Geometric and Graphics Programming Laboratory: Lecture 13

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Introduction

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- explicit or Cartesian, where the curve is given as the graph of a function;
- implicit, as the zero set of one or more global algebraic equations;
- parametric, associated with a vector function of one parameter;
- intrinsic, where points locally satisfy differential equations.

Outline: Curve1

Parametric representation

Polynomial and/or rational curves

When the coordinate maps are polynomial functions of a single variable the curve is called a polynomial (parametric) curve.

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- Both polynomial and rational curves are liked by shape designers, mainly because their coefficients have a precise geometric meaning and assess specific esthetic and geometric constraints on the generated free-form curves.
- By editing such coefficients, normally by using very simple and intuitive graphics tools, the designer is given strong control over shape generation.

Polynomial function

A polynomial of degree n is a function $f : \mathbb{R} \to \mathbb{R}$ of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$$

where the a's are real numbers (sometimes called the coefficients of the polynomial)

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- The set of polynomials of degree less or equal to n is a vector space of dimension $n + 1$.
- The powers of x give a basis for this space:

$$[x^n, x^{n-1}, ..., x^2, x, 1]$$

polynomial parametric curve

A polynomial parametric curve may be seen as a linear combination, with vector coefficients, of the elements of a suitable polynomial basis

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- depending on the choice of the polynomial basis, they may correspond either to points interpolated by the curve, or to approximated points, or to tangent vectors, etc
- vector coefficients are called geometric handles or control handles

Properties of parametric curves

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- Multiple points
- Affine and projective invariance
- Local and global control
- Variation diminishing

Lagrange curves

Linear curves

We know that the line segment between points $\mathbf{p}_1$ and $\mathbf{p}_2$ can be written, in any $\mathbb{E}^d$, as:

$$(\mathbf{p}_2 - \mathbf{p}_1)u + \mathbf{p}_1, \qquad u \in [0,1],$$

i.e. as a polynomial with vector coefficients in the u indeterminate.

In general, let consider the polynomial curve of first degree, said in algebraic form:

$$\mathbf{C}(u) = \mathbf{a}u + \mathbf{b}$$

and write it as a product of matrices:

$$\mathbf{C}(u) = \begin{pmatrix} u & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} = \mathbf{U}_1 \ \mathbf{M}_1$$

Geometric form

In order to specify a given curve, we may specify two vector constraints to be satisfied by **a** and **b**.

We can, for example, require that the curve interpolates, at the extreme values of the parameter, two given points $\mathbf{p}_1$ and $\mathbf{p}_2$:

$$\mathbf{C}(0) = \mathbf{p}_1; \tag{1}$$
$$\mathbf{C}(1) = \mathbf{p}_2. \tag{2}$$

By substituting 0 and 1 for u in equation~11, we get, respectively:

$$\mathbf{p}_1 = \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{M}_1 \tag{3}$$
$$\mathbf{p}_2 = \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{M}_1 \tag{4}$$

Geometric form

The above vector equations can be collected into the following matrix equation:

$$\begin{pmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \mathbf{M}_1,$$

from which we have

$$\mathbf{M}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \end{pmatrix},$$

and, by substitution into equation~11, we get

$$\mathbf{C}(u) = \begin{pmatrix} u & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \end{pmatrix} = \mathbf{U}_1 \ \mathbf{B}_1 \ \mathbf{G}_1. \tag{5}$$

Terminology

The matrix $\mathbf{B}_1$ is called the basis matrix, whereas the vector of points $\mathbf{G}_1$ is called the geometry vector or tensor of control handles of the geometric form.

The vector of polynomial functions given by

$$\mathbf{U}_1\mathbf{B}_1 = \begin{pmatrix} u & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1-u & u \end{pmatrix} = \begin{pmatrix} b_1(u) & b_2(u) \end{pmatrix}$$

defines the $\{$*polynomial basis$\}$ of the geometric form, i.e.~its basis functions, called also* $\{$blending functions$\}$ of the curve.

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The curve point $\mathbf{C}(u)$ written as a combination of blending function with control handles, is a sort of “blend" of such data:

$$\mathbf{C}(u) = \sum_{i=0}^{1} b_i(u)\ \mathbf{p}_i.$$

Quadratic curves

The algebraic form of the polynomial parametric curves of second degree depends on three free vector parameters:

$$\begin{aligned}\mathbf{C}(u) &= \mathbf{a}u^2 + \mathbf{b}u + \mathbf{c} \\ &= \begin{pmatrix} u^2 & u & 1 \end{pmatrix} \mathbf{M}_2 \\ &= U_2\ \mathbf{M}_2 \end{aligned} \tag{6}$$

The passage of the curve for three given points $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{p}_3$ can be imposed in order to specify the matrix $\mathbf{M}_2$.

Quadratic curves

$$\begin{pmatrix} \mathbf{C}(0) \\ \mathbf{C}(\frac{1}{2}) \\ \mathbf{C}(1) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{4} & \frac{1}{2} & 1 \\ 1 & 1 & 1 \end{pmatrix} \mathbf{M}_2 = \begin{pmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{pmatrix},$$

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from which we get

$$\mathbf{M}_2 = \begin{pmatrix} 2 & -4 & 2 \\ -3 & 4 & -1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{pmatrix},$$

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and hence

$$\mathbf{C}(u) = U_2\mathbf{M}_2 = \begin{pmatrix} u^2 & u & 1 \end{pmatrix} \begin{pmatrix} 2 & -4 & 2 \\ -3 & 4 & -1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{pmatrix} = U_2\mathbf{B}_2\mathbf{G}_2.$$

The interpolating form is called Lagrange's geometric form of the quadratic polynomial curve

Quadratic curves

If we want to underline the meaning of this quadratic curve as a polynomial blending of three given points, then we may write:

$$\mathbf{C}(u) = (2u^2 - 3u + 1)\mathbf{p}_1 + (-4u^2 + 4u)\mathbf{p}_2 + (2u^2 - u)\mathbf{p}_3.$$

Cubic curves

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The algebraic form of their equation contains four vector parameters:

$$
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&= \begin{pmatrix} u^3 & u^2 & u & 1 \end{pmatrix} \mathbf{M}_3 \\
&= \mathbf{U}_3 \; \mathbf{M}_3
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Every set of four vector constraints that allow specification of the four degrees of freedom defines a different geometric form of the cubic curve. Such constraints may impose the passage of the curve through four assigned points, or through two points with assigned tangents (derivatives) in those points, and so on

Cubic Lagrange curve

$$\mathbf{C}(0) = \mathbf{p}_1, \qquad \mathbf{C}\left(\tfrac{1}{3}\right) = \mathbf{p}_2, \qquad \mathbf{C}\left(\tfrac{2}{3}\right) = \mathbf{p}_3, \qquad \mathbf{C}(1) = \mathbf{p}_4,$$

so that, by substitution in equation~(7) we have

$$\begin{pmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \\ \mathbf{p}_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ \frac{1}{27} & \frac{1}{9} & \frac{1}{3} & 1 \\ \frac{8}{27} & \frac{4}{9} & \frac{2}{3} & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \mathbf{M}_3,$$

and hence

$$\mathbf{M}_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ \frac{1}{27} & \frac{1}{9} & \frac{1}{3} & 1 \\ \frac{8}{27} & \frac{4}{9} & \frac{2}{3} & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \\ \mathbf{p}_4 \end{pmatrix} = \mathbf{B}_L \ \mathbf{G}_L.$$

the Lagrange form of the cubic curve is so defined as:

$$\mathbf{C}(u) = \mathbf{U}_3 \ \mathbf{B}_L \ \mathbf{G}_L \tag{8}$$

where $\mathbf{U}_3$ is the cubic power basis, $\mathbf{B}_L$ is the matrix of the cubic Lagrange's basis and $\mathbf{G}_L$ is the Lagrange control tensor.

Hermite curves

Hermite cubic curve

In this case the cubic curve segment is forced to have assigned extreme points $\mathbf{p}_1$ and $\mathbf{p}_2$ and assigned extreme tangents $\mathbf{s}_1$ and $\mathbf{s}_2$, with $u \in [0, 1]$.

Such constraints are therefore:

$$\begin{aligned} \mathbf{C}(0) &= \mathbf{p}_1 \\ \mathbf{C}(1) &= \mathbf{p}_2 \\ \mathbf{C}'(0) &= \mathbf{s}_1 \\ \mathbf{C}'(1) &= \mathbf{s}_2 \end{aligned}$$

From the algebraic form~(7) of the curve we can compute the derivative with respect to the parameter, which gives the tangents to the curve:

$$\mathbf{C}'(u) = \begin{pmatrix} 3u^2 & 2u & 1 & 0 \end{pmatrix} \mathbf{M}_3 \tag{9}$$

Hermite cubic curve

by substitution of $0, 1$ for u in equations~(7) or (9), respectively:

$$\begin{pmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{s}_1 \\ \mathbf{s}_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{pmatrix} \mathbf{M}_3.$$

So, we get

$$\mathbf{M}_3 = \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{s}_1 \\ \mathbf{s}_2 \end{pmatrix}$$

so that the Hermite geometric form for the cubic polynomial curves results

$$\mathbf{C}(u) = \begin{pmatrix} u^3 & u^2 & u & 1 \end{pmatrix} \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{s}_1 \\ \mathbf{s}_2 \end{pmatrix}.$$

Hermite cubic curve

It can be written in matrix form as

$$\mathbf{C}(u) = U_3\ \mathbf{B}_h\ \mathbf{G}_h, \tag{10}$$

where $\mathbf{B}_h$ is the matrix of the Hermite Basis and $\mathbf{G}_h$ is the Hermite control tensor of cubic curves.

From equation~(10) we get

$$\mathbf{C}(u) = \begin{pmatrix} h_1(u) & h_2(u) & h_3(u) & h_4(u) \end{pmatrix} \begin{pmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{s}_1 \\ \mathbf{s}_2 \end{pmatrix}$$

where the Hermite basis polynomials are:

$$\begin{aligned} h_1(u) &= 2u^3 - 3u^2 + 1 \\ h_2(u) &= -2u^3 + 3u^2 \\ h_3(u) &= u^3 - 2u^2 + u \\ h_4(u) &= u^3 - u^2 \end{aligned}$$

Hermite cubic curve

Assignment

Generate in Python the cubic Hermite basis polynomials

Bezier curves

Title

text

References

GP4CAD book