SCC121
Fundamentals of Computer Science

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Recap: Binary Relations

Symbol	Symbol name	Meaning
<a, b>	ordered pair	a pair of elements with an order associated with them
R over A x B	binary relation	set of ordered pairs <a, b>, where a is paired with b through the relation R, with a $\in$ A and b $\in$ B

Recap: n-ary Relations

Symbol	**Symbol name**	**Meaning**
<x1, x2, ..., xn>	ordered n tuple	a set of *n* objects x1, x2,…, xn with an order associated with them
A1 x A2 x... x An	Cartesian product of n sets	the set of all possible ordered *n*-tuples <x1, x2, ... , xn> , where x1 ∈ A1, x2 ∈ A2, …, xn ∈ An
R over A1 x A2 x... x An	n-ary relation	set of ordered *n*-tuples <a1, a2, ..., an> where a1 ∈ A1, a2 ∈ A2, …, an ∈ An

Overview

Preliminary

- Ordered pairs
- Cartesian product

Binary and n-ary relations

- Definitions
- Representing relations
- Operations on relations
- Properties of relations

Objectives

- Facility with set operations on binary relations
- Understanding the relations’ properties

Overview

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Operations on Relations

Union of relations R1 and R2 from set A to B, is another relation from A to B: the set of all ordered pairs <a, b> that are in R1, or R2, or both.

Notation: R1 ∪ R2

Example: Let A = {1, 2} B = {x, y} , R1 and R2 are defined on A x B:

R1 = {<1, x>, <2, y>} and R2 = {<1, x>, <2, x>}

Then, R1 ∪ R2 = {<1, x>, <2, y>, <2, x>}, also defined on A x B.

Operations on Relations

Intersection of R1 and R2 from set A to B, is another relation from A to B: the set of all ordered pairs <a, b> that are common to R1 and R2

Notation: R1 ∩ R2

Example: Let A = {1, 2} B = {x, y} , R1 and R2 are defined on A x B:

R1 = {<1, x>, <2, y>} and R2 = {<1, x>, <2, x>}

Then, R1 ∩ R2 = {<1, x>}, also defined on A x B.

Operations on Relations

Difference of R1 and R2, is another relation from A to B: the set of all ordered pairs <a, b> that are in R1 but not in R2

Notation: R1 – R2

Example: Let A = {1, 2} B = {x, y} , R1 and R2 are defined on A x B:

R1 = {<1, x>, <2, y>} and R2 = {<1, x>, <2, x>}

Then, R1 - R2 = {<2, y>}, also defined on A x B.

Operations on Relations

Product of R1 and R2, is another relation from A to B: the set of all possible ordered n-tuples such as <a, b, c, d> concatenating n-tuples from R1 such as <a, b> with those from R2 such as <c, d>.

Notation: R1x R2

Example: Let A = {1, 2} B = {x, y} , R1 and R2 are defined on A x B:

R1 = {<1, x>, <2, y>} and R2 = {<1, x>, <2, x>}

Then, R1 x R2 = {<1, x, 1, x>, <1, x, 2, x>, <2, y, 1, x>, <2, y, 2, x>} also defined on A x B.

Subrelations

R1 is a subrelation of R2 if every ordered tuple that is an element of R1 is also an element of R2.

Notation: $R1 \subseteq R2$

Example: Let $A = \{1, 2\}$ $B = \{x, y\}$, R1 and R2 are defined on A x B:

$R1 = \{<1, x>, <2, y>\}$ and $R2 = \{<1, x>, <2, x>\}$

$R1 \not\subset R2$ because $<2, y> \notin R2$. $R2 \not\subset R1$ because $<2, x> \notin R1$.

Example:

- "Having a pet cat" is a subrelation of ? "having a pet"
- "Is a brother of" is a subrelation of? "is a sibling of"

Empty Relation

- The Empty relation is the relation that has no elements
- Written $\emptyset$
- Empty relation is a subrelation of any other relation

Example: Operations on Relations

- Let T be the relation that pairs students with courses that they have taken. Let G be the relation that pairs students with courses that they need to take to graduate.
- What do the relations $T \cup G$, $T \cap G$, and $G - T$ represent?

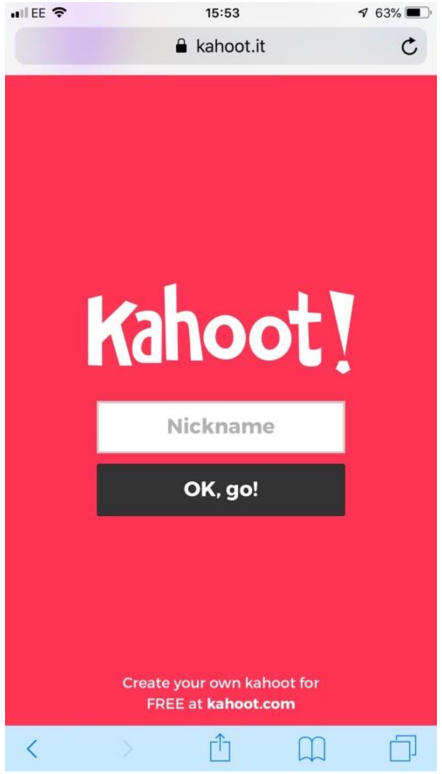
- $T \cup G$ = All pairs <a,b> where Student **a** has taken course **b** OR Student **a** needs to take course **b** to graduate
- $T \cap G$ = All pairs <a,b> where Student **a** has taken course **b** AND Student **a** needs course **b** to graduate
- $G - T$ = All pairs <a,b> where Student **a** needs to take course **b** to graduate BUT Student **a** has not yet taken course **b**

Summary: Operations on Relations

Symbol	Symbol name	Meaning
R1 ⋃ R2	union of relations	set of all ordered pairs <a, b> that are in R1, or R2, or both
R1 ∩ R2	intersection of relations	set of all ordered pairs <a, b> that are common to both R1 and R2
R1 – R2	difference of relations	set of all ordered pairs <a, b> that are in R1 but not in R2
R1 ⊆ R2	subrelation	R1 is subrelation of R2 if every ordered tuple that is an element of R1 is also an element of R2

Let’s playxercise!

- https://kahoot.it/



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- Properties of relations

Properties of Relations

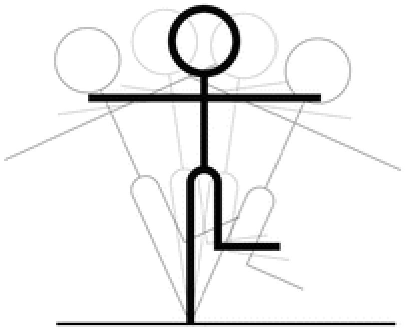
- Symmetry
- Transitivity
- Reflexivity
- Irreflexivity
- Equivalence

Properties of Relations

Intuitive understanding: Balance

Properties of Relations

Intuitive understanding: Balance



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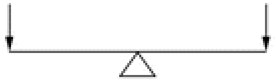

- Johnson, M. (2013). *The body in the mind: The bodily basis of meaning, imagination, and reason*. University of Chicago Press. p. 95-99

Properties of Relations

- Symmetry
- Transitivity
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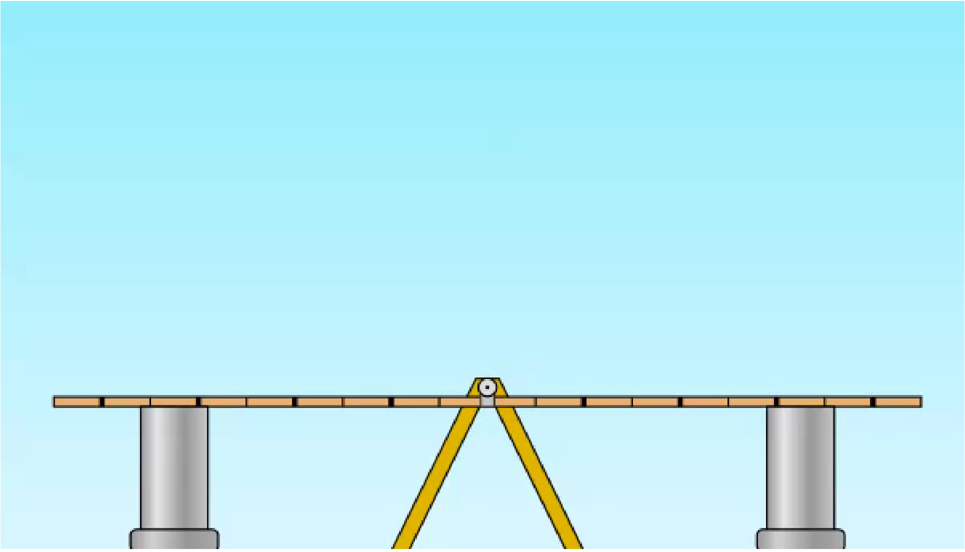
- Irreflexivity
- Equivalence

Properties of Relations: Symmetry

Intuitive understanding: Balance



Symmetry Definition

R $\subseteq$ A x A is symmetric if

for any **a** and **b** in ***A***,

if **<a, b>** $\in$ ***R*** then **<b, a>** $\in$ ***R*.**

Check symmetry:

- for any <a, b> in R, then <b, a> should also be in R
- If there is at least one ordered pair <a, b> in R, and <b, a> is not in R, then R is not symmetric

Symmetry Example

- “is sibling of”
- “is married to”
- “has the same height as”

Examples of relations which are not symmetric:

- “is sister of”: Mary is sister of John, but John is not sister of Mary
- “is smaller than”: 2 is smaller than 5, but 5 is not smaller than 2
- “x loves y”

Symmetry Example: equality (=)

Let A = {1, 2, 3}, <a, b> such that a = b

A x A = { <1,1>, <1, 2>, <1, 3>,
<2, 1>, <2, 2>, <2, 3>,
<3, 1>, <3, 2>, <3, 3> }

EQ = {<1, 1>, <2, 2>, <3, 3>}
and it is symmetric

Symmetry Example: equality (=)

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<3, 1>, <3, 2>, <3, 3> }

EQ = {<1, 1>, <2, 2>, <3, 3>}
and it is symmetric because
for any a, b $\in$ **A**
if <a, b> $\in$ EQ then <b, a> $\in$ EQ

<a, b>	<b, a>
<1, 1>	<1, 1>
<2, 2>	<2, 2>
<3, 3>	<3, 3>

symmetric pairs

Properties of Relations

- Symmetry
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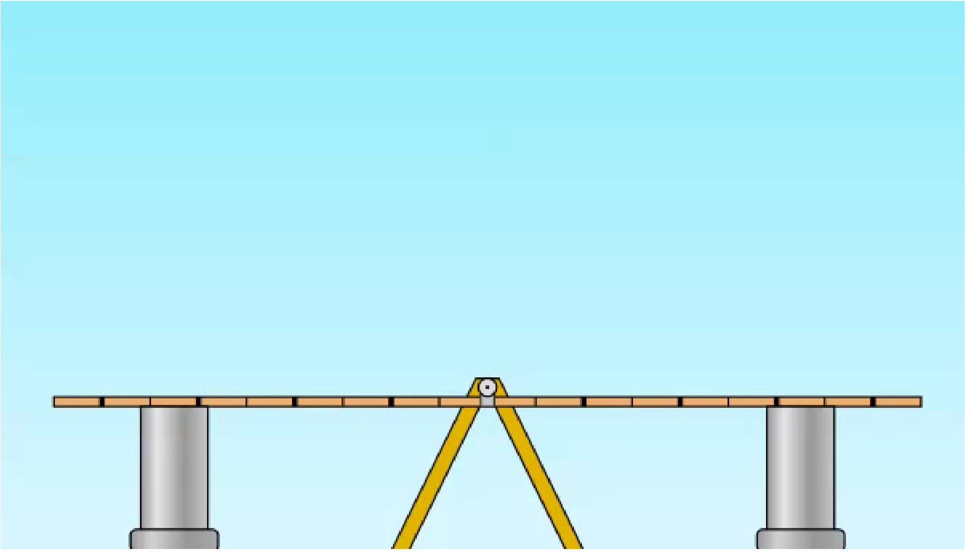
Properties of Relations: Transitivity

Intuitive understanding: Balance



Transitivity Example

- “is taller than”
- “is younger than”
- “is ancestor of”

Examples of relations which are not transitive:

“is parent of”: John is parent of Mike, Mike is parent of Mary, then John is not parent of Mary (but grandparent)

“is 1 inch shorter than”: Ann is 1 inch shorter than Paul, Paul is 1 inch shorter than Mark, then Ann is not 1 inch shorter than Mark (but 2 inch shorter)

Transitivity Definition

$\boldsymbol{R} \subseteq \boldsymbol{A} \times \boldsymbol{A}$ is **transitive**

for any **a**, **b**, and **c** $\in$ **A**,

if **<a, b>** $\in$ ***R*** and **<b, c>** $\in$ ***R***

then **<a, c>** $\in$ ***R*.**

Check transitivity:

- for any <a, b> and <b, c> in R, then <a, c> should also be in R
- If we have at least two ordered pairs <a, b> and <b, c> in R, but <a, c> is not in R, R is not transitive

Transitivity Exercise

Is "mother of" a transitive relation?

It is not transitive, because

if Alice is the mother of Brenda, and

Brenda is the mother of Claire,

then Alice is not the mother of Claire.

Is “=“ relation transitive?

Yes it is. If $a = b$ and $b = c$, then $a = c$

Transitivity Example: less than or equal to (≤)

Let A = {1, 2, 3}, <a, b> such that a ≤ b

A x A = { <1,1>, <1, 2>, <1, 3>,
<2, 1>, <2, 2>, <2, 3>,
<3, 1>, <3, 2>, <3, 3>}

Transitivity Example: less than or equal to (≤)

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<2, 2>, <2, 3>, <3, 3>}

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<3, 1>, <3, 2>, <3, 3>}

LE = {<1, 1>, <1, 2>, <1, 3>,
<2, 2>, <2, 3>, <3, 3>}
and it is transitive, because
for any a, b, c ∈ A
if <a, b> ∈ LE and <b, c> ∈ LE
then <a, c> ∈ LE

<1, 1>	<1, 1>	<1, 1>
<1, 1>	<1, 2>	<1, 2>
<1, 1>	<1, 3>	<1, 3>
<1, 2>	~~<2, 1>~~	
<1, 2>	<2, 2>	< 1, 2>
<1, 2>	<2, 3>	< 1, 3 >
<1, 3>	~~<3, 1>~~	
<1, 3>	~~<3, 2>~~	
<1, 3>	<3, 3>	<1, 3>
~~<2, 1>~~	<1, 1>	
~~<2, 1>~~	<1, 2>	
~~<2, 1>~~	<1, 3>	
<2, 2>	~~<2, 1>~~	
<2, 2>	<2, 2>	<2, 2>
<2, 2>	<2, 3>	<2, 3>
<2, 3>	~~<3, 1>~~	
<2, 3>	~~<3, 2>~~	
<2, 3>	<3, 3>	<2, 3>

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LE = {<1, 1>, <1, 2>, <1, 3>,
<2, 2>, <2, 3>, <3, 3>}
and it is transitive, because
for any a, b, c ∈ A
if <a, b> ∈ LE and <b, c> ∈ LE
then <a, c> ∈ LE

~~<3, 1>~~	<1, 1>	
~~<3, 1>~~	<1, 2>	
~~<3, 1>~~	<1, 3>	
~~<3, 2>~~	~~<2, 1>~~	
~~<3, 2>~~	<2, 2>	
~~<3, 2>~~	<2, 3>	
<3, 3>	~~<3, 1>~~	
<3, 3>	~~<3, 2>~~	
<3, 3>	<3, 3>	<3, 3>

Properties of Relations

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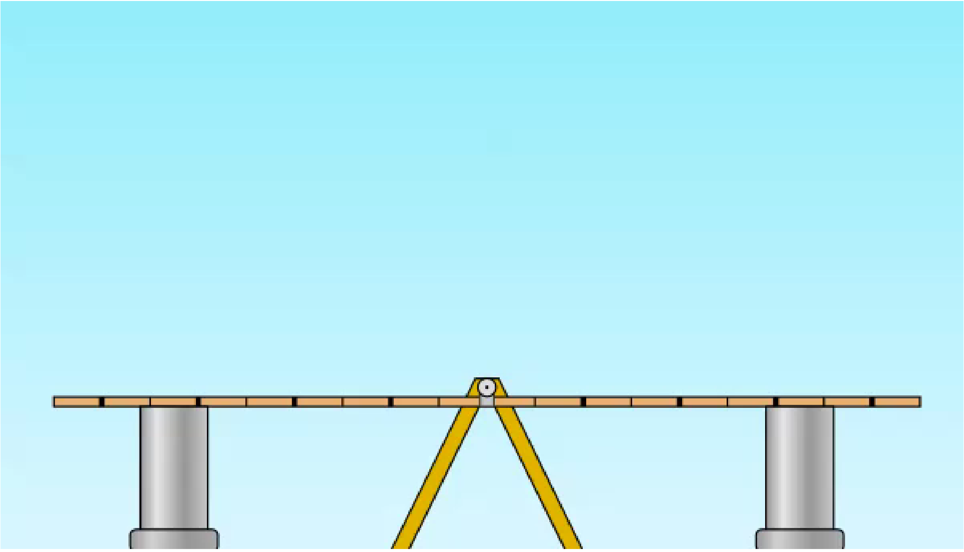
Properties of Relations: Reflexivity

Intuitive understanding: Balance



Reflexivity Definition

R $\subseteq$ **A x A** is **reflexive**

if **< a, a >** $\in$ ***R*** for

every element **a** of ***A***

Check reflexivity:

- for any a in A, then <a, a> should be in R
- every element of A is in relation with itself
- If we have at least one element a in A, and <a, a> is not in R, R is not reflexive

Reflexivity Example

Let B = {5, 6, 7}

- For R to be reflexive, it must contain
- <5, 5>, <6, 6>, and <7, 7>

- It can contain other elements as well, but it must have these three.
- If it has less than these three, than R is not reflexive
- R1 = {<5, 6>, <6, 7>, <7, 7>} is not reflexive.

Reflexivity Example

- “*x* lives within one mile of *y*”
- “*x* was born on the same day as y”
- “x shares the same first name as y”

Examples of relations which are not reflexive

- “greater than”: 3 is not greater than 3
- “is a child of”: no person is their own child

Reflexivity Example: less than or equal to (≤)

Let A = {1, 2, 3}, <a, b> such that a ≤ b

A x A = { <1,1>, <1, 2>, <1, 3>,
<2, 1>, <2, 2>, <2, 3>,
<3, 1>, <3, 2>, <3, 3>}

Reflexivity Example: less than or equal to (≤)

Let A = {1, 2, 3}, <a, b> such that a ≤ b

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LE = {<1, 1>, <1, 2>, <1, 3>,
<2, 2>, <2, 3>, <3, 3>}

and it is reflexive because
for any a, b, c ∈ A
<a, a> ∈ LE

Reflexivity Example: less than or equal to (≤)

Let A = {1, 2, 3}, <a, b> such that a ≤ b

A x A = { <1,1>, <1, 2>, <1, 3>,
<2, 1>, <2, 2>, <2, 3>,
<3, 1>, <3, 2>, <3, 3>}

LE = **{<1, 1>,** <1, 2>, <1, 3>,
<2, 2>, <2, 3>, **<3, 3>**}
and it is reflexive because
for any a, b, c ∈ A
<a, a> ∈ LE

a = 1	<1, 1>
a = 2	<2, 2>
a = 3	<3, 3>

Properties of Relations

- Symmetry
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Irreflexivity Definition

R $\subseteq$ ***A x A*** is **irreflexive**

if **<a, a>** $\notin$ ***R*** for

every element ***a*** of ***A*.**

Check irreflexivity:

- no element of A is in relation with itself
- If there is at least one element a in A and <a, a> is in R, then R is not irreflexive

Irreflexivity Example: less than (<)

A = {1, 2, 3}

A x A = {<1, 1>, <1, 2>, <1, 3>,
<2, 1>, <2, 2>, <2, 3>,
<3, 1>, <3, 2>, <3, 3>}

Irreflexivity Example: less than (<)

A = {1, 2, 3}

A x A = {<1, 1>, <1, 2>, <1, 3>,
<2, 1>, <2, 2>, <2, 3>,
<3, 1>, <3, 2>, <3, 3>}

LT = {<1, 2>, <1, 3>, <2, 3>}

and it is irreflexive because, for any element 1, 2, 3 in A,
<1, 1>, <2, 2>, <3, 3> are not in this relation.

Irreflexivity Example: greater than (>)

A = {1, 2, 3}

A x A = {<1, 1>, <1, 2>, <1, 3>,
<2, 1>, <2, 2>, <2, 3>,
<3, 1>, <3, 2>, <3, 3>}

Irreflexivity Example: greater than (>)

A = {1, 2, 3}

A x A = {<1, 1>, <1, 2>, <1, 3>,
<2, 1>, <2, 2>, <2, 3>,
<3, 1>, <3, 2>, <3, 3>}

GT = {<2, 1>, <3, 1>, <3, 2>}

and it is irreflexive because, for any element 1, 2, 3 in A,
<1, 1>, <2, 2>, <3, 3> are not in this relation.

Properties of Relations

- Symmetry
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Equivalence Definition

R $\subseteq$ **A x A** is an equivalence relation if:
R is reflexive, and
R is symmetric, and
R is transitive.

Example**:** The equality relation “=“
is an equivalence relation

Summary: Relations Properties

- Symmetric relation – for any a and b $\in A$,
  if $<a, b> \in R$, then $<b, a> \in R$.
- Transitive relation – for any a, b, and c $\in$ A,
  if $<a, b> \in R$ and $<b, c> \in R$, then $<a, c> \in R$.
- Reflexive relation – for any a $\in$ A, $<a, a> \in$ R
- Irreflexive relation – for any a $\in$ A, $<a, a> \notin$ R
- Equivalent relation – R is symmetric, transitive, and reflexive