Dynamic bicycle model

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1   Simple dynamic model
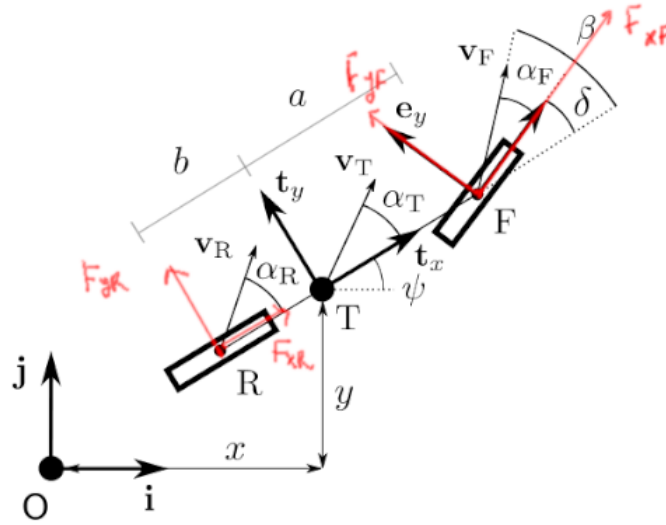

Figure 1: Bicycle model

$$m_T\ddot{x} = F_{xF}cos(\psi + \delta) + F_{xR}cos(\psi) - F_{yF}sin(\psi + \delta) - F_{yR}sin(\psi)$$
$$m_T\ddot{y} = F_{xF}sin(\psi + \delta) + F_{xR}sin(\psi) + F_{yF}cos(\psi + \delta) + F_{yR}cos(\psi)$$
$$I_T\ddot{\psi} = F_{xF}asin(\delta) + F_{yF}acos(\delta) - F_{yR}b \tag{1}$$

$$\alpha_F = arctan(\frac{\dot{y} + a\dot{\psi}cos(\psi)}{\dot{x} - a\dot{\psi}sin(\psi)}) - (\delta + \psi)$$
$$\alpha_R = arctan(\frac{\dot{y} - b\dot{\psi}cos(\psi)}{\dot{x} + b\dot{\psi}sin(\psi)}) - \psi \tag{2}$$

At first, as state vector this has been used:

$$
\begin{aligned}
z1 &= x \\
z2 &= y \\
z3 &= \psi \\
z4 &= \dot{x} \\
z5 &= \dot{y} \\
z6 &= \dot{\psi}
\end{aligned}
\tag{3}
$$

So that

$$
\begin{aligned}
\dot{z}_1 &= z_4 \\
\dot{z}_2 &= z_5 \\
\dot{z}_3 &= z_6 \\
\dot{z}_4 &= \frac{F_{xF}cos(z_3 + \delta) + F_{xR}cos(z_3) - F_{yF}sin(z_3 + \delta) - F_{yR}sin(z_3)}{m_T} \\
\dot{z}_5 &= \frac{F_{xF}sin(z_3 + \delta) + F_{xR}sin(z_3) + F_{yF}cos(z_3 + \delta) + F_{yR}cos(z_3)}{m_T} \\
\dot{z}_6 &= \frac{F_{xF}a\,sin(\delta) + F_{yF}a\,cos(\delta) - F_{yR}b}{I_T}
\end{aligned}
\tag{4}
$$

With slip angles

$$
\begin{aligned}
\alpha_F &= arctan(\frac{z_5 + az_6cos(z_3)}{z_4 - az_6sin(z_3)}) - (\delta + z_3) \\
\alpha_R &= arctan(\frac{z_5 - bz_6cos(z_3)}{z_4 + bz_6sin(z_3)}) - z_3
\end{aligned}
\tag{5}
$$

Now instead of using $\dot{x}$ and $\dot{y}$, v_T and α_T have been used. The trasformations are the following:

$$
\begin{aligned}
\dot{x} &= v_T cos(\psi + \alpha_T) \\
\dot{y} &= v_T sin(\psi + \alpha_T)
\end{aligned}
$$

$$
\begin{aligned}
\ddot{x} &= \dot{v}_T cos(\psi + \alpha_T) - v_T(\dot{\psi} + \dot{\alpha}_T)sin(\psi + \alpha_T) \\
\ddot{y} &= \dot{v}_T sin(\psi + \alpha_T) + v_T(\dot{\psi} + \dot{\alpha}_T)cos(\psi + \alpha_T)
\end{aligned}
\tag{6}
$$

Substituting and simplyfing with the help of Matlab

$$\dot{v_T} = \frac{F_{xF}cos(\alpha_T - \delta) + F_{xR}cos(\alpha_T) + F_{yF}sin(\alpha_T - \delta) + F_{yR}sin(\alpha_T)}{m_T}$$

$$\dot{\alpha_T} = \frac{-F_{xF}sin(\alpha_T - \delta) - F_{xR}sin(\alpha_T) + F_{yF}cos(\alpha_T - \delta) + F_{yR}cos(\alpha_T) - m_T v_T \dot{\psi}}{m_T v_T}$$

$$\ddot{\psi} = \frac{F_{xF}asin(\delta) + F_{yF}acos(\delta) - F_{yR}b}{I_T}$$

$$(7)$$

$$\alpha_F = arctan(\frac{v_T sin(\alpha_T) + a\dot{\psi}}{v_T cos(\alpha_T)}) - \delta$$

$$\alpha_F = arctan(\frac{v_T sin(\alpha_T) - b\dot{\psi}}{v_T cos(\alpha_T)})$$

$$(8)$$

The new state and the state equations are

$$x1 = x$$
$$x2 = y$$
$$x3 = \psi$$
$$x4 = v_T$$
$$x5 = \alpha_T$$
$$x6 = \dot{\psi}$$

$$\dot{x_1} = x_4 cos(x_3 + x_5)$$
$$\dot{x_2} = x_5 sin(x_3 + x_5)$$
$$\dot{x_3} = x_6$$
$$\dot{x_4} = \frac{F_{xF}cos(x_5 - \delta) + F_{xR}cos(x_5) + F_{yF}sin(x_5 - \delta) + F_{yR}sin(x_5)}{m_T}$$
$$\dot{x_5} = \frac{-F_{xF}sin(x_5 - \delta) - F_{xR}sin(x_5) + F_{yF}cos(x_5 - \delta) + F_{yR}cos(x_5) - m_T x_4 x_6}{m_T x_4}$$
$$\dot{x_6} = \frac{F_{xF}asin(\delta) + F_{yF}acos(\delta) - F_{yR}b}{I_T}$$

$$(9)$$

2 Pacejka tyre model

The following Pacejka tyre model (Magic Formula) has been used, taking as inputs the body slip angle α, the vertical load F_z and the lateral friction coefficient

μ_y

$$F_y = -\frac{\mu_y}{\mu_{y0}}(F_{y0} + Sv)$$

(10)

With

$$F_{y0} = Dsin(Carctan(B\alpha_{eq} - E(B\alpha_{eq} - arctan(B\alpha_{eq}))))$$
$$\alpha_{eq} = \frac{\mu_{y0}}{\mu_y}(\alpha + S_h)$$
$$C = a_0$$
$$D = (a_1 F_z + a_2)F_z BCD = a_3 sin(2arctan(\frac{F_z}{a_4}))(1 - a_5|\gamma|)$$
$$B = BCD/CD$$
$$E = a_6 F_z + a_7$$
$$S_h = a_8\gamma + a_9 F_z + a_{10}$$
$$S_v = a_{11}F_z\gamma + a_{12}F_z + a_{13}$$

(11)

Where a_i, i $\in \{0,..,13\}$, are the parameters of the Pacejka model, whose value and meaning can be seen in the Appendix.

3 Aerodynamic force

The following changes have been done in the previous model to take into account for the aerodynamic force $F_A = \frac{1}{2}\rho C_x S v^2$ in the same direction of v_T but in the opposite side, and $F_{Lift} = \frac{1}{2}\rho C_z S v^2$ that "pushes" the vehicle to be sticked on the ground

$$\ddot{x_4} = \frac{F_{xF}cos(x_5 - \delta) + F_{xR}cos(x_5) + F_{yF}sin(x_5 - \delta) + F_{yR}sin(x_5) - \frac{1}{2}\rho C_x S v^2}{m_T}$$

(12)

$$\ddot{F_z} = mg + \frac{1}{2}\rho C_z S v^2$$

(13)

4 Fuel consumption

The following changes have been done in the previous model to take fuel consumption into account. A simplified version has been used, in which the Power is

computed (P_e) and multiplied by a coefficient (C_{fuel}) that expresses the relation among mass loss (in terms of fuel consumption) and power provided

$$P_e = (F_{xF} + F_{xR})v_T$$
$$\dot{m} = P_e C_{fuel} \tag{14}$$

5 Tyre wear

The following have been added in the previous model to take into account for wear of the rubber compound of the tyre. The model is called Archard model, and it makes use of the vertical pression ($P_z = \frac{F_z}{Area}$), the longitudinal velocity of the wheels (v_{xF} and v_{xR}) and some parameters (such as $K_{wear}[-]$ and $H[\frac{N}{m^2}]$). The model output is the wear depth over the time ($\dot{h}$), that will be then converted into mm^3 of wasted material. Here the Archard model formulation is shown

$$\dot{h} = \frac{K_{wear} P_{load} v_{xi}}{H}$$
$$\text{where i} \in \{F, R\} \tag{15}$$

6 Banking

Taking into account the shape of the road we introduce other terms in the model equations. As can be seen in figure 2 and 3 we first consider the resultant lateral force F_{LAT} acting on the vehicle, after this step we are able to find the equilibrium on the Z axis that allows to compute the value of F_{Nx}.

$$F_{LAT} = F_{yR} + F_{xF}sin(\delta) + F_{yF}cos(\delta) \tag{16}$$

$$mg + F_{Lift}cos(\gamma) + F_{LAT}sin(\gamma) = F_{Nz}$$
$$F_{Nx} = F_{Nz}tan(\gamma) \tag{17}$$

As can be seen from this figure, the contribution of this lateral force acting on the vehicle will add some new terms in the equations of the model. Along the direction of v_T the contribution of the force is $F_{Nx}sin(\alpha_T)$, while on the orthogonal direction it is represented by the force $F_{Nx}cos(\alpha_T)$.
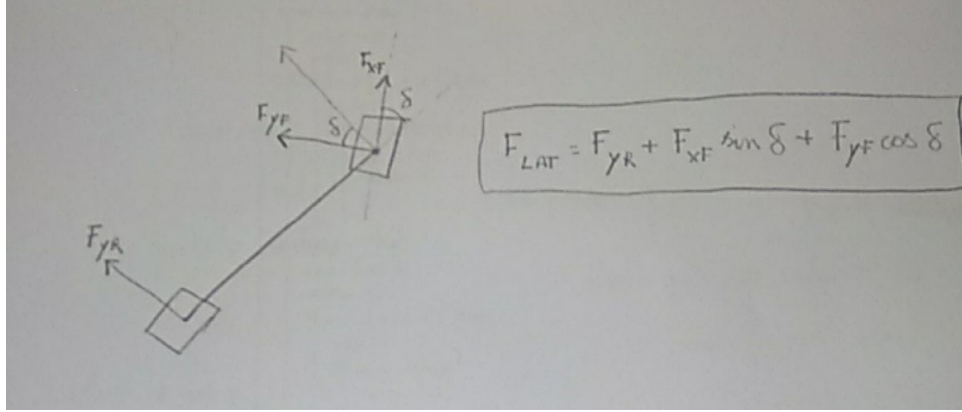
Figure 2: Lateral forces acting on the vehicle

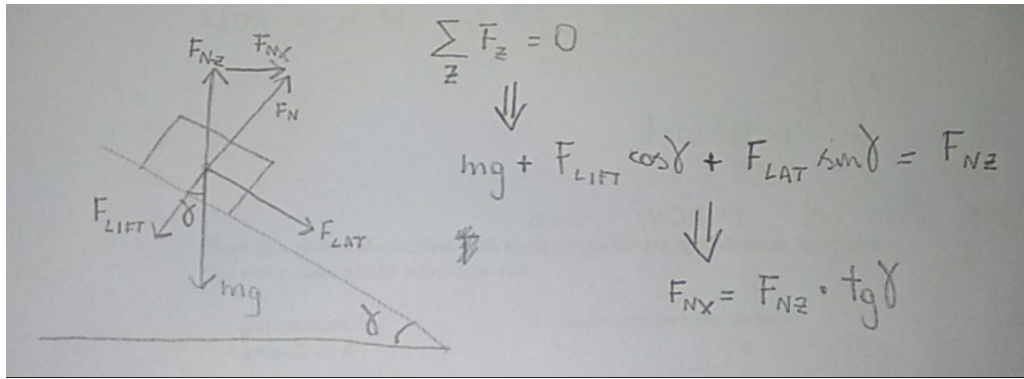$$F_{LAT} = F_{yR} + F_{xF}\sin\delta + F_{yF}\cos\delta$$

Figure 3: Equilibrium on Z axis

$$\sum_z \bar{F}_z = 0$$

$$mg + F_{LIFT}\cos\gamma + F_{LAT}\sin\gamma = F_{Nz}$$

$$F_{Nx} = F_{Nz} \cdot tg\gamma$$

In the end the final equations will be:

$$\dot{x}_4 = \frac{F_{xF}cos(x_5 - \delta) + F_{xR}cos(x_5) + F_{yF}sin(x_5 - \delta) + F_{yR}sin(x_5) - \frac{1}{2}\rho C_x S v^2 + F_{Nx}sin(x_5)}{m_T}$$

$$\dot{x}_5 = \frac{-F_{xF}sin(x_5 - \delta) - F_{xR}sin(x_5) + F_{yF}cos(x_5 - \delta) + F_{yR}cos(x_5) - m_T x_4 x_6 + F_{Nx}cos(x_5)}{m_T x_4}$$
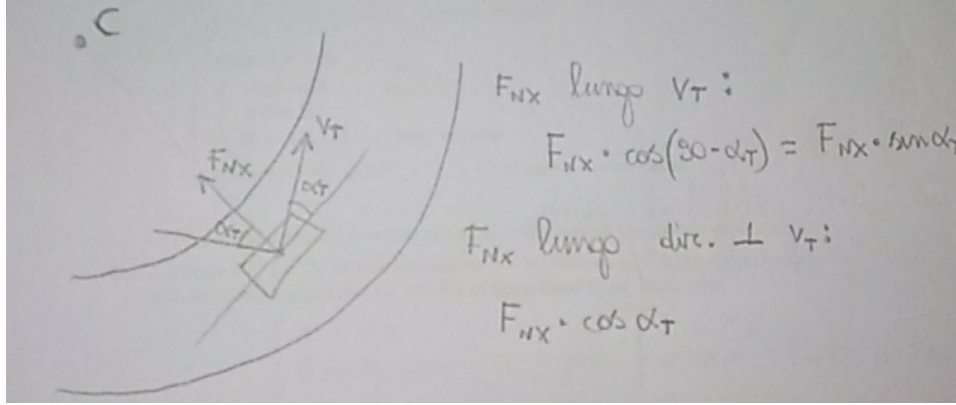
$$(18)$$

Figure 4: Terms affecting the model equations

7 Appendix

In this section we will define the values given to the different parameters.

$$a0 = 1.4[-] \text{——Shape factor}$$
$$a1 = 0[1/kN] \text{——Load dependency of lateral friction (*1000)}$$
$$a2 = 1100[-] \text{——Lateral friction level (*1000)}$$
$$a3 = 1100[N/deg] \text{——Maximum cornering stiffness}$$
$$a4 = 10[kN] \text{——Load at maximum cornering stiffness}$$
$$a5 = 0 \text{——Camber sensitivity of cornering stiffness}$$
$$a6 = 0 \text{——Load dependency of curvature factor}$$
$$a7 = -2[-] \text{——Curvature factor level}$$
$$a8 = 0 \text{——Camber sensitivity of horizontal shift}$$
$$a9 = 0 \text{——Load dependency of horizontal shift}$$
$$a10 = 0 \text{——Horizontal shift level}$$
$$a11 = 0 \text{——Combined load and camber sensitivity of vertical shift}$$
$$a12 = 0 \text{——Load dependency of vertical shift}$$
$$a13 = 0 \text{——Vertical shift level}$$

$$\gamma = 0[rad] \text{——Camber angle}$$
$$\mu_{lateral} = 0.3[-] \text{——Lateral friction coefficient}$$

$$m_T = 1500[kg] \text{———Mass of the vehicle}$$

$$g = 9.81[\frac{m}{s^2}] \text{———Gravity acceleration}$$

$$a = 1.2[m] \text{———Distance between center of vehicle and front wheel}$$

$$b = 1.6[m] \text{———Distance between center of vehicle and rear wheel}$$

$$I_T = 2875[\frac{kgm^2}{s}] \text{———Moment of Inertia of the vehicle}$$

$$C_x = 0.8[-] \text{———Drag coefficient}$$

$$C_z = 1.5[-] \text{———Lift coefficient}$$

$$rho = 1.225[\frac{kg}{m^3}] \text{———Density of air}$$

$$Area = 2[m] \text{———Area of the vehicle on which the air goes through}$$

$$vel_{max} = 50[\frac{m}{s}] \text{———Maximum velocity of the vehicle}$$

$$vel_{init} = 20[\frac{m}{s}] \text{———Initial velocity of the vehicle}$$

$$yPos_{init} = -5.625[m] \text{———Initial y position of the vehicle}$$

$$C_{fuel} = 3x10^-7[\frac{s^2}{m^2}] \text{———Fuel consumption coefficient}$$

$$K_{wear} = 10^-9[-] \text{———Tyre wear parameter}$$

$$H = 60[\frac{N}{m^2}] \text{———Tyre wear parameter}$$

$$TyreContactArea = 0.04[m^2] \text{———Area of the tyre which is in contact to the asphalt}$$

$$TyreRadius = 0.17[m] \text{———Radius of the tyre}$$

$$TyreArea = TyreRadius^2\pi; [m^2] \text{———Area of the tyre}$$

$$TyreWidth = 0.2[m] \text{———Width of the tyre}$$