This means for a set of POVMs we must assign each outcome to either 0 or 1 in a way which will most closely replicate the desired statistics. In order to find the best assignment we need a figure of merit to determine how close an assignment is to the perfect measurement.

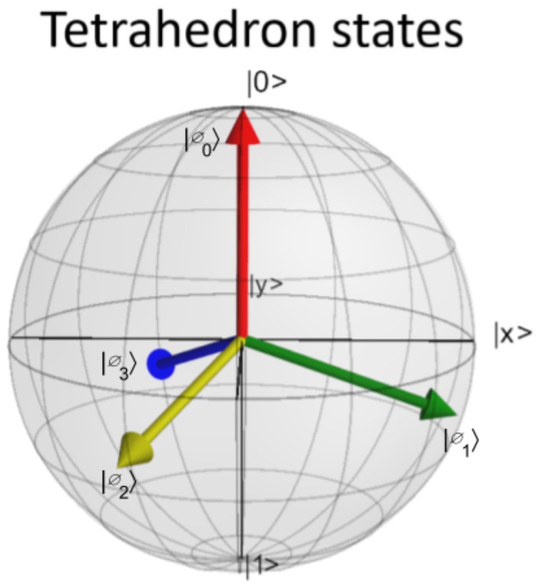


Fig. 1: The states making up the tetrahedron POVM from equation 12 plotted on a Bloch sphere.

The figure of merit used in [7] is biased towards states which have statistics close to the perfect measurement so we will use a slightly modified measurement function which removes this bias.

$$\epsilon = 2 \int d\psi |P(0) - \mathscr{P}(0)| \tag{14}$$

where $\int d\psi$ means to integrate over all input states, $P(0)$ is the probability our assignment returns 0 for state $|\psi\rangle$ and $\mathscr{P}(0)$ is the probability a perfect measurement returns 0 for state $|\psi\rangle$. It can be shown the solution to this equation is

$$\epsilon = \frac{(1-x)^2 + y^2}{1 - x + y} \tag{15}$$

where $x = Prob(0| \, |\psi\rangle = |0\rangle)$ is the probability our assignment returns 0 for an input state of $|0\rangle$ and $y = Prob(0| \, |\psi\rangle = |1\rangle)$ is the probability our assignment returns 0 for an input state of $|1\rangle$. A measurement is a perfect measurement if and only if $x = 1$ and $y = 0$ [7]. Using the probabilities of equations 13 for the tetrahedron measurement, an exhaustive search shows that the best assignment we can make is

$$\begin{aligned} &\{0,1\} \rightarrow 0 \\ &\{2,3\} \rightarrow 1 \end{aligned} \tag{16}$$

which has the associated probabilities

$$\begin{aligned} Prob(0) &= \frac{1}{3}(1 + |\alpha|^2 + \frac{1}{\sqrt{2}}(\alpha\beta^* + \alpha^*\beta)) \\ Prob(1) &= \frac{1}{3}(1 + |\beta|^2 - \frac{1}{\sqrt{2}}(\alpha\beta^* + \alpha^*\beta)). \end{aligned} \tag{17}$$

By substituting $\alpha = 1$, $\beta = 0$ into the equation for $Prob(0)$ we find $x = \frac{2}{3}$ and by substituting in $\alpha = 0$, $\beta = 1$ we find $y = \frac{1}{3}$. Putting these results into equation 15 we find our figure of merit to be $\epsilon = \frac{1}{3}$. In fact this assignment is degenerate as pairing any outcome 1, 2 or 3 with 0 will lead to the same result. x and y are no where near what we would expect to see for a perfect measurement but at least $y < x$ which is a feature we will continue to see when we make improvements to this measurement in the next section.

5 A protocol for improving the tetrahedron measurement

We will now show it is possible to improve on this assignment by introducing an ancillary qubit and performing a classical cloning procedure using the CNOT. We start with the arbitrary input state $|\psi\rangle$ we want to measure and an ancillary qubit initialised in the zero state, we then apply a CNOT gate from the input state to the ancillary to get $\left|\tilde{\psi}\right\rangle = \alpha|00\rangle + \beta|11\rangle$. Finally, we measure both qubits with the tetrahedron measurement and assign the result to either 0 or 1. The pre-measurement CNOT unitary gate along with the assignment is called a protocol[1]. To choose an optimal assignment for this protocol we should first find the probability of each outcome. The probability of measuring outcome (k, l) is

$$Prob(k,l) = \left\langle\tilde{\psi}\right| M_k \otimes M_l \left|\tilde{\psi}\right\rangle \tag{18}$$

where k is the outcome on the first qubit and l is the outcome on the second qubit and M_k are the tetrahedron POVMs from equation 11. The probabilities for all 16 outcomes are

$$Prob(0,0) = \frac{|\alpha|^2}{4}$$
$$Prob(0,1) = Prob(0,2) = Prob(0,3) = Prob(1,0) = Prob(2,0) = Prob(3,0) = \frac{|\alpha|^2}{12}$$
$$Prob(1,1) = Prob(2,3) = Prob(3,2) = \frac{1}{6}(\frac{1}{6} + \frac{|\beta|^2}{2} + \frac{1}{3}(\alpha\beta^* + \alpha^*\beta))$$

[1] In this report we will only be analysing deterministic protocols, where a given outcome is always assigned to the same value. There also exists probabilistic protocols where a given outcome is assigned to 0 with some probability and assigned to 1 otherwise. Probabilistic protocols will be briefly discussed in the conclusion, section 7.

$$
\begin{aligned}
Prob(1,2) = Prob(2,1) &= \frac{1}{6}(\frac{1}{6} + \frac{|\beta|^2}{2} + \frac{1}{3}(e^{i\frac{2\pi}{3}}\alpha\beta^* + e^{-i\frac{2\pi}{3}}\alpha^*\beta)) \\
Prob(1,3) = Prob(3,1) &= \frac{1}{6}(\frac{1}{6} + \frac{|\beta|^2}{2} + \frac{1}{3}(e^{-i\frac{2\pi}{3}}\alpha\beta^* + e^{i\frac{2\pi}{3}}\alpha^*\beta)) \\
Prob(2,2) &= \frac{1}{6}(\frac{1}{6} + \frac{|\beta|^2}{2} + \frac{1}{3}(e^{-i\frac{4\pi}{3}}\alpha\beta^* + e^{i\frac{4\pi}{3}}\alpha^*\beta)) \\
Prob(3,3) &= \frac{1}{6}(\frac{1}{6} + \frac{|\beta|^2}{2} + \frac{1}{3}(e^{i\frac{4\pi}{3}}\alpha\beta^* + e^{-i\frac{4\pi}{3}}\alpha^*\beta)). \qquad (19)
\end{aligned}
$$

By performing an exhaustive search we find the best assignment to be

$$
\begin{aligned}
&\{(0,0),(0,1),(0,2),(0,3),(1,0),(2,0),(3,0),(1,1)\} \to 0 \\
&\{(1,2),(1,3),(2,1),(2,3),(3,1),(3,2),(3,3),(2,2)\} \to 1 \qquad (20)
\end{aligned}
$$

where there is a degeneracy in that swapping out $(1,1)$ for any other outcome not including a 0 is also optimal. This assignment has the associated probabilities

$$
\begin{aligned}
Prob(0) &= \frac{7}{9} - \frac{2}{3}|\beta|^2 + \frac{1}{18}(\alpha\beta^* + \alpha^*\beta) \\
Prob(1) &= \frac{1}{9} + \frac{2}{3}|\beta|^2 - \frac{1}{18}(\alpha\beta^* + \alpha^*\beta) \qquad (21)
\end{aligned}
$$

and $\epsilon = \frac{5}{27} < \frac{1}{3}$ which is an improvement on the single qubit case. We also find again that $y < x$ as $x = \frac{7}{9}$ and $y = \frac{1}{9}$. Figure 2a shows the assignment from equation 20 plotted on a grid and figure 2b shows the ratio of $\frac{y}{x}$ for each outcome separately. The optimal assignment can be interpreted as following a filling algorithm where outcomes are chosen from lowest $\frac{y}{x}$ ratio to highest $\frac{y}{x}$ ratio until half of the outcomes have been selected. In this case, all of the outcomes with a 0 outcome have a ratio of 0 and all the of the outcomes without a 0 have a ratio of 4, hence the degeneracy in optimal assignments as any of the none zero outcomes can be chosen with equal likelihood.

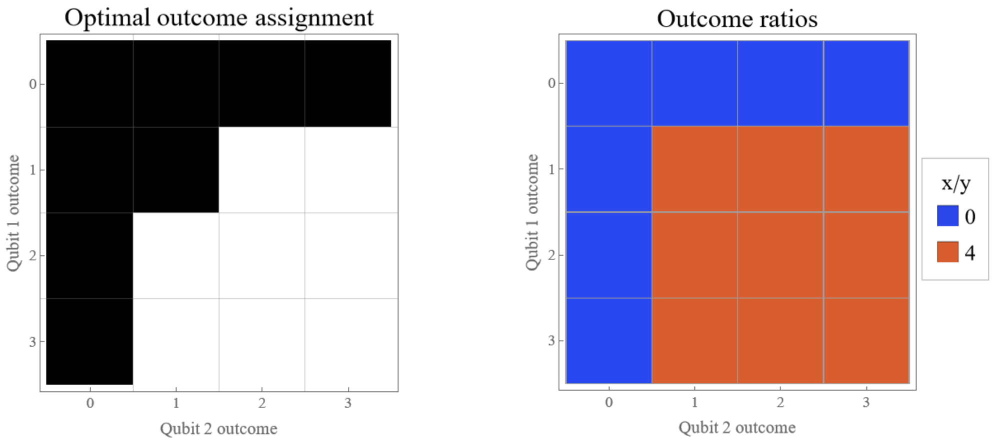
6 Optimising the protocol

We can further optimise the protocol from section 5 by optimising the unitary we perform between the qubits [7]. In general, applying a pre-measurement unitary U on an arbitrary state $|\psi\rangle$ with an ancillary qubit can be written as

$$
|\Psi\rangle = U|\psi\rangle|0\rangle = \alpha|\psi_0\rangle + \beta|\psi_1\rangle \qquad (22)
$$

where

$$
|\psi_0\rangle = U|0\rangle|0\rangle, \quad |\psi_1\rangle = U|1\rangle|0\rangle. \qquad (23)
$$



(a) (b)

Fig. 2: Figures illustrating the assignment of outcomes for the optimal CNOT protocol in equation 20. 2a shows a possible optimal outcome assignment for the pre-measurement CNOT unitary. The outcomes assigned to 0 are black and the outcomes assigned to 1 are white. 2b shows the ratio of y/x (defined in text) for each outcome for the pre-measurement CNOT unitary. All seven outcomes with a 0 have a ratio of 0 so are prioritised by the filling algorithm. This leaves one more outcome to choose but all remaining outcomes have the same ratio of 4 therefore you can choose any one of them, hence the degeneracy. In 2a, we chose $(1, 1)$.

By summing together the POVMs representing each of the outcomes in an assignment, we can define an assignment as a POVM

$$Q_i = \sum_{a_1, a_2 \in \{0,1,2,3\}} \delta_{i, f(a_1, a_2)} M_{a_1} \otimes M_{a_2} \tag{24}$$

where $i \in \{0, 1\}$, $Q_{0(1)}$ is the POVM for assignment to $0(1)$ and $f(a_1, a_2)$ defines the assignment, mapping outcomes to $0(1)$ if they are assigned to $0(1)$. We use equations 3 and 22 to find the probability of measuring an outcome i for a given assignment

$$Prob(i) = \langle \Psi | Q_i | \Psi \rangle = \langle \psi | \langle 0 | U^\dagger Q_i U | \psi \rangle | 0 \rangle \, . \tag{25}$$

It was shown in [7] that for a given assignment we can optimise ϵ, equation 15, to find an optimal (x^*, y^*)[2] which can then be used to derive the optimal pre-measurement unitary U. To do this we only need the minimum and maximum eigenvalues of the assignment POVM, equation 24, for $i = 0$. Our figure of merit is slightly different to

[2] (x^*, y^*) means x and y are optimal.

that used in [7] and has optimal x and y of

$$(x^*, y^*) = \begin{cases} (\lambda_{max}, (\sqrt{2}-1)(1-\lambda_{max})) & \text{if } \lambda_{max} < 1-(\sqrt{2}+1)\lambda_{min}, \\ (\lambda_{max}, \lambda_{min}) & \text{if } 1-(\sqrt{2}+1)\lambda_{min} \leq \lambda_{max} \\ & \quad \leq 1-(\sqrt{2}-1)\lambda_{min}, \\ (1-(\sqrt{2}-1)\lambda_{min}, \lambda_{min}) & \text{if } 1-(\sqrt{2}-1)\lambda_{min} < \lambda_{max}, \end{cases} \tag{26}$$

where $\lambda_{max}, \lambda_{min}$ are the maximum and minimum eigenvalues of Q_0 respectively. If we calculate x and y in this way it is guaranteed ϵ will be minimal. We can use this optimising procedure on the assignment we found in equation 20 to see if we can improve on the CNOT pre-measurement unitary. For this assignment we find $(\lambda_{max}, \lambda_{min}) = (0.8057, 0.08486)$[3] which puts us in the second case of equation 26, $(x^*, y^*) = (\lambda_{max}, \lambda_{min})$. On solving equation 15 we find $\epsilon = 0.1610 < \frac{5}{27} = 0.1852$ which is an improvement on the CNOT gate. To find the pre-measurement unitary U associated with this result we use equation 23 to see that

$$\begin{aligned} (x^*, y^*) = (\langle\psi_0| Q_0 |\psi_0\rangle, \langle\psi_1| Q_0 |\psi_1\rangle) = (\lambda_{max}, \lambda_{min}) \\ \implies |\psi_0\rangle = |\lambda_{max}\rangle, \quad |\psi_1\rangle = |\lambda_{min}\rangle \end{aligned} \tag{27}$$

where $|\lambda_{max}\rangle$ is the normalised eigenvector associated with the maximum eigenvalue and $|\lambda_{min}\rangle$ is the normalised eigenvector associated with the minimum eigenvalue. By equation 23 this partially defines the pre-measurement unitary with the remainder being an arbitrary choice, therefore we have an infinite family of possible unitaries to choose from.

Although we have found the most optimal pre-measurement unitary for the assignment of equation 20 this does not necessarily mean we have found the most optimal protocol. Performing an exhaustive search of all 65536 possible assignments and calculating the optimal figure of merit using the procedure outlined above we find that we can improve on the protocol yet again! There is a degeneracy of 144 optimal protocols each with a figure of merit of $\epsilon = \frac{6-\sqrt{2(5+\sqrt{21})}}{12} = 0.1352 < 0.1610$. One of these protocols,

$$\begin{aligned} \{(0,0),(0,1),(0,2),(0,3),(1,0),(1,1),(2,0),(2,1)\} \to 0 \\ \{(1,2),(1,3),(2,2),(2,3),(3,0),(3,1),(3,2),(3,3)\} \to 1, \end{aligned} \tag{28}$$

is shown in figure 3 along with the $\frac{y}{x}$ ratio for each outcome. By comparing figures 3a and 3b we can see that the filling algorithm has been used for this assignment, however there is no freedom in the choice of outcome like in the CNOT case, which is true for all the optimal protocols, implying each protocol has a different pre-measurement unitary. We also note that the outcome ratios are much more varied than in the CNOT case. We can construct all the assignments of the degenerate optimal protocols by following this construction:

[3] We give the numerical result as the analytical result is quite burdensome.

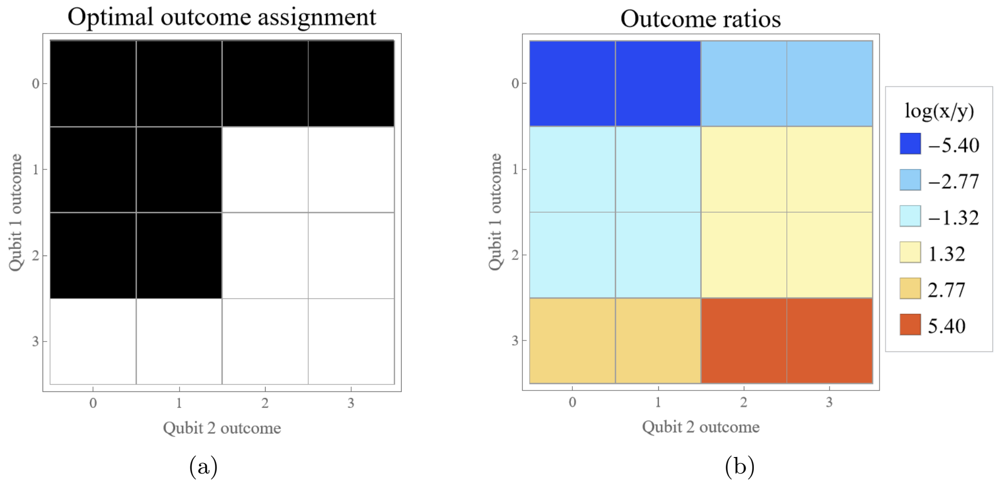


Fig. 3: Figures illustrating the assignment of outcomes for a maximally optimal protocol. 3a shows the outcome assignment in equation 28. The outcomes assigned to 0 are black and the outcomes assigned to 1 are white. 3b shows the logarithm (base 10) of the ratio of y/x (defined in text) for each outcome for the optimised unitary. All the blue tinted outcomes have a higher x than y so are prioritised by the filling algorithm. Unlike the CNOT case, there is no freedom in the choice of outcome with this specific pre-measurement unitary.

- Choose any row and include these outcomes in the assignment.
- Choose any other row and exclude these outcomes in the assignment.
- Choose any 2 columns and include these outcomes in the assignment, except those outcomes in the excluded row.

The transpose of any assignment constructed in this way is also valid. The fact we can construct all maximally optimal assignments in this way implies all maximally optimal protocols belong to the same symmetry group. Note that the assignment shown in figure 3a can be constructed in this way however none of the optimal assignments for the CNOT pre-measurement unitary, eg. figure 2a, belong to this symmetry.

For some further insight into the structure of these optimal protocols we can examine the eigenvectors and eigenvalues of the assignment POVMs. The eigenvectors and eigenvalues for the assignment given in equation 28 are summarised in table 1. For all maximally optimal protocols $(x^*, y^*) = (\lambda_{max}, \lambda_{min}) = (\frac{1}{6}(3 + \sqrt{\frac{5+\sqrt{21}}{2}}), \frac{1}{6}(3 - \sqrt{\frac{5+\sqrt{21}}{2}}))$ so always lie in the second case of equation 26. This means equation 27 also applies here, so we can use the eigenvectors to define an optimal pre-measurement

Table 1: Eigenvectors and eigenvalues of an optimal protocol

Variable	Eigenvector	Eigenvalue
λ_0	$\|\phi_{min}\rangle \mathbf{Y} \|\chi\rangle$	$\frac{1}{6}(3-\sqrt{\frac{5+\sqrt{21}}{2}})$
λ_1	$\mathbf{YR} \|\phi_{max}\rangle \|\chi\rangle$	$\frac{1}{6}(3-\sqrt{\frac{5-\sqrt{21}}{2}})$
λ_2	$\mathbf{YR} \|\phi_{min}\rangle \mathbf{Y} \|\chi\rangle$	$\frac{1}{6}(3+\sqrt{\frac{5-\sqrt{21}}{2}})$
λ_3	$\|\phi_{max}\rangle \|\chi\rangle$	$\frac{1}{6}(3+\sqrt{\frac{5+\sqrt{21}}{2}})$

$$|\chi\rangle = \sqrt{\frac{\sqrt{3}+1}{2\sqrt{3}}}\,|0\rangle + \sqrt{\frac{9-3\sqrt{3}}{18}}\,|1\rangle$$
$$|\phi_{min}\rangle = \frac{1}{\sqrt{42}}(\sqrt{21-6\sqrt{7}-\sqrt{21}}\,|0\rangle + \sqrt{21+6\sqrt{7}+\sqrt{21}}e^{i\frac{2\pi}{3}}\,|1\rangle)$$
$$|\phi_{max}\rangle = \frac{1}{\sqrt{36\sqrt{7}}}(\sqrt{36-6\sqrt{3}+18\sqrt{7}}\,|0\rangle + \sqrt{-36+6\sqrt{3}+18\sqrt{7}}e^{-i\frac{\pi}{3}}\,|1\rangle)$$

Table illustrating the eigenvectors and eigenvalues of the POVM defined by the assignment in equation 28. $\mathbf{Y}$ is the Pauli Y matrix and $\mathbf{R} = |0\rangle\langle 0| + e^{i\frac{2\pi}{3}}|1\rangle\langle 1|$. Some quantities to note are that $\lambda_0 = \lambda_{min}$, $\lambda_3 = \lambda max$ and $\langle\phi_{max}|\phi_{min}\rangle = \frac{1}{\sqrt{7}}$. $\langle\phi_{max}|\,\mathbf{YR}\,|\phi_{max}\rangle = \langle\phi_{min}|\,\mathbf{YR}\,|\phi_{min}\rangle = \langle\chi|\,\mathbf{Y}\,|\chi\rangle = 0$ therefore all the eigenvectors are orthogonal.

unitary operator

$$U := \begin{cases} |0\rangle|0\rangle \to |\phi_{max}\rangle|\chi\rangle \\ |1\rangle|0\rangle \to |\phi_{min}\rangle\mathbf{Y}|\chi\rangle \\ |0\rangle|1\rangle \to \mathbf{YR}|\phi_{max}\rangle|\chi\rangle \\ |1\rangle|1\rangle \to \mathbf{YR}|\phi_{min}\rangle\mathbf{Y}|\chi\rangle\,. \end{cases} \tag{29}$$

where the variables are defined in table 1. The eigenvectors for all the optimal protocols are completely separable implying the pre-measurement unitary is a non-entangling gate. Only the first two mappings of $|0\rangle|0\rangle$ and $|1\rangle|0\rangle$ are required by our protocol, the final two mappings are arbitrary up to orthogonality but the ones given define a basis for the space in which the final two mappings must lie.

Below is a summary of the optimal protocol we have just discussed.

1. We start with an arbitrary state we wish to measure accurately using the tetrahedron measurement, $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$.
2. We introduce an ancillary qubit in state $|0\rangle$ and apply the unitary operator from equation 29, $U|\psi\rangle|0\rangle = \alpha|\phi_{max}\rangle|\chi\rangle + \beta|\phi_{min}\rangle\mathbf{Y}|\chi\rangle$.
3. We measure both qubits using the tetrahedron measurement getting a result of (l,k), $l,k \in \{0,1,2,3\}$.
4. We return either 0 or 1 based on the assignment in equation 28.

7 Conclusion

In this report we have discussed the theory of POVMs and their application to modelling noisy measurements. We used the tetrahedron measurement as an example of a POVM and demonstrated how we could map it onto the standard basis by attempting to replicate the statistics of a perfect measurement. We introduced a figure of merit ϵ to quantify how close an arbitrary measurement was to a perfect measurement and showed the best we could do using only the qubit we were given and the tetrahedron measurement was $\epsilon = \frac{1}{3}$. We then showed that by introducing a ancillary qubit we could improve this result, eventually ending up at $\epsilon = \frac{6-\sqrt{2(5+\sqrt{21})}}{12}$, a significant improvement of a factor of $\frac{6-\sqrt{2(5+\sqrt{21})}}{4}$ on measuring the qubit directly. We achieved this by iteratively working through a number of examples, leaving many open questions along the way. First, we showed how by using a basic CNOT gate and finding an ideal assignment, we were already able to improve the result to $\epsilon = \frac{5}{7}$, we then improved on this result again to $\epsilon = 0.1610$ by optimising the unitary associated with the optimal CNOT assignment. It would be interesting to investigate the optimisation further as we only looked at a single assignment from the 9 possible optimal assignments of the CNOT. We could analyse all these assignments to see what they have in common, as well as any other assignments which may have the same the same ϵ, once optimised, and then we can see how and if their pre-measurement unitaries relate back to the CNOT. The final optimisation involved searching all of the possible assignments to find the ones with the lowest ϵ. Surprisingly, these most optimal assignments did not include the CNOT assignment we had originally found. As with the CNOT case it would be interesting to investigate these fully optimised protocols further to see what they have in common, beyond the symmetry shown in the report, with the goal of understanding why it is these assignments and this symmetry specifically which is most optimal.

Beyond what has already been discussed there are many exciting directions in which to take this project. It was shown in [7] that if we assign the outcomes probabilistically, instead of deterministically as we have been doing, then unintuitvely it is possible to improve on ϵ even further. We can study this effect for the simple two qubit tetrahedron measurement we have already analysed thoroughly in the deterministic setting. It was also shown in [7] that it is possible to improve ϵ still further by introducing more ancillary qubits, in fact it was shown as we increase the number of qubits linearly ϵ improves exponentially. It would be interesting to study this effect using the tetrahedron to help get a better understanding of what is going on. We can also study other examples of POVMs which have direct practical application, such as the imperfect Z measurement, which is relevant in superconducting quantum computers.

At the beginning of this project, analysis of the tetrahedron measurement in the two qubit case was meant to be a simple warm up exercise, however this example proved to be much more rich in detail than originally expected and is testament to the amount we have yet to understand in this exciting new area of research. Over the next few years of my PhD I hope to make a significant dent in this problem.

Appendix A Code

The code used for all the calculations and plots in this paper can be accessed from: https://github.com/custal/noisy_quantum_measurements

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