Computational Graphics: Lecture 3

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Outline: Algebra reminders

Linear spaces

Definition

A linear (or vector) space $\mathcal{V}$ over a field $\mathcal{F}$ is a set with two composition rules, such that, for each $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathcal{V}$ and for each $\alpha, \beta \in \mathcal{F}$, the rules $+, \cdot$ satisfy the following axioms:

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8. $1 \cdot \mathbf{v} = \mathbf{v}$. (neutral element of product)

Example: vector space of real matrices

Let $\mathcal{M}_n^m(\mathbb{R})$ be the set of $m \times n$ matrices with elements in the field $\mathbb{R}$. An element A in such a set is denoted as

$$A = (\alpha_{ij})$$

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Addition and multiplication by a scalar are defined component-wise:

$$A + B = (\alpha_{ij}) + (\beta_{ij}) = (\alpha_{ij} + \beta_{ij})$$

$$\gamma A = \gamma(\alpha_{ij}) = (\gamma\alpha_{ij})$$

Linear combinations

Linear combination

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in \mathcal{V}$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathcal{F}$,
The vector

$$\alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n = \sum_{i=1}^{n} \alpha_i \mathbf{v}_i \in \mathcal{V}$$

is called a linear combination of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ with scalars $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathcal{F}$

Subspaces

Subspace

Let $(\mathcal{V}, +, \cdot)$ be a vector space on the field $\mathcal{F}$.

$\mathcal{U} \subset \mathcal{V}$ is a subspace of $\mathcal{V}$ if $(\mathcal{U}, +, \cdot)$ is a vector space with respect to the same operations.

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$\mathcal{U} \neq \emptyset$;

for each $\alpha \in \mathcal{F}$ and $\mathbf{u}_1, \mathbf{u}_2 \in \mathcal{U}$, $\alpha\mathbf{u}_1 + \mathbf{u}_2 \in \mathcal{U}$

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Question

Examples of codimension? in 1D, 2D, 3D

Spans

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- If a subspace $\mathcal{U}$ of $\mathcal{V}$ can be generated as the span of a set S of vectors in $\mathcal{V}$, then S is called a generating set or a spanning set for $\mathcal{U}$.

Linear independence

- A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is linearly independent if

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  implies that $\alpha_i = 0$ for each i
- As a consequence, a set of vectors is linearly independent when none of them belongs to the span of the others.

Bases

Bases and coordinates

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When working with vector spaces, the concept of basis, a discrete subset of linearly independent elements, is probably the most useful to deal with.

- each element of the space can be represented uniquely as linear combination of basis elements
- this leads to a parametrization of the space, i.e. to represent each element by a sequence of scalars, called its coordinates with respect to the chosen basis.

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2. $\mathcal{V} = \mathrm{lin}\,\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$

Bases

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  2. each minimal spanning set is a basis;
  3. each linearly independent set of vectors is contained in a basis;
  4. each maximal set of linearly independent vectors is a basis;

Example: vector space of polynomials of degree $\leq n$

A linear space we make often use of in Computer Graphics and Geometric modeling is the space of dimension $n+1$:

$$\mathbb{P}^n(\mathbb{R}) = \{p : \mathbb{R} \to \mathbb{R} : u \mapsto \sum_{i=0}^{n} a_i u^i, a_i \in \mathbb{R}\}$$

of univariate polynomials of degree $\leq n$ on the real field (with real coefficients), with $p^i \in P_n$, where

$$P_n = (p^n, p^{n-1}, ..., p^1, p^0) \quad \text{and} \quad p^i : u \mapsto u^i$$

is the power basis.

Components

If $(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n)$ is an ordered basis for $\mathcal{V}$, then for each $\mathbf{v} \in \mathcal{V}$ there exists a unique n-tuple of scalars $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathcal{F}$ such that

$$\mathbf{v} = \sum_{i=1}^{n} \alpha_i \mathbf{e}_i .$$

Components

The n-tuple of scalars (α_i) is called the components of $\mathbf{v}$ with respect to the ordered basis $(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n)$.

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- i.e. $\alpha_i = \beta_i$, for every i.

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- and hence:

$$[T] = [V]^{-1}$$

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$$[B_3]_{P_3} = \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

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- WHY ?

Affine spaces

Affine space

The idea of affine space corresponds to that of a set of points where the displacement from a point $\mathbf{x}$ to another point $\mathbf{y}$ is obtained by summing a vector $\mathbf{v}$ to the $\mathbf{x}$ point.

Definition

A set $\mathcal{A}$ of points is called an affine space modeled on the vector space $\mathcal{V}$ if there is a function

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2. $\mathbf{x} + \mathbf{0} = \mathbf{x}$ for each $\mathbf{x} \in \mathcal{A}$, where $\mathbf{0} \in \mathcal{V}$ is the null vector;
3. for each pair $\mathbf{x}, \mathbf{y} \in \mathcal{A}$ there is a unique $(\mathbf{y} - \mathbf{x}) \in \mathcal{V}$ such that

$$\mathbf{x} + (\mathbf{y} - \mathbf{x}) = \mathbf{y}.$$

Dimension

The affine space $\mathcal{A}$ is said of dimension n if modeled on a vector space $\mathcal{V}$ of dimension n.

Vector sum vs affine action

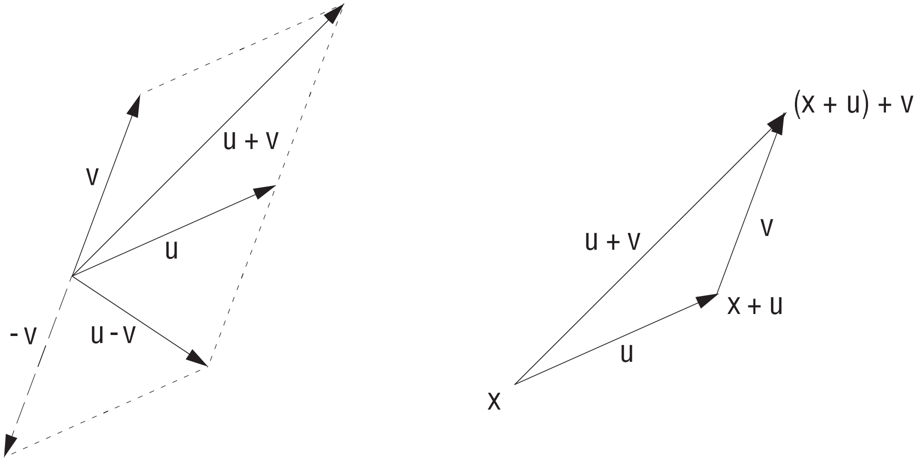

Figure 1: (a) Vector sum and difference are given by the parallelogram rule (b) associativity of displacement (point and vector sum) in an affine space

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- Addition and difference of vectors are geometrically produced by the parallelogram rule

Operations on vectors and points

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- Addition and difference of vectors are geometrically produced by the parallelogram rule
- notice also the associative property of the affine action on a point space.

Operations on vectors and points

The sum of a set $\{\mathbf{v}_i\}$ of vectors $(i = 1, \ldots, n)$ can be geometrically obtained, in an affine space:

- by setting $\mathbf{p}_0 = \mathbf{0}$

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Remark: operations on points

1. the addition of points is not defined;
2. the difference of two points is a vector;
3. the sum of a point and a vector is a point.

Affine combinations

Positive, affine and convex combinations

Three types of combinations of vectors or points can be defined. They lead to the concepts of cones, hyperplanes and convex sets, respectively.

Positive combination

Let $\mathbf{v}_0, \ldots, \mathbf{v}_d \in \mathbb{R}^n$ and $\alpha_0, \ldots, \alpha_d \in \mathbb{R}^+ \cup \{0\}$.
The vector

$$\alpha_0 \mathbf{v}_0 + \cdots + \alpha_d \mathbf{v}_d = \sum_{i=0}^{d} \alpha_i \mathbf{v}_i$$

is called a positive combination of such vectors.
The set of all the positive combinations of $\{\mathbf{v}_0, \ldots, \mathbf{v}_d\}$ is called the positive hull of $\{\mathbf{v}_0, \ldots, \mathbf{v}_d\}$ and denoted $\mathrm{pos}\,\{\mathbf{v}_0, \ldots, \mathbf{v}_d\}$.
This set is also called the cone generated by the given vectors

Affine combination

Let $\mathbf{p}_0, \dots, \mathbf{p}_d \in \mathbb{E}^n$ and $\alpha_0, \dots, \alpha_d \in \mathbb{R}$, such that $\alpha_0 + \cdots + \alpha_d = 1$.
The point

$$\sum_{i=0}^{d} \alpha_i \mathbf{p}_i := \mathbf{p}_0 + \sum_{i=1}^{d} \alpha_i (\mathbf{p}_i - \mathbf{p}_0)$$

is called an affine combination of the points $\mathbf{p}_0, \dots, \mathbf{p}_d$.

Affine combination

The set of all affine combinations of $\{\mathbf{p}_0, \ldots, \mathbf{p}_d\}$ is an affine subspace, denoted by $\mathrm{aff}\,\{\mathbf{p}_0, \ldots, \mathbf{p}_d\}$
It is easy to verify that:

$$\mathrm{aff}\,\{\mathbf{p}_0, \ldots, \mathbf{p}_d\} = \mathbf{p}_0 + \mathrm{lin}\,\{\mathbf{p}_1 - \mathbf{p}_0, \ldots, \mathbf{p}_d - \mathbf{p}_0\}.$$

Affine combination

1. The dimension of an affine subspace is the dimension of the corresponding linear vector space.

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Double description

Every affine subspace can be described either as

- the intersection of affine hyperplanes, or as
- the affine hull of a finite set of points.

Convex combinations

Convex combination

Let $\mathbf{p}_0, \ldots, \mathbf{p}_d \in \mathbb{E}^n$ and $\alpha_0, \ldots, \alpha_d \geq 0$, with $\alpha_0 + \cdots + \alpha_d = 1$.
The point

$$\alpha_0 \mathbf{p}_0 + \cdots + \alpha_d \mathbf{p}_d = \sum_{i=0}^{d} \alpha_i \mathbf{p}_i$$

is called a convex combination of points $\mathbf{p}_0, \ldots, \mathbf{p}_d$.
A convex combinations is both positive and affine.

Convex hull

The set of all convex combinations of $\{\mathbf{p}_0, \dots, \mathbf{p}_d\}$ is a convex set, called convex hull of $\{\mathbf{p}_0, \dots, \mathbf{p}_d\}$, and is denoted by $\mathrm{conv}\,\{\mathbf{p}_0, \dots, \mathbf{p}_d\}$.

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Properties

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Properties

- the convex hull of a set of points is the intersection of all convex sets that contain them
- the convex hull of a set of points is the smallest set that contains them

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- Produce (and draw) 100 random points within the unit square $[0, 1]^2$;
- Produce (and draw) 1000 random points within S_1, the 1D sphere (circle) of unit radius centered at the origin $(0, 0)$;

References

Linear Algebra Done Right book
NumPy tutorial