Lagrangian mixing in Plane Couette flow

a blog

`elton/blog/blog.tex`, rev. 65: last edit by Predrag Cvitanović, 05/14/2008

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Contents

Chapter 1

Research blog on Lagrangian mixing

[1] [2]

1.1 JE May 14, 2008: Eigenvalues and velocity gradient tensor

For a perturbation $\delta\mathbf{x}$ the change in the velocity field is given by $\delta\mathbf{u} = V\delta\mathbf{x}$ where V is the nine component velocity gradient tensor defined by $V_{ij} = \frac{\partial u_i}{\partial x_j}$. Since $\mathbf{u}$ is given by (1.15), it is a relatively simple extension of this formula to evaluate these partials. To find $\partial\mathbf{u}/\partial y$, one needs to use the relation $\frac{\partial}{\partial y}T_n(y) = nU_{n-1}(y)$ where T_n is the nth Chebyshev polynomial of the first kind and U_n is the nth Chebyshev polynomial of the second kind. Everything else is straightforward.

The eigenvalues of V, evaluated at a stagnation point , give local stability and reveal the qualitative nature of the motions nearby the stagnation point. For the four stagnation points we have so far the eigenvalues, eigenvectors, and velocity gradient tensors are as follows.

[1] `elton/blog/mixing.tex`, rev. 64: last edit by Predrag Cvitanović, 05/14/2008
[2] PC: why svnfileauthor shows up in the footer, but not svnfilerev?

(Lx/2,0,Lz/4): There are 3 real eigenvalues, two positive and one negative.

$$\lambda_1 = -0.50835, \mathbf{v}_1 = \begin{pmatrix} 0.98792 \\ 0.15305 \\ 0.02446 \end{pmatrix} \tag{1.1}$$

$$\lambda_2 = 0.49197, \mathbf{v}_2 = \begin{pmatrix} 0.69027 \\ -0.67972 \\ 0.24802 \end{pmatrix} \tag{1.2}$$

$$\lambda_3 = 0.07756, \mathbf{v}_3 = \begin{pmatrix} 0.29654 \\ 0.66152 \\ 0.68880 \end{pmatrix} \tag{1.3}$$

The velocity gradient tensor is

$$V = \begin{pmatrix} -0.43054 & -0.63462 & 0.82824 \\ -0.12213 & 0.30680 & -0.16758 \\ 0.00017 & -0.11187 & 0.18493 \end{pmatrix} \tag{1.4}$$

The point is a saddle; Close by it has 1 stable dimension and a 2D plane of instability spanned by $\mathbf{v}_2$ and $\mathbf{v}_3$. The stagnation point at (0,0,3Lz/4) has the same eigenvalues as this point. It's eigenvectors and velocity gradient tensor differ by a minus sign on the third column and third row (except for T_{33}) where the 2 minuses cancel.

(Lx/2,0,3Lz/4): There is one real, negative eigenvalue and a complex pair with positive real part.

$$\lambda_1 = -0.08427, \mathbf{v}_1 = \begin{pmatrix} 0.96801 \\ 0.16017 \\ 0.19316 \end{pmatrix} \tag{1.5}$$

$$\lambda_2 = 0.03866 + 0.08644i, \mathbf{v}_2 = \begin{pmatrix} 0.81734 \\ -0.11589 - 0.22763i \\ 0.40624 - 0.31886i \end{pmatrix} \tag{1.6}$$

$$\lambda_3 = 0.03866 - 0.08644i, \mathbf{v}_3 = \begin{pmatrix} 0.81734 \\ -0.11589 + 0.22763i \\ 0.40624 + 0.31886i \end{pmatrix} \tag{1.7}$$

The velocity gradient tensor is

$$V = \begin{pmatrix} -0.03169 & -0.36345 & 0.03788 \\ -0.02505 & -0.02882 & 0.07959 \\ 0.00147 & -0.17516 & 0.05358 \end{pmatrix} \tag{1.8}$$

This stagnation point spirals out in a plane given by the complex pair of eigenvectors. It is stable in one dimension that is dominantly along the x direction. As with the first two points, the point at (0,0,Lz/4) has the same eigenvalues and the velocity gradient tensor is the same except with sign changes in the third component. This follows from the symmetry. We now want to understand the connections between the manifolds

1.2 JE May 12, 2008: Notes and rough sketch of topics

After using the sum formula discussed in sect. 1.6 to compute **u** at every point along a trajectory, I have switched to an interpolation method because of run-time issues. Using the previous method, I create an arbitrarily fine set of grid-point values for **u** and then use a bilinear(trilinear?) interpolation method. (**Note:** To the eye it looks like this works pretty well but I need to look at the numerical values more closely and check the accuracy of the interpolation. How fine should I make the grid? Always need to be careful about approximations in the chaotic regime).

The starting point is clear because we already have four stagnation points predicted from translational symmetries of plane Couette flow. Starting a small sphere of initial conditions around the stagnation points and evolving them forward and backward in time gives a good estimate of the stable and unstable manifolds. Results are shown in Fig. 1.1. From these we begin to get a feel for the dynamics. Also, I can create movies to show the evolution of a ball of little ink droplets moving through the fluid, but I need to get these from .mat format to mp3's before I can post them anywhere. Being able to visualize the stretching and folding of these material lines and surfaces will be a key point.

This rough visualization of the manifolds is nice, but much better can be done. Since we have a sum formula for computing velocities at any point, by differentiating under the sum it should be a simple matter to compute the [3×3] velocity gradients matrix (**??**) eigenvalues/eigenvectors of this matrix will give linear stability and allow for exact computation of the stable and unstable manifolds. There are several expectations/predictions of what we'll find: (1) We will have one real eigenvalue and one complex pair. Judging from Fig. 1.1 I'm not sure about this one yet. It certainly looks true for the black/blue stagnation point, but for the other one it seems unclear what is going on. The stretching is strong around this point so it may be that the plane is quickly dominated in one direction and appears to collapse to a line, or it may be that all the eigenvalues are real. (**JE May 14:** It turns out one point has all real eigenvalues and the other has one real and a complex pair.) (2) The eigenvalues for the two points should be the same but with signs reversed resulting from translational symmetry, and this amounts to time reversal invariance. I'm a little confused on this one, I would appreciate a comment from anyone with an explanation.(**JE May 14:** Of the 4 total stagnation points, there are two pairs that have the same eigenvalues and their eigenvectors differ in sign in one component. I'm not sure if that might have been what was meant here). (3) There is a heteroclinic connection between the stagnation points. Will find out soon, this would be great for using chaos. Apparently the time reversal would force this connection.

After exhausting these four points we will want to find other stagnation points, either by using other symmetries or numerically. The numerical task would involve computing u^2 all along the grid and spotting regions where it is below a given threshold. Then using an interpolation in the small regions the stagnation points can be pinned down. The same eval/evec stable/unstable manifold analysis can then be done for any other stagnation point.

The long term goal after all of this is of course to compute mixing and diffusion properties; Lyapunov exponents, material stretching, striation thickness, time to mix etc... Most investigations like this tend to be in two dimensional closed systems, but if we find we have good Lagrangian chaos, there is no reason not to do it here.

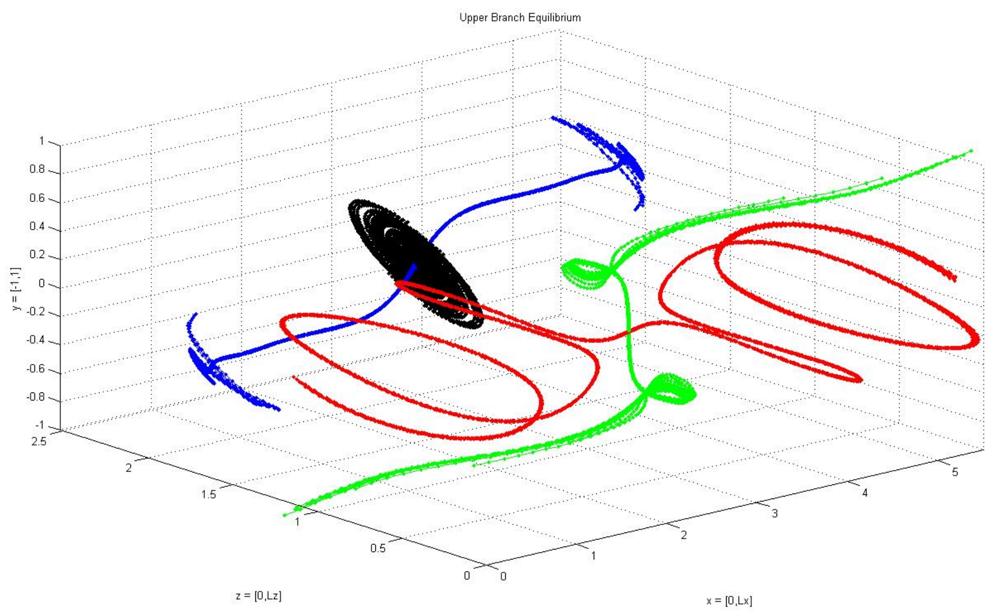


Figure 1.1: Stable and unstable manifolds of two stagnation points at $(L_x/2, 0, L_z/4)$ and $(L_x/2, 0, 3L_z/4)$. Black and green are unstable.

1.3 Notes on Mixing

General discussion and notes about mixing in fluids, largely taken from the book by J.M. Ottino (see reading assignments).

Coming soon.

1.4 JE April 29, 2008: Mixing and fixed points in the Upper Branch

To begin looking at the evolution of Lagrangian tracers in the Upper Branch, I have first integrated a grid of initial points. The grid is chosen to lie in the y-z plane, centered at $x = L_x/2$. The initial points are equally spaced, and offset by one position from the edge of the box. If the number of points is chosen to be one less than a multiple of 4, there will be points starting at $(L_x/2, 0, L_z/4)$ and $(L_x/2, 0, 3L_z/4)$. Similarly, if we make the grid be centered at x = 0, we will have points starting at $(0, 0, L_z/4)$ and $(0, 0, 3L_z/4)$. The trajectories are run for 15 seconds, and the results of this are shown in Fig. 1.2(a) and Fig. 1.3(a). (Note: The figures need some editing. It's almost impossible to read the axis labels. Also, they won't go where I want them to!)

Invariance under the symmetry group **S**, explained by JH in sect. 1.5, implies the existence of 4 stagnation points:

$$\begin{aligned}
(x, y, z) &= (0, 0, L_z/4) \\
(x, y, z) &= (0, 0, 3L_z/4) \qquad (1.9)\\
(x, y, z) &= (L_x/2, 0, L_z/4) \\
(x, y, z) &= (L_x/2, 0, 3L_z/4)
\end{aligned}$$
$$(1.10)$$

In Fig. 1.2(b) and Fig. 1.3(b) the figures from part (a) have been rotated to almost a y-z projection in order to reveal these stagnation points. The behavior of trajectories near these fixed points seems to reveal "what kind" of fixed points they are. The point at $3L_z/4$ in Fig. 1.2(b) appears to be an unstable spiral, whereas the point at $L_z/4$ is probably hyperbolic. There is also some other interesting behavior going on near the point at $L_z/4$. The next step is probably to look at eigenvalues and stable/unstable manifold of these stagnation points.

JFG April 30, 2008 The plots of tracers and the derivation of stagnation points from symmetries are very interesting.

I see from Fig. 1.3 and Fig. 1.2 that you're plotting tracers for the difference from laminar flow. (You can see $u \to 0$ as $y \to \pm 1$.) I don't know if this is intentional. If it's not, sorry, we haven't been sufficiently clear on the definitions of data we gave you. We usually work with $\mathbf{u}$ defined as the difference from laminar flow, so that the total velocity field $\mathbf{u}_{\rm tot} = \mathbf{u} + y\hat{\mathbf{x}}$. So you might want to add the laminar flow $y\hat{\mathbf{x}}$ on to $\mathbf{u}$ before computing tracers. That'll produce $u \to \pm 1$ as $y \to \pm 1$. The stagnation points are all at $y = 0$, so they will not change.

JE May 02, 2008 No, that was not intentional. I had forgotten that the velocity fields were computed *from* laminar. That will be an easy fix to get $\mathbf{u}_{\rm tot}$.

1.5 JH 2008-4-28: Symmetry and Lagrangian equilibria

All of the Eulerian equilibria that we know of so far are invariant under the symmetry group $\mathbf{S}$:

$$\begin{aligned} s_1\,[u,v,w](x,y,z) &= [u,v,-w](x+L_x/2,\ y,\ -z) \\ s_2\,[u,v,w](x,y,z) &= [-u,-v,w](-x+L_x/2,\ -y,\ z+L_z/2) \\ s_3\,[u,v,w](x,y,z) &= [-u,-v,-w](-x,\ -y,\ -z+L_z/2)\,. \end{aligned} \tag{1.11}$$

From the form of s_3, we can see that any Eulerian *equilibria* that is invariant under has 4 Lagrangian *equilibria* located at the points which satisfy the condition:

$$(x,y,z) = (-x,-y,-z+L_z/2) \tag{1.12}$$

There are 4 points which satisfy this constraint:

$$\begin{aligned} (x,y,z) &= (0,0,L_z/4) \\ (x,y,z) &= (0,0,3L_z/4) \\ (x,y,z) &= (L_x/2,0,L_z/4) \\ (x,y,z) &= (L_x/2,0,3L_z/4) \end{aligned} \tag{1.13}$$

$$\tag{1.14}$$

Also of note is the fact that there no s_3 symmetric relative equilibria, since non-uniform s_3-symmetric lose symmetry under small translations.

1.6 JE April 25, 2008: Numerical Procedures

In order to integrate streamlines of plane Couette flow and follow the paths of tracer particles, it is first necessary to have a correctly computed equilibrium velocity field.

The starting point for this task is to obtain the previously computed FlowField data for a given equilibrium, e.g. upper branch, lower branch, etc... These are made available at the website http://channelflow.org/ as is most of the information I am about to summarize about FlowFields. Essentially, the FlowField data contains a long array of numbers which are the spectral coefficients of the expansion of a velocity field $\mathbf{u}(\mathbf{x})$. The form of the expansion is

$$\mathbf{u}(\mathbf{x}) = \sum_{m_y=0}^{M_y-1}\sum_{m_x=0}^{M_x-1}\sum_{m_z=0}^{M_z-1} \hat{\mathbf{u}}_{m_x,m_y,m_z}\bar{T}_{m_y}(y)e^{2\pi i(k_x x/L_x+k_z z/L_z)} + (\mathrm{c.c.}) \tag{1.15}$$

The $\hat{\mathbf{u}}$'s are the spectral coefficients - the information stored in a FlowField. The $\bar{T}(y)$'s are Chebyshev polynomials defined on the interval [a,b] (in most cases [-1,1]). The order of the summations, although mathematically irrelevant, reflects the order in which the spectral coefficients are stored as a data array. z is the innermost loop, then x, then y, and finally the vector component of $\mathbf{u}(\mathbf{x})$ is the outermost loop. For a given FlowField the upper bounds on the sums are known from the geometry, and the k's are related to the m's through the following relations:

$$k_x = \begin{cases} m_x & 0 \le m_x \le M_x/2 \\ m_x - M_x & M_x < m_x < M_x \end{cases} \tag{1.16}$$

$$k_z = m_z \qquad 0 \le m_z < M_z \tag{1.17}$$

Hence, with a knowledge of the spectral coefficients we can evaluate this sum at a particular $\mathbf{x} = (x, y, z)$ and correctly compute $\mathbf{u}(\mathbf{x})$.

Various internal functions within Channelflow have been written to compute $\mathbf{u}$ on a set of gridpoints. It is possible, by interpolation of the velocity fields on these gridpoint values, to integrate a trajectory with great computational speed. However this will not be nearly as accurate as evaluating the sum (1.15), and currently we don't really know whether the first method would give a reasonable approximation at all. For this reason the current strategy is to evaluate (1.15) to give the exact velocity field at every point along a trajectory. Summing over 10^5 coefficients at every step sounds slow and inefficient, and it surely is compared to the interpolation method. But luckily it doesn't seem to be *too* slow. I have written a function in Matlab that performs this computation for a single point in about 0.01 seconds. It is certainly possible that this could be made faster. The code has been checked to be correct by picking an (x,y,z) coordinate that *happens* to lie on a gridpoint value and then comparing the result to the value given by the internal Channelflow functions. If, for example, we wanted to compute trajectories for 50 initial points for 500 time steps each this would still only take less than 5 minutes (ignoring the time needed to perform a Runge-Kutta step, or whatever).

1.6.1 Specifics

The new Channelflow function "field2ascii-spectral.cpp" converts the spectral coefficients to ascii format, which is readable by Matlab. The command
./field2ascii-spectral.x u u-whatev
takes in the FlowField u.ff and produces the files u-whatev.asc and u-whatev-geom.asc. In Matlab, the commands load('u-whatev.asc') and load('u-whatev-geom.asc') create vectors containing all of the necessary data. The newly written Matlab script "trajectory.m" takes this information and performs the sum (1.15). (Note that all of the hyphens in these file names should actually be underscores, I just don't know how to display underscores in LaTEX). So I am now basically ready to start playing with tracers.

1.7 Notational conventions

Predrag, May 12, 2008: In Lagrangian mixing we need to distinguish between $3D$ physical fluid flow (for a given invariant solution) and the dynamical ∞-dimensional state space flow.

We distinguish the two by using physically motivated nomenclature «««< .mine for $3D$ physical fluid flow: We shall refer to the $3D$ point $\mathbf{x}$ for which $\mathbf{u}(\mathbf{x}) = 0$ as a *stagnation point*, and the moving point $\mathbf{x}(\mathrm{t})$ for which $\mathbf{u}(\mathbf{x}(t)) = 0$, $\mathbf{x} - \mathbf{c}t = \mathbf{x}_{\mathrm{tw}}(0)$ as a *traveling stagnation point.* ======= for $3D$ physical fluid flow: We shall refer to the $3D$ point $\mathbf{x}$ for which $\mathbf{u}(\mathbf{x}_{\mathrm{eq}}) = 0$ as the *stagnation point* $\mathbf{x}_{\mathrm{eq}}$, and the moving point $\mathbf{x}(\mathrm{t})$ for which $\mathbf{u}(\mathbf{x}_{\mathrm{tw}}(t)) = 0$, $\mathbf{x}_{\mathrm{tw}}(t) - \mathbf{c}t = \mathbf{x}_{\mathrm{tw}}(0)$ as the *traveling stagnation point* $\mathbf{x}_{\mathrm{tw}}(\mathrm{t})$. »»»> .r65

(to be continued: velocity gradients matrix, etc..)

Predrag to Jonathan, Oct 13, 2007: Relative equilibria are not periodic, they are stationary in the velocity $\mathbf{c}$ co-moving frame. Rather than using "period of T" description (such as "x traveling with a period of $T = 169.62747092815$"), state that TW$\pm$1 has velocity $c_x = L_x/T$?

1.8 Integrating velocity fields

Predrag Nov 2, 2007 to Kai Schneider:
John Elton is planning to use some of the exact plane Couette flow solutions computed by Waleffe, Viswanath, Gibson and Halcrow (data sets are on channelfow.org) and study Lagrangian tracer trajectories for such solutions. Marie Farge (farge@lmd.ens.fr) tells me that many people do this inaccurately, but you know how to do it right. Let us know what we should read not to waste time on not doing it right?

Kai Schneider: (kschneid@cmi.univ-mrs.fr)
Concerning Lagrangian particles: it is important to use the right techniques for time integration and for interpolation of the velocity (and acceleration) for computing them accurately.

For time integration we are using a second order Runge-Kutta scheme and for space interpolation a bicubic (in 2d) scheme.

There is a nice recent paper by Homann, Dreher and Grauer [**?**] and an older one by P.K. Yeung and Pope [**?**] (you have the specialist on that just next 10 buildings away).

1.9 PC Nov 2, 2007 – passive scalar advection?

The other thing we might try is passive scalar transport using these velocity fields (but that I really have barely started thinking about).

Very sketchy:

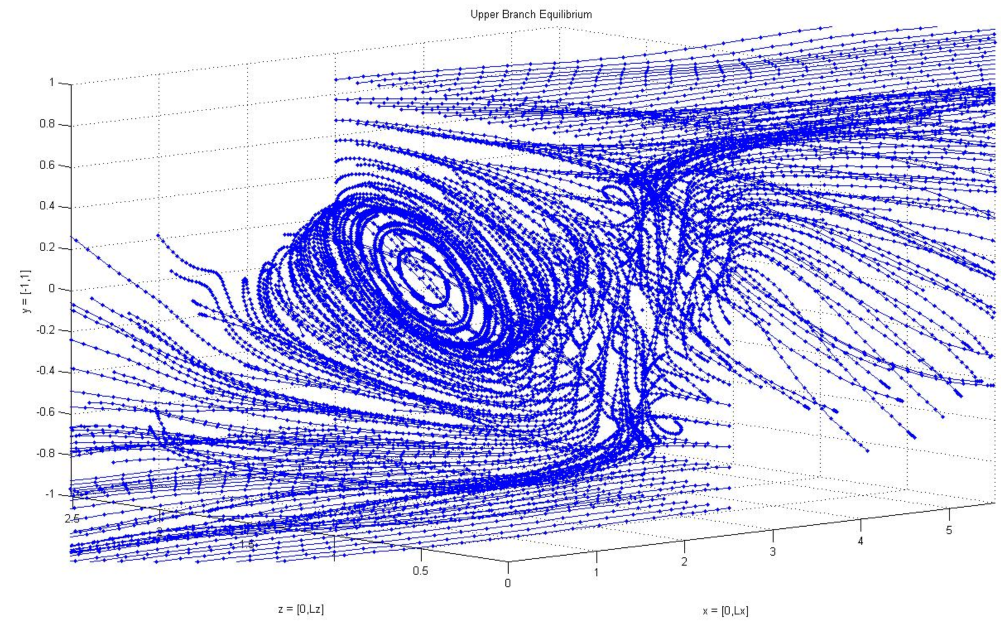
Given a velocity field, densities (passive scalars?) are advected by th *Fokker-Planck equation*

$$\partial_t \rho + \partial_i(\rho v_i) = D\,\partial^2 \rho. \tag{1.18}$$

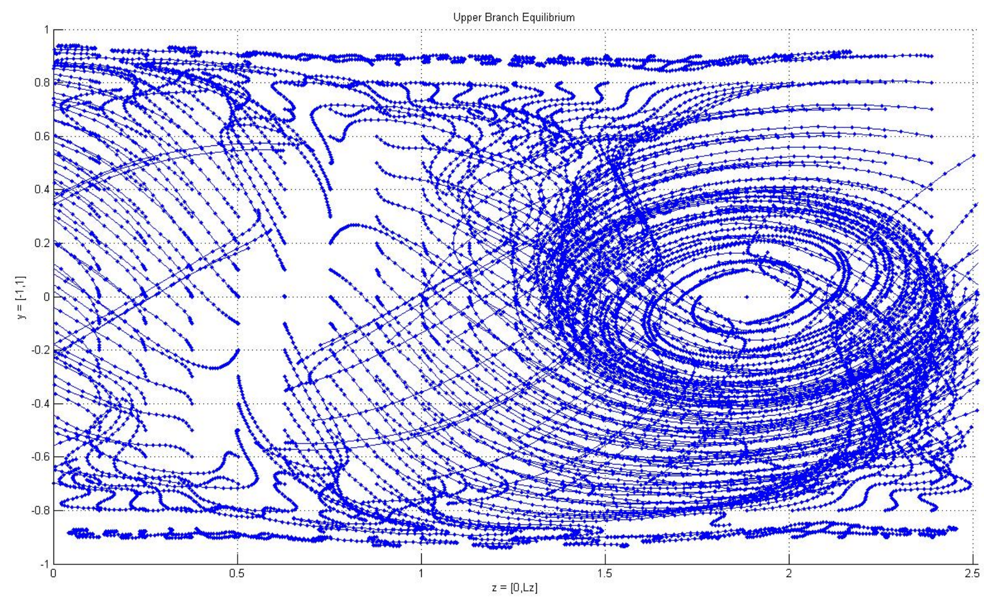
The left hand side, $d\rho/dt = \partial_t\rho + \partial \cdot (\rho v)$, is deterministic, with the continuity equation recovered in the weak noise limit $D \to 0$. The right hand side describes the diffusive transport in or out of the material particle volume. If the density is lower than in the immediate neighborhood, the local curvature is positive, $\partial^2\rho > 0$, and the density grows. Conversely, for negative curvature diffusion lowers the local density, thus smoothing the variability of ρ.

Not sure that this is the thing we want to investigate, and sure do not know how to think about the diffusive part $D\,\partial^2\rho$. Easier to try playing with tracer particles first...

JFG 2008-04-29: If we want to do this, it would not be hard to integrate the Fokker-Planck equation using channelflow, at least with an explicit time-stepping method. Express the probability density as a 1d FlowField, compute the $\partial_i(\rho v_i)$ and $D\,\partial^2\rho$ terms using differential operators, and add them together using Adams-Bashforth, Runge-Kutta, or similar formulae to get an update equation for the density. It would not take many lines of code.

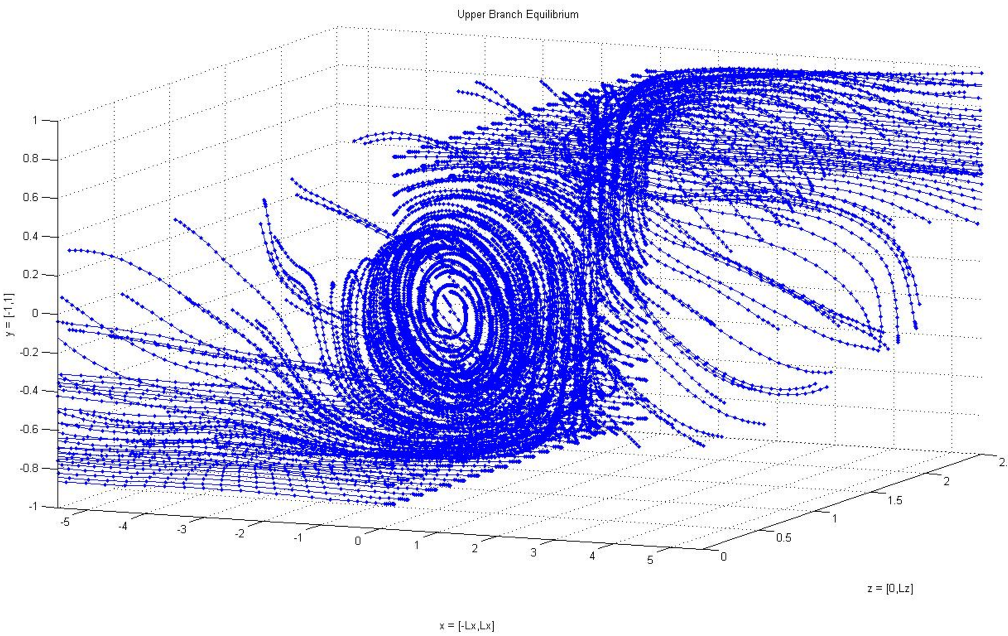


(a)

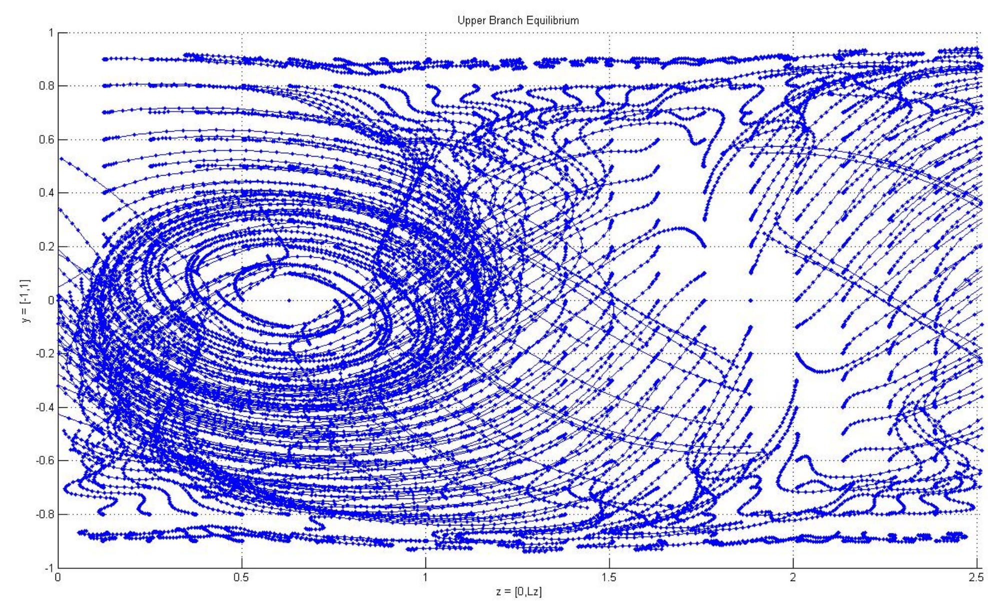


(b)

Figure 1.2: (a) Grid of 19 x 19 initial points in the yz plane, centered at $x = L_x/2$; Then integrated for 15 time units. (b) Rotated to show the 2 stagnation points



(a)



(b)

Figure 1.3: (a) Grid of 19 x 19 initial points in the yz plane, centered at $x = 0$; Then integrated for 15 time units. (b) Rotated to show the other 2 stagnation points

Chapter 2

Bluesky research

[1]

Throughout: $ on the margin indicates that the text has already been incorporated into the Elton *et al.* article gibson/mixing, or another article, or the Elton project. $

2.1 Reading assignments

2.1.1 Articles and books of potential interest

Ottino [?]

Chaté, H.; Villermaux, E.; and Chomaz, J.M. Mixing: Chaos and Turbulence. New York: Kluwer Academic/Plenum Publishers, 1999.

Solomon,T.H. *et al.*"Lagrangian chaos and multiphase processes in vortex flows". Comm. Nonlinear Sci. and Numer. Simul.. 8,(2003): 239-252.

Fogleman, M.A.; Fawcett, M.J.; and Solomon, "T.H. Lagrangian chaos and correlated Lévy flights in a non-Beltrami flow: Transient versus long-term transport". Phys. Rev. E. 63,(2001).

Du, Yunson and Ott, Edward. "Growth rates for fast kinematic dynamo instabilities of chaotic fluid flows". J. Fluid Mech.. (1993), 257: 265-288.

Castelian, Cathy; Mokrani, Asen; Le Guer, Yves; Peerhossaini, Hassan. "Experimental study of chaotic advection regime in a twisted duct flow". Euro. J. of Mechanics. B. 20, (2001): 205-232.

[1] `elton/blog/strategy.tex`, rev. 64: last edit by Predrag Cvitanović, 05/14/2008

PC 2007-11-2: Read (and explain to Predrag) Schneider recommendations - Homann, Dreher and Grauer [?] and P.K. Yeung and Pope [?] (the P.K. is just next 10 buildings away).
PC 2007-12-22: Read (and explain to Predrag) Mathur, Haller, Peacock, Ruppert-Felsot, and Swinney, "Uncovering the Lagrangian Skeleton of Turbulence" [?]. Predrag put a copy into ChaosBook.org/library.

PC 2007-12-22: Read (and explain to Predrag) Arneodo *et al.* [?], "Universal intermittent properties of particle trajectories in highly turbulent flows."

2.1.2 Keywords: Lagrangian mixing in turbulence

Do literature review for Lagrangian mixing: possible keywords to google:

- tracer particles in turbulent ...
- Lagrangian dynamics in turbulence
- inertial particles
- Lyapunov exponents of heavy particles in turbulent flows

Possible authors (still to check)

- Krzysztof Gawedzky (Lyon)
- B. Eckhardt (Marburg): Geometry of particle paths in chaotic and turbulent flows
- Jean-Francois Pinton (Lyon): Lagrangian experiments

2.1.3 Diverse literature

PC 2006-06-18: Not directly relevant to us now, but ref. [?] has exact analytic solutions of GOY model - perhaps of interest to make connections with the Kolmogorov obssessed.

2.2 Analyze this: John Elton land

Check out its symmetry, translate if that brings it back into the $\mathbf{u}_{\mathrm{EQ2}}$, $\mathbf{u}_{\mathrm{EQ4}}$ fold.

2.3 Fish

2.4 Find nontrivial stable coherent state

JFG 2008-04-29: We have one already - the upper branch right after bifurcation.

2.4.1 Divakar's suggestions

Visualize Lagrangian transport in Divakar's and Jonathan's relative periodic orbit co-moving frames for plane Couette flow.

2.5 Bluesky, completed

2.5.1 Nothing yet

Chapter 3

IAQs

Infrequently Asked Questions [IAQs].

3.1 Strategy

3.1.1 Advertise arXiv steady paper

1. crosslink the paper with nonlin dynamics
2. email individually arXiv paper link to colleagues who might comment on the paper
3. PC contacts ?? and makes them aware of our work.
4. snailmail the paper to ??, with a handwritten personal letter.

3.2 Spruce up personal websites

JRE homepage with publication list, movies

3.3 Papers

3.3.1 JFM mixing paper

1. JRE DONE jan 2008: incorporate Divakar's advice into the JFM paper
2. JRE april 2008: Submit JFM paper

3.4 Conferences

3.5 Outreach

Whistle news item
get announcement approved for GaTech SoP research page

3.6 JRE grad school

Chapter 4

Channelflow

[1]

4.1 Lagrangian streamlines

4.1.1 Streamline plot and movie production

4.1.2 OpenMP-parallelize channelflow

4.1.3 Benchmark channelflow against similar codes

4.2 Channelflow, COMPLETED

4.2.1 Get channelflow running on cluster (as is)

Aug 2007: DONE now runs on PACE cluster

[1] `elton/blog/channelflow.tex`, rev. 54: last edit by Predrag Cvitanović, 05/02/2008