SCC121
Fundamentals of Computer Science

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Overview

- Why predicate logic:
  - Rationale and distinction from propositional logic
- Predicate logic's syntax:
  - Logical concepts: predicates, terms, formulae
  - Operators: connectives, quantifiers
- Predicate logic's semantics:
  - Interpretation, satisfiable formulae

Objectives

- Understanding basic ideas about predicate logic
- Facility to use predicate logic notations
- Facility to operate with quantifiers
- Understanding the semantics of predicate logic

Compound Formulae

Propositional logic:

- Atomic propositions
- Compound propositions – propositions obtained by applying to atomic propositions logical connectives: NOT, AND, OR, XOR, implication, equivalence

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- Compound propositions – propositions obtained by applying to atomic propositions logical connectives: NOT, AND, OR, XOR, implication, equivalence

Predicate logic:

- Atomic formulae
- Compound formulae – formulae obtained by applying to atomic formulae logical connectives: NOT, AND, OR, XOR, implication, equivalence and quantifiers: existential and universal.

Compound Formulae – Examples

- “Wizard of Oz is not a SCC121 student”
  - Predicate = “is a SCC121 student” – Symbol S

- “Wizard of Oz is not a SCC121 student” ~S(w)
- “Jay and Kay are SCC121 students” S(j) AND S(k)
- “Either Jay or Kay is a SCC121 student” S(j) XOR S(k)
- “If Jay is a SCC121 student, then Kay is also” S(j) → S(k)

Compound Formulae – Exercise

Is the following a compound formula, and if so, which are its atomic formulae?

- R(x) is if $x^2 = 4$ and $x>0$, then $x = sqrt(4)$

Compound Formulae – Exercise

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Answer;

R(x) consists of 2 atomic formulae linked through the implication connective $\rightarrow$

- R(x) is if $x^2=4$ and $x>0$, then $x = \text{sqrt}(4)$
- Let's say A(x) is $x^2=4$ and $x>0$ and C(x) is $x = \text{sqrt}(4)$
- Then we can write R(x) is $A(x) \rightarrow C(x)$

Compound Formulae – Exercise

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Compound Formulae – Exercise

Is the following a compound formula, and if so, which are its atomic formulae?

- $T(x, y)$ is $x + y = 4$ if and only if $y = 4 - x$

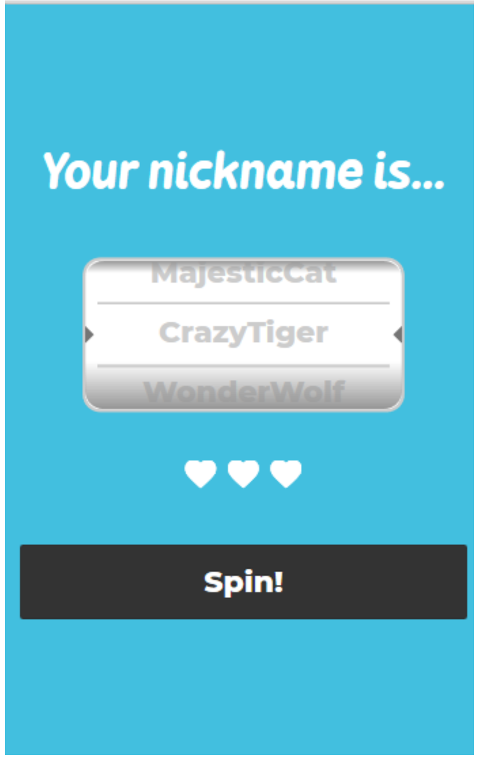
Answer;

$T(x, y)$ consists of 2 formulae combined through the biconditional connective $\leftrightarrow$

- $T(x, y)$ is $x + y = 4$ if and only if $y = 4 - x$
- Let's say $C(x, y)$ is $x + y = 4$, $D(x, y)$ is $y = 4 - x$
- Then we can write $T(x, y)$ is $C(x, y) \leftrightarrow D(x, y)$

Let’s playxercise!

- https://kahoot.it/



Atomic Formulae' Arity

Arity of an atomic formula – the number of variables it takes as arguments

- Those with one variable as argument are called 1-place, or unary, i.e., P(x) involves 1 variable
- Those with two variables as arguments are called 2-place, or binary, i.e., Q(x, y) involves 2 variables
- Those with n variables as arguments are called n-place, or n-ary, i.e., M(x1, x2, …, xn) involves n variables

Atomic Formulae’ Arity

- Atomic formulae of arity 0 (i.e., take no arguments) are propositions
- Atomic formulae of arity 1, are called properties, i.e., P(x)
- Atomic formulae of arity >=2 denote relationships, i.e., P(x, y, z)

arity - number of terms as variables to fill in the argument positions in open atomic formulae to get closed ones

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Quantifiers

Quantifiers – operators denoting the quantity of values from the Universe of discourse for which an open formula is satisfied.

English terms signaling quantifiers:

- all, each, any, both, everyone, everything
- none, neither, no one, nothing
- at least one, at least two, etc.
- at most one, at most two, etc.
- exactly one, exactly two, etc.
- some, most, many, several, a few
- someone, something

"every"

"at least one"

Quantifiers

Quantifiers relate to variables

Statements like “everyone”, “someone” do not refer to individual entities:

- “Every man is mortal”
- “No man chased Socrates”
- “Kermit kissed someone”

Quantifiers

Quantifiers are combined with terms (variables) in open formulae, to create quantified formulae.

- Every SCC121 student is clever
- At least one SCC121 student will pass with distinction
- None SCC121 students will fail the exam

Quantifiers

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- Every SCC121 student is clever
- At least one SCC121 student will pass with distinction
- None SCC121 students will fail the exam

Let us have variable x - students from the Universe of all students taking SCC121

- Every SCC121 student is clever — C(x) = "x is clever"
- At least one SCC121 student will pass — P(x) = "x will pass"
- None SCC121 students will fail — F(x) = "x will fail"

Quantifiers

Two types of quantifiers:

- Universal quantifier: every – everything is such that…
- Existential quantifier: at least one – at least one thing is such that…

Example: x – a student from the Universe of all students taking SCC121

- Every SCC121 student is clever — C(x) = “x is clever”
- At least one SCC121 student will pass — P(x) = “x will pass”
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Quantifiers are unary operators, just like negation.

- They require a single argument in order to form a formula, i.e., x.

Quantifiers

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- Universal quantifier: every – everything is such that…
- Existential quantifier: at least one – at least one thing is such that…

Example: x – a student from the Universe of all students taking SCC121

- Every SCC121 student is clever $\forall x\ C(x)$
- At least one SCC121 student will pass $\exists x\ P(x)$
- None SCC121 students will fail $\forall x\ {\sim}F(x)$

Quantifiers are unary operators, just like negation.

- They require a single argument in order to form a formula, i.e., x.

Universal Quantifier

- Symbol: $\forall$ is read “for all”
- $\forall x\ P(x)$ means that for all values of variable x in the Universe of Discourse, the formula P is True
  - $\forall x\ P(x)$ is not satisfied if there is one value of x for which P is False

Universal quantifier and connective AND

- If all elements in the Universe of discourse can be listed, then the universal quantification $\forall x\ P(x)$ is equivalent to the conjunction:

$$P(x_1) \wedge P(x_2) \wedge P(x_3) \wedge \ldots \wedge P(x_n)$$

Universal Quantifier

- Symbol: ∀ is read "for all"
- ∀x P(x) means that for all values of variable x in the Universe of Discourse, the formula P is True
- In English:
  - "For every x, P(x) is True"
  - "For all x, P(x) is True "
  - "For each x, P(x) is True "
- Example:
  - "Every SCC121 student is clever" ∀x C(x)
  - "All SCC121 students are clever" ∀x C(x)
  - "Each SCC121 students are clever" ∀x C(x)

Universal Quantifier - Examples

Express the following formulae in predicate logic:

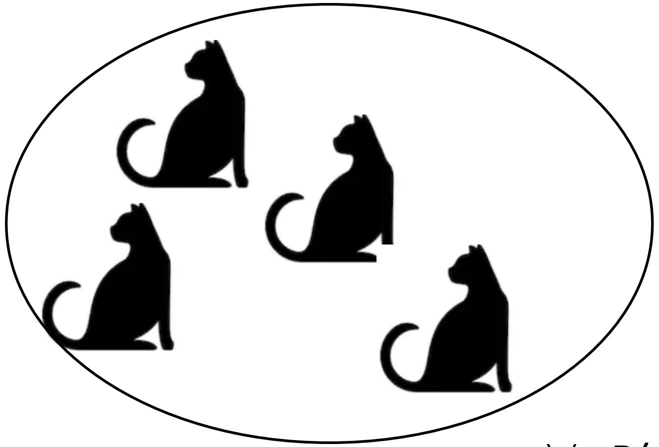
- "All cars have wheels"
  - Formula HW(x) = “x has wheels”, and the Universe of Discourse is cars.
  - All – means that HW(x) is true for all values of x.
  - So $\forall x\ HW(x)$
- "Everyone gets a break once in a while“
  - Formula GB(x) = “x gets a break”, and the Universe of Discourse is people.
  - Everyone – means that GB(x) is true for all values of x.
  - So: $\forall x\ GB(x)$

Negation of Universal Quantifier

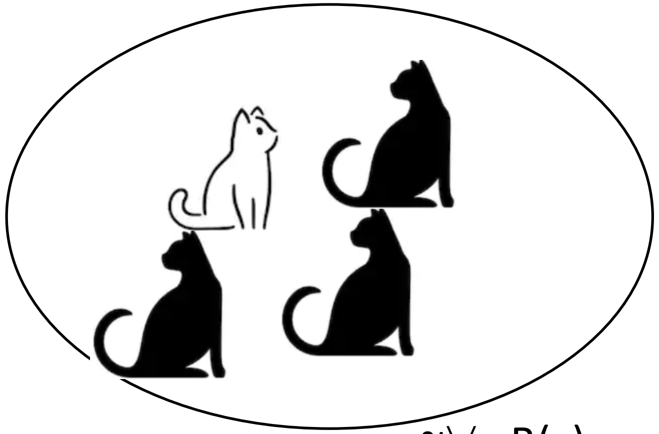
- Let B(x) = “the cat x is black”, and ∀x B(x) = “all cats are black” on the Universe of Discourse of cats.

What is the negation of ∀x B(x)?

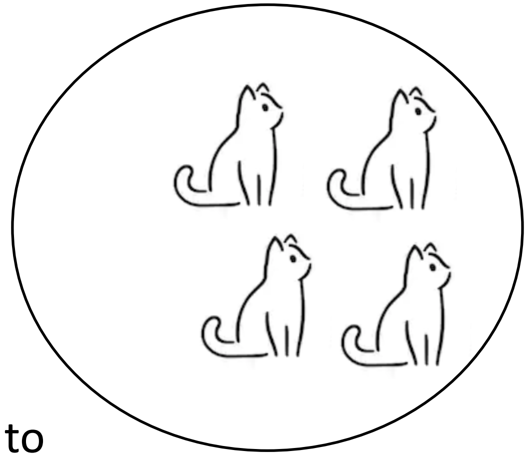
- ~∀x B(x) = “not all cats are black” is equivalent to
- “there is at least one cat who is not black” ∃x ~B(x)



∀x B(x)



~∀x B(x) equivalent to
∃x ~B(x)



∀x ~B(x) - this is the **opposite of ∀x B(x)** and is stronger: “no cats are black”

Negation of Universal Quantifier

- Let $B(x)$ = "the cat x is black", and $\forall x\ B(x)$ = "all cats are black" on the Universe of Discourse of cats.

What is the negation of $\forall x\ B(x)$?

- $\sim\forall x\ B(x)$ = "not all cats are black", which is equivalent to
- "there is at least one cat who is not black"

If a universally quantified formula is False, then there is at least one value of its variable that makes it False:

$\sim\forall x\ P(x)$ is equivalent to $\exists x \sim P(x)$

Existential Quantifier

- Symbol: $\exists$ is read "there exists"
- $\exists x\, P(x)$ means that for at least one value of variable x in the Universe of Discourse, the formula P is True
  - $\exists x\, P(x)$ is not satisfied if for each value of x, P is false.

Existential quantifier and connective OR

- If all the elements in the universe of discourse can be listed, then the existential quantification $\exists x\, P(x)$ is equivalent to the disjunction:

$$P(x_1) \vee P(x_2) \vee P(x_3) \vee \ldots \vee P(x_n)$$

Existential Quantifier

- Symbol: ∃ is read “there exists”
- $\exists x\, P(x)$ means that for at least one value of variable x in the Universe of Discourse, the formula $P(x)$ is True.
  - $\exists x\, P(x)$ is not satisfied if for each value of x, P is false.

- In English:
  - "There is at least one x such that P(x) is true”
  - "There exist some x such that P(x) is true"

- Example:
  - “There is at least one SCC121 student will pass with distinction” $\exists x\, S(x)$
  - “There exist some SCC121 students who will pass with distinction” $\exists x\, S(x)$

Existential Quantifier - Examples

Express the following formulae in predicate logic:

- "Someone loves you"
  - Formula L(x) = “x loves you”, and the Universe of Discourse is living creatures.
  - Someone – means that L(x) is true for at least one value of x
  - So ∃x L(x)

- “Some people like their meat raw"
  - Formula L(x) = “x likes their meat raw”, and the Universe of Discourse is people.
  - Some – means that L(x) is true for some values of x
  - So ∃x L(x)

Negation of Existential Quantifier

- Let P(x) = “the cat x is purple”, and $\exists x\ P(x)$ = “there is a cat who is purple” on the Universe of Discourse of cats.
- What is the negation of $\exists x\ P(x)$?
- This would be $\sim \exists x\ P(x)$ = “it is not the case that there is a cat who is purple”. This is equivalent to: “every cat is not purple”: $\forall x \sim P(x)$

If we claim that a formula with an existential quantifier is False, the only way that can occur is if the formula is never True:

$\sim\exists x\ Q(x)$ is equivalent to $\forall x \sim Q(x)$

De Morgan's Laws for Quantifiers

Each quantifier can be expressed using the other.

We have the logical equivalences:

$\sim\forall x\ P(x) \equiv \exists x\ \sim P(x)$

$\sim\forall x$ Likes(x, cake) $\exists x$ ~Likes(x, cake)

$\sim\exists x\ P(x) \equiv \forall x\ \sim P(x)$.

$\sim\exists x$ Likes(x, broccoli) $\forall x$ ~Likes(x, broccoli)

Restricted Quantifiers

Unrestricted quantifiers - all elements in the Universe of Discourse

Restricted quantifiers - some elements in the Universe of Discourse (a subset)

For restricted universal quantifier we use implication

- Example: “Every man has two legs” with Universe of Discourse of men.

  $M(x)$ = “x is a man” $L(x)$ = “x has two legs”

  $\forall x\ (M(x) \rightarrow L(x))$ (every man with the property of having two legs)

For restricted existential quantifier we use conjunction

- Example: “There is a man who has two legs” with Universe of Discourse

  $M(x)$ = “x is a man” $L(x)$ = “x has two legs”

  $\exists x\ (M(x) \wedge L(x))$ (there are some things in the world that are both man and have two legs)

Rules of Inference

Inference rules for propositional logic:

- modus ponens, modus tollens, addition, simplification
- hypothetical syllogism, disjunctive syllogism, absorption

Specific inference rules for quantified formulae:

- Universal instantiation
- Universal generalization
- Existential instantiation
- Existential generalization

Rules of Inference

- Universal instantiation

$$\frac{\forall x\ P(x)}{\therefore\ P(c)}$$

- Example

  Every man is mortal

  Therefore: any specific man is mortal

Rules of Inference

- Universal generalization

$$\frac{P(a) \text{ for any arbitrary } a}{\therefore\ \forall x\, P(x)}$$

- Example

  Any arbitrary man is mortal

  Therefore: Every man is mortal

Rules of Inference

- Existential instantiation

$$\frac{\exists x\ P(x)}{\therefore\ P(a) \text{ for some element } a}$$

- Example

  There is someone who is mortal, let's call them a

  Therefore: a is mortal

Rules of Inference

- Existential generalization

$$\frac{P(c) \text{ for some element } c}{\therefore\ \exists x\, P(x)}$$

- Example

  Aristotle is mortal

  Therefore: there is someone who is mortal

Rules of Inference - Exercise

Using the rules of inference, construct a valid argument to show that:

Every man has two legs.

John is a man.

Then: John has two legs.

Rules of Inference - Exercise

Using the rules of inference, construct a valid argument to show that:

Every man has two legs.

John is a man.

Then: John has two legs.

Let $M(x)$ = “x is a man” and $L(x)$ = “x has two legs”

Let John be an element in the Universe of Discourse of all men.

Rules of Inference - Exercise

Using the rules of inference, construct a valid argument to show that:

Every man has two legs.

John is a man.

Then: John has two legs.

Let M(x) = “x is a man” and L(x) = “x has two legs”

Let John be an element in the Universe of Discourse of all men.

1. $\forall x\ (M(x) \rightarrow L(x))$ premise (every man with the property of having two legs)
2. $M(j) \rightarrow L(j)$ Universal Instantiation from (1)
3. $M(j)$ premise
4. $L(j)$ Modus ponens using (2) and (3)

Application of Quantifiers - Precedence

- Universal and existential quantifiers are unary operators.
- They have the highest precedence over all binary connectives.
- Connectors' precedence for predicate logic:

  Quantifiers: $\forall$, $\exists$

  $\neg$

  $\wedge$

  $\vee$

  $\rightarrow$

  $\leftrightarrow$

Example: $\forall x\ P(x) \vee Q(x)$ equivalent to $(\forall x\ P(x)) \vee Q(x)$ and not: $\forall x\ (P(x)) \vee Q(x))$

Applications of Quantifiers

Applied to open atomic formula, i.e., P(x), x variable over Universe the Discourse:

- prefix it with ∀x, to obtain a universally quantified formula: ∀x P(x)
- prefix it with ∃x to obtain an existentially quantified formula: ∃x P(x)

Applications of Quantifiers

Applied to open compound formula which uses connectives from propositional logic to obtain a quantified formula:

Open formula	Universally quantified formula	Existentially quantified formula
$\sim F(x)$	$\forall x \sim F(x)$	$\exists x \sim F(x)$
$F(x) \wedge G(x)$	$\forall x\ (F(x) \wedge G(x))$	$\exists x\ (F(x) \wedge G(x))$
$F(x) \vee G(x)$	$\forall x\ (F(x) \vee G(x))$	$\exists x\ (F(x) \vee G(x))$
$F(x) \rightarrow G(x)$	$\forall x\ (F(x) \rightarrow G(x))$	$\exists x\ (F(x) \rightarrow G(x))$
$F(x) \leftrightarrow G(x)$	$\forall x\ (F(x) \leftrightarrow G(x))$	$\exists x\ (F(x) \leftrightarrow G(x))$

Nested Quantifiers

More than one quantifier may be necessary to capture the meaning of a formula in predicate logic, i.e., $\forall x \, \exists y \, P(x, y)$

Nested Quantifiers: Order

More than one quantifier may be necessary to capture the meaning of a formula in predicate logic.

Order of nested quantifiers does not matter if **quantifiers are of the same type.**

Example:

- $P(x, y)$ = "x is a parent of y"
- $C(x, y)$ = "x is a child of y"

- Let's write: "For all x and y, if x is a parent of y then y is a child of x"
  - $\forall x \, \forall y \, P(x, y) \rightarrow C(y, x)$ equivalent formula to $\forall y \, \forall x \, P(x, y) \rightarrow C(y, x)$

Nested Quantifiers

The order of nested quantifiers matters if **quantifiers are of different types**

Example:

- Assume: $L(x, y)$ = "x loves y"
  - $\forall x\ \exists y\ L(x, y)$ = "everybody loves somebody"
  - $\exists y\ \forall x\ L(x, y)$ = "there is someone who is loved by everyone"
- The meaning of the two is different.

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Interpretation in Predicate Logic

Interpretation is used to identify the truth value of each formulae and includes:

- identifying the Universe of Discourse
- assigning to the variable each value from the Universe of Discourse

Interpretation in Predicate Logic: Assignment

- Assignment can be viewed as a table where variable x is substituted in an open formula by each constant value from the Universe of discourse.
- Atomic open formula P(x) assigns truth value for each value of x

Example: P(x) “x is a number smaller than 5”. Universe of discourse = {2,3,5}

For each value assigned to the variable x, P(x) becomes proposition which has truth value:

- P(2) = “2 is smaller than 5” True
- P(3) = “3 is smaller than 5” True
- P(5) = “5 is smaller than 5” False

Satisfiable Formulae

- Satisfaction of open formula
  - There is a least one value from the Universe of discourse for which the formula is True
  - These values satisfy the formula
- Example for one-place formula: R(x): x + 4 = 3

  if x = -1, -1 + 4 = 3; 3 = 3

  R(-1) is True, so x= -1 satisfies R(x)
- Example of two-place formula: R(x, y): x + y = 4

  R(x, y) = R(2,2), so 2 + 2 = 4 so, <(x, y> = <2,2> satisfies R(x, y)

  <x, y> = <3,1> is another tuple that satisfies R(x, y)
- An n-place formula is satisfiable if there is (at least) a n-tuple which satisfies it.

Summary

- Atomic formula - expression consisting of one predicate and one or more terms.
- Compound formula – formula obtained by applying to atomic formula logical connectives and/or quantifiers.
- Arity of an atomic formula – the number of variables it takes as arguments
- Universal quantifier: $\forall x\, P(x)$ - for all values of variable x in the Universe of Discourse, the formula $P(x)$ is True: “every x is such that”.
- Existential quantifier: $\exists x\, P(x)$ - for at least one value of variable x in the Universe of Discourse, the formula $P(x)$ is True: “there exists one or more x such that”.
- Assignment – substituting variable x in an open formula by each constant value from the Universe of discourse.