CS 3530: Assignment 7e

Fall 2023

Problem 7.35 (20 points)

Problem

A subset of the nodes of a graph G is a ***dominating set*** if every other node of G is adjacent to some node in the subset. Let

$$\text{DOMINATING-SET} = \{\langle G, k\rangle | G \text{ has a dominating set with } k \text{ nodes } \}$$

.

Show that DOMINATING-SET is NP-complete. You may assume that VERTEX-COVER is NP-complete.

Note: In order to receive credit for this assignment, you must complete the full NP-completeness proof process outlined here.

Prove Dominating-Set $\in$ NP

Describe a certificate for Dominating-Set

We get two sets G and D, G is the set of all nodes that are part of the dominant set, and D contains all nodes that are not part of the dominant set

Provide a polynomial verifier for Dominating-Set

Check that each node in D is adjacent to some node in G This is done in polynomial time because it will only take G*D steps

Prove Dominating-Set is NP-hard

Given that VERTEX-COVER is NP-complete, show that VERTEX-COVER $\leq_P$ DOMINATING-SET with the following steps.

Provide reduction from Vertex-Cover to Dominating-Set

Consider an instance $\langle (V, E), k\rangle$ of VERTEX-COVER

Construct G' by creating a new graph such that there is a vertex for each vertex that exists in G

For each edge in G add a vertext to G' such that for an edge e = (u,v), there now exists a new vertex z and the edges (u, v), (u, z), (u, z)

G = $\langle V, E\rangle$

G' = $\langle (V - S) \bigcup V', E \bigcup E'\rangle$ where $S \subseteq V$ are nodes of degree 0

Prove reduction from Vertex-Cover to Dominating-Set is polynomial

Because for each edge in G we are creating a vertex + 3 edges at most the operations with be V^4 where V is the amount of vertices in G

Show Vertex-Cover has a size k cover if Dominating-Set has a size k' dominating set

Show Dominating-Set has a size k' dominating set if Vertex-Cover has a size k cover

Suppose $\langle (V, E), k\rangle$ is in VERTEX-COVER

There exist $C \subseteq V$ of size k where each edge $(u, v) \in E$ has either $u \in C$ or $v \in C$

if $v \in (V - S)$ then the degree of v is one or more, then there exist a node u such that (u,v) $\in$ E which implies that either u or v is in C, which means v is covered

if $w \in V'$ then w is adjacent to both u and v where (u,v) $\in$ E which implies that at least either u or v is in C which means w is covered

Suppose that $\langle (((V - S) \bigcup V', E \bigcup E'), k\rangle$ is in DOMINATING-SET

Then there exist $C \subseteq ((V - S) \bigcup V')$ of size k

In cases where multiple such C exists we can say that at least one will include no vertices inside V'

This always exists since $w \in (C \bigcap V')$ that corresponds to edge (u,v) which only covers nodes u,v,w, but using u instead of w covers u,v,w and possibly more

Therefore $C \subseteq (V - S)$ and c is a vertext cover for G

This is because C is a DOMINATING-SET for G' implying that all nodes of V' are covered, so every edge (u,v) $\in$ E has at least one u,v in c

This shows that vertex cover reduces to dominating set

Conclude that Dominating-Set is NP-hard

Because VERTEX-COVER is NP-hard, by providing the reduction we can say DOMINATING-SET is also NP-hard

Conclude that Dominating-Set is NP-complete

Because DOMINATING-SET is in NP and is NP-hard, it is NP-complete