Dynamic bicycle model

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1 Simple dynamic model

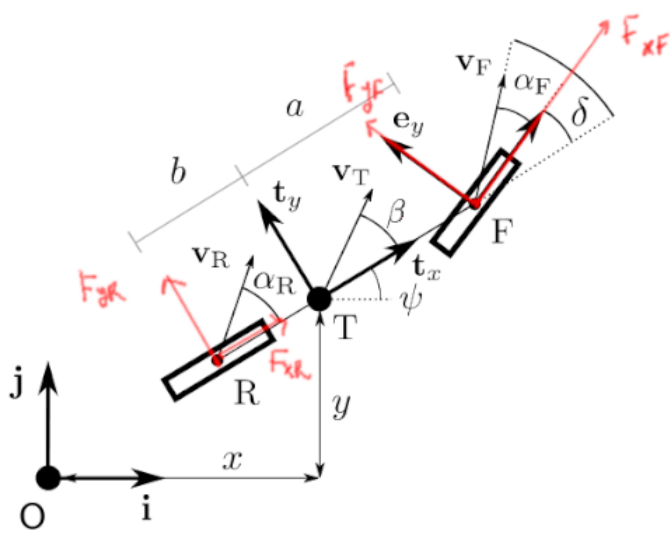

Figure 1: Bicycle model

$$
\begin{aligned}
m_T\ddot{x} &= F_{xF}cos(\psi+\delta) + F_{xR}cos(\psi) - F_{yF}sin(\psi+\delta) - F_{yR}sin(\psi) \\
m_T\ddot{y} &= F_{xF}sin(\psi+\delta) + F_{xR}sin(\psi) + F_{yF}cos(\psi+\delta) + F_{yR}cos(\psi) \\
I_T\ddot{\psi} &= F_{xF}asin(\delta) + F_{yF}acos(\delta) - F_{yR}b
\end{aligned} \tag{1}
$$

$$
\begin{aligned}
\alpha_F &= arctan(\frac{\dot{y}+a\dot{\psi}cos(\psi)}{\dot{x}-a\dot{\psi}sin(\psi)}) - (\delta+\psi) \\
\alpha_R &= arctan(\frac{\dot{y}-b\dot{\psi}cos(\psi)}{\dot{x}+b\dot{\psi}sin(\psi)}) - \psi
\end{aligned} \tag{2}
$$

At first, as state vector this has been used:

$$\begin{aligned} z1 &= x \\ z2 &= y \\ z3 &= \psi \\ z4 &= \dot{x} \\ z5 &= \dot{y} \\ z6 &= \dot{\psi} \end{aligned} \tag{3}$$

So that

$$\begin{aligned} \dot{z_1} &= z_4 \\ \dot{z_2} &= z_5 \\ \dot{z_3} &= z_6 \\ \dot{z_4} &= \frac{F_{xF}cos(z_3+\delta)+F_{xR}cos(z_3)-F_{yF}sin(z_3+\delta)-F_{yR}sin(z_3)}{m_T} \\ \dot{z_5} &= \frac{F_{xF}sin(z_3+\delta)+F_{xR}sin(z_3)+F_{yF}cos(z_3+\delta)+F_{yR}cos(z_3)}{m_T} \\ \dot{z_6} &= \frac{F_{xF}asin(\delta)+F_{yF}acos(\delta)-F_{yR}b}{I_T} \end{aligned} \tag{4}$$

With slip angles

$$\begin{aligned} \alpha_F &= arctan(\frac{z_5+az_6cos(z_3)}{z_4-az_6sin(z_3)})-(\delta+z_3) \\ \alpha_R &= arctan(\frac{z_5-bz_6cos(z_3)}{z_4+bz_6sin(z_3)})-z_3 \end{aligned} \tag{5}$$

Now instead of using $\dot{x}$ and $\dot{y}$, v_T and β have been used. The trasformations are the following:

$$\begin{aligned} \dot{x} &= v_Tcos(\psi+\beta) \\ \dot{y} &= v_Tsin(\psi+\beta) \\ \\ \ddot{x} &= \dot{v_T}cos(\psi+\beta)-v_T(\dot{\psi}+\dot{\beta})sin(\psi+\beta) \\ \ddot{y} &= \dot{v_T}sin(\psi+\beta)+v_T(\dot{\psi}+\dot{\beta})cos(\psi+\beta) \end{aligned} \tag{6}$$

Substituting and simplyfing with the help of Matlab

$$\dot{v_T} = \frac{F_{xF}cos(\beta-\delta) + F_{xR}cos(\beta) + F_{yF}sin(\beta-\delta) + F_{yR}sin(\beta)}{m_T}$$
$$\dot{\beta} = \frac{-F_{xF}sin(\beta-\delta) - F_{xR}sin(\beta) + F_{yF}cos(\beta-\delta) + F_{yR}cos(\beta) - m_T v_T\dot{\psi}}{m_T v_T}$$
$$\ddot{\psi} = \frac{F_{xF}asin(\delta) + F_{yF}acos(\delta) - F_{yR}b}{I_T} \tag{7}$$

$$\alpha_F = arctan(\frac{v_T sin(\beta) + a\dot{\psi}}{v_T cos(\beta)}) - \delta$$
$$\alpha_R = arctan(\frac{v_T sin(\beta) - b\dot{\psi}}{v_T cos(\beta)}) \tag{8}$$

The new state and the state equations are

$$x1 = x$$
$$x2 = y$$
$$x3 = \psi$$
$$x4 = v_T$$
$$x5 = \beta$$
$$x6 = \dot{\psi}$$

$$\dot{x_1} = x_4 cos(x_3 + x_5)$$
$$\dot{x_2} = x_5 sin(x_3 + x_5)$$
$$\dot{x_3} = x_6$$
$$\dot{x_4} = \frac{F_{xF}cos(x_5-\delta) + F_{xR}cos(x_5) + F_{yF}sin(x_5-\delta) + F_{yR}sin(x_5)}{m_T}$$
$$\dot{x_5} = \frac{-F_{xF}sin(x_5-\delta) - F_{xR}sin(x_5) + F_{yF}cos(x_5-\delta) + F_{yR}cos(x_5) - m_T x_4 x_6}{m_T x_4}$$
$$\dot{x_6} = \frac{F_{xF}asin(\delta) + F_{yF}acos(\delta) - F_{yR}b}{I_T} \tag{9}$$

2 Pacejka tyre model

The following Pacejka tyre model (Magic Formula '94) has been used, taking as inputs the tyre slip angle α (α_F and α_R) and the vertical load F_z on the tyre (respectively $l_F * F_z$ and $l_R * F_z$, where l_F and l_R are coefficients to distribute the load between front wheel and rear wheel, such that $l_F + l_R = 1, l_F >= 0, l_R >= 0$).

$$F_y = Dsin(Carctan(B_{x1} - E(B_{x1} - arctan(B_{x1})))) + V \tag{10}$$

With

$$\begin{aligned} C &= a_0 \\ D &= F_z(a_1 F_z + a_2)(1 - a_{15}\gamma^2) \\ BCD &= a_3 sin(2arctan(\frac{F_z}{a_4}))(1 - a_5|\gamma|) \\ B &= BCD/CD \\ E &= (a_6 F_z + a_7)(1 - (a_{16}\gamma + a_{17})sign(\alpha + H)) \\ H &= a_8 F_z + a_9 + a_{10}\gamma \\ V &= a_{11} F_z + a_{12} + (a_{13} F_z + a_{14})\gamma F_z \\ B_{x1} &= B(\alpha + H) \end{aligned} \tag{11}$$

Where a_i, i $\in$ {0,..,17}, are the parameters of the Pacejka model, whose value and meaning can be seen in the Appendix.

3 Aerodynamic force

The following changes have been done in the previous model to take into account for the aerodynamic force $F_A = \frac{1}{2}\rho C_x S v^2$ in the same direction of v_T but in the opposite side, and $F_{Lift} = \frac{1}{2}\rho C_z S v^2$ that "pushes" the vehicle to be sticked on the ground

$$\dot{x_4} = \frac{F_{xF}cos(x_5 - \delta) + F_{xR}cos(x_5) + F_{yF}sin(x_5 - \delta) + F_{yR}sin(x_5) - \frac{1}{2}\rho C_x S x_4^2}{m_T} \tag{12}$$

$$F_z = mg + \frac{1}{2}\rho C_z S x_4^2 \tag{13}$$

4 Fuel consumption

The following changes have been done in the previous model to take fuel consumption into account. A simplified version has been used, in which the Power is computed (P_e) and multiplied by a coefficient (C_{fuel}) that expresses the relation among mass loss (in terms of fuel consumption) and power provided

$$\begin{aligned} P_e = (F_{xF} + F_{xR})x_4, F_{xF} >= 0, F_{xR} >= 0 \\ \dot{m} = P_e C_{fuel} \end{aligned} \tag{14}$$

5 Tyre wear

The following have been added in the previous model to take into account for wear of the rubber compound of the tyre. The model is called Archard model, and it makes use of the vertical pression ($P_z = \frac{F_z}{Area}$), the longitudinal sliding velocity of the wheels (v_{xF} and v_{xR}) and some parameters (such as K_{wear} and H). The model output is the wear depth over the time ($\dot{h}$), that will be then converted into mm^3 of wasted material.
The model has been converted in order to take into account, instead of the sliding velocity, the forces on the wheels, thus the parameters have been re-modulated too.
Here the modified Archard model formulation is shown

$$\begin{gathered} \dot{h}_i = \frac{K_{wear} P_{load} \sqrt{F_{xi}^2 + F_{yi}^2}}{H} \\ \text{where i} \in \{\text{F, R}\} \end{gathered} \tag{15}$$

6 Banking

Taking into account the shape of the road we introduce other terms in the model equations. As can be seen in figure 2 and 3 we are able to find the term whose projection will be summed up in the previous dynamic equations, that is the mg term, that multiplied by $sin(\gamma)$ will be directed as the perpendicular to the vehicle direction.

As can be seen from those figures, the contribution of this lateral force acting on the vehicle will add some new terms in the equations of the model. Along the direction of v_T the contribution of the force is $mgsin(\gamma)sin(\beta)$, while on the orthogonal direction it is represented by the force $mgsin(\gamma)cos(\beta)$.

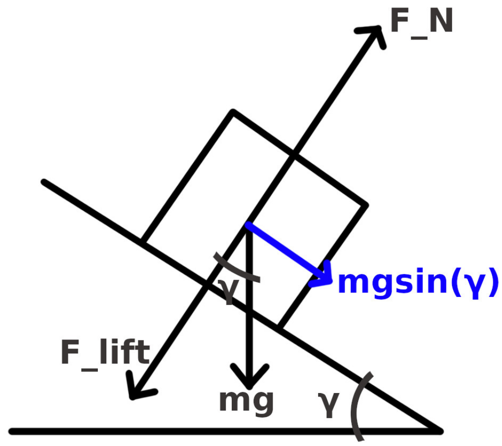

Figure 2: Contribution of mg

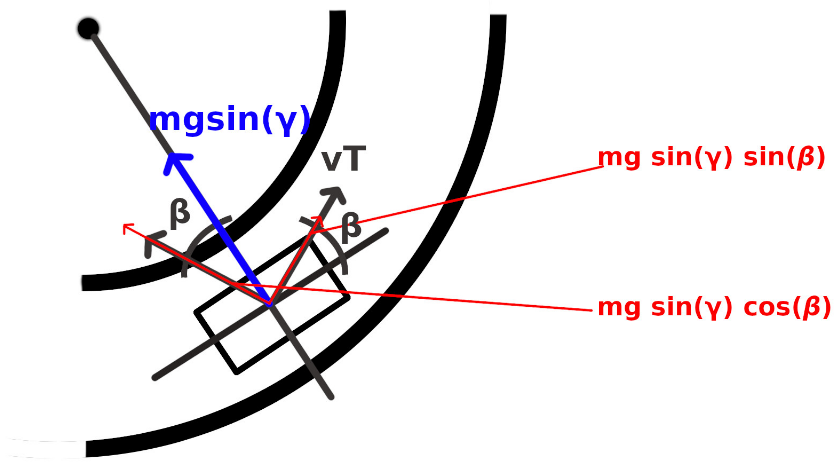

Figure 3: Terms affecting the model equations

In the end, the final equations will be:

$$\dot{x_4} = \frac{F_{xF}cos(x_5-\delta) + F_{xR}cos(x_5) + F_{yF}sin(x_5-\delta) + F_{yR}sin(x_5) - \frac{1}{2}\rho C_x S v^2 + mgsin(\gamma)sin(x_5)}{m_T}$$

$$\dot{x_5} = \frac{-F_{xF}sin(x_5-\delta) - F_{xR}sin(x_5) + F_{yF}cos(x_5-\delta) + F_{yR}cos(x_5) - m_T x_4 x_6 + mgsin(\gamma)cos(x_5)}{m_T x_4} \tag{16}$$

We also noticed that with banking the contribution of the vertical load to the normal force is multiplied by a $cos(\gamma)$ factor:

$$F_z = mg\,cos(\gamma) + \frac{1}{2}\rho C_z S x_4^2 \tag{17}$$

7 Friction ellipse and wear considerations

Coming to the conclusion of our model, we took in consideration also the physical relation between longitudinal and lateral forces through the friction ellipse

$$(\frac{F_x}{F_{x,max}})^2 + (\frac{F_y}{F_{y,max}})^2 = 1 \tag{18}$$

In particular, $F_{x,max}$ and $F_{y,max}$ are respectively the maximum longitudinal and lateral forces, that are are calculated through the Pacejka parameters, being $D + V$ the point of max of the tyre model. Here we recall that:

$$\begin{aligned} D_{lat} &= F_z(a_1 F_z + a_2)(1 - a_{15}\gamma^2) \\ V_{lat} &= a_{11}F_z + a_{12} + (a_{13}F_z + a_{14})\gamma F_z \\ \\ D_{long} &= F_z(b_1 F_z + b_2) \\ V_{long} &= b_{11}F_z + b_{12} \end{aligned} \tag{19}$$

Thus as we can see, the maximum longitudinal and lateral forces are functions of the vertical load F_z.
(Note: different parameters are used for longitudinal and lateral Pacejka, in particular a_i is referred to the lateral one while b_i to the longitudinal one).

In this way the ellipse is defined, but, taking in consideration the wear h the ellipse is scaled. Thus, at the end, $F_{x,max}$ and $F_{y,max}$ are functions of F_z and h, specifically:

$$\begin{aligned} F_{x,max} &= (D_{long} + V_{long})\frac{1}{w_1 h + w_2} \\ F_{y,max} &= (D_{lat} + V_{lat})\frac{1}{w_1 h + w_2} \end{aligned} \tag{20}$$

Where w_1 and w_2 are parameters opportunely chosen. In this way the more the wear, the more the ellipse is shrunk.

The ellipse is saying us which is the maximum lateral force wrt the given F_x in input. Thus, finally, the output of the lateral Pacejka is scaled, so to have the peak value $(D_{lat} + V_{lat})$ equal to the value given by the ellipse. To do so, the D value of the Pacejka is directly fed as input to the model through the output of

the ellipse.
Following image tries to clarify the steps detailed so far.

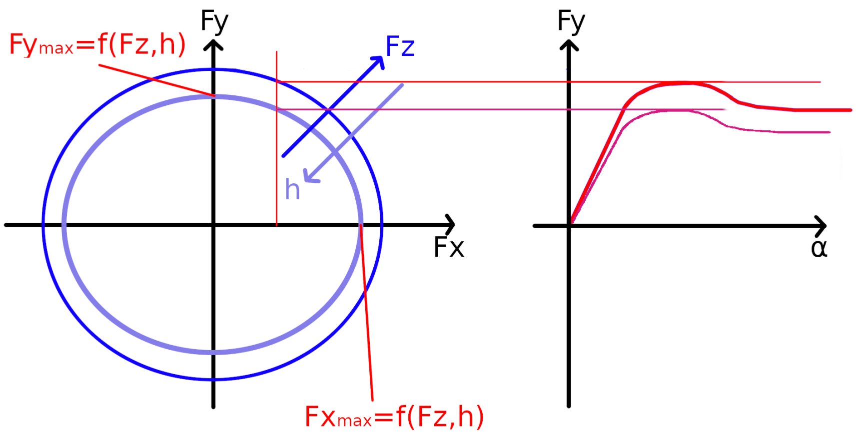

Figure 4: Friction ellipse and wear effects

8 Appendix

In this section we will define the values given to the different parameters.

$$\begin{aligned}
&\text{Parameters of lateral Pacejka tyre model}\\
a0 = 1.4[-]&\text{———Shape factor}\\
a1 = 0[1/kN]&\text{———Load influence on lateral friction coefficient (*1000)}\\
a2 = 500[-]&\text{———Lateral friction coefficient (*1000)}\\
a3 = 1100[N/deg]&\text{———Change of stiffness with slip}\\
a4 = 10[kN]&\text{———Change of progressivity of stiffness / load}\\
a5 = 0[\%/deg/100]&\text{———Camber influence on stiffness}\\
a6 = 0[-]&\text{———Curvature change with load}\\
a7 = -2[-]&\text{———Curvature factor}\\
a8 = 0[deg/kN]&\text{———Load influence on horizontal shift}\\
a9 = 0[deg]&\text{———Horizontal shift at load = 0 and camber = 0}\\
a10 = 0[-]&\text{———Camber influence on horizontal shift}\\
a11 = 0[N]&\text{———Vertical shift}\\
a12 = 0[N]&\text{———Vertical shift at load = 0}\\
a13 = 0[N/deg/kN]&\text{———Camber influence on vertical shift, load dependent}\\
a14 = 0[N/deg]&\text{———Camber influence on vertical shift}\\
a15 = 0[1/deg]&\text{———Camber influence on lateral friction coefficient}\\
a16 = 0[-]&\text{———Curvature change with camber}\\
a17 = 0[-]&\text{——— Curvature shift}\\
\\
\gamma = 0[rad]&\text{———Camber angle}\\
\\
&\text{Parameters of longitudinal tyre model - only the ones used}\\
b1 = 0[1/kN]&\text{———Load influence on longitudinal friction coefficient (*1000)}\\
b2 = 600[-]&\text{———Longitudinal friction coefficient (*1000)}\\
b11 = 0[N]&\text{———Vertical shift}\\
b12 = 0[N]&\text{———Vertical shift at load = 0}
\end{aligned}$$

$$m_T = 1500[kg]\text{———Mass of the vehicle}$$
$$g = 9.81[\frac{m}{s^2}]\text{———Gravity acceleration}$$
$$l_F = 0.48[-]\text{———Distribution of load on the front wheel}$$
$$l_R = 0.52[-]\text{———Distribution of load on the rear wheel}$$
$$a = 1.2[m]\text{———Distance between center of vehicle and front wheel}$$
$$b = 1.6[m]\text{———Distance between center of vehicle and rear wheel}$$
$$I_T = 2875[\frac{kgm^2}{s}]\text{———Moment of Inertia of the vehicle}$$

$$C_x = 0.8[-]\text{———Drag coefficient}$$
$$C_z = 1.5[-]\text{———Lift coefficient}$$
$$rho = 1.225[\frac{kg}{m^3}]\text{———Density of air}$$
$$Area = 2[m]\text{———Area on which the air goes through}$$

$$vel_{max} = 50[\frac{m}{s}]\text{———Maximum velocity of the vehicle}$$
$$vel_{init} = 20[\frac{m}{s}]\text{———Initial velocity of the vehicle}$$
$$yPos_{init} = -5.625[m]\text{———Initial y position of the vehicle}$$

$$C_{fuel} = 3x10^-7[\frac{s^2}{m^2}]\text{———Fuel consumption coefficient}$$

$$K_{wear} = 10^-9[-?]\text{———Tyre wear parameter}$$
$$H = 10^-9[-?]\text{———Tyre wear parameter}$$

$$TyreContactArea = 0.04[m^2]\text{———Contact area of the tyre}$$
$$TyreRadius = 0.17[m]\text{———Radius of the tyre}$$
$$TyreArea = TyreRadius^2\pi; [m^2]\text{———Area of the tyre}$$
$$TyreWidth = 0.2[m]\text{———Width of the tyre}$$

$$w_1 = 10^{-4.7}\text{———Parameter to scale ellipse due to wear}$$
$$w_2 = 1\text{———Parameter to scale ellipse due to wear}$$