6. Data Types

(a) The set of Booleans.

(b) The finite sets.

(c) Atomic formulae and the traffic light example. (The example will be omitted 2008).

(d) The disjoint union of sets and disjunction.

(e) The Σ-set. (Will be omitted 2008.)

(f) Natural Deduction and Dependent Type Theory. (Will be largely omitted 2008).

(g) The set of natural numbers.

(h) Lists. (Will probably be omitted 2008.)

(i) Universes. (Will probably be omitted 2008.)

(j) Algebraic types. (Will be omitted 2008.)

(a) The Set of Booleans

Formation Rule

$$\mathrm{Bool} : \mathrm{Set} \quad (\mathrm{Bool\text{-}F})$$

Introduction Rules

$$\mathrm{tt} : \mathrm{Bool} \quad (\mathrm{Bool\text{-}I}_{\mathrm{tt}}) \qquad\qquad \mathrm{ff} : \mathrm{Bool} \quad (\mathrm{Bool\text{-}I}_{\mathrm{ff}})$$

Elimination Rule

$$\frac{C : \mathrm{Bool} \to \mathrm{Set} \quad case_{\mathrm{tt}} : C\ \mathrm{tt} \quad case_{\mathrm{ff}} : C\ \mathrm{ff} \quad b : \mathrm{Bool}}{\mathrm{Case}_{\mathrm{Bool}}\ C\ case_{\mathrm{tt}}\ case_{\mathrm{ff}}\ b : C\ b}\ (\mathrm{Bool\text{-}El})$$

The Set of Booleans (Cont.)

Equality Rules

$$\frac{C : \mathrm{Bool} \to \mathrm{Set} \qquad case_{\mathrm{tt}} : C\ \mathrm{tt} \qquad case_{\mathrm{ff}} : C\ \mathrm{ff}}{\mathrm{Case}_{\mathrm{Bool}}\ C\ case_{\mathrm{tt}}\ case_{\mathrm{ff}}\ \mathrm{tt} = case_{\mathrm{tt}} : C\ \mathrm{tt}}\ (\mathrm{Bool\text{-}Eq}_{\mathrm{tt}})$$

$$\frac{C : \mathrm{Bool} \to \mathrm{Set} \qquad case_{\mathrm{tt}} : C\ \mathrm{tt} \qquad case_{\mathrm{ff}} : C\ \mathrm{ff}}{\mathrm{Case}_{\mathrm{Bool}}\ C\ case_{\mathrm{tt}}\ case_{\mathrm{ff}}\ \mathrm{ff} = case_{\mathrm{ff}} : C\ \mathrm{ff}}\ (\mathrm{Bool\text{-}Eq}_{\mathrm{ff}})$$

Further we have equality versions of the formation-, introduction- and elimination-rules.

Remarks

- $\mathrm{Case}_{\mathrm{Bool}}\ C\ \ case_{\mathrm{tt}}\ case_{\mathrm{ff}}\ b$ can be read as

  $$\mathrm{if}\ b\ \mathrm{then}\ case_{\mathrm{tt}}\ \mathrm{else}\ case_{\mathrm{ff}}$$

  where the additional argument C is required in order to determine the type of $case_{\mathrm{tt}}$, of $case_{\mathrm{ff}}$, and of the result of this construct.

Remarks (Cont.)

- The argument $C : \mathrm{Bool} \to \mathrm{Set}$ denotes the set into which we are eliminating.
  - Instead of $C : \mathrm{Set}$, we demand $C : \mathrm{Bool} \to \mathrm{Set}$, since the set into which we are eliminating might depend on the Boolean valued argument.
  - That is necessary in order to define functions $f : (b : \mathrm{Bool}) \to D$ where D depends on b.

Remarks (Cont.)

- If we define

$$\begin{array}{rcl} C & := & \lambda b^{\mathrm{Bool}}.D \\ & : & \mathrm{Bool} \longrightarrow \mathrm{Set} \\ f & := & \lambda b^{\mathrm{Bool}}.\mathrm{Case}_{\mathrm{Bool}}\ C\ case_{\mathrm{tt}}\ case_{\mathrm{ff}}\ b \\ & : & (b : \mathrm{Bool}) \longrightarrow C\ b \end{array}$$

  where

$$(b : \mathrm{Bool}) \longrightarrow C\ b = (b : \mathrm{Bool}) \longrightarrow D$$

  we have:
  - $f\ \mathrm{tt} : C\ \mathrm{tt}$.
  - $f\ \mathrm{ff} : C\ \mathrm{ff}$.
  - $f : (b : \mathrm{Bool}) \longrightarrow C\ b$.

Remarks (Cont.)

- The argument C above has no computational content.
  - It is not needed in order to compute $\mathrm{Case}_{\mathrm{Bool}}\ C\ case_{\mathrm{tt}}\ case_{\mathrm{ff}}\ \mathrm{tt}$ and $\mathrm{Case}_{\mathrm{Bool}}\ C\ case_{\mathrm{tt}}\ case_{\mathrm{ff}}\ \mathrm{ff}$.
- C is only needed in order to obtain decidable type checking:
  - In the presence of arguments like this we can decide whether a judgement $a : B$ is derivable.

Remarks (Cont.)

- We can write the elimination rule in a **more compact** but less readable way:
  - $$\begin{aligned}\mathrm{Case}_{\mathrm{Bool}} : (C : \mathrm{Bool} \rightarrow \mathrm{Set}) \qquad \qquad .\\ \rightarrow (case_{\mathrm{tt}} : C\ \mathrm{tt})\\ \rightarrow (case_{\mathrm{ff}} : C\ \mathrm{ff})\\ \rightarrow (b : \mathrm{Bool})\\ \rightarrow C\ b\end{aligned}$$
- tt, ff are the **constructors** of Bool.

Remarks (Cont.)

- Notice that we then get for $C : \mathrm{Bool} \to \mathrm{Set}$, $case_{\mathrm{tt}} : C\ \mathrm{tt}, case_{\mathrm{ff}} : C\ \mathrm{ff}$
  - $f := \mathrm{Case}_{\mathrm{Bool}}\ C\ case_{\mathrm{tt}}\ case_{\mathrm{ff}}$,
    $: (b : \mathrm{Bool}) \to C\ b$
  - $f\ \mathrm{tt} = \mathrm{Case}_{\mathrm{Bool}}\ C\ case_{\mathrm{tt}}\ case_{\mathrm{ff}}\ \mathrm{tt} = case_{\mathrm{tt}} : C\ \mathrm{tt}$,
  - $f\ \mathrm{ff} = \mathrm{Case}_{\mathrm{Bool}}\ C\ case_{\mathrm{tt}}\ case_{\mathrm{ff}}\ \mathrm{ff} = case_{\mathrm{ff}} : C\ \mathrm{ff}$.
- So we obtain functions from Bool into other sets **without having to write $\lambda \mathrm{b}^{\mathbf{Bool}}.\cdots$**.
- That's why we choose the argument to eliminate from as the **last one**.

Remarks (Cont.)

- This is similar to the definition of for instance (+) in **curried form** in Haskell
  - $(+) : \mathrm{int} \to \mathrm{int} \to \mathrm{int}$.
  - $(+)\ 3$ is the function which takes an integer and adds to it 3.
    - **Shorter** than writing $\lambda x^{\mathrm{int}}.3 + x$.

Remarks (Cont.)

- Note that we have the following **order of the arguments** of $\mathrm{Case}_{\mathrm{Bool}}$:
  - First we have the **set into which we eliminate**.
  - Then follow the **cases**, one for each constructor.
  - Finally we put the **element which we are eliminating**.
- In some sense $\mathrm{Case}_{\mathrm{Bool}}$ is a “then _else _if ” – the **condition** (if $\ldots$) **is the last one**.

Select Example

- Assume we have introduced in type theory

$$
\begin{array}{lcl}
\mathrm{Name} & : & \mathrm{Bool} \rightarrow \mathrm{Set} \quad , \\
\mathrm{Name}\ \mathrm{tt} & = & \mathrm{FemaleName} \quad , \\
\mathrm{Name}\ \mathrm{ff} & = & \mathrm{MaleName} \quad .
\end{array}
$$

Select Example

- Then we can define the function

$$\begin{array}{rcl}\mathrm{SelectBool} & : & (b : \mathrm{Bool}) \to \mathrm{Name}\ b \\ \mathrm{SelectBool}\ \mathrm{tt} & = & \mathrm{sara} \\ \mathrm{SelectBool}\ \mathrm{ff} & = & \mathrm{tom}\end{array}$$

  as follows:

$$\mathrm{SelectBool} = \mathrm{Case}_{\mathrm{Bool}}\ \mathrm{Name}\ \mathrm{sara}\ \mathrm{tom}$$

  - Note that by using twice the η-rule we get that

$$\begin{array}{l}\mathrm{SelectBool} \\ \quad = \quad \lambda b^{\mathrm{Bool}}.\mathrm{Case}_{\mathrm{Bool}}\ (\lambda d^{\mathrm{Bool}}.\mathrm{Name}\ d)\ \mathrm{sara}\ \mathrm{tom}\ b\end{array}$$

Select Example

- We verify the correctness of $\mathrm{SelectBool}$:

$$\begin{aligned}\mathrm{SelectBool}\ \mathrm{tt} &= \mathrm{Case}_{\mathrm{Bool}}\ \mathrm{Name}\ \mathrm{sara}\ \mathrm{tom}\ \mathrm{tt} = \mathrm{sara}\quad ,\\ \mathrm{SelectBool}\ \mathrm{ff} &= \mathrm{Case}_{\mathrm{Bool}}\ \mathrm{Name}\ \mathrm{sara}\ \mathrm{tom}\ \mathrm{ff} = \mathrm{tom}\quad .\end{aligned}$$

Jump over $\wedge_{\mathrm{Bool}}$

Example: $\wedge_{\mathrm{Bool}}$

- We want to introduce conjunction

$$\wedge_{\mathrm{Bool}} : \mathrm{Bool} \to \mathrm{Bool} \to \mathrm{Bool} \quad .$$

- This will be of the form

$$\wedge_{\mathrm{Bool}} = \ \lambda(b, c : \mathrm{Bool}).t$$

  for some term t.

- t will be defined by case distinction on b, so we get

$$\wedge_{\mathrm{Bool}} = \ \lambda(b, c : \mathrm{Bool}).\mathrm{Case}_{\mathrm{Bool}}\ C\ e\ f\ b$$

  for some e, f.

Example: $\wedge_{\mathrm{Bool}}$

$$\wedge_{\mathrm{Bool}} = \ \lambda(b, c : \mathrm{Bool}).\mathrm{Case}_{\mathrm{Bool}}\ C\ e\ f\ b$$

- C will be the set into which we are eliminating, depending on a Boolean value.
  - It need to be an element of $\mathrm{Bool} \to \mathrm{Set}$.
  - Therefore we have $C = \ \lambda d^{\mathrm{Bool}}.D$ for some D which might depend on d.
  - The set, into which we are eliminating, is always the same, namely Bool.
  - So $D = \mathrm{Bool}$ and therefore we have

$$C = \ \lambda d^{\mathrm{Bool}}.\mathrm{Bool} \ \ .$$

Example: $\bigwedge_{\mathrm{Bool}}$

- Note that in

$$\lambda d^{\mathrm{Bool}}.\mathrm{Bool}$$

Bool occurs in two different meanings:

  - The first occurrence is that of a set.
    - d is chosen here as an element of that set.
  - The second occurrence is that as an element of another type, namely Set.
    - So here Bool is a term.

Two Meanings of Elements of Set

- All elements A of Set have these two meanings:
  - They can be used as terms, which are elements of the type Set.
    - The corresponding judgements are $A : \mathrm{Set}$, $A = A' : \mathrm{Set}$.
  - And they can be used as sets, which have elements.
    - The corresponding judgements are $a : A$ and $a = a' : A$.

Example: $\wedge_{\mathrm{Bool}}$

- So

$$\wedge_{\mathrm{Bool}} = \ \lambda(b, c : \mathrm{Bool}).\mathrm{Case}_{\mathrm{Bool}}\ (\lambda d^{\mathrm{Bool}}.\mathrm{Bool})\ e\ f\ b$$

  for some e, f.

- For conjunction we have:
  - If b is true then

$$b \wedge c = \mathrm{tt} \wedge c = c$$

    - So the if-case e above is c.
  - If c is false then

$$b \wedge c = \mathrm{ff} \wedge c = \mathrm{ff}$$

    - So the else-case f above is ff.

Example: $\wedge_{\mathrm{Bool}}$

- In total we define therefore

$$\begin{array}{rcl} \wedge_{\mathrm{Bool}} & = & \lambda(b, c : \mathrm{Bool}).\mathrm{Case}_{\mathrm{Bool}}\ (\lambda d^{\mathrm{Bool}}.\mathrm{Bool})\ c\ \mathrm{ff}\ b \\ & : & \mathrm{Bool} \to \mathrm{Bool} \to \mathrm{Bool} \end{array}$$

- We verify the correctness of this definition:
  - $\wedge_{\mathrm{Bool}}\ \mathrm{tt}\ c = \mathrm{Case}_{\mathrm{Bool}}\ (\lambda d^{\mathrm{Bool}}.\mathrm{Bool})\ c\ \mathrm{ff}\ \mathrm{tt} = c$.
    as desired.
  - $\wedge_{\mathrm{Bool}}\ \mathrm{ff}\ c = \mathrm{Case}_{\mathrm{Bool}}\ (\lambda d^{\mathrm{Bool}}.\mathrm{Bool})\ c\ \mathrm{ff}\ \mathrm{ff} = \mathrm{ff}$.
    Correct as desired.

Jump over derivation of $\wedge_{\mathrm{Bool}}$

Derivation of $\wedge_{\mathrm{Bool}}$

- We derive in the following $\wedge_{\mathrm{Bool}} : \mathbf{Bool} \to \mathbf{Bool} \to \mathbf{Bool}$.
- We write Bool, if it
  - is a type in **boldface red**,
  - and if it is a term, in $\mathit{italic\ blue}$.

Derivation of $\wedge_{\mathrm{Bool}}$

- First we derive
  $b : \mathbf{Bool}, c : \mathbf{Bool} \Rightarrow \lambda(d^{\mathbf{Bool}}).\mathit{Bool} : \mathbf{Bool} \to \mathrm{Set}$:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathit{Bool} : \mathrm{Set}}{b : \mathbf{Bool} \Rightarrow \mathrm{Context}}\,(\mathrm{Context}_1)}{b : \mathbf{Bool} \Rightarrow \mathit{Bool} : \mathrm{Set}}\,(\mathrm{Bool\text{-}F})}{b : \mathbf{Bool}, c : \mathbf{Bool} \Rightarrow \mathrm{Context}}\,(\mathrm{Context}_1)}{b : \mathbf{Bool}, c : \mathbf{Bool} \Rightarrow \mathit{Bool} : \mathrm{Set}}\,(\mathrm{Bool\text{-}F})}{b : \mathbf{Bool}, c : \mathbf{Bool}, d : \mathbf{Bool} \Rightarrow \mathrm{Context}}\,(\mathrm{Context}_1)}{b : \mathbf{Bool}, c : \mathbf{Bool}, d : \mathbf{Bool} \Rightarrow \mathit{Bool} : \mathrm{Set}}\,(\mathrm{Bool\text{-}F})}{b : \mathbf{Bool}, c : \mathbf{Bool} \Rightarrow \lambda d^{\mathbf{Bool}}.\mathit{Bool} : \mathbf{Bool} \to \mathrm{Set}}\,(\to\text{-I})
$$

Derivation of $\wedge_{\mathrm{Bool}}$

- We derive

$$b : \mathbf{Bool}, c : \mathbf{Bool} \Rightarrow \mathbf{Bool} = (\lambda d^{\mathbf{Bool}}.\mathit{Bool})\ \mathrm{tt} : \mathrm{Set}$$

(using part of the derivation above):

$$\dfrac{\dfrac{\dfrac{\cdots}{b{:}\mathbf{Bool},c{:}\mathbf{Bool},d{:}\mathbf{Bool}\Rightarrow\mathrm{Context}}}{b{:}\mathbf{Bool},c{:}\mathbf{Bool},d{:}\mathbf{Bool}\Rightarrow\mathit{Bool}{:}\mathrm{Set}}\,(\mathrm{Bool}\text{-}\mathrm{F}) \qquad \dfrac{\dfrac{\cdots}{b{:}\mathbf{Bool},c{:}\mathbf{Bool}\Rightarrow\mathrm{Context}}}{b{:}\mathbf{Bool},c{:}\mathbf{Bool}\Rightarrow\mathrm{tt}{:}\mathbf{Bool}}\,(\mathrm{Bool}\text{-}\mathrm{I}}{\dfrac{b{:}\mathbf{Bool},c{:}\mathbf{Bool}\Rightarrow(\lambda d^{\mathbf{Bool}}.\mathit{Bool})\ \mathrm{tt}{=}\mathit{Bool}{:}\mathrm{Set}}{b{:}\mathbf{Bool},c{:}\mathbf{Bool}\Rightarrow\mathbf{Bool}{=}(\lambda d^{\mathrm{Bool}}.\mathit{Bool})\ \mathrm{tt}{:}\mathrm{Set}}\,(\mathrm{Sym}_{\mathrm{Elem}})}\,(\rightarrow\text{-}\mathrm{Eq})$$

Derivation of $\wedge_{\mathrm{Bool}}$

- Similarly follows

$$b : \mathbf{Bool}, c : \mathbf{Bool} \Rightarrow \mathbf{Bool} = (\lambda d^{\mathbf{Bool}}.Bool)\ \mathrm{ff} : \mathrm{Set}$$

Derivation of $\wedge_{\mathrm{Bool}}$

- Using part of the proof above, we derive

$$b : \mathbf{Bool}, c : \mathbf{Bool} \Rightarrow c : (\lambda d^{\mathbf{Bool}}.\mathit{Bool})\ \mathrm{tt}$$

$$\dfrac{\dfrac{\dfrac{\cdots}{b{:}\mathbf{Bool},c{:}\mathbf{Bool}\Rightarrow\mathrm{Context}}}{b{:}\mathbf{Bool},c{:}\mathbf{Bool}\Rightarrow c{:}\mathbf{Bool}}\,(\mathrm{Ass}) \qquad \dfrac{\cdots}{b{:}\mathbf{Bool},c{:}\mathbf{Bool}\Rightarrow\mathbf{Bool}=(\lambda d^{\mathbf{Bool}}.\mathit{Bool})\ \mathrm{tt}{:}\mathrm{Set}}}{b{:}\mathbf{Bool},c{:}\mathbf{Bool}\Rightarrow c{:}(\lambda d^{\mathbf{Bool}}.\mathit{Bool})\ \mathrm{tt}}\,(\mathrm{Transfer}$$

- We derive using $(\mathrm{Transfer}_0)$

$$b : \mathbf{Bool}, c : \mathbf{Bool} \Rightarrow \mathrm{ff} : (\lambda d^{\mathbf{Bool}}.\mathit{Bool})\ \mathrm{ff}$$

$$\dfrac{\dfrac{\dfrac{\cdots}{b{:}\mathbf{Bool},c{:}\mathbf{Bool}\Rightarrow\mathrm{Context}}}{b{:}\mathbf{Bool},c{:}\mathbf{Bool}\Rightarrow\mathrm{ff}{:}\mathbf{Bool}}\,(\mathrm{Bool}\text{-}\mathrm{I}_{\mathrm{ff}}) \qquad \dfrac{\cdots}{b{:}\mathbf{Bool},c{:}\mathbf{Bool}\Rightarrow\mathbf{Bool}=(\lambda d^{\mathbf{Bool}}.\mathit{Bool})\ \mathrm{ff}{:}\mathrm{Set}}}{b{:}\mathbf{Bool},c{:}\mathbf{Bool}\Rightarrow\mathrm{ff}{:}(\lambda d^{\mathbf{Bool}}.\mathit{Bool})\ \mathrm{ff}}\,(\mathrm{Tra}$$

Derivation of $\wedge_{\mathrm{Bool}}$

- We derive $b : \mathbf{Bool}, c : \mathbf{Bool} \Rightarrow b : \mathbf{Bool}$ using part of the proof above:

$$\frac{\begin{array}{c}\cdots\\ b : \mathbf{Bool}, c : \mathbf{Bool} \Rightarrow \mathrm{Context}\end{array}}{b : \mathbf{Bool}, c : \mathbf{Bool} \Rightarrow b : \mathbf{Bool}}\,(\mathrm{Ass})$$

Derivation of $\wedge_{\mathrm{Bool}}$

- Finally we obtain our judgement (we stack the premises of the rule because of lack of space):

$$
\dfrac{\dfrac{\dfrac{\begin{array}{c}
b{:}\mathbf{Bool},c{:}\mathbf{Bool} \Rightarrow \lambda d^{\mathbf{Bool}}.\mathit{Bool}{:}\mathbf{Bool}\rightarrow\mathrm{Set}\\
b{:}\mathbf{Bool},c{:}\mathbf{Bool} \Rightarrow c{:}(\lambda d^{\mathbf{Bool}}.\mathit{Bool})\ \mathrm{tt}\\
b{:}\mathbf{Bool},c{:}\mathbf{Bool} \Rightarrow \mathrm{ff}{:}(\lambda d^{\mathbf{Bool}}.\mathit{Bool})\ \mathrm{ff}\\
b{:}\mathbf{Bool},c{:}\mathbf{Bool} \Rightarrow b{:}\mathbf{Bool}
\end{array}}{b{:}\mathbf{Bool},c{:}\mathbf{Bool} \Rightarrow \mathrm{Case}_{\mathrm{Bool}}\ (\lambda d^{\mathbf{Bool}}.\mathit{Bool})\ c\ \mathrm{ff}\ b{:}\mathbf{Bool}}\ (\mathrm{Bool\text{-}El})}{b{:}\mathbf{Bool} \Rightarrow \lambda c^{\mathbf{Bool}}.\mathrm{Case}_{\mathrm{Bool}}\ (\lambda d^{\mathbf{Bool}}.\mathit{Bool})\ c\ \mathrm{ff}\ b{:}\mathbf{Bool}\rightarrow\mathbf{Bool}}\ (\rightarrow\text{-I})}{\lambda(b,c{:}\mathbf{Bool}).\mathrm{Case}_{\mathrm{Bool}}\ (\lambda d^{\mathbf{Bool}}.\mathit{Bool})\ c\ \mathrm{ff}\ b{:}\mathbf{Bool}\rightarrow\mathbf{Bool}\rightarrow\mathbf{Bool}}\ (\rightarrow\text{-I})
$$

Elimination into Type

We can extend add elimination and equality rules, having as result Type:

Elimination Rule into Type

$$\frac{C{:}\mathrm{Bool}{\rightarrow}\mathrm{Type} \qquad case_{\mathrm{tt}}{:}C\ \mathrm{tt} \qquad case_{\mathrm{ff}}{:}C\ \mathrm{ff} \qquad b{:}\mathrm{Bool}}{\mathrm{Case}_{\mathrm{Bool}}^{\mathrm{Type}}\ C\ case_{\mathrm{tt}}\ case_{\mathrm{ff}}\ b : C\ b}\ (\mathrm{Bool}\text{-}\mathrm{El}^{\mathrm{Type}})$$

Equality Rules into Type

$$\frac{C : \mathrm{Bool} \rightarrow \mathrm{Type} \qquad case_{\mathrm{tt}} : C\ \mathrm{tt} \qquad case_{\mathrm{ff}} : C\ \mathrm{ff}}{\mathrm{Case}_{\mathrm{Bool}}^{\mathrm{Type}}\ C\ case_{\mathrm{tt}}\ case_{\mathrm{ff}}\ \mathrm{tt} = case_{\mathrm{tt}} : C\ \mathrm{tt}}\ (\mathrm{Bool}\text{-}\mathrm{Eq}_{\mathrm{ff}}^{\mathrm{Type}})$$

$$\frac{C : \mathrm{Bool} \rightarrow \mathrm{Type} \qquad case_{\mathrm{tt}} : C\ \mathrm{tt} \qquad case_{\mathrm{ff}} : C\ \mathrm{ff}}{\mathrm{Case}_{\mathrm{Bool}}^{\mathrm{Type}}\ C\ case_{\mathrm{tt}}\ case_{\mathrm{ff}}\ \mathrm{ff} = case_{\mathrm{ff}} : C\ \mathrm{ff}}\ (\mathrm{Bool}\text{-}\mathrm{Eq}_{\mathrm{tt}}^{\mathrm{Type}})$$

Example Select

- Assume we have introduced

$$
\begin{array}{lcl}
\mathrm{FemaleName} & : & \mathrm{Set} \\
 & = & \{\mathrm{jill}, \mathrm{sara}\} \\
\mathrm{MaleName} & : & \mathrm{Set} \\
 & = & \{\mathrm{tom}, \mathrm{jim}\}
\end{array}
$$

- Then we can define

$$
\begin{array}{lcl}
\mathrm{Name} & : & \mathrm{Bool} \longrightarrow \mathrm{Set} \\
 & := & \lambda x^{\mathrm{Bool}}.\mathrm{Case}_{\mathrm{Bool}}^{\mathrm{Type}} \; (\lambda y.\mathrm{Set}) \\
 & & \qquad\qquad\qquad\; \mathrm{FemaleName} \; \mathrm{MaleName} \; x \\
 & : & \mathrm{Bool} \longrightarrow \mathrm{Set}
\end{array}
$$

Elimination into Type (Cont.)

We can extend this into an elimination rule **into Kind or other higher types**.

(b) The Finite Sets

Bool can be generalised to sets having n elements (n a fixed natural number):

Formation Rule

$$\mathrm{Fin}_n : \mathrm{Set} \quad (\mathrm{Fin}_n\text{-F})$$

Introduction Rules

$$\mathrm{A}_k^n : \mathrm{Fin}_n \quad (\mathrm{Fin}_n\text{-I}_k)$$

(for $k = 0, \ldots, n-1$)

Rules for Fin$_n$

Elimination Rule

$$\frac{\begin{array}{c} C : \mathrm{Fin}_n \to \mathrm{Set} \\ s_0 : C\ \mathrm{A}_0^n \\ s_1 : C\ \mathrm{A}_1^n \\ \cdots \\ s_{n-1} : C\ \mathrm{A}_{n-1}^n \\ a : \mathrm{Fin}_n \end{array}}{\mathrm{Case}_n\ C\ s_0\ \ldots\ s_{n-1}\ a : C\ a}\ (\mathrm{Fin}_n\text{-El})$$

The Finite Sets (Cont)

Equality Rules

$$\frac{\begin{array}{c} C : \mathrm{Fin}_n \longrightarrow \mathrm{Set} \\ s_0 : C\ \mathrm{A}^n_0 \\ s_1 : C\ \mathrm{A}^n_1 \\ \cdots \\ s_{n-1} : C\ \mathrm{A}^n_{n-1} \end{array}}{\mathrm{Case}_n\ C\ s_0\ \ldots\ s_{n-1}\ \mathrm{A}^n_k = s_k : C\ \mathrm{A}^n_k}\ (\mathrm{Fin}_n\text{-}\mathrm{Eq}_k)$$

(for $k = 0, \ldots, n-1$).

We add as well **equality versions** of the formation-, introduction-, and elimination rules.

Remark: Note that we have just introduced infinitely many rules (for each $n \in \mathbb{N}$ and $k = 0, \ldots, n-1$).

Omitting Premises in Equality Rules

- Since the premises of the equality rule can in most cases be determined from the introduction and elimination rules, we will **usually omit them**, when writing down equality rules.
- So we write for instance for the previous rule:

$$\mathrm{Case}_n\ C\ s_0\ \ldots\ s_{n-1}\ \mathrm{A}_k^n = s_k : C\ \mathrm{A}_k^n$$

- We sometimes even **omit the type**:

$$\mathrm{Case}_n\ C\ s_0\ \ldots\ s_{n-1}\ \mathrm{A}_k^n = s_k$$

More Compact Elimination Rules

- $$\begin{aligned}\mathrm{Case}_n : (C : \mathrm{Fin}_n \to \mathrm{Set}) \qquad\qquad\qquad\qquad .\\ \to (s_0 : C\ \mathrm{A}^n_0)\\ \to \cdots\\ \to (s_{n-1} : C\ \mathrm{A}^n_{n-1})\\ \to (a : \mathrm{Fin}_n)\\ \to C\ a\end{aligned}$$

Elimination into Type

- Similarly as for Bool we can write down **elimination rules**, where $\mathbf{C} : \mathbf{Fin_n} \to \mathbf{Type}$ (instead of $C : \mathrm{Fin}_n \to \mathrm{Set}$).
- This can be done for all sets defined later as well.

Rules for $\top$

$\top$ is the special case Fin_n for $n = 1$ (we write true for A_0^1):

Formation Rule

$$\top : \mathrm{Set} \quad (\top\text{-F})$$

Introduction Rules

$$\mathrm{true} : \top \quad (\top\text{-I})$$

Elimination Rule

$$\frac{C : \top \to \mathrm{Set} \qquad c : C\ \mathrm{true} \qquad t : \top}{\mathrm{Case}_\top\ c\ t : C\ t}\ (\top\text{-El})$$

Rules for $\top$

Equality Rule

$$\mathrm{Case}_{\top}\ c\ \mathrm{true} = c$$

We add as well **equality versions** of the formation-, introduction-, and elimination rules.

Jump over next slide (advanced material)

Rules for $\top$ (Cont.)

- $\mathrm{Case}_\top$ is **computationally not very interesting**.
  - $\mathrm{Case}_\top\ c$ is the constant function $\lambda x^\top.c$.
  - However, in Agda we might not be able to derive

$$\lambda t^\top.c : (t : \top) \to C\ t$$

- From a **logic point of view**, it expresses:
  From an element of $C\ \mathrm{true}$ we obtain an element of $C\ t$ **for every** $\mathrm{t} : \top$.
  - So there is no $C : \top \to \mathrm{Set}$ s.t. $C\ \mathrm{true}$ is inhabited, but $C\ x$ is not inhabited for some other $x : \top$.
  - This means that all elements of x of type $\top$ are **indistinguishable from true**, i.e. they are **identical to true**.
  - This equality is called **Leibnitz equality**.

Rules for $\perp$

$\perp$ is the special case Fin_n for $n = 0$:

Formation Rule

$$\perp : \mathrm{Set} \quad (\perp\text{-F})$$

There is no Introduction Rule

Elimination Rule

$$\frac{C : \perp \to \mathrm{Set} \qquad f : \perp}{\mathrm{Case}_{\perp}\ f : C\ f}\ (\perp\text{-El})$$

There is no Equality Rule

We add as well **equality versions** of the formation- and elimination rule.

(c) Atomic Formulae

Full title of this section:
Atomic formulae and the Traffic Light Example.

- Atom can be defined as follows:

$$\begin{aligned} \mathrm{Atom} &: \mathrm{Bool} \rightarrow \mathrm{Set} \\ \mathrm{Atom} &= \mathrm{Case}_{\mathrm{Bool}}^{\mathrm{Type}} \ (\lambda b^{\mathrm{Bool}}.\mathrm{Set}) \ \top \ \bot \end{aligned}$$

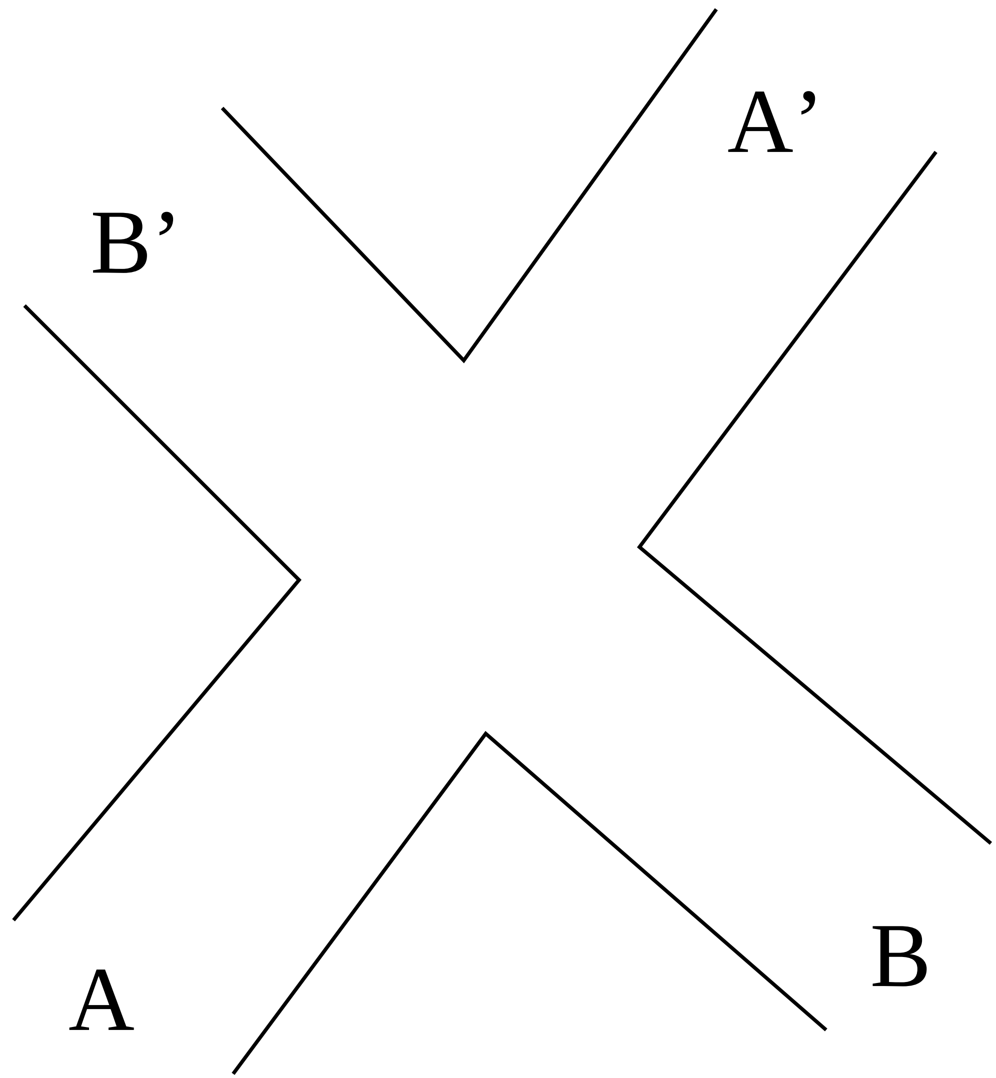
- So we have

$$\begin{aligned} \mathrm{Atom}\ \mathrm{tt} &= \top \\ \mathrm{Atom}\ \mathrm{ff} &= \bot \end{aligned}$$

Jump over Traffic Light Example.

The Traffic Light Example

- Assume a **road crossing**, controlled by **traffic lights**:



The Traffic Light Example

- Assume from each direction A, A', B, B' there is one traffic light,
  - but A and A' always coincide, similarly B and B'.

The Set of Physical States

- For simplicity assume that **each traffic light is either red or green**:

$$
\begin{array}{l}
\mathrm{data\ Colour} : \mathrm{Set\ where} \\
\quad \begin{array}{lcl} \mathrm{red} & : & \mathrm{Colour} \\ \mathrm{green} & : & \mathrm{Colour} \end{array}
\end{array}
$$

- The set of **physical states of the system** is given by a pair, determining the colour of A (and therefore as well A') and of B (and B')

$$
\begin{array}{l}
\mathrm{record\ PhysState} : \mathrm{Set\ where} \\
\quad \mathrm{field} \\
\qquad \begin{array}{lcl} \mathrm{sigA} & : & \mathrm{Colour} \\ \mathrm{sigB} & : & \mathrm{Colour} \end{array}
\end{array}
$$

The Set of Control States

- The set of **control states** is a set of states of the system, a controller of the system can choose.
  - Each of these states **should be safe**.
  - In our example, **all safe states will be captured** (this can usually be only achieved in small examples).
- A **complete set of control states** consists of:
  - allRed – all signals are red.
  - $\mathrm{onlyAGreen}$ – signal A (and A') is green, signal B is red.
  - $\mathrm{onlyBGreen}$ – signal B is green, signal A is red.

The Set of Control States (Cont.)

- We therefore define

$$
\begin{array}{l}
\mathrm{data\ ControlState} : \mathrm{Set\ where} \\
\quad \begin{array}{lcl}
\mathrm{allRed} & : & \mathrm{ControlState} \\
\mathrm{onlyAGreen} & : & \mathrm{ControlState} \\
\mathrm{onlyBGreen} & : & \mathrm{ControlState}
\end{array}
\end{array}
$$

Control States to Physical States

- We define the **state of signals A, B depending on a control state**:

$$\begin{array}{llcl}
\mathrm{toSigA} : \mathrm{ControlState} \longrightarrow \mathrm{Colour} \\
\mathrm{toSigA} & \mathrm{allRed} & = & \mathrm{red} \\
\mathrm{toSigA} & \mathrm{onlyAGreen} & = & \mathrm{green} \\
\mathrm{toSigA} & \mathrm{onlyBGreen} & = & \mathrm{red}
\end{array}$$

$$\begin{array}{llcl}
\mathrm{toSigB} : \mathrm{ControlState} \longrightarrow \mathrm{Colour} \\
\mathrm{toSigB} & \mathrm{allRed} & = & \mathrm{red} \\
\mathrm{toSigB} & \mathrm{onlyAGreen} & = & \mathrm{red} \\
\mathrm{toSigB} & \mathrm{onlyBGreen} & = & \mathrm{green}
\end{array}$$

Control States to Physical States

- Now we can define the **physical state corresponding to a control state**:

$$
\begin{array}{l}
\mathrm{toPhysState} : \mathrm{ControlState} \rightarrow \mathrm{PhysState} \\
\mathrm{toPhysState}\ c = \mathrm{record}\{\begin{array}{rcl}\mathrm{sigA} & = & \mathrm{toSigA}\ c\ ; \\ \mathrm{sigB} & = & \mathrm{toSigB}\ c\ \}\end{array}
\end{array}
$$

Safety Predicate

- We define now **when a physical state is safe**:
  - It is **safe iff not both signals are green**.
  - We define now a corresponding predicate **directly**, without defining first a Boolean function.
  - We first define a predicate depending on two signals:

$$
\begin{array}{llll}
\mathrm{CorAux} : \mathrm{Colour} \to \mathrm{Colour} \to \mathrm{Set} & & & \\
\mathrm{CorAux} \quad \mathrm{red} & \_ & = & \top \\
\mathrm{CorAux} \quad \mathrm{green} & \mathrm{red} & = & \top \\
\mathrm{CorAux} \quad \mathrm{green} & \mathrm{green} & = & \bot
\end{array}
$$

Safety Predicate (Cont.)

- Now we define

$$\mathrm{Cor} : \mathrm{PhysState} \to \mathrm{Set}$$
$$\mathrm{Cor}\ s = \mathrm{CorAux}\ (\mathrm{PhysState.sigA}\ s)\ (\mathrm{PhysState.sigB}\ s)$$

- **Remark:** In some cases in order to define a function from a **record type** into some other set, it is better first to **introduce an auxiliary function**, depending on the components of that product.

Safety of the System

- Now we show that **all control states are safe**:

$$
\begin{array}{llcl}
\multicolumn{4}{l}{\mathrm{corProof} : (s : \mathrm{ControlState}) \to \mathrm{Cor}\ (\mathrm{toPhysState}\ s)} \\
\mathrm{corProof} & \mathrm{allRed} & = & \mathrm{true} \\
\mathrm{corProof} & \mathrm{onlyAGreen} & = & \mathrm{true} \\
\mathrm{corProof} & \mathrm{onlyBGreen} & = & \mathrm{true}
\end{array}
$$

See **exampleTrafficLight1.agda**

Safety of the System (Cont.)

- The first element true was an element of $\mathbf{Cor\ (phys\_state\ Allred)}$, which reduces to $\top$.
- Similarly for the other two elements.
- This works only because **each control state corresponds to a correct physical state**.
  - If this hadn't been the case, we would have gotten instances where the goal to solve is $\perp$, which we can't solve.

Safety of the System (Cont.)

- If one makes a **mistake** which results in an unsafe situation
  - e.g. sets $\mathrm{toSigB\ onlyAGreen} = \mathrm{green}$,

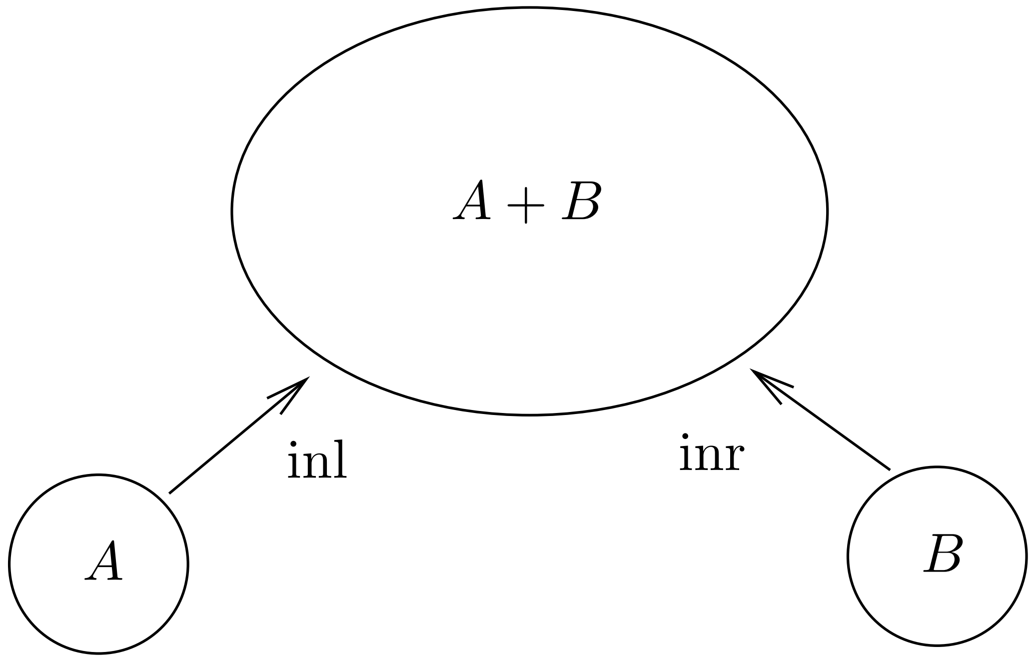
  then in the last step we obtain one goal of type $\perp$.
  - Then we can't solve this goal directly and **cannot prove the correctness**.
  - (We could in Agda solve this goal by using **full recursion**,
    - e.g. solve this goal as $\mathbf{corProof\ Agreen}$,

    but this would be rejected by the termination checker.)

(d) The Disjoint Union of Sets

- The **disjoint union** $A + B$ of two sets A and B is the union of A and B,
  - but defined in such a way that we can decide whether an element of this union is originally from A or B.
  - This is distinguished by having constructors $\mathrm{inl} : A \to A + B$ and inr.
    - Elements from $a : A$ are inserted into $A + B$ as $\mathrm{inl}\ a : A + B$.
    - elements from $b : B$ are inserted into $A + B$ as $\mathrm{inr}\ b : A + B$.
    - inl stands for “in-left”, inr for “in-right”.
  - If we have $a : A$ and $a : B$, then a is represented both as $\mathrm{inl}\ a$ and $\mathrm{inr}\ a$ in $A + B$.

Visualisation ($A + B$)



Disjoint Union

- Informally, if
$$A = \{1, 2\}$$
and
$$B = \{1, 2, 3\} \quad ,$$
then
$$A + B = \{\mathrm{inl}(1), \mathrm{inl}(2), \mathrm{inr}(1), \mathrm{inr}(2), \mathrm{inr}(3)\}$$
  - Each element of $A + B$ is
    - either of the form $\mathrm{inl}(a)$ for some $a : A$
    - or of the form $\mathrm{inr}(b)$ for $b : B$.

Jump over Comparison with Product

Comparison with the Product

- Note that if we have again

$$A = \{1, 2\}$$

and

$$B = \{1, 2, 3\} \quad ,$$

then for the product we have informally

$$A \times B = \{\mathrm{p}(1,1), \mathrm{p}(1,2), \mathrm{p}(1,3), \mathrm{p}(2,1), \mathrm{p}(2,2), \mathrm{p}(2,3)\}$$

  - Each element of $A \times B$ is of the form $\mathrm{p}(a, b)$ where $a : A$ and $b : B$.
  - So each element of $A \times B$ contains both an element of A and an element of B.

Disjoint Union vs. Product

- Note that, if A is empty, then
  - $A + B = \{\mathrm{inr}(b) \mid b : B\}$, which has a copy of each element of B,
  - $A \times B$ is empty, since we cannot form a pair $\mathrm{p}(a, b)$ where $a : A$, $b : B$, since there is no element $a : A$.

Rules for $A + B$

Formation Rule

$$\frac{A : \mathrm{Set} \qquad B : \mathrm{Set}}{A + B : \mathrm{Set}}\,(+\text{-F})$$

Introduction Rules

$$\frac{A : \mathrm{Set} \qquad B : \mathrm{Set} \qquad a : A}{\mathrm{inl}\ A\ B\ a : A + B}\,(+\text{-I}_{\mathrm{inl}})$$

$$\frac{A : \mathrm{Set} \qquad B : \mathrm{Set} \qquad b : B}{\mathrm{inr}\ A\ B\ b : A + B}\,(+\text{-I}_{\mathrm{inr}})$$

Rules for $A + B$

Elimination Rules

$$\frac{\begin{array}{c} A : \mathrm{Set} \\ B : \mathrm{Set} \\ C : (A + B) \to \mathrm{Set} \\ case_{inl} : (a : A) \to C\ (\mathrm{inl}\ A\ B\ a) \\ case_{inr} : (b : B) \to C\ (\mathrm{inr}\ A\ B\ b) \\ d : A + B \end{array}}{\mathrm{Case}_{+}\ A\ B\ C\ case_{inl}\ case_{inr}\ d : C\ d}\ (+\text{-El})$$

($case_{inl}$, $case_{inr}$ stand for “case left”, “case right”).

Rules for $A + B$

Equality Rules

$$\begin{array}{l}\mathrm{Case}_+ \ A\ B\ C\ case_{inl}\ case_{inr}\ (\mathrm{inl}\ A\ B\ a) \\ \quad = case_{inl}\ a : C\ (\mathrm{inl}\ A\ B\ a)\end{array} \qquad (+\text{-}\mathrm{Eq}_{\mathrm{inl}})$$

$$\begin{array}{l}\mathrm{Case}_+ \ A\ B\ C\ case_{inl}\ case_{inr}\ (\mathrm{inr}\ A\ B\ b) \\ \quad = case_{inr}\ b : C\ (\mathrm{inr}\ A\ B\ b)\end{array} \qquad (+\text{-}\mathrm{Eq}_{\mathrm{inr}})$$

Additionally, we have the **equality versions** of the formation-, introduction and elimination rules.

Logical Framework Version

- A **more compact notation** for the formation, introduction and elimination rules is:
  - $\_+\_ : \mathrm{Set} \to \mathrm{Set} \to \mathrm{Set}$, written infix.
  - $\mathrm{inl} : (A, B : \mathrm{Set}) \to A \to (A + B)$.
  - $\mathrm{inr} : (A, B : \mathrm{Set}) \to B \to (A + B)$.
  - $$\begin{aligned} \mathrm{Case}_{+} : {} & (A, B : \mathrm{Set}) \\ & \to (C : (A + B) \to \mathrm{Set}) \\ & \to ((a : A) \to C\ (\mathrm{inl}\ A\ B\ a)) \\ & \to ((b : B) \to C\ (\mathrm{inr}\ A\ B\ b)) \\ & \to (d : A + B) \\ & \to C\ d\ . \end{aligned}$$
  - Equality rule as before.

Disjoint Union in Agda

- The disjoint union can be defined as a "data"-set having **two constructors**
  - inl (in-left for left injection) and
  - inr (in-right for right injection):

$$
\begin{array}{l}
\mathrm{data}\ \_{+}\_\ (A\ B : \mathrm{Set}) : \mathrm{Set}\ \mathrm{where} \\
\quad \mathrm{inl} : A \to A + B \\
\quad \mathrm{inr} : B \to A + B
\end{array}
$$

Disjoint Union in Agda (Cont.)

- Elimination is represented by pattern matching.
  So if want to define for $A, B : \mathrm{Set}$ for instance

$$
\begin{aligned}
&f : A + B \to \mathrm{Bool} \\
&f\ x = \{!\ \ !\}
\end{aligned}
$$

we can define $f\ x$ by case distinction on x:

$$
\begin{aligned}
&f : A + B \to \mathrm{Bool} \\
&f\ (\mathrm{inl}\ a) = \mathrm{tt} \\
&f\ (\mathrm{inr}\ b) = \mathrm{ff}
\end{aligned}
$$

Use of Concrete Disjoint Sets

- It is usually **more convenient** to define concrete disjoint unions **directly** with more intuitive names for constructors, e.g.

$$
\begin{array}{l}
\mathrm{data}\ \mathrm{Plant} : \mathrm{Set}\ \mathrm{where} \\
\quad \begin{array}{lcl}
\mathrm{tree} & : & \mathrm{Tree} \longrightarrow \mathrm{Plant} \\
\mathrm{flower} & : & \mathrm{Flower} \longrightarrow \mathrm{Plant}
\end{array}
\end{array}
$$

- Now one can define for instance

$$
\begin{array}{l}
\mathrm{isFlower} : \mathrm{Plant} \longrightarrow \mathrm{Bool} \\
\begin{array}{lcl}
\mathrm{isFlower}\ (\mathrm{tree}\ t) & = & \mathrm{ff} \\
\mathrm{isFlower}\ (\mathrm{flower}\ f) & = & \mathrm{tt}
\end{array}
\end{array}
$$

Disjunction

- $A \lor B$ is true iff A is true or B is true.
- Therefore a **proof of A $\lor$ B consists of a proof of A or a proof of B, plus the information which one.**
  - It is therefore an element $\mathrm{inl}\ p$ for a proof $p : A$ or an element $\mathrm{inr}\ q$ for a proof $q : B$.
- Therefore the set of proofs of $A \lor B$ is the **disjoint union of A and B**, i.e. $\mathbf{A + B}$.
- We can **identify** $A \lor B$ with $A + B$.

Disjunction in Agda

- Or is represented as disjoint union in type theory.
- In Agda we can type in the symbol for $\vee$ using Leim as $\backslash$vee.

$$
\begin{array}{l}
\mathrm{data}\ \_\vee\_\ (A\ B : \mathrm{Set}) : \mathrm{Set}\ \mathrm{where} \\
\quad \begin{array}{lcl}
\mathrm{or1} & : & A \rightarrow A \vee B \\
\mathrm{or2} & : & B \rightarrow A \vee B
\end{array}
\end{array}
$$

- See **exampleproofproplogic7.agda**.
- On the blackboard $A \rightarrow A \vee B$ and $A \vee A \rightarrow A$ will now be shown in Agda.

Example (Disjunction)

- The following derives $(A \vee B) \to (B \vee A)$:

$$\begin{array}{lcl} \multicolumn{3}{l}{\mathrm{lemma3} : A \vee B \to B \vee A} \\ \mathrm{lemma3}\ (\mathrm{or1}\ a) & = & \mathrm{or2}\ a \\ \mathrm{lemma3}\ (\mathrm{or2}\ b) & = & \mathrm{or1}\ b \end{array}$$

- See **exampleproofproplogic9.agda**.

Disjunction with more Args.

- As for the conjunction, it is useful to introduce special ternary versions of the disjunction (and versions with higher arities):

$$
\begin{array}{l}
\mathrm{data}\ \mathrm{OR3}\ (A\ B\ C : \mathrm{Set}) : \mathrm{Set}\ \mathrm{where} \\
\quad \begin{array}{lcl}
\mathrm{or1} & : & A \longrightarrow \mathrm{OR3}\ A\ B\ C \\
\mathrm{or2} & : & B \longrightarrow \mathrm{OR3}\ A\ B\ C \\
\mathrm{or2} & : & C \longrightarrow \mathrm{OR3}\ A\ B\ C
\end{array}
\end{array}
$$

- See **exampleproofproplogic8.agda**.

Jump over Σ-Type.

(e) The Σ-Set

- The Σ-set is a second version of the **dependent product** of two sets.
- It depends on
  - a set A,
  - and a second set B depending on A, i.e. on $B : A \to \mathrm{Set}$.
- Similar to the standard product $(x : A) \times (B\ x)$.
- In Agda
  - $(x : A) \times (B\ x)$ is a in Agda a builtin construct,
  - the Σ-set is introduced by the user using a constructor, similar to the previous sets.
- The Σ-set behaves sometimes better than the standard product.

Rules for Σ

Formation Rule

$$\frac{A : \mathrm{Set} \qquad B : A \to \mathrm{Set}}{\Sigma\ A\ B : \mathrm{Set}}\ (\Sigma\text{-F})$$

Introduction Rule

$$\frac{\begin{array}{c} A : \mathrm{Set} \\ B : A \to \mathrm{Set} \\ a : A \\ b : B\ a \end{array}}{\mathrm{p}\ A\ B\ a\ b : \Sigma\ A\ B}\ (\Sigma\text{-I})$$

Rules for Σ

Elimination Rule

$$\frac{\begin{array}{c} A : \mathrm{Set} \\ B : A \to \mathrm{Set} \\ C : (\Sigma\ A\ B) \to \mathrm{Set} \\ c : (a : A) \to (b : B\ a) \to C\ (\mathrm{p}\ A\ B\ a\ b)) \\ d : \Sigma\ A\ B \end{array}}{\mathrm{Case}_{\Sigma}\ A\ B\ C\ c\ d : C\ d}\ (\Sigma\text{-El})$$

Equality Rule

$$\mathrm{Case}_{\Sigma}\ A\ B\ C\ c\ (\mathrm{p}\ A\ B\ a\ b) = c\ a\ b : C\ (\mathrm{p}\ A\ B\ a\ b) \quad (\Sigma\text{-Eq})$$

Additionally we have the **Equality versions** of the formation-, introduction- and elimination-rules.

The Σ-Set using the Log. Framew.

- The more compact notation is:
  - $$\begin{aligned}\Sigma : (A : \mathrm{Set}) \\ \longrightarrow (A \longrightarrow \mathrm{Set}) \\ \longrightarrow \mathrm{Set} \quad .\end{aligned}$$
  - $$\begin{aligned}\mathrm{p} : (A : \mathrm{Set}) \\ \longrightarrow (B : A \longrightarrow \mathrm{Set}) \\ \longrightarrow (a : A) \\ \longrightarrow (B\ a) \\ \longrightarrow \Sigma\ A\ B \quad .\end{aligned}$$

The Σ-Set using the Log. Framew.

- Case_Σ :

$$
\begin{aligned}
&(A : \mathrm{Set})\\
&\to (B : A \to \mathrm{Set})\\
&\quad \to (C : (\Sigma\ A\ B) \to \mathrm{Set})\\
&\qquad \to ((a : A, b : B\ a) \to C\ (\mathrm{p}\ A\ B\ a\ b))\\
&\qquad\quad \to (d : \Sigma\ A\ B)\\
&\qquad\qquad \to C\ d\ \ .
\end{aligned}
$$

- Equality rule as before.

The Σ-Set and the Dep. Prod.

- Both the Σ-set and the dep. product have similar introduction rules.
  - For the Σ-set, the constructors have additional arguments A, B necessary for bureaucratic reasons only.
- One can define the projections π_0, π_1 using Case_Σ:

$$\begin{aligned}
\pi_0 &= \mathrm{Case}_\Sigma\ A\ B\ (\lambda x^{(\Sigma\ A\ B)}.A)\ (\lambda x^A.\lambda y^{(B\ x)}.x) \\
\pi_1 &= \mathrm{Case}_\Sigma\ A\ B\ (\lambda x^{(\Sigma\ A\ B)}.B\ \pi_0(x))\ (\lambda x^A.\lambda y^{(B\ x)}.y)
\end{aligned}$$

- On the other hand, from π_0, π_1 we can define Case_Σ as follows:

$$\begin{aligned}
&\lambda A^{\mathrm{Set}}.\lambda B^{A\to\mathrm{Set}}.\lambda C^{(\Sigma\ A\ B)\to\mathrm{Set}}. \\
&\quad \lambda s^{(a:A)\to(b:B\ a)\to C\ (\mathrm{p}\ a\ b)}.\lambda d^{(\Sigma\ A\ B)}.s\ \pi_0(d)\ \pi_1(d)\ .
\end{aligned}$$

The Σ-Set and the Dep. Prod.

- However the dependent product has the **η-rule** (which is however not implemented in Agda).
- Because of the lack of η-rule, Σ works usually **better than the dependent product** in Agda.
  - I personally **don't use the dependent product** of Agda much.

The Σ-Set in Agda

- Σ can be defined as a "data"-set with a constructor, e.g. p:

$$\mathrm{data}\ \Sigma\ (A : \mathrm{Set})\ (B : A \to \mathrm{Set}) : \mathrm{Set}\ \mathrm{where}$$
$$\mathrm{p} : (a : A) \to B\ a \to \Sigma\ A\ B$$

- Elimination uses **case-distinction**:

$$f : \Sigma\ A\ B \to D$$
$$f\ (\mathrm{p}\ a\ b) = \{!\ \ !\}$$

sigmaset.agda

The Σ-Set in Agda (Cont.)

- Again one usually defines concrete Σ-sets more directly.
- **Example:** Assume we have defined
  - a set $\mathrm{PlantGroup}$ for **groups of plants** (e.g. “tree”, “flower”),
  - depending on $g : \mathrm{PlantGroup}$, sets $(\mathrm{PlantsInGroup}\ g)$ for **plants in that group**.
- The **set of plants** can then be defined as

$$
\begin{array}{l}
\mathrm{data\ Plant : Set\ where} \\
\mathrm{plant} : (g : \mathrm{PlantGroup}) \rightarrow \mathrm{PlantsInGroup}\ g \rightarrow \mathrm{Plant}
\end{array}
$$

The Σ-Set in Agda (Cont.)

- Not surprisingly, for **elimination** we use **pattern matching**, e.g.:

$$
\begin{aligned}
&f : \mathrm{Plant} \to \mathrm{PlantGroup} \\
&f\ (\mathrm{plant}\ g\ \_) = g
\end{aligned}
$$

(f) Natural Ded. and Dep. Type Theo

- In this section we study, how derivations in dependent type theory correspond to derivations in natural deduction. (Omitted 2008)
- We will as well introduce constructive logic.
  Jump to constructive logic.

Conjunction

- We have seen before that we can identify in type theory conjunction with the non-dependent product.
- With this interpretation, the **introduction rule** for the product allows to form a proof of $A \wedge B$ from a proof of A and a proof of B:

$$\frac{p : A \qquad q : B}{\langle p, q \rangle : A \wedge B} \, (\times\text{-I})$$

- This means that we can **derive A $\wedge$ B from A and B**.

Conjunction and Natural Ded.

- In so called natural deduction, one has rules for deriving and eliminating formulas formed using the standard connectives.
- There the rule for introducing proofs of $A \wedge B$ is

$$\frac{A \qquad B}{A \wedge B}\,(\wedge\text{-I})$$

- The type theoretic introduction rule corresponds exactly to this rule.

Omit Example1

Example 1

- For instance, assume we want to prove that a function sort from lists to lists is a sorting algorithm.
- Then we have to show that for every list l the application of sort to l is sorted, and has the same elements of l.
- In order to show this, one would assume a list l and show
  - first that $\mathrm{sort}\ l$ is sorted,
  - then, that $\mathrm{sort}\ l$ has the same elements as l
  - and finally conclude that it fulfils the conjunction of both properties.
  - The last operation uses the introduction rule for $\wedge$.

Conjunction (Cont.)

- The **elimination rule** for $\land$ allows to project a proof of $A \land B$ to a proof of A and a proof of B:

$$\frac{p : A \land B}{\pi_0(p) : A}\,(\times\text{-El}_0) \qquad \frac{p : A \land B}{\pi_1(p) : B}\,(\times\text{-El}_1)$$

- This means that we can **derive from A $\land$ B both A and B**.
- This corresponds to the **natural deduction elimination rule for $\land$**:

$$\frac{A \land B}{A}\,(\land\text{-El}_0) \qquad \frac{A \land B}{B}\,(\land\text{-El}_1)$$

Omit Example 2

Example 2

- Assume we have defined a function f, which takes a list of natural numbers l, a proof that l is sorted, and a natural number n, and returns the Boolean value tt or ff indicating whether n is in this list or not.
- Assume now a sorting function sort from lists of natural numbers to natural numbers, plus a proof that it is a sorting function, i.e. that $\mathrm{sort}\ l$ is sorted and has the same elements as l for every list l.
- We want to apply f to $\mathrm{sort}\ l$ and need therefore a proof that $\mathrm{sort}\ l$ is sorted.
- We have that the conjunction of "$\mathrm{sort}\ l$ is sorted" and "$\mathrm{sort}\ l$ has the same elements as l" holds.
- Using the elimination rule for $\wedge$ one can conclude the desired property, that $\mathrm{sort}\ l$ is sorted.

Example 3

- Assume a proof of $A \wedge B$.
- We want to show $B \wedge A$.
  - By $\wedge$-elimination we obtain from $A \wedge B$ that B holds.
  - Similarly we conclude that A holds.
  - Using $\wedge$-introduction we conclude $B \wedge A$.
  - In natural deduction, this proof is as follows:

$$\dfrac{\dfrac{A \wedge B}{B}\,(\wedge\text{-El}_0) \qquad \dfrac{A \wedge B}{A}\,(\wedge\text{-El}_1)}{B \wedge A}\,(\wedge\text{-I})$$

- We have seen in the previous section how to derive this in Agda.

Disjunction

- We have seen before that we can identify in type theory disjunction can be identified with the disjoint union.
- With this identification, the **introduction rules** for $+$ allows to form a proof of $A \vee B$ from a proof of A or from a proof of B.

$$\frac{A : \mathrm{Set} \qquad B : \mathrm{Set} \qquad p : A}{\mathrm{inl}\ A\ B\ p : A + B}\ (+\text{-}\mathrm{I}_{\mathrm{inl}})$$

$$\frac{A : \mathrm{Set} \qquad B : \mathrm{Set} \qquad p : B}{\mathrm{inr}\ A\ B\ p : A + B}\ (+\text{-}\mathrm{I}_{\mathrm{inr}})$$

Disjunction (Cont.)

- Omitting the premises $A, B : \mathrm{Set}$ and omitting them as arguments of inl and inr (which is needed only for type checking purposes in the presence of the identity type – this type is not treated in this module) we get:

$$\frac{A : \mathrm{Set} \qquad B : \mathrm{Set} \qquad p : A}{\mathrm{inl}\ p : A + B}\ (+\text{-}\mathrm{I}_{\mathrm{inl}})$$

$$\frac{A : \mathrm{Set} \qquad B : \mathrm{Set} \qquad p : B}{\mathrm{inr}\ p : A + B}\ (+\text{-}\mathrm{I}_{\mathrm{inr}})$$

Disjunction (Cont.)

- This means that we can **derive A $\vee$ B from A and from B**.
- This is what is expressed by the **natural deduction introduction rules for $\vee$**:

$$\frac{A}{A \vee B}\,(\vee\text{-I}_{\text{inl}}) \qquad\qquad \frac{B}{A \vee B}\,(\vee\text{-I}_{\text{inr}})$$

Omit Example 1

Example 1

- Assume we want to show that every prime number is equal to 2 or odd.
- In order to show this one assumes a prime number.
  - If it is 2, it is trivially equal to 2.
    - Using the introduction rule for $\vee$ one concludes that it is equal to 2 or odd.
  - Otherwise, one argues (using some proof) that it is odd.
    - Using the introduction rule for $\vee$ one concludes again that it is equal to 2 or odd.

Disjunction (Cont.)

- The **elimination rule** for $+$ allows to form from an element of $A + B$ an element of any set C provided we can compute such an element from A and from B:

$$\frac{\begin{array}{c} A : \mathrm{Set} \\ B : \mathrm{Set} \\ C : (A \vee B) \rightarrow \mathrm{Set} \\ sl : (a : A) \rightarrow C\ (\mathrm{inl}\ A\ B\ a) \\ sr : (b : B) \rightarrow C\ (\mathrm{inr}\ A\ B\ b) \\ d : A \vee B \end{array}}{\mathrm{Case}_{+}\ A\ B\ C\ sl\ sr\ d : C\ d}\ (+\text{-El})$$

Disjunction (Cont.)

- Omitting the dependency of C on $A \vee B$, the premises A, B and C, and the arguments A, B and C, we get:

$$\frac{d : A \vee B \qquad sl : A \rightarrow C \qquad sr : B \rightarrow C}{\mathrm{Case}_{+}\ sl\ sr\ d : C}\ (+\text{-El})$$

- This means that we can **derive from A $\vee$ B a formula C, if we can derive C from A and from B**.

Disjunction (Cont.)

- This is what is expressed by the **natural deduction elimination rules for** $\vee$:

$$\frac{A \vee B \qquad A \vdash C \qquad B \vdash C}{C}\,(\vee\text{-El})$$

- In the above rule we have written

$$A \vdash C$$

for

from assumption A we can derive C.

  - This is written sometimes in the following form

$$\begin{array}{c} A \\ \vdots \\ C \end{array}$$

Disjunction (Cont.)

- Note that in natural deduction, from the premise $A \vdash C$ we obtain $A \rightarrow C$, which is the premise used in the corresponding rule in dependent type theory.

Omit Example 2

Example 2

- Assume we want to show that every prime number is equal to 2, equal to 3, or ≥ 5.
- We want to make use of the proof above that every prime number is equal to 2 or odd.
- We assume a prime number.
  - We know that it is equal to 2 or odd.
  - In case it is equal to 2 we conclude that it is equal to 2, equal to 3, or ≥ 5.
  - In case it is odd, we conclude using the fact that it is prime and 1 is not prime, that it is equal to 3 or ≥ 5. Therefore it is equal to 2, equal to 3, or ≥ 5.
  - Now from the elimination rule for $\vee$ we conclude that the prime number chosen is equal to 2, equal to 3, or ≥ 5.

Example 3

- Assume a proof of $A \lor B$.
- We want to show $B \lor A$.
  - We have $A \lor B$.
  - From assumption A we obtain A and therefore by $\lor$-introduction $B \lor A$.
  - From assumption B we obtain B and therefore by $\lor$-introduction $B \lor A$.
  - By $\lor$-elimination we obtain from these three premises $B \lor A$ without any premises.

Example 3 (Cont.)

- In natural deduction, this proof is as follows (we write $A_1, \ldots, A_n \vdash B$ for B follows under assumptions $A_1, \ldots, A_n$):

$$\dfrac{A \lor B \qquad \dfrac{A \vdash A}{A \vdash B \lor A}\,(\lor\text{-I}_{\text{inr}}) \qquad \dfrac{B \vdash B}{B \vdash B \lor A}\,(\lor\text{-I}_{\text{inr}})}{B \lor A}\,(\lor\text{-El})$$

- We have seen in the previous section how to derive this in Agda.

Implication

- We have seen before that we can identify in type theory implication with the non-dependent function type.
- In order to distinguish between the function type and the logical implication we will write in this subsection $\supset$ instead of $\rightarrow$ for logical implication.

Implication (Cont.)

- With this identification of logical implication and the function type, the **introduction rule for $\rightarrow$** allows to form a proof of $A \supset B$ from a proof of B depending on a proof p of A:

$$\frac{p : A \Rightarrow q : B}{\lambda p^A . q : A \supset B} \,(\rightarrow \text{-I})$$

- This means that, if we, **from assumptions p:A can prove B**
  - (i.e. we can make use of a context $p : A$ for proving $q : B$)

  then we can derive A $\supset$ B without assuming p:A.

Implication (Cont.)

- This is what is expressed by the **introduction rule for $\supset$ in natural deduction**:

$$\frac{A \vdash B}{A \supset B}\,(\supset\text{-I})$$

Example

- We extend the proof that, if we have $A \vee B$, then we have $B \vee A$, to a proof of

$$(A \vee B) \supset (B \vee A)$$

- The previous proof can be easily transformed into a proof of $A \vee B \vdash B \vee A$.
- By $\supset$-introduction, it follows $(A \vee B) \supset (B \vee A)$.

Example

- The complete proof in natural deduction is as follows is as follows.

$$
\dfrac{A \vee B \vdash A \vee B \qquad \dfrac{A \vdash A}{A \vdash B \vee A}\,(\vee\text{-I}_{\mathrm{inr}}) \qquad \dfrac{B \vdash B}{B \vdash B \vee A}\,(\vee\text{-I}_{\mathrm{inl}})}{\dfrac{A \vee B \vdash B \vee A}{(A \vee B) \supset (B \vee A)}\,(\supset\text{-I})}\,(\vee\text{-El})
$$

Implication (Cont.)

- The **elimination rule for $\supset$** allows to apply a proof p of $A \supset B$ to a proof of q of A in order to obtain a proof of B:

$$\frac{p : A \supset B \qquad q : A}{p\,q : B}\,(\rightarrow\text{-El})$$

- This means that we can **derive from A $\supset$ B and A that B holds**.
- This is what is expressed by the **natural deduction elimination rule for $\supset$**:

$$\frac{A \supset B \qquad A}{B}\,(\supset\text{-El})$$

Example

- Assume we want to show $A \supset (A \supset B) \supset B$.
- We can show this as follows:
  - From assumptions A and $A \supset B$ we can conclude $A \supset B$.
  - From assumptions A and $A \supset B$ we can conclude as well A.
  - Using the elimination rule for $\supset$, we conclude that under the same assumptions we get B.
  - Using the introduction rule for $\supset$ we conclude from assumption A that $(A \supset B) \supset B$ holds.
  - Using again the introduction rule for $\supset$ we conclude that $A \supset (A \supset B) \supset B$ holds without any assumptions.

Example

- A proof in natural deduction is as follows:

$$\dfrac{\dfrac{\dfrac{A, A \supset B \vdash A \supset B \qquad A, A \supset B \vdash A}{A, A \supset B \vdash B}\,(\supset\text{-El})}{A \vdash (A \supset B) \supset B}\,(\supset\text{-I})}{A \supset (A \supset B) \supset B}\,(\supset\text{-I})$$

Universal Quantification

- We have seen before that we can identify in type theory universal quantification with the dependent function type.
- With this identification, the **introduction rule** for the dependent function type allows to form a proof of $\forall x^A.B$ from a proof of B depending on an element $x : A$:

$$\frac{x : A \Rightarrow p : B}{\lambda x^A.p : \forall x^A.B} \; (\to\text{-I})$$

- This means that, if we, **from x:A can prove B**, then we get a proof of $\forall x^A.B$ which doesn't depend on $x : A$.

Universal Quantification (Cont.)

- This is what is expressed by the **natural deduction introduction rule for** $\forall$:

$$\frac{x : A \vdash B}{\forall x^A . B} \,(\forall\text{-I})$$

  where

  - **x might not occur free in any assumption of the proof**.
    - This is guaranteed in type theory, since $x : A$ must be the last element of the context, so any other assumptions must be located before it and can therefore **not depend on x:A.**

Universal Quantification (Cont.)

- Note that we have written

$$x : A \vdash B$$

for

we can derive B from variable $x : A$.

- This is usually not mentioned as such in natural deduction.
- We prefer this notation, since it
  - makes the variable x explicit,
  - and allows to deal with more complex types A.

Universal Quantification (Cont.)

- The **conclusion** of the introduction rule **will no longer depend on free variables x.**
  - This is made explicit by mentioning free variables $x : A$ in our notation.
  - In type theory this corresponds to the fact that **x:A does no longer occur in the context of the conclusion.**

Example

- Assume one wants to show that for every natural number n we have $n + 0 == n$.
- In order to show this one assumes a natural number n and shows then that $n + 0 == n$.
- then using the introduction rule for $\forall$ one concludes $\forall n^{\mathbb{N}}.n + 0 == n$.
- In natural deduction, this proof is as follows (where the prove of $n + 0 == n$ is not carried out):

$$\frac{n + 0 == n}{\forall n^{\mathbb{N}}.n + 0 == n} \ (\forall\text{-I})$$

Universal Quantification (Cont.)

- The **elimination rule** for the dependent function type allows to apply a proof p of $\forall x^A.B$ to an element $a : A$ in order to obtain a proof of $B[x := a]$:

$$\frac{p : \forall x^A.B \qquad a : A}{p\ a : B[x := a]}\ (\rightarrow\text{-El})$$

- This means that we can **derive from $\forall x^A.B$ and an element of a:A that B[x:=a] holds**.

Universal Quantification (Cont.)

- This is what is expressed by the **natural deduction elimination rule for** $\forall$
  - For the simple languages used in natural deduction, there is no need to derive that $a : A$;
    in more **complex type theories we have to carry out this derivation**.

$$\frac{\forall x^A.B \qquad a : A}{B[x := a]}\ (\forall\text{-El})$$

Example

- Assume a proof of $\forall n^{\mathbb{N}}.0 + n == n$.
- We want to conclude that
  $\forall n, m : \mathbb{N}.0 + (n + m) == (n + m)$.
- This can be done as follows:
  - One assumes $n, m : \mathbb{N}$.
  - Then one can conclude $n + m : \mathbb{N}$.
  - Using $\forall n^{\mathbb{N}}.0 + n == n$ and the elimination rule for $\forall$ one concludes $0 + (n + m) == (n + m)$ under assumption $n, m : \mathbb{N}$.
  - Now using the introduction rule for $\forall$ twice it follows $\forall n, m : \mathbb{N}.0 + (n + m) == (n + m)$.

Example

- In natural deduction, this proof is written as follows:

$$\dfrac{\forall n^{\mathbb{N}}.0 + n == n \qquad \dfrac{n{:}\mathbb{N}, m{:}\mathbb{N} \vdash n{:}\mathbb{N} \qquad n{:}\mathbb{N}, m{:}\mathbb{N} \vdash m{:}\mathbb{N}}{n : \mathbb{N}, m : \mathbb{N} \vdash n + m : \mathbb{N}}\,(\mathbb{N}\text{-El}_+)}{\dfrac{n : \mathbb{N}, m : \mathbb{N} \vdash 0 + (n + m) == (n + m)}{\dfrac{n : \mathbb{N} \vdash \forall m^{\mathbb{N}}.0 + (n + m) == (n + m)}{\forall n, m : \mathbb{N}.0 + (n + m) == (n + m)}\,(\forall\text{-I})}\,(\forall\text{-I})}\,(\forall\text{-El})$$

Existential Quantification

- We have seen before that we can identify in type theory existential quantification with the dependent product.
- With this identification, the **introduction rule** for the dependent product allows to form a proof of $\exists x^A.B$ from an element $a : A$ and a proof $p : B[x := a]$:

$$\frac{a : A \qquad p : B[x := a]}{\langle a, p \rangle : \exists x^A.B}\,(\times\text{-I})$$

- This is what is expressed by the **natural deduction introduction rule for $\exists$**:

$$\frac{a : A \qquad B[x := a]}{\exists x^A.B}\,(\exists\text{-I})$$

Example

- Assume we want to show $\forall n^{\mathbb{N}}.\exists m^{\mathbb{N}}.m > n$.
  - In order to prove this one assumes first $n : \mathbb{N}$.
  - Then one concludes $\mathrm{S}\ n : \mathbb{N}$ and $\mathrm{S}\ n > n$.
  - Using the introduction rule for $\exists$ one concludes $\exists m^{\mathbb{N}}.m > n$ under the assumption $n : \mathbb{N}$.
  - Using the introduction rule for $\forall$ one concludes $\forall n^{\mathbb{N}}.\exists m^{\mathbb{N}}.m > n$.

Example

- In natural deduction, this proof reads as follows:

$$\dfrac{\dfrac{n : \mathbb{N} \vdash n : \mathbb{N}}{n : \mathbb{N} \vdash \mathrm{S}\; n : \mathbb{N}}\,(\mathbb{N}\text{-I}_{\mathrm{S}}) \qquad n : \mathbb{N} \vdash \mathrm{S}\; n > n}{\dfrac{n : \mathbb{N} \vdash \exists m^{\mathbb{N}}.m > n}{\forall n^{\mathbb{N}}.\exists m^{\mathbb{N}}.m > n}\,(\forall\text{-I})}\,(\exists\text{-I})$$

Existential Quantification (Cont.)

- The **elimination rule** for the dependent product allows to project a proof p of $\exists x^A.B$ to an element $\pi_0(p) : A$ and proof $\pi_1(p) : B[x := \pi_0(p)]$.
- This kind of rule works only if we have **explicit proofs**.
- From this we can derive a rule which is essentially that used in natural deduction (in which one doesn't have explicit proofs):
  - Assume:
    - $C : \mathrm{Set}$, which does not depend on $x : A$,
    - $p : \exists x^A.B$ and
    - $x : A, y : B \Rightarrow c : C$.
  - Then we have $\mathbf{c[x := \pi_0(p), y := \pi_1(p)] : C}$, **not depending on x:A or y:B**.

Existential Quantification (Cont.)

- Therefore the **rule in natural deduction** follows from the type theoretic rules:

$$\frac{\exists x^A.B \qquad x^A, B \vdash C}{C}\,(\exists\text{-El})$$

  where C does not depend on $x : A$ and B.

- Here $x : A, B \vdash C$ means that from $x : A$ and assumption B we can derive C.
  - As in the introduction rule for natural deduction, $x : A$ is usually not mentioned explicitly, since the type structure there is very simple.

Example

- Assume we have shown $\forall n^{\mathbb{N}}.\exists m^{\mathbb{N}}.m > n \wedge \mathrm{Prime}(m)$.
- We want to show that for all n there exist two primes above it, i.e.

$$\forall n^{\mathbb{N}}.\exists m, k : \mathbb{N}.m > k \wedge k > n \wedge \wedge \mathrm{Prime}(m) \wedge \mathrm{Prime}(k) \quad .$$

- We can derive this as follows:
  - Assume $n : \mathbb{N}$.
  - We have $\exists m^{\mathbb{N}}.m > n \wedge \mathrm{Prime}(m)$.
  - So assume $m : \mathbb{N}$ and $m > n \wedge \mathrm{Prime}(m)$.
  - We have as well $\exists k^{\mathbb{N}}.k > m \wedge \mathrm{Prime}(k)$.
  - So assume $k : \mathbb{N}$ and $k > m \wedge \mathrm{Prime}(k)$.

Example

- Then we can conclude

$$m > k \land k > n \land \mathrm{Prime}(m) \land \mathrm{Prime}(k)$$

and therefore as well

$$\exists m, k : \mathbb{N}. m > k \land k > n \land \mathrm{Prime}(m) \land \mathrm{Prime}(k)$$

- Now by $\exists$-elimination twice follows

$$n : \mathbb{N} \vdash \exists m, k : \mathbb{N}. m > k \land k > n \land \mathrm{Prime}(m) \land \mathrm{Prime}(k)$$

without assuming m, k as above.

- By $\forall$-introduction follows

$$\forall n^{\mathbb{N}}. \exists m, k : \mathbb{N}. m > k \land k > n \land \mathrm{Prime}(m) \land \mathrm{Prime}(k)$$

Example

- The formal proof in natural deduction is as follows (some of the premises can be shown easily in natural deduction):

Example

- First step: Under the global assumption

$$n : \mathbb{N}, m : \mathbb{N}, m > n \wedge \mathrm{Prime}(m), k : \mathbb{N}, k > m \wedge \mathrm{Prime}(k)$$

  we prove the following

$$\dfrac{m : \mathbb{N} \qquad \dfrac{k : \mathbb{N} \qquad m > n \wedge k > m \wedge \mathrm{Prime}(m) \wedge \mathrm{Prime}(k)}{\exists k^{\mathbb{N}}.m > n \wedge k > m \wedge \mathrm{Prime}(m) \wedge \mathrm{Prime}(k)}\,(\exists}{\exists m, k : \mathbb{N}.m > n \wedge k > m \wedge \mathrm{Prime}(m) \wedge \mathrm{Prime}(k)}\,(\exists\text{-I})$$

- So we have shown

$$n : \mathbb{N}, m : \mathbb{N}, m > n \wedge \mathrm{Prime}(m), k : \mathbb{N}, k > m \wedge \mathrm{Prime}(k) \vdash$$
$$\exists m, k : \mathbb{N}.m > n \wedge k > m \wedge \mathrm{Prime}(m) \wedge \mathrm{Prime}(k)$$

Example

- Second step: Under the assumption

$$n : \mathbb{N}, m : \mathbb{N}, m > n \land \mathrm{Prime}(m)$$

we can conclude

$$\exists k^{\mathbb{N}}.k > m \land \mathrm{Prime}(k)$$

and then conclude by $\exists$-elimination and Step 1

$$\frac{\begin{array}{c}\exists k^{\mathbb{N}}.k>m\land\mathrm{Prime}(k)\\ k{:}\mathbb{N},k>m\land\mathrm{Prime}(k)\vdash\exists m,k{:}\mathbb{N}.m>n\land k>m\land\mathrm{Prime}(m)\land\mathrm{Prime}(k)\end{array}}{\exists m, k : \mathbb{N}.m > n \land k > m \land \mathrm{Prime}(m) \land \mathrm{Prime}(k)}\,(\exists\text{-I})$$

Example

- Third step: Again we can conclude

$$n : \mathbb{N} \vdash \exists m^{\mathbb{N}}.m > n \wedge \mathrm{Prime}(m)$$

and then conclude by $\exists$-elimination and Step 2

$$\dfrac{\dfrac{\begin{array}{c} n{:}\mathbb{N}{\vdash}\exists m^{\mathbb{N}}.m{>}n{\wedge}\mathrm{Prime}(m) \\ n{:}\mathbb{N},m{:}\mathbb{N},m{>}n{\wedge}\mathrm{Prime}(m){\vdash}\exists m,k{:}\mathbb{N}.m{>}n{\wedge}k{>}m{\wedge}\mathrm{Prime}(m){\wedge}\mathrm{Prime}(k) \end{array}}{n : \mathbb{N} \vdash \exists m, k : \mathbb{N}.m > n \wedge k > m \wedge \mathrm{Prime}(m) \wedge \mathrm{Prime}(k)} (\exists\text{-I})}{\forall n^{\mathbb{N}}.\exists m, k : \mathbb{N}.m > n \wedge k > m \wedge \mathrm{Prime}(m) \wedge \mathrm{Prime}(k)} (\forall\text{-I})$$

Construct. (or Intuit.) Logic

- From type theoretic proofs we can **directly extract programs**.
- For instance, if $p : \forall x^A.\exists y^B.C[x, y]$, then we have
  - for $x : A$ it follows $b := \pi_0(p\ x) : B$ and $\pi_1(p\ x) : C[x, b]$.
  - Therefore $f := \ \lambda x^A.\pi_0(p\ x)$ is a **function of type $\mathbf{A} \rightarrow \mathbf{B}$**, and we have

    $$\boldsymbol{\lambda x^A.\pi_1(p\ x) : \forall x^A.C[x, f\ x]}$$

    i.e. we have a proof that $\mathbf{\forall x^A.C[x, f\ x]}$ **holds**.
  - Therefore, from a proof of $\forall x^A.\exists y^B.C[x, y]$, we can **extract a function**, which computes the y from the x.

Constructive Logic (Cont.)

- We can derive as well a function which **depending on $\mathrm{p} : \mathrm{A} + \mathrm{B}$ decides whether p= inl(a) or p = inr(b)**.
- Therefore we can decide, from a proof of a disjunction, **which of the disjuncts holds**.
- This has consequences due to the **undecidability of the Turing halting problem.**
  - Before continuing, I introduce briefly this result for those who haven't been in the module on computability theory.

Turing Machines

- A **Turing machine** (in short **TM**) is a program language which is according to **Church's thesis** universal:
  - Every computable function can be computed by a TM.
  - TMs can have one input string, no interaction, and have as output one output string.
    - Both these strings are usually interpreted as natural numbers.
  - To run a TM with no input means to run it with the empty input string.

Turing Complete Languages

- Any programming language, which can simulate a TM, shares this property and is called **Turing complete**.
  - Most standard programming languages, e.g. Java, Pascal, C, C++ are **Turing complete**.
  - **Agda**, restricted to termination checked programs, is **not Turing complete**.
    - No (decidable) language, which allows to write terminating programs only, can be Turing complete.

Turing Halting Problem

- The **Turing halting problem** is the question, whether a TM (with no inputs) terminates.
  - An essentially equivalent form is the question whether a TM with one input terminates.
- One can introduce a predicate **halts x** depending on a TM x (which can be represented as a string, as a natural number, or as a specific data type) expressing that "TM x holds, if given no inputs".
- Therefore the Turing halting problem is the question whether we can decide

$$\text{halts } x \lor \neg\text{halts } x \quad .$$

Unprovability in Type Theory

- It is known that the Turing halting problem is undecidable:
  - We cannot decide in a computable way for every x the Turing halting problem for x.
- Similarly we cannot decide whether a Java program with no input and no interaction terminates or not.
- Because of the undecidability of the Turing halting problem, the following formula is unprovable in Martin-Löf Type Theory and as well in Agda:

$$\forall x^{\mathrm{TM}}.\mathrm{halts}\ x \vee \neg\mathrm{halts}\ x \quad .$$

  - Here TM is a data type which allows to encode all TM in a standard way.

Unprovability in Constructive Logic

- If we could prove it, we could get a function, which determines for $x : \mathrm{TM}$ whether $\mathrm{halts}\ x$ or not.
- But such a function needs to be computable, and such a computable function doesn't exist.

Constructive Logic (Cont.)

- In classical logic we **can prove the above**, since we can derive $\mathbf{A} \vee \neg \mathbf{A}$ (tertium non datur) for any formula A.
- In type theory, this law **cannot hold**, unless we don't want that all programs can be evaluated.
  - The logic of type theory is **intuitionistic (constructive) logic**, in which $A \vee \neg A$ and $\neg\neg A \supset A$ are not provable for all formulae A.
- Jump over remaining slides

Constructive Logic (Cont.)

- In **classical logic**,
  - $\exists \mathbf{x}^{\mathbf{A}}.\mathrm{B}$ **is equivalent to** $\neg\forall \mathbf{x}^{\mathbf{A}}.\neg\mathrm{B}$,
  - $\mathbf{A} \vee \mathrm{B}$ **is equivalent to** $\neg(\neg\mathbf{A} \wedge \neg\mathrm{B})$.
- If we take decidable atomic formulae only and

$$\text{replace} \quad \exists x^A.B \quad \text{by} \quad \neg\forall x^A.\neg B$$
$$\text{replace} \quad A \vee B \quad \text{by} \quad \neg(\neg A \wedge \neg B)$$

  then **all formulae provable in classical logic are derivable** in type theory.
  - All we need is $\neg\neg A \supset A$, which can be shown for all formulae built from decidable atomic formulae using $\neg$, $\supset$, $\wedge$, $\forall$.

Constructive Logic (Cont.)

- Especially, the tertium non datur formula

$$A \vee \neg A$$

translates into

$$\neg(\neg A \wedge \neg\neg A)$$

which trivially holds, since $\neg A$ and $\neg\neg A$ implies $\perp$.

- In this sense, **type theory contains classical logic**.

Weak vs. strong Disj./Quant.

- But type theory is **richer**, since it has as well so called **strong disjunction and existential quantification**.
  - **Strong disjunction** and **strong existential quantification** are the formulae

$$A \vee B \text{ and } \exists x^A.B$$

whereas **weak disjunction** and **weak existential quantification** are the formulae

$$\neg(\neg A \wedge \neg B) \text{ and } \neg\forall x^A.\neg B$$

Weak vs. strong Disj./Quant.

- From a proof $p : \exists x^A.B$ we can extract an element x of A s.t. B holds, namely
$$\pi_0(x)$$
This is in general **not possible for weak existential quantification.**
- From a proof $p : A \vee B$ we can determine which one of A or B holds (the other disjunct might hold as well).
From a proof of **weak disjunction** this is in general **not possible.**

Constructive Logic (Cont.)

- **Remark:** One can always obtain classical logic in Agda for arbitrary formulae by **postulating** tertium non datur for the formulae for which one needs it:

  postulate p : A $\vee$ $\neg$A

- Jump over the following proofs.

Constructive Logic (Cont.)

- Proof (using classical logic) of

$$\exists \mathbf{x}^{\mathbf{A}}.\mathbf{B} \leftrightarrow (\neg\forall \mathbf{x}^{\mathbf{A}}.\neg\mathbf{B}) \quad :$$

  - We have classically:

$$\neg\neg\mathbf{A} \supset \mathbf{A} \quad :$$

    - If A is true, then $\neg\neg A \supset A$ holds.
    - If A is false, then $\neg\neg A$ is false, therefore $\neg\neg A \supset A$ holds.

Constructive Logic (Cont.)

- We show intuitionistically $\neg\exists\mathbf{x}^{\mathbf{A}}.\mathbf{B} \leftrightarrow \forall\mathbf{x}^{\mathbf{A}}.\neg\mathbf{B}$:
  - Assume $\neg\exists x^A.B$, $x : A$ and show $\neg B$.
    If we had B, then we had $\exists x^A.B$, contradicting $\neg\exists x^A.B$. Therefore $\neg B$.
  - Assume $\forall x^A.\neg B$. Show $\neg\exists x^A.B$:
    Assume $\exists x^A.B$. Assume x s.t. B holds.
    By $\forall x^A.\neg B$ we get $\neg B$, therefore a contradiction.
- Now it follows (classically):

$$(\exists\mathbf{x}^{\mathbf{A}}.\mathbf{B}) \leftrightarrow (\neg\neg\exists\mathbf{x}^{\mathbf{A}}.\mathbf{B}) \leftrightarrow (\neg\forall\mathbf{x}^{\mathbf{A}}.\neg\mathbf{B})$$

Constructive Logic (Cont.)

- Proof of

$$\mathbf{A} \lor \mathbf{B} \leftrightarrow \neg(\neg\mathbf{A} \land \neg\mathbf{B}) \quad :$$

  - We show intuitionistically $\neg(\mathbf{A} \lor \mathbf{B}) \leftrightarrow (\neg\mathbf{A} \land \neg\mathbf{B}) \quad :$
    - Assume $\neg(A \lor B)$. If A then $A \lor B$, a contradiction, therefore $\neg A$.
      Similarly we get $\neg B$, therefore $\neg A \land \neg B$.
    - Assume $\neg A \land \neg B$, show $\neg(A \lor B)$.
      Assume $A \lor B$. If A then a contradiction with $\neg A$, similarly with B.
  - Now it follows (classically):

$$(\mathbf{A} \lor \mathbf{B}) \leftrightarrow \neg\neg(\mathbf{A} \lor \mathbf{B}) \leftrightarrow \neg(\neg\mathbf{A} \land \neg\mathbf{B})$$

Class. Logic for $\exists$, $\vee$-free Formulae

- We show that for formulas A built from $\neg$, $\supset$, $\wedge$, $\forall$ and decidable prime formulae we have

$$\neg\neg A \supset A \quad .$$

- The formula $\neg\neg A \supset A$ is called **stability for A**.
- This is done by induction over the buildup of these formulae.

Class. Logic for $\exists$, $\vee$-free Formulae

- Case $A \equiv \mathrm{Atom}\ c$.
  - We make case distinction on c.
  - If $c = \mathrm{tt}$, then we have $A \equiv \top$, A is provable, therefore as well $\neg\neg A \supset A$.
  - If $c = \mathrm{ff}$, then we have $A \equiv \perp$.
    - Assume $\neg\neg A \equiv (\perp \supset \perp) \supset \perp$.
    - $\perp \supset \perp$ is provable.
    - Therefore we obtain $\perp$, which is A.
    - So we have
$$\neg\neg A \vdash A$$
      and obtain
$$\neg\neg A \supset A \quad .$$

Class. Logic for $\exists$, $\vee$-free Formulae

- Case $A \equiv B \supset C$, and assume we have already shown stability for B and C.
- We have to show that from $\neg\neg A$ we obtain A, which is $B \supset C$.
- So assume $\neg\neg A$, B and show C.
- We show $\neg\neg C$, then by stability of C we obtain C.
- $\neg\neg C \equiv \neg C \supset \bot$.
- Therefore assume $\neg C$ and show $\bot$.
  - We show $\neg A$ which is $A \supset \bot$.
    - So assume A and show $\bot$. $A \equiv B \supset C$, therefore by B we get C, and by $\neg C$ therefore $\bot$.
  - By $\neg\neg A$, which is $\neg A \supset \bot$, we get therefore $\bot$, which completes the proof for this case.

Class. Logic for $\exists$, $\vee$-free Formulae

- Case $A \equiv B \wedge C$, and assume we have already shown stability for B and C.
- Assume $\neg\neg A$ and show A.
  - We show $\neg\neg B$, which implies by the stability of B that B holds.
    - Since $\neg\neg B \equiv \neg B \supset \bot$, we assume $\neg B$ and have to show $\bot$.
    - We show $\neg A$, i.e. show that A implies $\bot$.
      - Assume A, which is $B \wedge C$. Then we get B, and by $\neg B$ therefore $\bot$.
    - By $\neg\neg A$ we obtain $\bot$.
  - Therefore we have shown B.
  - A similar proof shows C, and therefore we get $B \wedge C$, i.e. A.

Class. Logic for $\exists$, $\vee$-free Formulae

- Case $A \equiv \forall x^B.C$, and assume we have already shown stability for C.
- Assume $\neg\neg A$ and show A.
- So assume $x : B$, and show C.
- We show $\neg\neg C$, which by the stability of C implies C.
  - So assume $\neg C$ and show $\perp$.
  - We show $\neg A$.
    - Assume A, which is $\forall x^B.C$.
    - Then we obtain C, and by $\neg C$ therefore $\perp$.
  - By $\neg\neg A$ we therefore get $\perp$, and are done.

Class. Logic for $\exists$, $\vee$-free Formulae

- Case $A \equiv \neg B$, and we have stability for B.
- $\neg B \equiv B \supset \bot$.
- $\bot \equiv \bot = \mathrm{Atom\ false}$.
- By stability for decidable prime formulae we get stability for $\bot$.
- Together with the stability for B we obtain by case $\supset$ the stability for $B \supset \bot \equiv \neg B$.

(g) The Set of Natural Numbers

- The set $\mathbb{N}$ is the type theoretic representation of the set $\mathbb{N} := \{0, 1, 2, \ldots, \}$.
- $\mathbb{N}$ can be generated by
  - starting with the empty set,
  - adding 0 to it, and
  - adding, whenever we have x in it $x + 1$ to it.

The Set of Natural Numbers (Cont.)

- Let S be a type theoretic notation for the operation $x \mapsto x + 1$.
- Then the type theoretic rules are

$$\mathbb{N} : \mathrm{Set} \quad (\mathbb{N}\text{-F})$$

$$0 : \mathbb{N} \quad (\mathbb{N}\text{-I}_0)$$

$$\frac{n : \mathbb{N}}{\mathrm{S}\ n : \mathbb{N}}\ (\mathbb{N}\text{-I}_{\mathrm{S}})$$

Primitive Recursion

- **Primitive Recursion expresses:**
  Assume we have
  - $a : \mathbb{N}$.
  - and, if $n : \mathbb{N}$, $x : \mathbb{N}$ then $g\ n\ x : \mathbb{N}$.

  Then we can define $\mathrm{f} : \mathbb{N} \to \mathbb{N}$, s.t.
  - $f\ 0 = a$,
  - $f\ (\mathrm{S}\ n) = g\ n\ (f\ n)$.

Primitive Recursion (Cont.)

- The **computation of $\mathbf{f\ n}$** proceeds now as follows:
  - Compute n.
  - If $n = 0$, then the result is a.
  - Otherwise $n = \mathrm{S}\ n'$.
    - We assume that we have determined already how to compute $f\ n'$.
    - Now $f\ n$ reduces to $g\ n'\ (f\ n')$.
    - $g\ n'\ (f\ n')$ can be computed, since we know how to compute
      - g
      - $f\ n'$.

Example

- The function $f : \mathbb{N} \to \mathbb{N}$ with $f\ n = 2 \cdot n$ can be defined **primitive recursively** by:
  - $f\ 0 = 0$.
  - $f\ (\mathrm{S}\ n) = \mathrm{S}\ (\mathrm{S}\ (f\ n))$.
- Therefore take in the definition above:
  - $a = 0$,
  - $g\ n\ x = \mathrm{S}\ (\mathrm{S}\ x)$.

Generalised Primitive Recursion

- We can **generalise primitive recursion** as follows:
  - First we can **replace the range of f by an arbitrary set C**
    - i.e. we allow for any set C

$$f : \mathbb{N} \to C$$

  - Further, C can now **depend on $\mathbb{N}$**.
- We obtain the following set of rules:

Rules for the Natural Numbers

Formation Rule

$$\mathbb{N} : \mathrm{Set} \quad (\mathbb{N}\text{-F})$$

Introduction Rules

$$0 : \mathbb{N} \quad (\mathbb{N}\text{-I}_0)$$

$$\frac{n : \mathbb{N}}{\mathrm{S}\ n : \mathbb{N}}\,(\mathbb{N}\text{-I}_{\mathrm{S}})$$

Rules for the Natural Numbers

Elimination Rule

$$\frac{\begin{array}{c} C : \mathbb{N} \to \mathrm{Set} \\ a : C\ 0 \\ g : (x : \mathbb{N}) \to C\ x \to C\ (\mathrm{S}\ x) \\ n : \mathbb{N} \end{array}}{\mathrm{P}\ C\ a\ g\ n : C\ n}\ (\mathbb{N}\text{-El})$$

Equality Rules

$$\begin{array}{rcll} \mathrm{P}\ C\ a\ g\ 0 & = & a & (\mathbb{N}\text{-Eq}_0) \\ \mathrm{P}\ C\ a\ g\ (\mathrm{S}\ n) & = & g\ n\ (\mathrm{P}\ C\ a\ g\ n) & (\mathbb{N}\text{-Eq}_{\mathrm{S}}) \end{array}$$

Additionally we have the **Equality versions** of the formation-, introduction- and elimination-rules.

Jump over Elimination into Type

Elimination into Type

- In order to define predicates on the natural numbers by prim. recursion, we need sometimes elimination into type:

  Strong elimination Rule

$$\frac{\begin{array}{c} n : \mathbb{N} \Rightarrow C[n] : \mathrm{Type} \\ a : C[0] \\ g : (x : \mathbb{N}) \to C[x] \to C[\mathrm{S}\ x] \\ n : \mathbb{N} \end{array}}{\mathrm{P}_C^{\mathrm{Type}}\ a\ g\ n : C[n]}\ (\mathbb{N}\text{-}\mathrm{El}^{\mathrm{Type}})$$

  Strong Equality Rules

$$\begin{array}{rcll} \mathrm{P}_C^{\mathrm{Type}}\ a\ g\ 0 & = & a & (\mathbb{N}\text{-}\mathrm{Eq}_0^{\mathrm{Type}}) \\ \mathrm{P}_C^{\mathrm{Type}}\ a\ g\ (\mathrm{S}\ n) & = & g\ n\ (\mathrm{P}_C^{\mathrm{Type}}\ a\ g\ n) & (\mathbb{N}\text{-}\mathrm{Eq}_\mathrm{S}^{\mathrm{Type}}) \end{array}$$

Rules for the Natural Numbers

- Note that if we define in the elimination rule $f := \mathrm{P}\ C\ g$ (which is η-equal to $\lambda n^{\mathbb{N}}.\mathrm{P}\ C\ g\ n$) then
  - The conclusion of the elimination rule reads:

$$f\ n : C\ n$$

    which means that

$$f : (n : \mathbb{N}) \to C\ n \quad .$$

  - The equality rules read:

$$\begin{array}{rcl} f\ 0 & = & a \\ f\ (\mathrm{S}\ n) & = & g\ n\ (f\ n) \end{array}$$

Logical Framework Rules for $\mathbb{N}$

- The more compact notation is:
  - $\mathbb{N} : \mathrm{Set}$,
  - $0 : \mathbb{N}$,
  - $S : \mathbb{N} \to \mathbb{N}$,
  - $$\begin{aligned} \mathrm{P} \ : & (C : \mathbb{N} \to \mathrm{Set}) \\ & \to C\ 0 \\ & \quad \to ((x : \mathbb{N}) \to C\ x \to C\ (S\ x)) \\ & \qquad \to (n : \mathbb{N}) \\ & \qquad\quad \to C\ n\ . \end{aligned}$$
  - The same equality rules as before.

Natural Numbers in Agda

- $\mathbb{N}$ is defined using **data**:

$$\begin{array}{l}\mathrm{data}\ \mathbb{N} : \mathrm{Set\ where}\\ \quad \mathrm{Z} : \mathbb{N}\\ \quad \mathrm{S} : \mathbb{N} \to \mathbb{N}\end{array}$$

  Here $\mathbb{N}$ can be typed in using Leim as $\backslash$Bbb$\{$N$\}$.
  (We cannot use 0 for zero, since this denotes the builtin native natural number 0 in Agda).

- Therefore we have

$$\begin{array}{rcl}\mathrm{Z} & : & \mathbb{N}\\ \mathrm{S} & : & \mathbb{N} \to \mathbb{N}\end{array}$$

Elimination Rules for $\mathbb{N}$ in Agda

- Elimination is represented in Agda as before via case distinction.
- Assume we want to define

$$
\begin{array}{l}
f : (n : \mathbb{N}) \to A \\
f\ n = \{!\ \ !\}
\end{array}
$$

  - A possibly depending on n,
- Then we can distinguish the cases $n = \mathrm{Z}$ and $n = \mathrm{S}\ m$ and obtain:

$$
\begin{array}{l}
f : (n : \mathbb{N}) \to A \\
\begin{array}{lllll}
f & \mathrm{Z} & = & \{!\ \ !\} \\
f & (\mathrm{S}\ n) & = & \{!\ \ !\}
\end{array}
\end{array}
$$

Elimination Rules for $\mathbb{N}$ in Agda

- For solving the goals, we can now **make use of f**. That will be **accepted by the type checker.**
- However, if we use of full f, and then type check the file, the termination checker will complain, and we obtain for instance

$$
\begin{aligned}
&\boxed{f} : (n : \mathbb{N}) \to A \\
&f\ n = \boxed{f}\ n
\end{aligned}
$$

exampleNat1.agda

Elimination Rules for $\mathbb{N}$ in Agda

- If we, in

$$\begin{array}{llcl} g : (n : \mathbb{N}) \to A \\ g & \mathrm{Z} & = & \{!\ \ !\} \\ g & (\mathrm{S}\ n) & = & \{!\ \ !\} \end{array}$$

  - **do not make use of g when defining** $g\ \mathrm{Z}$ and
  - **only use of** $\mathrm{g\ n}$ **when defining** $g\ (\mathrm{S}\ n)$

  then the termination check succeeds (once the definition is complete).

Elimination Rules for $\mathbb{N}$ in Agda

- If we haven't completed the definition of g, the termination checker might complain, as long as not all details are known.
  - For instance, if we have the following we get an error:

$$
\begin{array}{llll}
\boxed{g} : \mathbb{N} \longrightarrow \mathbb{N} \\
g & \mathrm{Z} & = & \mathrm{Z} \\
g & (\mathrm{S}\ n) & = & \boxed{g}\ \{!\ \ !\}
\end{array}
$$

  - If we complete it as follows the error vanishes (one might need to load the agda code again):

$$
\begin{array}{llll}
g : \mathbb{N} \longrightarrow \mathbb{N} \\
g & \mathrm{Z} & = & \mathrm{Z} \\
g & (\mathrm{S}\ n) & = & g\ n
\end{array}
$$

Elimination Rules for $\mathbb{N}$ in Agda

- If **check-termination succeeds**, the definition should be **correct.**
  - (The lecturer hasn't checked the algorithm).
- However, **if check-termination fails**, the **definition might still be correct**.
  Jump over Limitations of Termination Checker.

Power of Termination Check

- The following definition of the **Fibonacci numbers** can't be defined this way directly using the rules of type theory, but it **can be defined in Agda** as follows and **check-termination accepts it**:
  $(\mathrm{one} := \mathrm{S}\ \mathrm{Z})$:

$$
\begin{array}{llll}
\multicolumn{4}{l}{\mathrm{fib} : \mathbb{N} \to \mathbb{N}} \\
\mathrm{fib} & \mathrm{Z} & = & \mathrm{one} \\
\mathrm{fib} & (\mathrm{S}\ \mathrm{Z}) & = & \mathrm{one} \\
\mathrm{fib} & (\mathrm{S}\ (\mathrm{S}\ n)) & = & \mathrm{fib}\ n + \mathrm{fib}\ (\mathrm{S}\ n)
\end{array}
$$

fib1.agda

Limitations of Termination Checker

- Assume we define the **predecessor function**

$$
\begin{aligned}
&\mathrm{pred} : \mathbb{N} \longrightarrow \mathbb{N} \\
&\mathrm{pred} \quad \mathrm{Z} \qquad\quad = \quad \mathrm{Z} \\
&\mathrm{pred} \quad (\mathrm{S}\ n) \quad = \quad n
\end{aligned}
$$

i.e.

$$
\mathrm{pred}(n) = \begin{cases} 0 & \text{if } n = 0 \\ n - 1 & \text{otherwise.} \end{cases}
$$

Limitations of Termination Checker

- Then the function

$$\begin{array}{llcl} \boxed{f} : \mathbb{N} \to \mathbb{N} & & & \\ f & \mathrm{Z} & = & \mathrm{Z} \\ f & (\mathrm{S}\ n) & = & \boxed{f}\ (\mathrm{pred}\ n) \end{array}$$

  terminates always
  - (it returns for all $n : \mathbb{N}$ the value Z).
- However, **check-termination fails**.
  terminationnat1.agda

Limitations of Termination Checker

- Because of the **undecidability of the Turing halting problem**
  - it is undecidable, whether a recursively defined function terminates or not,
- therefore there is no **extension of check-termination**, **which accepts exactly all in Agda definable functions, which terminate for all inputs**.

Example: Addition

- Definition of $+$ in Agda:

$$\begin{array}{lllll} \multicolumn{5}{l}{\text{infixr } 10 \_+\_} \\ \multicolumn{5}{l}{\_+\_ : \mathbb{N} \to \mathbb{N} \to \mathbb{N}} \\ n & + & \mathrm{Z} & = & n \\ n & + & \mathrm{S}\ m & = & \mathrm{S}\ (n+m) \end{array}$$

- The definitition is correct, since when defining $n + \mathrm{S}\ m$, $n + m$ is defined before $n + \mathrm{S}\ m$.
- Because of the line

$$\text{infixr } 10 \_+\_ \ ,$$

$n + m + k$ is interpreted as $n + (m + k)$.

Example: Multiplication

- Definition

$$\begin{array}{l}\mathrm{infixr}\ 20\ \_*\_\\ \_*\_ : \mathbb{N} \to \mathbb{N} \to \mathbb{N}\\ n \quad * \quad \mathrm{Z} \qquad = \quad \mathrm{Z}\\ n \quad * \quad \mathrm{S}\ m \quad = \quad n * m + n\end{array}$$

- Because of the line

$$\mathrm{infixr}\ 20\ \_*\_ \ ,$$

$\_*\_$ binds more than $\_+\_$
  - Remember we had $\mathrm{infixr}\ 10\ \_+\_$.
- We can use in the definition of $\_*\_$ $+$, and can refer in case of $n * \mathrm{S}\ m$ to $n * m$, which is defined before $n * \mathrm{S}\ m$.

Equality on $\mathbb{N}$

- We can define a Boolean valued equality on $\mathbb{N}$ as follows:

$$
\begin{array}{l}
\_{==}_{\mathrm{Bool}}\_ : \mathbb{N} \to \mathbb{N} \to \mathrm{Bool} \\
\begin{array}{lllll}
\mathrm{Z} & ==_{\mathrm{Bool}} & \mathrm{Z} & = & \mathrm{tt} \\
\mathrm{S}\ n & ==_{\mathrm{Bool}} & \mathrm{S}\ m & = & n\ ==_{\mathrm{Bool}}\ m \\
\_ & ==_{\mathrm{Bool}} & \_ & = & \mathrm{ff}
\end{array}
\end{array}
$$

- Note that the third case expresses: in all other cases (i.e. when defining $n ==_{\mathrm{Bool}} m$ and neither both n, m are Z nor both are of the form S _) we obtain the result ff.

Equality on $\mathbb{N}$

- Then we can define equality _$==$_ on $\mathbb{N}$ as follows

$$
\begin{array}{l}
\_{==}\_ : \mathbb{N} \to \mathbb{N} \to \mathrm{Set} \\
n == m = \mathrm{Atom}\ (n\ \ {==}_{\mathrm{Bool}}\ \ m)
\end{array}
$$

Equality on $\mathbb{N}$ (Cont.)

- Alternatively we could have defined $\_\!==\!\_$ directly (this uses in fact large elimination on $\mathbb{N}$):

$$
\begin{array}{llll}
\multicolumn{4}{l}{\_==\_ : \mathbb{N} \to \mathbb{N} \to \mathrm{Set}} \\
\mathrm{Z} & == \; \mathrm{Z} & = & \top \\
\mathrm{S}\; n & == \; \mathrm{S}\; m & = & n == m \\
\_ & == \; \_ & = & \bot
\end{array}
$$

nat1.agda

Reflexivity of ==

- **Reflexivity** of $==$ is the formula:

$$\forall n^{\mathbb{N}}.n == n$$

- **Type theoretically** this means that we have to prove

$$\begin{aligned}
\mathrm{refl} &: \mathrm{Refl} \\
\mathrm{refl} &= \{!\ \ !\}
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{Refl} &: \mathrm{Set} \\
\mathrm{Refl} &= (n : \mathbb{N}) \to n == n
\end{aligned}$$

Reflexivity of ==

$$\mathrm{Refl} : \mathrm{Set}$$
$$\mathrm{Refl} = (n : \mathbb{N}) \to n == n$$

$$\mathrm{refl} : \mathrm{Refl}$$
$$\mathrm{refl} = \{!\ \ !\}$$

- Since refl is an element of a function type, we replace the definition of refl by

$$\mathrm{refl} : \mathrm{Refl}$$
$$\mathrm{refl}\ n = \{!\ \ !\}$$

where the type of the goal is $n == n$.

Reflexivity of == (Cont.)

$$\mathrm{Refl} : \mathrm{Set}$$
$$\mathrm{Refl} = (n : \mathbb{N}) \to n == n$$

$$\mathrm{refl} : \mathrm{Refl}$$
$$\mathrm{refl}\ n = \{!\ \ !\}$$

- This can now be shown using **pattern matching**:

$$\begin{array}{lll} \mathrm{refl} : \mathrm{Refl} & & \\ \mathrm{refl}\ \ \mathrm{Z} & = & \{!\ \ !\} \\ \mathrm{refl}\ \ (\mathrm{S}\ n) & = & \{!\ \ !\} \end{array}$$

Reflexivity of == (Cont.)

- In order to prove $\mathrm{refl}\ \mathrm{Z}$, we observe

$$
\begin{array}{rcl}
(\mathrm{Z} == \mathrm{Z}) & = & \mathrm{Atom}\ (\mathrm{Z}\ ==_{\mathrm{Bool}}\ \mathrm{Z}) \\
& = & \mathrm{Atom}\ \mathrm{tt} \\
& = & \top
\end{array}
$$

  - Therefore the goal can be solved by taking $\mathrm{true} : \top$.

Reflexivity of == (Cont.)

- In order to prove $\mathrm{refl}\ (\mathrm{S}\ n)$, we observe

$$
\begin{aligned}
(\mathrm{S}\ n == \mathrm{S}\ n) &= \mathrm{Atom}\ (\mathrm{S}\ n\ \ {==}\mathrm{Bool}\ \ \mathrm{S}\ n) \\
&= \mathrm{Atom}\ (n\ \ {==}\mathrm{Bool}\ \ n) \\
&= (n == n)
\end{aligned}
$$

  - Therefore the goal can be solved by taking $\mathrm{refl}\ n : (n == n)$.

Reflexivity of == (Cont.)

- The complete proof is as follows:

$$
\begin{array}{llcl}
\multicolumn{4}{l}{\mathrm{refl} : \mathrm{Refl}} \\
\mathrm{refl} & \mathrm{Z} & = & \mathrm{true} \\
\mathrm{refl} & (\mathrm{S}\ n) & = & \mathrm{refl}\ n
\end{array}
$$

- Note that this is not a black hole recursion, since in the second equation $\mathrm{refl}\ n$ is defined before $\mathrm{refl}\ (\mathrm{S}\ n)$.
  reflnat.agda

Symmetry of ==

- **Symmetry** of $==$ is the formula:

$$\forall n, m : \mathbb{N}. n == m \rightarrow m == n$$

- **Type theoretically** this means that we have to prove

$$\begin{array}{l} \mathrm{Sym} : \mathrm{Set} \\ \mathrm{Sym} = (n\ m : \mathbb{N}) \rightarrow n == m \rightarrow m == n \end{array}$$

In Agda this is shown by defining

$$\begin{array}{l} \mathrm{sym} : \mathrm{Sym} \\ \mathrm{sym}\ n\ m\ nm = \{!\ \ !\} \end{array}$$

Symmetry of == (Cont.)

$$\mathrm{Sym} : \mathrm{Set}$$
$$\mathrm{Sym} = (n\ m : \mathbb{N}) \to n == m \to m == n$$

- This can now be shown using **case distinction** on both n and m:

$$\begin{array}{llllll}
\multicolumn{6}{l}{\mathrm{sym} : \mathrm{Sym}} \\
\mathrm{sym} & \mathrm{Z} & \mathrm{Z} & nm & = & \{!\ \ !\} \\
\mathrm{sym} & \mathrm{Z} & (\mathrm{S}\ m) & nm & = & \{!\ \ !\} \\
\mathrm{sym} & (\mathrm{S}\ n) & \mathrm{Z} & nm & = & \{!\ \ !\} \\
\mathrm{sym} & (\mathrm{S}\ n) & (\mathrm{S}\ m) & nm & = & \{!\ \ !\}
\end{array}$$

- For convenience we spell out the type of sym in the following.

Symmetry of == (Cont.)

$$
\begin{array}{llllll}
\mathrm{sym} : (n\ m : \mathbb{N}) \to n == m \to m == n \\
\mathrm{sym} & \mathrm{Z} & \mathrm{Z} & nm & = & \{!\ \ !\} \\
\mathrm{sym} & \mathrm{Z} & (\mathrm{S}\ m) & nm & = & \{!\ \ !\} \\
\mathrm{sym} & (\mathrm{S}\ n) & \mathrm{Z} & nm & = & \{!\ \ !\} \\
\mathrm{sym} & (\mathrm{S}\ n) & (\mathrm{S}\ m) & nm & = & \{!\ \ !\}
\end{array}
$$

- In case $\mathrm{sym}\ \mathrm{Z}\ \mathrm{Z}\ nm$, the goal is

$$(\mathrm{Z} == \mathrm{Z}) = \top$$

  which can be solved by using true.
  - The argument nm is irrelevant and can be replaced by _.

Symmetry of == (Cont.)

$$
\begin{array}{llllll}
\mathrm{sym} : (n\ m : \mathbb{N}) \to n == m \to m == n \\
\mathrm{sym} & \mathrm{Z} & \mathrm{Z} & \_ & = & \mathrm{true} \\
\mathrm{sym} & \mathrm{Z} & (\mathrm{S}\ m) & nm & = & \{!\ \ !\} \\
\mathrm{sym} & (\mathrm{S}\ n) & \mathrm{Z} & nm & = & \{!\ \ !\} \\
\mathrm{sym} & (\mathrm{S}\ n) & (\mathrm{S}\ m) & nm & = & \{!\ \ !\}
\end{array}
$$

- In case $\mathrm{sym}\ \mathrm{Z}\ (\mathrm{S}\ m)\ nm$, we have

$$nm : \mathrm{Z} == \mathrm{S}\ m = \bot$$

so there is no element in nm, we can solve it as

$$\mathrm{sym}\ \mathrm{Z}\ (\mathrm{S}\ m)\ ()$$

Symmetry of == (Cont.)

$$\begin{array}{llllll}
\mathrm{sym} : (n\ m : \mathbb{N}) \to n == m \to m == n \\
\mathrm{sym} & \mathrm{Z} & \mathrm{Z} & _ & = & \mathrm{true} \\
\mathrm{sym} & \mathrm{Z} & (\mathrm{S}\ m) & () & & \\
\mathrm{sym} & (\mathrm{S}\ n) & \mathrm{Z} & nm & = & \{!\ \ !\} \\
\mathrm{sym} & (\mathrm{S}\ n) & (\mathrm{S}\ m) & nm & = & \{!\ \ !\}
\end{array}$$

- In case $\mathrm{sym}\ (\mathrm{S}\ n)\ \mathrm{Z}\ nm$, we have

$$nm : \mathrm{S}\ m == \mathrm{Z} = \bot$$

so there is no element in nm, we can solve it as

$$\mathrm{sym}\ (\mathrm{S}\ n)\ \mathrm{Z}\ ()$$

Symmetry of == (Cont.)

$$
\begin{array}{llllll}
\mathrm{sym} : (n\ m : \mathbb{N}) \to n == m \to m == n \\
\mathrm{sym} & \mathrm{Z} & \mathrm{Z} & \_ & = & \mathrm{true} \\
\mathrm{sym} & \mathrm{Z} & (\mathrm{S}\ m) & () \\
\mathrm{sym} & (\mathrm{S}\ n) & \mathrm{Z} & () \\
\mathrm{sym} & (\mathrm{S}\ n) & (\mathrm{S}\ m) & nm & = & \{!\ \ !\}
\end{array}
$$

- In case $\mathrm{sym}\ (\mathrm{S}\ n)\ (\mathrm{S}\ m)\ nm$, we have that the type of the goal is
$$(\mathrm{S}\ m == \mathrm{S}\ n) = (m == n)$$
- This goal can be solved by
$$\mathrm{sym}\ n\ m\ nm : m == n$$
which is type correct since
$nm : (\mathrm{S}\ n == \mathrm{S}\ m) = (n == m)$

Symmetry of == (Cont.)

- The complete proof is as follows:

$$
\begin{array}{llllll}
\multicolumn{6}{l}{\mathrm{sym} : (n\ m : \mathbb{N}) \to n == m \to m == n} \\
\mathrm{sym} & \mathrm{Z} & \mathrm{Z} & \_ & = & \mathrm{true} \\
\mathrm{sym} & \mathrm{Z} & (\mathrm{S}\ m) & () & & \\
\mathrm{sym} & (\mathrm{S}\ n) & \mathrm{Z} & () & & \\
\mathrm{sym} & (\mathrm{S}\ n) & (\mathrm{S}\ m) & nm & = & \mathrm{sym}\ n\ m\ nm
\end{array}
$$

- Note that this code termination checks, since in the last equation $\mathrm{sym}\ n\ m\ nm$ is defined before $\mathrm{sym}\ (\mathrm{S}\ n)\ (\mathrm{S}\ m)\ nm$.
  symnat.agda

Symmetry of == (Cont.)

$$
\begin{array}{llllll}
\mathrm{sym} : (n\ m : \mathbb{N}) \to n == m \to m == n \\
\mathrm{sym} & \mathrm{Z} & \mathrm{Z} & \_ & = & \mathrm{true} \\
\mathrm{sym} & \mathrm{Z} & (\mathrm{S}\ m) & () \\
\mathrm{sym} & (\mathrm{S}\ n) & \mathrm{Z} & () \\
\mathrm{sym} & (\mathrm{S}\ n) & (\mathrm{S}\ m) & nm & = & \mathrm{sym}\ n\ m\ nm
\end{array}
$$

- In the cases

$$
\begin{array}{lllll}
\mathrm{sym} & \mathrm{Z} & (\mathrm{S}\ m) & nm & \quad \text{and} \\
\mathrm{sym} & (\mathrm{S}\ n) & \mathrm{Z} & nm
\end{array}
$$

  we have that nm is an element of $\perp$, and the goal is $\perp$.

- So we can, instead of using empty case distinction on nm, return the proof nm and obtain the following:

Symmetry of == (Cont.)

$$
\begin{array}{llllll}
\multicolumn{6}{l}{\mathrm{sym} : (n\ m : \mathbb{N}) \to n == m \to m == n} \\
\mathrm{sym} & \mathrm{Z} & \mathrm{Z} & \_ & = & \mathrm{true} \\
\mathrm{sym} & \mathrm{Z} & (\mathrm{S}\ m) & nm & = & nm \\
\mathrm{sym} & (\mathrm{S}\ n) & \mathrm{Z} & nm & = & nm \\
\mathrm{sym} & (\mathrm{S}\ n) & (\mathrm{S}\ m) & nm & = & \mathrm{sym}\ n\ m\ nm
\end{array}
$$

symnat2.agda

Example: $<$ on $\mathbb{N}$

- The following introduces $<$ on $\mathbb{N}$:

$$\begin{array}{llll}
{<}\mathrm{Bool} : \mathbb{N} \to \mathbb{N} \to \mathrm{Bool} & & & \\
_ \quad <\mathrm{Bool} & \mathrm{Z} & = & \mathrm{ff} \\
\mathrm{Z} \quad <\mathrm{Bool} & \mathrm{S}\ m & = & \mathrm{tt} \\
\mathrm{S}\ n \quad <\mathrm{Bool} & \mathrm{S}\ m & = & n\ <\mathrm{Bool}\ m
\end{array}$$

$$\begin{array}{l}
_{<}_ : \mathbb{N} \to \mathbb{N} \to \mathrm{Set} \\
n < m = \mathrm{Atom}\ (n\ <\mathrm{Bool}\ m)
\end{array}$$

lessnat1.agda

Example: $<$ on $\mathbb{N}$

- Alternatively, we can define $<$ using large elimination:

$$
\begin{array}{llll}
\_<\_ : \mathbb{N} \to \mathbb{N} \to \mathrm{Set} & & & \\
\_ & < \; \mathrm{Z} & = & \bot \\
\mathrm{Z} & < \; \mathrm{S}\; m & = & \top \\
\mathrm{S}\; n & < \; \mathrm{S}\; m & = & n < m
\end{array}
$$

lessnat2.agda

Example: Tuples of Length n

- We define tuples (or vectors) of length n in Agda:

$$
\begin{array}{l}
\mathrm{data\ Nil} : \mathrm{Set\ where} \\
\quad [\,] : \mathrm{Nil} \\
\mathrm{data\ Cons}\ (A\ B : \mathrm{Set}) : \mathrm{Set\ where} \\
\quad \_{::}\_ : A \rightarrow B \rightarrow \mathrm{Cons}\ A\ B
\end{array}
$$

- Now we can define

$$
\begin{array}{llll}
\multicolumn{4}{l}{\mathrm{Tuple} : \mathrm{Set} \rightarrow \mathbb{N} \rightarrow \mathrm{Set}} \\
\mathrm{Tuple}\ \ A & \mathrm{Z} & = & \mathrm{Nil} \\
\mathrm{Tuple}\ \ A & (\mathrm{S}\ n) & = & \mathrm{Cons}\ A\ (\mathrm{Tuple}\ A\ n)
\end{array}
$$

Tuples of Length n

- Therefore,

$$\mathrm{Tuple}\ A\ n = \underbrace{\mathrm{Cons}\ A\ (\mathrm{Cons}\ A\ \cdots (\mathrm{Cons}\ A}_{n \text{ times}}\ \mathrm{Nil}) \cdots)) \ .$$

- The elements of $\mathrm{Tuple}\ A\ n$ are

$$a_1\ :: (a_2\ \cdots (a_n :: [\,]) \cdots)$$

  for elements $a_1, \ldots, a_n$ of A.
- In ordinary mathematical notation, we would write $\langle a_1, \ldots, a_n \rangle$ for such an element.
- Jump over next slides.

Remarks on Tuples of Length n

- In **ordinary mathematics**, we would define

$$\begin{aligned}\mathrm{Tuple}(A,0) &:= \{\langle\rangle\} \ , \\ \mathrm{Tuple}(A,n+1) &:= \{\langle a_1,\ldots,a_{n+1}\rangle \quad \mid a_1,\ldots,a_{n+1} \in A\} \ .\end{aligned}$$

- If we define

$$\begin{aligned}[\,] &:= \langle\rangle \ , \\ a_1 :: \langle a_2,\ldots,a_{n+1}\rangle &:= \langle a_1,\ldots,a_{n+1}\rangle \ ,\end{aligned}$$

then this reads:

$$\begin{aligned}\mathrm{Tuple}(A,0) &:= \{[\,]\} \ , \\ \mathrm{Tuple}(A,n+1) &:= \{a :: b \quad \mid a \in A \wedge b \in \mathrm{Tuple}(A,n)\} \ .\end{aligned}$$

Remarks on Tuples of Length n

- In the type theoretic definition we have **constructors**

$$
\begin{array}{lcl}
[\,] & : & \mathrm{Tuple}\ A\ \mathrm{Z} \\
\_::\_ & : & A \rightarrow \mathrm{Tuple}\ A\ n \rightarrow \mathrm{Tuple}\ A\ (\mathrm{S}\ n)
\end{array}
$$

- This is the **type theoretic analogue** of the previous definitions.

Componentwise Sum of n-Tuples

- We define **component-wise sum of tuples of length n**.
  - Using mathematical notation, this sum for instance as follows:

$$\langle 2, 3, 4 \rangle + \langle 5, 6, 7 \rangle = \langle 7, 9, 11 \rangle \quad .$$

Componentwise Sum of n-Tuples

$$
\begin{array}{l}
\mathrm{sum}\mathbb{N}\mathrm{Tuple} : (n : \mathbb{N}) \to \mathrm{Tuple}\ \mathbb{N}\ n \to \mathrm{Tuple}\ \mathbb{N}\ n \to \mathrm{Tuple}\ \mathbb{N}\ n \\
\begin{array}{llllll}
\mathrm{sum}\mathbb{N}\mathrm{Tuple} & \mathrm{Z} & [\,] & [\,] & = & [\,] \\
\mathrm{sum}\mathbb{N}\mathrm{Tuple} & (\mathrm{S}\ n) & (m :: l) & (m' :: l') & = &
\end{array} \\
\qquad\qquad (m + m') :: (\mathrm{sum}\mathbb{N}\mathrm{Tuple}\ n\ l\ l')
\end{array}
$$

tuple.agda

(h) Lists

- We define the set of lists of elements of type A in Agda.
- We have two constructors:
  - $[\,]$, generating the empty list.
  - $\_::\_$, adding an element of A in front of a list
- So we define lists as follows:

$$\mathrm{infixr}\ 20\ \_::\_$$

$$\begin{array}{l} \mathrm{data\ List}\ (A : \mathrm{Set}) : \mathrm{Set\ where} \\ \quad \begin{array}{lll} [\,] & : & \mathrm{List}\ A \\ \_::\_ & : & A \rightarrow \mathrm{List}\ A \rightarrow \mathrm{List}\ A \end{array} \end{array}$$

Elimination Principle for Lists

- The elimination principle is structural recursion on lists: Assume
  - $A : \mathrm{Set}$
  - $C : \mathrm{Set}$, depending on $l : \mathrm{List}\ A$.

  Then we can define

$$
\begin{array}{l}
f : (l : \mathrm{List}\ A) \rightarrow C \\
\begin{array}{lllll}
f & [\,] & = & \{! & !\} \\
f & (a :: l) & = & \{! & !\}
\end{array}
\end{array}
$$

  and in the second goal we can make use of $f\ l$.

Example: Length of a List

$$\mathrm{length} : \mathrm{List}\ \mathbb{N} \rightarrow \mathbb{N}$$

$$\begin{array}{lll} \mathrm{length}\ \ [\,] & = & \mathrm{Z} \\ \mathrm{length}\ \ (\_ :: l) & = & \mathrm{S}\ (\mathrm{length}\ l) \end{array}$$

Example: sumlist

- $\mathrm{sumlist}\ l$ will compute the sum of the elements of list l.

$$
\begin{array}{l}
\mathrm{sumlist} : \mathrm{List}\ \mathbb{N} \rightarrow \mathbb{N} \\
\begin{array}{lllll}
\mathrm{sumlist} & [\,] & = & \mathrm{Z} \\
\mathrm{sumlist} & (n :: l) & = & n + \mathrm{sumlist}\ l
\end{array}
\end{array}
$$

Interesting Exercise

- Define

$$\_{+}{+}\_ : \{A : \mathrm{Set}\} \to \mathrm{List}\ A \to \mathrm{List}\ A \to \mathrm{List}\ A \quad ,$$

  s.t. $l +\!+\, l'$ is the result of appending the list l' at the end of list l.

- E.g., if $\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}$ are elements of A, then

$$\begin{aligned} &\mathrm{a} :: \mathrm{b} :: [\,] \quad +\!+ \quad \mathrm{c} :: \mathrm{d} :: [\,] \\ &= \mathrm{a} :: \mathrm{b} :: \mathrm{c} :: \mathrm{d} :: [\,] \end{aligned}$$

  list.agda

(i) Universes

- A universe U is a set, the elements of which are codes for sets.
- So we have
  - $\mathrm{U} : \mathrm{Set}$,
  - $\mathrm{T} : \mathrm{U} \to \mathrm{Set}$ (the decoding function).
- We consider in the following a universe closed under
  - $\bot$, $\top$, Bool, $\mathbb{N}$,
  - $+$,
  - Σ,
  - the dependent function type.

Rules for the Universe

Formation Rule

$$\mathrm{U} : \mathrm{Set} \quad (\text{U-F})$$

$$\frac{a : \mathrm{U}}{\mathrm{T}\ a : \mathrm{Set}}\ (\text{T-F})$$

Rules for the Universe

Introduction and Equality Rules

$\widehat{\bot} : \mathrm{U}$	$(\text{U-I}_{\widehat{\bot}})$	$\mathrm{T}\,(\widehat{\bot}) = \bot : \mathrm{Set}$	$(\text{T-Eq}_{\widehat{\bot}})$
$\widehat{\top} : \mathrm{U}$	$(\text{U-I}_{\widehat{\top}})$	$\mathrm{T}\,(\widehat{\top}) = \top : \mathrm{Set}$	$(\text{T-Eq}_{\widehat{\top}})$
$\widehat{\mathrm{Bool}} : \mathrm{U}$	$(\text{U-I}_{\widehat{\mathrm{Bool}}})$	$\mathrm{T}\,(\widehat{\mathrm{Bool}}) = \mathrm{Bool} : \mathrm{Set}$	$(\text{T-Eq}_{\widehat{\mathrm{Bool}}})$
$\widehat{\mathbb{N}} : \mathrm{U}$	$(\text{U-I}_{\widehat{\mathbb{N}}})$	$\mathrm{T}\,(\widehat{\mathbb{N}}) = \mathbb{N} : \mathrm{Set}$	$(\text{T-Eq}_{\widehat{\mathbb{N}}})$

Rules for the Universe

Introduction and Equality Rules (Cont.)

$$\frac{a : \mathrm{U} \qquad b : \mathrm{U}}{a \mathbin{\widehat{+}} b : \mathrm{U}} \; (\text{U-I}_{\widehat{+}})$$

$$\mathrm{T}\ (a \mathbin{\widehat{+}} b) = \mathrm{T}\ a + \mathrm{T}\ b : \mathrm{Set} \qquad (\text{T-Eq}_{\widehat{+}})$$

$$\frac{a : \mathrm{U} \qquad b : \mathrm{T}\ a \to \mathrm{U}}{\widehat{\Sigma}\ a\ b : \mathrm{U}} \; (\text{U-I}_{\widehat{\Sigma}})$$

$$\mathrm{T}\ (\widehat{\Sigma}\ a\ b) = \Sigma\ (\mathrm{T}\ a)\ (\lambda x^{\mathrm{T}\ a}.\mathrm{T}\ (b\ x)) : \mathrm{Set} \qquad (\text{T-Eq}_{\widehat{\Sigma}})$$

Rules for the Universe

Introduction and Equality Rules (Cont.)

$$\frac{a : \mathrm{U} \qquad b : \mathrm{T}\ a \to \mathrm{U}}{\widehat{\Pi}\ a\ b : \mathrm{U}}\ (\text{U-I}_{\widehat{\Pi}})$$

$$\mathrm{T}\ (\widehat{\Pi}\ a\ b) = (x : \mathrm{T}\ a) \to \mathrm{T}\ (b\ x) : \mathrm{Set} \qquad (\text{T-Eq}_{\widehat{\Pi}})$$

Elimination and Equality Rules

- There exist as well elimination rules and corresponding equality rules for the universe.
- They are very long (one step for each of constructor of U) and are not very much used.
- They follow the principles present in previous rules.
- We have of course as well the equality versions of the formation-, introduction- and equality rules.

Applications of the Universe

- Ordinary elimination rules don't allow to eliminate into Set.
- However often, one can verify, that all sets needed are "elements of a universe",
  - i.e. there are codes in the universe representing them.
- Then one can eliminate into the universe instead of Set and use T to obtain the required function.

Applications of the Universe

- Example: Define

$$\begin{array}{rcl}\widehat{\mathrm{Atom}} & : & \mathrm{Bool} \to \mathrm{U} \quad , \\ \widehat{\mathrm{Atom}} & := & \mathrm{Case}_{\mathrm{Bool}}\ (\lambda x^{\mathrm{Bool}}.\mathrm{U})\ \widehat{\top}\ \widehat{\bot} \quad , \\ & & \\ \mathrm{Atom} & : & \mathrm{Bool} \to \mathrm{Set} \quad , \\ \mathrm{Atom} & : & \lambda x^{\mathrm{Bool}}.\mathrm{T}\ (\widehat{\mathrm{Atom}}\ x) \quad , \end{array}$$

  Then
  - $\mathrm{Atom}\ \mathrm{tt} = \top$,
  - $\mathrm{Atom}\ \mathrm{ff} = \bot$.

Universes in Agda

- U and T need to be defined simultaneously.
  - Usually Agda type checks definitions in sequence, so no reference to later definitions possible.
  - Special construct **mutual**.
    - Everything in the scope of it is type checked simultaneously.
    - Scope determined by indentation.
  - It is necessary, since the definition of U refers to that of T, and the definition of T refers to that of U.
  - In general mutual allows simultaneous inductive and/or recursive definitions.
  - The termination checker can handle certain terminating simultaneous inductive and/or recursive definitions like the universe.

Universes in Agda (Cont.)

$$
\begin{array}{l}
\mathrm{mutual} \\
\quad \mathrm{data\ U : Set\ where} \\
\qquad \begin{array}{lcl}
\bot\mathrm{hat} & : & \mathrm{U} \\
\mathrm{tophat} & : & \mathrm{U} \\
\mathrm{Boolhat} & : & \mathrm{U} \\
\mathbb{N}\mathrm{hat} & : & \mathrm{U} \\
\_{+}\mathrm{hat}\_ & : & \mathrm{U} \rightarrow \mathrm{U} \rightarrow \mathrm{U} \\
\Sigma\mathrm{hat} & : & (a : \mathrm{U}) \rightarrow (\mathrm{T}\ a \rightarrow \mathrm{U}) \rightarrow \mathrm{U} \\
\Pi\mathrm{hat} & : & (a : \mathrm{U}) \rightarrow (\mathrm{T}\ a \rightarrow \mathrm{U}) \rightarrow \mathrm{U}
\end{array}
\end{array}
$$

Universes in Agda (Cont.)

T in the following is to be intended the same as U:

$$
\begin{array}{lllll}
\mathrm{T} : \mathrm{U} \longrightarrow \mathrm{Set} \\
\mathrm{T} & \bot\mathrm{hat} & = & \bot \\
\mathrm{T} & \mathrm{tophat} & = & \top \\
\mathrm{T} & \mathrm{Boolhat} & = & \mathrm{Bool} \\
\mathrm{T} & \mathbb{N}\mathrm{hat} & = & \mathbb{N} \\
\mathrm{T} & (a + \mathrm{hat}\ b) & = & \mathrm{T}\ a + \mathrm{T}\ b \\
\mathrm{T} & (\Sigma\mathrm{hat}\ a\ b) & = & \Sigma\ (\mathrm{T}\ a)\ (\lambda x \longrightarrow \mathrm{T}\ (b\ x)) \\
\mathrm{T} & (\Pi\mathrm{hat}\ a\ b) & = & \Pi\ (\mathrm{T}\ a)\ (\lambda x \longrightarrow \mathrm{T}\ (b\ x))
\end{array}
$$

(j) Algebraic Types

- The construct **data** in Agda is much more powerful than what is covered by type theoretic rules.
- In general we can define now sets having arbitrarily many constructors with arbitrarily many arguments of arbitrary types.

$$
\begin{array}{l}
\mathrm{data\ A : Set\ where} \\
\qquad \mathrm{C}_1 \ : (a_1 : \mathrm{A}^1_1) \to (a_2 : \mathrm{A}^1_2) \to \cdots \ (a_{n_1} : \mathrm{A}^1_{\mathrm{n}_1}) \to A \\
\qquad \mathrm{C}_2 \ : (a_1 : \mathrm{A}^2_1) \to (a_2 : \mathrm{A}^2_2) \to \cdots (a_{n_2} : \mathrm{A}^2_{\mathrm{n}_2}) \ \to A \\
\qquad \cdots \\
\qquad \mathrm{C}_m : (a_1 : \mathrm{A}^{\mathrm{m}}_1) \to (a_2 : \mathrm{A}^{\mathrm{m}}_2) \to \cdots (a_{n_m} : \mathrm{A}^{\mathrm{m}}_{\mathrm{n}_{\mathrm{m}}}) \to A
\end{array}
$$

Meaning of "data"

- The idea is that A as before is the least set A s.t. we have constructors:

$$
\begin{aligned}
\mathrm{C_i} : (a_{i1} : \mathrm{A_{i1}}) \\
&\longrightarrow \cdots \\
&\longrightarrow (a_{in_i} : \mathrm{A_{in_i}}) \\
&\longrightarrow \mathrm{A}
\end{aligned}
$$

  where a constructor always constructs new elements.
- In other words the elements of A are exactly those constructed by those constructors.

Strictly Positive Algebraic Types

- In the types $\mathrm{A_{ij}}$ we can make use of A.
  - However, it is difficult to understand A, if we have **negative** occurrences of A.
  - Example:

$$\begin{array}{l} \mathrm{data\ A : Set\ where} \\ \qquad \mathrm{C : (A \longrightarrow A) \longrightarrow A} \end{array}$$

  - What is the least set A having a constructor

$$\mathrm{C : (A \longrightarrow A) \longrightarrow A} \qquad ?$$

Strictly Positive Algebraic Types

- If we
  - have constructed some elements of A already,
  - find a function $f : \mathrm{A} \to \mathrm{A}$, and
  - add $\mathrm{C}\ \mathrm{f}$ to A,

  then f might no longer be a function $\mathrm{A} \to \mathrm{A}$.
  (f applied to the new element $\mathrm{C}\ f$ might not be defined).
- In fact, the termination checker issues a warning, if we define A as above.
- We shouldn't make use of such definitions.

Strictly Positive Algebraic Types

- A "good" definition is the set of lists of natural numbers, defined as follows:

$$\begin{array}{l}\mathrm{data}\ \mathbb{N}\mathrm{List} : \mathrm{Set}\ \mathrm{where} \\ \qquad \begin{array}{lcl} [\,] & : & \mathbb{N}\mathrm{List} \\ \_{::}\_ & : & \mathbb{N} \to \mathbb{N}\mathrm{List} \to \mathbb{N}\mathrm{List} \end{array}\end{array}$$

- The constructor $\_{::}\_$ of $\mathbb{N}\mathrm{List}$ refers to $\mathbb{N}\mathrm{List}$, but in a positive way:
We have: if $a : \mathbb{N}$ and $l : \mathbb{N}\mathrm{List}$, then

$$(a :: l) : \mathbb{N}\mathrm{List} \quad .$$

Strictly Positive Algebraic Types

- If we add $a :: l$ to $\mathbb{N}\mathrm{List}$, the reason for adding it (namely $l : \mathbb{N}\mathrm{List}$) is not destroyed by this addition.
- So we can "construct" the set $\mathbb{N}\mathrm{List}$ by
  - starting with the empty set,
  - adding $[\,]$ and
  - closing it under _$::$_ whenever possible.

- Because we can "construct" $\mathbb{N}\mathrm{List}$, the above is an acceptable definition.

Strictly Positive Algebraic Types

- In general:

$$
\begin{array}{l}
\mathrm{data}\ \mathrm{A} : \mathrm{Set}\ \mathrm{where} \\
\quad \mathrm{C}_1 \ : (a_1 : \mathrm{A}^1_1) \to (a_2 : \mathrm{A}^1_2) \to \cdots\ (a_{n_1} : \mathrm{A}^1_{\mathrm{n}_1}) \to A \\
\quad \mathrm{C}_2 \ : (a_1 : \mathrm{A}^2_1) \to (a_2 : \mathrm{A}^2_2) \to \cdots (a_{n_2} : \mathrm{A}^2_{\mathrm{n}_2}) \ \to A \\
\quad \cdots \\
\quad \mathrm{C}_m : (a_1 : \mathrm{A}^{\mathrm{m}}_1) \to (a_2 : \mathrm{A}^{\mathrm{m}}_2) \to \cdots (a_{n_m} : \mathrm{A}^{\mathrm{m}}_{\mathrm{n}_\mathrm{m}}) \to A
\end{array}
$$

  is a strictly positive algebraic type, if all A_{ij} are
  - either types which don't make use of A
  - or are A itself.
- And if A is a strictly positive algebraic type, then A is acceptable.

Strictly Positive Algebraic Types

- The definitions of finite sets, $\Sigma\ A\ B$, $A + B$ and $\mathbb{N}$ were strictly positive algebraic types.

One further Example

- The set of binary trees can be defined as follows:

$$
\begin{array}{l}
\mathrm{data\ BinTree : Set\ where} \\
\quad \begin{array}{lcl}
\mathrm{leaf} & : & \mathrm{BinTree} \\
\mathrm{branch} & : & \mathrm{Bintree} \rightarrow \mathrm{Bintree} \rightarrow \mathrm{Bintree}
\end{array}
\end{array}
$$

- This is a strictly positive algebraic type.
  bintree.agda

Extensions of Strict. Pos. Alg. Type

- An often used extension is to define several sets simultaneously inductively.
- Example: the even and odd numbers:

$$
\begin{array}{l}
\mathrm{mutual} \\
\qquad \mathrm{data\ Even} : \mathrm{Set\ where} \\
\qquad\quad \mathrm{Z} \quad : \quad \mathrm{Even} \\
\qquad\quad \mathrm{S} \quad : \quad \mathrm{Odd} \longrightarrow \mathrm{Even} \\
\qquad \mathrm{data\ Odd} : \mathrm{Set\ where} \\
\qquad\quad \mathrm{S}' \quad : \quad \mathrm{Even} \longrightarrow \mathrm{Odd}
\end{array}
$$

- In such examples the constructors refer strictly positive to all sets which are to be defined simultaneously.
  evenodd.agda

Extensions of Strict. Pos. Alg. Type

- We can even allow $\mathrm{A}_{ij} = B_1 \to A$ or even $\mathrm{A}_{ij} = B_1 \to \cdots \to B_l \to A$, where A is one of the types introduced simultaneously.
- Example (called "Kleene's O"):

$$
\begin{array}{l}
\mathrm{data}\ \mathrm{O} : \mathrm{Set}\ \mathrm{where} \\
\qquad \begin{array}{lcl}
\mathrm{leaf} & : & \mathrm{O} \\
\mathrm{succ} & : & \mathrm{O} \to \mathrm{O} \\
\mathrm{lim} & : & (\mathbb{N} \to \mathrm{O}) \to \mathrm{O}
\end{array}
\end{array}
$$

- The last definition is unproblematic, since, if we have $f : \mathbb{N} \to \mathrm{O}$ and construct $\lim f$ out of it, adding this new element to O doesn't destroy the reason for adding it to O.
- So again O can be "constructed".

Elimination Rules for data

- Functions f from strictly positive algebraic types can now be defined by case distinction as before.
- For termination we need only that in the definition of f, when have to define $f\ (\mathrm{C}\ a_1\ \cdots\ a_n)$, we can refer only to f applied to elements used in $\mathrm{C}\ a_1\ \cdots\ a_n$.

Examples

- For instance
  - in the Bintree example, when defining

$$\mathrm{f} : \mathrm{Bintree} \to A$$

by case-distinction, then the definition of

$$\mathrm{f}\ (\mathrm{branch}\ l\ r)$$

can make use of $\mathrm{f}\ l$ and $\mathrm{f}\ r$.

Examples

- In the example of O, when defining

$$\mathrm{g} : \mathrm{O} \rightarrow A$$

by case-distinction, then the definition of

$$\mathrm{g}\ (\mathrm{lim}\ f)$$

can make use of $\mathrm{g}\ (f\ n)$ for all $n : \mathbb{N}$.