Linear Algebraic Representation

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A novel representation scheme

Evolution of geometric representations

Past: from boundary to interior

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- our algebraic models are very compact, and there is no overhead to answer topological queries w.r.t. traditional graph-based methods
- Advanced implementations on modern hardware use open standards for parallel programming of heterogeneous systems (OpenCL)

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Provide a novel computer representation of big geometric data for the IEEE P3333.2 Standard for 3D medical

- usable with both images and geometric data

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- The 0-skeleton coincides with the set $V(P)$ of vertices of P.

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- A CW-structure on the space X is a filtration

$$\emptyset = X^{-1} \subset X_0 \subset X_1 \subset \cdots \subset X = \cup_n X_n,$$

such that, for each n, the space X_n is homeomorphic to a space obtained from X_{n-1} by attachment of n-cells of X in $\Lambda_n = \Lambda_n(X)$.

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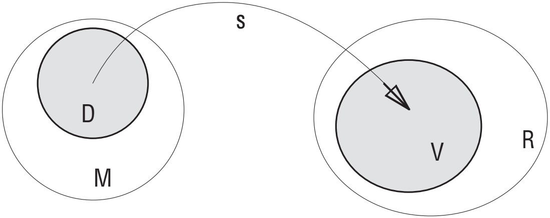
- A cellular complex is finite when it contains a finite number of cells.

Representation scheme

mapping $s : M \to R$ from a space M of mathematical models to a space R of computer representations



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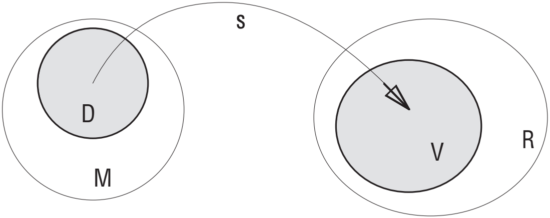

1. The M set contains the mathematical models of the class of solid objects the scheme aims to represent
2. The R set contains the symbolic representations, i.e. the proper data structures, built according to a suitable grammar

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Definition (Computer representations)

The LAR codomain is the set R of d-tuples of CSR sparse binary matrices[a]

$$\mathrm{CSR}(M_n^m(\mathbb{Z}_2)), \qquad m = \sharp\Lambda_p, \quad n = \sharp\Lambda_0,$$

[a] Compressed Sparse Row (CSR) format, for which efficient implementations on high-performance hardware exist. See [?] and [?].

Linear Algebraic Representation (LAR) scheme

may support (at least in principle) all topological and geometric queries and constructions that may be asked of the corresponding model

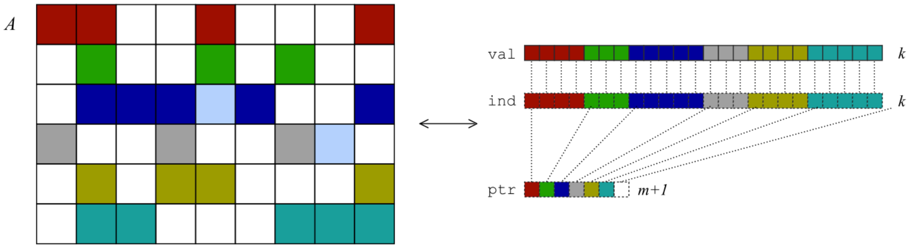
Remark (The interesting point)

for a given cellular d-complex $\Lambda(X)$, all of the M_p matrices $(1 \leq p \leq d)$ have the same number of columns

This simple fact provides a convenient tool (matrix algebra) for computing boundary and coboundary operators and topological relations between cells.

Sparse binary matrices

Compressed Sparse Row (CSR) storage




```
// Basic SpMV implementation,
// y <- y + A*x, where A is in CSR.
 for (i = 0; i < m; ++i) {
   double y0 = y[i];
   for (k = ptr[i]; k < ptr[i+1]; ++k)
     y0 += val[k] * x[ind[k]];
   y[i] = y0;
 }
```

a basic CSR-based SpMV implementation

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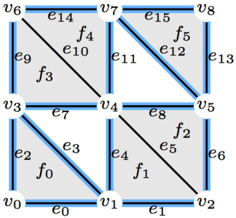

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(ex: simplicial or cuboidal)
only the CSR(M_d) is needed:

	$M_1^t \quad \Leftarrow \quad M_q^t \quad \Leftarrow \quad M_d^t$
M_1	$\Downarrow$
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M_d	

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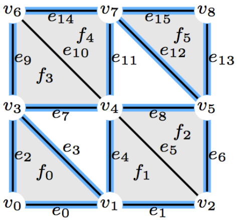

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$M_1^tM_1 = \mathtt{VV} =$
```
[[0,1,3],
 [0,1,2,3,4],
 [1,2,4,5],
 [0,1,3,4,6],
 [1,2,3,4,5,6,7],
 [2,4,5,7,8],
 [3,4,6,7],
 [4,5,6,7,8],
 [5,7,8]]
```

$M_1^t = \mathtt{VE} =$
```
[[0,2],
 [0,1,3,4],
 [1,5,6],
 [2,3,7,9],
 [4,5,7,8,10,11],
 [6,8,12,13],
 [9,10,14],
 [11,12,14,15],
 [13,15]]
```

$M_2^t = \mathtt{VF} =$
```
[[0],
 [0,1],
 [1,2],
 [0,3],
 [1,2,3,4],
 [2,5],
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```

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```
[[0,1],
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 [5,7],
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$M_1M_1^t = \mathtt{EE} =$
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LAR computations examples

download `https://github.com/cvdlab/larpy`

Linear Algebraic Representation in Python

Remarks about LAR

- A large class of geometric representations involve incidence structures", informally called topology" in boundary representations, but also scenes in graphics, assembly and tolerance graphs, FE meshes, multimaterial structures, and so on

- Incidence structures are logical/symbolic (as opposed to numeric and geometric) relationships We can broadly refer to them as topological (incidence) structures.

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  3. they are not suitable for parallel processing, cloud, etc.
- At the other extreme, are purely geometric/numerical representations, such as point clouds, polygons, triangles
- They can be compressed and efficiently processed, but they do not deal with semantics, logical data, and are intrinsically non-robust (in a numerical sense) So, even if we get a 3D JPEG, it will not be sufficient for advanced applications.

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- LAR enjoys a neat mathematical format, being based on chains (the domains of discrete integration) and cochains (the discrete prototype of differential forms).

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- We can demonstrate how LAR can be used with
  - convex decompositions,

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- Due to its wide coverage and high simplicity, LAR may be proposed as a new standard for topological incidence structures—they can be used with any and all geometric representations, and can be applied in all of the above applications

- We can demonstrate how LAR can be used with
  - convex decompositions,
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Remarks about LAR

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- We can demonstrate how LAR can be used with
  - convex decompositions,
  - B'ezier curves and surfaces, or
  - images because it can be used with different geometric embeddings and put on top of any metric structure.

Chain and cochain complex

applies to most domains characterized as cell complexes, without any restrictions on their type, dimension, codimension, orientability, manifoldness, and connectedness

$$\begin{tikzcd}
\cdots & C^3 \arrow[l, "\delta^3"'] & C^2 \arrow[l, "\delta^2"'] & C^1 \arrow[l, "\delta^1"'] & C^0 \arrow[l, "\delta^0"'] \\
\cdots \arrow[r, "\partial_4"] & C_3 \arrow[u, "\cong"'] \arrow[r, "\partial_3"] & C_2 \arrow[u, "\cong"'] \arrow[r, "\partial_2"] & C_1 \arrow[u, "\cong"'] \arrow[r, "\partial_1"] & C_0 \arrow[u, "\cong"']
\end{tikzcd}$$

Definition (Chain complex)

Chain complex is a sequence of Abelian groups $\ldots, C_3, C_2, C_1, C_0$ connected by homomorphisms (boundary operators) $\partial_n : C_n \to C_{n-1}$, such that for all n:

$$\partial_{n-1} \circ \partial_n = 0$$

Definition (Cochain complex)

Cochain complex is a sequence of Abelian groups $C^0, C^1, C^2, C^3, \ldots$ connected by homomorphisms (coboundary operators) $\delta^n : C^n \to C^{n+1}$, such that for all n:

$$\delta^{n+1} \circ \delta^n = 0$$

[6] A. DiCarlo, F. Milicchio, A. Paoluzzi, and V. Shapiro. Chain-Based Representations for Solid and Physical Modeling. 2009. IEEE Transactions on Automation Science and Engineering

Incidence relations7 vs linear operators

Change of paradigm in shape representation

Use symbols V, E, F for K_0, K_1, K_2, the bases of linear spaces C_0, C_1, C_2 of chains in 2D.

Incidence/adjacency relations:
$$XY \subset X \times Y$$

	V	E	F
V	VV	VE	VF
E	EV	EE	EF
F	FV	FE	FF

Linear operators:
$$\mathcal{XY} : \mathcal{Y} \to \mathcal{X}$$

$$\begin{tikzcd}
C_0 \arrow[r, Rightarrow, "\mathcal{FV}"] & C_2 \arrow[d, "\mathcal{EF}"] \arrow[dl, "\mathcal{VF}"'] \\
C_0 \arrow[r, "\mathcal{EV}"'] & C_1
\end{tikzcd}$$

$$\mathcal{EF} = \mathcal{EV} \circ \mathcal{VF} = \mathcal{EV} \circ \mathcal{FV}^\top$$

[7] See: Woo, A combinatorial analysis of boundary data structure schemata, IEEE CG&A, 1985

2D simplicial complex (non manifold pointset)

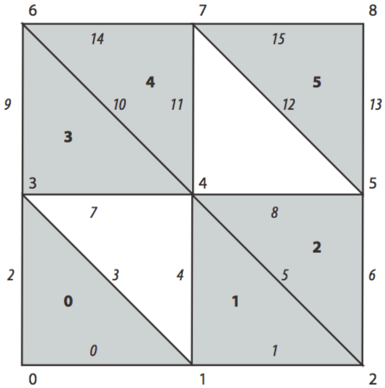
$K = K_0 \cup K_1 \cup K_2$ $\qquad \#K_0 =: k_0 = 9; \quad \#K_1 =: k_1 = 16; \quad \#K_2 =: k_2 = 6$

1-cells by 0-cells

```
EV = [[0,1],
      [1,2],
      [0,3],
      [1,3],
      [1,4],
      [2,4],
      [2,5],
      [3,4],
      [4,5],
      [3,6],
      [4,6],
      [4,7],
      [5,7],
      [5,8],
      [6,7],
      [7,8]]
```

2-cells by 0-cells

```
FV = [[0,1,3],
      [1,2,4],
      [2,4,5],
      [3,4,6],
      [4,6,7],
      [5,7,8]]
```



Example

2D non-manifold simplicial complex

```
V = [[0,0],[1,0],[2,0],[0,1],[1,1],[2,1],[0,2],[1,2],[2,2]]
FV = [[0,1,3],[1,2,4],[2,4,5],[3,4,6],[4,6,7],[5,7,8]]
```

faces extraction

```
model = (V,FV)
V,faces = larSkeletons(model,dim=2,grid=False)
F0V, F1V, F2V = faces
```

extruded 3D model

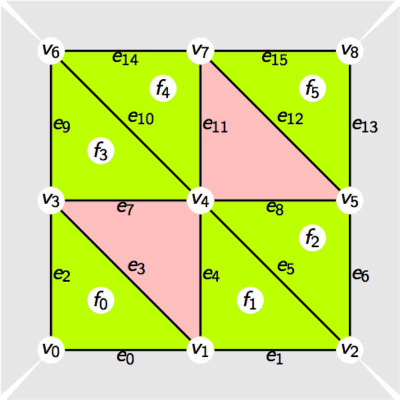
```
model = larExtrude((V,FV),2*[1,2,-3])
VIEW(EXPLODE(1,1,1.2)(MKPOLS(model)))
```

Extraction of facets

start from the characteristic matrix M_d of a cellular partition Λ_d of $\mathbb{E}^d$, empty cells included

$2\text{D} \Rightarrow d = 2$

$$M_2 = \begin{pmatrix}
1&1&0&1&0&0&0&0&0\\
0&1&1&0&1&0&0&0&0\\
0&0&1&0&1&1&0&0&0\\
0&0&0&1&1&0&1&0&0\\
0&0&0&0&1&0&1&1&0\\
0&0&0&0&0&1&0&1&1\\
0&1&0&1&1&0&0&0&0\\
0&0&0&0&1&1&0&1&0\\
1&1&1&0&0&0&0&0&0\\
0&0&1&0&0&1&0&0&1\\
0&0&0&0&0&0&1&1&1\\
1&0&0&1&0&0&1&0&0
\end{pmatrix}$$



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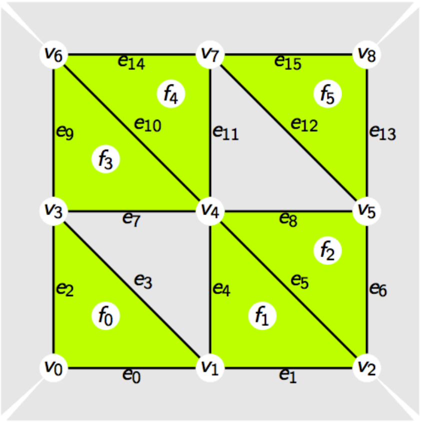
$$M_2 = \begin{pmatrix} 1\,1\,0\,1\,0\,0\,0\,0\,0 \\ 0\,1\,1\,0\,1\,0\,0\,0\,0 \\ 0\,0\,1\,0\,1\,1\,0\,0\,0 \\ 0\,0\,0\,1\,1\,0\,1\,0\,0 \\ 0\,0\,0\,0\,1\,0\,1\,1\,0 \\ 0\,0\,0\,0\,0\,1\,0\,1\,1 \\ 0\,1\,0\,1\,1\,0\,0\,0\,0 \\ 0\,0\,0\,0\,1\,1\,0\,1\,0 \\ 1\,1\,1\,0\,0\,0\,0\,0\,0 \\ 0\,0\,1\,0\,0\,1\,0\,0\,1 \\ 0\,0\,0\,0\,0\,0\,1\,1\,1 \\ 1\,0\,0\,1\,0\,0\,1\,0\,0 \end{pmatrix}$$



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start from the characteristic matrix M_d of a cellular partition Λ_d of $\mathbb{E}^d$, empty cells included

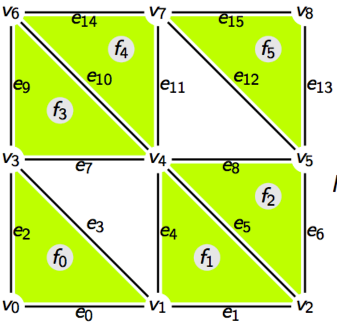
$$A = M_2 M_2^t = \begin{pmatrix}
3 & 1 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 2 & 2 & 0 \\
 & 3 & 2 & 1 & 1 & 0 & 2 & 1 & 0 & 0 & 2 & 1 \\
 & & 3 & 1 & 1 & 1 & 1 & 2 & 0 & 0 & 1 & 2 \\
 & & & 3 & 2 & 0 & 0 & 0 & 1 & 2 & 2 & 1 \\
 & & & & 3 & 1 & 0 & 0 & 2 & 1 & 1 & 2 \\
 & & & & & 3 & 0 & 2 & 2 & 0 & 0 & 2 \\
 & & & & & & 3 & 1 & 0 & 1 & 1 & 0 \\
 & & & & & & & 3 & 1 & 0 & 0 & 1 \\
 & & & sym & & & & & 3 & 1 & 0 & 1 \\
 & & & & & & & & & 3 & 1 & 0 \\
 & & & & & & & & & & 3 & 1 \\
 & & & & & & & & & & & 3
\end{pmatrix}$$

$$\sharp(\lambda_d^i \cap \lambda_d^j) = A(i,j) \geq d \quad (i \neq j) \ \Rightarrow \ \exists\ \lambda_{d-1} = M_d(i) \wedge M_d(j)$$

Extraction of facets

start from the characteristic matrix M_d of a cellular partition Λ_d of $\mathbb{E}^d$, empty cells included

$$A(0,6) = 2 \quad \Rightarrow \quad (110100000) \wedge (010110000) = (010100000) \quad \Rightarrow \quad e = (v_1, v_3)$$



$$M_2 = \begin{pmatrix}
1&1&0&1&0&0&0&0&0\\
0&1&1&0&1&0&0&0&0\\
0&0&1&0&1&1&0&0&0\\
0&0&0&1&1&0&1&0&0\\
0&0&0&0&1&0&1&1&0\\
0&0&0&0&0&1&0&1&1\\
0&1&0&1&1&0&0&0&0\\
0&0&0&0&1&1&0&1&0\\
1&1&1&0&0&0&0&0&0\\
0&0&1&0&0&1&0&0&1\\
0&0&0&0&0&0&1&1&1\\
1&0&0&1&0&0&1&0&0
\end{pmatrix} \Rightarrow M_1 = \begin{pmatrix}
1&1&0&0&0&0&0&0&0\\
1&0&0&1&0&0&0&0&0\\
0&1&1&0&0&0&0&0&0\\
0&1&0&1&0&0&0&0&0\\
0&1&0&0&1&0&0&0&0\\
0&0&1&0&1&0&0&0&0\\
0&0&1&0&0&1&0&0&0\\
0&0&0&1&1&0&0&0&0\\
0&0&0&1&0&0&1&0&0\\
0&0&0&0&1&1&0&0&0\\
0&0&0&0&1&0&1&0&0\\
0&0&0&0&1&0&0&1&0\\
0&0&0&0&0&1&0&1&0\\
0&0&0&0&0&1&0&0&1\\
0&0&0&0&0&0&1&1&0\\
0&0&0&0&0&0&0&1&1
\end{pmatrix}$$

Extraction of facets

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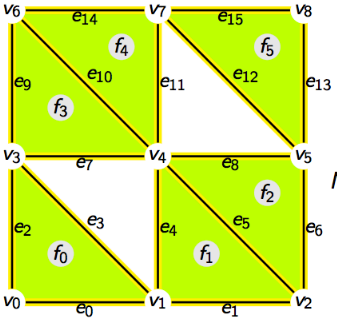

$$M_2 = \begin{pmatrix} 1&1&0&1&0&0&0&0&0 \\ 0&1&1&0&1&0&0&0&0 \\ 0&0&1&0&1&1&0&0&0 \\ 0&0&0&1&1&0&1&0&0 \\ 0&0&0&0&1&0&1&1&0 \\ 0&0&0&0&0&1&0&1&1 \\ 0&1&0&1&1&0&0&0&0 \\ 0&0&0&0&1&1&0&1&0 \\ 1&1&1&0&0&0&0&0&0 \\ 0&0&1&0&0&1&0&0&1 \\ 0&0&0&0&0&0&1&1&1 \\ 1&0&0&1&0&0&1&0&0 \end{pmatrix} \Rightarrow M_1 = \begin{pmatrix} 1&1&0&0&0&0&0&0&0 \\ 1&0&0&1&0&0&0&0&0 \\ 0&1&1&0&0&0&0&0&0 \\ 0&1&0&1&0&0&0&0&0 \\ 0&1&0&0&1&0&0&0&0 \\ 0&0&1&0&1&0&0&0&0 \\ 0&0&1&0&0&1&0&0&0 \\ 0&0&0&1&1&0&0&0&0 \\ 0&0&0&1&0&0&1&0&0 \\ 0&0&0&0&1&1&0&0&0 \\ 0&0&0&0&1&0&1&0&0 \\ 0&0&0&0&1&0&0&1&0 \\ 0&0&0&0&0&1&0&1&0 \\ 0&0&0&0&0&1&0&0&1 \\ 0&0&0&0&0&0&1&1&0 \\ 0&0&0&0&0&0&0&1&1 \end{pmatrix}$$