SCC121
Fundamentals of Computer Science

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Overview

Preliminary

- Ordered pairs
- Cartesian product

Binary and n-ary relations

- Definitions
- Representing relations
- Operations on relations
- Properties of relations

Objectives

- Understanding the basic ideas about relations
- Ability to represent relations

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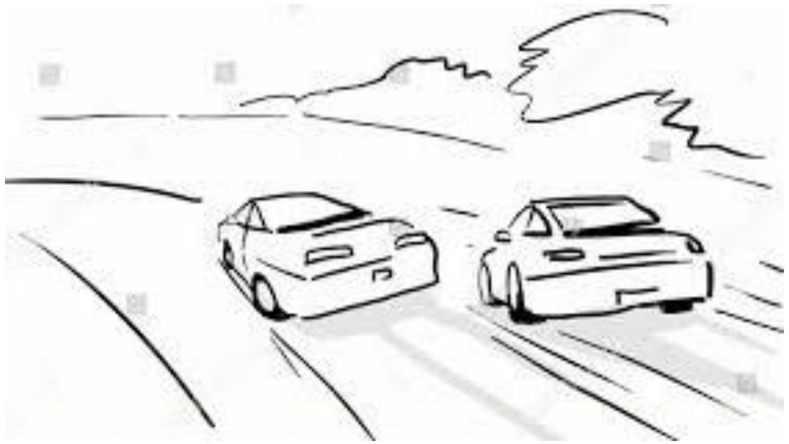
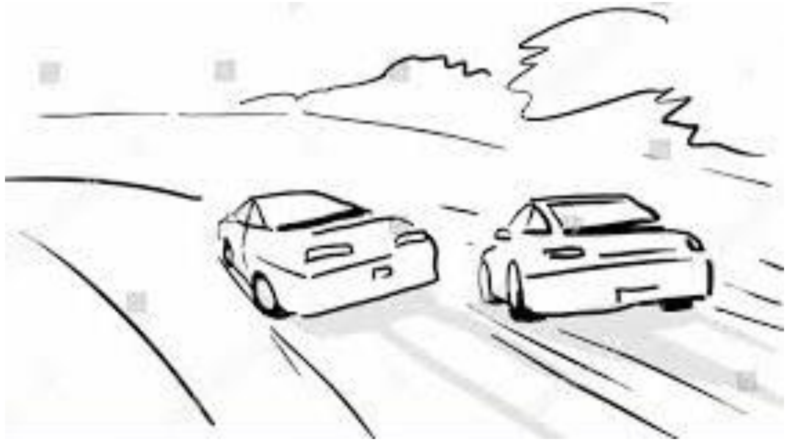
Relations

Relations



Relations



Ordered Pairs

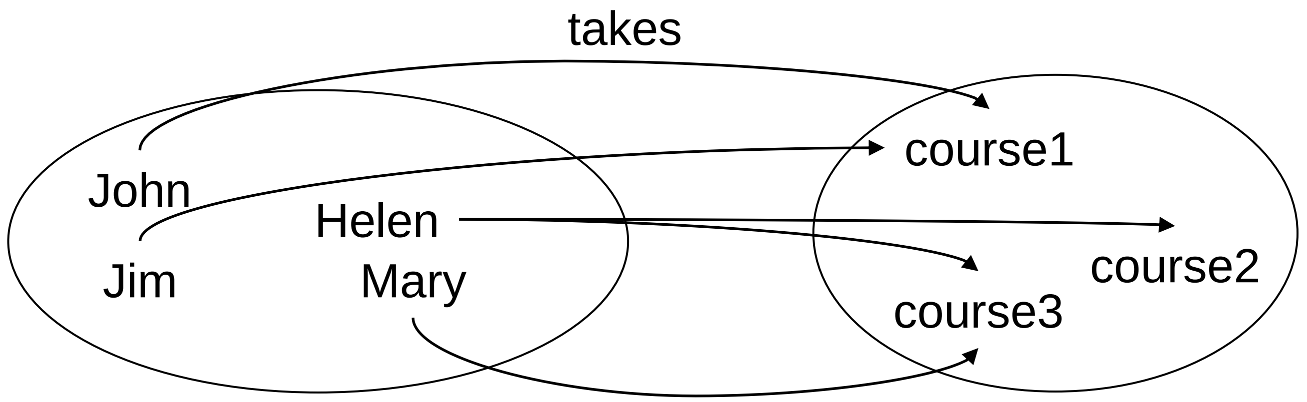
- An ordered pair
  - pair of objects
  - with an order associated with them.
  - written: <x, y>

Cartesian Product

- The set of all ordered pairs <*a*, *b*>
  - where a $\in$ A and b $\in$ B,
  - written: A x B
- Example: A = {a, b, c, d} and B = {1, 2, 3}
  - A x B = { <a, 1>, <a, 2>, <a, 3>,
<b, 1>, <b, 2>, <b, 3>,
<c, 1>, <c, 2>, <c, 3>,
<d, 1>, <d, 2>, <d, 3> }

Associations

- Example:



John takes course1, Jim takes course1, Mary takes course3, Helen takes course2 and course3

- sets of related objects <John, course1>
- order matters <John, course1> ≠ <course1, John>

Cartesian Product

- S = {Helen, Jim, John, Mary}
- C = {course1, course2, course 3}
- 4 * 3 elements = 12 pairs

<Helen, course 1>

<Helen, course 2>

<Helen, course 3>

<Jim, course 1>

<Jim, course 2>

<Jim, course 3>

<John, course 1>

<John, course 2>

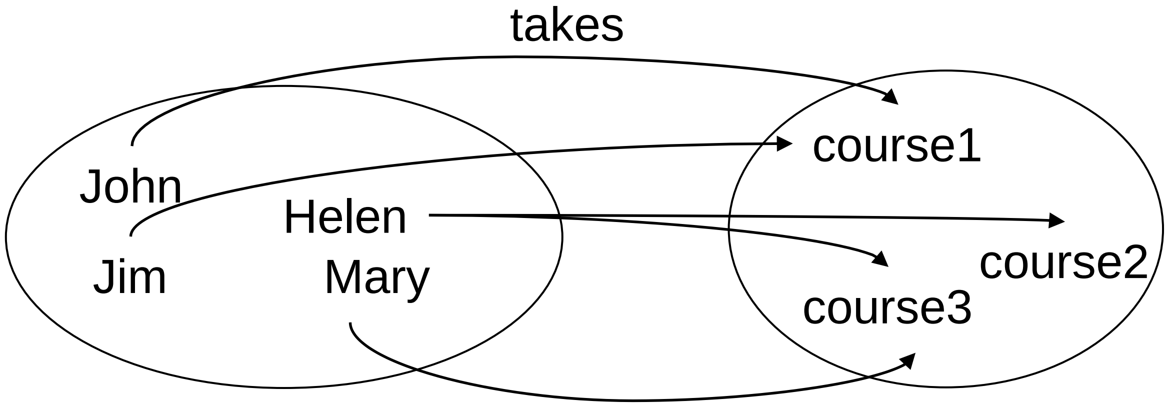
<John, course 3>

< Mary, course 1>

<Mary, course 2>

<Mary, course 3>

Associations



- John takes course1
- Jim takes course1
- Mary takes course3
- Helen takes course2 and course3

<Helen, course 1>
<Helen, course 2>
<Helen, course 3>
<Jim, course 1>
<Jim, course 2>
<Jim, course 3>
<John, course 1>
<John, course 2>
<John, course 3>
< Mary, course 1>
<Mary, course 2>
<Mary, course 3>

Binary Relation: Definition

Binary relation is defined from one set to another (over 2 sets)

Binary relation ***R*** from a set *A* to a set *B*, or over A x B

- a set of ordered pairs **<*a*, *b*>**, $a \in A$ and $b \in B$.

Written: **<*a*, *b*>** $\in$ ***R*** or ***a R b***

- an ordered pair **<*a*, *b*>** is in a relation ***R***
- element ***a*** is related to element ***b*** through the relation ***R***

Binary Relation: Definition

What is the relationship between R and A x B?

- A x B is the set of all ordered pairs <*a, b*>
- R is a subset of A x B: $R \subseteq A \times B$

We can also have A = B, so that the binary relation R is

a set of ordered pairs <a, b>, $a \in A$ and $b \in A$.

- If *A* = *B*, the relation from ***A*** to ***B*** is also called **relation on *A***

Example: Binary Relation

- Ordered pairs
  - <John, course1>
  - <Jim, course1>
  - <Mary, course3>
  - <Helen, course2>
  - <Helen, course3>
- Relation: T (from Takes)
  - John takes course1; < John, course1> $\in$ T
  - Jim takes course1; < Jim, course1> $\in$ T
  - Mary takes course3; <Mary, course3> $\in$ T
  - …

Binary Relation: Exercise

- Let $A = \{0, 1, 2, 3\}$ and $B = \{0, 1, 2, 3, 4\}$.
- List the ordered pairs in the relation R from A to B where $\langle a, b \rangle \in R$ if $b – a = 1$

Answer:

for $a = 0$, what is the value of b?

$b – a = 1$ means $b = a + 1$

or $\langle a, b\rangle = \langle a, a + 1\rangle$

$R = \{\langle 0, 1\rangle, \langle 1, 2\rangle, \langle 2, 3\rangle, \langle 3, 4\rangle\}$

Binary Relation: Exercise

- Let *A* = {0, 1, 2, 3} and *B* = {0, 1, 2, 3, 4}.
- A x B gives us a pool of possible answers
- 4 * 5 elements = 20 ordered pairs

A x B = { <0, 0>, <0, 1>, <0, 2>, <0, 3>, <0, 4>,
<1, 0>, <1, 1>, <1, 2>, <1, 3>, <1, 4>,
<2, 0>, <2, 1>, <2, 2>, <2, 3>, <2, 4>,
<3, 0>, <3, 1>, <3, 2>, <3, 3>, <3, 4>}

Binary Relation: Exercise

- Let *A* = {0, 1, 2, 3} and *B* = {0, 1, 2, 3, 4}.
- A x B gives us a pool of possible answers
- 4 * 5 elements = 20 ordered pairs
- <a, b> = <a, a +1>

A x B = { <0, 0>, **<0, 1>,** <0, 2>, <0, 3>, <0, 4>,
<1, 0>, <1, 1>, **<1, 2>,** <1, 3>, <1, 4>,
<2, 0>, <2, 1>, <2, 2> ,**<2, 3>**, <2, 4>,
<3, 0>, <3, 1>, <3, 2>, <3, 3>, **<3, 4>**}

Thus, R = {<0, 1>, <1, 2>, <2, 3>, <3, 4>}. <*a*, *b*> ∈ *R* if b =a + 1

Equality of Binary Relations

Binary relations:

- R1 $\subseteq$ A1 x A2 and R2 $\subseteq$ B1 x B2

- When are two relations equal?
  - R1 = R2 if
    - the same sets: A1 = B1 , A2 = B2;
    - the set of things related are the same: R1 = R2 as sets

Equality of Binary Relations

- $R1 \subseteq A1 \times A2$ and $R2 \subseteq B1 \times B2$

- $R1 = \{\langle 1, 2\rangle , \langle 2, 2\rangle\} \subseteq \{1, 2\} \times \{1, 2\}$

  $R2 = \{\langle a, b\rangle , \langle b, b\rangle\} \subseteq \{a, b\} \times \{a, b\}$

  R1 = R2 if

  - set $\{1, 2\} = \{a, b\}$, and
  - $\{\langle 1, 2\rangle , \langle 2, 2\rangle\} = \{\langle a, b\rangle , \langle b, b\rangle\}$

Equality of Binary Relations

- R1 = {<1, 2> , <2, 2>} $\subseteq$ {1, 2} x {1, 2} ,
  R2 = {<a, b> , <b, b>} $\subseteq$ {a, b} x {a, b} .

  R1 = R2 if

  - set {1, 2} = {a, b}, and
  - {<1, 2> , <2, 2>} = {<a, b> , <b, b>}

  So a = 1; and b = 2

Ordered n-tuples: Definition

Ordered n-tuple

- on n sets $A1, A2, \ldots, An$.
- ordered *n*-tuple is a set of *n* objects with an order associated with them

Written: **<x1, x2, ..., xn>**

- n sets and n elements in the n-tuple
- $x1 \in A1$, $x2 \in A2$, … $xn \in An$

Equality of Ordered n-tuples

Equality of ordered *n*-tuples:

$<x1, x2, x3, \ldots, xn> = <y1, y2, y3, \ldots, yn>$ if

$x1 = y1$,

$x2 = y2$,

$x3 = y3, \ldots,$

$xn = yn$

$(xi = yi$ for all $i, \ 1 \le i \le n)$

- Example: ordered *3*-tuple

  $<1, 2, 3> = <1, 2, 3>$ and

  $<1, 2, 3> \neq <2, 3, 1>$ because $1 \neq 2$, $2 \neq 3$, and $3 \neq 1$.

Cartesian Product of n Sets

Cartesian product of n sets $A_1, \ldots, A_n$

- the set of all possible ordered *n*-tuples

  $\langle x_1, x_2, \ldots, x_n \rangle$, where $x_1 \in A_1, x_2 \in A_2, \ldots, x_n \in A_n$ ($x_i \in A_i$, for all i, $1 \leq i \leq n$)

Written: $A_1 \times A_2 \times \ldots \times A_n$.

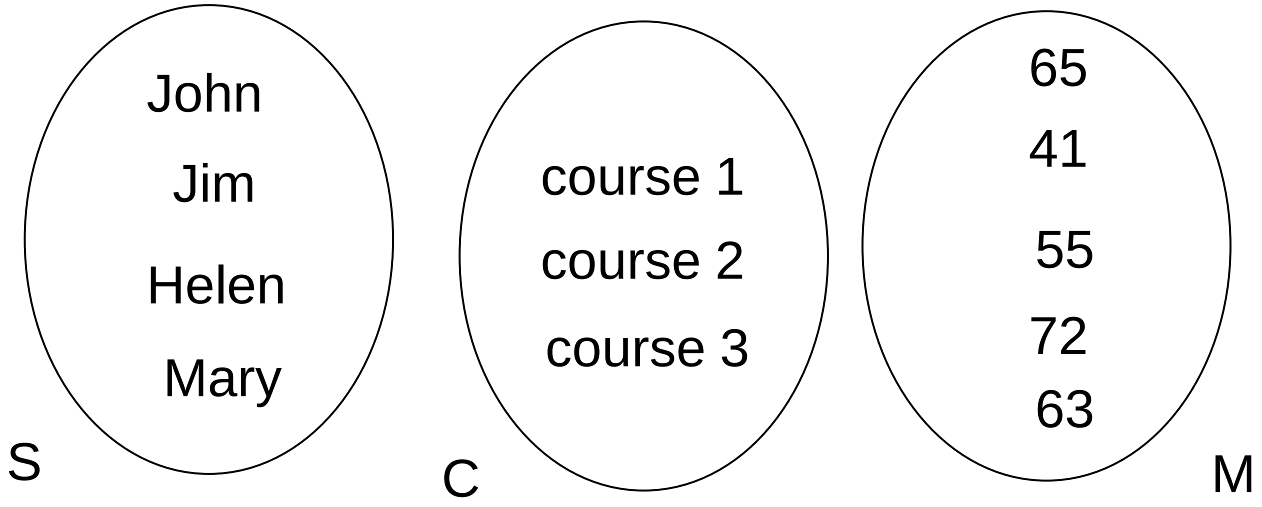
Ordered 3-tuples: Examples

- S = {John, Jim, Helen, Mary}
- C = {course1, course2, course3}
- M = {65, 41, 55, 72, 63}

<John, course1, 65>
<Jim, course1, 41>
<Mary, course3, 55>
<Helen, course2, 72>
<Helen, course3, 63>

Cartesian Product of 3 Sets: Example

- How many n-tuples in S x C x M?
- Every possible combination of the values:
- 4 * 3 * 5 = 60



Cartesian Product of 3 Sets: Example

- S ={John, Jim, Helen, Mary}
- C = {course1, course2, course3}
- M = {65, 41, 55, 72, 63}

A x B x C = {<**John**,**course1,65**>,<John,course1,**41**>, <John,course1,**55**>,<John,course1,**72**>,<John,course1,**63**>,
<**John**,course2,65>,<John,course2,41>, <John,course2,55>,John,course2,72>,<John,course2,63>,
<**John**,course3,65>,John, course3,41>, <John,course3,55>,<John, course3,72>,<John,course3,63>,
<**Jim**,course1,65>,<Jim,course1,41>, <Jim,course1,55>,<Jim,course1,72>,<Jim,course1,63>,
<**Jim**,course2,65>,<Jim,course2,41>, <Jim,course2,55>,<Jim,course2,72>,<Jim,course2,63>,
<**Jim**,course3,65>,<Jim, course3,41>, <Jim,course3,55>,<Jim, course3,72>,<Jim,course3,63>,
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<**Helen**,course3,65>,<Helen,course3,41>,<Helen,course3,55>, <Helen,course3,72>,<Helen,course3,63>,
<**Mary**,course1,65>,< Mary,course1,41>,<Mary,course1,55>, < Mary,course1,72>,<Mary,course1,63>,
<**Mary**,course2,65>,<Mary,course2,41>,<Mary,course2,55>, <Mary,course2,72>,<Mary,course2,63>,
<**Mary**,course3,65>,<Mary,course3,41>,<May,course3,55>, <Mary,course3,72>,<Mary,course3,63>}

n-ary Relation: Definition

- A binary relation involves 2 sets and can be described by a set of pairs
- A ternary relation involves 3 sets and can be described by a set of triples
- …
- An n-ary relation involves n sets and can be described by a set of n-tuples

- n-ary relation (R) on n sets A1, A2, …, An:
  - R is a set of ordered *n*-tuples <a1, a2, ..., an>

  where a1 $\in$ A1, a2 $\in$ A2, …, an $\in$ An, (ai $\in$ Ai for all i, 1 ≤ i ≤ n)

  R $\subseteq$ A1 x A2 x A3 x ... x An, subset of Cartesian product A1x A2 x A3 x ... x An.

n-ary Relation: Example

- S = {John, Jim, Helen, Mary}
- C = {course1, course2, course3}
- M = {65, 41, 55, 72, 63}

T = {<John, course1, 65>,
<Jim, course1, 41>,
<Mary, course3, 55>,
<Helen, course2, 72>,
<Helen, course3, 63>}

- T $\subseteq$ S x C x M

n-ary Relation: Example

Let A = {0, 1, 2, 3, 4} and R on A x A x A consisting of 3-tuples: <a, b, c> such that a <b < c. List the ordered pairs in the relation *R*

n-ary Relation: Example

Let A = {0, 1, 2, 3, 4} and R on A x A x A consisting of 3-tuples:

<a, b, c> such that a <b < c. List the ordered pairs in the relation *R*

Let’s start with: a = 0, then b can be 1, 2, 3 or 4, but not 0, and c can be 2, 3, or 4 but not 0 or 1.

R = {<0, 1, 2>, <0, 1, 3>, <0, 1, 4>, <0, 2, 3>, <0, 2, 4>, <0, 3, 4>, <1, 2, 3>, <1, 2, 4>, <1, 3, 4>, <2, 3, 4>}

n-ary Relation: Example

- A = {0, 1, 2, 3, 4}; A x A x A gives us a pool of possible answers
- 5 * 5 * 5 elements = 125 ordered pairs <a, b, c>

A x A x A = { <0, 0, 0>, <0, 0, 1 >, <0, 0, 2>, <0, 0, 3>, <0, 0, 4>,
<0, 1, 0>, <0, 1, 1 >, <0, 1, 2>, <0, 1, 3>, <0, 1, 4>,
<0, 2, 0>, <0, 2, 1 >, <0, 2, 2>, <0, 2, 3>, <0, 2, 4>,
<0, 3, 0>, <0, 3, 1 >, <0, 3, 2>, <0, 3, 3>, <0, 3, 4>,
<0, 4, 0>, <0, 4, 1 >, <0, 4, 2>, <0, 4, 3>, <0, 4, 4>,…
<4, 4, 0>, <4, 4, 1>, <4, 4, 2>, <4, 4, 3>, <4, 4, 4>}

n-ary Relation: Example

- A = {0, 1, 2, 3, 4}; A x A x A gives us a pool of possible answers
- 5 * 5 * 5 elements = 125 ordered pairs <a, b, c> such that a <b < c

A x A x A = { <0, 0, 0>, <0, 0, 1 >, <0, 0, 2>, <0, 0, 3>, <0, 0, 4>,
<0, 1, 0>, <0, 1, 1 >, <0, 1, 2>, <0, 1, 3>, <0, 1, 4>,
<0, 2, 0>, <0, 2, 1 >, <0, 2, 2>, <0, 2, 3>, <0, 2, 4>,
<0, 3, 0>, <0, 3, 1 >, <0, 3, 2>, <0, 3, 3>, <0, 3, 4>,
<0, 4, 0>, <0, 4, 1 >, <0, 4, 2>, <0, 4, 3>, <0, 4, 4>,…
<4, 4, 0>, <4, 4, 1>, <4, 4, 2>, <4, 4, 3>, <4, 4, 4>}

R = {<0, 1, 2>, <0, 1, 3>, <0, 1, 4>, <0, 2, 3>, <0, 2, 4>, <0, 3, 4>,
<1, 2, 3>, <1, 2, 4>, <1, 3, 4>}

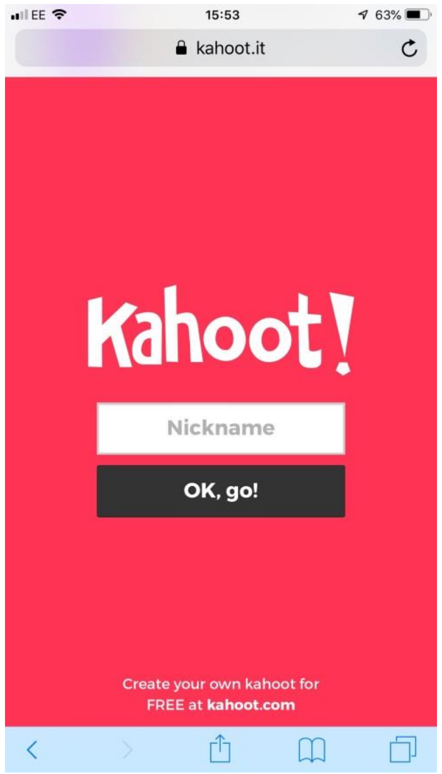
n-ary Relations: Equality

- n-ary relation R1 $\subseteq$ A1 x ... x An
- m-ary relation R2 $\subseteq$ B1 x ... x Bm

- R1 = R2 if
  - m = n
  - Ai = Bi for each i, 1 ≤ i ≤ n,
  - R1 = R2 as a set of ordered n-tuples

Let’s playxercise!

- https://kahoot.it/



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- Properties of relations

Representing Relations

- Tables
- Directed graphs - diagraphs

n-ary Relation: Example

T (takes relation) defined on S x C x M

- S = {John, Jim, Helen, Mary}
- C = {course1, course2, course3}
- M = {65, 41, 55, 72, 63}

T = {<John, course1, 65>,
<Jim, course1, 41>,
<Mary, course3, 55>,
<Helen, course2, 72>,
<Helen, course3, 63>}

- $T \subseteq S \times C \times M$

Representing Relations: Tables

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- T $\subseteq$ S x C x M

Student	**Course**	**Marks**
John	course1	65
Jim	course1	41
Mary	course3	55
Helen	course2	72
Helen	couse3	63

Representing Relations: Tables

T (takes relation) defined on S x C x M

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- T $\subseteq$ S x C x M

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Representing Relations: Diagraphs

Directed graph

- A diagram composed of:
  - points (i.e., vertices, nodes)
  - arrows (i.e., arcs) which connect points to other points

- Diagraph is an ordered pair of sets $G = <P, A>$:
  - P is a set of points
  - A is a set of ordered pairs (called arcs) of points of P.

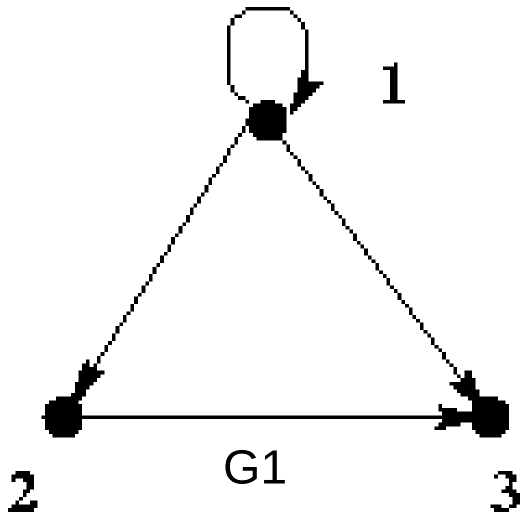
Representing Relations: Diagraphs

Example

P = { 1, 2, 3 }

A = {<1, 1>, <1, 2>, <1, 3>, <2, 3>}

G1 = <P, A>



Representing binary relations through diagraphs

- R $\subseteq$ P x P
- elements of set P are points of the diagraph G
- <p1, p2> is an arc of G from
  point p1 to point p2 if <p1, p2> is in R

Representing Relations: Diagraphs

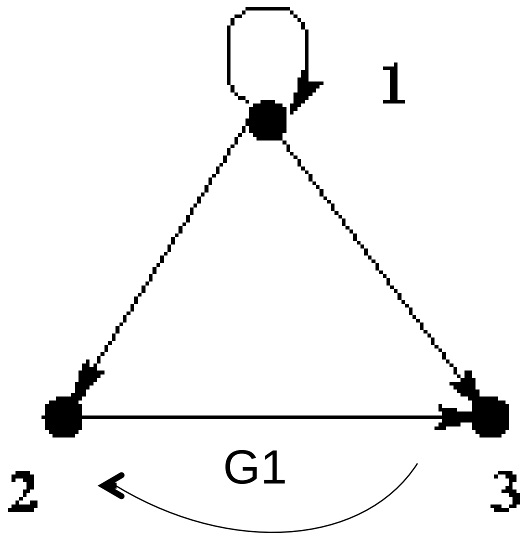
Example

P = { 1, 2, 3 }

A = {<1, 1>, <1, 2>, <1, 3>, <2, 3>,

<3, 2> }

G1 = <P, A>



Representing Relations: Diagraphs

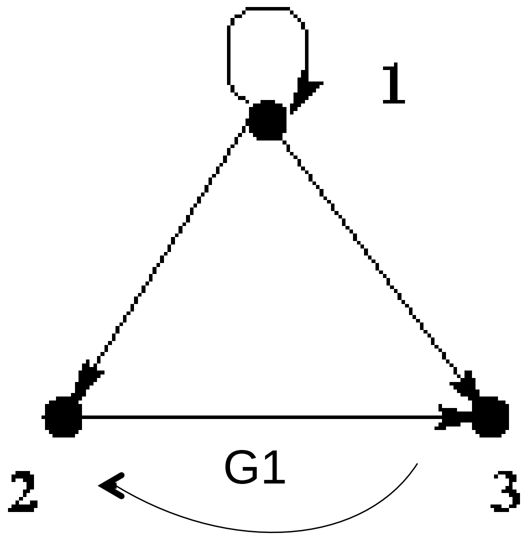
Example

P = { 1, 2, 3 }

A = {<1, 1>, <1, 2>, <1, 3>, <2, 3>,

<3, 2> }

G1 = <P, A>



If you want a bi-directional arc, say from point 3 to point 2, you need to add the ordered pair <3, 2> to set A.

Example

- For the set A = {1, 2, 3, 4},
- any relation R we define on A will be a subset of A x A
- R $\subseteq$ A x A

A x A = {<1, 1>, <1, 2>, <1, 3>, <1, 4>,
<2, 1>, <2, 2>, <2, 3>, <2, 4>,
<3, 1>, <3, 2>, <3, 3>, <3, 4>,
<4, 1>, <4, 2>, <4, 3>, <4, 4>}

Example

Draw the diagraphs of the following relations on the set

$A = \{1, 2, 3, 4\}$

- equal (=)
- less than (<)
- different ($\neq$)

Example: equal (=)

$$
\begin{aligned}
A \text{ x } A = \{&<1, 1>, <1, 2>, <1, 3>, <1, 4>, \\
&<2, 1>, <2, 2>, <2, 3>, <2, 4>, \\
&<3, 1>, <3, 2>, <3, 3>, <3, 4>, \\
&<4, 1>, <4, 2>, <4, 3>, <4, 4>\}
\end{aligned}
$$

Example: equal (=)

$A \times A = \{<1, 1>, <1, 2>, <1, 3>, <1, 4>,$

$<2, 1>, <2, 2>, <2, 3>, <2, 4>,$

$<3, 1>, <3, 2>, <3, 3>, <3, 4>,$

$<4, 1>, <4, 2>, <4, 3>, <4, 4>\}$

$R = \{<1, 1>, <2, 2>, <3, 3>, <4, 4>\}$

Example: equal (=)

A x A = {<1, 1>, <1, 2>, <1, 3>, <1, 4>,

<2, 1>, <2, 2>, <2, 3>, <2, 4>,

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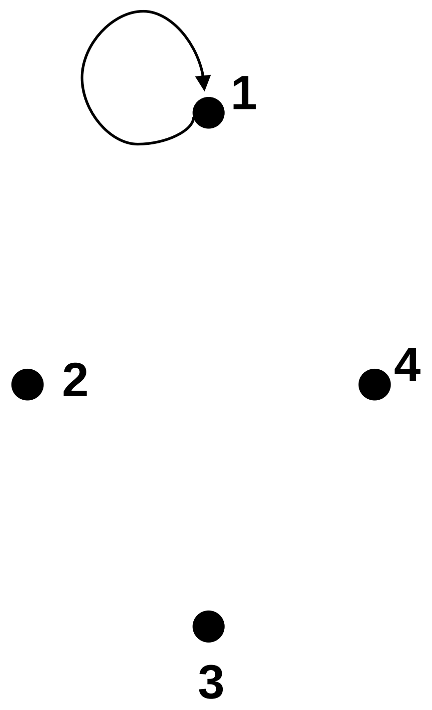
R = {<1, 1>, <2, 2>, <3, 3>, <4, 4>}



Example: equal (=)

A x A = {<1, 1>, <1, 2>, <1, 3>, <1, 4>,
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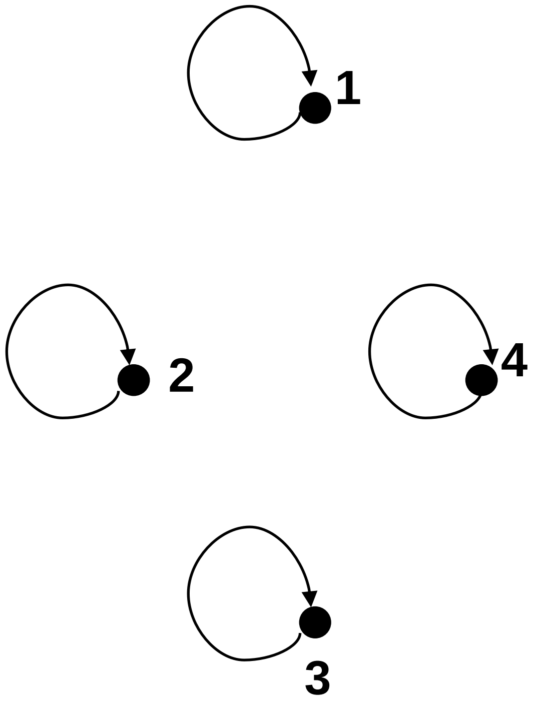
R = {<1, 1>, <2, 2>, <3, 3>, <4, 4>}



Example: less than (<)

A x A = {<1, 1>, <1, 2>, <1, 3>, <1, 4>,
<2, 1>, <2, 2>, <2, 3>, <2, 4>,
<3, 1>, <3, 2>, <3, 3>, <3, 4>,
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Example: less than (<)

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<4, 1>, <4, 2>, <4, 3>, <4, 4>}

R = {<1, 2>, <1, 3>, <1, 4>, <2, 3>, <2, 4>, <3, 4>}

Example: less than (<)

A x A = {<1, 1>, <1, 2>, <1, 3>, <1, 4>,
<2, 1>, <2, 2>, <2, 3>, <2, 4>,
<3, 1>, <3, 2>, <3, 3>, <3, 4>,
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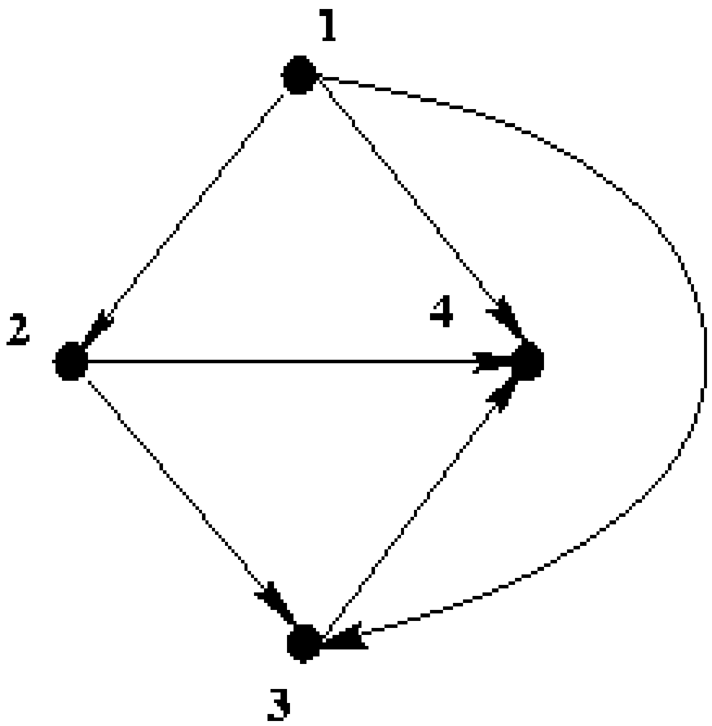
R = {<1, 2>, <1, 3>, <1, 4>, <2, 3>, <2, 4>, <3, 4>}



Example: different (≠)

A x A = {<1, 1>, <1, 2>, <1, 3>, <1, 4>,
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<3, 1>, <3, 2>, <3, 3>, <3, 4>,
<4, 1>, <4, 2>, <4, 3>, <4, 4>}

Example: different ($\neq$)

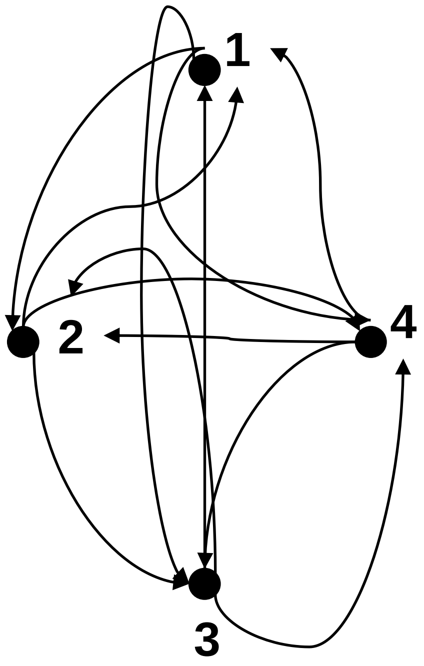
$A \times A = \{<1, 1>, <1, 2>, <1, 3>, <1, 4>,$
$<2, 1>, <2, 2>, <2, 3>, <2, 4>,$
$<3, 1>, <3, 2>, <3, 3>, <3, 4>,$
$<4, 1>, <4, 2>, <4, 3>, <4, 4>\}$

$R = \{<1, 2>, <1, 3>, <1, 4>,$
$<2, 1>, <2, 3>, <2, 4>,$
$<3, 1>, <3, 2>, <3, 4>,$
$<4, 1>, <4, 2>, <4, 3>\}$

Example: different (≠)

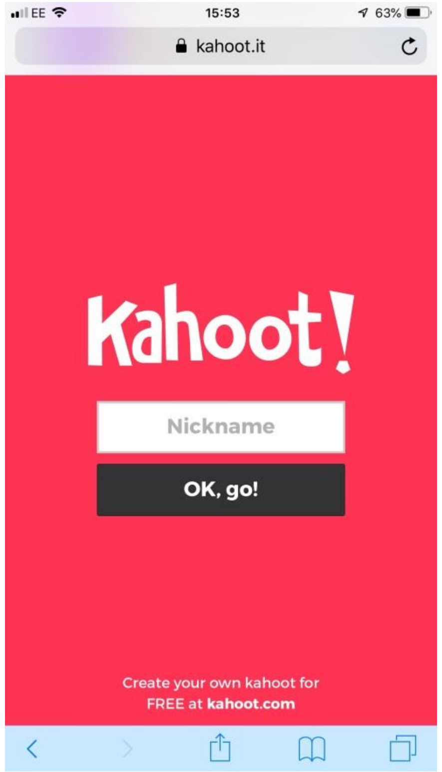
A x A = {<1, 1>, <1, 2>, <1, 3>, <1, 4>,
<2, 1>, <2, 2>, <2, 3>, <2, 4>,
<3, 1>, <3, 2>, <3, 3>, <3, 4>,
<4, 1>, <4, 2>, <4, 3>, <4, 4>}

R ={<1, 2>, <1, 3>, <1, 4>,
<2, 1>, <2, 3>, <2, 4>,
<3, 1>, <3, 2>, <3, 4>,
<4, 1>, <4, 2>, <4, 3>}



Let’s playxercise!

- https://kahoot.it/



Summary: Binary Relations

Symbol	**Symbol name**	**Meaning**
<a, b>	ordered pair	a pair of elements with an order associated with them
R over A x B	binary relation	set of ordered pairs <a, b>, where a is paired with b through the relation R, with $a \in A$ and $b \in B$

Summary: n-ary Relations

Symbol	Symbol name	Meaning
<x1, x2, ..., xn>	ordered n tuple	a set of *n* objects x1, x2,…, xn with an order associated with them
A1 x A2 x... x An	Cartesian product of n sets	the set of all possible ordered *n*-tuples <x1, x2, ... , xn> , where x1 $\in$ A1, x2 $\in$ A2, …, xn $\in$ An
	n-ary relation	set of ordered *n*-tuples <a1, a2, ..., an> where a1 $\in$ A1, a2 $\in$ A2, …, an $\in$ An