SCC121
Fundamentals of Computer Science

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Overview

- Logical reasoning
- Inference rules
- Replacement rules

Objectives

- Understanding inference and replacement rules in logical reasoning
- Facility to apply inference and replacement rules

Overview

- Logical reasoning
- Inference rules
- Replacement rules

Logical reasoning

- Argument – a sequence of propositions that end with a conclusion
  - Premises - the basis on which we try to establish the conclusion
  - Conclusion - the claim that we are trying to establish as true

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Logical reasoning

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  - Premises - the basis on which we try to establish the conclusion
  - Conclusion - the claim that we are trying to establish as true
- Arguments are valid if given that the premises are true, then the conclusion is also true.
- Written:

  Premise 1

  Premise 2…

  Premise n

  ∴ Conclusion

Propositional logic

Building blocks:

- Atomic propositions; compound propositions
- Fundamental connectives

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Building blocks:

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- Fundamental connectives

Rules - functions which take some propositions as premises and returns others as conclusions.

- Inference rules - templates for building valid arguments

Propositional logic

Building blocks:

- Atomic propositions; compound propositions
- Fundamental connectives

Rules - functions which take some propositions as premises and returns others as conclusions.

- Inference rules - templates for building valid arguments
- Replacement rules – rules for replacing parts of propositions with logically equivalent expressions

Overview

- Logical reasoning
- Inference rules
- Replacement rules

Rules of inference

Methods to evaluate the validity of arguments:

- Truth tables - reliable method but inconvenient, and less intuitive

Rules of inference

Methods to evaluate the validity of arguments:

- Truth tables - reliable method but inconvenient, and less intuitive
- Rules of inference
  - Highlight the logical reasoning behind a valid argument
  - All steps from premises to conclusion are justified

Rules of inference

- modus ponens
- modus tollens

- addition
- simplification
- hypothetical syllogism
- disjunctive syllogism

- absorption

Rules of inference

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Rules of inference - Modus Ponens

$P \to Q$	**premise**
P	**premise**
$\therefore Q$	**conclusion**

Rules of inference - Modus Ponens

$P \rightarrow Q$	**premise**
P	**premise**
$\therefore$ Q	**conclusion**

Example:

$P \rightarrow Q$	If it is raining, then it is cloudy
P	It is raining
$\therefore$ Q	It is cloudy

Rules of inference - Modus Ponens

			premise 1	premise 2	conclusion
#	**P**	**Q**	**$P \rightarrow Q$**	**P**	**Q**
1	T	T	T	T	T
2	T	F	F	T	F
3	F	T	T	F	T
4	F	F	T	F	F

Rules of inference - Modus Ponens

			premise 1	premise 2	conclusion
#	**P**	**Q**	**$P \rightarrow Q$**	**P**	**Q**
1	T	T	T	T	T
2	T	F	F	T	F
3	F	T	T	F	T
4	F	F	T	F	F

Rules of inference

modus ponens

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Rules of inference - Modus Tollens

$$\begin{array}{l} \mathbf{P \rightarrow Q} \\ \underline{\mathbf{\sim Q}\qquad\qquad} \\ \therefore \mathbf{\sim P} \end{array}$$

Rules of inference - Modus Tollens

$$\begin{array}{l} P \rightarrow Q \\ \underline{\sim Q \qquad\quad} \\ \therefore \sim P \end{array}$$

Example:

$P \rightarrow Q$	If it is raining, then it is cloudy
$\sim Q$	It is not cloudy
$\therefore \sim P$	Therefore, it is not raining

Rules of inference - Modus Tollens

			premise 1	premise 2	conclusion
#	**P**	**Q**	**$P \to Q$**	**~Q**	**~P**
1	T	T	T	F	F
2	T	F	F	T	F
3	F	T	T	F	F
4	F	F	T	T	T

Rules of inference - Modus Tollens

			premise 1	premise 2	conclusion
#	**P**	**Q**	**$P \to Q$**	**~Q**	**~P**
1	T	T	T	F	F
2	T	F	F	T	F
3	F	T	T	F	F
4	F	F	T	T	T

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Rules of inference – Addition (disjunction introduction)

$$\frac{P}{\therefore P \lor Q}$$

#	P	Q	P v Q
1	F	F	F
2	F	T	T
3	T	F	T
4	T	T	T

Example

P	It is raining
∴ **P v Q**	It is raining or it is cloudy

Rules of inference

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Rules of inference – Simplification (conjunction elimination)

$$\frac{P \wedge Q}{\therefore P}$$

#	P	Q	P ∧ Q	P
1	F	F	F	F
2	F	T	F	F
3	T	F	F	T
4	T	T	T	T

Example

$P \wedge Q$	It is raining and it is cloudy
$\therefore P$	It is raining

Rules of inference

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Rules of inference - Hypothetical syllogism (transitivity of implication)

$P \rightarrow Q$

$\underline{Q \rightarrow R}$

$\therefore P \rightarrow R$

Example:

$\mathbf{P \rightarrow Q}$	If it is raining, then it is cloudy
$\mathbf{Q \rightarrow R}$	If it is cloudy, then I'll be sad.
$\therefore \mathbf{P \rightarrow R}$	If it is raining, then I'll be sad.

Rules of inference - Hypothetical syllogism (transitivity of implication)

$P \rightarrow Q$
$\underline{Q \rightarrow R}$
$\therefore P \rightarrow R$

#	P	Q	R	**$P \rightarrow Q$**	**$Q \rightarrow R$**	**$P \rightarrow R$**
1	F	F	F	T	T	T
2	F	T	F	T	F	T
3	F	F	T	T	T	T
4	F	T	T	T	T	T
5	T	F	F	F	T	F
6	T	F	T	F	T	T
7	T	T	F	T	F	F
8	T	T	T	T	T	T

Rules of inference - Hypothetical syllogism (transitivity of implication)

$P \rightarrow Q$
$\underline{Q \rightarrow R}$
$\therefore P \rightarrow R$

#	P	Q	R	**P → Q**	**Q → R**	**P → R**
1	F	F	F	T	T	T
2	F	T	F	T	F	T
3	F	F	T	T	T	T
4	F	T	T	T	T	T
5	T	F	F	F	T	F
6	T	F	T	F	T	T
7	T	T	F	T	F	F
8	T	T	T	T	T	T

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Rules of inference - Disjunctive syllogism (disjunction elimination)

Note: disjunctive syllogism works for both OR and XOR

$P \lor Q$

$\sim P$

$\therefore Q$

Example:

$\mathbf{P \lor Q}$	I will choose soup or I will choose salad.
$\mathbf{\sim P}$	I will not choose soup.
$\therefore \mathbf{Q}$	Therefore, I will choose salad.

Rules of inference - Disjunctive syllogism

#	P	Q	P ∨ Q	~ P	Q
1	F	F	F	T	F
2	F	T	T	T	T
3	T	F	T	F	F
4	T	T	T	F	T

#	P	Q	P XOR Q	~ P	Q
1	F	F	F	T	F
2	F	T	T	T	T
3	T	F	T	F	F
4	T	T	F	F	T

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Rules of inference - Absorption

$$\frac{P \rightarrow Q}{\therefore P \rightarrow (P \land Q)}$$

Example:

$P \rightarrow Q$	If it will rain, then I will wear my coat.
$\therefore P \rightarrow (P \land Q)$	Therefore, if it will rain then it will rain and I will wear my coat

Rules of inference - Absorption

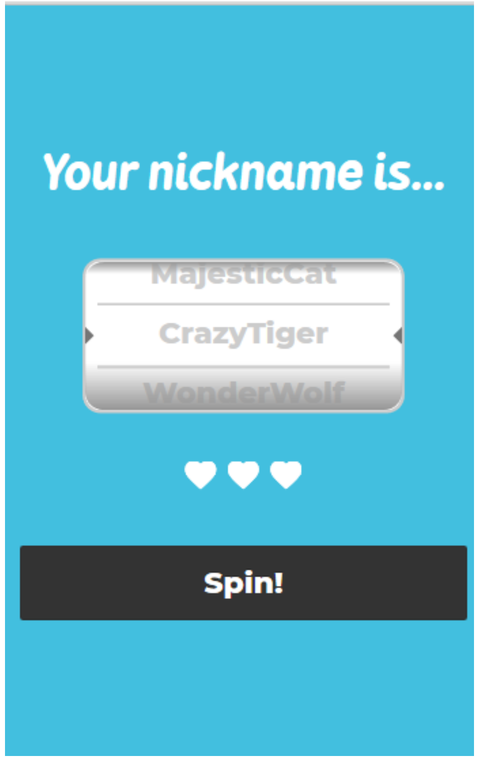
$P \rightarrow Q$

$\therefore P \rightarrow (P \land Q)$

#	P	Q	P → Q	P ∧ Q	P → (P ∧ Q)
1	F	F	T	F	T
2	F	T	T	F	T
3	T	F	F	F	F
4	T	T	T	T	T

Let’s playxercise!

- https://kahoot.it/



Summary: rules of inference

- Modus ponens – one premise is a conditional statement, the other premise affirms the antecedent, and the conclusion affirms the consequent.
- Modus tollens – one premise is a conditional statement, the other premise denies the consequent, and the conclusion denies the antecedent.
- Addition – the premise is a proposition, and the conclusion is a disjunction formed by that proposition and any other proposition.
- Simplification - the premise is a conjunction, and the conclusion is either of the propositions forming the conjunction.
- Hypothetical syllogism – the premises are two conditionals such as $P \rightarrow Q$ and $Q \rightarrow R$ so that one's antecedent matches the consequent of the other, and the conclusion is another conditional which results from the chain of reasoning: $P \rightarrow R$.
- Disjunctive syllogism – one premise is a disjunction, the other premise denies one of the propositions in the disjunction, and the conclusion affirms the other proposition in the disjunction.
- Absorption – the premise is a conditional: $P \rightarrow Q$, and the conclusion is also a conditional whose consequent is a conjunction of the consequent and antecedent: $P \rightarrow (P \wedge Q)$.

Overview

- Logical reasoning
- Inference rules
- Replacement rules

Rules of replacement

- Commutative law
- Associative law
- Distributive law
- De Morgan’s laws
- Absorption law
- Identity law
- Idempotence law
- Negation law
- Double negation law
- Implication law
- Contraposition law
- Equivalence law

Rules of replacement – commutative

Commutative law - the order of the propositions does not affect the result of the conjunction or disjunction

OR $P \vee Q$ is equivalent to $Q \vee P$; AND $P \wedge Q$ is equivalent to $Q \wedge P$

Rules of replacement – commutative

Commutative law - the order of the propositions does not affect the result of the conjunction or disjunction

OR $P \vee Q$ is equivalent to $Q \vee P$; AND $P \wedge Q$ is equivalent to $Q \wedge P$

P	Q	$P \vee Q$	$Q \vee P$
F	F	F	F
F	T	T	T
T	F	T	T
T	T	T	T

P	Q	$P \wedge Q$	$Q \wedge P$
F	F	F	F
F	T	F	F
T	F	F	F
T	T	T	T

Rules of replacement – associative

Associative law - the grouping of the propositions does not affect the result of the conjunction or disjunction

OR $(P \vee Q) \vee R$ is equivalent to $P \vee (Q \vee R)$ $(x + y) + z = x + (y + z)$

AND $(P \wedge Q) \wedge R$ is equivalent to $P \wedge (Q \wedge R)$

Example:
P = “It will rain” Q = “It will snow” R = “It will hail”

$(P \vee Q) \vee R$ = “It will rain or it will snow, or it will hail” is equivalent with:
$P \vee (Q \vee R)$ = “it will snow or, it will rain or it will hail”

$(P \wedge Q) \wedge R$ = “It will rain and it will snow, and it will hail” is equivalent with:
$P \wedge (Q \wedge R)$ = “it will snow and, it will rain and it will hail”

Rules of replacement – associative

Associative law - the grouping of the propositions does not affect the result of the conjunction or disjunction

OR $(P \vee Q) \vee R$ is equivalent to $P \vee (Q \vee R)$ $(x + y) + z = x + (y + z)$

AND $(P \wedge Q) \wedge R$ is equivalent to $P \wedge (Q \wedge R)$

P	Q	R	P v Q	$(P \vee Q) \vee R$	$Q \vee R$	$P \vee (Q \vee R)$
F	F	F	F	F	F	F
F	T	F	T	T	T	T
F	F	T	F	T	T	T
F	T	T	T	T	T	T
T	F	F	T	T	F	T
T	F	T	T	T	T	T
T	T	F	T	T	T	T
T	T	T	T	T	T	T

Rules of replacement – distributive

Distributive law – distribution (or separate application) of conjunction over disjunction; or distribution of disjunction over conjunction

$P \wedge (Q \vee R)$ is equivalent to $(P \wedge Q) \vee (P \wedge R)$ $x * (y + z) = (x * y) + (x * z)$

$P \vee (Q \wedge R)$ is equivalent to $(P \vee Q) \wedge (P \vee R)$

Example:
P = "It will rain" Q = "It will snow" R = "It will hail"

$P \wedge (Q \vee R)$ = "It will rain and, it will snow or hail" is equivalent with:
$(P \wedge Q) \vee (P \wedge R)$ = "it will rain and snow, or, it will rain and hail"

$P \vee (Q \wedge R)$ = "It will rain or, it will snow and hail" is equivalent with:
$(P \vee Q) \wedge (P \vee R)$ = "it will rain or snow, and, it will rain or hail"

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$P \vee (Q \wedge R)$ is equivalent to $(P \vee Q) \wedge (P \vee R)$

P	Q	R	Q v R	P ∧ (Q ∨ R)	P ∧ Q	P ∧ R	(P ∧ Q) ∨ (P ∧ R)
F	F	F	F	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	T	T	T	F	F	F	F
T	F	F	F	F	F	F	F
T	F	T	T	T	F	T	T
T	T	F	T	T	T	F	T
T	T	T	T	T	T	T	T

Rules of replacement – De Morgan’s laws

De Morgan’s laws

- the conjunction of negations is the negation of a disjunction

$\sim P \wedge \sim Q$ is equivalent to $\sim(P \vee Q)$

- the disjunction of negations is the negation of a conjunction

$\sim P \vee \sim Q$ is equivalent to $\sim(P \wedge Q)$

Augustus De Morgan

Rules of replacement – De Morgan's laws

Augustus De Morgan

De Morgan's laws

- the conjunction of negations is the negation of a disjunction

$\sim P \wedge \sim Q$ is equivalent to $\sim(P \vee Q)$

- the disjunction of negations is the negation of a conjunction

$\sim P \vee \sim Q$ is equivalent to $\sim(P \wedge Q)$

Example: P = "It will rain"   Q = "It will snow"

$\sim P \wedge \sim Q$ = "It will not rain and it will not snow" " is equivalent with:

$\sim(P \vee Q)$ = "It will not rain or snow"

$\sim P \vee \sim Q$ = "It will not rain or it will not snow" " is equivalent with:

$\sim(P \wedge Q)$ = "It will not rain and snow"

Rules of replacement – De Morgan's laws

De Morgan's laws

- the conjunction of negations is the negation of a disjunction

$\sim P \wedge \sim Q$ is equivalent to $\sim(P \vee Q)$

- the disjunction of negations is the negation of a conjunction

$\sim P \vee \sim Q$ is equivalent to $\sim(P \wedge Q)$

Augustus De Morgan

P	Q	~P	~Q	$\sim P \wedge \sim Q$	$P \vee Q$	$\sim(P \vee Q)$
F	F	T	T	T	F	T
F	T	T	F	F	T	F
T	F	F	T	F	T	F
T	T	F	F	F	T	F

Rules of replacement – absorption

Absorption law

- the disjunction of any proposition P with (P ∧ Q) has the same truth value as P

P ∨ (P ∧ Q) is equivalent to P

P	Q	P ∧ Q	P ∨ (P ∧ Q)
F	F	F	F
F	T	F	F
T	F	F	T
T	T	T	T

- the conjunction of any proposition P with (P ∨ Q) has the same truth value as P

P ∧ (P ∨ Q) is equivalent to P

P	Q	P ∨ Q	P ∧ (P ∨ Q)
F	F	F	F
F	T	T	F
T	F	T	T
T	T	T	T

Rules of replacement – identity

Identity law

- the conjunction of any proposition P with an arbitrary tautology T (proposition which is always true) has the same truth value as P

P ∧ T is equivalent to P

P	T	P ∧ T
F	T	F
T	T	T

- the disjunction of any proposition P with an arbitrary contradiction F (proposition which is always false) has the same truth value as P

P ∨ F is equivalent to P

P	F	P ∨ F
F	F	F
T	F	T

Rules of replacement – idempotence

Idempotence law

- the property of a conjunction or disjunction to be applied multiple times on a proposition without changing the proposition

- $P \wedge P$ is equivalent to P

P	P	$P \wedge P$
F	F	F
T	T	T

- $P \vee P$ is equivalent to P

P	P	$P \vee P$
F	F	F
T	T	T

Rules of replacement – negation

Negation law

- the disjunction of any proposition P and its negation is a tautology

P ∨ ~P is a tautology

P	~P	P ∨ ~P	T
F	T	T	T
T	F	T	T

- the conjunction of any proposition P and its negation is a contradiction

P ∧ ~P is a contradiction

P	~P	P ∧ ~P	F
F	T	F	F
T	F	F	F

Rules of replacement – double negation

Double negation law

~(~P) is logically equivalent to P

P	~P	~ (~P)
F	T	F
T	F	T

Example:

P = “It will rain”

~(~P) = “It will not, not rain”

or “It is not the case that it will not rain”

are logically equivalent to P = “It will rain”

Rules of replacement – implication

Implication law

Implication can be expressed by disjunction and negation

$P \rightarrow Q$ is logically equivalent to $\sim P \vee Q$

P	Q	$P \rightarrow Q$	$\sim P$	$\sim P \vee Q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Rules of replacement – contraposition

Contraposition law

A conditional $P \rightarrow Q$ is equivalent to its contrapositive (implication of negations): $\sim Q \rightarrow \sim P$

$P \rightarrow Q$ is logically equivalent to $\sim Q \rightarrow \sim P$

P	Q	$P \rightarrow Q$	$\sim Q$	$\sim P$	$\sim Q \rightarrow \sim P$
F	F	T	T	T	T
F	T	T	F	T	T
T	F	F	T	F	F
T	T	T	F	F	T

Rules of replacement – equivalence

Equivalence law

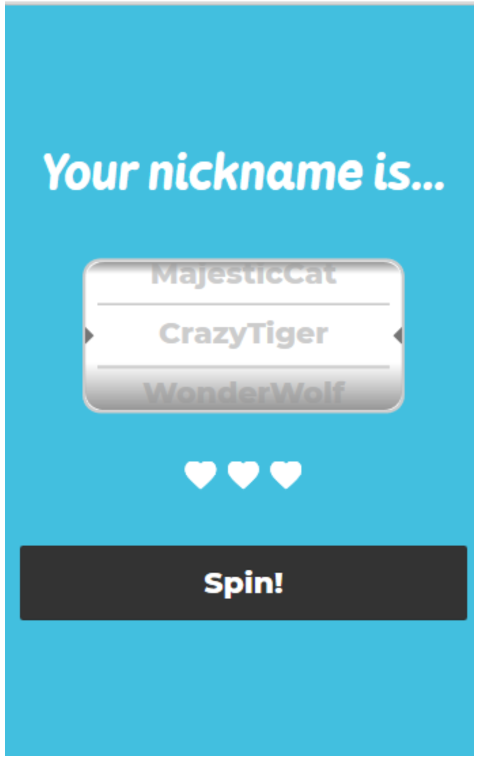
A biconditional $P \leftrightarrow Q$ is equivalent to the conjunction of two conditionals: $(P \rightarrow Q) \wedge (Q \rightarrow P)$

$P \leftrightarrow Q$ is logically equivalent to $(P \rightarrow Q) \wedge (Q \rightarrow P)$

P	Q	$P \leftrightarrow Q$	$P \rightarrow Q$	$Q \rightarrow P$	$(P \rightarrow Q) \wedge (Q \rightarrow P)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

Let’s playxercise!

- https://kahoot.it/



Inference rules - exercise

- *Either cat fur or dog fur was found at the scene of the crime. If dog fur was found at the scene of the crime, officer Thompson had an allergy attack. If cat fur was found at the scene of the crime, then Moriarty is responsible for the crime. But officer Thompson didn't have an allergy attack, and so therefore Moriarty must be responsible for the crime*.

- What inference rules are used in the above argument?

Inference rules - exercise

1	*Either cat fur was found at the scene of the crime, or dog fur was found at the scene of the crime.*	Premise 1
2	*If dog fur was found at the scene of the crime, then officer Thompson had an allergy attack*	Premise 2
3	*If cat fur was found at the scene of the crime, then Moriarty is responsible for the crime*	Premise 3
4	*Officer Thompson did not have an allergy attack*	Premise 4
5	*Dog fur was not found at the scene of the crime*	Intermediary 5 - follows from 2 and 4
6	*Cat fur was found at the scene of the crime.*	Intermediary 6- follows from 1 and 5.
7	*Moriarty is responsible for the crime*	Conclusion - follows from 3 and 6

Inference rules - exercise

P = “cat fur was found at the scene of the crime”
Q = “dog fur was found at the scene of the crime”
A = “officer Thompson had an allergy attack”
B = “Moriarty is responsible for the crime”

P XOR Q
Q -> A
P -> B
~A
~Q
P

B

$((Q \rightarrow A) \wedge \sim A) \Rightarrow \sim Q$ modus tollens

Q -> A
~A

~Q

If dog fur was found at the scene of the crime, then officer Thompson had an allergy attack
Officer Thompson did not have an allergy attack

Dog fur was not found at the scene of the crime

$((P \veebar Q) \wedge \sim Q) \Rightarrow P$ disjunctive syllogism

P XOR Q
~Q

P

Either cat fur was found at the scene of the crime, or dog fur was found at the scene of the crime
Dog fur was not found at the scene of the crime

Cat fur was found at the scene of the crime

Inference rules - exercise

P = “cat fur was found at the scene of the crime”
Q = “dog fur was found at the scene of the crime”
A = “officer Thompson had an allergy attack”
B = “Moriarty is responsible for the crime”

P XOR Q
Q -> A
P -> B
~A
~Q
P

B

$(P \wedge (P \rightarrow B)) \Rightarrow B$ **modus ponens**

P -> B — *If cat fur was found at the scene of the crime, then Moriarty is responsible for the crime*
P — *Cat fur was found at the scene of the crime.*

B — *Moriarty is responsible for the crime.*

Summary: replacement rules

- Commutative law – states that a compound proposition involving exclusively ANDs, or exclusively ORs is unaltered by reordering its atomic propositions.
- Associative law – states that a compound proposition involving exclusively ANDs or exclusively ORs, is unaltered by regrouping its atomic propositions.
- Distributive law – states that a compound proposition involving AND, OR and parentheses, is unaltered by distributing the first connective to link the first proposition separately with each proposition in the parentheses.
- De Morgan law states that the conjunction of negations is the negation of a disjunction: $\sim P \wedge \sim Q$ is equivalent to $\sim(P \vee Q)$.
- Absorption law states that the disjunction of any proposition P with $(P \wedge Q)$ has the same truth value as P: $P \vee (P \wedge Q)$ is equivalent to P; the conjunction of any proposition P with $(P \vee Q)$ has the same truth value as P: $P \wedge (P \vee Q)$ is equivalent to P.
- Identity law states that the conjunction of any proposition P with an arbitrary tautology has the same truth value as P; the disjunction of any proposition P with an arbitrary contradiction F (proposition which is always false) has the same truth value as P.

Summary: replacement rules

- Idempotence law states the property of a conjunction or disjunction to be applied multiple times on a proposition without changing the proposition: $P \wedge P$ is logically equivalent to P, $P \vee P$ is logically equivalent to P.
- Negation law states that the disjunction of any proposition P and its negation is a tautology; the conjunction of any proposition P and its negation is a contradiction.
- Doble negation law states that any proposition P is logically equivalent to its double negation $\sim(\sim P)$
- Implication law states that any implication $P \rightarrow Q$ is logically equivalent to $\sim P \vee Q$.
- Contraposition law states that a conditional $P \rightarrow Q$ is logically equivalent to its contrapositive (implication of negations): $\sim Q \rightarrow \sim P$.
- Equivalence law states that a biconditional $P \leftrightarrow Q$ is logically equivalent to the conjunction of two conditionals: $(P \rightarrow Q) \wedge (Q \rightarrow P)$.