Entropy and Mutual Information in a Bernoulli Process

Computational Neuroscience Notes

1 Entropy of a Bernoulli Random Variable

The entropy $H(F)$ of a Bernoulli random variable F with probability $P(F = 1) = p$ is given by:

$$H(F) = -p \log_2 p - (1 - p) \log_2(1 - p) \tag{1}$$

Given:

$$p = 0.1, \quad P(F = 0) = 1 - 0.1 = 0.9$$

Substituting values:

$$H(F) = -\left(0.1 \log_2 0.1 + 0.9 \log_2 0.9\right) \tag{2}$$

Using logarithm values:

$$\log_2 0.1 \approx -3.3219, \quad \log_2 0.9 \approx -0.152$$

$$\begin{aligned} H(F) &= -\left(0.1 \times (-3.3219) + 0.9 \times (-0.152)\right) \\ &= -\left(-0.33219 - 0.1368\right) \\ &= 0.33219 + 0.1368 \\ &= 0.469 \end{aligned}$$

Thus, the entropy $H(F)$ is approximately:

$$H(F) \approx 0.469 \text{ bits}$$

2 Mutual Information Between Stimulus S and Neuron Firing F

The mutual information between the stimulus S and the neuron firing F is given by:

$$MI(S, F) = H(F) - H(F|S) \tag{3}$$

2.1 Step 1: Compute $P(F = 1)$

Using the law of total probability:

$$P(F = 1) = P(F = 1|S = 1)P(S = 1) + P(F = 1|S = 0)P(S = 0) \tag{4}$$

Given:

$$P(S = 1) = 0.1, \quad P(S = 0) = 0.9$$

$$P(F = 1|S = 1) = \frac{1}{2} = 0.5, \quad P(F = 1|S = 0) = \frac{1}{18} \approx 0.0556$$

$$\begin{aligned} P(F = 1) &= (0.5 \times 0.1) + (0.0556 \times 0.9) \\ &= 0.05 + 0.05 \\ &= 0.1 \end{aligned}$$

Thus, from our previous result:

$$H(F) \approx 0.469 \text{ bits}$$

2.2 Step 2: Compute $H(F|S)$

The conditional entropy is given by:

$$H(F|S) = P(S = 1)H(F|S = 1) + P(S = 0)H(F|S = 0) \tag{5}$$

Each term is computed as:

$$H(F|S = i) = -P(F = 1|S = i)\log_2 P(F = 1|S = i) - P(F = 0|S = i)\log_2 P(F = 0|S = i) \tag{6}$$

2.2.1 Compute $H(F|S = 1)$

$$H(F|S = 1) = -\left(0.5\log_2 0.5 + 0.5\log_2 0.5\right) \tag{7}$$

Since $\log_2 0.5 = -1$,

$$\begin{aligned} H(F|S = 1) &= -\left(0.5 \times (-1) + 0.5 \times (-1)\right) \\ &= 1.0 \end{aligned}$$

2.2.2 Compute $H(F|S=0)$

$$H(F|S=0) = -\left(0.0556\log_2 0.0556 + 0.9444\log_2 0.9444\right) \tag{8}$$

Approximating:

$$\log_2 0.0556 \approx -4.17, \quad \log_2 0.9444 \approx -0.0815$$

$$\begin{aligned}
H(F|S=0) &= -\left(0.0556 \times (-4.17) + 0.9444 \times (-0.0815)\right) \\
&= -\left(-0.232 - 0.0769\right) \\
&= 0.309
\end{aligned}$$

2.2.3 Compute $H(F|S)$

$$\begin{aligned}
H(F|S) &= (0.1 \times 1.0) + (0.9 \times 0.309) \\
&= 0.1 + 0.278 \\
&= 0.378
\end{aligned}$$

2.3 Step 3: Compute $MI(S,F)$

$$\begin{aligned}
MI(S,F) &= H(F) - H(F|S) \\
&= 0.469 - 0.378 \\
&\approx 0.091
\end{aligned}$$

Thus, the closest value for the mutual information $MI(S,F)$ is:

0.091 bits