SCC121
Fundamentals of Computer Science

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Overview

- Functions
  - Definitions
  - Types
  - Operations

Objectives

- Understanding the basic ideas about different types of functions
- Ability to work with different types of functions

Function types

- Inverse
- Bijective
- Surjective
- Injective

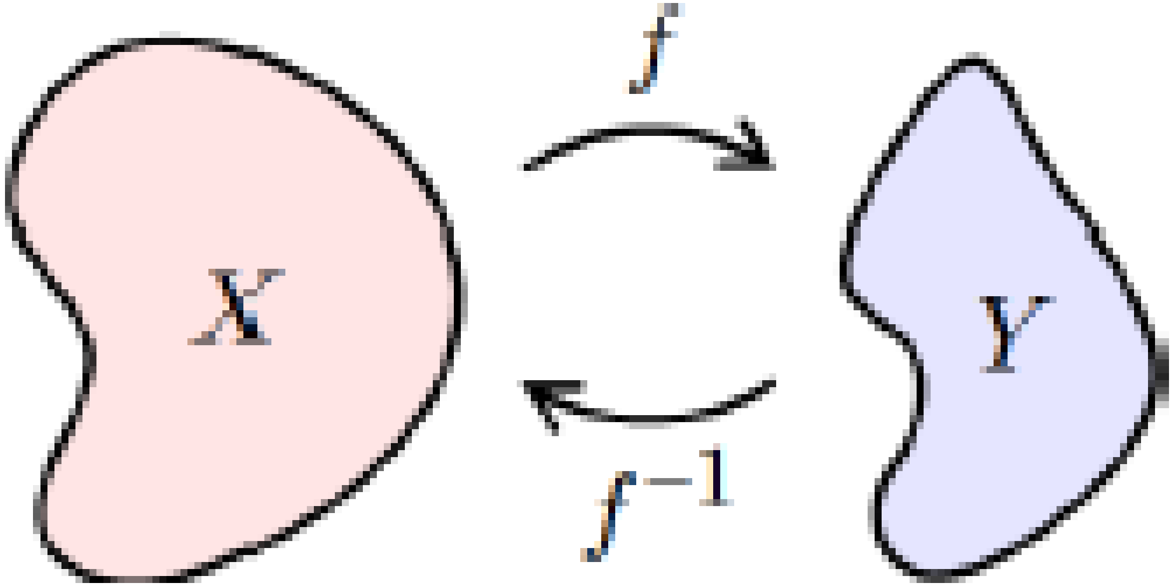
Function types

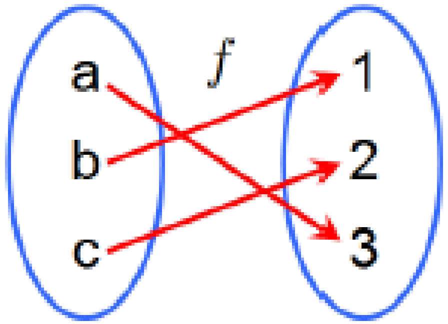
- Inverse
- Bijective
- Surjective
- Injective

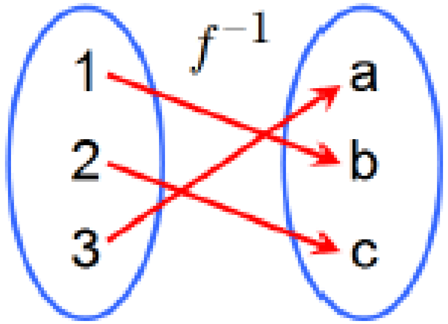
Function types: inverse function

- Each UK citizen has a unique National Insurance number, i.e., *QQ123456C*
  - Mary Smith has the NI number: *QQ123456C*
- Each student at Lancaster University has an eight digit student ID
  - John Brown has the student ID: 12345678
- Reversing the function:
  - identifying a UK citizen based on a NI number
  - identifying a Lancaster University student based on a student ID

Function types: inverse function







$f = \{\langle a, 3\rangle, \langle b, 1\rangle, \langle c, 2\rangle\}$

$f^{-1} = \{\langle 3, a\rangle, \langle 1, b\rangle, \langle 2, c\rangle\}$

Function types: inverse function

Finding the inverse function by reversing the operations

- $f(x) = x + 4$
- $f^{-1}(x) = x - 4$

- $g(x) = 4x$
- $g^{-1}(x) = x / 4$

Function types: inverse function

Finding the inverse function algebraically

- If $f(x) = x + 4$; let's name $f(x) = y$. Then: $y = x + 4$. Now let's express x as a function of y: $x = y – 4$. Now let's call $x = g(y)$. Then $g(y) = y – 4$.
- Now let's swap the letter y for x: or $g(x) = x – 4$.
- $g(x)$ is our inverse function which can also be written as $f^{-1}(x)$. So $f^{-1}(x) = x - 4$.

- If $f(x) = 4x$; $y = 4x$, or $x = y / 4$.
- $g(y) = y / 4$, or $g(x) = x / 4$, or $f^{-1}(x) = x / 4$.

Process:

- Let $y = f(x)$ and solve for x, getting $x = g(y)$.
- Then we use the convention of swapping the letters x and y, so that x represents the input, and y represents the output of the inverse function: $g(x) = f^{-1}(x)$

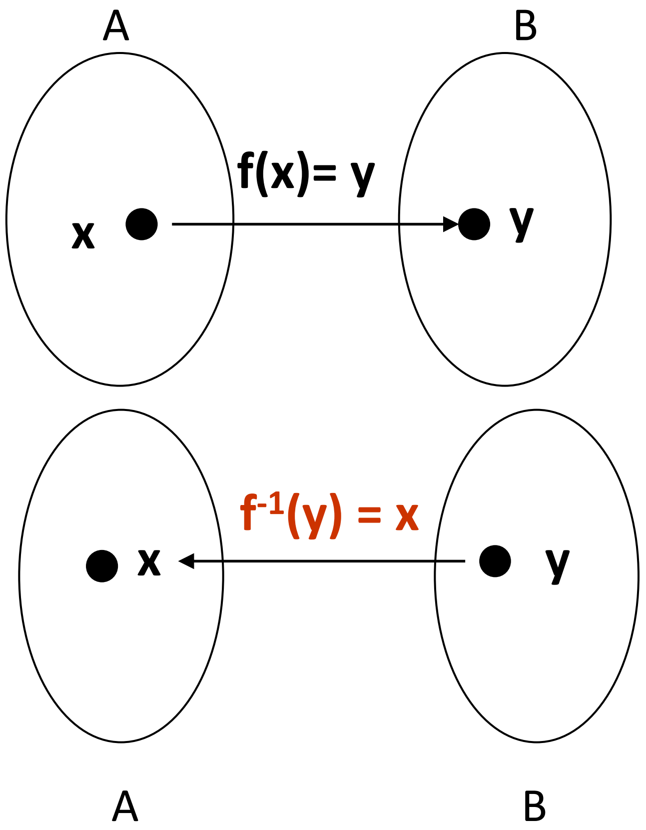
Function types: inverse function definition

- Let f be a function, $f : A \rightarrow B$

- $g : B \rightarrow A$ is called the inverse function of f,
  if for every element y of B, $g(y) = x$, whenever $f(x) = y$.

- The function g(x) is the inverse function of f(x) and is denoted by $f^{-1}(x)$

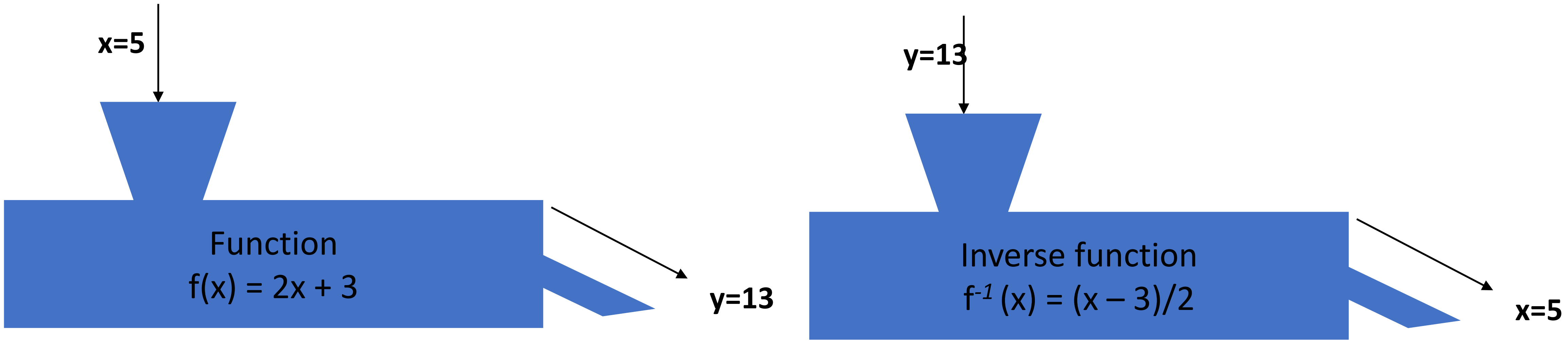
- Note that such an x has to be is unique for each y, because the inverse is also a function



Function types: inverse function definition

- Let f be a function, $f : A \to B$.

- $f^{-1}(x) : B \to A$ is called the inverse function of f,
  if for every element y of B, $f^{-1}(y) = x$, whenever $f(x) = y$.



Function types: inverse function examples

Let f : R $\rightarrow$ R, f(x) = 2x + 5

$f(x) = 2x + 5$

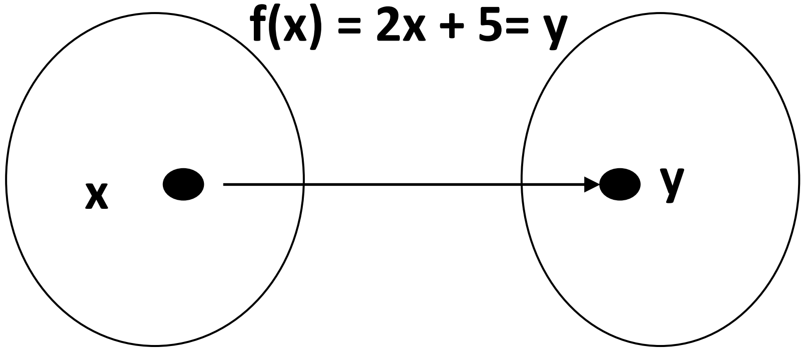
$y = 2x + 5$

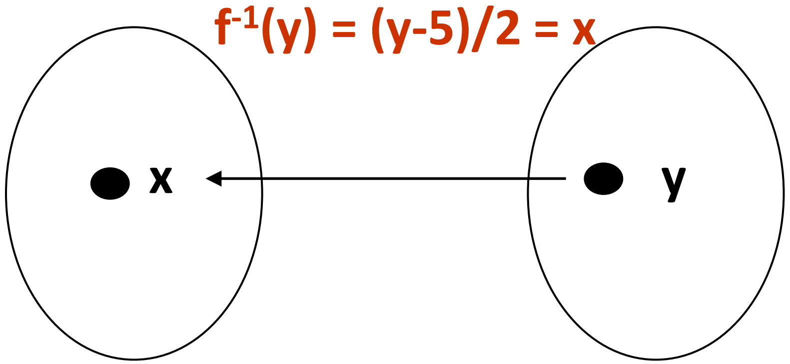
$y - 5 = 2x$

$x = (y - 5) / 2$

$f^{-1}(y) = (y - 5) / 2$, or

$f^{-1}(x) = (x - 5) / 2$





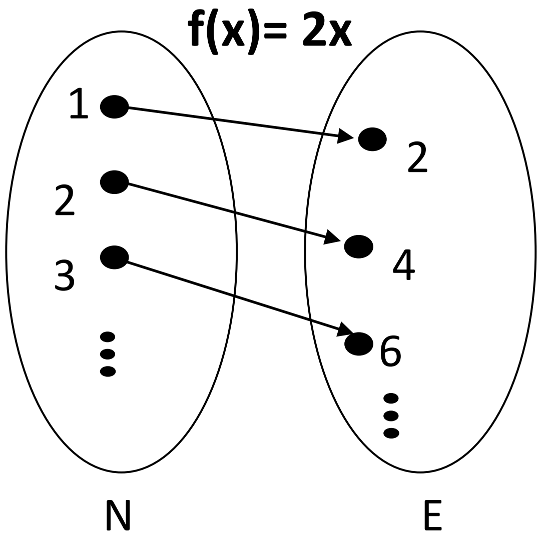
Function types: inverse function examples

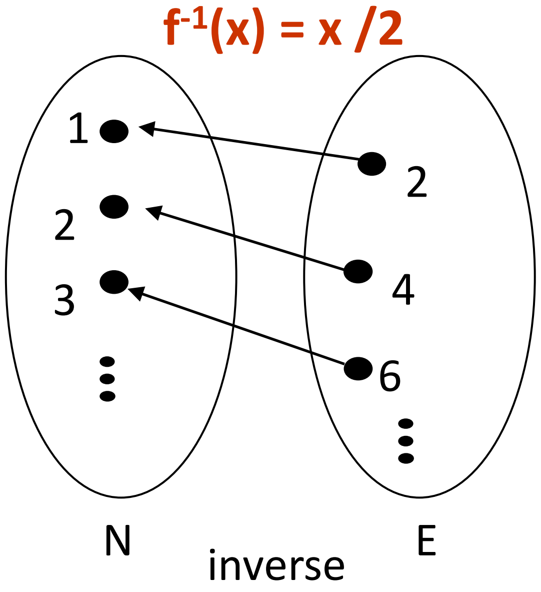
Let $f : N \rightarrow E$, $f(x) = 2x$.

N = set of natural numbers, E = set of even natural numbers

$f(x) = 2x$, $f(x) = y$, $2x = y$, $x = y / 2$. So $f^{-1}(y) = y / 2$

The inverse function: $f^{-1} : E \rightarrow N$, $f^{-1}(x) = x / 2$





Function types: inverse function example

Not every function has an inverse function.

Example: $f : R \rightarrow R$, $f(x) = x^2$

For a particular value $f(x) = 4$, can we reverse the function and find a **unique x**?

No, as both $x = 2$, and $x = -2$ are mapped to $f(x) = 4$; $f(-2) = f(2) = 4$

Thus, $f : R \rightarrow R$, $f(x) = x^2$ does not have an inverse.

Function types

- Inverse
- Bijective
- Surjective
- Injective

Function types: bijective function

- When does a function f have an inverse?
  - The function needs to be:
    - injective, or “one-to-one“, and
    - surjective, or “onto“

- When a function is both injective and surjective, it is called bijective

Function types: bijective function

- Every bijective function has an inverse function
- $f : A \rightarrow B$ - the initial function
- $f^{-1} : B \rightarrow A$ – the inverse function

- f associates **each** element of set A with a **unique** element set B.

- f^{-1} associates **each** element of set B with a **unique** element set A
  - Surjectivity of f function ensures the **each** condition of f^{-1}
  - Injectivity of f function ensures the **unique** condition of f^{-1}

Function types

- Inverse
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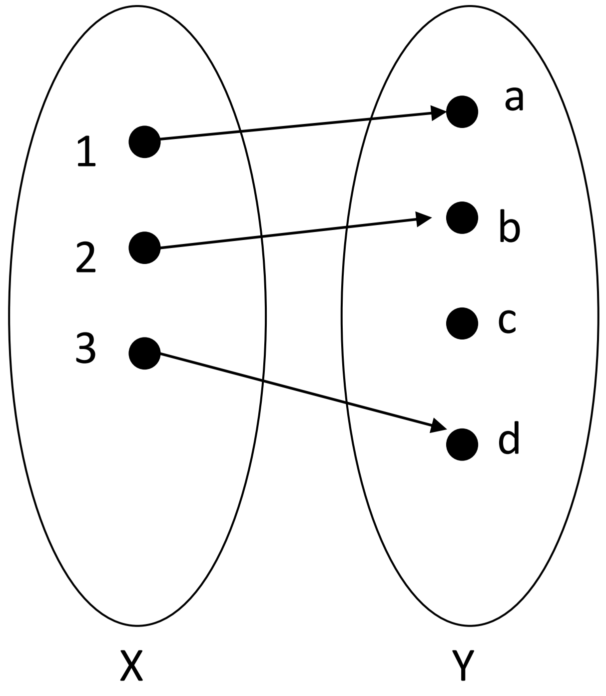
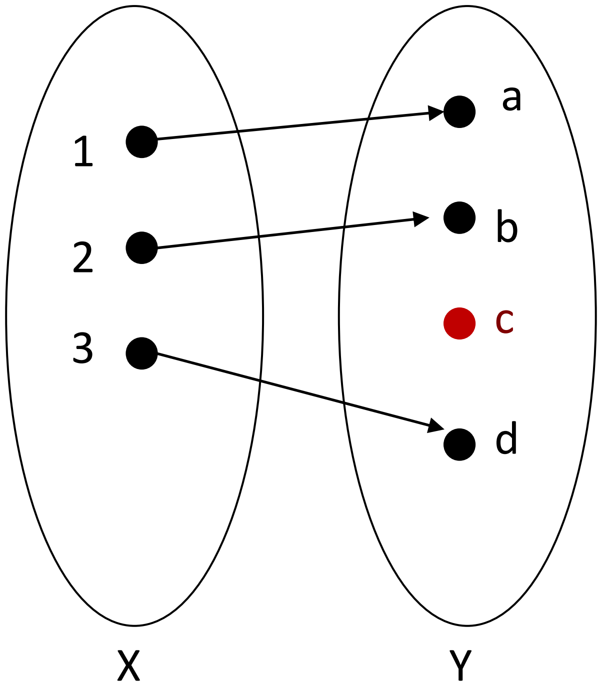
Function types: surjective function

- Is this a function?



Function types: surjective function

- Is this a function?

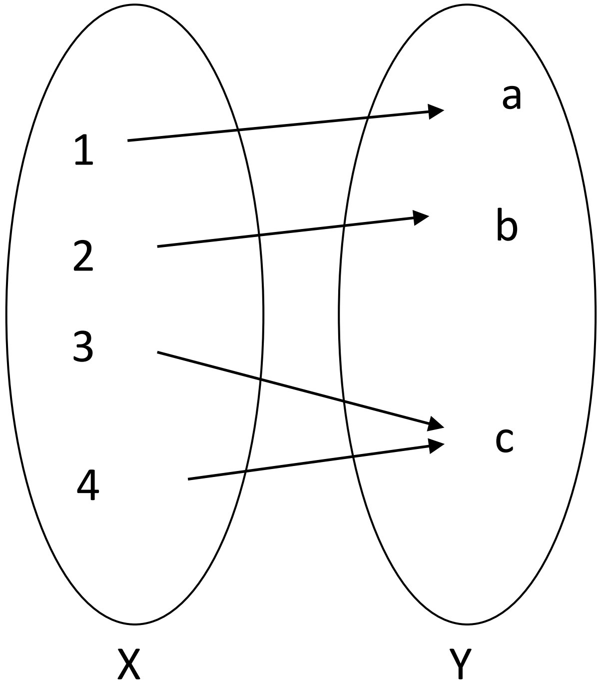


- It is a function but not a surjective one!

Function types: surjective function definition

A function ***f*** is said to be **surjective or onto**

- if for every element ***y*** of ***Y*** **,** there is at least one element ***x*** in ***X*** such that ***f(x) = y.***
- the range of ***f*** equals the codomain of ***f***.



Example:

- f(1) = a
- f(2) = b
- f(3) = c
- f(4) = c

Function types: surjective function example

Let's show that the function f(x) : R→ R, f(x) = 3x + 5 is surjective.

- If f is surjective, then for every real number **y**, there is a real number **x** so that *f(x) = y.*
- Let’s check if that is the case
  - *y = f(x)* means: y = 3x + 5, and if we subtract 5: y – 5 = 3x. If we divide by 3: (y - 5) / 3 = x
  - Now that we found x, we need to check if x is a real number;
  - We know that y is a real number, then (y – 5) is also a real number, which means that x = (y – 5) / 3 is also a real number
  - We showed that for any real number y, there is a real number x = (y – 5) / 3 so that f(x) = y
  - Thus f(x) is surjective

Function types: surjective function example

- Determine if the function $f: Z \rightarrow Z$, $f(x) = x + 1$ is surjective

- If f is surjective, then for every integer **y**, there is an integer **x** so that *f(x) = y.*
- Let’s check if that is the case
  - *y = f(x)* means: $y = x + 1$, and if we subtract 1: $y – 1 = x$.
  - Now that we found x, we need to check if x is an integer:
  - We know that y is an integer, then $y – 1$ is also an integer, which means that $x = y – 1$ is an integer
  - We showed that for any integer y, there is an integer $x = y – 1$ so that $f(x) = y$
  - Thus f(x) is surjective

Function types: surjective function exercise

Is the following function surjective?

$g : R \to R$, $g(x) = x^2$.

Let's check.

$y = g(x)$ means: $y = x^2$. This means that $x = \pm\sqrt{y}$

Now let's check if x is a real number for any y real number.

This is not the case, for negative y.

For instance, there is no real number x such that $x^2 = -1$.

We can redefine the domain as R+, so that that function is surjective.

Function types

- Inverse
- Bijective
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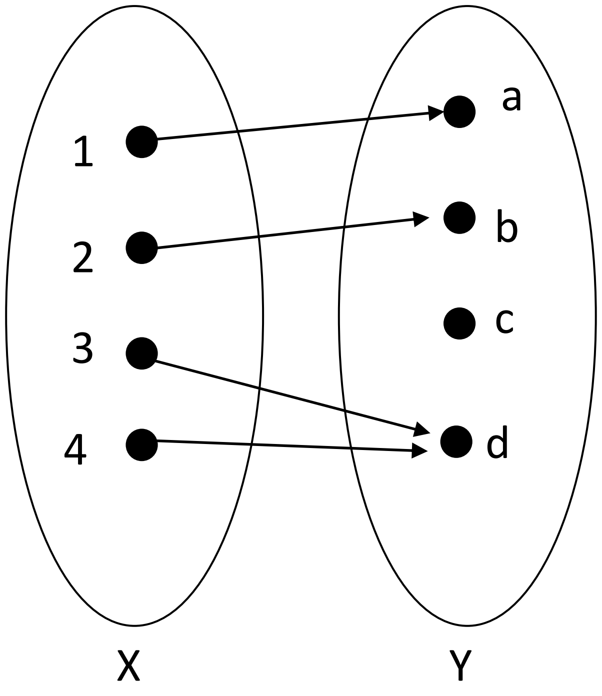
- Injective

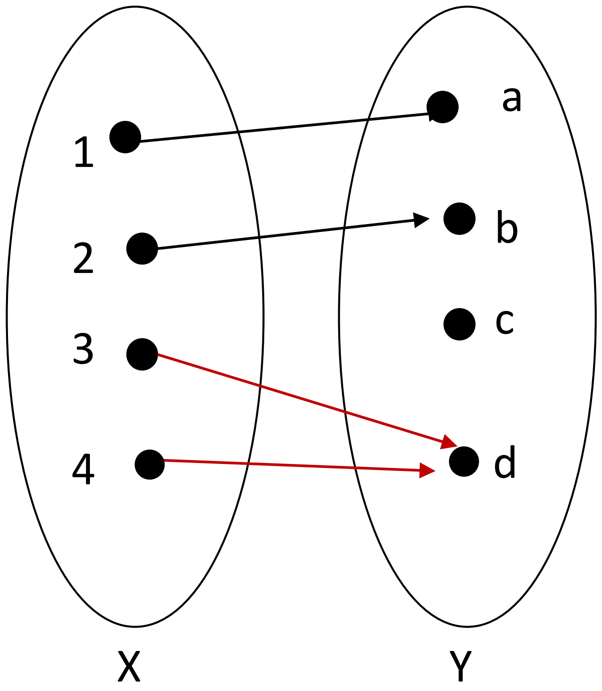
Function types: injective function

- Is this a function?



Function types: injective function

- Is this a function?



- It is a function but not an injective one!

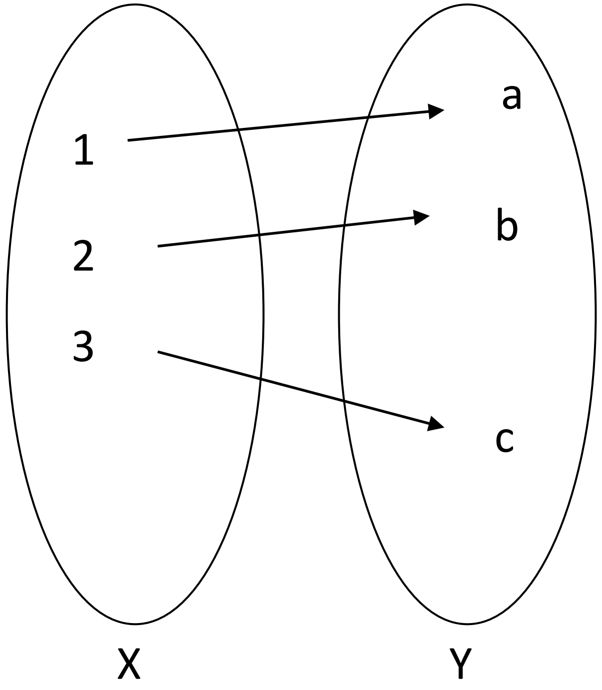
Function types: injective function definition

A function ***f*** : X → Y is said to be **injective** or **one-to-one**

- if no member of Y is the image under ***f*** of two distinct elements of X.
- In other words: if whenever ***f(x) = f(y)*****,** then ***x = y*****.**

Example:

- f(1) = a
- f(2) = b
- f(3) = c



Function types: injective function example

Let's show that the function $f(x) : R \rightarrow R$, $f(x) = 3x + 5$ is injective.

1. Suppose that $f(a) = f(b)$, and we will show that $a = b$.
2. If $f(a) = f(b)$, then $3a + 5 = 3b + 5$
3. Subtract 5 from both sides to get: $3a = 3b$
4. Divide both sides by 3 to get: $a = b$

We shown that $f(a) = f(b)$ implies $a = b$,
Thus the function $f(x)$ is injective

Function types: injective function example

Determine if the function f: Z → Z, f(x) = x + 1 is injective

- If f is injective, from **f(x) = f(y),** we should have **x = y**
- Let’s check if that is the case
  - f(x) = f(y) means: x + 1 = y + 1, and if we subtract 1, then x = y.
- So f(x) = f(y) leads to x = y, which is what is required for f(x) to be injective
  - So f(x) is injective

Function types: injective function exercise

Is the following function injective?

$g : R \to R$, $g(x) = x^2$.

Function types: injective function exercise

Is the following function injective?

$g : R \rightarrow R$, $g(x) = x^2$.

Let's check – assume $g(x) = g(y)$. This means that $x^2 = y^2$. for any real number x.

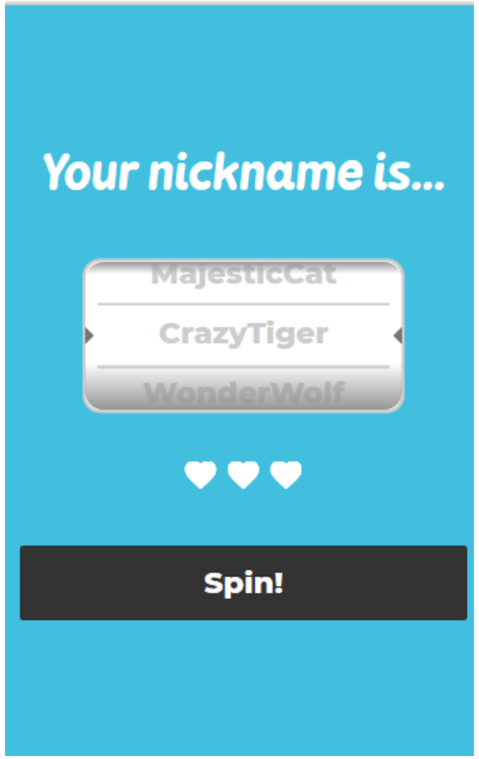
- If we take the square roots we get two values for x:
- $x = y$ or $x = -y$
- Hence, we found that for $g(x) = g(y)$, $x = -y$ which is $x \neq y$
- In other words $g(x) = x^2$, and $g(-x) = (-x)^2 = (-1)^2 * x^2 = x^2$ hence g is not injective.

We can redefine the domain as R+, so that that function is injective.

We also know that on R+ this function is surjective, so is bijective and its inverse is $f^{-1}(x) = \sqrt{x}$

Let’s playxercise!

- https://kahoot.it/



Summary: Function types

- Inverse function of function *f*, $f : A \rightarrow B$ is the function $f^{-1} : B \rightarrow A$ that reverses f, so that for every element *y* of *B*, $f^{-1}(y) = x$, whenever $f(x) = y$.
- Bijective function is a function which is both injective and surjective.
- Injective function (or one-to-one) is a function $f : A \rightarrow B$ for which; for every element ***y*** in the codomain B there is at most one element **x** in the domain A, or if $\boldsymbol{f(x) = f(y)}$**,** then $\boldsymbol{x = y}$**.**
- Surjective function (or onto) is a function $f : A \rightarrow B$ for which, for every element **y** in the codomain B, there is at least one element **x** in domain A such that $\boldsymbol{f(x) = y}$**.**

Overview

- Functions
  - Definitions
  - Types
  - Operations

Operations on functions

- Sum, difference
- Product, quotient
- Composition

Operations on functions

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Operations on functions

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Operations on functions: sum

Let $\boldsymbol{f} : A \to R$ and $\boldsymbol{g} : A \to R$.

- Sum of $\boldsymbol{f}$ and $\boldsymbol{g}$:
  - $(\boldsymbol{f} + \boldsymbol{g})(x) = \boldsymbol{f}(x) + \boldsymbol{g}(x)$, for all x in A

- Example:
  - $\boldsymbol{f}(x) = 3x + 1$ and $\boldsymbol{g}(x) = x^2$
  - $(\boldsymbol{f} + \boldsymbol{g})(x) = ?$

Operations on functions: sum

Let $\boldsymbol{f} : A \to R$ and $\boldsymbol{g} : A \to R$.

- Sum of $\boldsymbol{f}$ and $\boldsymbol{g}$:
  - $(\boldsymbol{f} + \boldsymbol{g})(x) = \boldsymbol{f}(x) + \boldsymbol{g}(x)$, for all x in A

- Example:
  - $\boldsymbol{f}(x) = 3x + 1$ and $\boldsymbol{g}(x) = x^2$
  - $(\boldsymbol{f} + \boldsymbol{g})(x) = \boldsymbol{f}(x) + \boldsymbol{g}(x) = 3x + 1 + x^2 = x^2 + 3x + 1$

Operations on functions: sum

- Let *f* = {<-3, 2>, <-2, 4>, <-1, 6>, <0, 8>}, and
- *g* = {<-2, 5>, <0, 7>, <2, 9>} and *h* = {<-3, 0>, <-2, 1>}.
- Find the following function and state its domain: f + g

Answer:

- Intersection of f and g domains = {-2, 0}
- (f + g) (-2) = f (-2) + g (-2) = 4 + 5 = 9
- (f + g) (0) = f(0) + g(0) = 8 + 7 = 15
- f + g = {<-2, 9>, <0,15>}

Operations on functions: difference

Let $f : A \rightarrow R$ and $g : A \rightarrow R$.

- Difference of f and g :
  - **$(f - g)(x) = f(x) - g(x)$,** for all x in A

- Example:
  - $f(x) = 3x + 1$ and $g(x) = x^2$
  - $(f - g)(x) = ?$

Operations on functions: difference

Let $\boldsymbol{f} : A \rightarrow R$ and $\boldsymbol{g} : A \rightarrow R$.

- Difference of $\boldsymbol{f}$ and $\boldsymbol{g}$:
  - $(\boldsymbol{f} - \boldsymbol{g})(x) = \boldsymbol{f}(x) - \boldsymbol{g}(x)$, for all x in A

- Example:
  - $\boldsymbol{f}(x) = 3x + 1$ and $\boldsymbol{g}(x) = x^2$
  - $(\boldsymbol{f} - \boldsymbol{g})(x) = \boldsymbol{f}(x) - \boldsymbol{g}(x) = 3x + 1 - x^2 = -x^2 + 3x + 1$

Operations on functions: difference

- Let *f* = {<-3, 2>, <-2, 4>, <-1, 6>, <0, 8>}, and
- *g* = {<-2, 5>, <0, 7>, <2, 9>} and *h* = {<-3, 0>, <-2, 1>}.
- Find the following function and state its domain: f - g

Answer:

- Intersection of f and g domains = {-2, 0}
- (f - g) (-2) = f(-2) - g (-2) = 4 - 5 = -1
- (f - g) (0) = f(0) - g(0) = 8 - 7 = 1
- f - g = {<-2, -1>, <0, 1>}

Operations on functions

- Sum, difference
- Product, quotient
- Composition

Operations on functions: product

Let $\boldsymbol{f} : A \rightarrow R$ and $\boldsymbol{g} : A \rightarrow R$.

- Product of $\boldsymbol{f}$ and $\boldsymbol{g}$:
  - **$(f * g)(x) = f(x) * g(x)$,** for all x in A

- Example:
  - $\boldsymbol{f}(x) = 3x + 1$ and $\boldsymbol{g}(x) = x^2$
  - $(\boldsymbol{f} * \boldsymbol{g})(x) = ?$

Operations on functions: product

Let ***f*** : A $\rightarrow$ R and ***g*** : A $\rightarrow$ R.

- Product of ***f*** and ***g*** :
  - (***f*** * ***g***)(x) = ***f*** (x) * ***g*** (x), for all x in A

- Example:
  - ***f*** (x) = 3x +1 and ***g*** (x) = x^2
  - (***f*** * ***g***)(x) = ***f*** (x) * ***g*** (x) = (3x + 1) * x^2 = $3x^3 + x^2$

Operations on functions: product

- Let *f* = {<-3, 2>, <-2, 4>, <-1, 6>, <0, 8>}, and
- *g* = {<-2, 5>, <0, 7>, <2, 9>} and *h* = {<-3, 0>, <-2, 1>}.
- Find the following function and state its domain: f * g

Answer:

- Intersection of f and g domains = {-2, 0}
- (f * g) (-2) = f (-2) * g (-2) = 4 * 5 = 20
- (f * g) (0) = f(0) * g(0) = 8 * 7 = 56
- f * g = {<-2, 20>, <0, 56>}

Operations on functions: quotient

Let $\boldsymbol{f} : A \rightarrow R$ and $\boldsymbol{g} : A \rightarrow R$.

- Quotient of ***f*** and ***g*** :
  - **($\boldsymbol{f}$ / $\boldsymbol{g}$)(x) = $\boldsymbol{f}$ (x) / $\boldsymbol{g}$ (x),** for all x in A

- Example:
  - $\boldsymbol{f}(x) = 3x + 1$ and $\boldsymbol{g}(x) = x^2$
  - $(\boldsymbol{f} / \boldsymbol{g})(x) = ?$

Operations on functions: quotient

Let ***f*** : A $\rightarrow$ R and ***g*** : A $\rightarrow$ R.

- Quotient of ***f*** and ***g*** :
  - (***f*** / ***g***)(x) = ***f*** (x) / ***g*** (x), for all x in A

- Example:
  - ***f*** (x) = 3x +1 and ***g*** (x) = x^2
  - (***f*** **/** ***g***)(x) = ***f*** (x) / ***g*** (x) = (3x + 1) / x^2 , with $x^2 \neq 0$

Operations on functions: quotient

- Let *f* = {<-3, 2>, <-2, 4>, <-1, 6>, <0, 8>}, and
- *g* = {<-2, 5>, <0, 7>, <2, 9>} and *h* = {<-3, 0>, <-2, 1>}.
- Find the following function and state its domain: f / h

Answer:

- Intersection of f and h domains = {-3, -2} for which h(x) ≠ 0
- x = -3 implies h(-3) = 0 which cannot be denominator of the quotient function.
- x = -2 implies h(-2) = 1 ≠ 0. Hence the domain of the quotient = {-2}

- (f /h) (-2) = f (-2) / h (-2) = 4 / 1 = 4
- f / h = {<-2, 4>}

Operations on functions

- Sum, difference
- Product, quotient
- Composition

Operations on functions: composition

Composite function

Let $g : A \to B$, and $f : B \to C$.

The composition of functions *f* and *g*

- written as $f \circ g : A \to C$
- is $\boldsymbol{(f \circ g)(x) = f(\ g(x)\ )}$ for all x in A

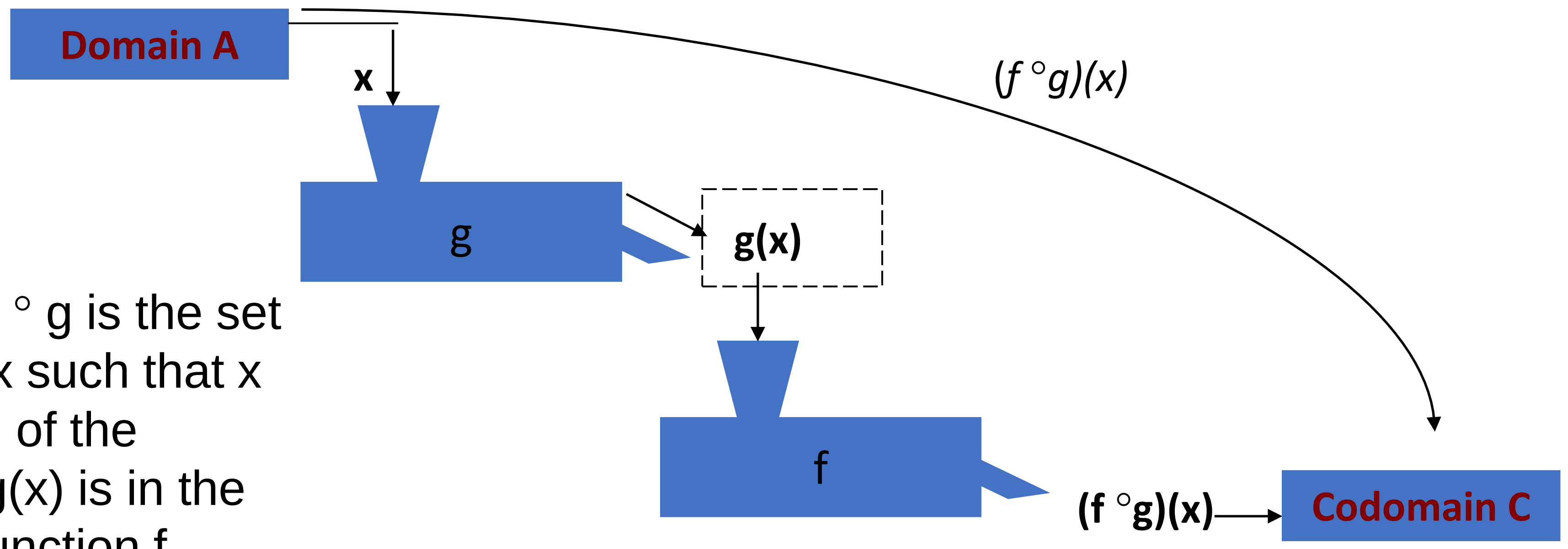
Note: the function *g* is applied first and then *f*.

Operations on functions: composition

A composite function is a function within a function

- Let $g : A \to B$, and $f : B \to C$
- $f \circ g : A \to C$ is $(f \circ g)(x) = f(\,g(x)\,)$ for x in A

- In terms of "function machines", the composition f $\circ$ g is the function which feeds an input to g and feeds the output of g to f



The domain of f $\circ$ g is the set of all elements x such that x is in the domain of the function g and g(x) is in the domain of the function f.

Operations on functions: composition

Example:

Let $g : A \rightarrow B$ with $g(x) = x + 1$

and $f : B \rightarrow C$ with $f(x) = 2x$

Then $(f \circ g)(x) : A \rightarrow C$

$(f \circ g)(x) = ?$

$(f \circ g)(x) = f\,(g(x)\,) = 2\;g(x) = 2\;(x + 1) = 2x + 2$

Operations on functions: composition

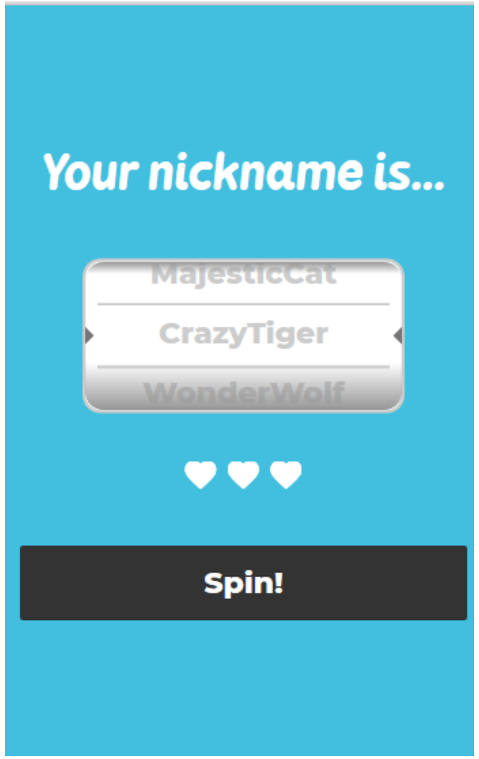
- Let f = {<-3, 2>, <-2, 4>, <-1, 6>, <0, 8>}, and
- g = {<-2, 5>, <0, 7>, <2, 9>} and h = {<-3, 0>, <-2, 1>}.
- Find the following function and state its domain: $g \circ h$

Answer:

- The domain of $g \circ h$ is a subset of the domain of the inside function h: {-3, -2}.
- We also need to check that these elements, when plugged into the outside function produce valid ordered pairs, i.e. h(-3) and h(-2) are in the domain of g.
- for x = -3, g(h(-3)) = g(0) = 7 is defined; for x = -2, $g(h(-2)) = g(1)$, which is undefined.
- Hence the domain of $g \circ h$ = {-3}
- g(h(-3)) = g (0) = 7
- $g \circ h$ ={<-3, 7>}

Let’s playxercise!

- https://kahoot.it/



Summary: operations on functions

- $(f + g)(x) = f(x) + g(x)$
- $(f - g)(x) = f(x) - g(x)$
- $(f * g)(x) = f(x) * g(x)$
- $(f / g)(x) = f(x) / g(x)$, $g(x) \neq 0$
- $(f \circ g)(x) = f(\, g(x) \,)$