Dynamic bicycle model

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July 16, 2020

1 Simple dynamic model

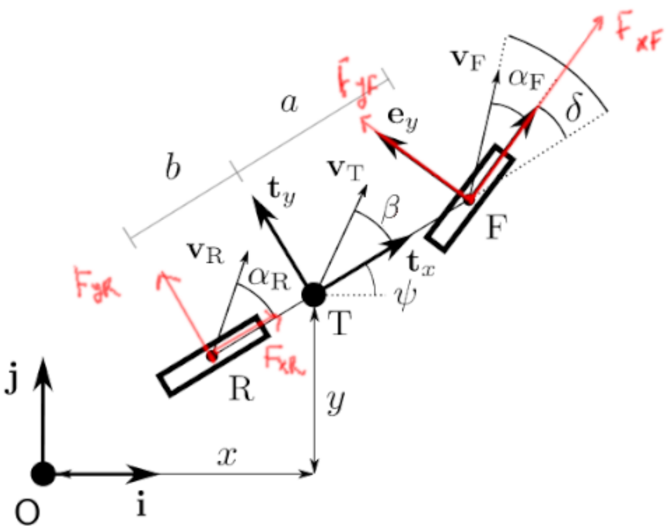

Figure 1: Bicycle model

$$
\begin{aligned}
m_T\ddot{x} &= F_{xF}cos(\psi+\delta) + F_{xR}cos(\psi) - F_{yF}sin(\psi+\delta) - F_{yR}sin(\psi) \\
m_T\ddot{y} &= F_{xF}sin(\psi+\delta) + F_{xR}sin(\psi) + F_{yF}cos(\psi+\delta) + F_{yR}cos(\psi) \\
I_T\ddot{\psi} &= F_{xF}asin(\delta) + F_{yF}acos(\delta) - F_{yR}b
\end{aligned}
\tag{1}
$$

$$
\begin{aligned}
\alpha_F &= arctan(\frac{\dot{y}+a\dot{\psi}cos(\psi)}{\dot{x}-a\dot{\psi}sin(\psi)}) - (\delta+\psi) \\
\alpha_R &= arctan(\frac{\dot{y}-b\dot{\psi}cos(\psi)}{\dot{x}+b\dot{\psi}sin(\psi)}) - \psi
\end{aligned}
\tag{2}
$$

At first, as state vector this has been used:

$$\begin{aligned} z1 &= x \\ z2 &= y \\ z3 &= \psi \\ z4 &= \dot{x} \\ z5 &= \dot{y} \\ z6 &= \dot{\psi} \end{aligned} \tag{3}$$

So that

$$\begin{aligned} \dot{z_1} &= z_4 \\ \dot{z_2} &= z_5 \\ \dot{z_3} &= z_6 \\ \dot{z_4} &= \frac{F_{xF}cos(z_3+\delta) + F_{xR}cos(z_3) - F_{yF}sin(z_3+\delta) - F_{yR}sin(z_3)}{m_T} \\ \dot{z_5} &= \frac{F_{xF}sin(z_3+\delta) + F_{xR}sin(z_3) + F_{yF}cos(z_3+\delta) + F_{yR}cos(z_3)}{m_T} \\ \dot{z_6} &= \frac{F_{xF}asin(\delta) + F_{yF}acos(\delta) - F_{yR}b}{I_T} \end{aligned} \tag{4}$$

With slip angles

$$\begin{aligned} \alpha_F &= arctan(\frac{z_5 + az_6cos(z_3)}{z_4 - az_6sin(z_3)}) - (\delta + z_3) \\ \alpha_R &= arctan(\frac{z_5 - bz_6cos(z_3)}{z_4 + bz_6sin(z_3)}) - z_3 \end{aligned} \tag{5}$$

Now instead of using $\dot{x}$ and $\dot{y}$, v_T and β have been used. The trasformations are the following:

$$\begin{aligned} \dot{x} &= v_T cos(\psi+\beta) \\ \dot{y} &= v_T sin(\psi+\beta) \\ \\ \ddot{x} &= \dot{v_T}cos(\psi+\beta) - v_T(\dot{\psi}+\dot{\beta})sin(\psi+\beta) \\ \ddot{y} &= \dot{v_T}sin(\psi+\beta) + v_T(\dot{\psi}+\dot{\beta})cos(\psi+\beta) \end{aligned} \tag{6}$$

Substituting and simplyfing with the help of Matlab

$$
\begin{aligned}
\dot{v_T} &= \frac{F_{xF}cos(\beta-\delta)+F_{xR}cos(\beta)+F_{yF}sin(\beta-\delta)+F_{yR}sin(\beta)}{m_T} \\
\dot{\beta} &= \frac{-F_{xF}sin(\beta-\delta)-F_{xR}sin(\beta)+F_{yF}cos(\beta-\delta)+F_{yR}cos(\beta)-m_Tv_T\dot{\psi}}{m_Tv_T} \\
\ddot{\psi} &= \frac{F_{xF}asin(\delta)+F_{yF}acos(\delta)-F_{yR}b}{I_T}
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
\alpha_F &= arctan(\frac{v_Tsin(\beta)+a\dot{\psi}}{v_Tcos(\beta)})-\delta \\
\alpha_R &= arctan(\frac{v_Tsin(\beta)-b\dot{\psi}}{v_Tcos(\beta)})
\end{aligned}
\tag{8}
$$

The new state and the state equations are

$$
\begin{aligned}
x1 &= x \\
x2 &= y \\
x3 &= \psi \\
x4 &= v_T \\
x5 &= \beta \\
x6 &= \dot{\psi}
\end{aligned}
$$

$$
\begin{aligned}
\dot{x_1} &= x_4cos(x_3+x_5) \\
\dot{x_2} &= x_5sin(x_3+x_5) \\
\dot{x_3} &= x_6 \\
\dot{x_4} &= \frac{F_{xF}cos(x_5-\delta)+F_{xR}cos(x_5)+F_{yF}sin(x_5-\delta)+F_{yR}sin(x_5)}{m_T} \\
\dot{x_5} &= \frac{-F_{xF}sin(x_5-\delta)-F_{xR}sin(x_5)+F_{yF}cos(x_5-\delta)+F_{yR}cos(x_5)-m_Tx_4x_6}{m_Tx_4} \\
\dot{x_6} &= \frac{F_{xF}asin(\delta)+F_{yF}acos(\delta)-F_{yR}b}{I_T}
\end{aligned}
\tag{9}
$$

2 Pacejka tyre model

The following Pacejka tyre model (Magic Formula) has been used, taking as inputs the tyre slip angle α (α_F and α_R), the vertical load F_z on the tyre (respectively $l_F * F_z$ and $l_R * F_z$, where l_F and l_R are coefficients to distribute the load between front wheel and rear wheel, such that $l_F + l_R = 1, l_F >= 0, l_R >= 0$) and the lateral friction coefficient μ_y

$$F_y = -\frac{\mu_y}{\mu_{y0}}(F_{y0} + Sv) \tag{10}$$

With

$$\begin{aligned} \mu_{y0} &= a_1 F_z + a_2 \\ F_{y0} &= Dsin(Carctan(B\alpha_{eq} - E(B\alpha_{eq} - arctan(B\alpha_{eq})))) \\ \alpha_{eq} &= \frac{\mu_{y0}}{\mu_y}(\alpha + S_h) \\ C &= a_0 \\ D &= \mu_{y0} F_z = (a_1 F_z + a_2) F_z \\ BCD &= a_3 sin(2arctan(\frac{F_z}{a_4}))(1 - a_5|\gamma|) \\ B &= BCD/CD \\ E &= a_6 F_z + a_7 \\ S_h &= a_8\gamma + a_9 F_z + a_{10} \\ S_v &= a_{11} F_z \gamma + a_{12} F_z + a_{13} \end{aligned} \tag{11}$$

Where a_i, i $\in$ {0,..,13}, are the parameters of the Pacejka model, whose value and meaning can be seen in the Appendix.

3 Aerodynamic force

The following changes have been done in the previous model to take into account for the aerodynamic force $F_A = \frac{1}{2}\rho C_x S v^2$ in the same direction of v_T but in the opposite side, and $F_{Lift} = \frac{1}{2}\rho C_z S v^2$ that "pushes" the vehicle to be sticked on the ground

$$\dot{x_4} = \frac{F_{xF} cos(x_5 - \delta) + F_{xR} cos(x_5) + F_{yF} sin(x_5 - \delta) + F_{yR} sin(x_5) - \frac{1}{2}\rho C_x S x_4^2}{m_T} \tag{12}$$

..

..

$$F_z = mg + \frac{1}{2}\rho C_z S x_4^2 \tag{13}$$

4 Fuel consumption

The following changes have been done in the previous model to take fuel consumption into account. A simplified version has been used, in which the Power is computed (P_e) and multiplied by a coefficient (C_{fuel}) that expresses the relation among mass loss (in terms of fuel consumption) and power provided

$$\begin{aligned} P_e = (F_{xF} + F_{xR})x_4, F_{xF} >= 0, F_{xR} >= 0 \\ \dot{m} = P_e C_{fuel} \end{aligned} \tag{14}$$

5 Tyre wear

The following have been added in the previous model to take into account for wear of the rubber compound of the tyre. The model is called Archard model, and it makes use of the vertical pression ($P_z = \frac{F_z}{Area}$), the longitudinal sliding velocity of the wheels (v_{xF} and v_{xR}) and some parameters (such as K_{wear} and H). The model output is the wear depth over the time ($\dot{h}$), that will be then converted into mm^3 of wasted material.
The model has been converted in order to take into account, instead of the sliding velocity, the forces on the wheels, thus the parameters have been remodulated too.
Here the modified Archard model formulation is shown

$$\begin{gathered} \dot{h_i} = \frac{K_{wear} P_{load} \sqrt{F_{xi}^2 + F_{yi}^2}}{H} \\ \text{where i} \in \{\text{F, R}\} \end{gathered} \tag{15}$$

6 Banking

Taking into account the shape of the road we introduce other terms in the model equations. As can be seen in figure 2 and 3 we are able to find the term whose projection will be summed up in the previous dynamic equations, that is the mg term, that multiplied by $sin(\gamma)$ will be directed as the perpendicular to the vehicle direction.

As can be seen from those figures, the contribution of this lateral force acting on the vehicle will add some new terms in the equations of the model. Along the direction of v_T the contribution of the force is $mgsin(\gamma)sin(\beta)$, while on the orthogonal direction it is represented by the force $mgsin(\gamma)cos(\beta)$.

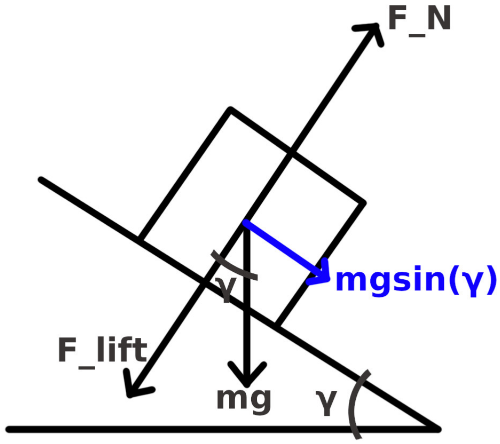

Figure 2: Contribution of mg

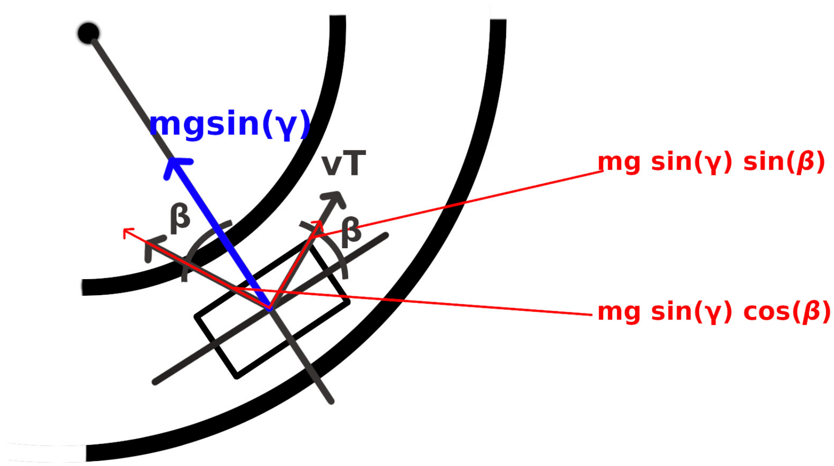

Figure 3: Terms affecting the model equations

In the end, the final equations will be:

$$
\begin{aligned}
\dot{x_4} &= \frac{F_{xF}cos(x_5-\delta)+F_{xR}cos(x_5)+F_{yF}sin(x_5-\delta)+F_{yR}sin(x_5)-\frac{1}{2}\rho C_x S v^2 \color{red}{+ mgsin(\gamma)sin(x_5)}}{m_T} \\
\dot{x_5} &= \frac{-F_{xF}sin(x_5-\delta)-F_{xR}sin(x_5)+F_{yF}cos(x_5-\delta)+F_{yR}cos(x_5)-m_T x_4 x_6 \color{red}{+ mgsin(\gamma)cos(x_5)}}{6 \qquad m_T x_4}
\end{aligned}
\tag{16}
$$

We also noticed that with banking the contribution of the vertical load to the normal force is multiplied by a $cos(\gamma)$ factor:

$$F_z = mg\,cos(\gamma) + \frac{1}{2}\rho C_z S x_4^2 \tag{17}$$

7 Appendix

In this section we will define the values given to the different parameters.

$$a0 = 1.4[-]\text{———Shape factor}$$
$$a1 = 0[1/kN]\text{———Load dependency of lateral friction (*1000)}$$
$$a2 = 1100[-]\text{———Lateral friction level (*1000)}$$
$$a3 = 1100[N/deg]\text{———Maximum cornering stiffness}$$
$$a4 = 10[kN]\text{———Load at maximum cornering stiffness}$$
$$a5 = 0\text{———Camber sensitivity of cornering stiffness}$$
$$a6 = 0\text{———Load dependency of curvature factor}$$
$$a7 = -2[-]\text{———Curvature factor level}$$
$$a8 = 0\text{———Camber sensitivity of horizontal shift}$$
$$a9 = 0\text{———Load dependency of horizontal shift}$$
$$a10 = 0\text{———Horizontal shift level}$$
$$a11 = 0\text{———Combined load and camber sensitivity of vertical shift}$$
$$a12 = 0\text{———Load dependency of vertical shift}$$
$$a13 = 0\text{———Vertical shift level}$$

$$\gamma = 0[rad]\text{———Camber angle}$$
$$\mu_{lateral} = 0.3[-]\text{———Lateral friction coefficient}$$

$$m_T = 1500[kg]\text{———Mass of the vehicle}$$
$$g = 9.81[\frac{m}{s^2}]\text{———Gravity acceleration}$$
$$l_F = 0.48[-]\text{———Distribution of load on the front wheel}$$
$$l_R = 0.52[-]\text{———Distribution of load on the rear wheel}$$
$$a = 1.2[m]\text{———Distance between center of vehicle and front wheel}$$
$$b = 1.6[m]\text{———Distance between center of vehicle and rear wheel}$$
$$I_T = 2875[\frac{kgm^2}{s}]\text{———Moment of Inertia of the vehicle}$$

$$C_x = 0.8[-] \text{———Drag coefficient}$$
$$C_z = 1.5[-] \text{———Lift coefficient}$$
$$rho = 1.225[\frac{kg}{m^3}] \text{———Density of air}$$
$$Area = 2[m] \text{———Area of the vehicle on which the air goes through}$$

$$vel_{max} = 50[\frac{m}{s}] \text{———Maximum velocity of the vehicle}$$
$$vel_{init} = 20[\frac{m}{s}] \text{———Initial velocity of the vehicle}$$
$$yPos_{init} = -5.625[m] \text{———Initial y position of the vehicle}$$

$$C_{fuel} = 3x10^-7[\frac{s^2}{m^2}] \text{———Fuel consumption coefficient}$$

$$K_{wear} = 10^-9[-?] \text{———Tyre wear parameter}$$
$$H = 10^-9[-?] \text{———Tyre wear parameter}$$

$$TyreContactArea = 0.04[m^2] \text{———Area of the tyre which is in contact to the asphalt}$$
$$TyreRadius = 0.17[m] \text{———Radius of the tyre}$$
$$TyreArea = TyreRadius^2\pi; [m^2] \text{———Area of the tyre}$$
$$TyreWidth = 0.2[m] \text{———Width of the tyre}$$