Cooperative Manipulation via Internal Force Regulation: A Rigidity Theory Perspective

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***Abstract*—This paper considers the integration of rigid cooperative manipulation with rigidity theory. Motivated by rigid models of cooperative manipulation systems, i.e., where the grasping contacts are rigid, we introduce first the notion of bearing and distance rigidity for graph frameworks in $\mathsf{SE}(3)$. Next, we associate the nodes of these frameworks to the robotic agents of rigid cooperative manipulation schemes and we express the object-agent interaction forces by using the graph rigidity matrix, which encodes the infinitesimal rigid body motions of the system. Moreover, we show that the associated cooperative manipulation grasp matrix is related to the rigidity matrix via a range-nullspace relation, based on which we provide novel results on the relation between the arising interaction and internal forces and consequently on the energy-optimal force distribution on a cooperative manipulation system. Finally, simulation results enhance the validity of the theoretical findings.**

***Index Terms*—Cooperative manipulation, infinitesimal rigidity, distance rigidity, bearing rigidity.**

I. INTRODUCTION

MULTI-robot systems have received a considerable amount of attention during the last decades, due to the advantages they offer with respect to single-robot setups. Example problems include consensus/rendezvous, connectivity maintenance, formation control, and robotic manipulation. In the latter case, multi-robot frameworks can yield significant advantages due to the potentially heavy payloads or challenging maneuvers. This work focuses on bridging the fields of cooperative robotic manipulation and robot formation control by associating the inter-agent interaction forces of the first to inter-agent geometric relations of the latter.

The goal of robot formation control is to control each robot using local information from neighboring agents so that the entire team forms a desired spatial geometric pattern [1]. A special instance of formation control with numerous applications is *rigid formations*. Two cases of rigid formation control have been widely studied in the literature, namely *distance rigidity* and *bearing rigidity*. Rigidity theory, a branch of discrete mathematics, explores under what conditions can the geometric pattern of a network be determined given that the length (distance) or bearing of each edge in a network of nodes is fixed. This theory has been applied in distance and bearing formation control and localization problems [2]–[8].

In this paper, we introduce the notion of *distance and bearing rigidity*, which studies under what conditions can the geometric pattern of a multi-agent system be uniquely determined if both the *distance* and the *bearing* of each edge is fixed. Moreover, we combine the latter with *rigid* cooperative manipulation, i.e., configurations where a number of robots carry an object via rigid contact points.

Cooperative manipulation is a special form of constrained dynamical systems [9]–[14]. The majority of related works assume that the robotic agents are attached to the object via *rigid grasps*, and hence the overall system can be considered as a closed-chain robotic agent. In terms of control design, most works consider decentralized schemes, where there is no communication between the agents, and use impedance and/or force control [15]–[18], possibly with contact force/torque measurements (e.g., [19], [20]). In addition, numerous works consider unknown dynamics/kinematics of the agents and the object and/or external disturbances [21]–[24].

An important property in rigid cooperative manipulation systems that has been studied thoroughly in the related literature is the regulation of internal forces. Internal forces are forces exerted by the agents at the grasping points that do not contribute to the motion of the object. While a certain amount of such forces is required in many cases (e.g., to avoid contact loss in multi-fingered manipulation), they need to be minimized in order to prevent object damage and unnecessary effort of the agents. Most works in rigid cooperative manipulation assume a certain decomposition of the interaction forces in motion-inducing and internal ones, without explicitly showing that the actual internal forces will be indeed regulated to the desired ones (e.g., [19], [20]); [11], [14], [25]–[27] analyze specific load decompositions based on whether they provide internal force-free expressions, whereas [28] is concerned with the cooperative manipulation interaction dynamics. The decompositions in the aforementioned works, however, are based on the inter-agent distances and do not take into account the actual dynamics of the agents. The latter, as we show in this paper, are tightly connected to the internal forces as well as their relation to the total force exerted by the agents.

More specifically, the contribution of this paper is twofold. Firstly, we integrate rigid cooperative manipulation with rigidity theory. Motivated by rigid cooperative manipulation systems, where the inter-agent distances *and* bearings are fixed, we introduce the notion of *distance and bearing rigidity* in

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the special Euclidean group $\mathsf{SE}(3)$. Based on recent results, we show next that the interaction forces in a rigid cooperative manipulation system depend on the distance and bearing rigidity matrix, a matrix that encodes the allowed coordinated motions of the multi-agent-object system. Moreover, we prove that the cooperative manipulation grasp matrix, which relates the object and agent velocities, is connected via a range-nullspace relation to the rigidity matrix. Secondly, we rely on the aforementioned findings to provide new results on the internal force-based rigid cooperative manipulation. We derive novel results on the relation between the arising interaction and internal forces in a cooperative manipulation system. This leads to novel conditions on the internal force-free object-agents force distribution and consequently to optimal, in terms of energy resources, cooperative manipulation. Finally, we verify the theoretical findings through simulation results on the V-REP environment. This paper extends our preliminary conference version [29], which tackles optimal cooperative manipulation by regulating the internal forces. That work, however, does not associate cooperative manipulation with rigidity theory or provide *explicit* results on the optimal object-agents force distribution.

The rest of the paper is organized as follows. Section II provides notation and necessary background. Section III provides the rigid cooperative manipulation model and Section IV discusses the details of distance and bearing rigidity. The main results of the paper are given in Section V. Section VI presents simulation results and Section VII concludes the paper. Proofs are given in the Appendix.

II. Preliminaries

A. Notation

The set of positive integers is denoted by $\mathbb{N}$ and the real n-coordinate space, with $n \in \mathbb{N}$, by $\mathbb{R}^n$. The $n \times n$ identity matrix is denoted by I_n, the n-dimensional zero vector by 0_n and the $n \times m$ matrix with zero entries by $0_{n\times m}$. We write 0 instead of 0_n when n is clear from the context. The vectors of the canonical basis of $\mathbb{R}^d$ are indicated as $\mathbf{e}_i,\ i \in \{1\dots d\}$, and they have a one in the $(i \bmod d)$-th entry and zeros elsewhere. Given a matrix $A \in \mathbb{R}^{n\times m}$, we use $A^\dagger$ for its Moore-Penrose inverse, and null(A), range(A) for its nullspace and range space, respectively. For a discrete set $\mathcal{N}$, $|\mathcal{N}|$ denotes its cardinality. Given $a, b \in \mathbb{R}^3$, $S(a)$ is the skew-symmetric matrix defined according to $S(a)b = a \times b$. In addition, S^n denotes the $(n+1)$-dimensional sphere and $\mathsf{SO}(3)$ $\mathsf{SE}(3)$ the rotation and special Euclidean group, respectively; $P_r(x) \coloneqq I_n - \frac{xx^\top}{\|x\|^2}$ projects vector $x \in \mathbb{R}^n$ onto the orthogonal complement of x. A graph $\mathcal{G}$ is a pair $(\mathcal{N}, \mathcal{E})$, where $\mathcal{N}$ is a finite set of $N = |\mathcal{N}| \in \mathbb{N}$ nodes, and $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$ is a finite set of $|\mathcal{E}|$ edges. The complete graph on N nodes is denoted by $\mathcal{K}_N$. We also make use of some notions from linear algebra.

Definition 1. *For $A, B \in \mathbb{R}^{n\times m}$, A is* left equivalent *(or* row equivalent*) to B if and only if there exists an invertible matrix $P \in \mathbb{R}^{n\times n}$ such that $A = PB$.*

The following propositions can be proved:

Proposition 1. *Let $A, B \in \mathbb{R}^{n\times m}$ Then A and B are left equivalent if and only if* null$(A) =$ null(B).

Proposition 2. *Let $A \in \mathbb{R}^{n\times m}$, and $B \coloneqq KA$, where $K \in \mathbb{R}^{n\times n}$ is an invertible matrix. Then it holds that $A^\dagger A = B^\dagger B$.*

Proposition 3. *Let $A, B \in \mathbb{R}^{n\times m}$ such that* range$(A^\top) =$ null(B). *Then it holds that $A^\dagger A + B^\dagger B = I_m$.*

III. Cooperative Manipulation Modeling

We provide in this section the dynamic modeling of the rigid cooperative manipulation system. A key feature of the model is the grasp matrix, which, as will be clarified, motivates the introduction of the notion of distance and bearing rigidity in the next section and the association between the two.

Consider N robotic agents, indexed by the set $\mathcal{N} \coloneqq \{1,\dots,N\}$, rigidly grasping an object. We denote by $q_i, \dot{q}_i \in \mathbb{R}^{n_i}$, with $n_i \in \mathbb{N}, \forall i \in \mathcal{N}$, the generalized joint-space variables and their time derivatives of agent i. The overall joint configuration is then $q \coloneqq [q_1^\top,\dots,q_N^\top]^\top, \dot{q} \coloneqq [\dot{q}_1^\top,\dots,\dot{q}_N^\top]^\top \in \mathbb{R}^n$, with $n \coloneqq \sum_{i\in\mathcal{N}} n_i$. In addition, we denote the inertial position and orientation of the ith end-effector by $p_i \in \mathbb{R}^3$ and $\eta_i \in \mathbb{T}$, respectively, where $\mathbb{T}$ is an orientation space (e.g., S^3 for the case of unit quaternions). Similarly, the velocity of the ith end-effector is denoted by $v_i \coloneqq [\dot{p}_i^\top, \omega_i^\top]^\top$, where $\omega_i \in \mathbb{R}^3$ is the respective angular velocity, and it holds that $v_i = J_i(q_i)\dot{q}_i$, where $J_i : \mathbb{S}_i \to \mathbb{R}^{6\times n_i}$ is the robot Jacobian, and $\mathbb{S}_i \coloneqq \{q_i \in \mathbb{R}^{n_i} : \dim(\text{null}(J_i(q_i))) = 0\}$ is the set away from kinematic singularities [30], $\forall i \in \mathcal{N}$. Moreover $R_i \in \mathsf{SO}(3)$ is the ith end-effector's rotation matrix, associated with η_i, $\forall i \in \mathcal{N}$, and we denote $x_i \coloneqq (p_i, R_i) \in \mathsf{SE}(3)$ and $x \coloneqq (x_1,\dots,x_N) \in \mathsf{SE}(3)^N$. The joint- and task-space dynamics of the agents are [30]:

$$B(q)\ddot{q} + N(q,\dot{q})\dot{q} + g_q(q) = \tau - J(q)^\top h, \tag{1a}$$
$$M(q)\dot{v} + C(q,\dot{q})v + g(q) = u - h, \tag{1b}$$

where $B \coloneqq \text{diag}\{[B_i]_{i\in\mathcal{N}}\}$, $N \coloneqq \text{diag}\{[N_i]_{i\in\mathcal{N}}\} \in \mathbb{R}^{nN\times nN}$, $J \coloneqq \text{diag}\{[J_i]_{i\in\mathcal{N}}\} \in \mathbb{R}^{6N\times nN}$, $g_q \coloneqq [g_{q_1}^\top,\dots,g_{q_N}^\top]$, $\tau \coloneqq [\tau_1^\top,\dots,\tau_N] \in \mathbb{R}^{nN}$, $M \coloneqq \text{diag}\{[M_i]_{i\in\mathcal{N}}\}$, $C \coloneqq \text{diag}\{[C_i]_{i\in\mathcal{N}}\} \in \mathbb{R}^{6N\times 6N}$, $v \coloneqq [v_1^\top,\dots,v_N^\top] \in \mathbb{R}^{6N}$, $h \coloneqq [h_1^\top,\dots,h_N^\top]^\top$, $u \coloneqq [u_1^\top,\dots,u_N^\top]^\top$, $g \coloneqq [g_1^\top,\dots,g_N^\top]^\top \in \mathbb{R}^{6N}$; the terms $B_i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i\times n_i}$, $N_i : \mathbb{R}^{2n_i} \to \mathbb{R}^{n_i\times n_i}$, $g_{q_i} : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$, are the positive definite inertia, Coriolis, and gravity terms, respectively, and $M_i \coloneqq (J_i B_i^{-1} J_i^\top)^{-1} : \mathbb{S}_i \to \mathbb{R}^{6\times 6}$, $C_i v_i \coloneqq M_i(J_i B_i^{-1} N_i - \dot{J}_i)\dot{q}_i : \mathbb{S}_i \times \mathbb{R}^{n_i} \to \mathbb{R}^6$, $g_i \coloneqq M_i J_i B_i^{-1} g_{q_i} : \mathbb{S}_i \to \mathbb{R}^6$ their task-space counterparts; $h_i \in \mathbb{R}^6$ are the forces exerted by the agents to the object at the grasping points, $\forall i \in \mathcal{N}$, and finally, $u_i \in \mathbb{R}^6$ and $\tau_i = J_i^\top u_i + \tau_{i,0} \in \mathbb{R}^{n_i}$ are the task- and joint-space inputs, respectively, with $\tau_{i,0}$ being a redundant term for overactuated agents.

Regarding the object, we denote by $x_O \coloneqq [p_O^\top, \eta_O^\top]^\top \in \mathbb{M} \coloneqq \mathbb{R}^3 \times \mathbb{T}$, $v_O \coloneqq [\dot{p}_O^\top, \omega_O^\top]^\top \in \mathbb{R}^{12}$ the pose and generalized velocity of the object's center of mass; $R_O(\eta_O) \in \mathsf{SO}(3)$ is the object's rotation matrix, associated with its orientation

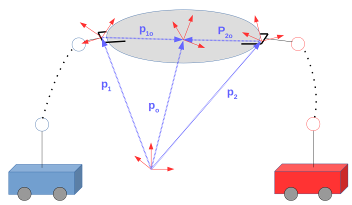

Fig. 1: Two robotic agents rigidly grasping an object.

η_O. We consider the following second order dynamics, which can be derived based on the Newton-Euler formulation:

$$\dot{R}_O = S(\omega_O)R_O \tag{2a}$$
$$M_O(x_O)\dot{v}_O + C_O(x_O,\dot{x}_O)v_O + g_O(x_O) = h_O, \tag{2b}$$

where $M_O : \mathbb{M} \to \mathbb{R}^{6\times 6}$ is the positive definite inertia matrix, $C_O : \mathbb{M}\times\mathbb{R}^6 \to \mathbb{R}^{6\times 6}$ is the Coriolis matrix, $g_O : \mathbb{M} \to \mathbb{R}^6$ is the gravity vector, and $h_O \in \mathbb{R}^6$ is the vector of generalized forces acting on the object's center of mass.

In view of Fig. 1 and the grasping rigidity, one obtains [21]

$$v_i = J_{O_i}(x_i)v_O,\ \forall i \in \mathcal{N}, \tag{3}$$

where $J_{O_i} : \mathsf{SE}(3) \to \mathbb{R}^{6\times 6}$ is the object-to-agent Jacobian matrix, with

$$J_{O_i}(x_i) = \begin{bmatrix} I_3 & -S(p_{iO}) \\ 0_{3\times 3} & I_3 \end{bmatrix}, \tag{4}$$

and $p_{iO} := p_i - p_O$, $\forall i \in \mathcal{N}$; J_{O_i} is always full-rank, due to the rigidity of the grasping contacts. The grasp matrix is formed by stacking $J_{O_i}^\top$ as

$$G(x) := [J_{O_1}(x_1)^\top, \dots, J_{O_N}(x_N)^\top] \in \mathbb{R}^{6\times 6N}, \tag{5}$$

and has full column-rank due to the rigidity of the grasping contacts; (3) can now be written in stack vector form as

$$v = G(x)^\top v_O. \tag{6}$$

The kineto-statics duality [30] along with the grasp rigidity suggest that the force h_O acting on the object's center of mass and the generalized forces $h_i, i \in \mathcal{N}$, exerted by the agents at the grasping points, are related through:

$$h_O = G(x)h. \tag{7}$$

By using (1), (7), and (2), we obtain the coupled dynamics:

$$\widetilde{M}(\bar{x})\dot{v}_O + \widetilde{C}(\bar{x})v_O + \widetilde{g}(\bar{x}) = G(x)u, \tag{8}$$

where $\widetilde{M} := M_O + GMG^\top$, $\widetilde{C} := C_O + GCG^\top + GM\dot{G}^\top$, $\widetilde{g} := g_O + Gg$, $\bar{x}$ is the overall state $\bar{x} := [q^\top, \dot{q}^\top, x_O^\top, \dot{x}_O^\top]^\top \in \mathbb{R}^{2n+6} \times \mathbb{M}$, and we have omitted the arguments for brevity.

The interaction forces h among the agents and the object can be decoupled into motion-induced and internal forces

$$h = h_{\text{m}} + h_{\text{int}}. \tag{9}$$

The internal forces h_{int} are squeezing forces that the agents exert to the object and belong to the nullspace of $G(x)$ (i.e., $G(x)h_{\text{int}} = 0$). Hence, they do not contribute to the acceleration of the coupled system and result in internal stresses that might damage the object. A closed form analytic expression for h_{m} and h_{int} will be given in Section V.

Note from (6) that the agent velocities v belong to the range space of $G(x)^\top$. Therefore, since $G(x)$ is a matrix that encodes rigidity constraints, this motivates the association of $G(x)$ to the *rigidity matrix* used in formation rigidity theory, and of the rigid cooperative manipulation scheme to a multi-agent rigid formation scheme. To this end, we introduce in the next section the notion of Distance and Bearing Rigidity.

IV. Distance and Bearing Rigidity in $\mathsf{SE}(3)$

We begin by recalling that the range space of the grasp matrix $G(x)^T$ corresponds to the rigid body translations and rotations of the system. While G appears naturally in the context of dynamic modeling of rigid bodies, it is also indirectly related to the notion of structural rigidity in discrete geometry.

In the classical structural rigidity theory, one considers a collection of rigid bars connected by joints allowing free rotations around the joint axis (*bar-and-joint* frameworks). One is then interested in understanding what are the allowable motions of the framework, i.e., those motions that preserve the lenghts of the bars and their connections to the joints. The so-called *trivial motions* for these frameworks are precisely the rigid body translations and rotations of the system. For some frameworks, there may be additional motions, known as *flexes*, that also preserve the constraints. This is captured by the notion of *infinitesimal motions* of the framework and is characterized by the *rigidity matrix* of the framework [31].

In this work we consider frameworks that encode both the lengths of bars and pose of the joints, leading to a *distance and bearing*-type framework. This relates notions from distance rigidity, and recent works in bearing rigidity for frameworks embedded in $\mathsf{SE}(2)$ and $\mathsf{SE}(3)$ [32], [33]. In this direction, we introduce the concept of distance and bearing rigidity (abbreviated as D&B Rigidity). For this work we focus on the notion of infinitesimal rigidity for D&B frameworks. We first formally define a D&B framework in $\mathsf{SE}(3)$:

Definition 2. *A framework in* $\mathsf{SE}(3)$ *is a triple* $(\mathcal{G}, p_{\mathcal{G}}, R_{\mathcal{G}})$*, where* $\mathcal{G} := (\mathcal{N}, \mathcal{E})$ *is a graph,* $p_{\mathcal{G}} : \mathcal{N} \to \mathbb{R}^3$ *is a function mapping each node to a position in* $\mathbb{R}^3$*, and* $R_{\mathcal{G}} : \mathcal{N} \to \mathrm{SO}(3)$ *is a function associating each node with an orientation element of* $\mathrm{SO}(3)$ *(both with respect to an inertial frame).*

In this work we employ the Special Orthogonal Group (rotation matrices) $\{R \in \mathbb{R}^{3\times 3} : R^\top R = I_3, \det(R) = 1\}$ to express the orientation of the agents. Moreover, we use the shorthand notation $p_i := p_{\mathcal{G}}(i)$, $R_i := R_{\mathcal{G}}(i)$, $p := [p_1^\top, \dots, p_N^\top]^\top \in \mathbb{R}^{3N}$, $R := (R_1, \dots, R_N) \in \mathsf{SO}(3)^N$, $x_i := (p_i, R_i) \in \mathsf{SE}(3)$, and $x := (x_1, \dots, x_N) \in \mathsf{SE}(3)^N$.

The distances and bearings in a framework can be summarized through the following *SE(3) D&B rigidity function*, $\gamma_{\mathcal{G}}$, that encodes the rigidity constraints in the framework. Consider a directed graph $\mathcal{G}$, where $\mathcal{E} \subseteq \{(i,j) \in \mathcal{N}^2 : i \neq j\}$, as well as its *undirected part* $\mathcal{E} \supseteq \mathcal{E}_u := \{(i,j) \in \mathcal{E} : i < j\}$. Then $\gamma_{\mathcal{G}}$ can be formed by the distance and bearing functions $\gamma_{e,d} : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}_{\geq 0}$, $\gamma_{e,b} : \mathsf{SE}(3)^2 \to \mathsf{S}^2$, with

$$\gamma_{e,d}(p_i, p_j) := \frac{1}{2}\|p_i - p_j\|^2, \forall e = (i,j) \in \mathcal{E}_u, \tag{10a}$$
$$\gamma_{e,b}(x_i, x_j) := R_i^\top \frac{p_j - p_i}{\|p_i - p_j\|}, \forall e = (i,j) \in \mathcal{E}, \tag{10b}$$

which encodes the distance $\|p_i - p_j\|$ between two agents as well as the local bearing vector $R_i^\top \frac{p_j - p_i}{\|p_i - p_j\|}$, expressed in the frame of agent i. Note that the distance functions are considered only for the undirected part of $\mathcal{G}$, since $\gamma_{(i,j),d} = \gamma_{(j,i),d}$. Now $\gamma_{\mathcal{G}}$ is formed by stacking the aforementioned distance and bearing functions, i.e., $\gamma_{\mathcal{G}} : \mathsf{SE}(3)^N \to \mathbb{R}^{|\mathcal{E}_u|} \times \mathsf{S}^{2|\mathcal{E}|}$, with

$$\gamma_{\mathcal{G}} := \begin{bmatrix} \gamma_d(p) \\ \gamma_b(x) \end{bmatrix} := \begin{bmatrix} \gamma_{1,d}, \dots, \gamma_{|\mathcal{E}_u|,d}, \gamma_{1,b}^\top, \dots, \gamma_{|\mathcal{E}|,b}^\top \end{bmatrix}^\top. \quad (11)$$

Note that the aforementioned expressions for $\gamma_{e,d}$, $\gamma_{e,b}$ are not unique and other choices that capture the rigidity constraints can also be made. We also mention our slight abuse of notation, where the index k in $\gamma_{k,d}$ and $\gamma_{k,b}$ refers to a labeled edge in $\mathcal{E}_u$ and $\mathcal{E}_b$.

In this work, we are interested in the set of D&B *infinitesimal* motions of a framework in $\mathsf{SE}(3)$. These can be thought as perturbations to a framework in $\mathsf{SE}(3)$ that leave $\gamma_{\mathcal{G}}$ unchanged. This set is characterized by the nullspace of the matrix appearing in the rate-of-change of $\gamma_{\mathcal{G}}$ under the kinematic equations associated with rotational motion in $\mathsf{SE}(3)$ [34]. That is, the nullspace of the matrix $\nabla_{(p,R)}\gamma_{\mathcal{G}}$, termed the $\mathsf{SE}(3)$-D&B *rigidity matrix* $\mathcal{R}_{\mathcal{G}} : \mathsf{SE}(3)^N \to \mathbb{R}^{(|\mathcal{E}_u|+3|\mathcal{E}|)\times 6N} := \nabla_{(p,R)}\gamma_{\mathcal{G}}$, i.e.,

$$\mathcal{R}_{\mathcal{G}}(x) = \begin{bmatrix} \frac{\partial \gamma_{1,d}}{\partial p_1} & \frac{\partial \gamma_{1,d}}{\partial R_1} & \cdots & \frac{\partial \gamma_{1,d}}{\partial p_N} & \frac{\partial \gamma_{1,d}}{\partial R_N} \\ \vdots & & \ddots & & \vdots \\ \frac{\partial \gamma_{|\mathcal{E}_u|,d}}{\partial p_1} & \frac{\partial \gamma_{|\mathcal{E}_u|,d}}{\partial R_1} & \cdots & \frac{\partial \gamma_{|\mathcal{E}_u|,d}}{\partial p_N} & \frac{\partial \gamma_{|\mathcal{E}_u|,d}}{\partial R_N} \\ \frac{\partial \gamma_{1,b}}{\partial p_1} & \frac{\partial \gamma_{1,b}}{\partial R_1} & \cdots & \frac{\partial \gamma_{1,b}}{\partial p_N} & \frac{\partial \gamma_{1,b}}{\partial R_N} \\ \vdots & & \ddots & & \vdots \\ \frac{\partial \gamma_{|\mathcal{E}|,b}}{\partial p_1} & \frac{\partial \gamma_{|\mathcal{E}|,b}}{\partial R_1} & \cdots & \frac{\partial \gamma_{|\mathcal{E}|,b}}{\partial p_N} & \frac{\partial \gamma_{|\mathcal{E}|,b}}{\partial R_N} \end{bmatrix}, \quad (12)$$

with

$$\begin{aligned} \frac{\partial \gamma_{e,d}}{\partial x_i} &= \begin{bmatrix} \frac{\partial \gamma_{e,d}}{\partial p_i} & \frac{\partial \gamma_{e,d}}{\partial R_i} \end{bmatrix} = \begin{bmatrix} (p_i - p_j)^\top & 0_{1\times 3} \end{bmatrix}, \\ \frac{\partial \gamma_{e,d}}{\partial x_j} &= \begin{bmatrix} \frac{\partial \gamma_{e,d}}{\partial p_j} & \frac{\partial \gamma_{e,d}}{\partial R_j} \end{bmatrix} = \begin{bmatrix} (p_j - p_i)^\top & 0_{1\times 3} \end{bmatrix}, \\ \frac{\partial \gamma_{e,b}}{\partial x_i} &= \begin{bmatrix} \frac{\partial \gamma_{e,b}}{\partial p_i} & \frac{\partial \gamma_{e,b}}{\partial R_i} \end{bmatrix} = \begin{bmatrix} -\frac{P_r(\gamma_{e,b})}{\|p_j - p_i\|} R_i^\top & S(\gamma_{e,b}) \end{bmatrix}, \\ \frac{\partial \gamma_{e,b}}{\partial x_j} &= \begin{bmatrix} \frac{\partial \gamma_{e,b}}{\partial p_j} & \frac{\partial \gamma_{e,b}}{\partial R_j} \end{bmatrix} = \begin{bmatrix} \frac{P_r(\gamma_{e,b})}{\|p_j - p_i\|} R_i^\top & 0_{3\times 3} \end{bmatrix}. \end{aligned}$$

The projection operator $P_r(\cdot)$ [7] is defined as in Section II-A. Infinitesimal motions, therefore, are motions $x(t)$ produced by velocities $v(t)$ that lie in the nullspace of $\mathcal{R}_{\mathcal{G}}$, for which it holds that $\dot{\gamma}_{\mathcal{G}} = \mathcal{R}_{\mathcal{G}}(x(t))v(t) = 0$, where $v := [\dot{p}_1^\top, \omega_1^\top, \dots, \dot{p}_N^\top, \omega_N^\top]^\top$, as defined in Section III. The infinitesimal motions therefore depend on the number of motion degrees of freedom the entire framework possesses. This directly relates to the structure of the underlying graph.

Motions that preserve the distances and bearings of the framework for *any* underlying graph are called *D&B trivial motions*. This leads to the definition of *infinitesimal rigdity*.

Definition 3. *A framework $(\mathcal{G}, p_{\mathcal{G}}, R_{\mathcal{G}})$ is D&B infinitesimally rigid in* $\mathsf{SE}(3)$ *if every D&B infinitesimal motion is a D&B trivial motion.*

We now aim to identify what the trivial motions of a D&B framework are, and to determine conditions for a framework to be infinitesimally rigid based on properties of $\mathcal{R}_{\mathcal{G}}$. Before we proceed, we note that the D&B rigidity function in $\mathsf{SE}(3)$ can be seen as a superposition of the rigidity functions associated with the classic distance rigidity theory [31] and the $\mathsf{SE}(3)$ bearing rigidity theory [33]. In particular, we note that $\mathcal{R}_{\mathcal{G},d} : \mathbb{R}^{3N} \to \mathbb{R}^{|\mathcal{E}_u|\times 3N} := \nabla_p \gamma_d$ is the well-studied (distance) rigidity matrix, while $\mathcal{R}_{\mathcal{G},b} : \mathsf{SE}^{3N} \to \mathbb{R}^{3\mathcal{E}\times 6N} := \nabla_{(p,R)}\gamma_{\mathcal{G},b}$ is the $\mathsf{SE}(3)$ bearing rigidity matrix. Note that $\mathcal{R}_{\mathcal{G},d}$ is associated with the framework $(\mathcal{G}, p_{\mathcal{G}})$, which is the projection of $(\mathcal{G}, p_{\mathcal{G}}, R_{\mathcal{G}})$ to $\mathbb{R}^3$. With an appropriate permutation, P_R, of the columns of $\mathcal{R}_{\mathcal{G}}$, we have that

$$\begin{aligned} \widetilde{\mathcal{R}}_{\mathcal{G}} &:= \mathcal{R}_{\mathcal{G}} P_R \\ &= \begin{bmatrix} \frac{\partial \gamma_{1,d}}{\partial p_1} & \cdots & \frac{\partial \gamma_{1,d}}{\partial p_N} & \frac{\partial \gamma_{1,d}}{\partial R_1} & \cdots & \frac{\partial \gamma_{1,d}}{\partial R_N} \\ \vdots & & \ddots & & & \vdots \\ \frac{\partial \gamma_{M_{\mathcal{G}},d}}{\partial p_1} & \cdots & \frac{\partial \gamma_{M_{\mathcal{G}},d}}{\partial p_N} & \frac{\partial \gamma_{M_{\mathcal{G}},d}}{\partial R_1} & \cdots & \frac{\partial \gamma_{M_{\mathcal{G}},d}}{\partial R_N} \\ \frac{\partial \gamma_{1,b}}{\partial p_1} & \cdots & \frac{\partial \gamma_{1,b}}{\partial p_N} & \frac{\partial \gamma_{1,b}}{\partial R_1} & \cdots & \frac{\partial \gamma_{1,b}}{\partial R_N} \\ \vdots & & \ddots & & & \vdots \\ \frac{\partial \gamma_{M_{\mathcal{G}},b}}{\partial p_1} & \cdots & \frac{\partial \gamma_{M_{\mathcal{G}},b}}{\partial p_N} & \frac{\partial \gamma_{M_{\mathcal{G}},b}}{\partial R_1} & \cdots & \frac{\partial \gamma_{M_{\mathcal{G}},b}}{\partial R_N} \end{bmatrix}, \quad (13) \end{aligned}$$

which is equal to

$$\widetilde{\mathcal{R}}_{\mathcal{G}} = \begin{bmatrix} \begin{bmatrix} \mathcal{R}_{\mathcal{G},d} & 0_{|\mathcal{E}_u|\times 3N} \end{bmatrix} \\ \mathcal{R}_{\mathcal{G},b} \end{bmatrix} =: \begin{bmatrix} \bar{\mathcal{R}}_{\mathcal{G},d} \\ \mathcal{R}_{\mathcal{G},b} \end{bmatrix}.$$

The nullspace of $\widetilde{\mathcal{R}}_{\mathcal{G}}$, therefore, is the intersection of the nullspaces of $\bar{\mathcal{R}}_{\mathcal{G},d}$ and $\mathcal{R}_{\mathcal{G},b}$.

With the above interpretation, we can now understand the trivial motions to be the intersection of trivial motions associated to distance rigidity with those associated to $\mathsf{SE}(3)$ bearing rigidity. In particular, let

$$\mathcal{S}_d := \operatorname{span}\left\{ 1_N \otimes I_3, \mathcal{L}^{\circlearrowright}_{\mathbb{R}^3}(\mathcal{G}) \right\},$$

denote the trivial motions associated to a distance framework [31]. That is, $1_N \otimes I_3$ represents translations of the entire framework, and $\mathcal{L}^{\circlearrowright}_{\mathbb{R}^3}(\mathcal{G})$ is the rotational subspace induced by the graph $\mathcal{G}$ in $\mathbb{R}^3$, i.e.,

$$\mathcal{L}^{\circlearrowright}_{\mathbb{R}^3}(\mathcal{G}) = \operatorname{span}\left\{ (I_3 \otimes S(\mathbf{e}_h))\, p_{\mathcal{G}}, h = 1, 2, 3 \right\}.$$

These motions can be produced by the *linear* velocities of the agents. It is known that $\mathcal{S}_d \subseteq \operatorname{null}(\mathcal{R}_{\mathcal{G},d})$ for any underlying graph $\mathcal{G}$ [31]. For the matrix $\bar{\mathcal{R}}_{\mathcal{G},d}$, we can define the corresponding set

$$\bar{S}_d := \operatorname{span}\left\{ \begin{bmatrix} 1_N \otimes I_3 \\ \star \end{bmatrix}, \begin{bmatrix} \mathcal{L}^{\circlearrowright}_{\mathbb{R}^3}(\mathcal{G}) \\ \star \end{bmatrix} \right\} \subseteq \operatorname{null}(\bar{\mathcal{R}}_{\mathcal{G},d}).$$

Note that the distance rigidity does not explicitly depend on the orientation of the nodes when expressed as a point in $SE(3)$. This accounts for the free $\star$ entry in the subspace $\bar{S}_d$ corresponding to the rotations. Thus, the set of trivial motions in $\mathbb{R}^3$ can be seen as the projection of $\bar{S}_d$ in $\mathbb{R}^3$.

Similarly, for an $\mathsf{SE}(3)$ bearing framework one can define the subspace [33]

$$\mathcal{S}_b := \operatorname{span}\left\{ \begin{bmatrix} 1_N \otimes I_3 \\ 0_{3N\times 3}, \end{bmatrix}, \begin{bmatrix} p_{\mathcal{G}} \\ 0_{3N}, \end{bmatrix}, \mathcal{L}^{\circlearrowright}_{\mathsf{SE}(3)}(\mathcal{G}) \right\},$$

where the vector $[p_{\mathcal{G}}^T\, 0_{3N}^T]^T$ represents a scaling of the framework. The space $\mathcal{L}^{\circlearrowright}_{\mathsf{SE}(3)}(\mathcal{G})$ is the rotational subspace induced by $\mathcal{G}$, in $\mathsf{SE}(3)$,

$$\mathcal{L}^{\circlearrowright}_{\mathsf{SE}(3)}(\mathcal{G}) = \operatorname{span}\left\{ \begin{bmatrix} (I_3 \otimes S(\mathbf{e}_h))\, p_{\mathcal{G}} \\ 1_n \otimes \mathbf{e}_h \end{bmatrix}, h = 1,2,3 \right\}. \quad (14)$$

It is also known that $\mathcal{S}_b \subseteq \operatorname{null}(\mathcal{R}_{\mathcal{G},b})$. Thus $\mathcal{S}_b$ describes the trivial motions of an $\mathsf{SE}(3)$ bearing framework [33]. The above discussion leads to the following result.

Proposition 4. *The trivial motions of a D&B framework are characterized by the set*

$$\mathcal{S}_{db} := \bar{\mathcal{S}}_d \cap \mathcal{S}_b = \operatorname{span}\left\{ \begin{bmatrix} 1_N \otimes I_3 \\ 0_{3N\times 3} \end{bmatrix}, \mathcal{L}^{\circlearrowright}_{\mathsf{SE}(3)}(\mathcal{G}) \right\}.$$

Furthermore, it follows that $\mathcal{S}_{db} \subseteq \operatorname{null}(\widetilde{\mathcal{R}}_{\mathcal{G}})$.

Having characterized the trivial motions, it now follows from Definition 3 that for infinitesimal rigidity, we require that $\operatorname{null}(\widetilde{\mathcal{R}}_{\mathcal{G}}) = \mathcal{S}_{db}$. This is summarized as follows.

Proposition 5. *The framework* $(\mathcal{G}, p_{\mathcal{G}}, R_{\mathcal{G}})$ *is D&B infinitesimally rigid in* $\mathsf{SE}(3)$ *if and only if*

$$\begin{aligned} \operatorname{null}(\widetilde{\mathcal{R}}_{\mathcal{G}}) &= \operatorname{null}(\bar{\mathcal{R}}_{\mathcal{G},d}) \cap \operatorname{null}(\mathcal{R}_{\mathcal{G},b}) \\ &= \operatorname{span}\left\{ \begin{bmatrix} 1_N \otimes I_3 \\ 0_{3N\times 3,} \end{bmatrix}, \mathcal{L}^{\circlearrowright}_{\mathsf{SE}(3)}(\mathcal{G}) \right\} = \mathcal{S}_{db}. \quad (15) \end{aligned}$$

Equivalently, the D&B framework is infinitesimally rigid in $\mathsf{SE}(3)$ *if and only if*

$$\operatorname{rank}(\widetilde{\mathcal{R}}_{\mathcal{G}}) = \dim(\widetilde{\mathcal{R}}_{\mathcal{G}}) - \dim(\operatorname{null}(\widetilde{\mathcal{R}}_{\mathcal{G}})) = 6N-6. \quad (16)$$

Hence, all the motions produced by the nullspace of $\widetilde{\mathcal{R}}_{\mathcal{G}}$ for an infinitesimally rigid framework must correspond to trivial motions, i.e., coordinated translations and rotations. Moreover, given (13), it follows that $(\mathcal{G}, p_{\mathcal{G}}, R_{\mathcal{G}})$ is D&B infinitesimally rigid in $\mathsf{SE}(3)$ if and only if

$$\operatorname{null}(\mathcal{R}_{\mathcal{G}}) = \{x = P_R y \in \mathsf{SE}(3)^N : y \in \operatorname{null}(\widetilde{\mathcal{R}}_{\mathcal{G}})\}, \quad (17)$$

i.e., the nullspace of $\mathcal{R}_{\mathcal{G}}$ consists of the vectors of $\operatorname{null}(\widetilde{\mathcal{R}}_{\mathcal{G}})$ whose elements are permutated by P_R.

It is worth noting that the aforementioned results are not valid if the rigidity matrix loses rank, i.e., $\operatorname{rank}(\mathcal{R}_{\mathcal{G}}) < \max\{\operatorname{rank}(\mathcal{R}_{\mathcal{G}}(x)), x \in \mathsf{SE}(3)\}$. These are degenerate cases that correspond, for example, to when all agents are aligned along a direction $\mathbf{v} \in \mathsf{S}^2$. For more discussion on these degenerate cases, the reader is referred to [34].

As a last remark, we observe that frameworks over the complete graph, $(\mathcal{K}_N, p_{\mathcal{K}_N}, R_{\mathcal{K}_N})$, are (except for the degenerate configurations), infinitesimally rigid. That is, $\operatorname{rank}(\widetilde{\mathcal{R}}_{\mathcal{K}_N}) = 6N-6$. This leads to the following corollary.

Corollary 1. *Consider the D&B frameworks* $(\mathcal{G}, p_{\mathcal{G}}, R_{\mathcal{G}})$ *and* $(\mathcal{K}_N, p_{\mathcal{G}}, R_{\mathcal{G}})$ *for nondegenrate configurations* $(p_{\mathcal{G}}, R_{\mathcal{G}})$. *Then* $(\mathcal{G}, p_{\mathcal{G}}, R_{\mathcal{G}})$ *is D&B infinitesimally rigid if and only if* $\operatorname{rank}(\widetilde{\mathcal{R}}_{\mathcal{G}}) = \operatorname{rank}(\widetilde{\mathcal{R}}_{\mathcal{K}_N}) = 6N-6$.

In the next section, we use the aforementioned results to link the D&B rigidity matrix of a complete graph to the forces (9).

V. Main Results

In this section we provide the main results of this work. Firstly, we give a closed form expression for the interaction and internal forces of the coupled system object-robots. Next, we connect these forces with the D&B rigidity matrix introduced in Section IV. Next, we use these results to provide a novel relation between the arising interaction and internal forces and we give conditions on the agent force distribution for cooperative manipulation free from internal forces. For the rest of the paper, we use the following notation for the cooperative object-manipulation system introduced in Section III: $\widetilde{x} := [\dot{q}^\top, v_O^\top]^\top \in \mathbb{R}^{Nn+6}$, $\bar{B}(\bar{x}) := \operatorname{diag}\{B(q), M_O(x_O)\}$, $\bar{N}(\bar{x}) := \operatorname{diag}\{N(q,\dot{q}), C_O(x_O, \dot{x}_O)\} \in \mathbb{R}^{(Nn+6)\times(Nn+6)}$, $\bar{\tau} := [\tau^\top, 0_6^\top]^\top$, $\bar{g}_q := [g_q(q)^\top, g_O(x_O)^\top]^\top \in \mathbb{R}^{Nn+6}$, $\bar{M}(\bar{x}) := \operatorname{diag}\{M(q), M_O(x_O)\}$, $\bar{C} := \operatorname{diag}\{C(q,\dot{q}), C_O(x_O, \dot{x}_O)\} \in \mathbb{R}^{(6N+6)\times(6N+6)}$, $\bar{v} := [v^\top, v_O^\top]^\top$, $\bar{g}(\bar{x}) := [g(q)^\top, g_O(x_O)^\top]^\top$, $\bar{u} := [u^\top, 0_6^\top]^\top \in \mathbb{R}^{6N+6}$, $\bar{J}(q) := \operatorname{diag}\{J(q), I_6\}$.

A. Interaction Forces Based on the D&B Rigidity Matrix

In this section we provide closed form expressions for the interaction forces of the coupled object-agents system and link them to the D&B rigidity matrix notion introduced in Section IV. In particular, we consider that the robotic agents and the object form a graph that will be defined in the sequel. Note that, due to the rigidity of the grasping points, the forces exerted by an agent influence, not only the object, but all the other agents as well. Hence, since there exists interaction among all the pairs of agents as well as the agents and the object, we model their connection as a complete graph, as described rigorously below. Moreover, as will be clarified later, the rigidity matrix of this graph encodes the constraints of the agents-object system, imposed by the rigidity of the grasping points, and plays an important role in the expression of the agents-object interaction forces.

Let the robotic agents form a framework $(\mathcal{G}, p_{\mathcal{G}}, R_{\mathcal{G}})$ in $\mathsf{SE}(3)$, where $\mathcal{G} := (\mathcal{N}, \mathcal{E})$ is the complete graph, i.e., $\mathcal{E} = \{(i,j) \in \mathcal{N}^2 : i \neq j\}$, and $p_{\mathcal{G}} := [p_1^\top, \dots, p_N^\top]^\top$, $R_{\mathcal{G}} := (R_1, \dots, R_N)$. Consider also the undirected part $\mathcal{E}_u = \{(i,j) \in \mathcal{E} : i < j\}$ of $\mathcal{E}$, as also described in Section IV. Since the graph is complete, we conclude that $|\mathcal{E}| = N(N-1)$ and $|\mathcal{E}_u| = \frac{N(N-1)}{2}$. Moreover, consider the extended framework $(\bar{\mathcal{G}}, p_{\bar{\mathcal{G}}}, R_{\bar{\mathcal{G}}})$ of the robotic agents and the object, i.e., where the object is considered as the $(N+1)$th agent; $\bar{\mathcal{G}}$ is the complete graph $\bar{\mathcal{G}} := (\bar{\mathcal{N}}, \bar{\mathcal{E}})$, where $\bar{\mathcal{N}} := \{1, \dots, \bar{N}\}$, $\bar{N} := N+1$, and $\bar{\mathcal{E}} := \{(i,j) \in \bar{\mathcal{N}}^2 : i \neq j\}$, with $|\bar{\mathcal{E}}| = \bar{N}(\bar{N}-1)$. Let also $\bar{\mathcal{E}}_u := \{(i,j) \in \bar{\mathcal{N}} : i < j\}$ be the undirected edge part, with $|\bar{\mathcal{E}}_u| = \frac{\bar{N}(\bar{N}-1)}{2}$.

Consider now the rigidity functions $\gamma_{e,d} : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}_{\geq 0}$, $\forall e \in \bar{\mathcal{E}}_u$ and $\gamma_{e,b} : \mathsf{SE}(3)^2 \to \mathsf{S}^2$, $\forall e \in \bar{\mathcal{E}}$, as given in (10), as well as the stack vector $\gamma_{\bar{\mathcal{G}}} : \mathsf{SE}(3)^{\bar{N}} \to \mathbb{R}^{\frac{\bar{N}(\bar{N}-1)}{2}} \times \mathsf{S}^{2\bar{N}(\bar{N}-1)}$ as given in (11). The rigidity constraints of the framework are encoded in the constraint $\gamma_{\bar{\mathcal{G}}} = \text{const.}$. Since the rigidity of the framework stems from the rigidity of the grasping points, these constraints encode also the rigidity constraints of the object-agent cooperative manipulation. By differentiating $\gamma_{\bar{\mathcal{G}}} = \text{const.}$, one obtains $\mathcal{R}_{\bar{\mathcal{G}}}(x, x_O)\bar{J}(q)\dot{\widetilde{x}} = -(\dot{\mathcal{R}}_{\bar{\mathcal{G}}}(x, x_O)\bar{J}(q) +$

$\mathcal{R}_{\bar{\mathcal{G}}}(x, x_O)\dot{\bar{J}}(q))\widetilde{x}$, where $\mathcal{R}_{\bar{\mathcal{G}}} : \mathsf{SE}(3)^{\bar{N}} \to \mathbb{R}^{\frac{7\bar{N}(\bar{N}-1)}{2} \times (6\bar{N})}$ is the rigidity matrix associated to $\bar{\mathcal{G}}$ and has the form (12). We write the equations above as $A(\bar{x}, t)\dot{\widetilde{x}} = b(\bar{x}, t)$, where

$$A(\bar{x}, t) \coloneqq \mathcal{R}_{\bar{\mathcal{G}}}(x, x_O)\bar{J}(q) \tag{18a}$$

$$b(\bar{x}, t) \coloneqq -[\dot{\mathcal{R}}_{\bar{\mathcal{G}}}(x, x_O)\bar{J}(q) + \mathcal{R}_{\bar{\mathcal{G}}}(x, x_O)\dot{\bar{J}}(q)]\widetilde{x}. \tag{18b}$$

One can verify that the motion of the cooperative object-agents manipulation system that is enforced by the aforementioned constraints corresponds to rigid body motions (coordinated translations and rotations of the system). Hence, since $\bar{\mathcal{G}}$ is complete, the analysis of Section IV dictates that these motions are the infinitesimal motions of the framework and are the ones produced by the nullspace of $\mathcal{R}_{\bar{\mathcal{G}}}(x, x_O)$.

Next, we turn to the main focus of our results, which is the case of internal forces and we consider the framework comprising only of the robotic agents $(\mathcal{G}, p_{\mathcal{G}}, R_{\mathcal{G}})$. The inter-agent rigidity constraints are expressed by the D&B rigidity functions $\gamma_{e,d} : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}_{\geq 0}$, $\forall e \in \mathcal{E}_u$ and $\gamma_{e,b} : \mathsf{SE}(3)^2 \to \mathsf{S}^2$, $\forall e \in \mathcal{E}$, as given in (10), as well as the stack vector $\gamma_{\mathcal{G}} : \mathsf{SE}(3)^N \to \mathbb{R}^{\frac{N(N-1)}{2}} \times \mathsf{S}^{2N(N-1)}$ as given in (11). Differentiation of $\gamma_{\mathcal{G}}(x(q)) = \text{const.}$, which encodes the rigidity constraints of the system comprised by the robotic agents, yields $\mathcal{R}_{\mathcal{G}}(x(q))J(q)\ddot{q} = -[\mathcal{R}_{\mathcal{G}}(x(q))\dot{J}(q) + \dot{\mathcal{R}}_{\mathcal{G}}(x(q))J(q)]\dot{q}$, written more compactly as

$$A_{\text{int}}(q, \dot{q}, t)\ddot{q} = b_{\text{int}}(q, \dot{q}, t), \tag{19}$$

where

$$A_{\text{int}}(q, \dot{q}, t) \coloneqq \mathcal{R}_{\mathcal{G}}(x(q))J(q), \tag{20a}$$

$$b_{\text{int}}(q, \dot{q}, t) \coloneqq -[\mathcal{R}_{\mathcal{G}}(x(q))\dot{J}(q) + \dot{\mathcal{R}}_{\mathcal{G}}(x(q))J(q)]\dot{q}. \tag{20b}$$

Similarly to the case of $\bar{\mathcal{G}}$, we conclude that the agent motions produced by the aforementioned constraints correspond to rigid body motions, which are the infinitesimal motions produced by the nullspace of $\mathcal{R}_{\mathcal{G}}$.

After giving the rigidity constraints in the cooperative manipulation system, we are now ready to derive the expressions for the interaction forces, h, in terms of the aforementioned rigidity matrices. We follow the same methodology as in [11]. Consider first (1a) and (2b) written in vector form as

$$\bar{B}(\bar{x})\dot{\widetilde{x}} + \bar{N}(\bar{x})\widetilde{x} + \bar{g}(\bar{x}) = \bar{\tau} + \begin{bmatrix} -J(q)^\top h \\ h_O \end{bmatrix}, \tag{21}$$

with the barred terms as introduced in the beginning of this section. We use Gauss' principle [13] to derive closed form expressions for $J(q)^\top h$ and h_O. Let the *unconstrained* coupled object-robots system be $\bar{B}(\bar{x})\alpha(\bar{x}) \coloneqq \bar{\tau} - \bar{N}(\bar{x})\widetilde{x} - \bar{g}(\bar{x})$, where α is the unconstrained acceleration, i.e., the acceleration the system would have if the agents did not grasp the object. According to Gauss's principle [12], the actual accelerations $\dot{\widetilde{x}}$ of the system are the closest ones to $\alpha(\bar{x})$, while satisfying the rigidity constraints. More rigorously, $\dot{\widetilde{x}}$ are the solutions of the constrained minimization problem

$$\begin{aligned} \min \quad & [\dot{\widetilde{x}} - \alpha(\bar{x})]^\top \bar{B}(\bar{x})[\dot{\widetilde{x}} - \alpha(\bar{x})] \\ \text{s.t.} \quad & A(\bar{x}, t)\dot{\widetilde{x}} = b(\bar{x}, t). \end{aligned}$$

The solution to this problem is obtained by using the Karush-Kuhn-Tucker conditions [35] and has a closed-form expression. It can be shown that it satisfies

$$\bar{B}\dot{\widetilde{x}} = \alpha + A^\top(A\bar{B}^{-1}A^\top)^\dagger(b - A\alpha),$$

which is compliant with the one in [12],

$$\bar{B}\dot{\widetilde{x}} = \alpha + \bar{B}^{\frac{1}{2}}(A\bar{B}^{-\frac{1}{2}}) \dagger (b - A\alpha),$$

since it holds that $A^\top(A\bar{B}^{-1}A^\top)^\dagger = \bar{B}^{\frac{1}{2}}(A\bar{B}^{-\frac{1}{2}})^\dagger$. Indeed, according to Th. 3.8 of [36], it holds that $H^\dagger = H^\top(HH^\top)^\dagger$, for any $H \in \mathbb{R}^{x \times y}$. Then the aforementioned equality is obtained by setting $H = A\bar{B}^{-\frac{1}{2}}$.

Therefore, the forces, projected onto the joint-space of the agents, have the form

$$\begin{bmatrix} -J^\top h \\ h_O \end{bmatrix} = A^\top(A\bar{B}^{-1}A^\top)^\dagger(b - A\alpha) \tag{22a}$$

$$= \bar{B}^{\frac{1}{2}}(A\bar{B}^{-\frac{1}{2}})^\dagger(b - A\alpha). \tag{22b}$$

Note that inversion of $(A\bar{B}^{-1}A^\top)$ is not needed to obtain (22), as implied in [10]. Consider now that $h_O = h_{\text{m}} = 0 \Leftrightarrow h = h_{\text{int}}$, i.e., the agents produce only internal forces, without inducing object acceleration. Then, the agent dynamics are

$$B(q)\ddot{q} + N(q, \dot{q})\dot{q} + g_q(q) = \tau - J(q)^\top h_{\text{int}}, \tag{23}$$

and the respective *unconstrained* acceleration is given by

$$B(q)\alpha_{\text{int}}(q, \dot{q}) \coloneqq \tau - N(q, \dot{q})\dot{q} - g_q(q).$$

Hence, by proceeding in a similar fashion as for $\dot{\widetilde{x}}$, we derive an expression for the *internal* forces as

$$-J^\top h_{\text{int}} = A_{\text{int}}^\top(A_{\text{int}}B^{-1}A_{\text{int}}^\top)^\dagger(b_{\text{int}} - A_{\text{int}}\alpha_{\text{int}}) \tag{24a}$$

$$= B^{\frac{1}{2}}(A_{\text{int}}B^{-\frac{1}{2}})^\dagger(b_{\text{int}} - A_{\text{int}}\alpha_{\text{int}}). \tag{24b}$$

Therefore, one concludes that when the unconstrained motion of the system does not satisfy the constraints (i.e., when $b_{\text{int}} \neq A_{\text{int}}\alpha_{\text{int}}$), then the actual accelerations of the system are modified in a manner directly proportional to the extent to which these constraints are violated. Moreover, it is evident from the aforementioned expression that the internal forces depend, not only on the relative distances $p_i - p_j$, but also on the closed loop dynamics and the inertia of the unconstrained system (see the dependence on α_{int} and B). Therefore, given a desired force $h_{O,\text{d}}$ to be applied to the object, an internal force-free distribution to agent forces $h_{i,\text{d}}$ at the grasping points cannot be independent of the system dynamics. We stress that the derived expression concerns the internal forces produced exclusively by the redundancy of the multi-robot system (excluding, for instance, potential internal forces needed to keep the object from falling due to gravity, which would also arise in a single manipulation task).

Note that, as dictated in Section IV, the rigidity matrix $\mathcal{R}_{\mathcal{G}}$ is not unique, since different choices of $\gamma_{\mathcal{G}}$ that encode the rigidity constraints can be made. Hence, one might think that different expressions of $\mathcal{R}_{\mathcal{G}}$ will result in different rigidity constraints of the form (20) and hence different interaction and internal forces - which is unreasonable. Nevertheless, note that all different expressions of the rigidity matrix $\mathcal{R}_{\mathcal{G}}$ have the same nullspace (the coordinated translations and rotations of the framework), and that suffices to prove that this is not the case, as illustrated in Coroll. 2.

Corollary 2. *Let $\mathcal{R}_{\mathcal{G},1}$ and $\mathcal{R}_{\mathcal{G},2}$ such that* $\text{null}(\mathcal{R}_{\mathcal{G},1}(q)) = \text{null}(\mathcal{R}_{\mathcal{G},2}(q))$ *and let*

$$\begin{aligned} J^\top h_{\text{int},i} :=& B^{\frac{1}{2}}(\mathcal{R}_{\mathcal{G},i} J B^{-\frac{1}{2}})^\dagger [\big(\mathcal{R}_{\mathcal{G},i}\dot{J} + \dot{\mathcal{R}}_{\mathcal{G},i} J\big)\dot{q} + \\ & \mathcal{R}_{\mathcal{G},i} J B^{-1}\alpha_{\text{int}}], \ \forall i \in \{1,2\}, \end{aligned}$$

where we have used (24) *and* (20)*. Then* $h_{\text{int},1} = h_{\text{int},2}$.

One can verify that a similar argument holds for the interaction forces $\begin{bmatrix} -J^\top h \\ h_O \end{bmatrix}$ and $\mathcal{R}_{\bar{\mathcal{G}}}$ as well.

The aforementioned expressions concern the forces in the joint-space of the robotic agents. The next Coroll. gives the expression of the forces in task-space:

Corollary 3. *The internal forces h_{int} are given by*

$$h_{\text{int}} = \mathcal{R}_{\mathcal{G}}^\top (\mathcal{R}_{\mathcal{G}} M^{-1} \mathcal{R}_{\mathcal{G}}^\top)^\dagger (\dot{\mathcal{R}}_{\mathcal{G}} v + \mathcal{R}_{\mathcal{G}}\alpha_{\text{int}}^{\text{ts}}), \tag{25}$$

where $\alpha_{\text{int}}^{\text{ts}}$ is the acceleration vector of the task-space unconstrained system $M(q)\alpha_{\text{int}}^{\text{ts}}(q,\dot{q}) := u - C(q,\dot{q})v - g(q)$, and the forces h, h_O are given by

$$\begin{bmatrix} -h \\ h_O \end{bmatrix} = -\mathcal{R}_{\bar{\mathcal{G}}}^\top (\mathcal{R}_{\bar{\mathcal{G}}} \bar{M}^{-1} \mathcal{R}_{\bar{\mathcal{G}}}^\top)^\dagger (\dot{\mathcal{R}}_{\bar{\mathcal{G}}} \bar{v} + \mathcal{R}_{\bar{\mathcal{G}}}\alpha^{\text{ts}}), \tag{26}$$

where α^{ts} is the acceleration vector of the task-space unconstrained system $\bar{M}(\bar{x})\alpha^{\text{ts}}(x) := \bar{u} - \bar{C}(\bar{x})\bar{v} - \bar{g}(\bar{x})$.

We also show later that the derived forces (26) are consistent with the relation $h_O = G(x)h$ (see (7)).

We now give a more explicit expression for h. One can verify that, by appropriately arranging the rows of $\gamma_{\mathcal{G}}$, it holds

$$\mathcal{R}_{\bar{\mathcal{G}}} := \begin{bmatrix} \mathcal{R}_{\mathcal{G}} & 0_{\frac{7N(N-1)}{2}\times 6} \\ \mathcal{R}_{O_1} & \mathcal{R}_{O_2} \end{bmatrix} \in \mathbb{R}^{\frac{7\bar{N}(\bar{N}-1)}{2}\times(6N+6)}, \tag{27}$$

where $\mathcal{R}_{O_1} \in \mathbb{R}^{7N\times 6N}$ and $\mathcal{R}_{O_2} \in \mathbb{R}^{7N\times 6}$ are the matrices

$$\mathcal{R}_{O_1} := \begin{bmatrix} (p_1 - p_O)^\top & 0_{1\times 3} & \cdots & 0_3^\top & 0_3^\top \\ \vdots & \vdots & \cdots & \ddots & \vdots \\ 0_3^\top & 0_3^\top & \cdots & (p_N - p_O)^\top & 0_3^\top \\ \frac{\partial \gamma_{e_{1O},b}}{\partial p_1} & \frac{\partial \gamma_{e_{1O},b}}{\partial R_1} & \cdots & 0_{3\times 3} & 0_{3\times 3} \\ \frac{\partial \gamma_{e_{O1},b}}{\partial p_1} & \frac{\partial \gamma_{e_{O1},b}}{\partial R_1} & \cdots & 0_{3\times 3} & 0_{3\times 3} \\ \vdots & \cdots & \ddots & \vdots & \vdots \\ 0_{3\times 3} & 0_{3\times 3} & \cdots & \frac{\partial \gamma_{e_{NO},b}}{\partial p_N} & \frac{\partial \gamma_{e_{NO},b}}{\partial R_N} \\ 0_{3\times 3} & 0_{3\times 3} & \cdots & \frac{\partial \gamma_{e_{ON},b}}{\partial p_N} & \frac{\partial \gamma_{e_{ON},b}}{\partial R_N} \end{bmatrix}$$

$$\mathcal{R}_{O_2} := \begin{bmatrix} -(p_1 - p_O)^\top & 0_3^\top \\ \vdots & \vdots \\ -(p_N - p_O)^\top & 0_3^\top \\ \frac{\partial \gamma_{e_{1O},b}}{\partial p_O} & \frac{\partial \gamma_{e_{1O},b}}{\partial R_O} \\ \frac{\partial \gamma_{e_{O1},b}}{\partial p_O} & \frac{\partial \gamma_{e_{O1},b}}{\partial R_O} \\ \vdots & \vdots \\ \frac{\partial \gamma_{e_{NO},b}}{\partial p_O} & \frac{\partial \gamma_{e_{NO},b}}{\partial R_O} \\ \frac{\partial \gamma_{e_{ON},b}}{\partial p_O} & \frac{\partial \gamma_{e_{ON},b}}{\partial R_O} \end{bmatrix},$$

and $e_{iO} := (i, \bar{N})$, $e_{Oi} := (\bar{N}, i) \in \bar{\mathcal{E}}$ corresponding to the object-ith agent edge. Therefore, (26) can be written as

$$h = \begin{bmatrix} \mathcal{R}_{\mathcal{G}}^\top & \mathcal{R}_{O_1}^\top \end{bmatrix} (\mathcal{R}_{\bar{\mathcal{G}}} \bar{M}^{-1} \mathcal{R}_{\bar{\mathcal{G}}}^\top)^\dagger (\dot{\mathcal{R}}_{\bar{\mathcal{G}}} \bar{v} + \mathcal{R}_{\bar{\mathcal{G}}}\alpha^{\text{ts}}) \tag{28a}$$

$$h_O = -\begin{bmatrix} 0 & \mathcal{R}_{O_2}^\top \end{bmatrix} (\mathcal{R}_{\bar{\mathcal{G}}} \bar{M}^{-1} \mathcal{R}_{\bar{\mathcal{G}}}^\top)^\dagger (\dot{\mathcal{R}}_{O} \bar{v} + \mathcal{R}_{\bar{\mathcal{G}}}\alpha^{\text{ts}}). \tag{28b}$$

Note also that

$$G\mathcal{R}_{O_1}^\top = -\mathcal{R}_{O_2}^\top, \tag{29}$$

which will be used in the analysis to follow.

Another expression for the interaction forces h can be obtained by differentiating (6), which, after using (1) and (2) yields after straightforward manipulations

$$\begin{aligned} h =& (M^{-1} + G^\top M_O^{-1} G)^{-1}[M^{-1}(u - g - Cv) - \dot{G}^\top v_O \\ & + G^\top M_O^{-1}(C_O v_O + g_O)]. \end{aligned} \tag{30}$$

In order to show the consistency of our results, we prove next that (28a) and (30) are identical.

Corollary 4. *Let h_1 be given by* (28a) *and h_2 be given by* (30)*. Then $h_1 = h_2$.*

Remark 1. *According to Th. 3.8 of [36], the task-space internal forces can be also written as*

$$h_{\text{int}} = M^{\frac{1}{2}}(\mathcal{R}_{\mathcal{G}} M^{-\frac{1}{2}})^\dagger (\dot{\mathcal{R}}_{\mathcal{G}} v + \mathcal{R}_{\mathcal{G}}\alpha_{\text{int}}^{\text{ts}}), \tag{31}$$

which is compliant with the result in [11].

One concludes, therefore, that in order to obtain internal force-free trajectories, the term $\dot{\mathcal{R}}_{\mathcal{G}} v + \mathcal{R}_{\mathcal{G}}\alpha_{\text{int}}^{\text{ts}} = \dot{\mathcal{R}}_{\mathcal{G}} v + \mathcal{R}_{\mathcal{G}} M^{-1}(u - C\dot{v} - g)$ must belong to the nullspace of $M^{\frac{1}{2}}(\mathcal{R}_{\mathcal{G}} M^{-\frac{1}{2}})^\dagger$. The latter, however, is identical to the nullspace of $\mathcal{R}_{\mathcal{G}}$, since it holds that $\text{null}(\mathcal{R}_{\mathcal{G}} M^{-1/2})^\dagger = \text{null}(M^{-\frac{1}{2}}\mathcal{R}_{\mathcal{G}}^\top)$ and M is positive definite. This result is summarized in the following corollary.

Corollary 5. *The cooperative manipulation system is free of internal forces, i.e., $h_{\text{int}} = 0_{6N}$, if and only if*

$$\dot{\mathcal{R}}_{\mathcal{G}} v + \mathcal{R}_{\mathcal{G}} M^{-1}(u - C\dot{v} - g) \in \text{null}(\mathcal{R}_{\mathcal{G}}^\top)$$

In cooperative manipulation schemes, the most energy-efficient way of transporting an object is to exploit the full potential of the cooperating robotic agents, i.e., each agent does not exert less effort at the expense of other agents, which might then potentially exert more effort than necessary. For instance, consider a rigid cooperative manipulation scheme, with only one agent (a leader) working towards bringing the object to a desired location, whereas the other agents have zero inputs. Since the grasps are rigid, if the leader has sufficient power, it will achieve the task by "dragging" the rest of the agents, compensating for their dynamics, and creating non-negligible internal forces. In such cases, when the cooperative manipulation system is rigid (i.e., the grasps are considered to be rigid), the optimal strategy of transporting an object is achieved by regulating the internal forces to zero. Therefore, from a control perspective, the goal of a rigid cooperative manipulation system is to design a control protocol that achieves a desired cooperative manipulation task, while guaranteeing that the internal forces remain zero.

B. Cooperative Manipulation via Internal Force Regulation

In this section, we derive a new relation between the interaction and internal forces h and h_{int}, respectively. Moreover, we derive novel sufficient and necessary conditions on the agent force distribution for the provable regulation of the internal forces to zero, according to (25), and we show its application in a standard inverse-dynamics control law that guarantees trajectory tracking of the object's center of mass. This is based on the following main theorem, which links the complete agent graph rigidity matrix $\mathcal{R}_\mathcal{G}$ to the grasp matrix G:

Theorem 1. *Let N robotic agents, with configuration $x = (p,R) \in \mathsf{SE}(3)^N$, rigidly grasping an object and associated with a grasp matrix $G(x)$, as in* (5)*. Let also the agents be modeled by a framework on the complete graph $(\mathcal{K}_N, p_{\mathcal{K}_N}, R_{\mathcal{K}_N}) = (\mathcal{K}_N, p, R)$ in $\mathsf{SE}(3)$, which is associated with a rigidity matrix $\mathcal{R}_{\mathcal{K}_N}$. Let also x be such that* $\text{rank}(\mathcal{R}_{\mathcal{K}_N}(x)) = \max_{y\in\mathsf{SE}(3)^N}\{\text{rank}(\mathcal{R}_{\mathcal{K}_N}(y))\}$*. Then*

$$\text{null}(G(x)) = \text{range}(\mathcal{R}_{\mathcal{K}_N}(x)^\top). \tag{32}$$

Hence, since the internal forces belong to $\text{null}(G)$, one concludes that they are comprised of all the vectors z for which there exists a y such that $z = \mathcal{R}_\mathcal{G}^\top y$. This can also be verified by inspecting (31); one can prove that $\text{range}(M^{\frac{1}{2}}(\mathcal{R}_\mathcal{G}M^{-\frac{1}{2}})^\dagger) = \text{range}(\mathcal{R}_\mathcal{G}^\top)$. The aforementioned result provides significant insight regarding the control of the motion of the coupled cooperative manipulation system. In particular, by using (31) and Th. 1, we provide next new conditions on the agent force distribution for provable avoidance of internal forces. We first derive a novel relation between the agent forces h and the internal forces h_{int}.

In many related works, h is decomposed as

$$h = G^*Gh + (I - G^*G)h, \tag{33}$$

where G^* is a right inverse of G. The term G^*Gh is a projection of h on the range space of $G^\top$, whereas the term $(I - G^*G)h$ is a projection of h on the null space of G. A common choice is the Moore-Penrose inverse $G^* = G^\dagger = G^\top(GG^\top)^{-1}$. This specific choice yields the vector $G^*Gh = G^\dagger Gh \in \text{range}(G^\top)$ that is closest to h, i.e., $\|h - G^\dagger Gh\| \leq \|h - y\|$, $\forall y \in \text{range}(G^\top)$. However, as the next theorem states, if the second term of (33) must equal h_{int}, as defined in (31), G^* must equal $MG^\top(GMG^\top)^{-1}$.

Theorem 2. *Consider N robotic agents rigidly grasping an object with coupled dynamics* (8)*. Let $h \in \mathbb{R}^{6N}$ be the stacked vector of agent forces exerted at the grasping points. Then the agent forces h and the internal forces h_{int} are related as:*

$$h_{\text{int}} = (I_{6N} - MG^\top(GMG^\top)^{-1}G)h.$$

Based on Th. 2, we provide next new results on the internal force-free (optimal) distribution of a force to the agents.

Theorem 3. *Consider N robotic agents rigidly grasping an object with coupled dynamics* (8)*. Let a desired force to be applied to the object $h_{O,\text{d}} \in \mathbb{R}^6$, which is distributed to the agents' desired forces as $h_\text{d} = G^*h_{O,\text{d}}$, and where G^* is a right inverse of G, i.e., $GG^* = I_6$. Then it holds that*

$$h_{\text{int}} = 0 \Leftrightarrow G^* = MG^\top(GMG^\top)^{-1}.$$

The aforementioned theorem provides novel necessary and sufficient conditions for provable minimization of internal forces in a cooperative manipulation scheme. As discussed before, this is crucial for achieving energy-optimal cooperative manipulation, where the agents do not have to "waste" control input and hence energy resources that do not contribute to object motion. Related works that focus on deriving internal force-free distributions G^*, e.g., [11], [14], [25]–[27], are solely based on the inter-agent distances, neglecting the actual dynamics of the agents and the object. The expression (25), however, gives new insight on the topic and suggests that the dynamic terms of the system play a significant role in the arising internal forces, as also indicated by Coroll. 5. This is further exploited by Th. 3 to derive a right-inverse that depends on the inertia of the system. Note also that, as mentioned before and explained in [11], the internal forces depend on the acceleration of the robotic agents and hence the incorporation of M in G^* is something to be expected.

The forces h, however, are not the actual control input of the robotic agents, and hence we cannot simply set $h = h_\text{d} = MG^\top(GMG^\top)^{-1}Gh_{O,\text{d}}$ for a given $h_{O,\text{d}}$. Therefore, we design next a standard inverse-dynamics control algorithm controller that guarantees tracking of a desired trajectory by the object center of mass while provably achieving regulation of the internal forces to zero. Provable force regulation is also done in [10], requiring however the constraints matrix ($\mathcal{R}_\mathcal{G}$ in our case) to have positive singular values.

Let a desired position trajectory for the object center of mass be $p_\text{d} : \mathbb{R}_{\geq 0} \to \mathbb{R}^3$, and $e_p := p_O - p_\text{d}$. Let also a desired object orientation be expressed in terms of a desired rotation matrix $R_\text{d} : \mathbb{R}_{\geq 0} \to \mathsf{SO}(3)$, with $\dot{R}_\text{d} = S(\omega_\text{d})R_\text{d}$, where $\omega_\text{d} : \mathbb{R}_{\geq 0} \to \mathbb{R}^3$ is the desired angular velocity. Then an orientation error metric is [2] $e_O := \frac{1}{2}\text{tr}\left(I_3 - R_\text{d}^\top R_O\right) \in [0,2]$, which, after differentiation and by using (2a) and properties of skew-symmetric matrices, becomes [2]

$$\dot{e}_O = \frac{1}{2}e_R^\top R_O^\top\left(\omega_O - \omega_\text{d}\right), \tag{34}$$

where $e_R := S^{-1}\left(R_\text{d}^\top R_O - R_O^\top R_\text{d}\right) \in \mathbb{R}^3$. The equilibrium $e_R = 0$ corresponds to $e_O = 0$, implying $\text{tr}(R_\text{d}^\top R_O) = 3$ and $R_O = R_\text{d}$, as well as to $e_O = 2$ implying $\text{tr}(R_\text{d}^\top R_O) = -1$ and $R_O \neq R_\text{d}$ [2]. The second case represents an undesired equilibrium, where the desired and the actual orientation differ by 180 degrees. This issue is caused by topological obstructions on $\mathsf{SO}(3)$ and it has been proven that no continuous controller can achieve *global* stabilization [2]. We design next a control protocol that guarantees internal force-free convergence of e_p, e_O, while guaranteeing that $e_O(t) < 2$, $\forall t \in \mathbb{R}_{\geq 0}$, provided that the right inverse $G^* = MG^\top(GMG^\top)^{-1}$ is used.

Corollary 6. *Consider N robotic agents rigidly grasping an object, as described in Section III, with coupled dynamics* (8)*. Let a desired trajectory be defined by $p_\text{d} : \mathbb{R}_{\geq 0} \to \mathbb{R}^3$, $R_\text{d} : \mathbb{R}_{\geq 0} \to \mathsf{SO}(3)$, $\dot{p}_\text{d}, \omega_\text{d} \in \mathbb{R}^3$, and assume that $e_O(0) < 2$, with e_O. Consider the control law*

$$\begin{aligned} u = g + (CG^\top + M\dot{G}^\top)v_O + G^*\left(g_O + C_Ov_O\right) + \\ (MG^\top + G^*M_O)(\dot{v}_\text{d} - K_de_v - K_pe_x), \end{aligned} \tag{35}$$

where $e_v := v_O - v_\text{d}$, $v_\text{d} := [\dot{p}_\text{d}^\top, \omega_\text{d}^\top]^\top \in \mathbb{R}^6$, $e_x := [e_p^\top, \frac{1}{2(2-e_O)^2}e_R^\top R_O^\top]^\top$, $K_p := \text{diag}\{K_{p_1}, k_{p_2}I_3\}$, where $K_{p_1} \in \mathbb{R}^{3\times 3}, K_d \in \mathbb{R}^{6\times 6}$ are positive definite matrices, and

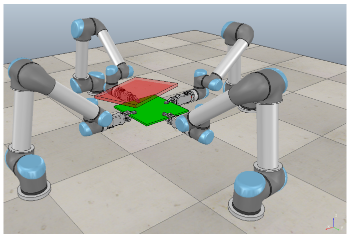

Fig. 2: Four UR5 robotic arms rigidly grasping an object. The red counterpart represents a desired object pose at $t = 0$.

$k_{p_2} \in \mathbb{R}_{>0}$ is a positive constant. Then the solution of the closed-loop coupled system satisfies the following:

1) $e_O(t) < 2$, $\forall t \in \mathbb{R}_{\geq 0}$
2) $\lim_{t\to\infty}(p_O(t) - p_\text{d}(t) = 0_3$, $\lim_{t\to\infty} R_\text{d}(t)^\top R_O(t) = I_3$
3) It holds that $h_\text{int}(t) = 0 \Leftrightarrow G^* = MG^\top(GMG^\top)^{-1}$.

Remark 2 (Uncertain dynamics and force sensing). *Note that the employed controller requires knowledge of the agent and object dynamics. In case of dynamic parameter uncertainty, standard adaptive control schemes that attempt to estimate potential uncertainties in the model (e.g., [21], [24]) would intrinsically create internal forces, since the dynamics of the system would not be accurately compensated. The same holds for schemes that employ force/torque sensors that provide the respective measurements at the grasp points (e.g., [19], [20]) in periodic time instants. Since the interaction forces depend explicitly on the control input, such measurements will unavoidably correspond to the interaction forces of the previous time instants due to causality reasons, creating thus small disturbances in the dynamic model.*

Remark 3 (Load-sharing). *Finally, note that $G^\star = MG^\top(GMG^\top)^{-1}$ induces an implicit and natural load-sharing scheme via the incorporation of M. More specifically, note that the force distribution to the robotic agents via $G^* h_{O,\text{d}}$ yields for each agent $M_i J_{O_i}(\sum_{i\in\mathcal{N}} J_{O_i}^\top M_i J_{O_i})^{-1}$, $\forall i \in \mathcal{N}$. Hence, larger values of M_i will produce larger inputs for agent i, implying that agents with larger inertia characteristics will take on a larger share of the object load. Note that this is also a desired load-sharing scheme, since larger dynamic values usually imply more powerful robotic agents. Previous works that implemented load-sharing schemes (e.g., [19]) used load-sharing coefficients, without relating the resulting force distribution with the arising internal forces.*

In case it is required to achieve a *desired* internal force $h_\text{int,d}$, one can add in (35) a term of the form described next.

Corollary 7. *Let $h_\text{int,d} \in$ null(G) be a desired internal force to be achieved. Then adding the extra term $u_\text{int,d} = (I_{6N} - MG^\top(GMG^\top)^\top)^{-1} h_\text{int,d}$ in (35) achieves $h_\text{int} = h_\text{int,d}$.*

Finally, in view of Th. 1, one can also verify the consistency of the expressions of h, h_O in (26) with the grasp-matrix rigidity constraint $h_O = G(x)h$ (see (7)). Indeed, Th. 1 dictates that $G\mathcal{R}_\mathcal{G}^\top = 0$. Therefore, by combining (29) and (28) we conclude that $h_O = G(x)h$. Note also that, in view of Coroll. 2, the result is still valid if different $\gamma_{\bar{\mathcal{G}}}$ and $\mathcal{R}_{\bar{\mathcal{G}}}$ are chosen.

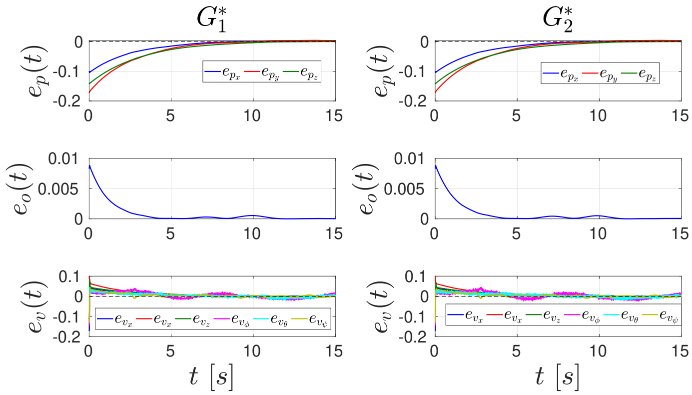

Fig. 3: The error metrics $e_p(t)$, $e_O(t)$, $e_v(t)$, respectively, top to bottom, for the two choices G_1^* and G_2^* and $t \in [0, 15]$ seconds.

VI. Simulation Results

This section provides simulation results using 4 identical UR5 robotic manipulators in the realistic dynamic environment V-REP [37]. The agents are rigidly grasping an object of 40 kg as shown in Fig. 2. In order to verify the findings of the previous sections, we apply the controller (35) to achieve tracking of a desired trajectory by the object's center of mass. We simulate the closed loop system for two cases of G^*, namely the proposed one $G_1^* = MG^\top(GMG^\top)^{-1}$ as well as the more standard choice $G_2^* = G^\top(GG^\top)^{-1}$, showing the validity of Coroll. 6 and 7.

The initial pose of the object is set as $p_O(0) = [-0.225, -0.612, 0.161]^\top$, $\eta_O(0) = [0, 0, 0]^\top$ and the desired trajectory as $p_\text{d}(t) = p_O(0) + [0.2\sin(w_p dt + \varphi_\text{d}), 0.2\cos(w_p t + \varphi_\text{d}), 0.09 + 0.1\sin(w_p t + \varphi_\text{d})]^\top$, $\eta_\text{d}(t) = [0.15\sin(w_\phi t + \varphi_\text{d}), 0.15\sin(w_\theta t + \varphi_\text{d}), 0.15\sin(w_\psi t + \varphi_\text{d})]^\top$ (in meters and rad, respectively), where $\varphi_\text{d} = \frac{\pi}{6}$, $w_p = w_\phi = w_\psi = 1$, $w_\theta = 0.5$, and $\eta_\text{d}(t)$ is transformed to the respective $R_\text{d}(t)$. The gains are set as $K_{p_1} = 15$, $k_{p_2} = 75$, $K_d = 40I_6$.

The results are given in Figs. 3-4 for 15 seconds. Fig. 3 depicts the pose and velocity errors $e_p(t)$, $e_O(t)$, $e_v(t)$, which converge to zero for both choices of G^*, as expected. The norms of the control inputs $\tau_i(t)$ of the agents are shown in Fig. 4. Moreover, the norm of the internal forces, $\|h_\text{int}(t)\|$, computed via (25), is shown in Fig. 5 (left). It is clear that G_2^* yields significantly larger internal forces, whereas G_1^* keeps them very close to zero, as proven in the theoretical analysis. The larger internal forces in the case of G_2^* are associated with the larger control inputs τ_i. This can be also concluded from Fig. 4; It is clear that inputs of larger magnitude occur in the case of G_2^*, which create internal forces.

Finally, we set a random force vector $h_\text{int,d}$ in the nullspace of G and we simulate the control law (35) with the extra component $u_\text{int,d} = h_\text{int,d}$ (see Coroll. 7). Fig. 5 (right) illustrates the error norm $\|e_\text{int}(t)\| := \|h_\text{int,d}(t) - h_\text{int}(t)\|$, which evolves close to zero. The minor observed deviations can be attributed to model uncertainties and hence the imperfect cancellation of the respective dynamics via (35).

VII. Conclusion and Future Work

We introduce the notion of distance and bearing rigidity in SE(3) and we use the associated rigidity matrix to express the

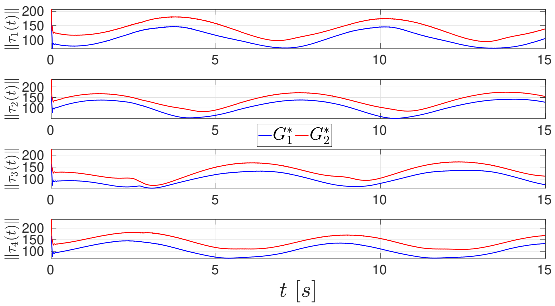

Fig. 4: The norms of the resulting control inputs, $\|\tau_i(t)\|$ for G_1^* (with blue) and G_2^* (with red), $\forall i \in \{1,\dots,4\}$, and $t \in [0, 15]$ seconds.

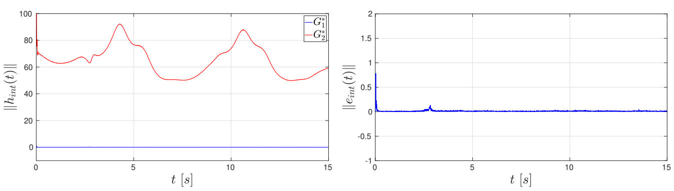

Fig. 5: Left: The signal $\|h_{\text{int}}(t)\|$ (as computed via (25)) for the two cases of G^* and $t \in [0, 15]$ seconds. Right: The signal $\|e_{\text{int}}(t)\|$, when using G_1^* and for $t \in [0, 15]$ seconds.

interaction forces that emerge in a cooperative manipulation scheme. Based on these results, we connect the rigidity and grasp matrices via a nullspace-range relation and we provide novel results on internal-forced based cooperative manipulation control and on the relation between the interaction and internal forces. Future efforts will be directed towards using rigidity theory for object pose estimation and robust control design that minimizes the arising internal forces.

Appendix A

Proof of Corollary 2: The poses and velocities in the terms $(\mathcal{R}_{\mathcal{G},i}\dot{J} + \dot{\mathcal{R}}_{\mathcal{G},i}J)\dot{q}$ are the actual ones resulting from the coupled system dynamics and hence they respect the rigidity constraints imposed by $R_{\mathcal{G},i}J\ddot{q} = (\mathcal{R}_{\mathcal{G},i}\dot{J} + \dot{\mathcal{R}}_{\mathcal{G},i}J)\dot{q}$, $\forall i \in \{1, 2\}$. Therefore, exploiting the positive definiteness of B, we need to prove that $(\mathcal{R}_{\mathcal{G},1}JB^{-\frac{1}{2}})^\dagger \mathcal{R}_{\mathcal{G},1}J = (\mathcal{R}_{\mathcal{G},2}JB^{-\frac{1}{2}})^\dagger \mathcal{R}_{\mathcal{G},2}J$. In view of Def. 1 and Prop. 1, since $\mathcal{R}_{\mathcal{G},1}$ and $\mathcal{R}_{\mathcal{G},2}$ have the same nullspace, they are left equivalent matrices and there exists an invertible matrix P such that $\mathcal{R}_{\mathcal{G},1} = P\mathcal{R}_{\mathcal{G},2}$. Hence, it holds that

$$
\begin{aligned}
&(\mathcal{R}_{\mathcal{G},2}JB^{-\frac{1}{2}})^\dagger \mathcal{R}_{\mathcal{G},2}J - (\mathcal{R}_{\mathcal{G},1}JB^{-\frac{1}{2}})^\dagger \mathcal{R}_{\mathcal{G},1}J = \\
&[(\mathcal{R}_{\mathcal{G},2}JB^{-\frac{1}{2}})^\dagger \mathcal{R}_{\mathcal{G},2}JB^{-\frac{1}{2}} - (P\mathcal{R}_{\mathcal{G},2}JB^{-\frac{1}{2}})^\dagger P\mathcal{R}_{\mathcal{G},2}JB^{-\frac{1}{2}}]B^{\frac{1}{2}},
\end{aligned}
$$

which is equal to 0, according to Prop. 2 and the positive definiteness of B. ∎

Proof of Corollary 3: By using the expressions of $M(q), C(q,\dot{q})v, g(q)$ from (1a) to expand (25), one can conclude that $J^\top h_{\text{int}}$, with h_{int} given by (25) and in view of (20), is equal to (24a). Similarly, by expanding (26) and using (18), one can verify that the vector $[-(J^\top h)^\top, h_O^\top]^\top$, with $[-h^\top, h_O^\top]^\top$ given by (26), is equal to (22). ∎

Proof of Corollary 4: By using (27), (28a) becomes

$$
\begin{aligned}
h_1 =& \mathcal{R}_{O_1}^\top(\mathcal{R}_{O_1}M^{-1}\mathcal{R}_{O_1}^\top + \mathcal{R}_{O_2}M_O^{-1}\mathcal{R}_{O_2}^\top)^\dagger[\dot{\mathcal{R}}_{O_1}v + \dot{\mathcal{R}}_{O_2}v_O \\
&+ \mathcal{R}_{O_1}M^{-1}(u - g - Cv) - \mathcal{R}_{O_2}M_O^{-1}(g_O + C_Ov_O)]
\end{aligned}
$$

which, after using (29) and $v = G^\top v_O$, becomes

$$
\begin{aligned}
h_1 =& \mathcal{R}_{O_1}^\top[\mathcal{R}_{O_1}(M^{-1} + G^\top M_O^{-1}G)\mathcal{R}_{O_1}^\top]^\dagger[\dot{\mathcal{R}}_{O_1}v - \\
&\dot{\mathcal{R}}_{O_1}G^\top v_O - \mathcal{R}_{O_1}\dot{G}^\top v_O + \mathcal{R}_{O_2}M^{-1}(u - g - Cv) + \\
&\mathcal{R}_{O_1}G^\top M_O^{-1}(g_O + C_Ov_O)] \\
=& \mathcal{R}_{O_1}^\top[\mathcal{R}_{O_1}(M^{-1} + G^\top M_O^{-1}G)\mathcal{R}_{O_1}^\top]^\dagger \mathcal{R}_{O_1}[-\dot{G}^\top v_O \\
&+ M^{-1}(u - g - Cv) + M_O^{-1}(g_O + C_Ov_O)].
\end{aligned}
$$

Denote now for convenience $M_G := M^{-1} + G^\top M_O^{-1}G$. According to Th. 3.8 of [36], it holds that $\mathcal{R}_{O_1}^\top(\mathcal{R}_{O_1}M_G\mathcal{R}_{O_1}^\top)^\dagger\mathcal{R}_{O_1} = M_G^{-\frac{1}{2}}(\mathcal{R}_{O_1}M_G^{\frac{1}{2}})^\dagger\mathcal{R}_{O_1}$. Next, note that $\mathcal{R}_{O_1}$ has linearly independent columns and hence $(\mathcal{R}_{O_1}M_G^{\frac{1}{2}})^\dagger = (M_G^{\frac{1}{2}}\mathcal{R}_{O_1}^\top\mathcal{R}_{O_1}M_G^{\frac{1}{2}})^{-1}M_G^{\frac{1}{2}}\mathcal{R}_{O_1}^\top = M_G^{-\frac{1}{2}}(\mathcal{R}_{O_1}^\top\mathcal{R}_{O_1})^{-1}\mathcal{R}_{O_1}^\top$, since M_G is symmetric and positive definite. Therefore, one obtains $M_G^{-\frac{1}{2}}(\mathcal{R}_{O_1}M_G^{\frac{1}{2}})^\dagger\mathcal{R}_{O_1} = M_G^{-1}$, and hence $h_1 = h_2$. ∎

Proof of Theorem 1: Since $\mathcal{R}_{\mathcal{K}_N}$ is over the complete graph and $\text{rank}(\mathcal{R}_{\mathcal{K}_N}(x)) = \max_{y\in\mathsf{SE}(3)^N}\{\text{rank}(\mathcal{R}_{\mathcal{K}_N}(y))\}$, the framework $(\mathcal{K}_N, p, R)$ is infinitesimally rigid. Hence, the nullspace of $\mathcal{R}_{\mathcal{K}_N}$ consists only of the infinitesimal motions of the framework, i.e., coordinated translations and rotations, as defined in Prop. 4. In particular, in view of (17), Prop. 5, and (14), one concludes that $\text{null}(\mathcal{R}_{\mathcal{K}_N})$ is the linear span of $1_N \otimes \begin{bmatrix} I_3 \\ 0_{3\times3} \end{bmatrix}$ and the vector space $[\chi_1^\top, \dots, \chi_N^\top]^\top \in \mathsf{SE}(3)^N$, with $\chi_i := [\chi_{i,p}^\top, \chi_{i,R}^\top]^\top \in \mathsf{SE}(3)$, satisfying

$$
\chi_{i,p} - \chi_{j,p} = -S(p_i - p_j)\chi_{i,R} \tag{36a}
$$

$$
\chi_{i,R} = \chi_{j,R}, \tag{36b}
$$

where $p_i := p_{\mathcal{K}_N}(i)$, $p_j := p_{\mathcal{K}_N}(j)$, $\forall i, j \in \mathcal{N}$, with $i \neq j$. In view of (6), one obtains $v = G(x)^\top v_O$. Note that the first 3 columns of $G^\top$ form the space $1_N \otimes \begin{bmatrix} I_3 \\ 0_{3\times3} \end{bmatrix}$ whereas its last 3 columns span the aforementioned rotation vector space. Indeed, for any $\dot{p}_O, \omega_O \in \mathbb{R}^6$ the range of these columns is

$$
[-\dot{p}_O^\top S(p_{1O})^\top, \omega_O^\top, \dots, -\dot{p}_O^\top S(p_{NO})^\top, \omega_O^\top]^\top,
$$

for which it is straightforward to verify that (36) holds. Hence, $\text{null}(\mathcal{R}_{\mathcal{K}_N}) = \text{range}(G^\top)$ and by using the rank-nullity theorem the result follows. ∎

Next, in order to prove Th. 2, we need the following result.

Proposition 6. *Consider the grasp and rigidity matrices G, $\mathcal{R}_\mathcal{G}$, of (5), (12), respectively. Then it holds that*

$$
MG^\top(GMG^\top)^{-1}G + M^{\frac{1}{2}}\mathcal{R}_\mathcal{G}M^{-\frac{1}{2}})^\dagger\mathcal{R}_\mathcal{G}M^{-1} = I. \tag{37}
$$

Proof: Let $A := \mathcal{R}_\mathcal{G}M^{-\frac{1}{2}}$ and $B := GM^{\frac{1}{2}}$. Then $\text{range}(A^\top) = \text{null}(B)$. Indeed, according to Th. 1, it holds that if $z = \mathcal{R}_\mathcal{G}^\top y$, for some $y \in \mathbb{R}^6$, then $Gz = 0_6$. By multiplying by $M^{-\frac{1}{2}}$, we obtain $M^{-\frac{1}{2}}z = M^{-\frac{1}{2}}\mathcal{R}_\mathcal{G}^\top y$, which implies that $\hat{z} := M^{-\frac{1}{2}}z \in \text{range}((\mathcal{R}_\mathcal{G}M^{\frac{1}{2}})^\top)$. It also holds that $B\hat{z} = GM^{\frac{1}{2}}\hat{z} = Gz = 0_6$, and hence $\hat{z} \in \text{null}(B)$. Therefore, in view of Prop. 3, Th. 3.8 of [36], according to which $G^\top(GMG^\top)^\dagger = M^{-\frac{1}{2}}(GM^{\frac{1}{2}})^\dagger$, and the fact that $GMG^\top$ is invertible, we conclude that

$$(GM^{\frac{1}{2}})^\dagger GM^{\frac{1}{2}} + (\mathcal{R}_\mathcal{G} M^{-\frac{1}{2}})^\dagger \mathcal{R}_\mathcal{G} M^{-\frac{1}{2}} = I \Leftrightarrow$$
$$M^{\frac{1}{2}} G^\top (GMG^\top)^\dagger GM^{\frac{1}{2}} + (\mathcal{R}_\mathcal{G} M^{-\frac{1}{2}})^\dagger \mathcal{R}_\mathcal{G} M^{-\frac{1}{2}} = I,$$

and by left and right multiplication by $M^{\frac{1}{2}}$ and $M^{-\frac{1}{2}}$, respectively, the result follows. ■

Proof of Theorem 2: We first show that

$$[I - MG^\top(GMG^\top)^{-1}G](M^{-1} + G^\top M_O^{-1} G)^{-1} = M^{\frac{1}{2}}(\mathcal{R}_\mathcal{G} M^{-\frac{1}{2}})^\dagger \mathcal{R}_\mathcal{G}.$$

Indeed, since $(M^{-1} + G^\top M_O^{-1} G)^{-1}$ has full rank, it suffices to show that

$$I - MG^\top(GMG^\top)^{-1}G = M^{\frac{1}{2}}(\mathcal{R}_\mathcal{G} M^{-\frac{1}{2}})^\dagger \mathcal{R}_\mathcal{G}(M^{-1} + G^\top M_O^{-1} G),$$

which can be concluded from the fact that $\mathcal{R}_\mathcal{G} G^\top = 0$ (due to Th. 1) and Prop. 6. Therefore, in view of (30), it holds that

$$\begin{aligned}&(I - MG^\top(GMG^\top)^{-1}G)h = \\ &[I - MG^\top(GMG^\top)^{-1}G](M^{-1} + G^\top M_O^{-1} G)^{-1}[-\dot{G}^\top v_O \\ &+ M^{-1}(u - g - Cv) + G^\top M_O^{-1}(C_O v_O + g_O)] = \\ &M^{\frac{1}{2}}(\mathcal{R}_\mathcal{G} M^{-\frac{1}{2}})^\dagger \mathcal{R}_\mathcal{G}[M^{-1}(u - g - Cv) + G^\top M_O^{-1}(C_O v_O - \dot{G}^\top v_O)]\end{aligned}$$

which, in view of the facts that $\mathcal{R}_\mathcal{G} G^\top = 0$, and hence $-\mathcal{R}_\mathcal{G}\dot{G}^\top = \dot{\mathcal{R}}_\mathcal{G} G^\top$, as well as $G^\top v_O = v$, becomes

$$M^{\frac{1}{2}}(\mathcal{R}_\mathcal{G} M^{-\frac{1}{2}})^\dagger[\dot{\mathcal{R}}_\mathcal{G} v + \mathcal{R}_\mathcal{G} M^{-1}(u - g - Cv)] = h_{\text{int}}.$$

■

Proof of Theorem 3: According to Th. 2, the derivation of h_d that yields zero internal forces can be formulated as a quadratic minimization problem:

$$\text{QP}: \min_{h_\text{d}} \quad \|h_{\text{int}}\|^2 = h_\text{d}^\top H h_\text{d}, \text{ s.t. } G h_\text{d} = h_{O,\text{d}}, \tag{38}$$

where $H := (I_{6N} - MG^\top(GMG^\top)^{-1}G)^\top(I_{6N} - MG^\top(GMG^\top)^{-1}G)$. Firstly, note that the choice $G^* = MG^\top(GMG^\top)^{-1} h_{O,\text{d}}$ is a minimizer of QP, since $GG^* = I_6$, and $HG^* h_{O,\text{d}} = 0_{6N}$, and therefore sufficiency is proved.

In order to prove necessity, we prove next that G^* is a strict minimizer, i.e., there is no other right inverse of G that is a solution to of QP. Note first that $G \in \mathbb{R}^{6 \times 6N}$ has full row rank, which implies that the dimension of its nullspace is $6N - 6$. Let $Z := [z_1, \ldots, z_{6N-6}] \in \mathbb{R}^{6 \times (6N-6)}$ be the matrix formed by the vectors $z_1, \ldots, z_{6N-6} \in \mathbb{R}^{6N}$ that span the nullspace of G. It follows that $\text{rank}(Z) = 6N - 6$ and $GZ = 0_{6\times 6}$. Let now the matrix $H' := Z^\top H Z \in \mathbb{R}^{(6N-6)\times(6N-6)}$. Since $GZ = 0_{6\times(6N-6)} \Rightarrow Z^\top G^\top = 0_{(6N-6)\times 6}$, it follows that $H' = Z^\top Z$. Hence, $\text{rank}(H') = \text{rank}(Z) = 6N - 6$, which implies that H' is positive definite. Therefore, according to [38, Th. 1.1], QP has a strong minimizer. ■

Proof of Corollary 6:
1) By substituting (35) in (8) and using $GG^* = I$ and $Gu_\mathcal{R} = 0_6$, we obtain, in view of (8) and the positive definiteness of $\widetilde{M}$ that $\widetilde{M}(\bar{x})(\dot{e}_v + K_d e_v + K_p e_x)$ implying

$$\dot{e}_v = -K_d e_v - K_p e_x. \tag{39}$$

Consider now the function $V := \frac{1}{2} e_p^\top K_{p1} e_p + \frac{k_{p2}}{2 - e_O} + \frac{1}{2} e_v^\top e_v$, for which it holds $V(0) < \infty$, since $e_O(0) < 2$. By differentiating V, and using (34) and (39), one obtains $\dot{V} = -e_v^\top K_d e_v \leq 0$. Hence, it holds that $V(t) \leq V(0) < \infty$, which implies that $\frac{k_{p2}}{2 - e_O(t)}$ is bounded and consequently $e_O(t) < 2$.
2) Since $V(t) \leq V(0) < \infty$, the errors e_p, e_v are bounded, which, given the boundedness of the desired trajectories p_d, R_d and their derivatives, implies the boundedness of the control law u. Hence, it can be proved that $\ddot{V}$ is bounded which implies the uniform continuity of $\dot{V}$. Therefore, according to Barbalat's lemma ([39], Lemma 8.2), we deduce that $\lim_{t\to\infty} \dot{V}(t) = 0 \Rightarrow \lim_{t\to\infty} e_v(t) = 0_6$. Since $e_x(t)$ is also bounded, it can be proved by using the same arguments that $\lim_{t\to\infty} \dot{e}_v(t) = 0_6$ and hence (39) implies that $\lim_{t\to\infty} e_x(t) = 0_6$.
3) Let the desired object force be

$$h_{O,\text{d}} = C_O v_O + g_O + M_O \alpha_\text{d}, \tag{40}$$

where $\alpha_\text{d} := \dot{v}_\text{d} - K_d e_v - K_p e_x$. In view of Th. 3, it suffices to prove $h = h_\text{d} = G^* h_{O,\text{d}}$. By substituting (35) in (30) and canceling terms, we obtain

$$\begin{aligned}h =&(M^{-1} + G^\top M_O^{-1} G)^{-1}[M^{-1} G^* h_{O,\text{d}} + G^\top \alpha_\text{d} + \\ &G^\top M_O^{-1}(C_O v_O + g_O)].\end{aligned}$$

Next, we add and subtract the term $G^\top M_O^{-1} G G^* h_{O,\text{d}}$ as

$$\begin{aligned}h =&(M^{-1} + G^\top M_O^{-1} G)^{-1}(M^{-1} + G^\top M_O^{-1} G) G^* h_{O,\text{d}} + \\ &(M^{-1} + G^\top M_O^{-1} G)^{-1}[G^\top M_O^{-1}(M_O \alpha_\text{d} + C_O v_O + \\ &g_O - G^\top M_O h_{O,\text{d}})],\end{aligned}$$

which, in view of (40), becomes $h = G^* h_{O,\text{d}}$. Completion of the proof follows by invoking Th. 3. ■

Proof of Corollary 7: Since $h_{\text{int,d}} \in \text{null}(G) = \text{range}(\mathcal{R}_\mathcal{G}^\top)$, it holds that $M^{-\frac{1}{2}} h_{\text{int,d}} \in \text{range}(M^{-\frac{1}{2}} \mathcal{R}_\mathcal{G}^\top) = \text{range}(\mathcal{R}_\mathcal{G} M^{-\frac{1}{2}})^\dagger$. Therefore, it holds that

$$\begin{aligned}&(\mathcal{R}_\mathcal{G} M^{-\frac{1}{2}})^\dagger \mathcal{R}_\mathcal{G} M^{-1} h_{\text{int,d}} = \\ &(\mathcal{R}_\mathcal{G} M^{-\frac{1}{2}})^\dagger \mathcal{R}_\mathcal{G} M^{-\frac{1}{2}}(M^{-\frac{1}{2}} h_{\text{int,d}}) = M^{-\frac{1}{2}} h_{\text{int,d}}.\end{aligned} \tag{41}$$

Hence, (31) yields the resulting internal forces

$$\begin{aligned}h_{\text{int}} =& M^{\frac{1}{2}}(\mathcal{R}_\mathcal{G} M^{-\frac{1}{2}})^\dagger \mathcal{R}_\mathcal{G} M^{-1}(I - MG^\top(GMG^\top)^{-1}) h_{\text{int,d}} \\ =& M^{\frac{1}{2}}(\mathcal{R}_\mathcal{G} M^{-\frac{1}{2}})^\dagger \mathcal{R}_\mathcal{G} M^{-1} h_{\text{int,d}} = M^{\frac{1}{2}} M^{-\frac{1}{2}} h_{\text{int,d}} = h_{\text{int,d}},\end{aligned}$$

where we have used (41) and the fact that $\mathcal{R}_\mathcal{G} G^\top = 0$. ■

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