

The join of two sets $P, Q \subset \mathbb{R}^n$ is the set $PQ = \{\gamma p + \lambda q, p \in P, q \in Q\}$, where $\gamma, \lambda \in \mathbb{R}, \gamma, \lambda \geq 0$, and $\gamma + \lambda = 1$. The join operation is associative and commutative. A simplex $\sigma \subset \mathbb{R}^n$ of order d , or d -simplex, is the join of $d + 1$ affinely independent points, called vertices. The $n + 1$ points p_0, \dots, p_n are affinely independent when the n vectors $\mathbf{p}_1 - \mathbf{p}_0, \dots, \mathbf{p}_n - \mathbf{p}_0$ are linearly independent. A d -simplex can be seen as a d -dimensional triangle: 0-simplex is a point, 1-simplex is a segment, 2-simplex is a triangle, 3-simplex is a tetrahedron, and so on. Any subset of $s + 1$ vertices ($0 \leq s \leq d$) of a d -simplex σ defines an s -simplex, which is called s -face of σ .