Parallel & Distributed Computing: Lecture 10

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1. Matrix products (continue)
2. Decomposition Techniques
3. Parallel Algorithm Models

Matrix products (continue)

The evolution of lower bounds for matrix multiplication

Until the late 1960s computing the product C of two $n \times n$ matrices required a cubic number of operations, as the fastest algorithm known was the naive algorithm which runs in $O(n^3)$ time

In 1969, Strassen gave the first subcubic time algorithm for matrix multiplication, running in $O(n^{2.808})$ time

This amazing discovery spawned a long line of research which gradually reduced the matrix multiplication exponent ω over time.

The first result to break 2.5 was by Coppersmith and Winograd who obtained $\omega < 2.496$.

Three years later, Coppersmith and Winograd combined Strassen's technique with a novel form of analysis based on large sets avoiding arithmetic progressions and obtained the bound of $\omega < 2.376$, unchanged for more than twenty years

Cache oblivious matrix multiplication

Source: Michael Bader and Christoph Zenger, Cache oblivious matrix multiplication using an element ordering based on the Peano curve, Linear Algebra and its Applications, 2006

Algorithm 1 multiplication of two n-by-n matrices

```
for i from 1 to n do
   for j from 1 to n do
      C[i,j] := 0;
      for k from 1 to n to
          C[i,j] := C[i,j] + A[i,k] * B[k,j];
      end do;
   end do;
end do;
```

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Algorithm 1 multiplication of two n-by-n matrices

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      for k from 1 to n to
          C[i,j] := C[i,j] + A[i,k] * B[k,j];
      end do;
   end do;
end do;
```

Algorithm 2 multiplication of two n-by-n matrices (revisited)

```
// matrix C is assumed to be initialized
for all triples (i,j,k) in {1..n}x{1..n}x{1..n} do
   C[i,j] := C[i,j] + A[i,k] * B[k,j];
end do;
```

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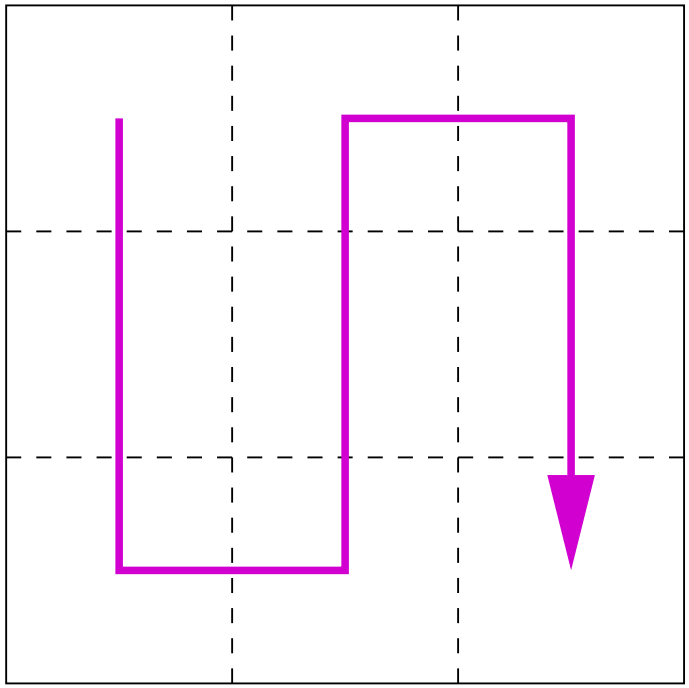
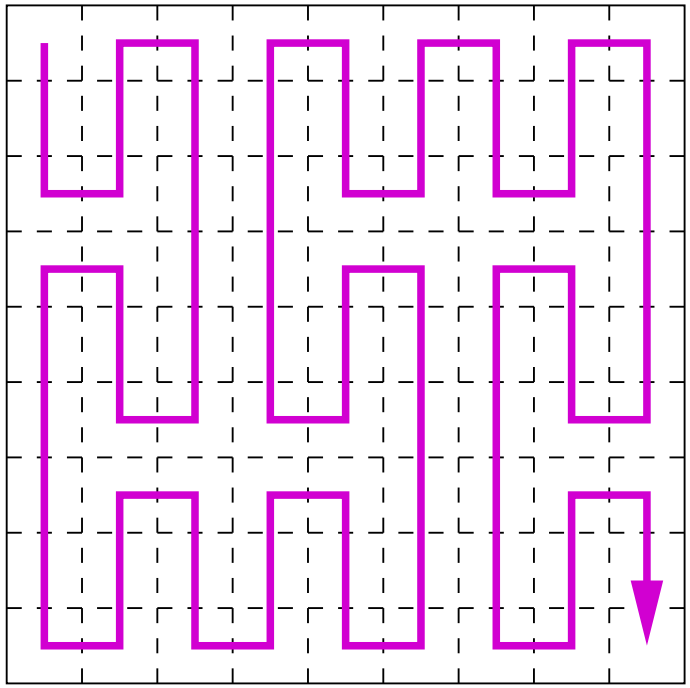
Algorithm 2 multiplication of two n-by-n matrices (revisited)

```
// matrix C is assumed to be initialized
for all triples (i,j,k) in {1..n}x{1..n}x{1..n} do
   C[i,j] := C[i,j] + A[i,k] * B[k,j];
end do;
```

- may be executed in any order, because of the commutativity
- find optimal serializations of the loop
- shows better locality of the element access, and can benefit from the presence of cache memory

Recursive construction of the Peano curve





iterations of the Peano curve are generated in a self-similar, recursive process

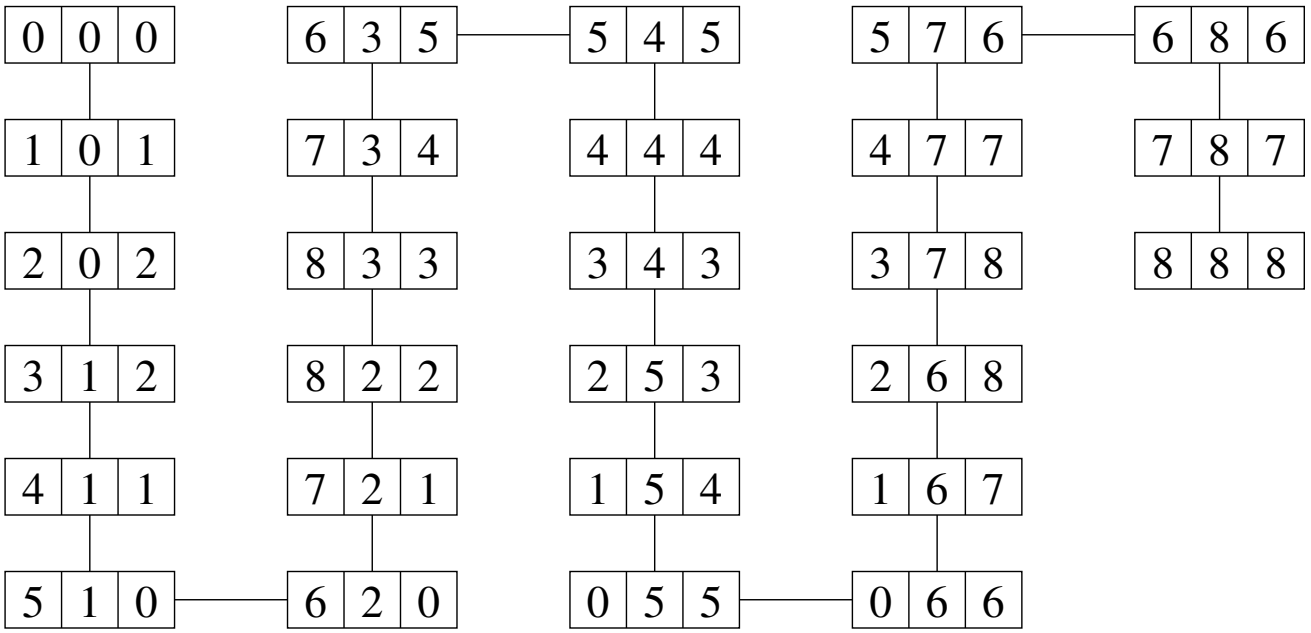
Elements of matrices are stored in Peano-like order

$$\underbrace{\begin{pmatrix} a_0 & a_5 & a_6 \\ a_1 & a_4 & a_7 \\ a_2 & a_3 & a_8 \end{pmatrix}}_{=:A} \underbrace{\begin{pmatrix} b_0 & b_5 & b_6 \\ b_1 & b_4 & b_7 \\ b_2 & b_3 & b_8 \end{pmatrix}}_{=:B} = \underbrace{\begin{pmatrix} c_0 & c_5 & c_6 \\ c_1 & c_4 & c_7 \\ c_2 & c_3 & c_8 \end{pmatrix}}_{=:C}$$

$$c_k = \sum_{(i,j)\in C_k} a_i b_j$$

$$C_0 = \{(a_0, b_0), (a_5, b_1), (a_6, b_2)\}, \ldots \text{ etc } \ldots$$

Graph representation of optimal serialization



Graph representation of the operations of a 3×3 (also odd-by-odd) matrix multiplication

after each element operation, either directly re-use a matrix element, or move to its direct neighbour

Recursive construction of the Peano curve

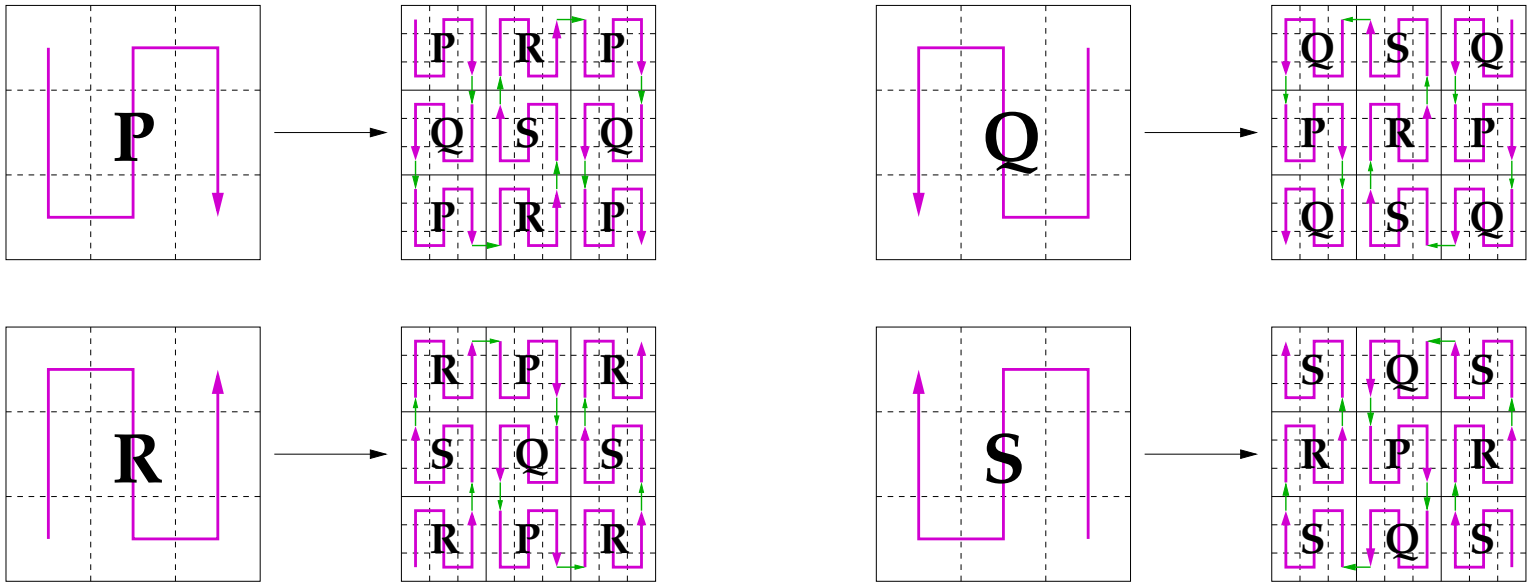
According to Algorithm 2 to compute the matrix-matrix product, we have to perform two steps:

1. initialize all $c_k := 0 fork = 0, ..., 8$
2. for all triples (k, i, j) where $(i, j) \in C_k$, and $k = 0, ..., 8$ execute:

$$c_k := c_k + a_i \, b_j$$

- optimally localized execution order of the operations, avoiding jumps in k, i, and j

Recursive block numbering scheme based on Peano curve



Implementation

$$\underbrace{\begin{pmatrix} P_{A0} & R_{A5} & P_{A6} \\ Q_{A1} & S_{A4} & Q_{A7} \\ P_{A2} & R_{A3} & P_{A8} \end{pmatrix}}_{=:A} \underbrace{\begin{pmatrix} P_{B0} & R_{B5} & P_{B6} \\ Q_{B1} & S_{B4} & Q_{B7} \\ P_{B2} & R_{B3} & P_{B8} \end{pmatrix}}_{=:B} = \underbrace{\begin{pmatrix} P_{C0} & R_{C5} & P_{C6} \\ Q_{C1} & S_{C4} & Q_{C7} \\ P_{C2} & R_{C3} & P_{C8} \end{pmatrix}}_{=:C}$$

We get the following operations on the matrix blocks:

$$\begin{aligned} P_{C0} &:= P_{A0}P_{B0} + R_{A5}Q_{B1} + P_{A6}P_{B2} \\ Q_{C1} &:= Q_{A1}P_{B0} + S_{A4}Q_{B1} + Q_{A7}P_{B2} \\ R_{C5} &:= P_{A0}R_{B5} + R_{A5}S_{B4} + P_{A6}R_{B3} \\ S_{C4} &:= Q_{A1}R_{B5} + S_{A4}S_{B4} + Q_{A7}R_{B3} \end{aligned}$$

Algorithmic scheme

$$
\begin{aligned}
&P_{C0} := 0 \\
&P_{C0} \xleftarrow{+} P_{A0}P_{B0} \qquad \text{(short notation for } P_{C0} := P_{C0} + P_{A0}P_{B0} \\
&P_{C0} \xleftarrow{+} R_{A5}Q_{B1} \\
&P_{C0} \xleftarrow{+} P_{A6}P_{B2}
\end{aligned}
\tag{8}
$$

If we just consider the ordering of the matrix blocks, we can see that there are exactly eight different types of block multiplications:

$$
\begin{aligned}
&P \xleftarrow{+} PP \qquad && P \xleftarrow{+} RQ \\
&Q \xleftarrow{+} QP \qquad && Q \xleftarrow{+} SQ \\
&R \xleftarrow{+} PR \qquad && R \xleftarrow{+} RS \\
&S \xleftarrow{+} QR \qquad && S \xleftarrow{+} SS.
\end{aligned}
\tag{9}
$$

Pseudocoding

Algorithm 3 Recursive implementation of the Peano matrix multiplication

```
/* global variables:
 * A, B, C: the matrices, C will hold the result of AB
 * a, b, c: indices of the matrix element of A, B, and C
 */
peanomult(int phsA, int phsB, int phsC, int dim)
{
   if (dim == 1) {
      C[c] += A[a] * B[b];
   }
   else
   {
      peanomult( phsA,  phsB,  phsC, dim/3); a += phsA; c += phsC;
      peanomult( phsA, -phsB,  phsC, dim/3); a += phsA; c += phsC;
      peanomult( phsA,  phsB,  phsC, dim/3); a += phsA; b += phsB;

      peanomult( phsA,  phsB, -phsC, dim/3); a += phsA; c -= phsC;
      peanomult( phsA, -phsB, -phsC, dim/3); a += phsA; c -= phsC;
      peanomult( phsA,  phsB, -phsC, dim/3); a += phsA; b += phsB;

      peanomult( phsA,  phsB,  phsC, dim/3); a += phsA; c += phsC;
      peanomult( phsA, -phsB,  phsC, dim/3); a += phsA; c += phsC;
      peanomult( phsA,  phsB,  phsC, dim/3); b += phsB; c += phsC;

      peanomult( phsA,  phsB,  phsC, dim/3); a -= phsA; c += phsC;
      peanomult( phsA, -phsB,  phsC, dim/3); a -= phsA; c += phsC;
      peanomult( phsA,  phsB,  phsC, dim/3); a -= phsA; b += phsB;

      peanomult( phsA,  phsB, -phsC, dim/3); a -= phsA; c -= phsC;
      peanomult( phsA, -phsB, -phsC, dim/3); a -= phsA; c -= phsC;
      peanomult( phsA,  phsB, -phsC, dim/3); a -= phsA; b += phsB;

      peanomult( phsA,  phsB,  phsC, dim/3); a -= phsA; c += phsC;
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      peanomult( phsA,  phsB,  phsC, dim/3); b += phsB; c += phsC;

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      peanomult( phsA,  phsB, -phsC, dim/3); a += phsA; c -= phsC;
      peanomult( phsA, -phsB, -phsC, dim/3); a += phsA; c -= phsC;
      peanomult( phsA,  phsB, -phsC, dim/3); a += phsA; b += phsB;

      peanomult( phsA,  phsB,  phsC, dim/3); a += phsA; c += phsC;
      peanomult( phsA, -phsB,  phsC, dim/3); a += phsA; c += phsC;
      peanomult( phsA,  phsB,  phsC, dim/3);
   };
}
```

Decomposition Techniques

Introduction

Some commonly used decomposition techniques for achieving concurrency

- recursive decomposition,
- data-decomposition,
- exploratory decomposition,
- speculative decomposition
- hybrid decomposition (mixing the above methods)

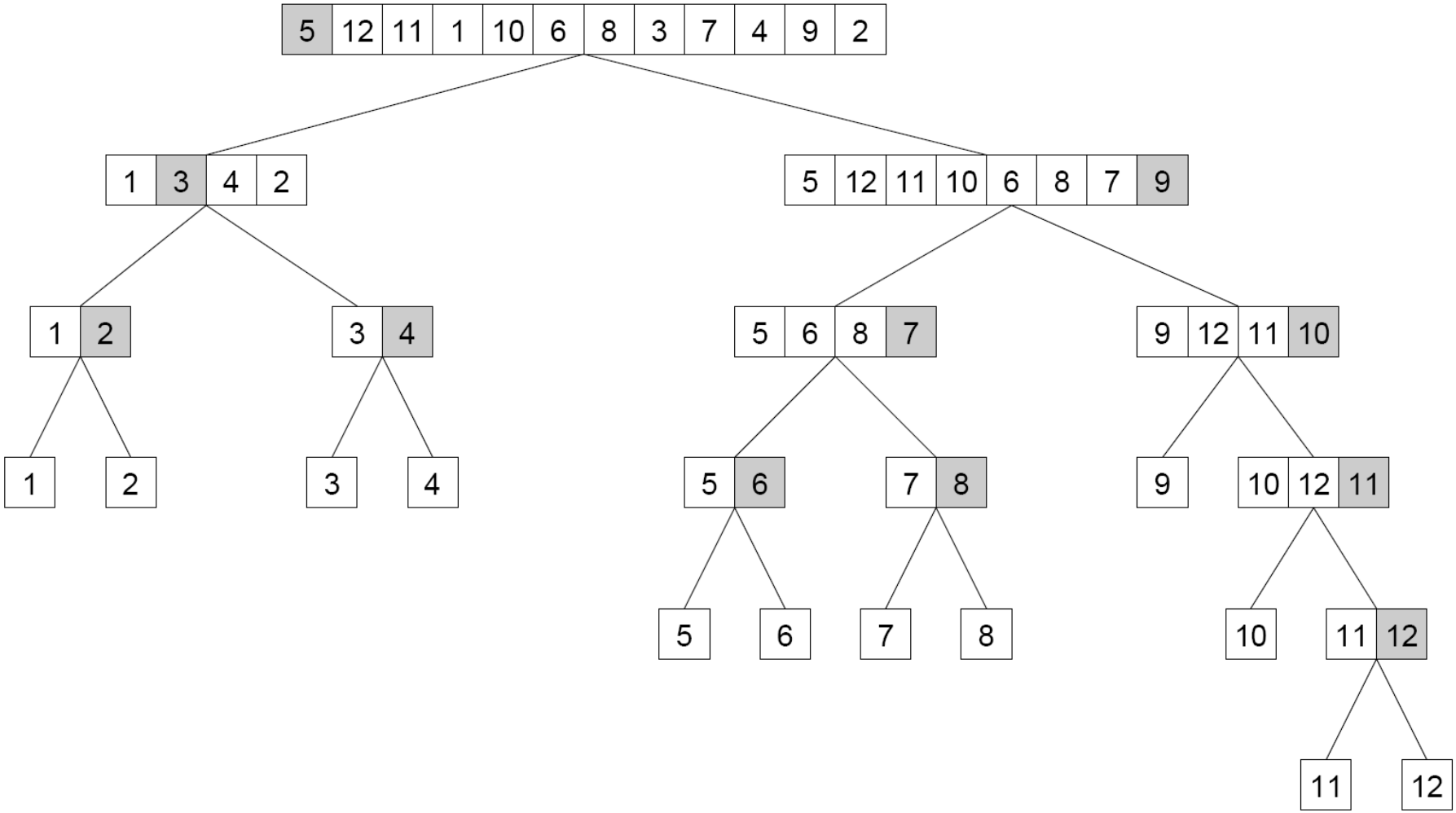
Good starting point for many problems

A combination of techniques can be used to obtain effective decompositions for a large set of problems

Recursive Decomposition (divide-and-conquer)

- Recursive decomposition is a method for inducing concurrency in problems that can be solved using the divide-and-conquer strategy
- A problem is solved by first dividing it into a set of independent subproblems.
- Each subproblem is solved by recursively applying a similar division into smaller subproblems followed by a combination of their results.

Recursive Decomposition (Quicksort)



also represents the task graph for the problem

Recursive Decomposition

A serial program for finding the minimum in an array of numbers A of length n

```
1.  procedure SERIAL_MIN (A, n)
2.  begin
3.  min = A[0];
4.  for i := 1 to n − 1 do
5.      if (A[i] < min) min := A[i];
6.  endfor;
7.  return min;
8.  end SERIAL_MIN
```

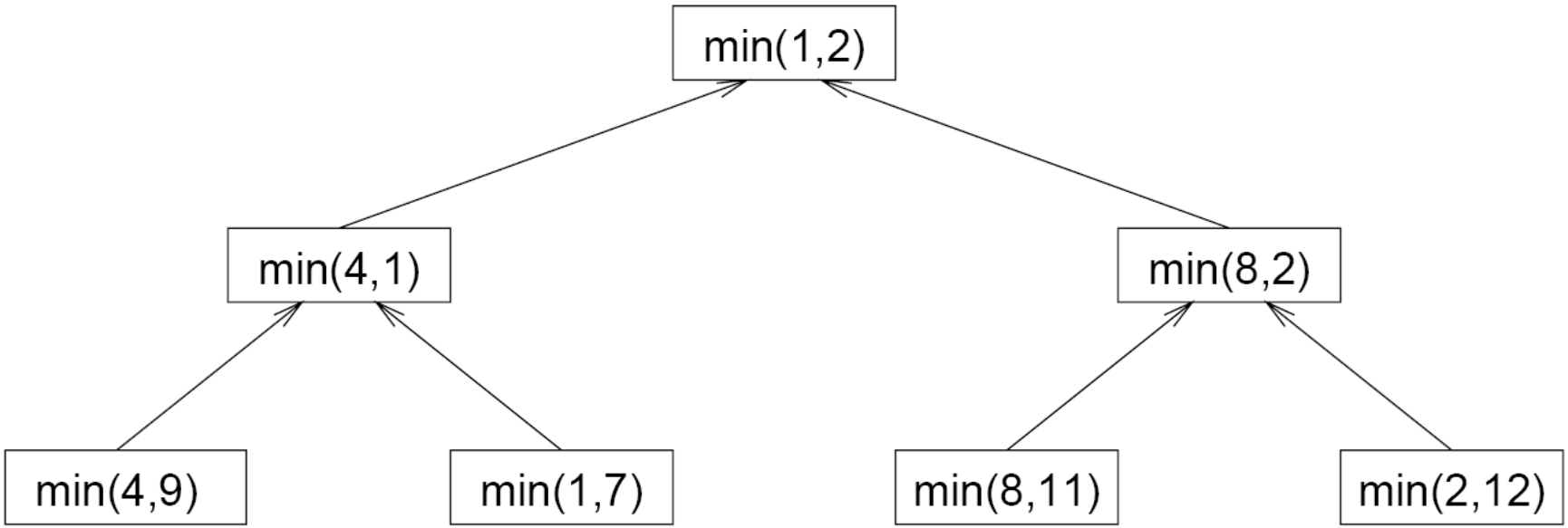
Algorithm 3.1 A serial program for finding the minimum in an array of numbers A of length n.

Recursive Decomposition

```
1.   procedure RECURSIVE_MIN (A, n)
2.   begin
3.   if (n = 1) then
4.       min := A[0];
5.   else
6.       lmin := RECURSIVE_MIN (A, n/2);
7.       rmin := RECURSIVE_MIN (&(A[n/2]), n − n/2);
8.       if (lmin < rmin) then
9.           min := lmin;
10.      else
11.          min := rmin;
12.      endelse;
13.  endelse;
14.  return min;
15.  end RECURSIVE_MIN
```

Recursive Decomposition

The task-dependency graph for finding the minimum number in the sequence $\{4, 9, 1, 7, 8, 11, 2, 12\}$.



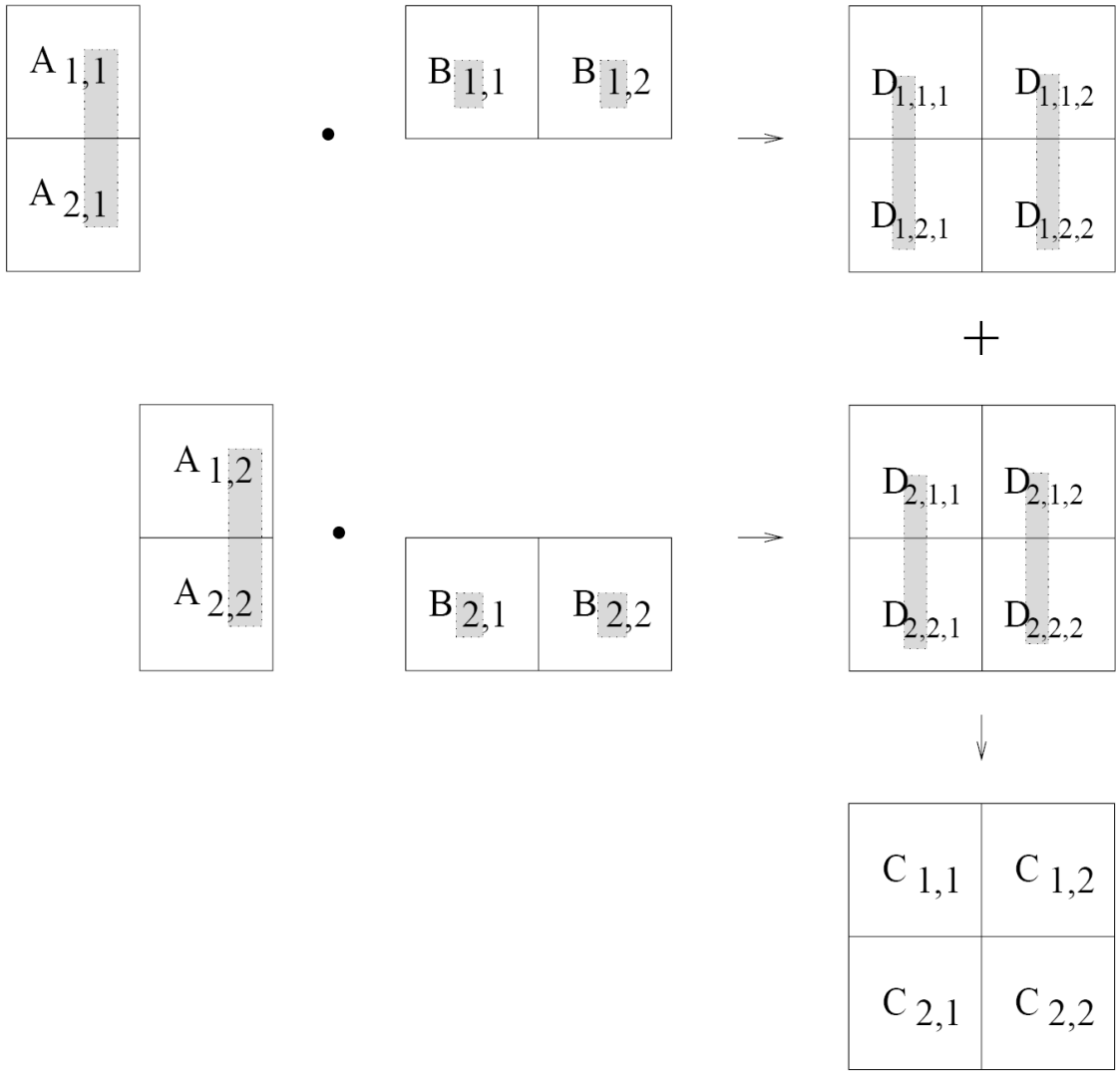
Each node in the tree represents the task of finding the minimum of a pair of numbers.

with n numbers and p tasks, complexity $O(n/p + \log(p))$

Recursive Decomposition

```
1.   procedure RECURSIVE_MIN (A, n)
2.   begin
3.   if (n = 1) then
4.       min := A[0];
5.   else
6.       lmin := RECURSIVE_MIN (A, n/2);
7.       rmin := RECURSIVE_MIN (&(A[n/2]), n - n/2);
8.       if (lmin < rmin) then
9.           min := lmin;
10.      else
11.          min := rmin;
12.      endelse;
13.  endelse;
14.  return min;
15.  end RECURSIVE_MIN
```

Data Decomposition (matrix product: intermediate decomposition)



Data Decomposition (matrix product: intermediate decomposition)

$$\begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix} \cdot \begin{pmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{pmatrix} \rightarrow \begin{pmatrix} \begin{pmatrix} D_{1,1,1} & D_{1,1,2} \\ D_{1,2,2} & D_{1,2,2} \end{pmatrix} \\ \begin{pmatrix} D_{2,1,1} & D_{2,1,2} \\ D_{2,2,2} & D_{2,2,2} \end{pmatrix} \end{pmatrix}$$

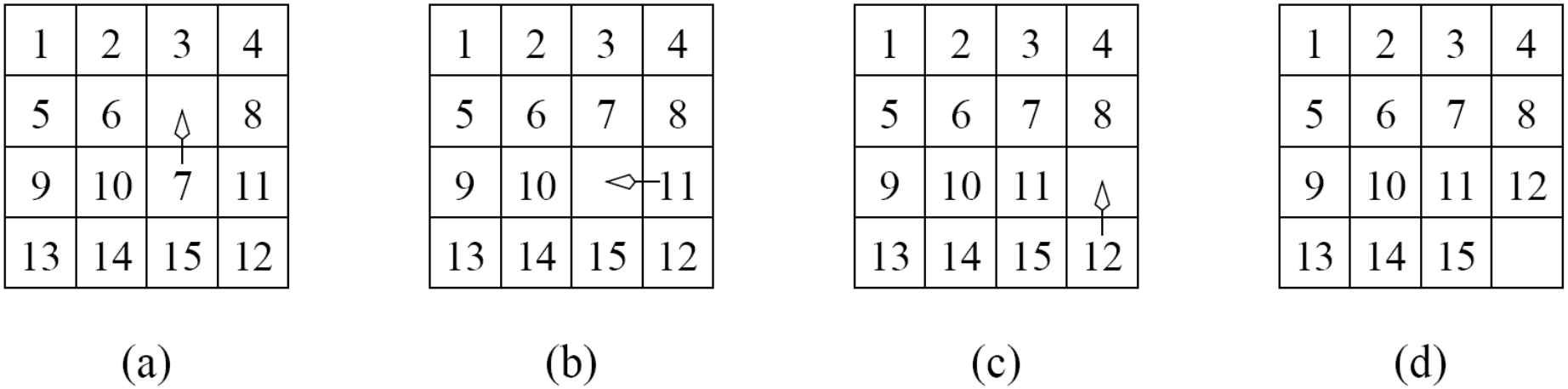
Stage II

$$\begin{pmatrix} D_{1,1,1} & D_{1,1,2} \\ D_{1,2,2} & D_{1,2,2} \end{pmatrix} + \begin{pmatrix} D_{2,1,1} & D_{2,1,2} \\ D_{2,2,2} & D_{2,2,2} \end{pmatrix} \rightarrow \begin{pmatrix} C_{1,1} & C_{1,2} \\ C_{2,1} & C_{2,2} \end{pmatrix}$$

A decomposition induced by a partitioning of D

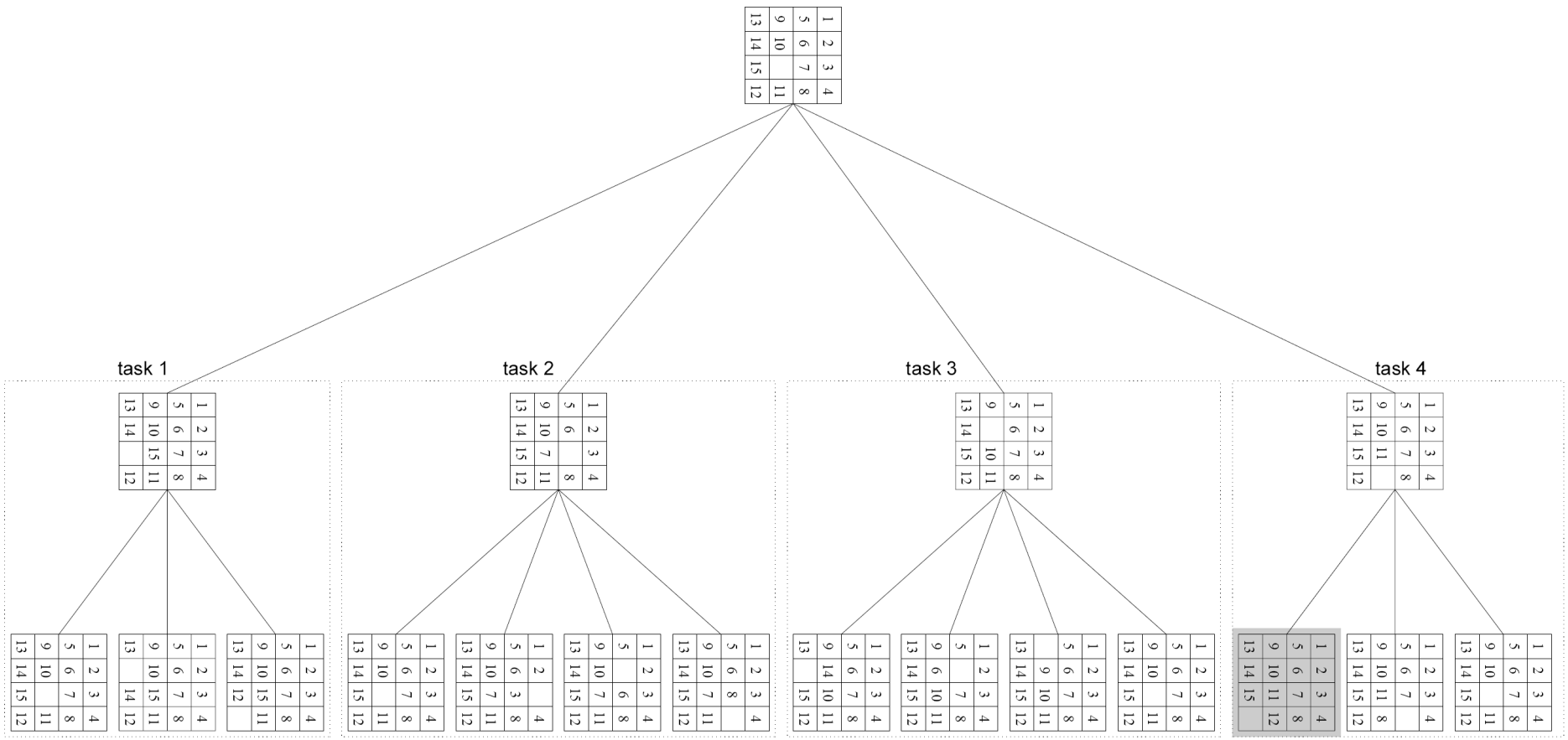
$$\begin{aligned}
\text{Task 01:} \quad & D_{1,1,1} = A_{1,1} B_{1,1} \\
\text{Task 02:} \quad & D_{2,1,1} = A_{1,2} B_{2,1} \\
\text{Task 03:} \quad & D_{1,1,2} = A_{1,1} B_{1,2} \\
\text{Task 04:} \quad & D_{2,1,2} = A_{1,2} B_{2,2} \\
\text{Task 05:} \quad & D_{1,2,1} = A_{2,1} B_{1,1} \\
\text{Task 06:} \quad & D_{2,2,1} = A_{2,2} B_{2,1} \\
\text{Task 07:} \quad & D_{1,2,2} = A_{2,1} B_{1,2} \\
\text{Task 08:} \quad & D_{2,2,2} = A_{2,2} B_{2,2} \\
\text{Task 09:} \quad & C_{1,1} = D_{1,1,1} + D_{2,1,1} \\
\text{Task 10:} \quad & C_{1,2} = D_{1,1,2} + D_{2,1,2} \\
\text{Task 11:} \quad & C_{2,1} = D_{1,2,1} + D_{2,2,1} \\
\text{Task 12:} \quad & C_{2,2} = D_{1,2,2} + D_{2,2,2}
\end{aligned}$$

Exploratory decomposition



Exploratory decomposition is used to decompose problems whose underlying computations correspond to a search of a space for solutions

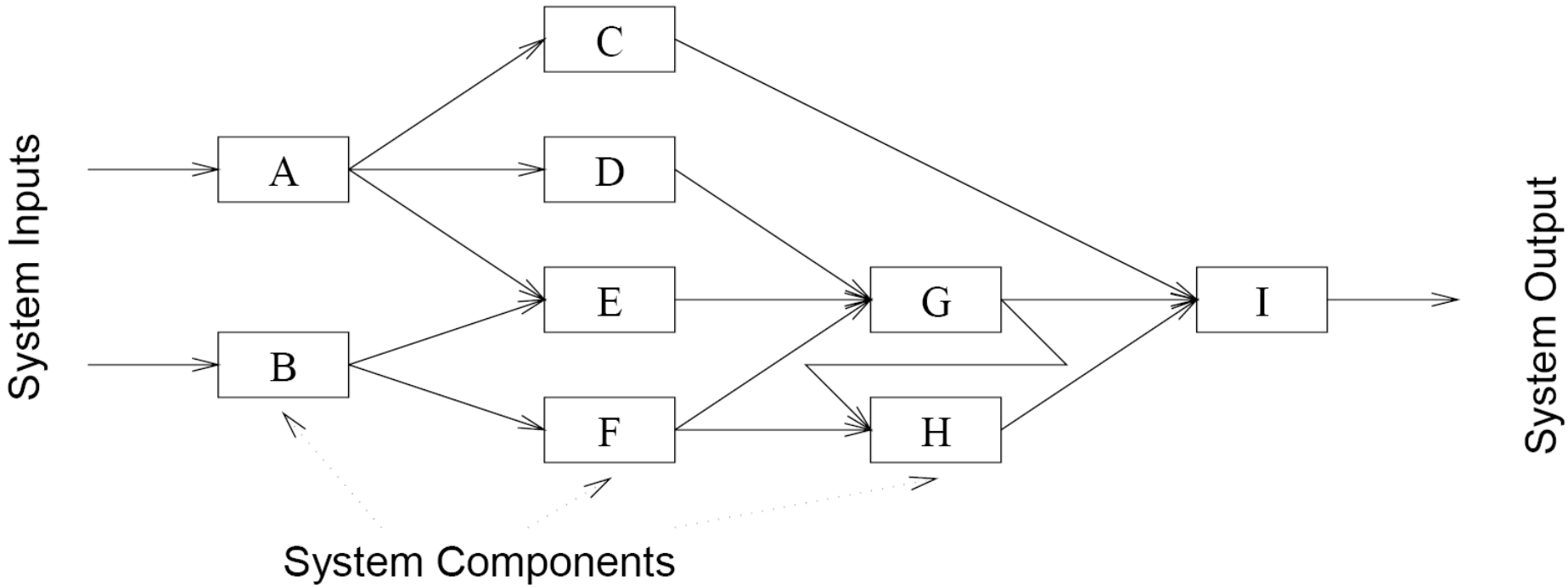
Exploratory decomposition



partition the search space into smaller parts, and search each concurrently, until the desired solutions are found

Speculative decomposition

Speculative decomposition is used when a program may take one of many possible computationally significant branches depending on the output of other computations



An example of an application in which speculative decomposition is useful is discrete event simulation.

Hybrid decomposition

```
1.    procedure COL_LU (A)
2.    begin
3.        for k := 1 to n do
4.            for j := k to n do
5.                A[j, k] := A[j, k]/A[k, k];
6.            endfor;
7.            for j := k + 1 to n do
8.                for i := k + 1 to n do
9.                    A[i, j] := A[i, j] − A[i, k] × A[k, j];
10.               endfor;
11.           endfor;
   /*
After this iteration, column A[k + 1 : n, k] is logically the kth
column of L and row A[k, k : n] is logically the kth row of U.
   */
12.       endfor;
13.   end COL_LU
```

Hybrid decomposition

$$\begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} \\ A_{2,1} & A_{2,2} & A_{2,3} \\ A_{3,1} & A_{3,2} & A_{3,3} \end{pmatrix} \rightarrow \begin{pmatrix} L_{1,1} & 0 & 0 \\ L_{2,1} & L_{2,2} & 0 \\ L_{3,1} & L_{3,2} & L_{3,3} \end{pmatrix} . \begin{pmatrix} U_{1,1} & U_{1,2} & U_{1,3} \\ 0 & U_{2,2} & U_{2,3} \\ 0 & 0 & U_{3,3} \end{pmatrix}$$

1: $A_{1,1} \rightarrow L_{1,1}U_{1,1}$
2: $L_{2,1} = A_{2,1}U_{1,1}^{-1}$
3: $L_{3,1} = A_{3,1}U_{1,1}^{-1}$
4: $U_{1,2} = L_{1,1}^{-1}A_{1,2}$
5: $U_{1,3} = L_{1,1}^{-1}A_{1,3}$

6: $A_{2,2} = A_{2,2} - L_{2,1}U_{1,2}$
7: $A_{3,2} = A_{3,2} - L_{3,1}U_{1,2}$
8: $A_{2,3} = A_{2,3} - L_{2,1}U_{1,3}$
9: $A_{3,3} = A_{3,3} - L_{3,1}U_{1,3}$
10: $A_{2,2} \rightarrow L_{2,2}U_{2,2}$

11: $L_{3,2} = A_{3,2}U_{2,2}^{-1}$
12: $U_{2,3} = L_{2,2}^{-1}A_{2,3}$
13: $A_{3,3} = A_{3,3} - L_{3,2}U_{2,3}$
14: $A_{3,3} \rightarrow L_{3,3}U_{3,3}$

Parallel Algorithm Models

Introduction: algorithm models

An algorithm model is typically a way of structuring a parallel algorithm by selecting a decomposition and mapping technique and applying the appropriate strategy to minimize interactions

- Data-Parallel
- Task Graph
- Work Pool
- Master-Slave
- Producer-Consumer
- Hybrid Models

Data-Parallel Model

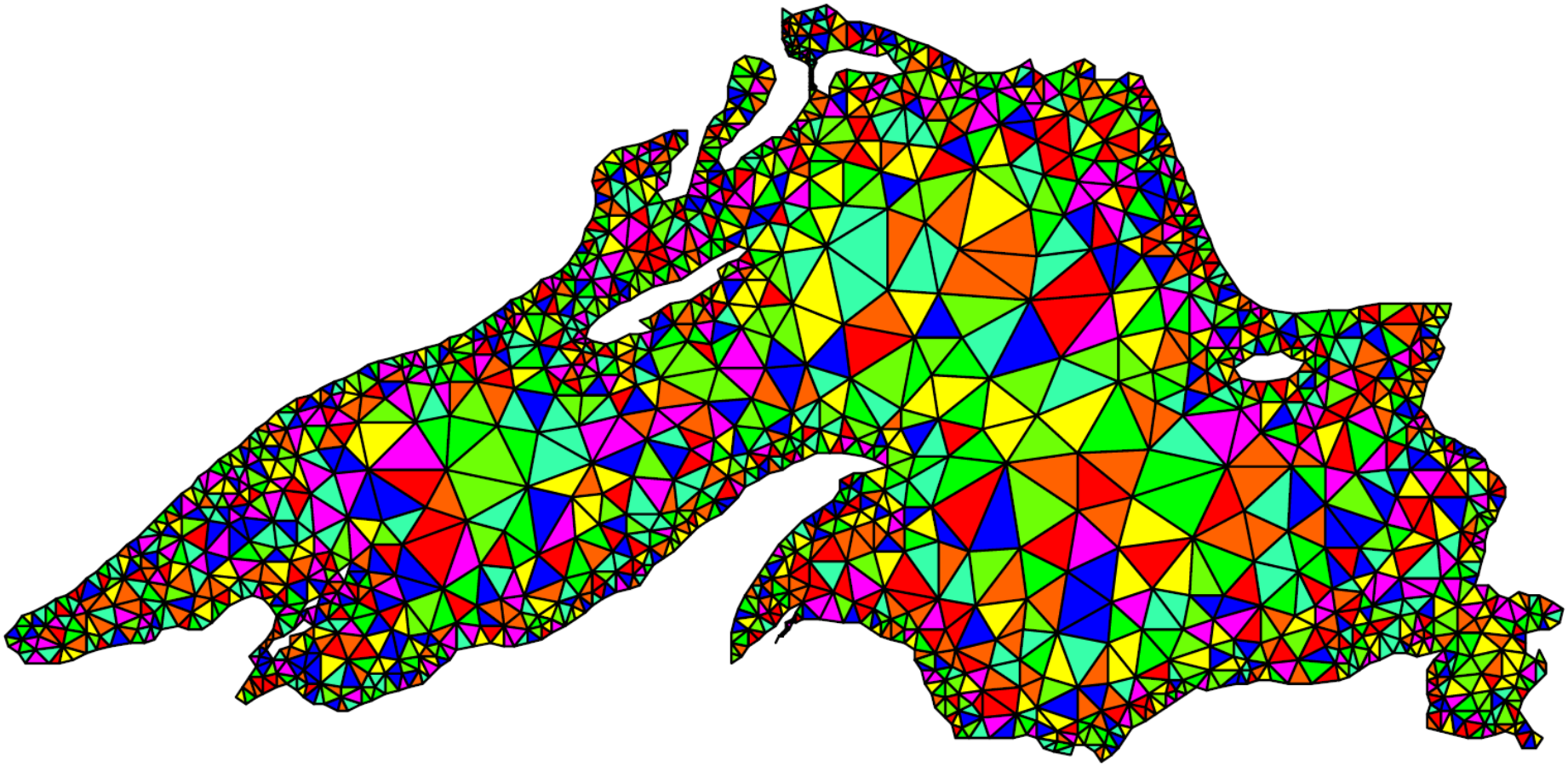
$$\begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix} \cdot \begin{pmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{pmatrix} \rightarrow \begin{pmatrix} C_{1,1} & C_{1,2} \\ C_{2,1} & C_{2,2} \end{pmatrix}$$

(a)

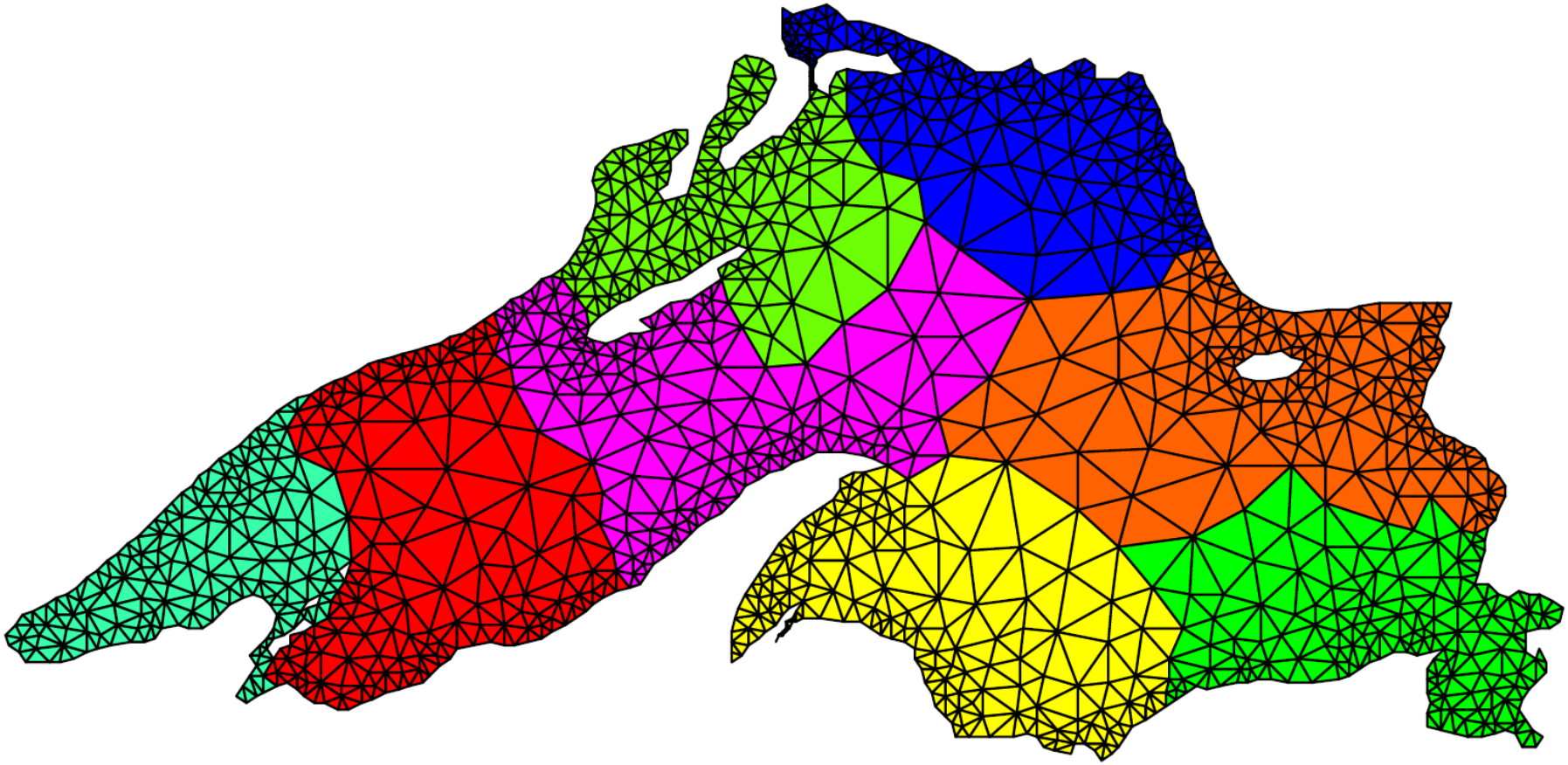
$$\begin{aligned} \text{Task 1: } C_{1,1} &= A_{1,1}B_{1,1} + A_{1,2}B_{2,1} \\ \text{Task 2: } C_{1,2} &= A_{1,1}B_{1,2} + A_{1,2}B_{2,2} \\ \text{Task 3: } C_{2,1} &= A_{2,1}B_{1,1} + A_{2,2}B_{2,1} \\ \text{Task 4: } C_{2,2} &= A_{2,1}B_{1,2} + A_{2,2}B_{2,2} \end{aligned}$$

(b)

Data-Parallel (geographical mesh)



Data-Parallel Model (geographical mesh)



Task Graph Model

(a)

1	2	3	4	5	6	7	8
33	21	13	54	82	33	40	72

(b) root = 4

(c)

	1	2	3	4	5	6	7	8
leftchild				1				
rightchild				5				

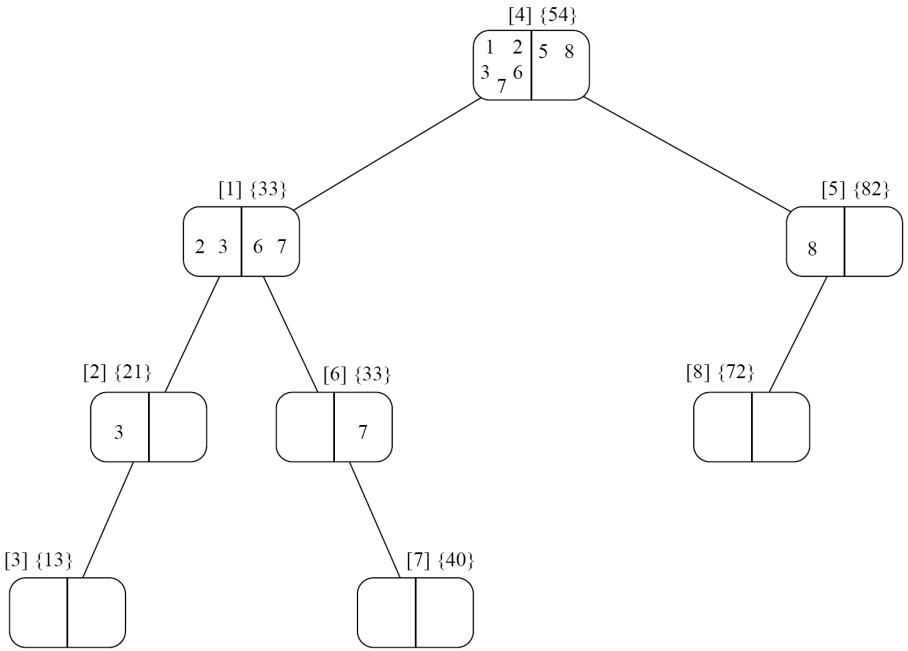
(d)

	1	2	3	4	5	6	7	8
leftchild	2			1	8			
rightchild	6			5				

(e)

	1	2	3	4	5	6	7	8
leftchild	2	3		1	8			
rightchild	6			5		7		

(f)



Work Pool Model

The work pool or the task pool model is characterized by a dynamic mapping of tasks onto processes for load balancing in which any task may potentially be performed by any process

There is no desired premapping of tasks onto processes.

In the message-passing paradigm, the work pool model is typically used when the amount of data associated with tasks is relatively small compared to the computation associated with the tasks.

Master-Slave Model

In the master-slave or the manager-worker model, master processes generate work and allocate it to worker processes.

The tasks may be allocated a priori if the manager can estimate the size of tasks or if a random mapping can do an adequate job of load balancing.

Usually, there is no desired premapping of work to processes, and any worker can do any job assigned to it.

This model is generally suitable to shared-address-space or message-passing paradigms since the interaction is naturally two-way:

- managers know that they need to give out work and
- workers know that they need to get work from the manager.

While using the master-slave model, care should be taken to ensure that the master does not become a bottleneck, which may happen if the tasks are too small (or the workers are relatively fast).

Pipeline or Producer-Consumer Model

In the pipeline model, a stream of data is passed on through a succession of processes, each of which perform some task on it.

This simultaneous execution of different programs on a data stream is called stream parallelism.

A pipeline is a chain of producers and consumers. Each process in the pipeline can be viewed as a consumer of a sequence of data items for the process preceding it in the pipeline and as a producer of data for the process following it in the pipeline.

The pipeline does not need to be a linear chain; it can be a directed graph.

The pipeline model usually involves a static mapping of tasks onto processes.

Hybrid Models

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