CS 3530: Assignment 7c

Fall 2023

Problem 7.23 (20 points)

Problem

Let HALF-CLIQUE = $\{\langle G\rangle | G$ is an undirected graph having a complete subgraph with at least $m/2$ nodes, where m is the number of nodes in $G\}$. Show that HALF-CLIQUE is NP-complete.

Note: In order to receive credit for this assignment, you must complete the full NP-completeness proof process outlined here.

Prove Half-Clique $\in$ NP

Describe a certificate for Half-Clique

A subset of all nodes in G which we will call n, and a subset of all edges in G which we will call e, these subset are our potential clique we will verify.

Provide a polynomial verifier for Half-Clique

N = "On input $\langle n, e\rangle$:

1. Verify that n is at least of size k/2 where k is the total number of nodes in G.
2. If n is smaller than k/2, reject.
3. Check that each node in n has an edge in e for each other node in n besides itself.
4. If all nodes are connected accept, else reject."

Therefore HALF-CLIQUE $\in$ NP

Prove Half-Clique is NP-hard

Given that CLIQUE is NP-complete, show that CLIQUE $\leq_P$ HALF-CLIQUE with the following steps.

Provide reduction from Clique to Half-Clique

On input $\langle G, k\rangle$ where G is a graph of n vertices and k is an integer:

1. If k = n/2 then output $\langle G\rangle$
2. If k < n/2 construct a new graph G' by adding a complete graph with n-2k vertices and connecting them to all vertices in G, and output $\langle G'\rangle$
3. If k > n/2 construct a new graph G" by adding 2k-n isolate vertices to G, and output $\langle G''\rangle$

Prove reduction from Clique to Half-Clique is polynomial

There are no steps that have the potential to be non-polynomial as the steps will always be at most n which is polynomial

Showing Clique and Half-Clique are satisfied

When k = n/2: it is clear that $\langle G, n/2 \rangle \in CLIQUE$ iff $\langle G \rangle \in HALF - CLIQUE$

When k < n/2: if G has a k-clique, then G' has a clique of size k+(n-2k)=(2n-2k)/2

$\langle G' \rangle \in HALF - CLIQUE$ as G' is a graph with 2n-2k vertices

if $\langle G' \rangle \in HALF - CLIQUE$ then at most n-2k of the clique come from the n-2k new vertices. Therefore the remaining at least k vertices from a clique in G.

So, $\langle G, k \rangle \in CLIQUE$

When k > n/2: if G has a k-clique then G" has a clique size k= 2k/2, and

$\langle G' \rangle \in HALF - CLIQUE$ as G" is a graph with n + 2k - n = 2k vertices.

if $\langle G'' \rangle \in HALF - CLIQUE$ then the clique does not contain any of the new vertices as they are isolated.

Thus, the clique is a k-clique of G, and $\langle G, k \rangle \in CLIQUE$

Conclude that Half-Clique is NP-hard

Because a language must be NP-hard in order to be NP-complete, and we know that CLIQUE is NP-complete we can also conclude that it is NP-hard

Because we provided the reduction from Clique to Half-Clique we can conclude that Half-Clique is NP-hard

Conclude that Half-Clique is NP-complete

Since we know CLIQUE is NP-complete we can conclude HALF-CLIQUE is also NP-complete as there is a reduction from CLIQUE to HALF-CLIQUE