Geometric & Graphics Programming Lab: Lecture 19

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Outline: Hierarchical structures

Introduction

Curves

Curves as point loci with a specific shape are often identified by a proper name, such as circle, parabola, hyperbola, spiral, helix, etc.

For applications of computer-aided design, some classes of free-form curves are more interesting, because they are able to satisfy both geometric and esthetic constraints set by designers, depending on the shape design problem at hand.

Even more useful, splines are piecewise-continuous composite curves used to either interpolate or approximate a discrete set of points.

Curve representations

Curves may be represented by using different kinds of equations. In particular, it is possible to distinguish representations:

1. explicit or Cartesian, where the curve is given as the graph of a function;
2. implicit, as the zero set of one or more global algebraic equations;
3. parametric, associated with a vector function of one parameter;
4. intrinsic, where points locally satisfy differential equations.

Explicit representation

An explicit or Cartesian representation of a plane curve is the graph $(x, f(x))$ of a function $f : \mathbb{R} \to \mathbb{R}$. We may also write, in this case:

$$y = f(x).$$

- This representation is not diffuse nor useful in geometric modeling, because it is unusable for closed curves
- more in general, is unusable for curves where more than one value of the dependent variable y is associated with the same value of the independent variable x$
- it does not easily support affine transformations.

Explicit representation example

A simple example of explicit representation in the 2D plane is the well-known Cartesian equation of the line:

$$y = mx + c,$$

where m is called the angular coefficient and coincides with the tangent of the angle that the line creates with the x axis, and c is the ordinate of the intersection point between the line and the y axis.

- this representation cannot be used for vertical lines, for which we must conversely use the equation $x = a$.

Implicit representation

The implicit representation denotes a plane curve as the locus of points that satisfy an equation, usually algebraic, of type:

$$f(x, y) = 0.$$

The simplest example of a plane curve is given by the implicit representation of the 2D line:

$$ax + by + c = 0.$$

Parametric representation

With a parametric representation, a curve $\mathbf{c}$ is given as a point-valued map of a single real parameter:

$$\mathbf{c} : D \to \mathbb{E}^n \quad \text{such that} \quad \mathbf{c}(u) = \mathbf{o} + \begin{pmatrix} x_1(u) & \dots & x_n(u) \end{pmatrix}^T$$

where $\mathbf{o}$ is the origin of the reference frame, and $D \subset \mathbb{R}$ often coincides with the standard real interval $[0, 1]$.

- When $n = 2, 3$ the curve is said to be a plane or a space curve, respectively.
- The component functions $x_i : \mathbb{R} \to \mathbb{R}$, $i = 1, \dots, n$, are called the coordinate maps of the curve.
- The curve as a locus of points should be properly called the image of the curve.

Polynomial parametric curves

Parametric curves in graphics and CAD

The use of parametric curves in graphics and CAD started at the end of the 1960s, first within the aerospatial and automotive industries, both in Europe and in the USA, and later in academia, where applied mathematicians were studying mathematical methods for computer-aided geometric design, actually by simulating the hand-crafting of physically modeled curves and surfaces.

Previously, various instruments were used in the industry for this purpose, and in particular some mechanical and graphical "ad hoc" tools, from pantograph to a flexible ruler called a spline, whose name was soon extrapolated to denote the mathematical interpolation/approximation of sequences of points

Properties of parametric curves

- Control handles
- Multiple points
- Affine and projective invariance
- Local and global control
- Variation diminishing

Linear curves

Algebraic form

In general, let consider the polynomial curve of first degree, said in algebraic form:

$$\mathbf{C}(u) = \mathbf{a}u + \mathbf{b}$$

and write it as a product of matrices:

$$\mathbf{C}(u) = \begin{pmatrix} u & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} = \mathbf{U}_1 \; \mathbf{M}_1 \tag{1}$$

Geometric form (1/2)

The matrix equation~(1) contains two vector degrees of freedom, i.e.~two "free" vector coefficients **a** and **b**. In order to specify a given curve, we may specify two vector constraints to be satisfied by **a** and **b**.

$$\begin{aligned} \mathbf{C}(0) &= \mathbf{p}_1; \\ \mathbf{C}(1) &= \mathbf{p}_2. \end{aligned}$$

By substituting 0 and 1 for u in equation~1, we get, respectively:

$$\mathbf{p}_1 = \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{M}_1$$

$$\mathbf{p}_2 = \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{M}_1$$

Geometric form (2/2)

The above vector equations can be collected into the following matrix equation:

$$\begin{pmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \mathbf{M}_1,$$

from which we have

$$\mathbf{M}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \end{pmatrix},$$

and, by substitution into equation~1, we get

$$\mathbf{C}(u) = \begin{pmatrix} u & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \end{pmatrix} = \mathbf{U}_1 \ \mathbf{B}_1 \ \mathbf{G}_1. \qquad (2)$$

Terminology

Equation~1 is called the algebraic form of the curve.

Conversely, equation~2 is called the geometric form of passage through two given points

The matrix $\mathbf{B}_1$ is called the basis matrix, whereas the vector of points $\mathbf{G}_1$ is called the geometry tensor or tensor of control handles of the geometric form.

Finally, the vector of polynomial functions given by

$$\mathbf{U}_1\mathbf{B}_1 = \begin{pmatrix} u & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1-u & u \end{pmatrix} = \begin{pmatrix} b_1(u) & b_2(u) \end{pmatrix}$$

defines the polynomial basis of the geometric form, i.e.~its basis functions, called also blending functions of the curve.

Blending functions

The generic curve point $\mathbf{C}(u)$ is written as a combination of blending function with control handles, i.e.~as a sort of "blend" of such data:

$$\mathbf{C}(u) = \sum_{i=0}^{1} b_i(u)\ \mathbf{p}_i.$$