Linear Algebraic Representation

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A novel representation scheme

Evolution of geometric representations

Past: from boundary to interior

- Geometry soups
- Geometric primitives
- Boundary schemes
- Decompositive schemes

Present: from geometry to topology

- Simplicial complex
- Brick complex
- Polytopal complex
- General cell complex (CW-complex)

Next: from discrete topology to discrete physics

- Gradient operator
- Divergence operator
- Curl operator
- Lagrangian operator

A novel representation scheme

- the topology of complex models is computable from simple mathematical data structures
- topological queries can be answered by straightforward sparse-matrix computations based on linear operators and elementary algebra (multiplying and transposing sparse matrices)
- our algebraic models are very compact, and there is no overhead to answer topological queries w.r.t. traditional graph-based methods
- Advanced implementations on modern hardware use open standards for parallel programming of heterogeneous systems (OpenCL)

LAR: Linear Algebraic Representation

Provide a novel computer representation of big geometric data for the IEEE P3333.2 Standard for 3D medical

- usable with both images and geometric data
- allowing boundary rep and cellular decomposition of the interior
- very compact and susceptible of further compression
- dimension-independent (2D, 3D, 4D)
- using simple parallel algorithms
- and basic computational methods (sparse matrix algebra)
- supporting heterogeneous distributed computing (OpenCL, WebCL)
- and using mobile standard graphics (OpenGL ES, WebGL)

Definitions

- A compact topological subspace is a convex cell if it is the set of solutions of affine equalities and inequalities.
- A face of a cell is the convex cell obtained by replacing some of the inequalities by equalities.
- A facet is a face defined by just one equality.
- The dimension n of a n-cell is that of its affine hull, the smallest affine subspace that contains it.
- A convex-cell complex or polytopal complex P is a finite union of convex cells such that:

1. if A is a cell of P, so are the faces of A;
2. the intersection of two cells of P is a common face of each of them.

- A simplicial (cuboidal) complex is a polytopal complex where all cells are simplices (cuboids).
- The dimension of P is the maximal cell dimension of P.
- The r-skeleton P_r is the subcomplex formed by the cells of dimension $\leq r$.
- The 0-skeleton coincides with the set $V(P)$ of vertices of P.

Cellular complex

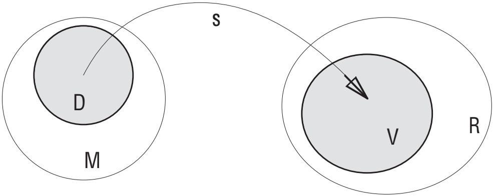
- Let X be a topological space.
- Let $\Lambda(X) = \cup_p \Lambda_{p\in\mathbb{N}}$ be a partition of X, with Λ_p the set of p-cells.
- A CW-structure on the space X is a filtration

$$\emptyset = X^{-1} \subset X_0 \subset X_1 \subset \cdots \subset X = \cup_n X_n,$$

  such that, for each n, the space X_n is homeomorphic to a space obtained from X_{n-1} by attachment of n-cells of X in $\Lambda_n = \Lambda_n(X)$.
- A space endowed with a CW-structure is a CW-complex, also said a cellular complex.
- A cellular complex is finite when it contains a finite number of cells.

Representation scheme

mapping $s : M \to R$ from a space M of mathematical models to a space R of computer representations



1. The M set contains the mathematical models of the class of solid objects the scheme aims to represent
2. The R set contains the symbolic representations, i.e. the proper data structures, built according to a suitable grammar

Linear Algebraic Representation (LAR)

- Let a Hausdorff space X of dimension d and a finite cellular complex $\Lambda(X)$ be given, such that $X = \Lambda_0 \cup \cdots \cup \Lambda_d$,
- and let $M_p \in \mathbb{Z}_2^{m \times n}$ be binary matrices, with $1 \leq p \leq d$, $m = \sharp\Lambda_p$ rows, and $n = \sharp\Lambda_0$ columns.

Definition (Math models)

The LAR domain is the set M of chain complexes on a finite cellular complex $\Lambda(X)$.

Definition (Computer representations)

The LAR codomain is the set R of d-tuples of CSR sparse binary matrices[1]

$$\mathrm{CSR}(M_n^m(\mathbb{Z}_2)), \qquad m = \sharp\Lambda_p, \quad n = \sharp\Lambda_0,$$

Linear Algebraic Representation (LAR) scheme

may support (at least in principle) all topological and geometric queries and constructions that may be asked of the corresponding model

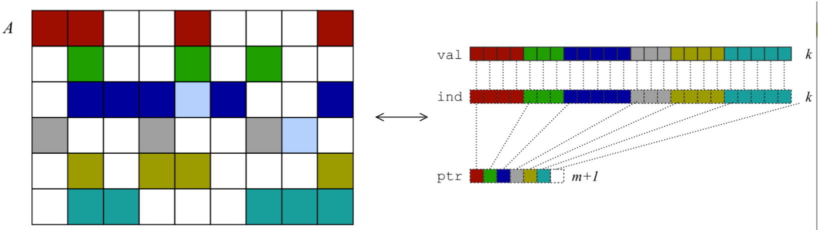
Remark (The interesting point)

for a given cellular d-complex $\Lambda(X)$, all of the M_p matrices $(1 \leq p \leq d)$ have the same number of columns

This simple fact provides a convenient tool (matrix algebra) for computing boundary and coboundary operators and topological relations between cells.

Sparse binary matrices

Compressed Sparse Row (CSR) storage

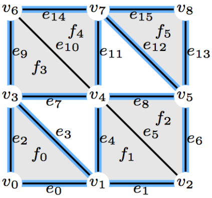



```
// Basic SpMV implementation,
// y <- y + A*x, where A is in CSR.
 for (i = 0; i < m; ++i) {
   double y0 = y[i];
   for (k = ptr[i]; k < ptr[i+1]; ++k)
     y0 += val[k] * x[ind[k]];
   y[i] = y0;
 }
```

a basic CSR-based SpMV implementation

S. Williams et al., Optimization of sparse matrix–vector multiplication on emerging multicore platforms Parallel Computing 35 (2009) 178–194

LAR topology (CSR representation)



for convex-cell complexes (ex: simplicial or cuboidal) only the CSR(M_d) is needed:

	M_1^t	$\Leftarrow$	M_q^t	$\Leftarrow$	M_d^t
M_1			$\Downarrow$		
$\Uparrow$					
M_p		$\Longrightarrow$	$M_pM_q^t$		
$\Uparrow$					
M_d					

$M_1^tM_1 = \texttt{VV} =$
```
[[0,1,3],
 [0,1,2,3,4],
 [1,2,4,5],
 [0,1,3,4,6],
 [1,2,3,4,5,6,7],
 [2,4,5,7,8],
 [3,4,6,7],
 [4,5,6,7,8],
 [5,7,8]]
```

$M_1^t = \texttt{VE} =$
```
[[0,2],
 [0,1,3,4],
 [1,5,6],
 [2,3,7,9],
 [4,5,7,8,10,11],
 [6,8,12,13],
 [9,10,14],
 [11,12,14,15],
 [13,15]]
```

$M_2^t = \texttt{VF} =$
```
[[0],
 [0,1],
 [1,2],
 [0,3],
 [1,2,3,4],
 [2,5],
 [3,4],
 [4,5],
 [5]]
```

$M_1 = \texttt{EV} =$
```
[[0,1],
 [1,2],
 [0,3],
 [1,3],
 [1,4],
 [2,4],
 [2,5],
 [3,4],
 [4,5],
 [3,6],
 [4,6],
 [4,7],
 [5,7],
 [5,8],
 [6,7],
 [7,8]]
```

$M_1M_1^t = \texttt{EE} =$
```
[[0,1,2,3,4],
 [0,1,3,4,5,6],
 [0,2,3,7,9],
 [0,1,2,3,4,7,9],
 [0,1,3,4,5,7,8,10,11],
 [1,4,5,6,7,8,10,11],
 [1,5,6,8,12,13],
 [2,3,4,5,7,8,9,10,11],
 [4,5,6,7,8,10,11,12,13],
 [2,3,7,9,10,14],
 [4,5,7,8,9,10,11,14],
 [4,5,7,8,10,11,12,14,15],
 [6,8,11,12,13,14,15],
 [6,8,12,13,15],
 [9,10,11,12,14,15],
 [11,12,13,14,15]]
```

$M_1M_2^t = \texttt{EF} =$
```
[[0,1],
 [0,1,2],
 [0,3],
 [0,1,3],
 [0,1,2,3,4],
 [1,2,3,4],
 [1,2,5],
 [0,1,2,3,4],
 [1,2,3,4,5],
 [0,3,4],
 [1,2,3,4],
 [1,2,3,4,5],
 [2,4,5],
 [2,5],
 [3,4,5],
 [4,5]]
```

$M_2 = \texttt{FV} =$
```
[[0,1,3],
 [1,2,4],
 [2,4,5],
 [3,4,6],
 [4,6,7],
 [5,7,8]]
```

$M_2M_1^t = \texttt{FE} =$
```
[[0,1,2,3,4,7,9],
 [0,1,3,4,5,6,7,8,10,11],
 [1,4,5,6,7,8,10,11,12,13],
 [2,3,4,5,7,8,9,10,11,14],
 [4,5,7,8,9,10,11,12,14,15],
 [6,8,11,12,13,14,15]]
```

$M_2M_2^t = \texttt{FF} =$
```
[[0,1,3],
 [0,1,2,3,4],
 [1,2,3,4,5],
 [0,1,2,3,4],
 [1,2,3,4,5],
 [2,4,5]]
```

LAR computations examples

download `https://github.com/cvdlab/larpy`

Linear Algebraic Representation in Python

Remarks about LAR

- A large class of geometric representations involve incidence structures", informally called topology" in boundary representations, but also scenes in graphics, assembly and tolerance graphs, FE meshes, multimaterial structures, and so on

- Incidence structures are logical/symbolic (as opposed to numeric and geometric) relationships We can broadly refer to them as topological (incidence) structures.

Remarks about LAR

- Topological incidence structures dominate and are becoming more important, because they correspond to higher-level representations and semantics in applications, but:
  1. there are no standards for such representations,
  2. they tend to dominate the size of representations, e.g in b-reps, and
  3. they are not suitable for parallel processing, cloud, etc.

- At the other extreme, are purely geometric/numerical representations, such as point clouds, polygons, triangles

- They can be compressed and efficiently processed, but they do not deal with semantics, logical data, and are intrinsically non-robust (in a numerical sense) So, even if we get a 3D JPEG, it will not be sufficient for advanced applications.

Remarks about LAR

- Motivated by the above considerations, we engaged in developing an innovative computer representation of topological and geometric data, in the framework of the IEEE-SA P3333.2 Working Group – Three-Dimensional Model Creation Using Unprocessed 3D Medical Data
- LAR (Linear Algebraic Representation) scheme is a simple algebraic rep for cell complexes that uses a CSR (Compressed Sparse Row) form for coding the characteristic matrices of linear spaces of (co)chains
- LAR enjoys a neat mathematical format, being based on chains (the domains of discrete integration) and cochains (the discrete prototype of differential forms).

Remarks about LAR

- Due to its wide coverage and high simplicity, LAR may be proposed as a new standard for topological incidence structures—they can be used with any and all geometric representations, and can be applied in all of the above applications

- We can demonstrate how LAR can be used with
  - convex decompositions,
  - B'ezier curves and surfaces, or
  - images because it can be used with different geometric embeddings and put on top of any metric structure.

Chain and cochain complex

applies to most domains characterized as cell complexes, without any restrictions on their type, dimension, codimension, orientability, manifoldness, and connectedness

$$
\begin{tikzcd}
\cdots & C^3 \arrow[l, "\delta^3"'] & C^2 \arrow[l, "\delta^2"'] & C^1 \arrow[l, "\delta^1"'] & C^0 \arrow[l, "\delta^0"'] \\
\cdots \arrow[r, "\partial_4"] & C_3 \arrow[u, "\cong"'] \arrow[r, "\partial_3"] & C_2 \arrow[u, "\cong"'] \arrow[r, "\partial_2"] & C_1 \arrow[u, "\cong"'] \arrow[r, "\partial_1"] & C_0 \arrow[u, "\cong"']
\end{tikzcd}
$$

Definition (Chain complex)

Chain complex is a sequence of Abelian groups $\ldots, C_3, C_2, C_1, C_0$ connected by homomorphisms (boundary operators) $\partial_n : C_n \to C_{n-1}$, such that for all n:

$$\partial_{n-1} \circ \partial_n = 0$$

Definition (Cochain complex)

Cochain complex is a sequence of Abelian groups $C^0, C^1, C^2, C^3, \ldots$ connected by homomorphisms (coboundary operators) $\delta^n : C^n \to C^{n+1}$, such that for all n:

$$\delta^{n+1} \circ \delta^n = 0$$

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[6] A. DiCarlo, F. Milicchio, A. Paoluzzi, and V. Shapiro. Chain-Based Representations for Solid and Physical Modeling. 2009. IEEE Transactions on Automation Science and Engineering

Incidence relations7 vs linear operators

Change of paradigm in shape representation

Use symbols V, E, F for K_0, K_1, K_2, the bases of linear spaces C_0, C_1, C_2 of chains in 2D.

Incidence/adjacency relations:

$$XY \subset X \times Y$$

	V	E	F
V	VV	VE	VF
E	EV	EE	EF
F	FV	FE	FF

Linear operators:

$$\mathcal{XY} : \mathcal{Y} \to \mathcal{X}$$

$$\begin{tikzcd} C_0 \arrow[r, Rightarrow, "\mathcal{FV}"] & C_2 \arrow[d, "\mathcal{EF}"] \arrow[dl, "\mathcal{VF}"'] \\ C_0 \arrow[r, "\mathcal{EV}"'] & C_1 \end{tikzcd}$$

$$\mathcal{EF} = \mathcal{EV} \circ \mathcal{VF} = \mathcal{EV} \circ \mathcal{FV}^\top$$

[7] See: Woo, A combinatorial analysis of boundary data structure schemata, IEEE CG&A, 1985

2D simplicial complex (non manifold pointset)

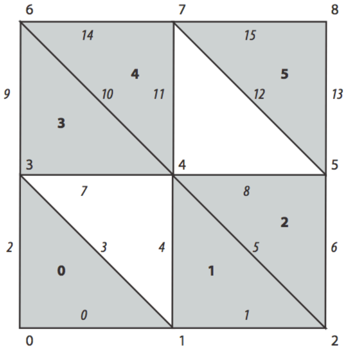
$K = K_0 \cup K_1 \cup K_2$ $\qquad \#K_0 =: k_0 = 9; \quad \#K_1 =: k_1 = 16; \quad \#K_2 =: k_2 = 6$

2-cells by 0-cells

```
FV = [[0,1,3],
      [1,2,4],
      [2,4,5],
      [3,4,6],
      [4,6,7],
      [5,7,8]]
```

1-cells by 0-cells

```
EV = [[0,1],
      [1,2],
      [0,3],
      [1,3],
      [1,4],
      [2,4],
      [2,5],
      [3,4],
      [4,5],
      [3,6],
      [4,6],
      [4,7],
      [5,7],
      [5,8],
      [6,7],
      [7,8]]
```



Example

2D non-manifold simplicial complex

```
V = [[0,0],[1,0],[2,0],[0,1],[1,1],[2,1],[0,2],[1,2
FV = [[0,1,3],[1,2,4],[2,4,5],[3,4,6],[4,6,7],[5,7,
```

faces extraction

```
model = (V,FV)
V,faces = larSkeletons(model,dim=2,grid=False)
F0V, F1V, F2V = faces
```

extruded 3D model

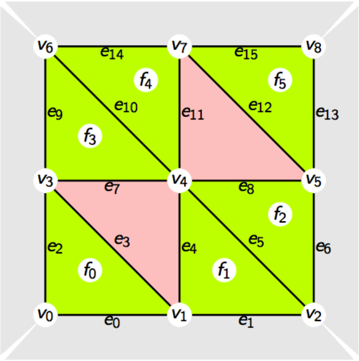
```
model = larExtrude((V,FV),2*[1,2,-3])
VIEW(EXPLODE(1,1,1.2)(MKPOLS(model)))
```

Extraction of facets

start from the characteristic matrix M_d of a cellular partition Λ_d of $\mathbb{E}^d$, empty cells included

$2D \Rightarrow d = 2$

$$M_2 = \begin{pmatrix}
1&1&0&1&0&0&0&0&0\\
0&1&1&0&1&0&0&0&0\\
0&0&1&0&1&1&0&0&0\\
0&0&0&1&1&0&1&0&0\\
0&0&0&0&1&0&1&1&0\\
0&0&0&0&0&1&0&1&1\\
0&1&0&1&1&0&0&0&0\\
0&0&0&0&1&1&0&1&0\\
1&1&1&0&0&0&0&0&0\\
0&0&1&0&0&1&0&0&1\\
0&0&0&0&0&0&1&1&1\\
1&0&0&1&0&0&1&0&0
\end{pmatrix}$$

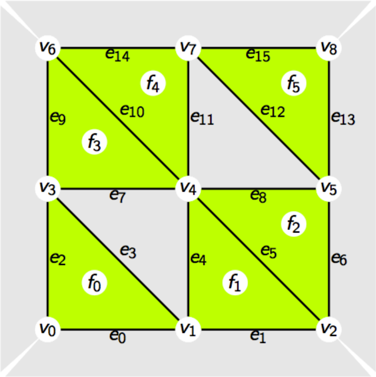


Extraction of facets

start from the characteristic matrix M_d of a cellular partition Λ_d of $\mathbb{E}^d$, empty cells included

$$2\text{D} \Rightarrow d = 2$$

$$M_2 = \begin{pmatrix}
1&1&0&1&0&0&0&0&0\\
0&1&1&0&1&0&0&0&0\\
0&0&1&0&1&1&0&0&0\\
0&0&0&1&1&0&1&0&0\\
0&0&0&0&1&0&1&1&0\\
0&0&0&0&0&1&0&1&1\\
0&1&0&1&1&0&0&0&0\\
0&0&0&0&1&1&0&1&0\\
1&1&1&0&0&0&0&0&0\\
0&0&1&0&0&1&0&0&1\\
0&0&0&0&0&0&1&1&1\\
1&0&0&1&0&0&1&0&0
\end{pmatrix}$$



Extraction of facets

start from the characteristic matrix M_d of a cellular partition Λ_d of $\mathbb{E}^d$, empty cells included

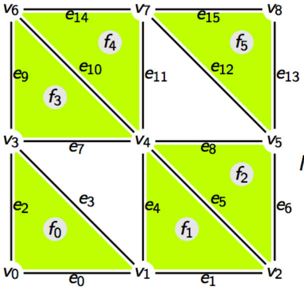
$$
A = M_2 M_2^t = \begin{pmatrix}
3 & 1 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 2 & 2 & 0 \\
 & 3 & 2 & 1 & 1 & 0 & 2 & 1 & 0 & 0 & 2 & 1 \\
 & & 3 & 1 & 1 & 1 & 1 & 2 & 0 & 0 & 1 & 2 \\
 & & & 3 & 2 & 0 & 0 & 0 & 1 & 2 & 2 & 1 \\
 & & & & 3 & 1 & 0 & 0 & 2 & 1 & 1 & 2 \\
 & & & & & 3 & 0 & 2 & 2 & 0 & 0 & 2 \\
 & & & & & & 3 & 1 & 0 & 1 & 1 & 0 \\
 & & & & & & & 3 & 1 & 0 & 0 & 1 \\
 & & & sym & & & & & 3 & 1 & 0 & 1 \\
 & & & & & & & & & 3 & 1 & 0 \\
 & & & & & & & & & & 3 & 1 \\
 & & & & & & & & & & & 3
\end{pmatrix}
$$

$$
\sharp(\lambda_d^i \cap \lambda_d^j) = A(i,j) \geq d \quad (i \neq j) \;\Rightarrow\; \exists\, \lambda_{d-1} = M_d(i) \wedge M_d(j)
$$

Extraction of facets

start from the characteristic matrix M_d of a cellular partition Λ_d of $\mathbb{E}^d$, empty cells included

$$A(0,6) = 2 \quad \Rightarrow \quad (\,110100000\,) \wedge (\,010110000\,) = (\,010100000\,) \quad \Rightarrow \quad e = (v_1, v_3)$$

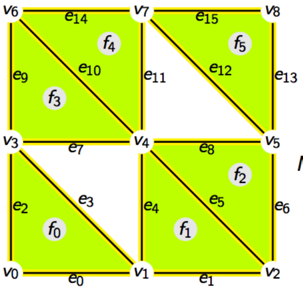


$$M_2 = \begin{pmatrix}
1&1&0&1&0&0&0&0&0\\
0&1&1&0&1&0&0&0&0\\
0&0&1&0&1&1&0&0&0\\
0&0&0&1&1&0&1&0&0\\
0&0&0&0&1&0&1&1&0\\
0&0&0&0&0&1&0&1&1\\
0&1&0&1&1&0&0&0&0\\
0&0&0&0&1&1&0&1&0\\
1&1&1&0&0&0&0&0&0\\
0&0&1&0&0&1&0&0&1\\
0&0&0&0&0&0&1&1&1\\
1&0&0&1&0&0&1&0&0
\end{pmatrix}
\Rightarrow M_1 = \begin{pmatrix}
1&1&0&0&0&0&0&0&0\\
1&0&0&1&0&0&0&0&0\\
0&1&1&0&0&0&0&0&0\\
0&1&0&1&0&0&0&0&0\\
0&1&0&0&1&0&0&0&0\\
0&0&1&0&1&0&0&0&0\\
0&0&1&0&0&1&0&0&0\\
0&0&0&1&1&0&0&0&0\\
0&0&0&1&0&0&1&0&0\\
0&0&0&0&1&1&0&0&0\\
0&0&0&0&1&0&1&0&0\\
0&0&0&0&1&0&0&1&0\\
0&0&0&0&0&1&0&1&0\\
0&0&0&0&0&1&0&0&1\\
0&0&0&0&0&0&1&1&0\\
0&0&0&0&0&0&0&1&1
\end{pmatrix}$$

Extraction of facets

start from the characteristic matrix M_d of a cellular partition Λ_d of $\mathbb{E}^d$, empty cells included

$$A(0,6) = 2 \quad \Rightarrow \quad (110100000) \wedge (010110000) = (010100000) \quad \Rightarrow \quad e = (v_1, v_3)$$



$$M_2 = \begin{pmatrix} 1\,1\,0\,1\,0\,0\,0\,0\,0 \\ 0\,1\,1\,0\,1\,0\,0\,0\,0 \\ 0\,0\,1\,0\,1\,1\,0\,0\,0 \\ 0\,0\,0\,1\,1\,0\,1\,0\,0 \\ 0\,0\,0\,0\,1\,0\,1\,1\,0 \\ 0\,0\,0\,0\,0\,1\,0\,1\,1 \\ 0\,1\,0\,1\,1\,0\,0\,0\,0 \\ 0\,0\,0\,0\,1\,1\,0\,1\,0 \\ 1\,1\,1\,0\,0\,0\,0\,0\,0 \\ 0\,0\,1\,0\,0\,1\,0\,0\,1 \\ 0\,0\,0\,0\,0\,0\,1\,1\,1 \\ 1\,0\,0\,1\,0\,0\,1\,0\,0 \end{pmatrix} \Rightarrow M_1 = \begin{pmatrix} 1\,1\,0\,0\,0\,0\,0\,0\,0 \\ 1\,0\,0\,1\,0\,0\,0\,0\,0 \\ 0\,1\,1\,0\,0\,0\,0\,0\,0 \\ 0\,1\,0\,1\,0\,0\,0\,0\,0 \\ 0\,1\,0\,0\,1\,0\,0\,0\,0 \\ 0\,0\,1\,0\,1\,0\,0\,0\,0 \\ 0\,0\,1\,0\,0\,1\,0\,0\,0 \\ 0\,0\,0\,1\,1\,0\,0\,0\,0 \\ 0\,0\,0\,1\,0\,0\,1\,0\,0 \\ 0\,0\,0\,0\,1\,1\,0\,0\,0 \\ 0\,0\,0\,0\,1\,0\,1\,0\,0 \\ 0\,0\,0\,0\,1\,0\,0\,1\,0 \\ 0\,0\,0\,0\,0\,1\,0\,1\,0 \\ 0\,0\,0\,0\,0\,1\,0\,0\,1 \\ 0\,0\,0\,0\,0\,0\,1\,1\,0 \\ 0\,0\,0\,0\,0\,0\,0\,1\,1 \end{pmatrix}$$