Geometric & Graphics Programming Lab: Lecture 11

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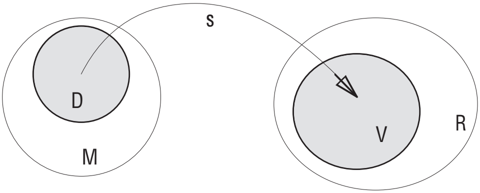
Outline: Cell complexes (LAR)

1. Solid Modeling
2. Space decompositions
3. Cellular complex
4. Simplicial mapping

Solid Modeling

Representation scheme: definition

mapping $s : M \rightarrow R$ from a space M of mathematical models to a space R of computer representations



1. The M set contains the mathematical models of the class of solid objects the scheme aims to represent
2. The R set contains the symbolic representations, i.e. the proper data structures, built according to a suitable grammar

A. Requicha, Representations for Rigid Solids: Theory, Methods, and Systems, ACM Comput. Surv., 1980.
V. Shapiro, Solid Modeling, In Handbook of Computer Aided Geometric Design, 2001

Representation schemes

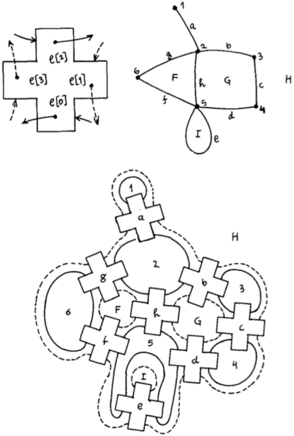
Most of such papers introduce or discuss one or more representation schemes . . .

1. Requicha, ACM Comput. Surv., 1980 [?]
2. Requicha & Voelcker, PEP TM-25, 1977, [?]
3. Rossignac & Requicha, Comput. Aided Des., 1991, [?]
4. Bowyer, SVLIS, 1994, [?]
5. Baumgart, Stan-CS-320, 1972, [?]
6. Braid, Commun. ACM, 1975, [?]
7. Dobkin & Laszlo, ACM SCG, 1987, [?]
8. Guibas & Stolfi, ACM Trans. Graph., 1985, [?]
9. Woo, IEEE Comp. Graph. & Appl., 1985, [?]
10. Yamaguchi & Kimura, Comp. Graph. & Appl., 1995, [?]
11. Gursoz & Choi & Prinz, Geom.Mod., 1990, [?]
12. S.S.Lee & K.Lee, ACM SMA, 2001, [?]
13. Rossignac & O'Connor, IFIP WG 5.2, 1988, [?]
14. Weiler, IEEE Comp. Graph. & Appl., 1985, [?]
15. Silva, Rochester, PEP TM-36, 1981, [?]
16. Shapiro, Cornell Ph.D Th., 1991, [?]
17. Paoluzzi et al., ACM Trans. Graph., 1993, [?]
18. Pratt & Anderson, ICAP, 1994, [?]
19. Bowyer, Djinn, 1995, [?]
20. Gomes et al., ACM SMA, 1999, [?]
21. Raghothama & Shapiro, ACM Trans. Graph., 1998, [?]
22. Shapiro & Vossler, ACM SMA, 1995, [?]
23. Hoffmann & Kim, Comput. Aided Des., 2001, [?]
24. Raghothama & Shapiro, ACM SMA, 1999, [?]
25. DiCarlo et al., IEEE TASE, 2008, [?]
26. Bajaj et al., CAD&A, 2006, [?]
27. Pascucci et al., ACM SMA, 1995, [?]
28. Paoluzzi et al., ACM Trans. Graph., 1995, [?]
29. Paoluzzi et al., Comput. Aided Des., 1989, [?]
30. Ala, IEEE Comput. Graph. Appl., 1992, [?]

and much more . . .

Representation scheme: Quad-Edge data structure

(Guibas & Stolfi, ACM Transactions on Graphics, 1985)



(a) Edge record showing Next links.
(b) A subdivision of the sphere.
(c) Data structure for the subdivision

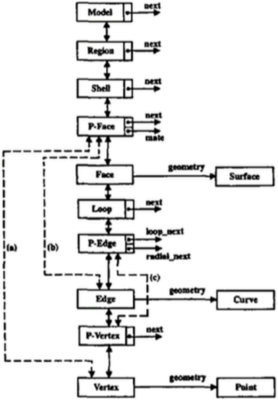
Primitives for the Manipulation of General Subdivisions and the Computation of Voronoi Diagrams

largely used in computational geometry algorithms and in geometric libraries

Representation scheme: Partial-Entity data structure

(Sang Hun Lee & Kunwoo Lee, ACM Solid Modeling, 2001)

Compact Non-Manifold Boundary Representation Based on Partial Topological Entities

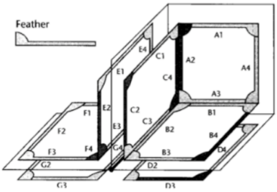



```
class Entity {
   int        _id;
   Attribute  *_attribute;
};
class Model : public Entity {
   Model      *_next;         // next model
   Region     *_region;       // list of regions
};
class Region : public Entity {
   Region     *_next;         // link field of the region list of a model
   Model      *_model;        // parent model
   Shell      *_shell;        // peripheral shell
};
class Shell : public Entity {
   Shell      *_next;         // next void shell
   Region     *_region;       // parent region
   Pface      *_pface;        // partial face
};
class Pface : public Entity {  // partial face (p-face) class
   Pface      *_next;         // next p-face
   Shell      *_shell;        // parent shell
   Entity     *_child;        // child entity: a face, an edge, or a vert
   Orient     _orient;        // orientation flag w.r.t. the face normal
   Pface      *_mate;         // mate p-face
};
class Face : public Entity {
   Pface      *_pface;        // one of two incident p-faces
   Loop       *_loop;         // peripheral loop
   Surface    *_geometry;     // surface
};
```

```
class Loop : public Entity {
   Loop       *_next;         // next hole loop
   Face       *_face;         // parent face
   Pedge      *_pedge;        // a p-edge in a loop
};
class Pedge : public Entity {  // partial edge (p-edge) class
   Loop       *_loop;         // parent loop
   Entity     *_child;        // child entity: an edge or a p-vertex
   Orient     *_orient;       // orientation flag w.r.t. the edge direction
   Pvertex    *_pvertex;      // start p-vertex
   Pedge      *_looped_prev;  // previous p-edge in the loop cycle
   Pedge      *_looped_next;  // next p-edge in the loop cycle
   Pedge      *_radial_prev;  // previous p-edge in the radial cycle
   Pedge      *_radial_next;  // next p-edge in the radial cycle
};
class Edge : public Entity {
   Entity     *_parent;       // parent entity: a p-edge or a p-face
   Pvertex    *_pvertex[2];   // two end p-vertices
   Curve      *_geometry;     // curve
};
class Pvertex : public Entity { // partial vertex (p-vertex) class
   Pvertex    *_next;         // another p-vertex associated with _vertex
   Entity     *_parent;       // parent entity: an edge or a p-edge
   Vertex     *_vertex;       // mother vertex
};
class Vertex : public Entity {
   Entity     *_parent;       // parent entity: a p-vertex or a p-face
   Point      *_geometry;     // position
};
```

Representation scheme: Coupling Entities data structure

(Yamaguchi & Kimura, IEEE Computer Graphics and Applications, 1995)



Fan
B4 & D3 :D3 = FM(B4), B4 = FM(D3)

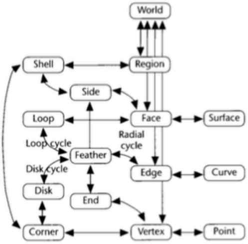
Blade
C2 & E2 :E2 = BM(C2), C2 = BM(E2)

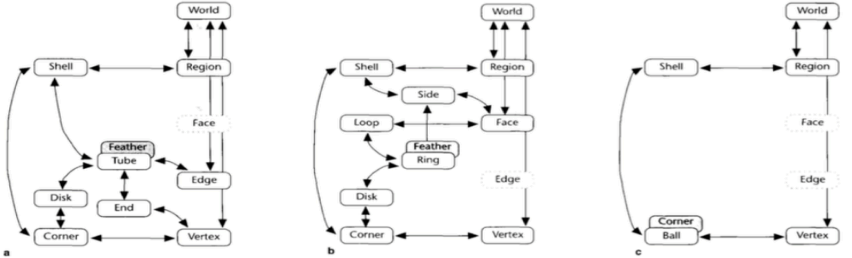
Wedge
A2 & C4 :C4 = WM(A2), A2 = WM(C4)

Loop cycle
A1>>A2>>A3>>A4 :A2 = CCL(A1), A3 = CCL(A2)
A4>>A3>>A2>>A1 :A1 = CL(A2), A2 = CL(A3)

Radial cycle
D2>>C3>>F4 :C3 = CCR(D2), F4 = CCR(C3)
F4>>C3>>D2 :D2 = CR(C3), C3 = CR(F4)

Disk cycle
A3>>C4>>B2 :C4 = CCD(A3), B2 = CCD(C4)
B2>>C4>>A3 :A3 = CD(C4), C4 = CD(B2)





Space decompositions

Join of pointsets

The join of two sets $P, Q \subset \mathbb{E}^n$ is the convex hull of their points:

$$PQ = join(P, Q) := \{\gamma p + \lambda q,\ p \in P,\ q \in Q\}$$
$$\gamma, \lambda \in \mathbb{R},\ \gamma, \lambda \geq 0,\ \gamma + \lambda = 1$$

.

The join operation is associative and commutative.

Join of pointsets: examples

```
pts = [[0,0],[.5,0],[0,.5],[.5,.5],
       [1,.5],[1.5,.5],[1.5,1],[.25,1]]      # coords
P = AA(MK)(pts)                               # 0-polyhedra
S = AA(JOIN)([P[0:4],P[4:7],P[7]])            # array of d-polyhedra
H = JOIN(S)                                   # 2-polyhedron

VIEW(STRUCT(AA(SKELETON(1))(S)))
VIEW(H)
```

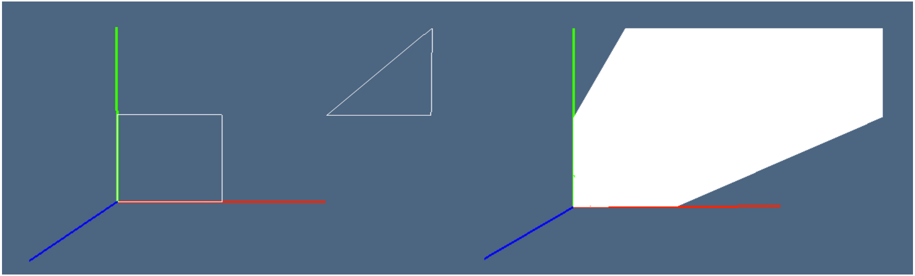


Figure 1: (a) 1-skeleton of pointsets in S; (b) convex hull H of pointset P

Simplex

A simplex $\sigma \subset \mathbb{E}^n$ of order d, or d-simplex, is the join of $d+1$ affinely independent points, called vertices.
The $n+1$ points $p_0, \ldots, p_n$ are affinely independent when the n vectors $p_1 - p_0, \ldots, p_n - p_0$ are linearly independent.
A d-simplex can be seen as a d-dimensional triangle: 0-simplex is a point, 1-simplex is a segment, 2-simplex is a triangle, 3-simplex is a tetrahedron, and so on.

Simplex: examples

```
s0,s1,s2,s3 = [SIMPLEX(d) for d in range(4)]     # array of standard d-simplices
VIEW(s1); VIEW(s2); VIEW(s3);

points = [[1,1,1],[0,1,1],[1,0,0],[1,1,0]]       # coords of 4 points
tetra = JOIN(AA(MK)(points))                     # 3-simplex
VIEW(tetra)
```

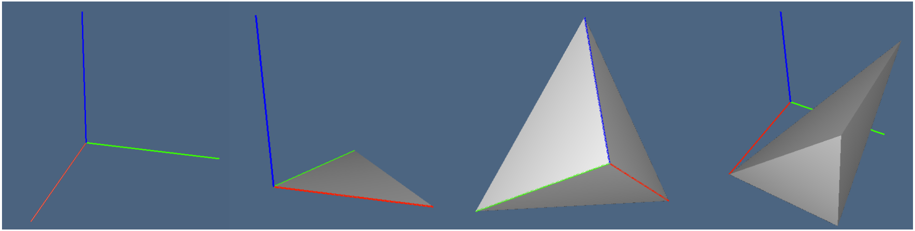


Figure 2: (a,b,c) 1-, 2-, and 3-standard simplex; (d) 3-simplex defined by 4 points

Simplicial complex

Any subset of $s + 1$ vertices $(0 \geq s \geq d)$ of a d-simplex σ defines an s-simplex, which is called s-face of σ.

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1. if $\sigma \in \Sigma$, then any s-face of σ belongs to Σ;
2. if $\sigma, \tau \in \Sigma$, then $\sigma \cap \tau \in \Sigma$.

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Geometric carrier $|\Sigma|$ is the pointset union of simplices in Σ.

Simplicial complex: examples

```
from larlib import *                                    # import LAR library
V,CV = larSimplexGrid1([5,5,5])                         # structured simplicial grid
FV = larSimplexFacets(CV)                               # 2-simplicial grid
EV = larSimplexFacets(FV)                               # 1-simplicial grid
VIEW(EXPLODE(1.5,1.5,1.5)(MKPOLS((V,CV))))
VIEW(EXPLODE(1.5,1.5,1.5)(MKPOLS((V,FV))))
VIEW(EXPLODE(1.5,1.5,1.5)(MKPOLS((V,EV))))

BV = [FV[t] for t in boundaryCells(CV,FV)]              # boundary 2-simplices
VIEW(EXPLODE(1.5,1.5,1.5)(MKPOLS((V,BV))))
```

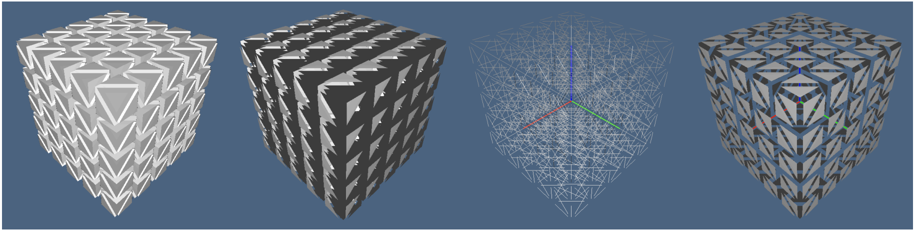

Figure 3: (a) 3-complex; (b) 2-subcomplex; (c) 1-subcomplex; (d) 2-boundary.

Simplicial complex: examples (see Disk Point Picking)

```
from larlib import *; from random import random as rand
points = [[2*PI*rand(),rand()] for k in range(1000)]
V = [[SQRT(r)*COS(alpha),SQRT(r)*SIN(alpha)] for alpha,r in points]
cells = [[k+1] for k,v in enumerate(V)]
VIEW(MKPOL([V,cells,None]))

from scipy.spatial import Delaunay
FV = Delaunay(array(V)).vertices
VIEW(EXPLODE(1.2,1.2,1)(MKPOLS((V,FV))))
VIEW(SKELETON(1)(STRUCT(MKPOLS((V,FV)))))
```

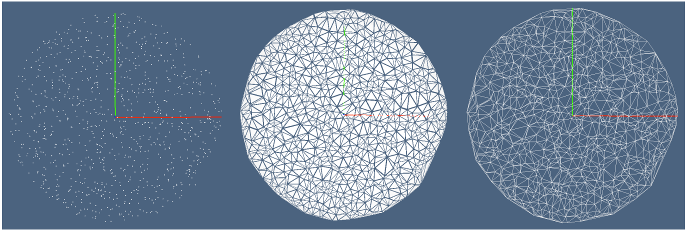

Figure 4: (a) Points; (b) Delaunay triangulation; (c) 1-skeleton.

Simplicial complex: examples

(see Disk Point Picking)

```
from larlib import *; from random import random as rand
points = [[2*rand()-1,2*rand()-1,2*rand()-1] for k in range(30000)]
V = [p for p in points if VECTNORM(p) <= 1]
VIEW(STRUCT(MKPOLS((V,AA(LIST)(range(len(V)))))))
```

```
from scipy.spatial import Delaunay
CV = Delaunay(array(V)).vertices
def test(tetra): return AND([v[-1] < 0 for v in tetra])
CV = [cell for cell in CV if test([V[v] for v in cell]) ]
VIEW(STRUCT(MKPOLS((V,CV))))
```

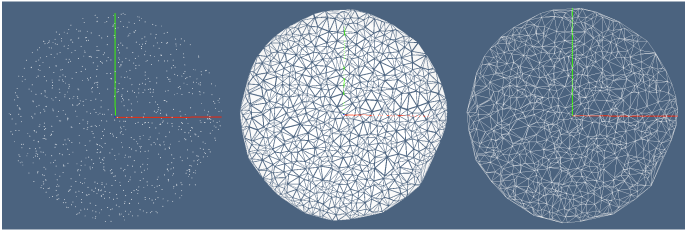

Figure 5: (a) Points; (b) Delaunay triangulation; (c) 1-skeleton.

Cellular complex

Politopal, simplicial, cuboidal complexes

Cellular complexes characterised by different types of cells:

1. In polytopal complexes cells are polytopes, i.e. bounded convex sets;
2. In simplicial complexes cells are simplices, i.e. d-polyedra with $d+1$ facets ($(d-1)$-faces) and $d+1$ vertices.
3. In cuboidal complexes cells are cuboids, (in general, sets homeomorphic to) Cartesian products of intervals, i.e. d-polyedra with $2d$ facets and $2d$ vertices.

Numbers of vertices and facets

A d-simplex, or d-dimensional simplex, has $d+1$ extremal points called vertices and $d+1$ facets.

- a point (0-simplex) has $0+1=1$ vertices and 1 facet ($\emptyset$);
- an edge (1-simplex) has $1+1=2$ vertices and 2 facets;
- a triangle (2-simplex) has $2+1=3$ vertices and 3 facets;
- a tetrahedron (3-simplex) has $3+1=4$ vertices and 4 facets, etc.

Numbers of vertices and facets

A *d*-simplex, or *d*-dimensional simplex, has $d+1$ extremal points called vertices and $d+1$ facets.

- a point (0-simplex) has $0+1=1$ vertices and 1 facet ($\emptyset$);
- an edge (1-simplex) has $1+1=2$ vertices and 2 facets;
- a triangle (2-simplex) has $2+1=3$ vertices and 3 facets;
- a tetrahedron (3-simplex) has $3+1=4$ vertices and 4 facets, etc.

A *d*-cuboid has conversely 2^d vertices and $2d$ facets:

- a point (0-cuboid) has $2^0=1$ vertices and 0 facets;
- an edge (1-cuboid) has $2^1=2$ vertices and 2 facets;
- a quadrilateral (2-cuboid) has $2^2=4$ vertices and 4 facets;
- a hexahedron (3-cuboids) has $2^3=8$ vertices and 8 facets, etc.

Cellular complex: properties

- Support $|K|$ of a cellular complex K is the union of points of its cells
- A triangulation of a polytope P is a simplicial complex K whose support is $|K| = P$
  - For example, a triangulation of a polygon is a subdivision in triangles

- Simplices and cuboids are polytopes.
- A polytope is always triangulable;
  - For example, a quadrilateral by be divided in two triangles, and a cube in either 5 or 6 tetrahedra without adding new vertices

Simplicial mapping

Simplicial mapping: definition

Definition

A simplicial map is a map between simplicial complexes with the property that the images of the vertices of a simplex always span a simplex.

Remarks

Simplicial maps are determined by their effects on vertices
for a precise definition of Simplicial Map look at Wolfram MathWorld

MAP operator in plasm

Map operator

`MAP(fun)(domain)`

Semantics

1. domain (HPC value) is decomposed into a simplicial complex
2. fun (a simplicial function) is applied to the domain vertices
3. the mapped domain is returned

MAP examples: 1-sphere (S^1) and 2-disk (D^2)

```
def sphere1(p): return [COS(p[0]), SIN(p[0])] # point function
def domain(n): return INTERVALS(2*PI)(n)      # generator of domain decomp
VIEW( MAP(sphere1)(domain(32)) )              # geometric value (HPC type)

def disk2D(p):                                            # point function
    u,v = p
    return [v*COS(u), v*SIN(u)]                           # coordinate functions
domain2D = PROD([INTERVALS(2*PI)(32), INTERVALS(1)(3)]) # 2D domain decompos
VIEW( MAP(disk2D)(domain2D) )
VIEW( SKELETON(1)(MAP(disk2D)(domain2D)) )
```

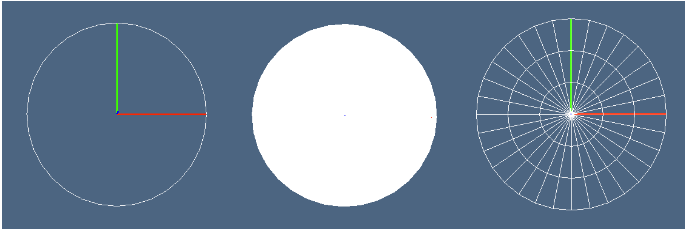

Figure 6: (a) sphere S^1 (b) disk D^2; (c) 1-skeleton.