

# Periodic orbit analysis of a system with continuous symmetry - a tutorial

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Dynamical systems with translational or rotational symmetry arise frequently in studies of spatially extended physical systems, such as Navier-Stokes flows on periodic domains. In these cases, it is natural to express the state of the fluid in terms of a Fourier series truncated to a finite number of modes. [Here, we study a 4-dimensional model with chaotic dynamics and  \$\text{SO}\(2\)\$  symmetry similar to those that appear in fluid dynamics problems.](#) A crucial step in the analysis of such a system is symmetry reduction. We use the model to illustrate different symmetry-reduction techniques. Its relative equilibria are conveniently determined by rewriting the dynamics in terms of a symmetry-invariant polynomial basis. However, for the analysis of its chaotic dynamics, the ‘method of slices’, which is applicable to very high-dimensional problems, is preferable. We show that a Poincaré section taken on the ‘slice’ can be used to further reduce this flow to what is for all practical purposes a unimodal map. This enables us to systematically determine all relative periodic orbits and their symbolic dynamics up to any desired period. We then present cycle averaging formulas adequate for systems with continuous symmetry and use them to compute dynamical averages using relative periodic orbits. The convergence of such computations is discussed.

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Periodic orbit theory provides a way to compute dynamical averages for chaotic flows by means of cycle averaging formulas that relate the time averages of observables to the spectra of unstable periodic orbits. Standard cycle averaging formulas are valid under the assumption that the stability multipliers of all periodic orbits have a single marginal direction corresponding to time evolution and are hyperbolic in all other directions. However, if a dynamical system has  $N$  continuous symmetries, periodic orbits are replaced by relative periodic orbits, invariant  $(N+1)$ -dimensional tori with marginal stability in  $(N+1)$  directions. Such exact invariant solutions arise in studies of turbulent flows, such as pipe flow or plane Couette flow, which have continuous symmetries. In practice, the translational invariance of these flows is approximated in numerical simulations by using periodic domains so that the state of the fluid is conveniently expressed as a Fourier series, truncated to a large but finite number (from tens to thousands) of Fourier modes. This paper is a tutorial on how such problems can be analyzed using periodic orbit theory. We illustrate all the necessary steps using a simple ‘two-mode’ model as an example.

## I. INTRODUCTION

Recent experimental observations of traveling waves in pipe flows have confirmed the intuition from dynamical systems theory that invariant solutions of Navier-Stokes equations play an important role in shaping the state space of turbulent flows.<sup>1</sup> When one casts fluid flow equations in a particular basis, the outcome is an infinite dimensional dynamical system that is often equivariant under transformations such as translations, reflections and rotations. For example, when periodic boundary conditions are imposed along the streamwise direction, the equations for pipe flow retain their form under the action of streamwise translations, azimuthal rotations and reflections about the central axis, i.e., they are equivariant under the actions of  $\text{SO}(2) \times \text{O}(2)$ . In this case it is natural to express the state of the fluid in a Fourier basis. However, as the system evolves, the nonlinear terms in the equations mix the various modes, so that the state of the system evolves not only along the symmetry directions, but also along directions transverse to them. This complicates the dynamics and gives rise to high dimensional coherent solutions such as relative equilibria and relative periodic orbits, which take on the roles played by equilibria and periodic orbits in flows without symmetry.

There is an extensive literature on equivariant dynamics, which can be traced back to Poincaré’s work on the 3-body problem.<sup>2</sup> Early references in the modern dynamical systems literature that we know of are works of Smale,<sup>3</sup> Field,<sup>4</sup> and Ruelle.<sup>5</sup> Our goal here is not to provide a comprehensive review of this literature, or study its techniques in generality. For those, we refer the reader

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to monographs by Golubitsky and Stewart,<sup>6</sup> and Field.<sup>7</sup> Our aim here is much more modest: We would like to provide a hands-on introduction to some of the concepts from equivariant dynamical systems theory, with an emphasis on those aspects relevant to the application of the periodic orbit theory to these systems. To this end, we undertake a step-by-step tutorial approach and illustrate each concept on a two-mode SO(2) equivariant normal form that has the minimal dimensionality required for chaotic dynamics. We provide visualizations of geometrical concepts, whenever possible. While the example studied here has no physical significance, such an analysis should ultimately be applicable to numerical solutions of turbulent flows on periodic domains, once sufficiently many exact invariant solutions become numerically accessible.

The rest of the paper is organized as follows: In Section II, we define basic concepts and briefly review the relevant symmetry reduction literature. In Section III, we introduce the two-mode model system, discuss several of its symmetry-reduced representations, and utilize a symmetry-reduced polynomial representation to find the only relative equilibrium of the system.<sup>1</sup> In Section IV, we show how the method of slices can be used to quotient the symmetry and reduce the dynamics onto a symmetry-reduced state space or ‘slice’. A Poincaré section taken on the slice then reduces the 4-dimensional chaotic dynamics in the full state space to an approximately one-dimensional, unimodal Poincaré return map. The return map is then used to construct a finite grammar symbolic dynamics for the flow and determine *all* relative periodic orbits up to a given period. In Section V, we present cycle averaging formulas adequate for systems with continuous symmetries and use the relative periodic orbits calculated in Section IV to calculate dynamically interesting observables. Finally, in Section VI, we discuss possible applications of the method of slices to various spatially extended systems.

The main text is supplemented by two appendices. Appendix A describes the multi-shooting method used to calculate the relative periodic orbits. Appendix B discusses how periodic Schur decomposition can be used to determine their Floquet multipliers, which can differ by 100s of orders of magnitude even in a model as simple as the two-mode system.

## II. CONTINUOUS SYMMETRIES

A dynamical system  $\dot{a} = v(a)$  is said to be *equivariant* under the group  $G$  of symmetry transformations if

$$v(a) = D(g)^{-1}v(D(g)a) \quad (1)$$

<sup>2</sup> DB 201-11-15: I would throw this out, at least as a highlight in the “review” of the paper. Our take-home message is that you could use our approach on turbulent flows, so the fact that we can cheat by using invariant polynomials shouldn’t be celebrated

for every point  $a$  in the state space  $\mathcal{M}$  and every element  $g \in G$ , where  $g$  is an abstract group element and  $D(g)$  is its  $[d \times d]$  matrix representation. Infinitesimally, the equivariance condition (1) can be expressed as a vanishing Lie derivative<sup>8</sup>

$$T v(a) - A(a) t(a) = 0, \quad (2)$$

where  $A(a)$  is the  $[d \times d]$  stability matrix with elements  $A_{ij}(a) = \partial v_i / \partial a_j$ ,  $t(a) = Ta$  is the group tangent at  $a$ , and  $T$  is the  $[d \times d]$  generator of infinitesimal transformations, such that  $D(\theta) = \exp(\theta T)$ , where the phase  $\theta \in [0, 2\pi)$  parametrizes the group action. (We shall interchangeably use notations  $D(g)$  and  $D(\theta)$ .) In general, there is a generator associated with each continuous symmetry. For the simple model considered here, which has a single SO(2) symmetry, there is only one parameter  $\theta$ , so we only have one generator  $T$ .

If the trajectory of a point  $a_q$  coincides with its group orbit, i.e., for every  $\tau$  there is a group transformation such that

$$a(\tau) = a_q + \int_0^\tau d\tau' v(a(\tau')) = D(\theta(\tau)) a_q, \quad (3)$$

$a_q$  is a point on *relative equilibrium*  $q$ . In our case, this is a 1-torus in state space. Expanding both sides of (3) for infinitesimal time verifies that the group tangent and the velocity vector are parallel, i.e.,  $v(a_q) = \dot{\theta}(0) t(a_q)$ . By symmetry, this must hold for all  $a(\tau) \in q$ , so for relative equilibria the *phase velocity* is constant,  $\dot{\theta}(\tau) = c$ . Multiplying the equivariance condition (2) by  $c$ , we find that velocity is a marginal stability eigenvector in the reference frame co-moving with the relative equilibrium,

$$(A(a) - cT)v(a) = 0, \quad a \in \mathcal{M}_q. \quad (4)$$

A state space point  $a_p$  lies on a *relative periodic orbit* of period  $T_p$  if its trajectory first intersects its group orbit after a finite time  $T_p$ ,

$$a(T_p) = a_p + \int_0^{T_p} d\tau' v(a(\tau')) = D(\theta_p) a_p, \quad (5)$$

with a phase  $\theta_p$ . In systems with SO(2) symmetry, relative periodic orbits are topologically 2-tori, where the trajectory of  $a_p$  generically traces out the torus ergodically by repeating the same path shifted by the group action  $D(\theta_p)$  after each prime period  $T_p$ . As we will see in Section IV, these tori can be very convoluted and difficult to visualize. In special cases where  $\theta_p = 0$ , the solution is a periodic orbit, a 1-dimensional loop in state space and the 2-torus is generated by all actions of the symmetry group on this loop.

The linear stability of relative periodic orbits is captured by their *Floquet multipliers*  $\Lambda_{p,j}$ , the eigenvalues of the Jacobian  $\hat{J}_p$  of the time-forward map  $a(\tau) = f^\tau(a(0))$ .  $\hat{J}_p$  is defined as

$$\hat{J}_p = D(-\theta_p) J^{T_p}(a_p), \quad \text{where } J_{ij}^{\tau}(a(0)) = \frac{\partial a_i(\tau)}{\partial a_j(0)}. \quad (6)$$

The magnitude of  $\Lambda_{p,j}$  determines whether a small perturbation along its corresponding eigendirection (or Floquet vector) will expand or contract after one period. If the magnitude of  $\Lambda_{p,j}$  is greater than 1, the perturbation expands; if it is less than 1, the perturbation contracts. In systems with  $N$  continuous symmetries, relative periodic orbits have  $(N+1)$  marginal directions ( $|\Lambda_{p,j}| = 1$ ), which correspond to the temporal evolution of the flow and the  $N$  symmetries. By applying symmetry reduction, the marginal Floquet multipliers corresponding to the symmetries are replaced by 0, so that periodic orbit theory, which requires that the flow have only one marginal direction, becomes applicable.

*Symmetry reduction* is a coordinate transformation that maps all the points on a group orbit  $D(\theta)a$ , which are equivalent from a dynamical perspective, to a single representative point in a symmetry reduced space. After symmetry reduction, relative equilibria and relative periodic orbits are converted to equilibria and periodic orbits in a reduced state space without loss of dynamical information; the full state space trajectory can always be retrieved via a reconstruction equation.

One well-studied technique for symmetry reduction, which works well for low-dimensional dynamical systems, such as the Lorenz system, is to recast the dynamical equations in terms of invariant polynomials<sup>9</sup>, i.e. *the orbit space*. However, there are multiple difficulties associated with using these techniques. Computing invariants is a non-trivial problem, and even for the simple case of  $\text{SO}(2)$ , computer algebra methods for finding invariants become impractical for systems with more than a dozen dimensions.<sup>10</sup> Moreover, the projection of a linear equivariant vector field onto orbit space is not necessarily a linear operation.<sup>11</sup> This means that even when conducting basic operations such as the linearization of nonlinear vector fields, special attention has to be paid to the choice of invariants, even when it is possible to find them. In contrast, the method of slices,<sup>12–18</sup> which we study in detail here, is a symmetry reduction scheme applicable to high-dimensional flows like the Navier-Stokes equations.<sup>19</sup>

### II.1. Method of slices

In a system with  $N$  continuous symmetries, a *slice*  $\hat{\mathcal{M}}$  is a codimension  $N$  submanifold of  $\mathcal{M}$  that cuts every group orbit once and only once. In the *method of slices*, the solution of a  $d$ -dimensional dynamical system is represented as a symmetry-reduced trajectory  $\hat{a}(\tau)$  within the  $(d-N)$ -dimensional slice and  $N$  time dependent group parameters  $\theta(\tau)$ , which map  $\hat{a}(\tau)$  to the full state space by the group action  $D(\theta(\tau))$  that defines a *moving frame*.

The idea goes back to Cartan,<sup>20</sup> and there is a rich literature on the method of slices (in variety of guises) and its applications to problems in dynamical systems theory: notable examples include the work of Field,<sup>21</sup>

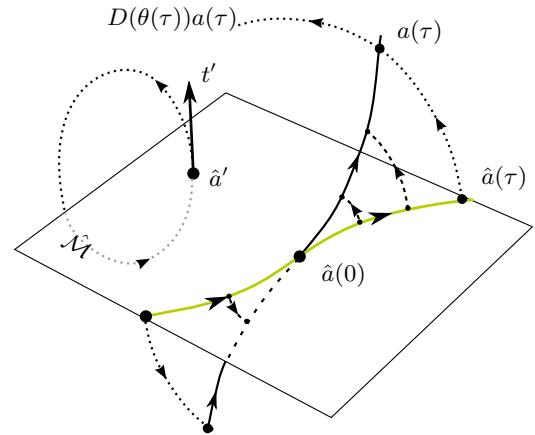


FIG. 1. (Color online) The slice hyperplane  $\hat{\mathcal{M}}$  is a hyperplane that contains the template point  $\hat{a}'$  and is normal to its group tangent  $t'$ . It intersects all group orbits (dotted lines) in an open neighborhood of  $\hat{a}'$ . The full state space trajectory  $a(\tau)$  (solid black line) and the reduced state space trajectory  $\hat{a}(\tau)$  (dashed green line) belong to the same group orbit  $\mathcal{M}_{a(\tau)}$  and are equivalent up to a group rotation  $D(\theta(\tau))$ .

Krupa,<sup>22</sup> and Ashwin and Melbourne,<sup>23</sup> who used slicing to prove rigorous results for equivariant systems. Fels and Olver<sup>24,25</sup> used the method of moving frames to compute invariant polynomials. Haller and Mezic<sup>26</sup> used the method of slices, under the name “orbit projection map”, to study three-dimensional volume-preserving flows. Our presentation closely follows that of Refs. 12 and 13 (the former derives the “reconstruction equation” for the template fitting method of Ref. 27).

A general definition of a slice puts no restriction on its shape and offers no guidance on how to construct it. In practice, computationally most convenient is a local, linear slice (a *slice hyperplane*) constructed in the neighborhood of a point  $\hat{a}'$  by using  $\hat{a}'$  as *template*. The slice hyperplane is then defined as the hyperplane that contains  $\hat{a}'$  and is perpendicular to its group tangent  $t' = T\hat{a}'$ . The relationship between a template, its slice hyperplane, and symmetry-reduced trajectories is illustrated in Fig. 1.

Reduced trajectories  $\hat{a}(\tau)$  can be obtained in two ways: by post-processing data or by reformulating the dynamics and integrating directly in the slice hyperplane. The post-processing method (also called the *method of moving frames*<sup>24,28</sup>) can be applied to both numerical and experimental data: one takes the data in the full state space and looks for the time dependent group parameter that brings the trajectory  $a(\tau)$  onto the slice. That is, one finds  $\theta(\tau)$  such that  $\hat{a}(\tau) = D(-\theta(\tau))a(\tau)$  satisfies the slice condition:

$$\langle \hat{a}(\tau) - \hat{a}' | t' \rangle = 0. \quad (7)$$

In the second implementation (valid only for abelian groups), one reformulates the dynamics as

$$\dot{v}(\hat{a}) = v(\hat{a}) - \dot{\theta}(\hat{a}) t(\hat{a}) \quad (8a)$$

$$\dot{\theta}(\hat{a}) = \langle v(\hat{a})|t' \rangle / \langle t(\hat{a})|t' \rangle, \quad (8b)$$

which can then be directly integrated to get the symmetry-reduced trajectory  $\hat{a}(\tau)$  and the reconstruction angle  $\theta(\tau)$ . In (8),  $\hat{v}$  is the projection of the full state space velocity  $v(a)$  onto the slice hyperplane. For a derivation see Ref. 8.

While early studies<sup>12,13,29</sup> applied the method of slices to a single solution at a time, studying the nonlinear dynamics of extended systems requires symmetry reduction of global objects, such as strange attractors and inertial manifolds. In this spirit, Ref. 14 used the method of slices to quotient the SO(2) symmetry from the chaotic dynamics of complex Lorenz flow. They showed that the singularity of the reconstruction equation that occurs when the denominator in (8b) vanishes (e.g., when the group tangents of the trajectory and the template are orthogonal) causes the reduced flow to make discontinuous jumps. The set of points  $\hat{a}^*$  where this occurs satisfy

$$\langle t(\hat{a}^*)|t' \rangle = 0 \quad (9)$$

and make up the *slice border* (studied in detail in Ref. 15).

Two strategies have been proposed in order to handle this problem: The first attempts to try to identify a template such that slice singularities are not visited by the dynamics.<sup>14</sup> The second uses multiple ‘charts’ of connected slice hyperplanes,<sup>12,15</sup> switching between charts when the dynamics approach the border of a particular chart. The latter approach was applied to complex Lorenz flow by Cvitanović *et al.*<sup>16</sup> and to pipe flow by Willis, Cvitanović, and Avila.<sup>17</sup> However, neither approach is straightforward to apply, particularly in high-dimensional systems.

## II.2. First Fourier mode slice

A third strategy has recently been proposed by Budanur *et al.*<sup>18</sup>, who considered Fourier space discretizations of partial differential equations (PDEs) with SO(2) symmetry. They showed that in these cases a simple choice of slice template, associated with the first Fourier mode, results in a slice in which it is highly unlikely that generic dynamics visit the neighborhood of the singularity. If the dynamics do occasionally come near the singularity, these close passages can be regularized by means of a time rescaling.

Here, we shall illustrate this approach, which we call the ‘first Fourier mode slice’, and apply it to a model system with two modes that will be described in Section III.

In the discussion so far, we have not specified any constraints on the symmetry group to be quotiented beyond the requirement that it be abelian as required for (8) to be valid.<sup>3</sup> Since we are interested in spatially extended

systems with translational symmetry, and in order to keep the notation compact, we restrict our discussion to one dimensional PDEs describing the evolution of a field  $u(x, t)$  in a periodic domain. By expressing the solutions in terms of a Fourier series

$$u(x, \tau) = \sum_{k=-\infty}^{\infty} u_k(\tau) e^{ikx}, \quad u_k = x_k + iy_k, \quad (10)$$

the translationally invariant PDE can be replaced by a system of coupled nonlinear ODEs for the Fourier coefficients equivariant under the 1-parameter compact group of SO(2) rotations.

Truncating the expansion to  $m$  modes, we write the real and imaginary parts of the Fourier coefficients with  $k \geq 1$  as the state vector  $a = (x_1, y_1, x_2, y_2, \dots, x_m, y_m)$ . The action of the SO(2) group on this vector can then be expressed as a block diagonal matrix:

$$D(\theta) = \begin{pmatrix} R(\theta) & 0 & \cdots & 0 \\ 0 & R(2\theta) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R(m\theta) \end{pmatrix}, \quad (11)$$

where

$$R(n\theta) = \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix} \quad (12)$$

is the rotation matrix for  $n$ th Fourier mode. The Lie algebra element for  $D(\theta)$  is given by

$$T = \begin{pmatrix} 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -2 & \cdots & 0 & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & -m \\ 0 & 0 & 0 & 0 & \cdots & m & 0 \end{pmatrix}. \quad (13)$$

In order to construct a slice hyperplane for such a system, we choose the following slice template:

$$\hat{a}' = (1, 0, \dots, 0). \quad (14)$$

The slice condition (7) then constrains points on the reduced trajectory to the hyperplane given by

$$\hat{a} = (\hat{x}_1, 0, \hat{x}_2, \hat{y}_2, \dots, \hat{x}_m, \hat{y}_m). \quad (15)$$

As discussed earlier, group orbits should cross the slice once and only once, which we achieve by restricting the slice hyperplane to the half-space where  $\hat{x}_1 > 0$ . In general, a slice hyperplane can be constructed by following a similar procedure for any choice of template. However,

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<sup>4</sup> ES 2015-05-20: Maybe you should discuss why you imposed the

restriction on Abelian groups.

the power of choosing template (14) becomes apparent by computing the border (9) of its slice hyperplane. The points on (15) lie on the slice border only if  $\hat{x}_1 = 0$ . This means that as long the dynamics are such that the magnitude of the first mode never vanishes, *every* group orbit is guaranteed to have a unique representative point on the slice hyperplane. By symmetry, any template of the form  $\hat{a}' = (\hat{x}'_1, \hat{y}'_1, 0, \dots, 0)$  would work just as well. The slice template (14) was chosen for notational and computational convenience.

More insight can be gained by writing the symmetry-reduced evolution equations (8) explicitly for template (14):

$$\hat{v}(\hat{a}) = v(\hat{a}) - \frac{\dot{y}_1(\hat{a})}{\hat{x}_1} t(\hat{a}), \quad (16a)$$

$$\dot{\theta}(\hat{a}) = \frac{\dot{y}_1(\hat{a})}{\hat{x}_1}. \quad (16b)$$

Since the argument  $\phi_1$  of a point  $(x_1, y_1)$  in the first Fourier mode plane is given by  $\phi_1 = \tan^{-1} \frac{y_1}{x_1}$ , its velocity is

$$\dot{\phi}_1 = \frac{x_1}{r_1^2} \dot{y}_1 - \frac{y_1}{r_1^2} \dot{x}_1, \quad (17)$$

where  $r_1^2 = x_1^2 + y_1^2$ . Therefore, on the slice hyperplane (15), where  $\dot{y}_1 = 0$ ,

$$\dot{\theta}(\hat{a}) = \dot{\phi}_1(\hat{a}). \quad (18)$$

That is, for our choice of template (14), the reconstruction phase coincides with the phase of the first Fourier mode. This makes this choice of template more natural from a group-theoretic point of view than the physically motivated templates used in Refs. 12–17.

<sup>5 7 9</sup> In general, additional care must be taken when the dynamics approach the slice border  $\hat{x}_1 = 0$ . Whenever this happens, the near-divergence of  $\hat{v}$  can be regularized by introducing a rescaled time coordinate<sup>18</sup> such that  $d\hat{\tau} = d\tau/\hat{x}_1$ . However, in our analysis of the two-mode system that we introduce below, we omit this step since points with a vanishing first mode are in an invariant subspace of the flow and, hence, are never visited by the dynamics.

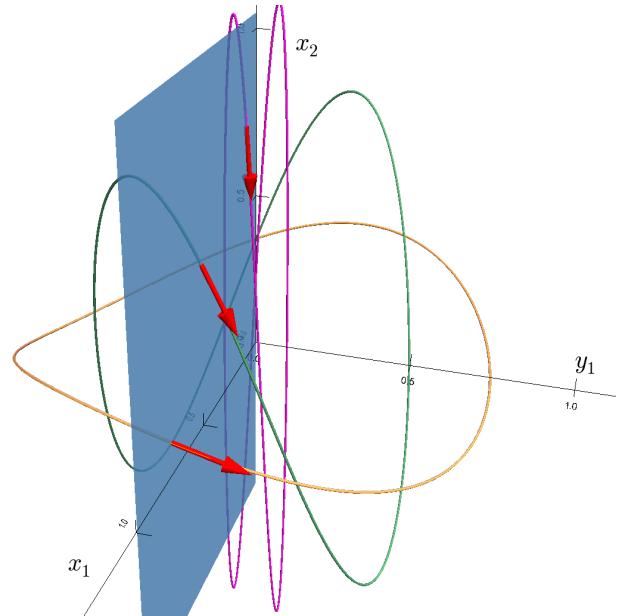


FIG. 2. (Color online) SO(2) group orbits of state space points  $(0.75, 0, 0.1, 0.1)$  (orange),  $(0.5, 0, 0.5, 0.5)$  (green)  $(0.1, 0, 0.75, 0.75)$  (pink) and the first mode (15) slice hyperplane (blue). The group tangents at the intersections with the slice hyperplane are shown as red arrows. As the magnitude of the first Fourier mode decreases relative to the magnitude of the second one, so does the group tangent angle to the slice hyperplane.

### II.3. Geometric interpretation of the first Fourier mode slice

Before moving on to our analysis of the two-mode model, we first discuss the geometrical interpretation of the first Fourier mode slice. The slice defined by (14), along with the directional constraint  $\hat{x}_1 > 0$ ,<sup>11</sup> fixes the phase of the first complex Fourier mode to 0. This can also be seen from (18), which shows that if the first Fourier mode slice (14) is used as a template, the reconstruction phase is the same as the phase of the first Fourier mode (18).

In complex representation, we can express the relationship between Fourier modes ( $z_n = x_n + iy_n$ ) and their representative points ( $\hat{z} = \hat{x}_n + i\hat{y}_n$ ) on the slice hyperplane by the U(1) action:

$$\hat{z}_n = e^{-in\phi_1} z_n. \quad (19)$$

This relation provides another interpretation for the slice border: For template (14), the slice border condition (9) defines the slice border as those points where  $|\hat{z}_1| = |z_1| = 0$ . At these points the phase of the first Fourier mode is

<sup>6</sup> ES 2014-05-20: What would you think about introducing a functional notation for  $y_1$  as in (16b)?

<sup>8</sup> PC: 2014-05-25: We should copy and paste from Ref. 18 all stuff about the in-slice time here?

<sup>10</sup> ES 2014-05-25: I agree with that suggestion, since rescaled time makes the method work for more general flows than 2-modes, and it should be explained here. However, we do not provide an example for its usefulness here. How about fishing for a second set of parameters where trajectories come consistently close to  $(0, 0, \dots)$ , and using rescaled time there? Even though  $(0, 0, \dots)$  would not be visited, there should still be apparent jumps that would go away by time-rescaling.

<sup>12</sup> DB 2014-05-15: Think this should be  $\hat{x}_1$ . Revert to  $x_1$  if I'm wrong.

not well-defined and hence neither is the transformation (19).

This is illustrated in Fig. 2, where the first Fourier mode slice hyperplane is shown along with the group orbits of points with decreasing  $|z_1|$ . When the magnitude of the first mode is small relative to that of the second (pink curve), the group tangent at the representative point for the group orbit (i.e., where the group orbit and the slice hyperplane intersect) has a larger component parallel to the slice hyperplane. If the magnitude of the first mode was exactly 0, the group tangent would lie entirely on the slice hyperplane, satisfying the slice border condition.

In Ref. 30, a polar coordinate representation of a two Fourier mode normal form is obtained by defining the  $G$ -invariant phase:  $\Phi = \phi_2 - 2\phi_1$  and three symmetry invariant coordinates  $\{r_1, r_2 \cos \Phi, r_2 \sin \Phi\}$ . One can see by direct comparison with (19), which yields  $\hat{z}_1 = r_1$  and  $\hat{z}_2 = r_2 e^{i\Phi}$ , that this representation is a special case ( $m = 2$ ), of the slice defined by (14). Corresponding ODEs for the polar representation were obtained in Ref. 30 by chain rule and substitution. Note that the method of slices provides a general form (16a) for symmetry reduced time evolution.<sup>13 15 17</sup>

### III. TWO-MODE SO(2)-EQUIVARIANT FLOW

Dangelmayr,<sup>31</sup> Armbruster, Guckenheimer and Holmes,<sup>32</sup> Jones and Proctor,<sup>33</sup> and Porter and Knobloch<sup>30</sup> (for more details, see Sect. XX.1 in Golubitsky *et al.*<sup>34</sup>) have investigated bifurcations in 1:2 resonance ODE normal form models to third order in the amplitudes. Here, we use this model as a starting point from which we derive what may be one of the simplest chaotic systems with continuous symmetry. We refer to this as the two-mode system:

$$\begin{aligned}\dot{z}_1 &= (\mu_1 - i e_1) z_1 + a_1 z_1 |z_1|^2 + b_1 z_1 |z_2|^2 + c_1 \bar{z}_1 z_2 \\ \dot{z}_2 &= (\mu_2 - i e_2) z_2 + a_2 z_2 |z_1|^2 + b_2 z_2 |z_2|^2 + c_2 z_1^2,\end{aligned}\quad (20)$$

where  $z_1$  and  $z_2$  are complex and all parameters real-valued. The parameters  $\{e_1, e_2\}$  break the reflectional

symmetry of the  $O(2)$ -equivariant normal form studied by Dangelmayr<sup>31</sup> leading to an  $SO(2)$ -equivariant system.<sup>19</sup> This complex two mode system can be expressed as a 4-dimensional system of real-valued first order ODEs by substituting  $z_1 = x_1 + i y_1$ ,  $z_2 = x_2 + i y_2$ , so that

$$\begin{aligned}\dot{x}_1 &= (\mu_1 + a_1 r_1^2 + b_1 r_2^2 + c_1 x_2) x_1 + c_1 y_1 y_2 + e_1 y_1, \\ \dot{y}_1 &= (\mu_1 + a_1 r_1^2 + b_1 r_2^2 - c_1 x_2) y_1 + c_1 x_1 y_2 - e_1 x_1, \\ \dot{x}_2 &= (\mu_2 + a_2 r_1^2 + b_2 r_2^2) x_2 + c_2 (x_1^2 - y_1^2) + e_2 y_2, \\ \dot{y}_2 &= (\mu_2 + a_2 r_1^2 + b_2 r_2^2) y_2 + 2c_2 x_1 y_1 - e_2 x_2, \\ \text{where } r_1^2 &= x_1^2 + y_1^2, \quad r_2^2 = x_2^2 + y_2^2.\end{aligned}\quad (21)$$

The large number of parameters  $(\mu_1, \mu_2, a_1, a_2, b_1, b_2, c_1, c_2, e_1, e_2)$  in this system makes full exploration of the parameter space impractical. Following in the tradition of Lorenz,<sup>35</sup> Hénon,<sup>36</sup> and Rössler,<sup>37</sup> we have tried various choices of parameters until settling on the following set of values, which we will use in all numerical calculations presented here:

$$\begin{array}{cccccccccc} \mu_1 & \mu_2 & e_1 & e_2 & a_1 & a_2 & b_1 & b_2 & c_1 & c_2 \\ -2.8 & 1 & 0 & 1 & -1 & -2.66 & 0 & 0 & -7.75 & 1 \end{array} \quad (22)$$

This choice of parameters is far from the bifurcation values studied by previous authors,<sup>30–33</sup> so that the model has no physical interpretation. However, these parameters yield chaotic dynamics, making the two-mode system a convenient minimal model for the study of chaos in the presence of a continuous symmetry: It is a 4-dimensional  $SO(2)$ -equivariant model, whose symmetry-reduced dynamics are chaotic and take place on a three-dimensional manifold. For another example of parameter values that result in chaotic dynamics, see Ref. 30.

It can be checked by inspection that Eqs. (20) are equivariant under the  $U(1)$  transformation

$$(z_1, z_2) \rightarrow (e^{i\theta} z_1, e^{i2\theta} z_2). \quad (23)$$

In the real representation (21), the  $SO(2)$  group action (23) on a state space point  $a$  is given  $\exp(\theta T) a$ ,<sup>21</sup> where  $a^\top = (x_1, y_1, x_2, y_2)$  and  $T$  is the Lie algebra element

$$T = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \end{pmatrix}. \quad (24)$$

One can easily check that the real two-mode system (21) satisfies the equivariance condition (2).

From (20), it is obvious that the equilibrium point  $(z_1, z_2) = (0, 0)$  is an invariant subspace and that  $z_1 = 0$ ,  $z_2 \neq 0$  is a 2-dimensional flow-invariant subspace

$$\dot{z}_1 = 0, \quad \dot{z}_2 = (\mu_2 - i e_2 + b_2 |z_2|^2) z_2 \quad (25)$$

<sup>14</sup> DB 2014-10-28: This section is kind of weird... doesn't seem to flow and we don't really use this stuff anywhere. Maybe consider deleting.

<sup>16</sup> BB 2014-10-30: In Ref. 30 they use this representation and their plots of chaotic dynamics (yes, they have parameters for which the two-mode system exhibits chaos) look very similar to ours. We have to state somewhere that this similarity is not an accident. I agree that it looks awkward as an individual subsection, I removed the subsection title for now, we may move it to the 'flow' section.

<sup>18</sup> DB 2015-1-9: Yeah... this definitely needs to go somewhere else... maybe where we discuss invariant polynomials since this is a coordinate transform that works because this system is easy but would never work for Navier-Stokes.

<sup>20</sup> DB 1-19-2015: Moved discussion of  $e$ 's to more appropriate spot later

<sup>22</sup> DB 1-19-2015: primes are reserved for template-related things... wrote out group action in words to avoid using primes in equation

with a single circular relative equilibrium of radius  $r_2 = \|z_2\| = \sqrt{-\mu_2/b_2}$  with phase velocity  $c = -e_2/2$ . At the origin the stability matrix  $A$  commutes with  $T$ , and so, can be block-diagonalized into two  $[2 \times 2]$  matrices. The eigenvalues of  $A$  at  $(0, 0, 0, 0)$  are  $\lambda_1 = \mu_1$  with multiplicity 2 and  $\lambda_2 = \mu_2 \pm ie_2$ . In the  $(x_1, y_1, x_2, y_2)$  coordinates, the eigenvectors for  $\lambda_1$  are  $(1, 0, 0, 0)$  and  $(0, 1, 0, 0)$  and the eigenvectors for  $\lambda_2$  are  $(0, 0, 1, 0)$  and  $(0, 0, 0, 1)$ .

In contrast,  $z_2 = 0$  is not, in general, a flow-invariant subspace since the dynamics

$$\dot{z}_1 = (\mu_1 - ie_1) z_1 + a_1 z_1 |z_1|^2, \quad \dot{z}_2 = c_2 z_1^2.$$

take the flow out of the  $z_2 = 0$  plane.

### III.1. Invariant polynomial bases

Before continuing our tutorial on the use of the method of slices using the first Fourier mode slice, we briefly discuss the symmetry reduction of the two-mode system using invariant polynomials. While representations of our model in terms of invariant polynomials and polar coordinates are useful for cross-checking our calculations in the full state space  $a^\top = (x_1, x_2, y_1, y_2)$ , their construction requires a bit of algebra even for this simple 4-dimensional flow. For very high-dimensional flows, such as Kuramoto-Sivashinsky and Navier-Stokes flows, we do not know how to carry out such constructions. As discussed in Refs. 30–32, for the two-mode system, it is easy to construct a set of four real-valued SO(2) invariant polynomials

$$\begin{aligned} u &= z_1 \bar{z}_1, & v &= z_2 \bar{z}_2 \\ w &= z_1^2 \bar{z}_2 + \bar{z}_1^2 z_2, & q &= (z_1^2 \bar{z}_2 - \bar{z}_1^2 z_2)/i. \end{aligned} \quad (26)$$

The polynomials  $[u, v, w, q]$  are linearly independent, but related through one syzygy,

$$w^2 + q^2 - 4u^2v = 0 \quad (27)$$

that confines the dynamics to a 3-dimensional manifold  $\hat{\mathcal{M}} = \mathcal{M}/\text{SO}(2)$ , which is a symmetry-invariant representation of the 4-dimensional SO(2) equivariant dynamics. We call this the reduced state space. By construction,  $u \geq 0$ ,  $v \geq 0$ , but  $w$  and  $q$  can be of either sign. That is explicit if we express  $z_1$  and  $z_2$  in polar coordinates ( $z_1 = |u|^{1/2} e^{i\phi_1}$ ,  $z_2 = |v|^{1/2} e^{i\phi_2}$ ), so that  $w$  and  $q$  take the form

$$\begin{aligned} w &= 2 \operatorname{Re}(z_1^2 \bar{z}_2) = 2u|v|^{1/2} \cos \psi \\ q &= 2 \operatorname{Im}(z_1^2 \bar{z}_2) = 2u|v|^{1/2} \sin \psi, \end{aligned} \quad (28)$$

where  $\psi = 2\phi_1 - \phi_2$ .

The dynamical equations for  $[u, v, w, q]$  follow from the chain rule, which yields

$$\begin{aligned} \dot{u} &= \bar{z}_1 \dot{z}_1 + z_1 \dot{\bar{z}}_1, & \dot{v} &= \bar{z}_2 \dot{z}_2 + z_2 \dot{\bar{z}}_2 \\ \dot{w} &= 2 \bar{z}_2 z_1 \dot{z}_1 + 2 z_2 \bar{z}_1 \dot{\bar{z}}_1 + z_1^2 \dot{\bar{z}}_2 + \bar{z}_1^2 \dot{z}_2 \\ \dot{q} &= (2 \bar{z}_2 z_1 \dot{z}_1 - 2 z_2 \bar{z}_1 \dot{\bar{z}}_1 + z_1^2 \dot{\bar{z}}_2 - \bar{z}_1^2 \dot{z}_2)/i \end{aligned} \quad (29)$$

Substituting (20) into (29), we obtain a set of four SO(2)-invariant equations,

$$\begin{aligned} \dot{u} &= 2\mu_1 u + 2a_1 u^2 + 2b_1 u v + c_1 w \\ \dot{v} &= 2\mu_2 v + 2a_2 u v + 2b_2 v^2 + c_2 w \\ \dot{w} &= (2\mu_1 + \mu_2) w + (2a_1 + a_2) u w + (2b_1 + b_2) v w \\ &\quad + 4c_1 u v + 2c_2 u^2 + (2e_1 - e_2) q \\ \dot{q} &= (2\mu_1 + \mu_2) q + (2a_1 + a_2) u q \\ &\quad + (2b_1 + b_2) v q - (2e_1 - e_2) w. \end{aligned} \quad (30)$$

Note that the O(2)-symmetry breaking parameters  $\{e_1, e_2\}$  of the Dangelmayr normal form system<sup>31</sup> appear only in the relative phase combination  $(2e_1 - e_2)$ , so one of the two can be set to zero without loss of generality. This consideration motivated our choice of  $e_1 = 0$  in (22). Using the syzygy (27), we can eliminate  $q$  from (30) to get<sup>23</sup>

$$\begin{aligned} \dot{u} &= 2\mu_1 u + 2a_1 u^2 + 2b_1 u v + c_1 w \\ \dot{v} &= 2\mu_2 v + 2a_2 u v + 2b_2 v^2 + c_2 w \\ \dot{w} &= (2\mu_1 + \mu_2) w + (2a_1 + a_2) u w + (2b_1 + b_2) v w \\ &\quad + 4c_1 u v + 2c_2 u^2 + (2e_1 - e_2)(4u^2 v - w^2)^{1/2} \end{aligned} \quad (31)$$

This invariant basis can be used either to investigate the dynamics directly or to visualize solutions<sup>9</sup> computed in the full equivariant basis (20).

### III.2. Equilibria of the symmetry-reduced dynamics

The first step in elucidating the geometry of attracting sets is the determination of their equilibria. We shall now show that the problem of determining the equilibria of the symmetry-reduced two-mode (30) system  $[u^*, v^*, w^*, q^*]$  can be reduced to finding the real roots of a multinomial expression. First, we define

$$A_1 = \mu_1 + a_1 u + b_1 v, \quad A_2 = \mu_2 + a_2 u + b_2 v \quad (32)$$

and rewrite (30) as

$$\begin{aligned} 0 &= 2A_1 u + c_1 w, & 0 &= 2A_2 v + c_2 w \\ 0 &= (2A_1 + A_2) w + 2(c_2 u + 2c_1 v) u \\ &\quad + (2e_1 - e_2) q \\ 0 &= (2A_1 + A_2) q - (2e_1 - e_2) w \end{aligned} \quad (33)$$

We already know that  $[0, 0, 0, 0]$  and  $[0, -\mu_2/b_2, 0, 0]$  are the only roots in the  $u = 0$  and  $v = 0$  subspaces, so we are looking only for the  $u > 0$ ,  $v > 0$ ,  $w, q \in \mathbb{R}$  solutions; there could be non-generic roots with either  $w = 0$  or  $q = 0$ , but not both simultaneously, since the syzygy

<sup>24</sup> PC: Note that  $4u^2 v - w^2 = 4u^2 v(1 - \cos^2 \psi)$ , so no serious singularity is introduced this way. Perhaps write equations of  $(u, v, \cos \psi)$  as in the ChaosBook exercises?

(27) precludes that. Either  $w$  or  $q$  can be eliminated by obtaining the following relations from (33):

$$\begin{aligned} w &= -\frac{2u}{c_1} A_1 = -\frac{2v}{c_2} A_2 \\ q &= \frac{2(-2e_1 + e_2)uv}{c_2 u + 2c_1 v}. \end{aligned} \quad (34)$$

Substituting (34)<sup>25</sup> into (33) we get two bivariate polynomials whose roots are the equilibria of the system (30):

$$\begin{aligned} f(u, v) &= c_2 u A_1 - c_1 v A_2 = 0, \\ g(u, v) &= (4A_1^2 u^2 - 4c_1^2 u^2 v)(c_2 u + 2c_1 v)^2 \\ &\quad + 4c_1^2 (-2e_1 + e_2)^2 u^2 v^2 = 0, \\ \deg(f) &= 2, \deg(g) = 6. \end{aligned} \quad (35)$$

We divide the common multiplier  $u^2$  from the second equation and by doing so, eliminate one of the two roots at the origin, as well as the  $(0, -\mu_2/b_2, 0, 0)$  root within the invariant subspace (25). Furthermore, we scale the parameters and variables as  $\tilde{u} = c_2 u$ ,  $\tilde{v} = c_1 v$ ,  $\tilde{a}_1 = a_1/c_2$ ,  $\tilde{b}_1 = b_1/c_1$ ,  $\tilde{a}_2 = a_2/c_2$ ,  $\tilde{b}_2 = b_2/c_1$  to get

$$\tilde{f}(\tilde{u}, \tilde{v}) = \tilde{u} \tilde{A}_1 - \tilde{v} \tilde{A}_2 = 0, \quad \deg(f) = 2, \quad (36)$$

$$\begin{aligned} \tilde{g}(\tilde{u}, \tilde{v}) &= (\tilde{A}_1^2 - c_1 \tilde{v})(\tilde{u} + 2\tilde{v})^2 + e_2^2 \tilde{v}^2 = 0, \\ \deg(g) &= 4, \end{aligned} \quad (37)$$

$$\begin{aligned} \text{where } \tilde{A}_1 &= \mu_1 + \tilde{a}_1 \tilde{u} + \tilde{b}_1 \tilde{v}, \\ \tilde{A}_2 &= \mu_2 + \tilde{a}_2 \tilde{u} + \tilde{b}_2 \tilde{v}, \end{aligned} \quad (38)$$

Solving coupled bivariate polynomials (36) is not, in general, a trivial task. However, for the choice of parameters given by (22), eq. (36) yields  $\tilde{v} = (\mu_1 + \tilde{a}_1 \tilde{u})/(\mu_2 + \tilde{a}_2 \tilde{u})$ . Substituting this into (37) makes it a fourth order polynomial in  $\tilde{u}$ , which we can solve. Only the non-negative, real roots of this polynomial correspond to relative equilibria in the two-mode state space since  $u$  and  $v$  are the squares of first and second mode amplitudes, respectively. Two roots satisfy this condition, the equilibrium at the origin:

$$p_{EQ} = [0, 0, 0, 0], \quad (39)$$

and the relative equilibrium:

$$p_{TW} = [0.193569, 0.154131, -0.149539, -0.027178]. \quad (40)$$

Note that by setting  $b_2 = 0$ , we send the relative equilibrium at  $(0, -\mu_2/b_2, 0, 0)$  to infinity. Thus, (40) is the only relative equilibrium of the two-mode system for our choice of parameters. While this is an equilibrium in the invariant polynomial basis, in the  $SO(2)$ -equivariant,

real-valued state space this is a 1-dimensional relative equilibrium group orbit. The point on this orbit that lies in first Fourier mode slice is (see Fig. 4(c)):

$$(x_1, y_1, x_2, y_2) = (0.439966, 0, -0.386267, 0.070204). \quad (41)$$

We computed the linear stability eigenvalues and eigenvectors of this relative equilibrium, by evaluating stability matrix within the first Fourier mode slice  $\hat{A}_{ij}(\hat{a}) = \partial \hat{v}_i / \partial \hat{a}_j|_{\hat{a}}$  on the relative equilibrium. Linear stability eigenvalues for the relative equilibrium (41)

$$\lambda_{1,2} = 0.05073 \pm i 2.4527, \quad \lambda_3 = -5.5055, \quad \lambda_4 = 0. \quad (42)$$

The 0 eigenvalue corresponds to the direction outside the slice, we expect this since the reduced trajectory equations (8a) keeps the solution within the slice. Imaginary part of the expanding complex pair sets the ‘winding time’ in the neighborhood of the equilibrium to  $T_w = 2\pi/\text{Im}(\lambda_1) = 2.5617$ . The large equilibrium of the contracting eigenvalue  $\lambda_3$  yields a very thin attractor in the reduced state space, thus, when looked at on a planar Poincaré section, the two-mode flow is almost one dimensional, see Fig. 5(a, b).

### III.3. No chaos when the reflection symmetry is restored

Before finishing our discussion of invariant polynomials, we make an important observation regarding the case when both of the reflection symmetry breaking parameters,  $e_1$  and  $e_2$  are set to 0. In this case,  $z_{1,2} \rightarrow \bar{z}_{1,2}$  symmetry is restored and the evolution equations for  $u$ ,  $v$ , and  $w$  in (30) become independent of  $q$ . Furthermore, the time evolution equation for  $q$  becomes linear in  $q$  itself, so that it can be expressed as

$$\dot{q} = \xi(u, v)q. \quad (43)$$

Hence, the time evolution of  $q$  can be written as

$$q(\tau) = e^{\int_0^\tau d\tau' \xi(u(\tau'), v(\tau'))} q(0). \quad (44)$$

If we assume that the flow is bounded, then we can also assume that a long time average of  $\xi$  exists. The sign of this average determines the long term behavior of  $q(\tau)$ ; it will either diverge or vanish depending on the sign of  $\langle \xi \rangle$  being positive or negative respectively. The former case leads to a contradiction: If  $q(\tau)$  diverges, the symmetry-invariant flow cannot be bounded since the syzygy (27) must be satisfied at all times. If  $q(t)$  vanishes, there are three invariant polynomials left, which are still related to each other by the syzygy. Thus, the flow is confined to a two dimensional manifold and cannot exhibit chaos. We must stress that this is a special result which holds for the two-mode normal form with terms up to third order.

<sup>27</sup> 2931

<sup>26</sup> DB : I think getting to the equation for  $q$  throws out a potential  $w = 0$  root. Do the first two equations then imply that  $u, v = 0$ ? If, so then  $q = 0$  and there's no problem, but I don't think that's the most general case.

<sup>28</sup> DB 2014-11-03: Do we know that it holds ONLY in this case? Or do we just not know if it holds in other cases?

<sup>30</sup> BB 2014-11-06: We know that it doesn't hold for PDEs and it is

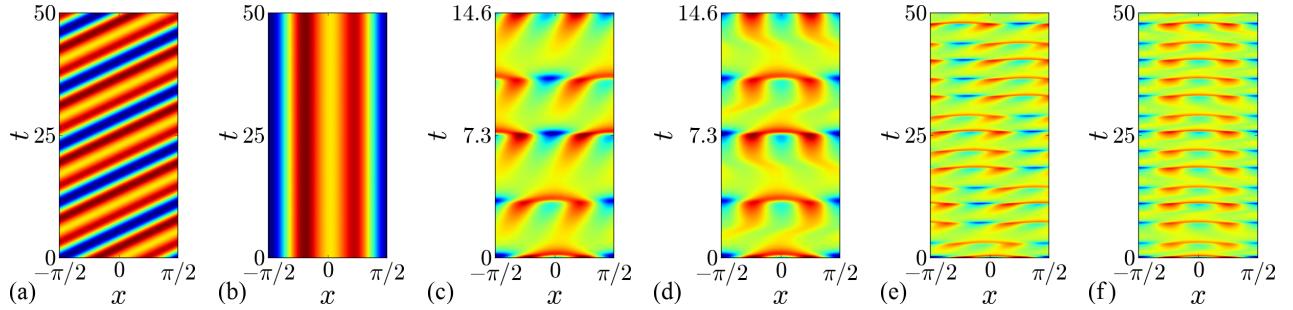


FIG. 3. (Color online) The relative equilibrium  $TW$  in (a) the system's configuration space becomes an equilibrium in (b) the symmetry-reduced configuration space. Two cycles of the relative periodic orbit  $\overline{01}$  in the (c) the symmetry-equivariant configuration space become a periodic orbit in (d) the symmetry-reduced configuration space. A typical ergodic trajectory of the two-mode system in the system's configuration space (e), in the symmetry-reduced configuration space (f). The color scale used in each figure is different to enhance contrast.

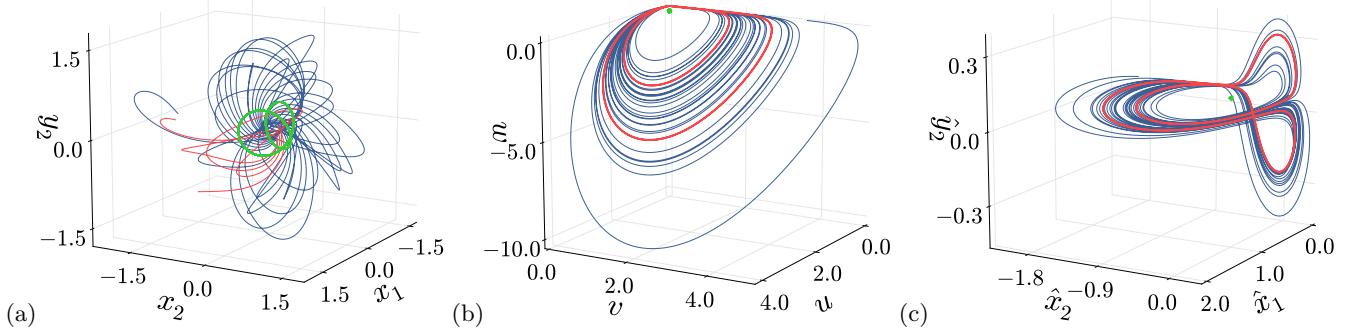


FIG. 4. (Color online) The same trajectories as in Fig. 3(a,c,e), colored green, red and blue respectively, (a) in a 3D projection of the 4-dimensional state space, (b) in a terms of 3 invariant polynomials, (c) in the 3-dimensional first Fourier mode slice hyperplane. Note that in the symmetry reduced representations (b and c), the relative equilibrium  $TW$  is reduced to an equilibrium, the green point; and the  $\overline{01}$  (red) closes onto itself after one repeat. In contrast to the invariant polynomial representation (b), in the first Fourier mode slice hyperplane(c), the qualitative difference between shifts by  $\approx \pi$  and  $\approx -\pi$  in near passages to the slice border is very clear, and it leads to the unimodal Poincaré return map of Fig. 5.

#### III.4. Visualizing two-mode dynamics

We now present visualizations of the dynamics of the two-mode system in four different representations: as 3D projections of the four-dimensional real-valued state space, as 3D projections in the invariant polynomial basis, as dynamics in the 3D slice hyperplane, and as two-dimensional spacetime diagrams of the color-coded field  $u(x, \tau)$ <sup>33</sup> is defined as follows:

$$u(x, \tau) = \sum_{k=-2}^2 z_k(\tau) e^{ikx},$$

where  $z_{-k} = \bar{z}_k$ ,  $z_0 = 0$ , and  $x \in [-\pi, \pi]$ . We can also define the symmetry reduced configuration space repre-

sentation as the inverse Fourier transform of the symmetry reduced Fourier modes:

$$\hat{u}(x, \tau) = \sum_{k=-2}^2 \hat{z}_k(\tau) e^{ikx},$$

where  $\hat{z}_{-k} = \bar{\hat{z}}_k$ ,  $\hat{z}_0 = 0$  and  $x \in [-\pi, \pi]$ . Fig. 3(a,b) show the sole relative equilibrium  $TW$  of the two-mode system in the symmetry-equivariant and symmetry-reduced configuration spaces, respectively. After the symmetry reduction, the relative equilibrium becomes an equilibrium. Fig. 3(c,d) show the relative periodic orbit  $\overline{01}$  again respectively in the symmetry-equivariant and symmetry-reduced configuration space representations. Similar to the relative equilibrium, the relative periodic orbit becomes a periodic orbit after symmetry reduction. Finally, Fig. 3(e,f) show a typical ergodic trajectory of the two-mode system in symmetry-equivariant and symmetry-reduced configuration space representations. Note that in each case, symmetry reduction cancels the ‘drifts’ along the symmetry ( $x$ ) direction.

As can be seen clearly in Fig. 4(a), these drifts show up in the Fourier mode representation as  $SO(2)$  rotations.

<sup>32</sup> DB 2014-11-10: Do you have a notion of why it might break down as you add more modes or higher order couplings? It'd be nice to include this here.

<sup>34</sup> DB 11-3-2014: Using  $u$  here is confusing since we've just spent the last few pages talking about  $u$  in the  $(u, v, w, q)$  basis

The relative equilibrium  $TW$  traces its  $SO(2)$  group orbit (green curve in Fig. 4(a)) as it drifts in the configuration space. The relative periodic orbit  $\overline{01}$  (red) and the ergodic trajectory (blue) rotate in the same fashion as they evolve. Fig. 4(b,c) show a three dimensional projection onto the invariant polynomial basis and the 3-dimensional trajectory on the slice hyperplane for the same orbits. In both figures, the relative equilibrium is reduced to an equilibrium and the relative periodic orbit is reduced to a periodic orbit.

#### IV. PERIODIC ORBITS

The simple structure of the symmetry-reduced dynamics allows us to determine the relative periodic orbits of the two-mode system by means of a Poincaré section and a return map. We illustrate this procedure in Fig. 5. Starting with an initial point close to the  $TW$ , we compute a long, symmetry-reduced ergodic trajectory by integrating (16a) and record where it crosses the Poincaré section, which we define as the plane that contains  $TW$  and is spanned the imaginary part of its unstable stability eigenvector and  $\hat{y}_2$ .<sup>35,37</sup> We then project these points onto a basis  $(v_1, v_2)$ , which spans the Poincaré section and fit cubic splines to the data as shown in Fig. 5 (b). We then construct a return map along this curve and express this in terms of the distance  $s$  from  $TW$  as measured by the arc length along the cubic spline fit. The resulting map, which is shown in Fig. 5 (c), is unimodal with a sharp cusp located at its critical point. Note that the region  $s \in (0, 0.6)$  corresponds to the neighborhood of the relative equilibrium and is only visited transiently. Once the dynamics fall onto the chaotic attractor, this region is never visited again. Removing this region from the return map, we obtain the return map shown in Fig. 5 (d), which we can then use to determine the accessible relative periodic orbits with their respective binary symbol sequences.

The unimodal return map of Fig. 5 diverges around  $s \approx 0.98$  and this neighborhood is visited very rarely by the flow. We took the furthest point that is visited by the ergodic flow,  $s_C = 0.98102264$  as the critical point of this map and coded points to the left and right hand sides of this point as ‘0’ and ‘1’, respectively, to construct binary symbolic dynamics. Accessible periodic orbits are then those with the topological coordinates less than that of this critical point. We skip the technical details regarding

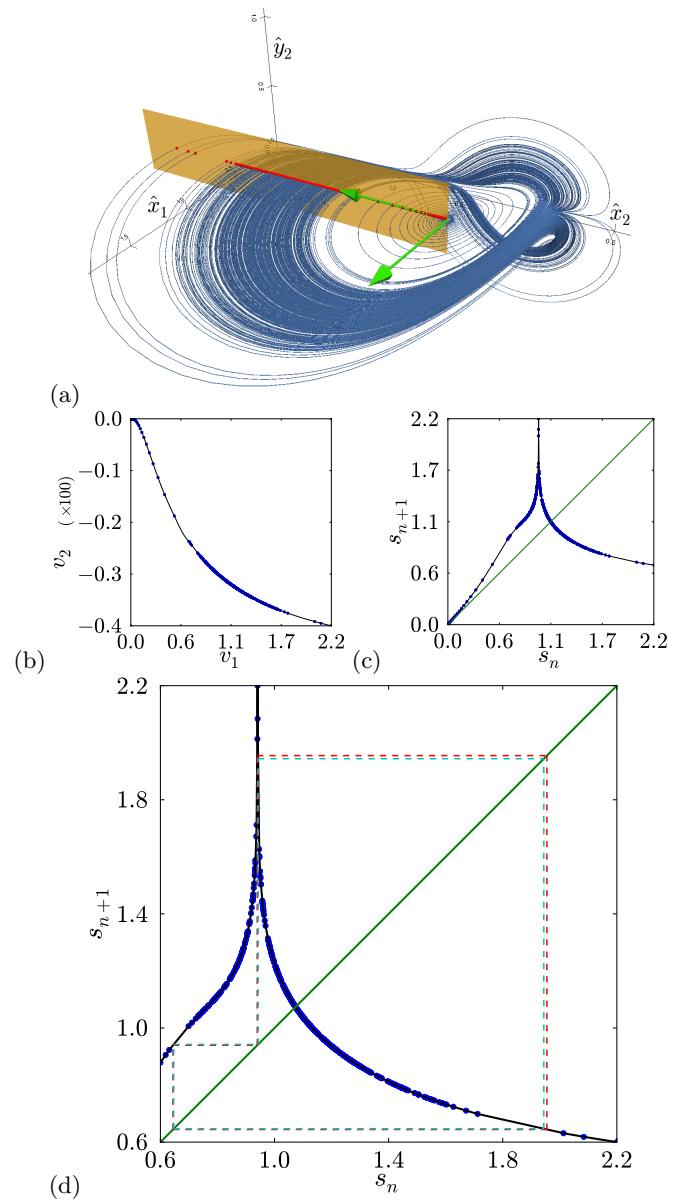


FIG. 5. (Color online) (a) Symmetry-reduced ergodic trajectory within the slice hyperplane (blue). Green arrows indicate the real and imaginary parts of the complex eigenvectors  $v_u$  which span the unstable manifold of  $TW$ . The Poincaré section, which contains  $TW$  and is spanned by  $\text{Im}[v_u]$  and  $\hat{y}_2$ , is visualized as a transparent plane. Points where the flow crosses the are marked in red. (b) Poincaré section shown in (a) projected on to two dimensions, where  $v_{1,2}$  are the basis which spans the section hyperplane. Note that the vertical axis, which corresponds to the direction parallel to  $\hat{y}_2$  is magnified by 100. All (blue) points are located relative to the  $TW$ , which it self is on the origin. Black curve is a cubic spline interpolation to this data set. (c) Return map of the arclengths (along the interpolation curve, measured from the origin  $TW$ ) of Poincaré section data shown in (b). Note that once the flow exits the neighborhood of the  $TW$  ( $s < 0.6$ ) it stays on the attractor and never comes back. Thus the data up to this is transient. (d) The return map without the transient points framed by orbit of the critical point. Dashed lines show the 3-cycles  $\overline{001}$  (red) and  $\overline{011}$  (cyan).

<sup>36</sup> BB 2014-11-06: It includes  $\hat{y}_2$  direction, a bit random but it doesn't matter that much because attractor is super thin, I think the figure does the job (everything is there, it's not a projection etc.), but you can add if you think it's necessary.

<sup>38</sup> DB 2014-11-10: Dropped color coding from main text since it's already in the figure. Reworded a little to make Poincaré section definition more precise and make our work repeatable. Also see comments in figure caption.

symbolic dynamics and kneading theory in this tutorial since there is a rich literature on these topics and we do not employ any novel symbolic dynamics technique here. For a pedagogical introduction to the subject, we refer the reader to Refs. 8 and 38.

We are now going to summarize the procedure of locating relative periodic orbits in the state space: Suppose the binary itinerary  $\overline{I_0} \overline{I_1} \dots \overline{I_{n-1}}$ , where,  $I_j = 0, 1$  corresponds to an admissible ‘n-cycle’, a relative periodic orbit that intersects our Poincaré section n-times. We first find arc-lengths  $\{s_0, s_1, \dots s_n\}$  that constitutes this cycle on the return map Fig. 5(d). We then find corresponding reduced state space points  $\{\hat{a}_0, \hat{a}_1, \dots \hat{a}_{n-1}\}$ . Finally we integrate the reduced flow and the phase (8) starting from each found reduced state space point  $\hat{a}_j$  until it returns to the Poincaré section, and divide this trajectory into  $N$  small pieces. As a result, we obtain  $n \times N$  state space points, durations and phase shifts  $\{a_i^{(0)}, \tau_i^{(0)}, \theta_i^{(0)}\}$  where  $i = 1, 2, \dots n \times N$ , which we feed into the multiple shooting Newton solver (see appendix A) to precisely determine the relative periodic orbit, its period and the associated phase shift. After finding  $n \times N$  state space points ( $a_i$ ), flight times ( $\tau_i$ ), and phase shifts ( $\theta_i$ ) associated with the  $n$  cycle, we then compute the flow Jacobian associated with each piece  $J^{\tau_i}(a_i)$ , using which we represent the Jacobian associated with the relative periodic orbit as

$$\hat{J} = D(-\theta_{n \times N}) J^{\tau_{n \times N}}(a_{n \times N}) \dots D(-\theta_2) J^{\tau_2}(a_2) D(-\theta_1) J^{\tau_1}(a_1). \quad (45)$$

This construction (45) of Jacobian is equivalent to our definition in (6), since both group action  $g$  and flow Jacobian  $J$  are multiplicative and they commute with each other as a consequence of  $g$ -equivariance of the flow. The form (45) is essential in determining its eigenvalues (Floquet multipliers) precisely for which we utilized periodic Schur decomposition (appendix B).

We found the admissible cycles of the two-mode system up to the topological length 12. We listed binary itineraries of shortest 7 relative periodic orbits (with topological lengths up to 5), along with their periods, phase shifts, Floquet multipliers, and Floquet exponents in table I. In Fig. 6 we show shortest 4 of the relative periodic orbits of the two-mode system within the first Fourier mode slice hyperplane. As seen from Fig. 6, trajectories of  $\overline{001}$  (red) and  $\overline{011}$  (cyan) almost overlap in a large region of the state space. This behavior is also manifested in the return map of Fig. 5 d), where we have shown cycles  $\overline{001}$  and  $\overline{011}$  with red and cyan respectively. This is a general property of the two-mode cycles with odd topological lengths: They come in pairs with almost equal leading (largest) Floquet exponents , see Fig. 7. Floquet exponents ( $\lambda_j$ ) characterize the rate of expansion/contraction of nearby perturbations to the relative periodic orbits and are related to Floquet multipliers ( $\Lambda_j$ ) by

$$\lambda_{p,j} = \frac{1}{T_p} \ln |\Lambda_{p,j}|, \quad j = 1, 2, \dots, d, \quad (46)$$

Itinerary	$T$	$\theta$	$\Lambda$	$\lambda$
1	3.64151221	0.08096967	-1.48372354	0.10834917
01	7.34594158	-2.94647181	-2.00054831	0.09439516
001	11.07967801	-5.64504385	-55.77844510	0.36295166
011	11.07958924	-2.50675871	54.16250810	0.36030117
0111	14.67951823	-2.74691247	-4.55966852	0.10335829
01011	18.39155417	-5.61529803	-30.00633820	0.18494406
01111	18.38741006	-2.48213868	28.41893870	0.18202976

TABLE I. Itinerary, period ( $T$ ), phase shift ( $\theta$ ), Floquet multiplier ( $\Lambda$ ), and Floquet exponent ( $\lambda$ ) of the found two-mode relative periodic orbits with topological lengths up to  $n = 5$ , more (up to  $n = 12$ ) available upon request.

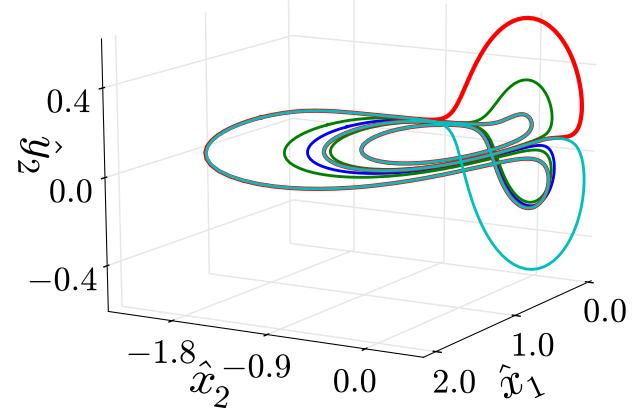


FIG. 6. (Color online) Shortest four relative periodic orbits of the two-mode system:  $\overline{1}$  (dark blue),  $\overline{01}$  (green),  $\overline{001}$  (red),  $\overline{011}$  (cyan). Note that relative periodic orbits  $\overline{001}$  and  $\overline{011}$  almost overlap everywhere except  $\hat{x}_1 \approx 0$ .

where the subscript  $p$  indicates ‘prime relative periodic orbit  $p$ ’ and  $T_p$  is its period. Having computed periods, phase shifts, and Floquet multipliers of relative periodic orbits, we are now ready to calculate dynamical averages and other statistical moments of observables using cycle averaging formulas.

## V. CYCLE AVERAGES

So far, we have explained how we find the relative periodic orbits of the two-mode system and compute their stability. However, we have not yet said anything about what to do with these numbers. We begin this section with an overview of the main results of the periodic orbit theory. Our review starts by recapitulating the presentation of Ref. 8, but then, in Section V.2, explains how the theory is modified in the presence of continuous symmetries.<sup>39</sup> In Section V.3, we present cycle expansions and explain how to approximate the Poincaré section in Fig. 5(d), in order to obtain a better convergence of the spectral determinants. The numerical results are discussed in Section V.4

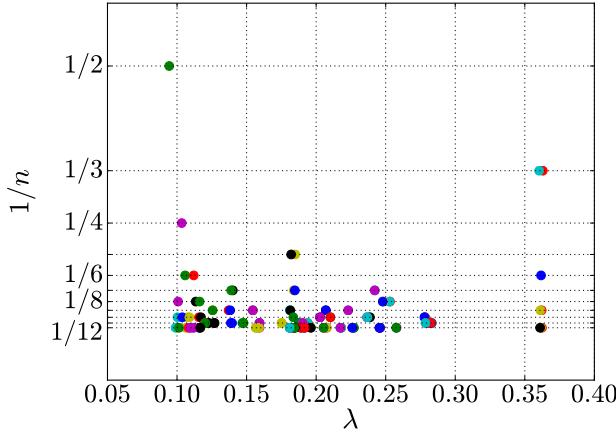


FIG. 7. (Color online) Distribution of the expanding Floquet exponents of all two-mode cycles with topological lengths  $n$  from 2 to 12.

### V.1. Classical trace formula

Consider the evolution operator  $\mathcal{L}^t$ , the action of which evolves a weighted density  $\rho(a, t)$  in the state space,

$$\begin{aligned}\rho(a', t) &= [\mathcal{L}^t \rho](a') = \int da \mathcal{L}^t(a', a) \rho(a, 0) \\ \mathcal{L}^t(a', a) &= \delta(a' - f^t(a)) e^{\beta \Omega^t(a)},\end{aligned}\quad (47)$$

where  $\beta$  is an auxiliary variable and  $\Omega^t(a)$  is the integrated value of an observable  $\omega(a)$  along the trajectory  $a(t) = f^t(a)$ ,

$$\Omega^t(a) = \int_0^t dt' \omega(f^{t'}(a)). \quad (48)$$

When  $\beta = 0$ , the evolution operator (47) evolves the initial density of state space points to its new form after time  $t$ ; this form of the evolution operator is known as the Perron-Frobenius operator. The multiplicative weight  $\exp(\beta \Omega^t(a))$  will enable us to compute the value of the observable  $\omega$  averaged over the natural measure.

As the integrated observable  $\Omega^t(a)$ , additive along the trajectory, is exponentiated in (47), the evolution operator is multiplicative along the trajectory:

$$\mathcal{L}^{t_1+t_2}(a', a) = \int da'' \mathcal{L}^{t_2}(a', a'') \mathcal{L}^{t_1}(a'', a). \quad (49)$$

This semigroup property allows us to define the evolution operator as the formal exponential of its infinitesimal generator  $\mathcal{A}$ :

$$\mathcal{L}^t = e^{\mathcal{A}t}. \quad (50)$$

Let  $\rho_\beta(a)$  be the eigenfunction of (47) corresponding to the leading eigenvalue of  $\mathcal{A}$  (i.e., the one with the largest real part) for a given  $\beta$ ,

$$[\mathcal{L}^t \rho_\beta](a) = e^{ts(\beta)} \rho_\beta(a). \quad (51)$$

If the system under study is ergodic, then an invariant ‘natural measure’  $\rho_0(a)$  with eigenvalue  $s(0) = 0$  exists, and the long time average of an observable is then its state space average over the natural measure:

$$\langle \omega \rangle = \int da \omega(a) \rho_0(a). \quad (52)$$

By evaluating the action of the evolution operator (47) for infinitesimal times, one finds that the long-time averages of observables, as well as of their higher moments, are given by derivatives of  $s(\beta)$ :

$$\begin{aligned}\langle \omega \rangle &= \lim_{t \rightarrow \infty} \frac{1}{t} \langle \Omega^t \rangle = \left. \frac{\partial s(\beta)}{\partial \beta} \right|_{\beta=0}, \\ \Delta &= \lim_{t \rightarrow \infty} \frac{1}{t} \langle (\Omega^t)^2 - \langle \Omega^t \rangle^2 \rangle = \left. \frac{\partial^2 s(\beta)}{\partial \beta^2} \right|_{\beta=0}, \\ &\vdots\end{aligned}\quad (53)$$

For example, if the observable  $\omega$  is a velocity  $\dot{a} = v(a)$ , the integrated observable  $\Omega^t$  is the displacement  $a(t)$  in  $d$  dimensions, and  $\Delta/2d$  is Einstein’s diffusion coefficient. In order to obtain  $s(\beta)$ , we construct the resolvent of  $\mathcal{A}$ , by taking the Laplace transform of (50):

$$\int_0^\infty dt e^{-st} \mathcal{L}^t = (s - \mathcal{A})^{-1}, \quad (54)$$

the trace of which peaks at the eigenvalues of  $\mathcal{A}$ . By taking the Laplace transform of  $\mathcal{L}^t$  and computing its trace by  $\text{tr } \mathcal{L}^t = \int da \mathcal{L}^t(a, a)$ , one obtains the classical trace formula<sup>40</sup>:

$$\sum_{\alpha=0}^{\infty} \frac{1}{s - s_\alpha} = \sum_p T_p \sum_{r=1}^{\infty} \frac{e^{r(\beta \Omega_p - s T_p)}}{|\det(\mathbf{1} - M_p^r)|} \quad (55)$$

that relates the spectrum of the evolution operator to the spectrum of periodic orbits. Here  $s$  is the auxiliary variable of the Laplace transform and  $s_\alpha$  are the eigenvalues of  $\mathcal{A}$ . The outer sum on the right hand side runs over the ‘prime cycles’  $p$  of the system, i.e., the shortest periodic orbits of period  $T_p$ .  $\Omega_p$  is the value of the observable integrated along the prime cycle and  $M_p$  is the transverse monodromy matrix, the eigenvalues  $\Omega^t(a)$  of which are the Floquet multipliers of  $p$  with the marginal ones excluded. In the derivation of (55), one assumes that the flow has a single marginal direction, namely the  $v(a)$  tangent to the periodic orbit, and evaluates the contribution of each periodic orbit to the trace integral by transforming to a local coordinate system where one of the coordinates is parallel to the flow, while the rest are transverse. Integral along the parallel direction contributes the factors of  $T_p$  in (55). The transverse integral over the delta function (47) contributes the factor of  $1/|\det(\mathbf{1} - M_p^r)|$ .

### V.2. Decomposition of the trace formula over irreducible representations

The classical trace formula (55) accounts for contributions from periodic orbits to long time dynamical av-

erages. However, relative periodic orbits of equivariant systems are almost never periodic in the full state space. In order to compute the contributions of relative periodic orbits to the trace of the evolution operator, one has to factorize the evolution operator into the irreducible subspaces of the symmetry group. For discrete symmetries, this procedure is studied in Ref. 41. For the quantum systems with continuous symmetries (Abelian and 3D rotations), the factorization of the semiclassical Green's operator is carried out in Ref. 42. Ref. 39 addresses the continuous factorization of the evolution operator and its trace; we provide a sketch of this treatment here. We start by stating, without proof, that a square-integrable field  $\psi(a)$  over a vector space can be factorized into its projections over the irreducible subspaces of a group  $G$ :

$$\psi(a) = \sum_m \mathbb{P}_m \psi(a), \quad (56)$$

where the sum runs over the irreducible representations of  $G$  and the projection operator onto the  $m$ th irreducible subspace, for a continuous group,<sup>39</sup>

$$\mathbb{P}_m = d_m \int_G d\mu(g) \chi_m(g(\theta)) \mathbb{D}(\theta). \quad (57)$$

Here,  $d_m$  is the dimension of the representation,  $d\mu(g)$  is the normalized Haar measure,  $\chi_m(g)$  is the character of  $m$ th irreducible representation and  $\mathbb{D}(\theta)$  is the operator that transforms a scalar field defined on the state space as  $\mathbb{D}(\theta)\rho(a) = \rho(D(\theta)^{-1}a)$ .<sup>41</sup> For our specific case of a single SO(2) symmetry,

$$d_m \rightarrow 1, \quad (58)$$

$$\int_G d\mu(g) \rightarrow \oint \frac{d\theta}{2\pi}, \quad (59)$$

$$\chi_m(g(\theta)) \rightarrow e^{-im\theta}. \quad (60)$$

Because the projection operator (57) decomposes scalar fields defined over the state space into their irreducible subspaces under action of  $G$ , it can be used to factorize the evolution operator. Thus, the kernel of the evolution operator transforms under the action of  $\mathbb{D}(\theta)$  as

$$\begin{aligned} \mathbb{D}(\theta)\mathcal{L}^t(a', a) &= \mathcal{L}^t(D(\theta)^{-1}a', a), \\ &= \mathcal{L}^t(a', D(\theta)a), \\ &= \delta(a' - D(\theta)f^t(a))e^{\beta\Omega^t(a)}, \end{aligned} \quad (61)$$

where the second step follows from the equivariance of the system under consideration. Relative periodic orbits contribute to  $\mathbb{P}_m \mathcal{L}^t = \mathcal{L}_m^t$  since when its kernel is modified as in (61), the projection involves an integral over

the group parameters that is non-zero when  $\theta = -\theta_p$ , the phase shifts of the relative periodic orbits. By computing the trace of  $\mathcal{L}_m^t$ , which in addition to the integral over state space, now involves another integral over the group parameters, one obtains the  $m$ th irreducible subspace contribution to the classical trace as

$$\sum_{\alpha=0}^{\infty} \frac{1}{s - s_{m,\alpha}} = \sum_p T_p \sum_{r=1}^{\infty} \frac{\chi_m(g^r(\theta_p))e^{r(\beta\Omega_p - sT_p)}}{\left| \det(\mathbf{1} - \hat{M}_p^r) \right|}. \quad (62)$$

The reduced trace formula (62) differs from the classical trace formula (55) by the group character term, which is evaluated at the relative periodic orbit phase shifts, and the reduced monodromy matrix  $\hat{M}$ , which is the  $(d - N - 1) \times (d - N - 1)$  reduced Jacobian for the relative periodic orbit evaluated on a Poincaré section in the reduced state space. The eigenvalues of  $\hat{M}$  are those of the relative periodic orbit Jacobian (6) excluding the marginal ones, i.e., the ones corresponding to time evolution and evolution along the continuous symmetry directions.

Since we are only interested in the leading eigenvalue of the evolution operator, we only consider contributions to the trace (55) from the projections (62) of the 0th irreducible subspace. For the SO(2) case at hand, these can be written explicitly as

$$\sum_{\alpha=0}^{\infty} \frac{1}{s - s_{0,\alpha}} = \sum_p T_p \sum_{r=1}^{\infty} \frac{e^{r(\beta\Omega_p - sT_p)}}{\left| \det(\mathbf{1} - \hat{M}_p^r) \right|}. \quad (63)$$

This form differs from the classical trace formula (55) only by the use of the reduced monodromy matrix instead of the full monodromy matrix<sup>43</sup> since the 0th irreducible representation of SO(2) has character 1. For this reason, cycle expansions,<sup>44</sup> which we cover next, are applicable to (63) after the replacement  $M \rightarrow \hat{M}$ .

### V.3. Cycle expansions

While the classical trace formula (55) and its factorization for systems with continuous symmetry (62) manifest the essential duality between the spectrum of an observable and that of the periodic orbits and relative periodic orbits, in practice, they are hard to work with since the eigenvalues are located at the poles of (55) and (62). The dynamical zeta function (66), which we derive below, provides a perturbative expansion form that enables us to order terms in decreasing importance while computing spectra for the two-mode system. As stated earlier, (63)

<sup>40</sup> PC: 2014-11-10: From now on, a problem - if I redefine  $D(\dots)$  as  $D$ , cannot easily revert to the concise  $g$  notation...

<sup>42</sup> DB 2014-11-10: Added missing parenthesis back on 2014-11-3 to this expression. Did I put it in the right place? I think so, but please double check and then delete this comment.

<sup>44</sup> DB 2014-11-10: What is the non-reduced monodromy matrix called? I put down full but don't know if this is the correct nomenclature

is equivalent to (55) via substitution  $M \rightarrow \hat{M}$ . We start by defining the ‘spectral determinant’:

$$\det(s - \mathcal{A}) = \exp \left( - \sum_p \sum_{r=1}^{\infty} \frac{1}{r} \frac{e^{r(\beta\Omega_p - sT_p)}}{|\det(\mathbf{1} - M_p^r)|} \right), \quad (64)$$

whose logarithmic derivative ( $(d/ds) \ln \det(s - \mathcal{A})$ ) gives the classical trace formula (55). The spectral determinant (64)<sup>45</sup> is easier to work with since the spectrum of  $\mathcal{A}$  is now located at the zeros of (64). The convergence of (64) is, however, still not obvious. More insight is gained by approximating  $|\det(\mathbf{1} - M_p^r)|$  by the product of expanding Floquet multipliers and then carrying out the sum over  $r$  in (64). This approximation yields

$$\begin{aligned} |\det(\mathbf{1} - M_p)| &= |(1 - \Lambda_{e,1})(1 - \Lambda_{e,2}) \dots \\ &\quad (1 - \Lambda_{c,1})(1 - \Lambda_{c,2}) \dots| \\ &\approx \prod_e |\Lambda_e| \equiv |\Lambda_p|, \end{aligned} \quad (65)$$

where  $|\Lambda_{e,i}| > 1$  and  $|\Lambda_{c,i}| < 1$  are expanding and contracting Floquet multipliers respectively. By making this approximation, the sum over  $r$  in (64) becomes the Taylor expansion of natural logarithm. Carrying out this sum, brings the spectral determinant (64) to a product (over prime cycles) known as the dynamical zeta function:

$$1/\zeta = \prod_p (1 - t_p) \text{ where, } t_p = \frac{1}{|\Lambda_p|} e^{\beta\Omega_p - sT_p} z^{n_p}. \quad (66)$$

Each ‘cycle weight’  $t_p$  is multiplied by the ‘order tracking term’  $z^{n_p}$ , where  $n_p$  is the topological length of the  $p$ th prime cycle. This polynomial ordering arises naturally in the study of discrete time systems where the Laplace transform is replaced by  $z$ -transform. Here, we insert the powers of  $z$  by hand, to keep track of the ordering, and then set its value to 1 at the end of calculation. Doing so allows us to write the dynamical zeta function (66) in the ‘cycle expansion’ form by grouping its terms in powers of  $z$ . For complete binary symbolic dynamics, where every binary symbol sequence is accessible, the cycle expansion reads

$$1/\zeta = 1 - t_0 - t_1 - (t_{01} - t_{01}t_1) \quad (67)$$

$$-[(t_{011} - t_{01}t_1) + (t_{001} - t_{01}t_0)] - \dots$$

$$= 1 - \sum_f t_f - \sum_n \hat{c}_n, \quad (68)$$

where we labeled each prime cycle by its binary symbol sequence. In (68) we grouped the contributions to the zeta function into two groups: ‘fundamental’ contributions  $t_f$  and ‘curvature’ corrections  $c_n$ . The curvature correction terms are denoted explicitly by parentheses in (67) and correspond to ‘shadowing’ combinations

where combinations of shorter cycle weights, also known as ‘pseudocycle’ weights, are subtracted from the weights of longer prime cycles. Since the cycle weights in (66) already decrease exponentially with increasing cycle period, the cycle expansion (67) converges even faster than exponentially when the terms corresponding to longer prime cycles are shadowed.<sup>47</sup>

For complete binary symbolic dynamics, the only fundamental contributions to the dynamical zeta function are from the cycles with topological length 1, and all longer cycles appear in the shadowing pseudocycle combinations. More generally, if the symbolic dynamics is a subshift of finite type,<sup>8</sup> with the grammar of admissible sequences described by a finite set of pruning rules, and the flow is uniformly hyperbolic, cycle expansions of spectral determinants are guaranteed to converge super-exponentially.<sup>44</sup> A generic unimodal map symbolic dynamics is not a subshift of finite type. However, we have shown in Section IV that the Poincaré return map for the two-mode system (Fig. 5 (d)) diverges at  $s \approx 0.98$  and approximated it as if its tip was located at the furthest point visited by an ergodic trajectory. Can we approximate the map in Fig. 5 (d) in such a way that corresponding symbolic dynamics has a finite grammar of pruning rules? The answer is yes.

As shown in Fig. 5 (d) the cycles  $\overline{001}$  and  $\overline{011}$  pass quite close to the tip of the cusp. Approximating the map as if its tip is located exactly at the point where  $\overline{001}$  cuts gives us what we are looking for: a single grammar rule, which says that the symbol sequence ‘00’ is inadmissible. This can be made rigorous by the help of kneading theory, however, the simple result is easy to see from the return map in Fig. 5 (d): Cover the parts of the return map, which are outside the borders set by the red dashed lines, the cycle  $\overline{001}$  and then start any point to the left of the tip and look at images. You will always land on a point to the right of the tip, unless you start at the lower left corner, exactly on the cycle  $\overline{001}$ . As we will show, this ‘finite grammar approximation’ is reasonable since the orbits that visit outside the borders set by  $\overline{001}$  are very unstable, and hence, less important for the description of invariant dynamics.

The binary grammar with only rule that forbids repeats of one of the symbols is known as the ‘golden mean’ shift,<sup>8</sup> named after its topological entropy which is  $\ln(1 + \sqrt{5})/2$ . Binary itineraries of golden mean cycles can be easily obtained from the complete binary symbolic dynamics by substitution  $0 \rightarrow 01$  in the latter. Thus, we can write the dynamical zeta function for the golden mean pruned symbolic dynamics by replacing 0s in (67) by 01:

$$\begin{aligned} 1/\zeta &= 1 - t_{01} - t_1 - (t_{011} - t_{01}t_1) \quad (69) \\ &\quad -[(t_{0111} - t_{011}t_1) + (t_{01011} - t_{01}t_{011})] - \dots \end{aligned}$$

<sup>46</sup> DB 2014-11-10: What’s up with G65?

<sup>48</sup> DB 2014-11-10: Tried to make this clearer. Think I didn’t change the content, but somebody more erudite than me please double check this.

Note that all the contributions longer than topological length 2 to the golden mean dynamical zeta function are in form of shadowing combinations. In Section V.4, we will compare the convergence of the cycle averages with and without the finite grammar approximation, but before moving on to numerical results, we explain the remaining details of computation.

While dynamical zeta functions are useful for investigating the convergence properties, they are not exact, and their computational cost is same as that of exact spectral determinants. For this reason, we expand the spectral determinant (64) ordered in the topological length of cycles and pseudocycles. We start with the following form of the spectral determinant (64)

$$\det(s - \mathcal{A}) = \prod_p \exp \left( - \sum_{r=1}^{n_p r < N} \frac{1}{r} \frac{e^{r(\beta \Omega_p - s T_p)}}{|\det(\mathbf{1} - M_p^r)|} z^{n_p r} \right), \quad (70)$$

where the sum over the prime cycles in the exponential becomes a product. We also inserted the order tracking term  $z$  and truncated the sum over cycle repeats at the expansion order  $N$ . For each prime cycle, we compute the sum in (70) and expand the exponential up to order  $N$ . We then multiply this expansion with the contributions from previous cycles and drop terms with order greater than  $N$ . This way, after setting  $z = 1$ , we obtain the spectral determinant truncated to cycles and pseudocycles of topological length up to  $n_\pi \leq N$ ,

$$F_N(\beta, s) = 1 - \sum_{n=1}^N Q_n(s, \beta). \quad (71)$$

In what follows, we shall drop the subscript,  $F_N \rightarrow F$ , but actual calculations are always done for a range of finite truncation lengths  $N$ . Remember that we are searching for the eigenvalues  $s(\beta)$  of the operator  $\mathcal{A}$  in order to compute the moments (53). Eigenvalues  $s(\beta)$  are located at the zeros of the spectral determinant, hence as function of  $\beta$  they satisfy the implicit equation

$$F(\beta, s(\beta)) = 0. \quad (72)$$

By taking derivative of (72) with respect to  $\beta$  and applying chain rule we obtain

$$\frac{ds}{d\beta} = - \frac{\partial F}{\partial \beta} / \frac{\partial F}{\partial s}. \quad (73)$$

Higher order derivatives can be evaluated similarly. Defining

$$\begin{aligned} \langle \Omega \rangle &= -\partial F / \partial \beta \\ \langle T \rangle &= \partial F / \partial s, \quad \langle T^2 \rangle = \partial^2 F / \partial s^2 \\ \langle \Omega^2 \rangle &= -\partial^2 F / \partial \beta^2, \quad \langle \Omega T \rangle = \partial^2 F / \partial \beta \partial s, \end{aligned} \quad (74)$$

we write the cycle averaging formulas as

$$\langle \omega \rangle = \langle \Omega \rangle / \langle T \rangle, \quad (75)$$

$$\begin{aligned} \Delta &= \frac{1}{\langle T \rangle} \left( \langle \Omega^2 \rangle - 2 \frac{ds}{d\beta} \langle \Omega T \rangle + \left( \frac{ds}{d\beta} \right)^2 \langle T^2 \rangle \right) \\ &= \frac{1}{\langle T \rangle} \langle (\Omega - T \langle \omega \rangle)^2 \rangle, \end{aligned} \quad (76)$$

with everything evaluated at  $\beta = 0$ ,  $s = s(0)$ . By probability conservation, we expect that for an invariant measure  $\rho_0(a)$ , the eigenvalue  $s(0)$  is 0. However, we did not make this substitution in cycle averaging formulas since, in practice, our approximations to the spectral determinant are always based on a finite number of periodic orbits, so that the solution of  $F_N(0, s(0)) = 0$  is small, but not exactly 0. This eigenvalue has a special meaning: It indicates how well the periodic orbits cover the strange attractor. Following this interpretation, we define  $\gamma = -s(0)$  as the ‘escape rate’: the rate at which the dynamics escape the region that is covered by the periodic orbits. Specifically, for our finite grammar approximation; the escape rate tells us how frequently the ergodic flow visits the part of the Poincaré map that we cut off by applying our finite grammar approximation.

We defined  $\langle T \rangle$  in (75) as a shorthand for a partial derivative, however, we can also develop an interpretation for it by looking at the definitions of the dynamical zeta function (66) and the spectral determinant (64). In both series, the partial derivative with respect to  $s$  turns them into weighted sum of the cycle periods; with this intuition, we define  $\langle T \rangle$  as the ‘mean cycle period’.

These remarks conclude our review of the periodic theory and its extension to the equivariant dynamical systems. We are now ready to present our numerical results and discuss their quality.

#### V.4. Numerical results

We constructed the spectral determinant (71) to different orders for two observables: phase velocity  $\dot{\theta}$  and the leading Lyapunov exponent. Remember that  $\Omega_p$  appearing in (70) is the integrated observable, so in order to obtain the moments of phase velocity and the leading Lyapunov exponent from (76) and (77), we respectively input  $\Omega_p = \theta_p$  phase shift of the prime cycle, and  $\Omega_p = \ln |\Lambda_{p,e}|$  logarithm of the expanding Floquet multiplier of  $\Lambda_{p,e}$  the prime cycle  $p$ .

In Section III.4, we explained that SO(2) phase shifts correspond to the drifts in the configuration space. We define the corresponding diffusion coefficient as

$$D = \frac{1}{2d} \lim_{t \rightarrow \infty} \frac{1}{t} \langle \theta(t)^2 - \langle \theta(t) \rangle^2 \rangle, \quad (77)$$

where  $d = 1$  since the configuration space is one dimen-

$N$	$\gamma$	$\langle T \rangle$	$\lambda$	$\langle \dot{\phi} \rangle$	$D$
1	0.249829963	3.6415122	0.10834917	0.0222352	<b>0.000000</b>
2	-0.011597609	5.8967605	0.10302891	-0.1391709	<b>0.143470</b>
3	0.027446312	4.7271381	0.11849761	-0.1414933	<b>0.168658</b>
4	-0.004455525	6.2386572	0.10631066	-0.2141194	<b>0.152201</b>
5	0.000681027	5.8967424	0.11842700	-0.2120545	<b>0.164757</b>
6	0.000684898	5.8968762	0.11820050	-0.1986756	<b>0.157124</b>
7	0.000630426	5.9031596	0.11835159	-0.1997353	<b>0.157345</b>
8	0.000714870	5.8918832	0.11827581	-0.1982025	<b>0.156001</b>
9	0.000728657	5.8897511	0.11826873	-0.1982254	<b>0.156091</b>
10	0.000728070	5.8898549	0.11826788	-0.1982568	<b>0.156217</b>
11	0.000727891	5.8898903	0.11826778	-0.1982561	<b>0.156218</b>
12	0.000727889	5.8898908	0.11826780	-0.1982563	<b>0.156220</b>

TABLE II. Cycle expansion estimates based on the golden mean approximation (69) to symbolic dynamics for the escape rate  $\gamma$ , average cycle period  $\langle T \rangle$ , Lyapunov exponent  $\lambda$ , average phase velocity  $\langle \dot{\phi} \rangle$  and the diffusion coefficient  $D$ , up to cycle length  $N$ .

$N$	$\gamma$	$\langle T \rangle$	$\lambda$	$\langle \dot{\phi} \rangle$	$D$
1	0.249829963	3.6415122	0.10834917	0.0222352	<b>0.000000</b>
2	-0.011597609	5.8967605	0.10302891	-0.1391709	<b>0.143470</b>
3	0.022614694	4.8899587	0.13055574	-0.1594782	<b>0.190922</b>
4	-0.006065601	6.2482261	0.11086469	-0.2191881	<b>0.157668</b>
5	0.000912644	5.7771642	0.11812034	-0.2128347	<b>0.168337</b>
6	0.000262099	5.8364534	0.11948918	-0.2007615	<b>0.160662</b>
7	0.000017707	5.8638210	0.12058951	-0.2021046	<b>0.160364</b>
8	0.000113284	5.8511045	0.12028459	-0.2006143	<b>0.159233</b>
9	0.000064082	5.8587350	0.12045664	-0.2006756	<b>0.158234</b>
10	0.000093124	5.8536181	0.12035185	-0.2007018	<b>0.158811</b>
11	0.000153085	5.8417694	0.12014700	-0.2004520	<b>0.158255</b>
12	0.000135887	5.8455331	0.12019940	-0.2005299	<b>0.158465</b>

TABLE III. Cycle expansion estimates of the escape rate  $\gamma$ , average cycle period  $\langle T \rangle$ , Lyapunov exponent  $\lambda$ , average phase velocity  $\langle \dot{\phi} \rangle$  and the diffusion coefficient  $D$  with respect to the expansion order  $N$ .

sional.<sup>49</sup>

Table IX and table III shows the cycle averages of the escape rate  $\gamma$ , mean period  $\langle T \rangle$ , leading Lyapunov exponent  $\lambda$ , mean phase velocity  $\langle \dot{\phi} \rangle$  and the diffusion coefficient  $D$  respectively with and without the finite grammar approximation. In the latter, we input all the relative periodic orbits we have found into the expansion (70), whereas in the former, we discarded the cycles with symbol sequence ‘00’.

In Section V.3, we motivated the finite grammar approximation by expecting a faster convergence due to the nearly exact shadowing combinations of the golden mean zeta function (69). This claim is supported by the data in table IX and table III. Take, for example, the Lyapunov exponent which converges to 7 digits for the 12<sup>th</sup> order

expansion when using the finite grammar approximation table IX, only converges to 4 digits at this order in table III. Other observables compare similarly in terms of their convergence in both cases. Note, however, that the escape rate in table IX converges to  $\gamma = 0.000727889$ , whereas in table III it gets smaller and smaller with an oscillatory behavior. This is due to the fact that in the finite grammar approximation, we threw out the part of attractor that corresponds to the cusp of the return map in Fig. 5 (d) above the point that is cut by 001.

In order to compare with the cycle averages, we numerically estimated the leading Lyapunov exponent of the two-mode system using the method of Wolf *et al.*<sup>45</sup> This procedure was repeated 100 times for different initial conditions, yielding a numerical mean estimate of  $\bar{\lambda} = 0.1198 \pm 0.0008$ . While the finite grammar estimate  $\lambda_{FG} = 0.1183$  is within 0.6% range of this value, the full cycle expansion agrees with the numerical estimate. This is not surprising, since in the finite grammar approximation, we discard the most unstable cycles, thus, obtaining a slight underestimate of the Lyapunov exponent while obtaining a significantly better convergence.

## VI. CONCLUSIONS AND DISCUSSION

In this tutorial, we have studied a simple dynamical system which exhibits chaos and is equivariant under a continuous symmetry transformation. We have shown that reducing this symmetry simplifies the qualitative dynamics to a great extent and enables one to find all relative periodic orbits of the systems via standard techniques such as Poincaré sections and return maps. In addition, we have shown that one can extract quantitative information from the relative periodic orbits by computing cycle averages.

We motivated our study of the two-mode system by the resemblance of its symmetry structure to that of the spatially extended systems; and the steps we outlined here are, in principle, applicable to physical systems that are described by  $N$ -Fourier mode truncations of PDEs such as 1D Kuramoto-Sivashinsky,<sup>46</sup> 3D pipe flows,<sup>19</sup> etc..

We showed in Section IV that two-mode dynamics can be completely described by a unimodal return map on the Poincaré section we constructed after continuous symmetry reduction. In a high-dimensional system, finding such an easy symbolic dynamics, or any symbolic dynamics at all is a challenging problem on its own. For desymmetrized (confined in the odd subspace) 1D spatio-temporally chaotic Kuramoto-Sivashinsky system Ref. 47 finds a bimodal return map after reducing the discrete symmetry of the problem, however, for turbulent fluids, we do not know any study that simplifies the flow to such an extent.

In Section V, we have shown that the symbolic dynamics and the associated grammar rules greatly affect the convergence of cycle averaging formulas. In general, finding a finite symbolic description of a flow is rarely as easy

<sup>50</sup> PC: 2014-11-11: this looks wrong. The full state space is 4-dimensional, this diffusion seems to be defined relative to slice, which is 3-dimensional. Have to seriously rethink this - you might well be right.

as it is in our model system. There exist other methods of ordering cycle expansion terms, for example, ordering pseudo-cycles by their stability and discarding terms that are above a threshold;<sup>48</sup> one expects the remaining terms to form shadowing combinations and converge exponentially. Whichever method of term ordering is deployed, the cycle expansions are only as good as the least unstable cycle that one fails to find. Symbolic dynamics solves both of problems at once since it puts the cycles into a topological order so that one cannot miss any accessible cycle and shadowing combinations naturally occur when the expansion is ordered in topological length. The question one might ask is: When there is no symbolic dynamics, how can we make sure that we find all periodic orbits of a flow up to some cycle period?

In the searches of the cycles of high-dimensional flows, one usually looks at the near recurrences of the ergodic flow, and then runs Newton searches starting nearby these recurrences to find if they are influenced by an exact recurrence. Such an approach does not answer the question we just asked to full confidence, however, one may argue that the dynamically important cycles influence recurrences of the ergodic flow, and hence cycles found this way are those that are relevant for computing averages.

To sum up, we have shown that the periodic orbit theory successfully extends to the systems with continuous symmetries. One still needs to think about remaining challenges, discussed above, associated with high dimensional systems. Once these are overcome, we would then be able to extract quantitative information from turbulence by using its exact unstable solutions, which would be big news.

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## Appendix A: Multiple shooting method for finding relative periodic orbits

Let us assume that we have a set of good guesses for a set of state space points, flight times and 1D symmetry

group parameter increments  $\{a_i^{(0)}, \tau_i^{(0)}, \theta_i^{(0)}\}$  such that the points  $\{a_i^{(0)}\}$  lie close to the relative periodic orbit  $p$ ,

$$a_{i+1}^{(0)} \approx D(-\theta_i^{(0)}) f^{\tau_i^{(0)}}(a_i^{(0)}) \quad \text{cyclic in } i = 1, \dots, n. \quad (\text{A1})$$

Here the period and the shift of the relative periodic orbit  $p$  are  $T_p \approx \sum \tau_i$ ,  $\theta_p \approx \sum \theta_i$ . and the Lagrangian description of the flow is  $a(\tau) = f^\tau(a(0))$  We want to determine  $(\Delta a_i, \Delta \tau_i, \Delta \theta_i)$  corresponding to the exact relative periodic orbit,

$$a_{i+1} + \Delta a_{i+1} = D(-\theta_i - \Delta \theta_i) f^{\tau_i + \Delta \tau_i}(a_i + \Delta a_i) \quad \text{cyclic in } i = 1, \dots, n. \quad (\text{A2})$$

To linear order in

$$\begin{aligned} & (\Delta a_i^{(m+1)}, \Delta \tau_i^{(m+1)}, \Delta \theta_i^{(m+1)}) \\ &= (a_i^{(m+1)} - a_i^{(m)}, \tau_i^{(m+1)} - \tau_i^{(m)}, \theta_i^{(m+1)} - \theta_i^{(m)}) \end{aligned} \quad (\text{A3})$$

the improved Newton guess  $(a_i^{(m+1)}, \tau_i^{(m+1)}, \theta_i^{(m+1)})$  is obtained by minimizing the effect of perturbations along the spatial, time and phase directions,<sup>51 53</sup>

$$\begin{aligned} & a_{i+1}' - D_{i+1} f^{\tau_i}(a_i) \\ &= D_{i+1} (J_{i+1} \Delta a_i + v_{i+1} \Delta \tau_i - t_{i+1} \Delta \theta_i), \end{aligned} \quad (\text{A4})$$

where, for brevity,  $a_i^{(m+1)} = a_i^{(m)} + \Delta a_i^{(m)} = a_i'$ ,  $a_i^{(m)} = a_i$ ,  $D(-\theta_i) = D_{i+1}$ ,  $v(a_i(\tau_i)) = v_{i+1}$ ,  $J^{\tau_i}(a_i) = J_{i+1}$ ,  $t(a_i(\tau_i)) = T a_i(\tau_i) = t_{i+1}$ , etc.. For sufficiently good initial guesses, the improved values converge under Newton iterations to the exact values  $(\Delta a_i, \Delta \tau_i, \Delta \theta_i) = (\Delta a_i^{(\infty)}, \Delta \tau_i^{(\infty)}, \Delta \theta_i^{(\infty)})$  at a super-exponential rate.<sup>55</sup> In order to deal with the marginal multipliers along the time and group orbit directions, one needs to apply a pair of constraints, which eliminate variations along the marginal directions on the relative periodic orbits 2D torus: a local Poincaré section orthogonal to the flow, and a local slice orthogonal to the group orbit at each point along the orbit,

$$\langle v(a_i) | \Delta a_i \rangle = 0, \quad \langle t(a_i) | \Delta a_i \rangle = 0. \quad (\text{A5})$$

We can rewrite everything as one matrix equation:

$$A \Delta = E, \quad \text{where,} \quad (\text{A6})$$

$$A = \begin{pmatrix} D_2 J_2 & D_2 v_2 & -TD_2 f^{\tau_1}(a_1) & -\mathbf{1} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ v(a_1) & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ t(a_1) & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & D_3 J_3 & D_3 v_3 & -TD_3 f^{\tau_2}(a_2) & -\mathbf{1} & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & v(a_2) & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & t(a_2) & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ -\mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & D_1 J_1 & D_1 v_1 & -TD_1 f^{\tau_1}(a_1) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & v(a_n) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & t(a_n) & 0 & 0 \end{pmatrix}, \quad (\text{A7})$$

$$\Delta = (\Delta a_1, \Delta \tau_1, \Delta \theta_1, \Delta a_2, \Delta \tau_2, \Delta \theta_2, \dots, \Delta a_n, \Delta \tau_n, \Delta \theta_n)^T, \quad (\text{A8})$$

$$E = (a_2 - D_2 f^{\tau_1}(a_1), 0, 0, a_3 - D_3 f^{\tau_2}(a_2), 0, 0, \dots, a_1 - D_1 f^{\tau_n}(a_n), 0, 0)^T. \quad (\text{A9})$$

We then solve (G55) for  $\Delta$  and update our initial guess by adding the vector of the computed  $\Delta$  values to it and iterate.

## Appendix B: Periodic Schur decomposition

Here we briefly summarize the periodic eigen decomposition<sup>49</sup> needed for evaluation of Floquet multipliers for two-mode periodic orbits. Due to the non-hyperbolicity of the return map of Fig. 5(d), Floquet multipliers can easily differ by 100s of orders of magnitude even in a model as simple as the two-mode system.<sup>57</sup>

We obtain the Jacobian of the relative periodic orbit as the following multiplication of short-time Jacobians from the multiple shooting computation of the previous section:

$$\begin{aligned} \hat{J} &= D_n J_n D_{n-1} J_{n-1} \dots D_1 J_1 \\ &= \hat{J}_n \hat{J}_{n-1} \dots \hat{J}_1 \end{aligned} \quad (\text{B1})$$

where,  $\hat{J}_i = D_i J_i \in \mathbb{R}^{4 \times 4}$ ,  $i = 1, 2, \dots, n$ .

This Jacobian is the same with our previous definition in (6) since  $J_i$  and  $D_i$  commute with each other, and are multiplicative respectively in time and phase. In order to determine the eigenvalues of  $\hat{J}$ , we bring each term

appearing in the product (B1) into periodic, real Schur form as follows:

$$\hat{J}_i = Q_i R_i Q_{i-1}^T, \quad (\text{B2})$$

where  $Q_i$  are orthogonal matrices that satisfy the cyclic property:  $Q_0 = Q_n$ . After this similarity transformation, we can define  $R = R_k R_{k-1} \dots R_1$  and re-write the Jacobian as:

$$\hat{J} = Q_n R Q_n^T. \quad (\text{B3})$$

The matrix  $R$  is block-diagonal, in general, with  $1 \times 1$  blocks for real eigenvalues and  $2 \times 2$  blocks for the complex pairs. It also has the same eigenvalues as  $\hat{J}$ . In our case, it is diagonal since all Floquet multipliers are real in the two-mode system relative periodic orbits. For each relative periodic orbit, we have two marginal Floquet multipliers corresponding to the time evolution direction and the continuous symmetry direction, in addition to one expanding and one contracting eigenvalue.

<sup>52</sup> PC: 2014-02-02 I have changed Burak's (G52) by  $Tf^{\tau_i}(a_i) \rightarrow TD(\theta_i)f^{\tau_i}(a_i)$ . The rest of the formulas need to be changed accordingly.

<sup>54</sup> PC: 2014-02-02 to Burak: please write up the detailed derivation, with what  $O(\Delta^2)$ 's are dropped as a problem / solution set for `cycles.tex`.

<sup>56</sup> PC: 2014-02-02 I am often running into the situation that the linear operators are better marked by the final rather than by the initial points of trajectory segments, as in  $J^{\tau_i}(a_i) = J_{i+1}$ . Implementing this requires a huge rewrite of the ChaosBook

<sup>58</sup> PC: 2014-07-14: cannot find anyplace in the blog numerical value of any of the allegedly very large unstable multipliers.  $\Lambda \approx 80,000$  does not seem so large compared to the numerical precision? I guess it shows I did not have to compute them myself :)

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## Appendix C: Draft write up flotsam

[2012-04-26 Predrag] Draft of Ref. 50 flotsam; most recent clippings at the top.

How to use this section? When you remove cogent text from the article proper - clip it and paste it into here, for possible reuse later.

Previous titles:

*A tutorial on the periodic orbit theory of the systems with continuous symmetries*

*Symmetry reduction of the 1:2 spatial resonance with broken reflection symmetry via the method of slices*

*Cartography of a 4-dimensional flow with a continuous symmetry: - How to slice it*

*Rotational symmetry reduction of a model of mode-interactions via the method of slices*

**2013-07-25 Predrag:** This is an example of Burak's edit, and here is an internal footnote<sup>59</sup> They are useful when you change something n the middle of a text, rather then just append the newest blog post. Please remove the color from my edits as you read them and approve them, and I'll do the same for yours.

**2015-05-03:** I have changed Burak's  $\Delta_\Omega$  to  $\Delta_\omega$ , as  $\Omega^\tau$  depends on time, but  $\omega$  does not.

**2015-05-06:** Why not just use  $\Delta\omega$ ? That notation is pretty typical in quantum mechanics, right? Like when we write the Heisenberg uncertainty principle?

**2014-05-27 Evangelos:** Ref. 10 is somewhat dated and probably nowdays people can find bases for  $d > 12$ , but I have not kept up with the literature. Do you have any newer reference?

**2013-08-14 Predrag:** Your definition of a relative equilibrium is correct for one-parameter  $G$ , but we will have to rethink it for  $N$ -dimensional Lie group  $G$ . In that case the time trajectory is a 1-dimensional curve, generically tracing out the  $N$ -dimensional group orbit quasiperiodically. Probably a better definition (click on magenta hyperlinks!) would be that  $v(a_a) \in \mathcal{M}_{EQ}$ , and than show that this is then true anywhere on the group orbit. Perhaps using Lie derivatives...

**2013-08-13 Burak: Invariant solutions:** <sup>61</sup> A point  $a_{EQ}$  is an equilibrium if the velocity function evaluated at this point is 0, namely,  $\dot{a}|_{a=a_{EQ}} = v(a_{EQ}) = 0$ . Orbit of a point  $a$  of a  $G$ -equivariant flow is called a relative equilibrium if for any time evolved point on the orbit  $a(\tau) = f^\tau(a_{REQ})$  there exists a parameter  $\phi$  such that  $a(\tau) = G(\phi)a$ , in other words, a time orbit is a relative equilibrium if it coincides with the group orbit. A point  $a$  is on a periodic orbit if its trajectory passes through itself after some certain time  $T$ , namely  $a = f^T(a)$ . Finally, a point  $a$  of a  $G$ -equivariant flow is said to be on a relative periodic orbit if there exists a point  $a(T) = f^T(a)$  on the orbit of  $a$ , such that it satisfies  $a(T) = g(\phi)a$  for a certain  $\phi$ .

[2013-09-30 Predrag] Experimenting with how to compare parameter sets, starting with the set (G19)

$$\begin{aligned} \mu_1 &= -2.8023, \mu_2 = 1, a_1 = -1, a_2 = -2.6636 \\ b_1 &= 0, b_2 = 0, c_1 = -4.1122, c_2 = 1.8854, e_2 = 1. \end{aligned} \quad (\text{C1})$$

	(G19)	.
$\mu_1$	-2.8023	.
$\mu_2$	1	.
$e_1$	0	0
$e_2$	1	.
$a_1$	-1	.
$a_2$	-2.6636	.
$b_1$	0	.
$b_2$	0	.
$c_1$	-4.1122	.
$c_2$	1.8854	.
$\#_{eqv}$	.	.

<sup>60</sup> BB An internal footnote by Burak.: [.](#)

<sup>62</sup> 2CB

ES2014-05-27: Dropped this from intro: the reconstruction equations developed in Ref. 12 and formulated them using a ‘rescaled time’ variable. Replaced with: Later on, Rowley *et al.*<sup>29</sup> generalized the method in order to handle self-similar solutions. – Consider returning to intro if we talk about rescaled time in more detail.

The basic equivariants include

$$\{z_1, \bar{z}_1 z_2\}, \quad \{z_2, z_1 \bar{z}_2\}. \quad (\text{C2})$$

We have equations (20) for  $\{z_1, \bar{z}_1, z_2, \bar{z}_2\}$  but perhaps need equations for the equivariants (??) - have not thought this through. [ChaosBook.org says:]

Differential operators acting on functions defined on a periodic domain usually diagonalized by representing the solution as a Fourier expansion.

<sup>63</sup> As at least 3 dimensions are required for a continuous time flow to exhibit chaos, in this case the state space of a  $G$ -equivariant flow has to be at least 4-dimensional. Under linear actions of  $\text{SO}(2)$  the state space decomposes into 2-dimensional irreducible subspaces (Fourier components) labeled by integers  $m = 0, 1, 2, \dots$ , which thus form the natural basis in which to study  $\text{SO}(2)$ -equivariant flows. Nonlinear flows, such as the 1 spatial dimension Kuramoto-Sivashinsky PDE for a ‘flame front velocity’ field  $u = u(x, t)$  on a periodic domain  $u(x, t) = u(x + L, t)$ , given by

$$u_t = F(u) = -\frac{1}{2}(u^2)_x - u_{xx} - u_{xxxx}, \quad x \in [-L/2, L/2], \quad (\text{C3})$$

can be expressed in terms of complex Fourier coefficients  $a_k(t)$ ,

$$u(x, t) = \sum_{k=-\infty}^{+\infty} a_k(t) e^{iq_k x}, \quad q_k = 2\pi k/L, \quad (\text{C4})$$

as

$$\dot{a}_k = v_k(a) = (q_k^2 - q_k^4) a_k - i \frac{q_k}{2} \sum_{m=-\infty}^{+\infty} a_m a_{k-m}. \quad (\text{C5})$$

( $t \geq 0$  is the time,  $x$  is the spatial coordinate, subscripts  $x$  and  $t$  denote partial derivatives with respect to  $x$  and  $t$ ). Nonlinear terms mix an infinity of Fourier components. In practice, they are represented by truncations to a finite number of Fourier modes; the most radical truncation that still might capture some qualitative features of a chaotic flow is to keep only a pair of Fourier modes.

$$\begin{aligned} w &= -\frac{2u}{c_1} A_1 = -\frac{2v}{c_2} A_2 \\ \rightarrow \quad 2A_1 + A_2 &= -\frac{w}{2uv} (c_2 u + 2c_1 v) \\ q &= \frac{1}{-2e_1 + e_2} (-w^2/2uv + 2u) (c_2 u + 2c_1 v) \\ w &= -\frac{1}{-2e_1 + e_2} (2A_1 + A_2) q, \quad \rightarrow \quad q = \frac{2(-2e_1 + e_2)uv}{c_2 u + 2c_1 v}, \end{aligned} \quad (\text{C6})$$

and thus we obtain **our main result**, two (bivariate) polynomials in two variables  $\{u, v\}$  with constant coefficients

$$\begin{aligned} f(u, v) &= c_2 u (\mu_1 + a_1 u + b_1 v) - c_1 v (\mu_2 + a_2 u + b_2 v) = 0, \quad \deg(f) = 2 \\ g(u, v) &= (w^2 - 4u^2v) (c_2 u + 2c_1 v)^2 + 4(-2e_1 + e_2)^2 u^2 v^2 = 0, \quad \deg(g) = 6, \end{aligned} \quad (\text{C7})$$

where  $w$  is defined by the first equation in (34).

Lots of coefficients, but we can absorb some into rescaled quantities  $\tilde{u} = c_2 u$ ,  $\tilde{v} = c_1 v$ ,  $\tilde{a}_1 = a_1/c_2$ ,  $\tilde{b}_1 = b_1/c_1$ ,  $\tilde{a}_2 = a_2/c_2$ ,  $\tilde{b}_2 = b_2/c_1$ , and by using  $A_1$  expression for  $w$  factored out the pair of  $u = 0$  roots:

so we are down to 8 coefficients. Note that  $e_2 \in \mathbb{R}_+$ , and if we agree that  $\mu_1 > -\mu_2 > 0$  we can rescale the first equation so that in new time units  $\tilde{\mu}_1 = 1$ ,  $\tilde{\mu}_2 = \mu_2/\mu_1 \in \mathbb{R}_-$ , so there are 7 coefficients in all. I see no further rescaling simplification.

<sup>64</sup> PC: notation for the translation group?

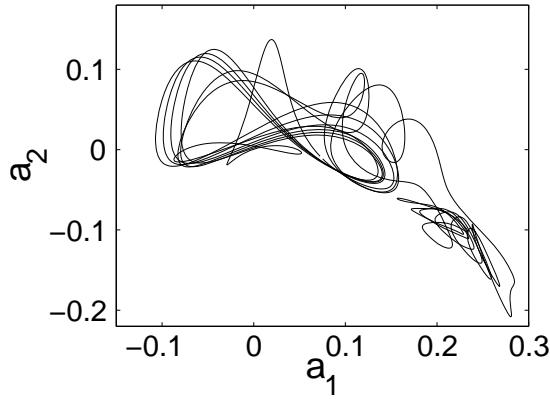


FIG. 8. A long Kuramoto-Sivashinsky periodic orbit of period  $T = 355.34$  that connects neighborhoods called ‘ $S_C$ ’ and ‘ $S_R$ ’, (c)  $[a_1, a_2]$  projection on the first two spatial Fourier modes (from Ref. 47).

As we already know  $[0, 0, 0, 0]$  and  $[0, v, 0, 0]$  roots, one should divide them out, and that is what we have done. Finding roots of bivariate polynomials is not easy.

**[2014-05-06 Burak]** I'll try to rewrite the abstract from the beginning, here is its current version:  
Dangelmayr<sup>31</sup> and Porter & Knobloch<sup>30</sup> have introduced a family of 2-Fourier mode  $\text{SO}(2)$ -equivariant ODEs in order to study bifurcations of solutions of dynamical systems in presence of symmetries. A 4-dimensional system of this kind is perhaps the simplest example of a system with a continuous symmetry that can exhibit chaos, so we use it to illustrate the role symmetries play in chaotic dynamics. We show that a continuous symmetry induces drifts in the full state space dynamics, drifts which obscure the chaotic dynamics. Change of equations of motions to a comoving frame does not eliminate these drifts: that is only attained by a *symmetry reduction* - reformulation of dynamics in a 3-dimensional symmetry-reduced state space, where every group orbit (set of all points reached by actions of the group of all symmetries of the equations of motion) is replaced by a point. Two-mode system is a particularly nice illustration of how this works, as in 3 dimensions we are able to visualize everything.

We compare three symmetry reduction methods: polar coordinates, invariant polynomial bases, and the ‘method of slices’. An invariant polynomial basis is convenient for determination of all relative equilibria of such system. Our conclusion, however, is that the most insight is offered by the method of slices. While in general a number of local slices are needed to cover a strange attractor<sup>16</sup>, for the two-mode system there we define a unique slice hyperplane that captures *all* symmetry-reduced dynamics. *I think this is misleading since we have shown that single slice treatment works fine for Kuramoto-Sivashinsky*

A Poincaré return map within the slice hyperplane enables us to reduce the dynamics further, essentially to a unimodal map, and determine, in principle, all relative periodic orbits of the system. We can visualize each step of this process without having to project solutions onto a submanifold since the slice hyperplane for this system is three dimensional.

**[2014-05-06 Burak]** Previously was in introduction, later commented out:  
Over the last decade, new insights into the dynamics of [blah blah]

Our goals here are two-fold: <sup>65</sup> (i) Illustrate method of slices in the lowest-dimensional setting possible. (ii) [blah blah].

As a motivation, consider the chaotic dynamics exhibited by the small-cell Kuramoto-Sivashinsky system studied in Ref. 47. Examination of typical long-time simulations shows that the spatio-temporal chaos arises from visits to two kinds of unstable patterns, a ‘central wobble’ region  $S_C$ , and a symmetric pair of right/left ‘drifts’  $\{S_L, S_R\}$ . In state space projections orbits stay in one neighborhood for a while, then hop to another neighborhood, as illustrated in Fig. 8. The strange attractor that they explore is curved and folded in such a way that a single local linear chart cannot cover the whole attractor, several charts are needed, as illustrated by Fig. 9 (d).

The state spaces of Kuramoto-Sivashinsky and fluid-dynamical flows are high-dimensional and difficult to visualize, so here we shall illustrate the key ideas by a much simpler example, the  $\text{SO}(2)$ -equivariant two-mode system.

[blah blah]

**[2014-05-06 Burak]** Previous outline items:

<sup>66</sup> PC: [2013-10-07] Burak's and Daniels outline of Das Artikel is in Fig. 50.

SO2-equivariant equations:

$$\begin{aligned}\dot{x}_1 &= (\mu_1 - r_1^2 + c_1 x_2) x_1 + c_1 y_1 y_2 \\ \dot{y}_1 &= (\mu_1 - r_1^2 - c_1 x_2) y_1 + c_1 x_1 y_2 \\ \dot{x}_2 &= (1 + a_2 r_1^2) x_2 + (x_1^2 - y_1^2) + y_2 \\ \dot{y}_2 &= (1 + a_2 r_1^2) y_2 + 2x_1 y_1 - x_2\end{aligned}\text{where } r_1^2 = x_1^2 + y_1^2, \quad r_2^2 = x_2^2 + y_2^2. \quad (\text{C8})$$

## 1. To do

10.11 Visualizations of the 4-dimensional two-mode system

10.1? draw a group orbit for the two-mode model

10.22 Two-mode system in polar coordinates (maybe skip)

10.23 The relative equilibria of the two-mode system

10.24 Plotting the relative equilibria of the two-mode system in invariant coordinates

10.25 Plotting the relative equilibria of the two-mode system in Cartesian coordinates (21)

10.2? construct a 2-chart atlas Fig. 9 for a two-mode system

- compute analytically the stability matrix  $A$  in polar coordinates
- Study eigenvalues, keep playing with parameters. We would like -preferably- no relative equilibrium to be attracting limit cycle, and several of the relative equilibria to be complex-pair unstable, leading to chaos, to be visualized and sliced in Cartesian coordinates.
- If you find a nice chaotic attractors, others can join in constructing an atlas for it. We just need one and only one example with non-trivial chart borders and at least 2 charts.

[blah blah]

- $TW_1 = (r_1, r_2, \psi) = (0.0516508, 1.26311, ?)$  and  $TW_2 = (0.467095, 0.2146, ?)$
- their plots in the Cartesian coordinates
- $\dot{\theta}$  to see how slow/fast are they.  $\dot{\theta}$  might be related to 4th eigenvalue, when you go back to Cartesian coordinates
- stability eigenvalues, eigenvectors of the equilibrium  $EQ_0$  at origin, at your parameter values - if it is stable, everything just might fall into it and die.
- plots of small perturbations of the above equilibrium and relative equilibria in the Cartesian coordinates to see whether the dynamics looks chaotic
- $TW_1$ : 2 large positive eigenvalues looks scary - probably nothing re-visits this relative equilibrium. A mildly unstable complex pair would have been sweeter. You get complex eigenvalue by Hopf-bifurcating off a stable orbit, typically.
- $TW_1$ : Does either unstable eigenvalue become a complex eigenvalue pair in Cartesian coordinates?
- $TW_2$ : contracting eigenvalues have very small imaginary part, so the presumably just rocket toward the relative equilibrium, not much spiraling there. At least the unstable eigenvalue seems slow compared to all other eigenvalues.
- $TW_1$ : Does the unstable eigenvalue become a complex eigenvalue pair in Cartesian coordinates?

[blah blah]

2011-09-09, 2012-03-30 Predrag: add BeThMovFr to continuous.tex overheads, and ChaosBook replace A27movFrame\*.\* everywhere

[blah blah]

[blah blah]

(a) (b) (c) (d)

FIG. 9. Two-mode,  $d = 4 \rightarrow 3$  dimensional  $\{x_1, x_2, z\}$  projections: (a) The strange attractor. (b) (c) In contrast to the 1-dimensional section borders of Fig. ??, here ... (d)

## Appendix D: Chart

[blah blah]

One can write the equations for the flow in the reduced state space  $\dot{\hat{a}} = \hat{v}(\hat{a})$  (for details see, for example, Ref. 8) as

$$\hat{v}(\hat{a}) = v(\hat{a}) - \dot{\theta}(\hat{a}) t(\hat{a}) \quad (\text{D1})$$

$$\dot{\theta}(\hat{a}) = \langle v(\hat{a}) | t' \rangle / \langle t(\hat{a}) | t' \rangle \quad (\text{D2})$$

which confines the motion to the slice hyperplane. Thus, the dynamical system  $\{\mathcal{M}, f^\tau\}$  with continuous symmetry  $G$  is replaced by the reduced state space dynamics  $\{\hat{\mathcal{M}}, \hat{f}^\tau\}$ : The velocity in the full state space  $v$  is the sum of  $\hat{v}$ , the velocity component in the slice hyperplane, and  $\dot{\theta}t$ , the velocity component along the group tangent space. The integral of the *reconstruction equation* for  $\dot{\theta}$  keeps track of the group shift in the full state space.

[blah blah]

[2014-05-07 Burak] Taken out from Section II.1:

Fig. 2 shows a 3D projection of the 4D state space corresponding to an  $m = 2$  truncation of (10), for which (11) and (13) are  $4 \times 4$  matrices. The slice hyperplane defined by (14), three different group orbits and the group tangents evaluated at their intersections with the slice hyperplane are visualized in Fig. 2. One can see as the magnitude of the second mode (the vertical axis) relative to the first mode increases, the group tangent gets closer to being parallel to the slice hyperplane, however, it still has a non zero perpendicular component. The vertical axis ( $x_2$ ) in Fig. 2 lies on the slice border of the slice hyperplane.

[2014-06-25 Burak] Taken out from Section ??:

We start with the first set of parameters<sup>67</sup>, (22).

Starting close to the relative equilibrium  $TW_1$ , we integrate the  $SO(2)$ -equivariant equations (21) for 500 time units and plot two projections of the 4D state space in Fig. ?? (a) and Fig. 3 (b). In order to compare the symmetry reduction techniques, we plotted the corresponding flow in the invariant polynomial basis on Fig. 3(c) and the symmetry reduced flow using method of slices on Fig. ???. While Fig. ?? is generated by simply integrating (30), we obtained Fig. ?? by integrating (8) within the slice hyperplane of the template,

$$\hat{a}' = (1, 0, 0, 0) \quad (\text{D3})$$

with the same initial condition  $x_0$  (note that it satisfies (7) for (D3) and (24)) and

The Poincaré section plane in Fig. ?? includes the origin<sup>69</sup> and is perpendicular to

$$\hat{n}_{0,GS} = (0, -0.54030, 0.84147) \quad (\text{D4})$$

relative periodic orbits and their binary itineraries, the two shortest cycles  $\bar{1}$  and  $\bar{0}\bar{1}$  are plotted in Fig. 10.

[2014-06-25 Burak] Taken out from :

An immediate generalization of the transformation (19), is to fix the phase of a higher Fourier mode rather than that of the first one; this, however, requires some extra care as we shall explain for the second Fourier mode. Consider the phase-fixing transformation,

$$\hat{z}_n = e^{-in\frac{\phi_2}{2}} z_n, \quad (\text{D5})$$

which fixes the phase of the second mode to 0. Note, however, that since  $\phi_2 \in (0, 2\pi]$  it can have discontinuities of  $2\pi$ . This means that when the first Fourier mode is transformed, it will have a phase discontinuity of  $\pi$ . This discontinuity

<sup>68</sup> ES 2014-05-15: Will you also show the second set? If not, you might as well drop it and stick with a single set. If there is

nothing interesting to be seen, then maybe keep it simple.  
<sup>70</sup> PC: ??

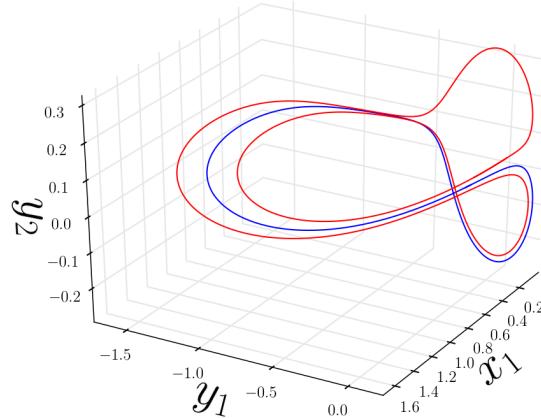


FIG. 10. Relative periodic orbits  $\bar{T}$  and  $\bar{O}\bar{T}$  embedded in the strange attractor of Fig. 5 (a).

can be fixed by another transformation:

$$\begin{aligned}\tilde{z}_1 &= e^{-i2\hat{\phi}_1} |\hat{z}_1|, \\ \tilde{z}_{n \neq 1} &= \hat{z}_n,\end{aligned}\quad (\text{D6})$$

where we simply doubled the phase  $\hat{\phi}_1$  of the symmetry-reduced first mode  $\hat{z}_1$ , obtained by (D5) and left the rest of the modes unchanged. Combination of (D5) and (D6) is a valid symmetry reduction scheme since every group orbit is represented by a single point, furthermore, it is also continuous and revertible hence one can make further computations, such as constructing Poincaré sections, using this form. For the 2-mode case at hand, representation (D6) does not have any particular advantage against (19), however, for higher dimensional flows, second (or higher) Fourier mode subspaces can have dynamical importance as shown in Ref. 46. In order to capture those regions of the state space this representation would be a useful alternative.

[2014-07-06 Burak] Previous abstract:

In nonlinear flows drifts along symmetry directions often obscure the physical, ‘shape changing’ dynamics. Symmetry reduction eliminates such drifts through a coordinate transformation that parametrizes motions along symmetry directions by ‘phase’ coordinates and maps symmetry-related state space points to single points in a symmetry-reduced state space.

Here, we illustrate and compare three different symmetry reduction methods: polar coordinates, invariant polynomial bases, and the ‘method of slices’ by applying them to a simple dynamical system - a chaotic flow based on the normal form of the 1:2 spatial resonance with broken reflection symmetry<sup>71 73</sup> - that can be thought of as a truncation of a spatially extended PDE. We find that the method of slices offers the most insight. Because the flow is four dimensional, the slice hyperplane for this system is three dimensional, which allows the visualization of each step of the symmetry-reduction process without having to project the dynamics onto lower-dimensional submanifolds. A Poincaré return map within the slice hyperplane enables further reduction of the dynamics to what is effectively a unimodal map, allowing (in principle) the determination of all relative periodic orbits of the system.

[2014-07-07 Burak] Previous introduction:

Recent experimental observations of travelling waves in pipe flows have confirmed predictions based on dynamical systems theory that coherent structures play an important role in shaping the state space of turbulent flows<sup>74</sup>. Identifying such solutions can lead to a better understanding of the transition to turbulence in fluid flows and may, in the long-term, allow the computation of dynamical averages using periodic orbits. In order to find coherent solutions such

<sup>72</sup> DB 2014-05-15: Is there anything called a normal form flow?

I don't think so but I could be wrong. I know 'normal forms' and I know 'flows' but not 'normal form flows'. Referring to 1:2 resonance might also peak a certain crowd's interest, even though we don't really do anything in a regime that they would care about. Talking about broken symmetry is sexier than talking

about SO(2).

<sup>74</sup> ES 2014-05-20: Talking about broken symmetry in the title of a paper on symmetry reduction could also be misleading. Important is not which symmetry is broken, but which symmetry we reduce, right? 1:2 resonance sounds cryptic to me, I would prefer to say mode interaction. Have a look at edited title.

as relative equilibria and relative periodic orbits it is essential, as we shall demonstrate, to use continuous symmetry reduction.

One well-studied technique for symmetry reduction, which works well for low-dimensional dynamical systems such as Lorenz and Rössler flows, is to recast the dynamical equations in terms of invariant polynomials<sup>9</sup>. However, establishing an invariant polynomial basis becomes impractical for systems with more than 12 dimensions<sup>1075</sup>. The method of slices<sup>12–18</sup>, which we study in detail here, offers a symmetry reduction scheme applicable to high-dimensional flows like the Navier-Stokes equations.

While the method of slices goes back to Cartan<sup>20</sup>, to the best to our knowledge, Rowley and Marsden<sup>12</sup> were first to apply it to spatially extended nonlinear flows. They demonstrated their method by applying it to the one-dimensional Kuramoto-Sivashinsky equation in a periodic domain and used it to study its dynamics in the neighborhood of a relative equilibrium by using the relative equilibrium itself as the slice ‘template’. Later on, Rowley *et al.*<sup>29</sup> generalized the method in order to handle self-similar solutions. Beyn and Thümmler<sup>13</sup> applied the method of slices to ‘freeze’ spiral waves in reaction-diffusion systems.

These early studies applied the method of slices to a single solution at a time, but studying the nonlinear dynamics of extended systems requires symmetry reduction of global objects, such as strange attractors or invariant manifolds. In this spirit, Siminos and Cvitanović<sup>14</sup> used the method of slices to quotient the SO(2) symmetry of chaotic dynamics in complex Lorenz flow and showed that the slice-dependent singularity of the reconstruction equation causes the reduced flow to make discontinuous jumps. This singularity was studied in detail by Froehlich and Cvitanović<sup>15</sup>.

Two strategies have been proposed in order to handle this problem: either to try to identify a template such that slice singularities are not visited by dynamics<sup>14</sup> or to use multiple ‘charts’ of connected slices<sup>12,15</sup>. The latter approach was applied to complex Lorenz flow by Cvitanović *et al.*<sup>16</sup> and to pipe flow by Willis, Cvitanović, and Avila<sup>17</sup>. However, neither approach is straightforward to apply, particularly in high-dimensional dynamical systems. More recently, Budanur *et al.*<sup>18</sup> considered Fourier space discretizations of PDEs, in which translational symmetry naturally appears as the rotation group SO(2). They showed that a simple choice of slice, associated to the first Fourier mode, results in a slice singularity that generic dynamics is extremely improbable to visit.<sup>77</sup> Close-passages to the singularity were regularized by means of a time rescaling. Here, we follow this approach and apply this so-called “first Fourier mode slice” method to a 2-mode ODE normal form, which may be the simplest system with SO(2) equivariant dynamics that exhibits chaos.

[2014-07-14 Burak] Taken out from Section III.2:

In order to find relative equilibria of the two-mode system, one has to solve two bivariate polynomials (36) which, in general, is not a trivial task. However, as we shall see in the examples of the next section, for particular choices of parameters, equations(36) symplify significantly allowing us to determine all relative equilibria of the two-mode system.

[2014-07-14 Burak] Taken out from Section IV:

To illustrate the method of slices on the two-mode system we choose a simple set of parameters for which we observe interesting dynamics. These parameters are listed in (22). With this set of parameters, we can write two-mode ODEs (20) in terms of three parameters  $\{\mu_1, c_2, a_2\}$ :

$$\begin{aligned} z_1 &= \mu_1 z_1 - z_1 |z_1|^2 + c_1 \bar{z}_1 z_2 \\ z_1 &= (1 - i) z_2 + a_2 z_2 |z_1|^2 + z_1^2, \end{aligned} \quad (\text{D7})$$

Note that by setting  $b_2 = 0$ , we send the relative equilibrium at  $(0, -\mu_2/b_2, 0, 0)$  to infinity. Moreover, (36) yields  $\tilde{v} = (\mu_1 + \tilde{a}_1 \tilde{u})/(\mu_2 + \tilde{a}_2 \tilde{u} - \tilde{u} \tilde{b}_1)$ . Substitution into (37) allows one to solve for a single variable. By solving (35) with the parameter set (22), we get two real roots, with non-negative  $u$  and  $v$ :

$$p_{EQ} = (0, 0, 0, 0)^T$$

which is a double root and corresponds to an equilibrium of (20), and

$$p_{TW} = (0.193569, 0.154131, -0.149539, -0.027178)^T,$$

<sup>76</sup> ES 2014-05-27: Ref. 10 is somewhat dated and probably nowadays people can find bases for  $d > 12$ , but I have not kept up with the literature. Do you have any newer reference?

<sup>78</sup> ES 2014-06-03: rephrased this to make easier to follow for the

casual reader: followed a geometrical approach and showed that for SO(2) a specific choice of a slice template can be used to move the slice singularity to the border of the symmetry-reduced submanifold.

which is a relative equilibrium. In real representation, a representative point on  $TW$  may be chosen as

$$(x_1, y_1, x_2, y_2) = (0.439966, 0, -0.386267, 0.070204)$$

[2014-07-15 Burak] Taken out from Section ??:

<sup>79 81</sup>

[2014-07-15 Burak] Took out from the lead paragraph: Burak, have a look at Chaos Journal articles. This section is required, and most of your abstract goes into it, the abstract itself should be concise: “Today, it is possible to [blah blah].”

[2014-07-15 Burak] Abstract before division:

Periodic orbit theory provides estimates for dynamical averages, such as dissipation rates or diffusion constants, of nonlinear flows by means of cycle averaging formulas which relate the spectra of observables to the spectra of unstable periodic orbits. These formulas are valid under the assumption that the periodic orbits have a single marginal direction along the time evolution, and hyperbolic in all other directions. Dynamical systems with continuous symmetries, however, have relative periodic orbits which are invariant ( $N + 1$ ) dimensional tori, where  $N$  is the number of continuous symmetries. These systems arise in the study of turbulent flows, such as Navier-Stokes in a pipe or plane Couette flow, where one often imposes periodic boundary conditions along the stream direction and truncates the corresponding infinite Fourier series to a finite but large number (from tens to thousands) Fourier modes. In this paper we study a ‘two-mode’ model of this type, the smallest possible truncation with a 4-dimensional state space, which has the same continuous symmetry structure as the 1D PDE while just high-dimensional enough to allow for chaotic dynamics. The crucial step in analysis of such systems is symmetry reduction, here a coordinate transformation that separates physical, ‘shape changing’ dynamics from the drifts along the symmetry direction. We start by reviewing continuous symmetries and symmetry reduction methods with a focus on the ‘method of slices’, which, to the best of our knowledge, is the only symmetry reduction method that can be applied to the infinite dimensional problems. We then define our two-mode  $\text{SO}(2)$ -equivariant model, compare different symmetry-reduction schemes, and determine its relative equilibria using invariant polynomials. We show that a Poincaré section within the ‘slice’ can be used to further reduce this flow to what is for all practical purposes a unimodal map; and hence we can find all relative periodic orbits and their binary symbolic dynamics up to any desired period. We finally compute dynamical averages using relative periodic orbits, and discuss convergence of the spectral determinants.

[2014-08-04 Burak] Took out from Section D:

We immediately see that the reconstruction phase in (8) is, for this particular slice, the phase of the first mode without the symmetry reduction.<sup>83 85 87 89</sup> As the magnitude of the first mode,  $\sqrt{\hat{x}_1^2 + \hat{y}_1^2}$ , relative to that of the second mode becomes smaller, its group tangent has a larger component parallel to the slice hyperplane. The slice border condition (9) is satisfied when the group tangent becomes parallel to the slice hyperplane.<sup>91</sup>

[2014-08-05 Burak] Took out from Section III:

Our goal is to illustrate and compare continuous symmetry reduction methods applicable to high-dimensional systems exhibiting chaos.

[2014-10-30 Burak] From introduction: This mode mixing phenomenon<sup>93 95 97</sup>

Section *Post-processing approach* refers to the “method of moving frames” (postprocessing approach) and as such belongs here. It has to be generalized a bit (I will do it if you agree with the change). Second mode slice can be used also for integration on the slice and can be introduced earlier.

[2014-05-15 Evangelos] Why do you count complex solutions as equilibria?

<sup>80</sup> PC: 2014-07-14 Why  $\pm L/2$  in Fig. 3 and Fig. ?? and not  $\pm\pi$ ?

<sup>82</sup> ES 2014-05-15: I have replaced the second-mode slice, double-angled figure in Fig. ??(b) with one resulting by integrating on the  $(0, 0, 1, 0)$  slice, for consistency with panel (c). I hope Burak will replace it with a publication quality figure of the same representation. The trick of angle doubling will be introduced in its own section.

<sup>84</sup> DB 2014-05-15: I don’t immediately see what we are saying here. How does this relate to (8)? I think this needs better explanation.

<sup>86</sup> ES 2014-05-15: It’s not straightforward to see this, also for me, unless I think of the postprocessing approach. So I have added (18) above. Please check and discuss in the blog if needed.

<sup>88</sup> DB 2014-05-15: This phrase has no meaning.

<sup>90</sup> BB 2014-05-15: Is it better now?

<sup>92</sup> DB 2014-05-15: Don’t love my wording of “parallel to the slice hyperplane” since some people might interpret the slice hyperplane as being defined by a vector normal to its surface and hence a perpendicular to the group tangent at the slice border. However, I think that “component that lies within the slice” is sort of vague and not well defined mathematically. What do you guys think?

<sup>94</sup> DB 9/16/2014: Is the mixing really the source of relative periodic orbits and relative equilibria? It seems to me that this statement isn’t necessarily true right?

<sup>96</sup> BB 9/17/2014: You are right, mode mixing is just a consequence of the nonlinearity, not necessarily the cause of the relative periodic orbits and relative equilibria. You can change as you see fit

<sup>98</sup> BB 10/30/2014: Changed myself.

[2014-11-07 Burak] Took out the partial derivative from this sentence (Mohammad's suggestion: 'Everybody knows the chain rule.'):

$$\dot{u}_i = \sum_j (\partial u_i / \partial a_j) \dot{a}_j,$$

More explicit form, does not fit in a column:

$$g(\theta) = \begin{pmatrix} \cos \theta & \sin \theta & 0 & 0 & \cdots & 0 & 0 \\ -\sin \theta & \cos \theta & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cos 2\theta & \sin 2\theta & \cdots & 0 & 0 \\ 0 & 0 & -\sin 2\theta & \cos 2\theta & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \cos m\theta & \sin m\theta \\ 0 & 0 & 0 & 0 & \cdots & -\sin m\theta & \cos m\theta \end{pmatrix}$$

[2014-11-08 Predrag] clipped some commented text from the manuscript and to here:

author Keith M. Carroll

author Bryce Robbins

author Evangelos Siminos

author Lei Zhang

[2014-09-16 Daniel] Should stability matrices and Jacobians be bolded like  $T$ ?

[2014-09-16 Burak] Tis bolded because in general it is  $T = (T_1, T_2, \dots, T_N)$ . Chaosbook convention is to use small letters for the vectors in the state space, capital letters for the matrices that act on the state space vectors (stability matrix, Jacobian etc), and bold capital letters for set of matrices like  $T$ . Current notation here is not perfect. In general we have  $N$  group parameters that determines the group element as  $g(\theta) = e^{\theta \cdot T}$ . I think easiest way to get rid off this ambiguity is to say that we are formulating everything for a single parameter group and than remove 'bf' from the macro for  $T$ .

[2014-11-09 Predrag] Daniel is right, ChaosBook.org choice of fonts is inconsistent. Now I've revamped all matrices in blackboard font, but not the vectors that really does not look good.

[2014-11-11 Daniel] There is some notational nastiness in the averages section, since there are  $a'$ 's floating around. We have previously used  $a'$  for the template and used primes in general mark template related things.

[2014-11-11 Predrag] Grin and bear it.

[2014-11-11 Daniel] What is  $T$  here... at this point in the paper  $T$  has been used to represent both period of cycles and Lie group generator. Okay... reading further it's clearly a period, but we may want to fix this notational ambiguity.

[2014-11-11 Predrag] We are lucky that Chinese did not invent modern mathematics. Otherwise we would have a unique ideogram for every concept.

[2014-11-11 Daniel] We say "only the combination  $(2e_1 - e_2)$  matters in the symmetry reduced dynamics, so for simplicity we set  $e_1 = 0$ " and then never really do it in any of the analysis except in numerics. Should we get rid of  $e_1$  everywhere or keep it?

[2014-11-11 Predrag] might revisit this in the next revision, no time for it now...

[2014-11-11 Daniel] What does  $\simeq$  mean in this context? Aren't they actually the same?

[2014-11-11 Predrag] They are isomorphic, but defining reps are different - one is 1d complex, the other 2d real

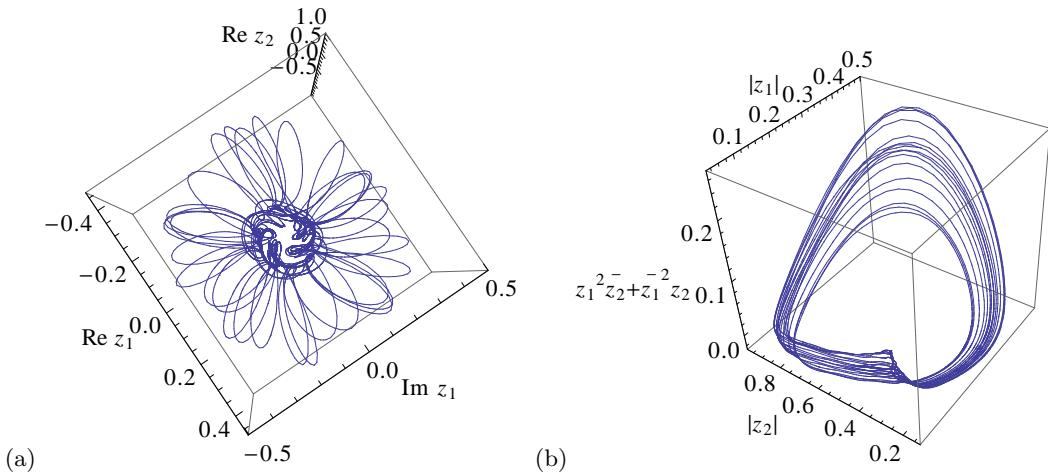


FIG. 11. Projections of Dangelmayr system (20) “attractor” for parameters (E1) in (a) equivariant state space coordinates  $\{x_1, y_1, x_2, y_2\}$  (b) invariant coordinates  $\{\sqrt{u}, \sqrt{v}, w\}$

## Appendix E: Two-mode simulations blog

### 1. Blogging simulations progress

**2012-06-11 Predrag:** started a new blog, for Chaos Gang two-mode simulations.

`siminos/blog/Mathematica.tex`

**2012-04-25 Evangelos:** Wrote an interactive Mathematica program, with sliders for parameters:

`siminos/cgang/Evangelos/dangelmayr_so2_int.nb`

**2012-04-25 Evangelos:** Dangelmayr system with Predrag’s modification becomes very interesting. It’s very easy to find chaotic behavior and in many cases it seems that asymptotic dynamics are 3-dimensional. After spending some time playing with parameters, I’ve found some for which it seems we have chaos (at least we can see some stretching and folding) while a 2-dimensional description might be possible (a simple 2-d branched manifold). The parameters are

$$\begin{aligned} \mu_1 &= -0.337, \mu_2 = 0.27, c_1 = 1, c_2 = -1, a_1 = -1.5, \\ a_2 &= -6.19, b_1 = 1.583, b_2 = -0.115, e_2 = 1.211. \end{aligned} \quad (\text{E1})$$

I would suggest playing with `siminos/cgang/Evangelos/dangelmayr_so2_int.nb` to locate *nearby* parameters which you think will give more interesting results or just explore the system with the parameters given here.

**2012-04-21 Evangelos:** The Dangelmayr system<sup>31</sup> written in complex coordinates  $z_1, z_2$  is given in (20) I have replaced symmetry breaking term  $e$  here and in (F6) with  $e_2$  since in this constant is naturally paired with  $\mu_2$ . See also Armbruster, Guckenheimer and Holmes<sup>32</sup> flow with  $O(2)$  symmetry, here (F1). In `siminos/cgang/Evangelos/dangelmayr_so2_int.nb` I integrate (20) rather than the polar form (F6), as the former has no dangerous denominators.

**2012-04-25 Evangelos:** Yet another snapshot of two-mode system with Predrag’s modification, for different parameters. This looks like an attractor made of two pieces, so it might be complicated enough to demonstrate slicing. However I have yet no insight on whether there are really two distinct topological mechanisms hidden here and what controls them. Once relative equilibria are located, we should add them here. The parameters are

$$\begin{aligned} \mu_1 &= -0.337, \mu_2 = 0.27, c_1 = 1, c_2 = 1, a_1 = -1.4, \\ a_2 &= -6, b_1 = 1.6, b_2 = -0.1, e_2 = 1.217, e_1 = 0. \end{aligned} \quad (\text{E2})$$

**2012-08-10 Predrag:** Not a good choice - please follow Chossat<sup>51</sup> and always use  $c_1 c_2 < 0$ , as suggested by [2012-08-06 Knobloch].

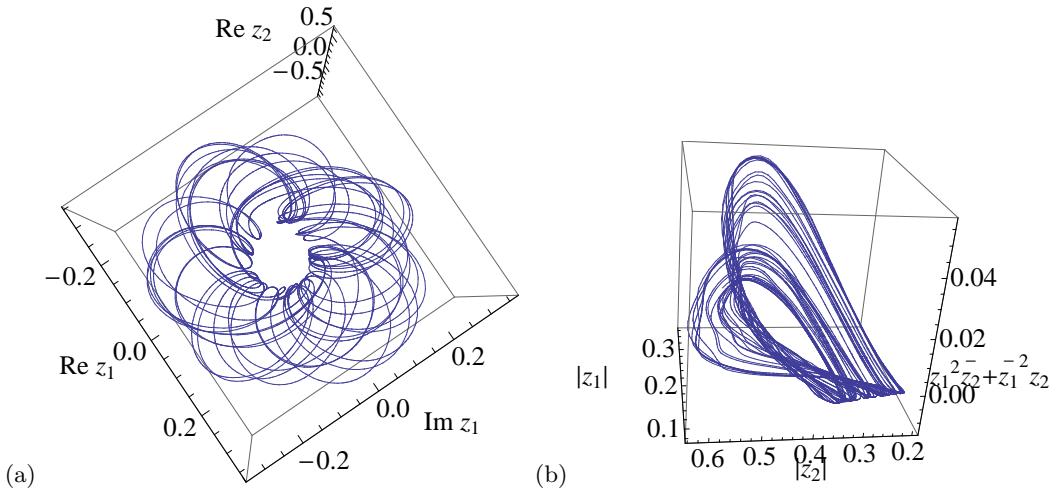


FIG. 12. Projections of Dangelmayr system (20) “attractor” for (not a good choice) parameters (E2), in (a) original state space variables  $\text{Re } z_1, \text{Im } z_1, \text{Re } z_2$ . (b) invariant coordinates used by Armbruster *et al.*<sup>32</sup>  $|z_1|, |z_2|, z_1^2 z_2 + z_1^2 \bar{z}_2$ .

**2012-04-27 Predrag:** Fig. 12(b) is one of 4 projections from the 4-dimensional invariant polynomials  $\{u, v, w, q\}$  dynamics on any three of them. I would prefer that all such projections are linear in coordinates (otherwise small  $z_i$  really get scrunched), so please plot  $\{|u|^{1/2}, |v|^{1/2}, w/u|v|^{1/2}, q/u|v|^{1/2}\} = \{r_1, r_2, \cos \psi, \sin \psi\}$ , or similar...

**2012-04-25 Bryce:** In the spirit of “experimenting” I flipped the sign of  $b_2$  from -0.1 to .01 and got the following plots. Not sure what to make of them though... My brain needs some fuel and my eyes are starting to strain so I’m throwing the towel in for now in favor of some food.

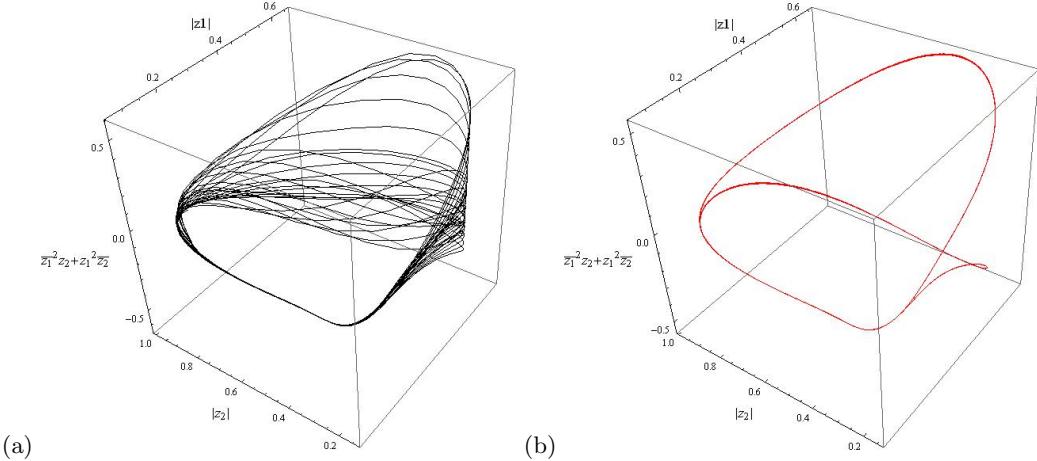


FIG. 13. Projection of Dangelmayr system (F10) “attractor” for parameters (E3). (a) Projection onto original state space:  $(|z_1|, |z_2|, z_1^2 z_2 + z_1^2 \bar{z}_2)$ . (b) Symmetry reduced trajectory using the following template for the slice  $(0.220299, 0.784274, -1.03596, 2.46552)$ .

$$\begin{aligned} \mu_1 &= -0.337, \mu_2 = 0.27, c_1 = 1, c_2 = 1, a_1 = -1.4, \\ a_2 &= -6, b_1 = 1.6, b_2 = 0.1, e_2 = 1.217. \end{aligned} \quad (\text{E3})$$

**2012-08-10 Predrag:** Not a good choice - please follow Chossat<sup>51</sup> and always use  $c_1 c_2 < 0$ , as suggested by [2012-08-06 Knobloch].

**2012-04-27 Predrag:** Experimenting is great, but please read what I have written, and either show me that I am wrong, or experiment accordingly. Here are clips of above unread text:

(clip 1) As we want hyperbolicity, pick  $\mu_j$  of opposite signs. In order to avoid the complex Lorenz equations near visits to the invariant subspace (there it was the  $z$  axis, here it is the  $m = 2$  subspace) pick contraction within the subspace,  $\mu_2 < 0$ , but make it overall repelling by making sure that  $\mu_1 > -\mu_2 > 0$ . No parameters that you have reported chaos for so far satisfy that.

(clip 2) Fig. 17 (b) is one of 4 projections from the 4-dimensional invariant polynomials  $\{u, v, w, q\}$  dynamics on any three of them. I would prefer that all such projections are linear in coordinates (otherwise small  $z_i$  really get scrunched), so please plot  $\{|u|^{1/2}, |v|^{1/2}, w/u|v|^{1/2}, q/u|v|^{1/2}\} = \{r_1, r_2, \cos \psi, \sin \psi\}$ , or similar...

**2012-04-27 Keith to Predrag:** I know what your suggestions are for  $\mu_1$  and  $\mu_2$ , but I found the origin never to have much an effect on any of the dynamics, so I went and tried some other stuff. What I found when I symmetry reduced was what appears to be two Rössler like trajectories floating around two equilibrium and a single trajectory visiting both: Fig. 14. I post processed this trajectory rather than running the dynamics in the slice, and the slice was:  $[1, 0, 0, 0]$  which is rather unoriginal, but it provided the easiest condition for rotating the frame at the time, because once I found these parameters, I thought there might be some interesting events to these. The problem, of course with this slice is that the invariant space  $r_1 = 0$  remains a circle. The parameters are (not a good choice) (E1).

**2012-04-28 Predrag:** Wow, this is beautiful. Victoria Secret's Pink line. The three dots are the equilibrium and two rather similar relative equilibria, both spiral-out unstable? Is the  $m = 2$  subspace ( $r_1 = 0, r_2 > 0$ ) relative equilibrium where Daniel says it is? Stable, unstable? Whatever 'slice' means (there are rumors of a good paper defining it, but's not on arXiv yet), it is not  $[1, 0, 0, 0]$ . No habla esta idioma. You mean template  $\hat{a}' = [1, 0, 0, 0]$  or group tangent  $t' = [1, 0, 0, 0]$ ? Presumably  $t' = [1, 0, 0, 0]$ . Not obvious to me that relative equilibrium in  $m = 2$  remain a circle, but there surely is one (attractive?) that seems to give shape to the hot pants. You seem to have set  $x_2 = 0$  (that is  $m = 2$  coordinate?)

Time to go to bed...

**2012-04-28 Keith:** I reposted the images for parameters I gave last night. I changed the colors of all the (relative equilibria) to be able to talk about them. I have given you a point in terms of  $[x_1, x_2, y_1, y_2]$  where  $x_1$  and  $x_2$  are the first mode and  $y_1$  and  $y_2$  are the 2nd mode.

Black  $[0, 0, 0, 0]$  with eigenvalues:  $(1 + 2i, 1 - 2i, 2, 2)$

Green  $[0, 0, -0.14462, -0.650791]$  or in  $(r_1, r_2, \psi)$  coordinates  $[0, 0.6667, ?]$  the  $\psi$  doesn't matter; with eigenvalues:  $(-6.44444, 6.88889, -1 \pm 1.73205i)$

Blue  $[0.358405, 0, 0.00733041, -0.526708]$  with eigenvalues:  $(-3.57283 \pm 10.3177i, 2.03058 \pm 7.89832i)$

Cyan  $[0.39517, 0, 0.00742352, 0.475084]$  with eigenvalues:  $(-3.05375 \pm 10.5197i, 1.48008 \pm 9.80073i)$

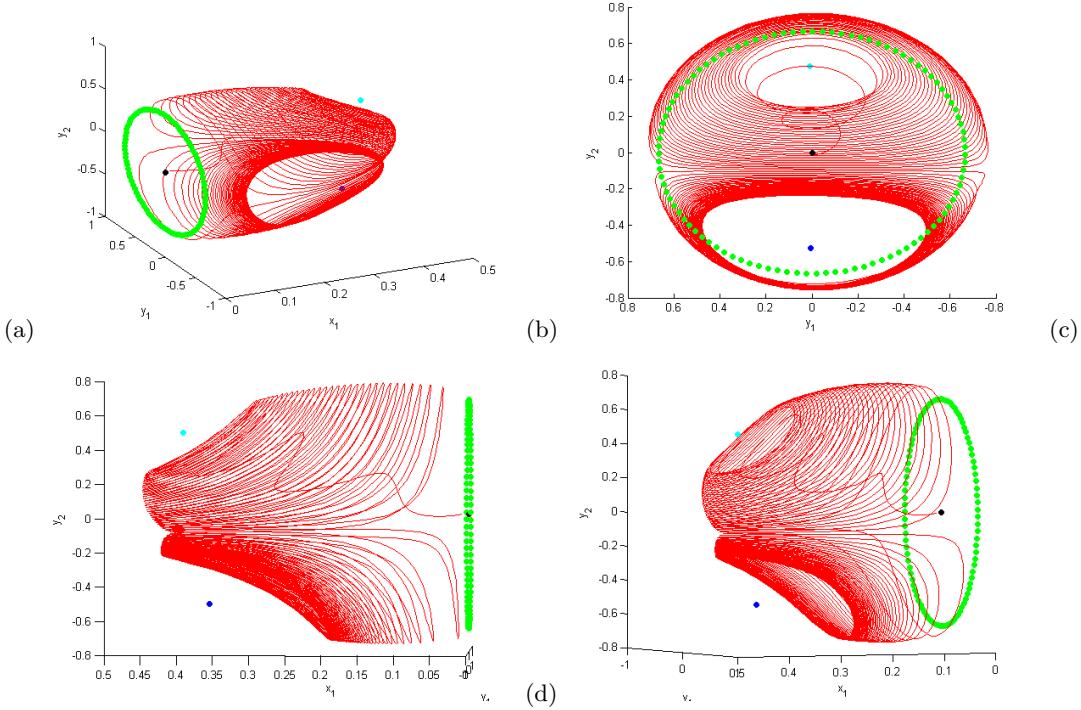


FIG. 14. (a) Symmetry reduced Hot Pants trajectory, side perspective. (b) Symmetry reduced along the  $r_1$  axis. (c) Another perspective around one of the relative equilibria. (d) Perspective around the other relative equilibrium. Finding the best angles to do this was difficult, so if you want an interactive version, I can post it. The parameters are given as:  $\mu_1 = a_2 =$

$$\begin{aligned} \mu_1 &= 2.0, \mu_2 = 1.0, a_1 = -7.5, \\ a_2 &= -2, b_1 = -4, b_2 = -2.25, c_1 = 10, c_2 = -35, e_2 = 2. \end{aligned} \quad (\text{E4})$$

**2012-08-10 Predrag:** Not a good choice we use  $c_j = \pm 1$ .

**2012-04-28 Predrag:** Fig. 14 attractor (what is it ?) is sexy and she knows it.

According to [2012-04-27 Daniel], the  $[0, 0, 0, 0]$  eigenvalues are  $\lambda_1 = \mu_1$  with multiplicity 2 and  $\lambda_3 = \mu_2 \pm ie_2$ . The eigenvectors for  $\lambda_1$  are  $(1, 0, 0, 0)$  and  $(0, 1, 0, 0)$  in the  $(x_1, x_2, y_1, y_2)$  basis. The eigenvectors for  $\lambda_2$  are  $(0, 0, 1, 0)$  and  $(0, 0, 0, 1)$  For your parameter values  $\lambda_1 = \lambda_2 = 2, \lambda_{3,4} = 1 \pm 2i$ , it checks.

[Green]  $m = 2$  invariant subspace: which of the eigenvalues' eigenvectors point into the  $m = 2$  world? Presumably  $(-6.44444, 6.88889)$ . According to my suggestions for invariant subspaces, we would like repelling directions to win, which I think means  $\lambda_1 + \lambda_2 > 0$ . If so,  $(-6.44444, 6.88889, -1 \pm 1.73205i)$  barely wins. However, if  $\sum \lambda_j > 0$  is the correct criterion, than contraction wins, which might be the reason that you seem to be coming closer and closer to the green relative periodic orbit.

[Blue] and [Cyan] equilibria might look cooler if the real part of the expanding eigenvalue was a bit closer to zero - but maybe it does not matter, as they rotate very fast (large imaginary parts). It could be that because the contraction  $-3.57283$  wins over the expansion  $2.03058$  your attractor look like the heteroclinic connections of Armbruster, Guckenheimer and Holmes,<sup>32</sup>.

It's your [Green] unreduced relative periodic orbit that blows wind into these Hot Pants - green should be a point, and than these surfaces will shrink into heteroclinic trajectories in reduced state space?

And I really would be much happier if you used the full state space equilibria as templates, as is the current religion, even though it might mean losing your Pants. You really want a slice which also reduces the rotations within the  $m_1 = 0$  invariant subspace.

**2012-04-28 Predrag:** The Hot Pants do not look chaotic. You'll have to wade deeper. If Fig. 14 is in a slice hyperplane, how wild is the full state space attractor?

**2012-04-28 Evangelos to Keith:** Fig. 14 looks like a transient to a stable limit cycle to me. You do plot  $t = 0$  to  $t = t_{max}$ , right? Why don't you post a figure going from  $t = t_{max}/2$  to  $t_{max}$ ?

**2012-04-28 Keith to Evangelos:** Yes I actually did that this morning, it turns out that it is a limit cycle. So sadly this is not a good situation to use.

**2012-04-28 Evangelos to Chaos Gang:** Predrag and I do not like a reduced state space in which a relative equilibrium (such as the green orbit in Fig. 14) does not appear as a single point. I read the explanation in [2012-04-28 Keith to Predrag] but are we sure we don't want to symmetry reduce within invariant subspaces? [2012-04-28 Keith] I never intended to use this as the invariant subspace, I just needed something to cut through the symmetries to see what was happening instead of the full space. I regret not having used the insights that I gained in the chaos course. [2012-04-29 Keith] I did not write that last sentence, but find it rather amusing.

**2012-04-29 Predrag to Daniel:** To summarize: two-mode relative equilibria are the zeros of:

$$\begin{aligned}\tilde{f}(\tilde{u}, \tilde{v}) &= \tilde{u} A_1 - \tilde{v} A_2 = 0, & \deg(\tilde{f}) &= 2 \\ \tilde{g}(\tilde{u}, \tilde{v}) &= (A_1^2 - c_1 \tilde{v})(\tilde{u} + 2\tilde{v})^2 + e_2^2 \tilde{v}^2 = 0, & \deg(\tilde{g}) &= 4 \\ \text{where } A_1 &= \mu_1 + \tilde{a}_1 \tilde{u} + \tilde{b}_1 \tilde{v}, & A_2 &= \mu_2 + \tilde{a}_2 \tilde{u} + \tilde{b}_2 \tilde{v},\end{aligned}$$

in the positive  $\tilde{u}, \tilde{v} > 0$  quadrant (see (36) for details).  $g(u, v)$  and especially  $f(u, v)$  look pretty simple, so I'm thinking one can simply color-code their signs (zeros are then 'nodal lines'). The relative equilibria are the intersections of two sets of curves - might be a quick way to develop an intuition about roles of various parameters, especially if you code it as something interactive where parameters are sliders.

**2012-04-30 Predrag to C Gang:** Evangelos Evangelos/dangelmayr\_so2\_int.nb is a really cute toy - try it!

**2012-04-30 Predrag to Evangelos:** I suggest using the 8 (or 7) parameters of the rescaled (36) (hope I introduced no further errors), and showing the value of a parameter under its slider (at least, I cannot see it). There will be a bunch of constraints on the parameters, Daniel and I will list them in as we find them.

**2012-04-30 Evangelos to Predrag:** You can see the value of a parameter by clicking on the cross next to the sidebar. I'll see if I can make it appear permanently. I switched to unscaled variables while debugging the code but I'm thinking not to switch back. The reason is that  $c_1$  and  $c_2$  in Dangelmayr's paper can only take the values  $\pm 1$  and I follow this in dangelmayr\_so2\_int.nb, so the gain in search space reduction is not huge compared to the potential errors if I use 2 sets of parameters. Also it keeps the code more readable. If we relax restrictions on  $c_1, c_2$ , I will certainly switch to scaled variables.

**2012-05-02 Evangelos to Predrag:** Now parameter values appear below sliders by default (thanks to local Mathematica expert Achilleas Lazarides).

**2012-04-30 Evangelos:** In siminos/cgang/Evangelos/dangelmayr\_so2\_int.nb I plot nullclines of (35) to help with counting the number of zeros. You can use some sliders to vary parameters and experiment. You can also experiment with phase-space plots of a typical trajectory and any relative equilibria found numerically.

**2012-04-30 Evangelos:** In dangelmayr\_so2\_int.nb, apart from the nullclines, I plot the zero contour of  $w^2 + q^2 - 4u^2v$  to check whether the syzygy is satisfied. It appears that the latter coincides with the  $g(u, v) = 0$  contour. Where does this come from? In deriving  $f(u, v) = g(u, v) = 0$  we did not use the syzygy (at least not when I rederived it), so I thought it would impose a further constraint satisfied exactly at the intersection of  $f(u, v) = 0$ ,  $g(u, v) = 0$  for valid solutions. Why does it instead coincide with  $g(u, v) = 0$ ?

**2012-04-30 Evangelos to Predrag:** I've tried your  $\mu_1 > -\mu_2 > 0$  suggestion but I cannot detect an interesting case. On the other hand with  $\mu_2 > -\mu_1 > 0$  I find chaos but no relative equilibria. We need to get more insight into stability of relative equilibria.

**2012-05-04 Sarah:** Here's a first stab at an interactive Rössler web page. It works on IE, but only somewhat on Firefox.

**2012-05-06 Predrag to Sarah:** You might want to try to turn Evangelos/dangelmayr\_so2\_int.nb into a two-mode Wolfram CDF demonstration - it has sliders for various parameters, and it should just work off the bat. We still have to settle on a good example of chaos that requires two templates.

**2012-07-30 Evangelos:** In `Evangelos/dangelmayr_so2_int.nb` I now color-code the  $(u - v)$ -plane according to the sign of (F34) when plotting the zero-contours of  $f(u, v)$  and  $g(u, v)$  to help in detecting a parameter region with two hyperbolic relative equilibria.

**2012-07-30 Evangelos:** I've allowed for  $e_1 \neq 0$  in `Evangelos/dangelmayr_so2_int.nb`, in order to allow for a second "frequency" in the system, maybe generating richer dynamics. Have not played with it yet. Expressions used to locate (relative) equilibria need to be modified accordingly.

**2012-08-06 Predrag:** Attractor for

$$\begin{aligned} \mu_1 &= -0.38, \mu_2 = 0.38, c_1 = -1, c_2 = 1, a_1 = -1.31, \\ a_2 &= -2.6, b_1 = 1.504, b_2 = 0.22, e_2 = 1.61. \end{aligned} \quad (\text{E5})$$

looks pretty chaotic. Can you check what equilibria it has, and their stabilities?

**2012-08-07 Daniel:** Here's a breakdown of the stability of the equilibria that we know analytically for the parameter values suggested by Predrag. I first converted these 4-dimensional Cartesian coordinates and then calculated their stability in that representation.

The origin is hyperbolic with eigenvalues  $\lambda_{1,2} = \mu_2 \pm ie_2 = 0.38 \pm i1.61$  and  $\lambda_{3,4} = \mu_1 = -0.38$ .

I believe relative equilibrium at  $(r_1, r_2, \psi) = \left(0, \sqrt{-\mu_2/b_2}, \sin^{-1}\left(\frac{e_2}{2c_1}\sqrt{\frac{-b_2}{\mu_2}}\right)\right)$  does not exist (if  $(x_1, x_2, y_1, y_2) \in \mathbb{R}$ ) for these parameter values as per my discussion in the blog on 2012-04-27.

Relative equilibrium (F24) is hyperbolic with eigenvalues  $\lambda_{1,2} = 0.0923 \pm i2.0682$ ,  $\lambda_3 = -0.7630$ , and  $\lambda_4 = 0.2851$ . This is kind of confusing... shouldn't there be one eigenvector in the symmetry direction with eigenvalue 0? Maybe I'm messing up converting from  $(u, v, w, q)$  to  $(x_1, x_2, y_1, y_2)$ ? I get  $(0.1809, 0.4102, 0, -0.1437) \rightarrow (0.3007, 0.3007, 0.6405, 0)$  where I have converted between the two basis by first converting the invariant coordinates to polar coordinates and then converting the polar coordinates to Cartesian coordinates.

Not sure how to use Evangelos's Mathematica code to find the other equilibria.

**2012-08-10 Evangelos:** There is a function `reqbPar` which finds relative equilibria as a function of the parameters. It solves (35) numerically, so I am not sure whether it finds all roots.

**2012-08-08 Daniel:** Tried finding further equilibria by finding numerical solutions to  $[\dot{u}, \dot{v}, \dot{w}, \dot{q}] = 0$  for the parameters suggested by Predrag. Only find Bryce's equilibrium and the origin. There are a couple of roots  $([-0.3492, 0.0351, -0.0909, 0.0940] \text{ and } [0.0000, -1.7273, -0.0000, -0.0000])$  that do not satisfy the constraints that  $u, v \geq 0$ . There are four further roots where  $[u, v, w, q]$  are complex. For more details see, `/cgang/Daniel/Matlab/PKRoots.m`

Then I went through all the roots that I found and checked to see if they satisfy the syzygy. The origin and all the unphysical (both real and complex) solutions do, but Bryce's solution doesn't. Evaluating the syzygy gives me  $-0.033$ . What does that mean? My gut feeling is that this equilibrium is accessible to the system if you start it at some generic point in  $(u, v, w, q)$ -space but not if you start in  $(x_1, x_2, y_1, y_2)$ -space.

**2012-08-08 Predrag:** The syzygy (27) is sacred, so there is an error in Bryce's solution.

**2012-08-09 Daniel to Predrag:** You are more learned than I but I don't know... If I search for the equilibria of (30) with the parameters (E5) numerically, I find a root that corresponds to the value of Bryce's analytical expression (F22) evaluated for (E5). This is without telling the computer anything about Bryce's solution. Plugging in his analytical solution into (30) symbolically also satisfies the conditions for an equilibrium, except it doesn't satisfy the syzygy.

**2012-08-10 Predrag:** See (F25): it is no solution.

**2012-08-08 Daniel:** I know we don't like Lyapunov exponents, but since I already had the code written up from a long time ago and all I had to do was give it the equations of motion and the Jacobian, I calculated them out anyway for Predrag's parameters (E5). The code uses the method proposed by Eckmann and Ruelle<sup>52</sup>. Anyway, the Lyapunov spectrum is  $\approx (0.1, 0, 0, -0.2)$  with a Kaplan-Yorke dimension of  $D_{KY} \approx 3.4$ .

**2012-08-08 Predrag:** Not that prejudiced (read `siminios`/Lyapunov blog). That looks right, this has a chance of being a strange attractor - two dimensions are trivial (symmetry has not been reduced), the remaining 1.4 feels too big.

But why don't you do the easy thing that you already have code for - take a point on the strange attractor as a template, and run in the slice hyperplane until the angle (F37) gets small? It's easier than thinking, and we just need a 2-chart example.

**2012-08-09 Daniel:** Maybe I'm calculating  $D_{KY}$  incorrectly. I'm not sure what to do with the degenerate 0 eigenvalue. My code does not account for this degeneracy (since it actually gets really small but distinct eigenvalues). If the correct thing to do with eigenvalues with multiplicities greater than one is to count them as a single eigenvalue then  $D_{KY} \approx 2.5$ .

I'll work on the on the slice later tonight. There's some things that need to be tweaked in my code for complex Lorenz.

**2012-08-08 Daniel:** Is  $[0, v, 0, 0]$  really a root and not just an invariant subspace? I don't think  $\dot{v}$  vanishes for all  $v$ .

**2012-08-08 Predrag:** Set  $x_1 = x_2 = 0$  in (20):

$$\dot{z}_2 = (\mu_2 - i e_2) z_2 + b_2 |z_2|^2 z_2, \quad (\text{E6})$$

In polar coordinates  $z_2 = r_2 e^{i\theta_2}$ ,  $r_2 = |v|^{1/2}$ .

$$\begin{aligned} \dot{r}_2 &= \mu_2 r_2 + b_2 r_2^3 \\ \dot{\theta}_2 &= e_2. \end{aligned} \quad (\text{E7})$$

By rescaling time and  $r_2$  one can get rid of  $\mu_2$ ,  $b_2$ , so only the sign of  $b_2/\mu_2$  matters. 2-dimensional problems are boring - this one is worked out as problem 5.1 in ChaosBook (version13.7.2); perhaps it should be moved to the chapter on continuous symmetry reduction, as the simplest, trivial example.. With SO(2) symmetry, the symmetry reduced space is 1-dimensional. It always has equilibrium at the origin (whose linear stability has to be computed in the full 2-dimensional space), and at most one relative equilibrium, whose stability can be given in analytic form. All other dynamics in the  $[y_1, y_2]$  plane is either attracted or repelled by these two invariant solutions.

Equivalently, set  $x_1 = x_2 = 0$  in (21):

$$\begin{aligned} \dot{y}_1 &= \mu_2 y_1 + e_2 y_2 + b_2 y_1^3 + b_2 y_1 y_2^2 \\ \dot{y}_2 &= \mu_2 y_2 - e_2 y_1 + b_2 y_1^2 y_2 + b_2 y_2^3 \end{aligned} \quad (\text{E8})$$

(E6) is clearly U(1) equivariant. (30) reduces to

$$\dot{v} = 2 \mu_2 v + 2 b_2 v^2. \quad (\text{E9})$$

This has equilibrium  $v^* = -\mu_2/b_2$ , provided this quantity is positive, otherwise not. It corresponds to a relative equilibrium in equivariant coordinates, with radius  $\sqrt{-\mu_2/b_2}$ . Parameters (E5) do not have this relative equilibrium.

**2012-08-09 Daniel:** Ok... so I wasn't crazy and it agrees with my result from 2012-04-27.

**2012-08-10 Evangelos:** As an example I've chosen

$$\begin{aligned} \mu_1 &= -0.234, \mu_2 = 0.28, c_1 = 1, c_2 = 1, a_1 = -1.4, \\ a_2 &= -2.18, b_2 = -0.1, e_2 = 1.217, e_1 = 0. \end{aligned} \quad (\text{E10})$$

**2012-08-10 Predrag:** Not a good choice - please follow Chossat<sup>51</sup> and always use  $c_1 c_2 < 0$ , as suggested by [2012-08-06 Knobloch].

**2012-08-10 Daniel to Evangelos:** What value of  $b_1$  did you use? I want to see if my code returns the same solutions.

**2012-08-10 Evangelos:** My Mathematica code uses (35) to find relative equilibria. There are two relative equilibria with  $(u, v, w, q)$  coordinates,

$$TW_1 : (0.0757999, 0.503198, -0.0648485, 0.0857872) \quad (\text{E11})$$

$$TW_2 : (0.313905, 0.233171, 0.199422, 0.228329), \quad (\text{E12})$$

the first stable, and the second one hyperbolic. Using (F33) (invariant polynomial basis) we get eigenvalues:

$$\lambda_i^{(1)} : (0.872034, -0.10026 \pm 0.516938i, 1.83992) \quad (\text{E13})$$

$$\lambda_i^{(2)} : (-0.191519, -1.1123 \pm 0.497766i, -2.12585) \quad (\text{E14})$$

Using (F35) (invariant polynomial basis, with syzygy incorporated) we get eigenvalues:

$$\lambda_i^{(1)} : (0.872034, -0.10026 \pm 0.516938i) \quad (\text{E15})$$

$$\lambda_i^{(2)} : (-0.191519, -1.1123 \pm 0.497766i) \quad (\text{E16})$$

I do not understand what the missing eigenvalue really means.

**2012-08-10 Daniel:** I'm confused. You are finding the eigenvalues of a  $4 \times 4$  matrix numerically and the computer only spits out three values? This is without ever telling the computer about the syzygy. When you "tell" it about the syzygy by reducing the system to  $3 \times 3$ , it then only returns two eigenvalues? How is this possible?

**2012-08-11 Predrag:** 2 real + 1 complex pair = 4.

**2012-08-10 Predrag:** The syzygy defines a 3-dimensional manifold imbedded in 4 dimensions. Can you try this: compute the normal to it by taking a gradient of the syzygy evaluated at  $TW_j$ , then dot it with the three eigenvectors. The first three should be normal to it (tangent to the syzygy 3-dimensional manifold), the third one is perturbation into Bryce space, outside the law (the lands where the syzygy rules).

**2012-08-10 Daniel:** So is the conclusion that  $(u, v, w, q)$ -space is divided into PK-space (where the syzygy is satisfied) and Bryce-space (where it is not), right? Can the  $(\dot{u}, \dot{v}, \dot{w}, \dot{q})$  dynamics connect the two? Bryce-space has its own equilibria (like Bryce's solution, which gives  $(\dot{u}, \dot{v}, \dot{w}, \dot{q})$  but does not respect the syzygy) but do these affect the dynamics in PK-space?

**2012-08-11 Predrag:** Bryce world is outside law and of no interest.

**2012-08-10 Evangelos:** Using (F11) (original coordinates) we get

$$\lambda_i^{(1)} : (1.51099, -0.177221 \pm 1.42156i, -0.121078) \quad (\text{E17})$$

$$\lambda_i^{(2)} : (0.608294, -0.853009, -1.0441 \pm 1.29555i) \quad (\text{E18})$$

Note that in this case the eigenvalues are not related in any way to the ones we find using invariant polynomials. If one wants to compute stability in original coordinates, she has to go to a comoving frame. As discussed in Appendix B of Cvitanović *et al.*<sup>46</sup>, this is equivalent to using the matrix  $A + cT$  where  $c$  is the phase velocity and  $T$  the generator of the group. Matrices  $A$  and  $A + cT$  will in general have different eigenvalues, so I disagree with [2012-08-01 Predrag] above ([2012-08-08 Predrag] My bad, sorry). As a next step I will try to use this formalism.

**2012-08-10 Daniel:** So just to double-check... this means that we should be looking for equilibria and checking their stability only in  $(u, v, q, w)$ -space? Or should be using the reduced space  $(u, v, w)$ ? Or do we need to go to co-rotating frames? Doesn't that presuppose that we already know the traveling waves to begin with so that we can go into a frame with the correct velocity?

**2012-08-11 Predrag:** In invariant coordinates there are only equilibria. Going back to equivariant coordinates, they become relative equilibria with known phase velocity  $c$ . When we slice, this is given by the reconstruction equation. In equivariant state space coordinates  $\{x_1, y_1, x_2, y_2\}$  the correct stability matrix is (4).

**2012-08-10 Predrag:** <sup>99</sup>This correct relative equilibrium stability matrix should has one zero eigenvalue, with the group tangent as eigenvector. See paragraph around (4) copied from ChaosBook.org. Is anything smart to be said for the corresponding left eigenvectors of (4)? As  $A$  is not self-adjoint,  $v$  is not a left eigenvector. In the theory of rotating spirals they<sup>53–55</sup> call them “Response Functions” (and capitalize them), while the right group tangent vectors they call “Goldstone modes” and they make a big deal out of them. <sup>101</sup>

**2012-05-07 Predrag to Chaos Gang:** It's not over until it is over.

Time to go to bed...

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<sup>100</sup> 2CB

<sup>102</sup> PC: recheck if Biktashev<sup>53,54</sup> are the right references

## Appendix F: Two-mode daily blog

Do not edit this chapter, edit only the svn version

### 1. How to read me

Here is a novice's guide to desymmetrization bloggery:

- How to read the running blog: go first to the latest blog post, end of Section ??.
- Guys writing the ultimate guide to slicing for the woman on the street, go to `siminos/atlas/`, blog in chapter ??*Atlas*.
- If you are reading an article of common interest (which does not fit into one of the specialized topics), enter your notes into Section ??.
- Comments to ChaosBook.org go into ChaosBook.tex section of `siminos/blog/`.
- Plumbers who ponder how to slice experimental data also blog there, in *Symmetry reduction of experimental data*
- Save all figures (pdf or png, not eps) used in the blog in `siminos/figs/` sv rm the figures no longer used (they all exist on the repository, if you need to recover them any time later)
- Save useful source programs for figures used in the blog in `siminos/figSrc/`
- Enter all bibliography items into `siminos/bibtex/siminos.bib` alphabetically by first author, create no other \*.bib file
- Save useful simulation programs in (for example) `siminos/matlab/`
- Never commit large data sets, movies, blog.pdf or anything large that you can easily recreate.

### 2. How to share literature

**2012-03-28 PC:** *Saved articles.* If you put an article into Dropbox, I can put it into [ChaosBook.org/library](#). You can fetch it from there by clicking on [ChaosBook.org/library/BibTexName.pdf](#) (login in as: student Lautrup), where BibTexName is the name you gave it in `siminos/bibtex/siminos.bib`.

**2012-01-01 PC:** In the long run that is a very awkward system, as we do not have manpower for entering the articles into the [ChaosBook.org/library/index.html](#), so it is hit and miss finding out whether a paper you want is there, unless linked it to this blog, like this:

“Porter and E. Knobloch<sup>30</sup> ([click here](#)) .”

Much better solution is to join [www.zotero.org/groups/cns](#) in order to be able to access the papers we are saving there, and save your own downloads there of copyright protected publications. Search for zotero in `siminos/blog/blog.pdf` to find a bit more info about it. There you can see what each article is, use BibTeX to describe it, etc. Ask Gable to help you get started.

**2012-03-28 PC:** Put 'our' articles into zotero CNS [library] folder [symmetry] unless the fit in another section better, enter them into `siminos/bibtex/siminos.bib`, and note somewhere in this blog that you have added them (and why?)

### 3. Blogging Two-mode progress

**2012-04-24 Predrag:** started a new blog, for Chaos Gang term project

`siminos/cgang/2modes.tex`

**2012-03-10 Predrag to Daniel:** I put soluChap10.pdf in Dropbox. In the current edition the programs are renumbered. So far you have done:

10.8 An SO(2)-equivariant flow with two Fourier modes

10.10 SO(2) equivariance of the two-mode system for infinitesimal angles.

Next bunch of exercises to do:

10.11 Visualizations of the 5-dimensional two-mode system

10.22 Two-mode system in polar coordinates.

10.23 The relative equilibria of the two-mode system

10.24 Plotting the relative equilibria of the two-mode system in polar coordinates

10.25 Plotting the relative equilibria of the two-mode system in Cartesian coordinates

Still have to make up exercises on group orbits, templates, slicing

**2012-03-21 Daniel:** From some of the most recent comments copied from pipes/blog suggest that Rich (I'm assuming Kerswell?) has been looking at periodic orbits in 2D Kolmogorov flow. Perhaps, the third floor 2D crowd should be in touch with these guys. Do Mike and Roman know about this?

**2012-03-21 Predrag:** Sorry, maybe it is my fault - I just assume everybody tracks the current literature, I should have told them in person. Here is another relevant blog entry:

**2011-05-18 Rich Kerswell** Read Irene Moroz, Geophys. Astrophys. Fluid Dynamics vol 105, 273-286, 2011, *Unstable periodic orbits in 4D Faraday disk dynamo*. Might give you a 4-dimensional dynamical system which is chaotic and can be reduced to a 3-dimensional state space; easier to visualize than complex Lorenz flow.

Kerswell papers are published and easy to find; if you learn something, please summarize it here for the rest of us.

**2012-03-26 Daniel:** Looked through various publications by I. Moroz et al. about a 4D model for a dynamo model that Rich Kerswell had mentioned as a possible candidate to replace complex Lorenz. This model has chaotic regimes and some terms that break its discrete symmetries. There are no continuous symmetries (although don't explicitly say that they aren't there either), so this might not be a good candidate to replace complex Lorenz as a toy model for slicing and dicing. However, the fact that they can find (exact, rather than relative) periodic orbits probably means that the system does not have continuous symmetries. If it did it'd be very hard to find exact periodic orbits. **2012-03-27 Predrag** I agree, this baby is DOA. Scratch that.

**2009-08-28 Predrag:** Evangelos in his thesis and Kohler in his [ChaosBook.org project](#) got nothing of interest out of Armbruster *et al.*<sup>56</sup>.

**2012-03-26 Predrag:** There is Armbruster, Guckenheimer and Holmes<sup>32</sup> flow with O(2) symmetry:

$$\begin{aligned}\dot{z}_1 &= \bar{z}_1 z_2 + z_1 (\mu_1 + e_{11}|z_1|^2 + e_{12}|z_2|^2) \\ \dot{z}_2 &= \pm z_1^2 + z_2 (\mu_2 + e_{21}|z_1|^2 + e_{22}|z_2|^2)\end{aligned}\tag{F1}$$

where  $z_1, z_2 \in \mathbf{C}$  and  $\mu_j$  and  $e_{jk}$  real parameters. This system corresponds to the first few terms in the center manifold reduction of a O(2)-symmetric partial differential equation near a codimension two bifurcation. It is a two-mode system, so group orbits are more interesting than for complex Lorenz flow. **[2009-08-28 Predrag]** Evangelos in his thesis and **Kohler** in his [ChaosBook.org project](#) got nothing of interest out of Armbruster *et al.*<sup>32</sup>. As far as we can tell, it exhibits no chaos. I would prefer some modification of it with SO(2) symmetry only, and exhibiting chaos (done! see 2012-03-27 remark below). Or some version of two-mode model Daniel has been playing with which does not behave singularly. He has an unstable cycle close to the origin that maybe does the trick.

**2012-03-26 Daniel:** Ref. 30 and two others may provide some candidate systems. Haven't had time to enter them into siminos.bib or read through them but the look promising.

**2012-03-27 Predrag to Keith:** Two-mode<sup>30</sup> ([click here](#)) write: "Chossat<sup>57</sup> has shown that breaking the symmetry  $O(2)$  down to  $SO(2)$  through the addition of small terms that break reflection symmetry generically destroys the heteroclinic cycles and replaces them by a quasiperiodic orbit characterized by two small frequencies, one associated with the broken heteroclinic connection and one with a slow drift along the group orbit of translations. Ashwin *et al.*<sup>58</sup> showed that this perturbation must be dispersive: if reflection symmetry is broken by adding a constant through flow the cycle will persist."

This shows us how to get an  $SO(2)$ -equivariant model rather than the Armbruster, Guckenheimer and Holmes<sup>32</sup>: "When the reflection symmetry is broken, the coefficients in the normal form equations are no longer forced to be real and hence can be expected to acquire imaginary parts", their Eq. (10). Play with that one.

**2012-03-28 Daniel:** Rodriguez and Schell<sup>59</sup> may actually be a good system. They are using a truncation of the Ginsberg-Landau equation, which results in a 4D system of ODE's. These equations have parameters regimes that are chaotic (as reported by Rodriguez and Schell and at least one other researcher). They are  $SO(2)$ -equivariant. They only have a small number of parameters (3). The truncation is a two-mode truncation so it should provide more interesting group orbits than the one-mode Complex Lorenz system.

"Within the two-mode approximation, and using the transformation

$$a_k = (b_k, c_k) = b_k + ic_k = r_k e^{i\theta_k},$$

the LG equation reduces to a set of four ordinary differential equations for the variables  $r_0, r_1, \theta_0, \theta_1$ ; however, it is more convenient to replace the last two variables with the linear combinations  $\psi_1 = \theta_1 - \theta_0$ , which is a phase difference, and  $\zeta = \theta_1 + \theta_0$ , which is a phase sum. "

Their system of equations is:

$$\begin{aligned} \dot{r}_0 &= \rho r_0 (1 - r_0^2) - r_0 r_1^2 \left( \rho + \frac{1}{2} \rho \cos(2\pi\psi_1) + \frac{1}{2} \sin(2\pi\psi_1) \right) \\ \dot{r}_1 &= (\rho - q^2 c_0) r_1 - \frac{3}{4} r_1^3 - r_1 r_0^2 (2\rho + \rho \cos(2\pi\psi_1) - \sin(2\pi\psi_1)) \\ \dot{\psi}_1 &= \left[ -2q^2 + 2r_0^2 - \frac{1}{2} r_1^2 + (2r_0^2 - r_1^2) \cos(2\pi\psi_1) + \rho (2r_0^2 + r_1^2) \sin(2\pi\psi_1) \right] / 2\pi \end{aligned} \quad (\text{F2})$$

$$\dot{\zeta} = \left[ -2q^2 + 6r_0^2 + \frac{7}{2} r_1^2 + (2r_0^2 + r_1^2) \cos(2\pi\psi_1) + \rho (2r_0^2 - r_1^2) \sin(2\pi\psi_1) \right] / 2\pi \quad (\text{F3})$$

At least two chaotic regimes are reported for this system. These occur for  $c_0 = \rho = 0.25$  and  $q < 0.9787$  or  $1.029039 < q < 1.029041$ .

**2012-03-28 Predrag:** notes on Rodríguez, J. D. and Schell<sup>59</sup>. What baffles me is that there are no ratios of form  $r_0/r_1$  in (F2). In contrast, Daniel 2012-03-13 derived for the two-mode system in ChaosBook the polar coordinates form

$$\begin{aligned} \dot{r}_1 &= -\sigma r_1 + \sigma r_1 r_2 \cos(\theta) \\ \dot{r}_2 &= -r_2 + r_1^2 ((\rho_1 - z) \cos(\theta) - \rho_2 \sin(\theta)) \\ \dot{\theta}_1 &= -\sigma r_2 \sin(\theta), \quad \dot{\theta}_2 = -e + \frac{r_1^2}{r_2} ((\rho_1 - z) \sin(\theta) + \rho_2 \cos(\theta)) \\ \dot{z} &= -bz + \frac{r_1^2}{r_2} \cos(\theta), \end{aligned} \quad (\text{F4})$$

where  $\theta = 2\theta_1 - \theta_2$ , and rewriting the angular part as  $\dot{\theta} = 2\dot{\theta}_1 - \dot{\theta}_2$ ,

$$\dot{\theta} = e - \frac{r_1^2}{r_2} ((\rho_1 - z) \sin(\theta) + \rho_2 \cos(\theta)) - 2r_2 \sigma \sin(\theta) \quad (\text{F5})$$

Simulation of these equations (as well as of complex Lorenz equations in polar coordinates and Dangelmayr and Armbruster, Guckenheimer and Holmes (F6) below) runs into both  $r_j = 0$  singularities, while (F2) seems to have none. What gives?

**2012-03-31 Predrag:** “[...] we consider a subclass of systems with circular symmetry: those invariant under the rotations in the plane, i.e., operations of the symmetry group  $\text{SO}(2)$ <sup>34</sup>.<sup>103</sup>[...] We show that even though phase locking cannot occur in systems with rotational symmetry, the locking of phase differences does occur.” [...] we have also discovered one exception in the correspondence found in the previous investigation. [...] There exists a small interval between the regions of quasiperiodic behavior ( $\Delta q < 10^{-6}$ ) in which chaos occurs. Although we were unable to locate this asymptotic chaos in the appropriate parameter range of the LG equation, we did observe “transient chaos”.

[...] it can be verified by substitution that phase differences, which are defined as  $\psi_1 = (\theta_1 - \theta_0)/\pi, \dots, \psi_j = (\theta_j - \theta_0)/\pi, \dots$ , can become fixed and are represented in a subspace by the fixed points,  $\psi = \text{constant}, r_0 = 1, r_j = 0$ , where  $j = 1, 2, \dots$ .

[Predrag: why would they want to lock? The locking the study happens prior to bifurcations to chaos, I do not think we care. What is the point of looking at all  $r_j = 0$ ?]

[...] It may seem somewhat strange to speak of a locked phase difference between two modes, when one of the modes has zero amplitude. However the quantity  $\theta_1$  is rigorously defined as the phase of a Fourier component whose amplitude asymptotically decays to zero and hence, the locked phase difference is well defined as an asymptotic limit for the phase difference.”

**2012-03-28 Daniel:** At first glance it seems like Mercader and Prat<sup>60</sup> might be a candidate if we decide to go with  $O(2)$  symmetry since really the Rayleigh-Bénard problem has  $O(2) \times Z_2$  symmetry and they are really talking about breaking the  $Z_2$  part. They have a reduced model, but I’m not sure that it is chaotic.

**2012-03-31 Predrag:** notes on Mercader and Prat<sup>60</sup>. The effects of weak breaking of the midplane reflection symmetry on the 1:2 steady state mode interaction in Rayleigh-Bénard convection are discussed, with bifurcations galore, ala Knobloch. They do it for the full PDEs, but in Sect. 4 motivate results by discussing them in the context of  $O(2)$ -equivariant amplitude equations for 1:2 mode interactions. We note that the Dangelmayr<sup>31</sup> and Armbruster, Guckenheimer and Holmes<sup>32</sup> third order in the amplitudes normal form flow with  $O(2)$  symmetry (F1) does have ratios of form  $r_0/r_1$ , when rewritten in the polar form (Eq. (4) in the paper),

$$\begin{aligned}\dot{r}_1 &= \mu_1 r_1 + a_1 r_1^3 + b_1 r_1 r_2^2 + c_1 r_1 r_2 \cos(\psi) \\ \dot{r}_2 &= \mu_2 r_2 + a_2 r_1^2 r_2 + b_2 r_2^3 + c_2 r_1^2 \cos(\psi) \\ \dot{\theta}_1 &= -c_1 r_2 \sin(\psi), \quad \dot{\theta}_2 = -e_2 + c_2 \frac{r_1^2}{r_2} \sin(\psi),\end{aligned}\tag{F6}$$

where  $\psi = 2\theta_1 - \theta_2$ , and Predrag stuck in tentatively an ‘ $e_2$ ’ term because something like that is needed to break  $O(2) \rightarrow \text{SO}(2)$ . Rewriting the angular part as  $\psi = 2\theta_1 - \theta_2$ :

$$\dot{\psi} = 2\dot{\theta}_1 - \dot{\theta}_2 = e_2 - \left( c_2 \frac{r_1^2}{r_2} + 2c_1 r_2 \right) \sin(\psi).\tag{F7}$$

The equations possess pure  $n = 2$  solutions but no pure  $n = 1$  solutions (except for the discrete ‘midplane reflection’ symmetry invariant subspace?). I do not think we care about the ‘midplane reflection’ symmetry (with 5th order normal form), so we only need (F6) to determine the relative equilibria.

**2012-03-31 Predrag:** The battle plan suggestion to ChaosGang:

- Two of you finalize drawings and text for Rössler sections and their borders.
- One of you finalizes drawings and text for Daniel’s 2-chart sliced complex Lorenz equations (preferably not Daniel).
- One of you computes (presumably numerically) relative equilibria of the  $\text{SO}(2)$ -equivariant version of Dangelmayr two-mode (F6), and (hopefully) their stability matrix  $A$  eigenvalues, plays with parameters. We would like the relative equilibria to be complex-pair unstable, leading to chaos, to be visualized and sliced in Cartesian coordinates (F1).

**2012-04-03 Predrag to Evangelos:** We have learned several important things since 2010.

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1. please study and either understand chart borders and ridges, or show why the construction is wrong. A chart *is not* the hyperplane, it is flat tile carved out of the hyperplane that *by construction* slices group orbits in template's neighborhood *only once*.
2. it appears that the complex Lorenz equations strange attractor we are studying never gets as far as the chart border (where (??) changes sign) but only gets squished close to the  $z$ -axis, with (??) maintaining sign. So there is a dynamical reason for large  $\dot{\theta}$  which is not an artifact of slicing. That's not what we are trying to illustrate here. That is why I keep suggesting that Gang of Chaos constructs a more interesting 2-chart atlas for two-mode systems such as (F1) and(F6).

**2012-04-03 Predrag to Lei:** We have to move on with ChaosGang paper. Please follow ChaosBook teachings on how to analyze a dynamical system (it is not by blindly simulating):

1. determine *numerically* the relative equilibria of the SO(2)-equivariant Dangelmayr two-mode system in polar coordinates

$$\begin{aligned} 0 &= r_1(\mu_1 + a_1 r_1^2 + \color{red}{b_1 r_2^2} + c_1 r_2 \cos(\psi)) \\ 0 &= \mu_2 r_2 + a_2 r_1^2 r_2 + b_2 r_2^3 + c_2 r_1^2 \cos(\psi) \\ 0 &= e - \left( c_2 \frac{r_1^2}{r_2} + 2 c_1 r_2 \right) \sin(\psi), \end{aligned} \quad (\text{F8})$$

Here I stuck in tentatively an ' $e$ ' term because something like that is needed to break  $O(2) \rightarrow SO(2)$ , verify that it really does that. The first two equations are cubic, the third one you can use to eliminate  $\cos(\psi)$ , so my guess is that there could be up to six real roots, but I have not thought it through. Once you have found parameters for which there are interesting relative equilibrium solutions, then

2. compute analytically the stability matrix  $A$  in polar coordinates
3. Study eigenvalues, keep playing with parameters. We would like -preferably- no relative equilibrium to be attracting limit cycle, and several of the relative equilibria to be complex-pair unstable, leading to chaos, to be visualized and sliced in Cartesian coordinates (F1).
4. If you find a nice chaotic attractors, others can join in constructing an atlas for it. We just need one and only one example with non-trivial chart borders and at least 2 charts.

Please read the blog entries above, my writing is starting to get a bit repetitive - a look at [Kohler](#) in his [Chaos-Book.org project](#) might be helpful, but we *do not* want to study  $O(2)$  discrete symmetry invariant subspaces. The papers discussed above exhibit some stable relative equilibria and relative periodic orbits for specific (out-of the hat) parameter choices. You want to screw up some parameters higher so these solutions go unstable.

**2012-04-05 Lei to Predrag:** I can see the  $e$  added really breaks the  $O(2)$  symmetry to  $SO(2)$  symmetry. As rotations don't change the equations and reflections change the sign of the last equation, adding a nonzero  $e$  will break the reflection symmetry.

By eliminating  $\psi$ , I got two polynomial equations of order 10. So there are 20 complex solutions. I tried parameters  $\mu_1 = \mu_2 = -1$ ,  $a_1 = a_2 = b_1 = b_2 = c_1 = c_2 = 1$ , and got 8 real solutions. All the 8 solutions I got for  $(r_1, r_2)$  are  $(\pm 0.537655, \pm 0.537655)$  and  $(\pm 0.980269, \pm 0.980269)$ . In this case, 6 equilibria points have only real eigenvalues, two equilibria points have complex eigenvalues with positive real parts. Will this case work? Others please check whether the calculation is correct or not.

**2012-04-05 Keith to Lei:** Lei what was your  $e$  value?

**2012-04-06 Predrag to Lei:** By definition,  $r_i \geq 0$ , so you have only two roots:  $(0.537655, 0.537655)$  and  $(0.980269, 0.980269)$ . I find it surprising that  $r_1 = r_2$ , as equations look asymmetric in  $r_i$ ; might be consequence of  $a_1 = a_2 = b_1 = b_2 = c_1 = c_2 = 1$  (what value  $e$ ? also  $e = 1$ ?), you want to break this artificial symmetry if it is the cause. If  $r = r_1 = r_2$  you have

$$\begin{aligned} 0 &= r(\mu_1 + (a_1 + b_1)r^2 + c_1 r \cos(\psi)) \\ 0 &= r(\mu_2 + (a_2 + b_2)r^2 + c_2 r \cos(\psi)) \\ 0 &= e - r(c_2 + 2c_1) \sin(\psi), \end{aligned} \quad (\text{F9})$$

which looks degenerate for your coefficient values.

Keep fishing...

**2012-04-10 Lei to Predrag:** I've been playing with the parameters of the two-mode system for a while. What I did is simply randomly chosen all the parameters and see what kinds of eigenvalues we can get. Under the condition  $r_1$  and  $r_2$  being nonnegative real numbers, it seems that I always got two equilibria points. One possible set of parameters I found may be of interest are  $\mu_1 = -1, \mu_2 = -4, a_1 = 1, a_2 = 1.5, b_1 = 3, b_2 = 2.5, c_1 = 3, c_2 = 3.5, e = 0.1$ . The equilibria points are  $(r_1, r_2) = (0.0516508, 1.26311)$  and  $(0.467095, 0.2146)$ . The corresponding eigenvalues are  $(19.9398, 0.8495, -11.9818)$  and  $(1.5352, -4.7992 + 0.0327i, -4.7992 - 0.0327i)$

**2012-04-10 Predrag to Lei:** I think we need  $\psi$  as well, i.e.,

- $TW_1 = (r_1, r_2, \psi) = (0.0516508, 1.26311, ?)$  and  $TW_2 = (0.467095, 0.2146, ?)$
- their plots in the Cartesian coordinates
- $\dot{\theta}$  to see how slow/fast are they.  $\dot{\theta}$  might be related to 4th eigenvalue, when you go back to Cartesian coordinates
- stability eigenvalues, eigenvectors of the equilibrium  $EQ_0$  at origin, at your parameter values - if it is stable, everything just might fall into it and die.
- plots of small perturbations of the above equilibrium and relative equilibria in the Cartesian coordinates to see whether the dynamics looks chaotic
- $TW_1$ : 2 large positive eigenvalues looks scary - probably nothing re-visits this relative equilibrium. A mildly unstable complex pair would have been sweeter. You get complex eigenvalue by Hopf-bifurcating off a stable orbit, typically.
- $TW_1$ : Does either unstable eigenvalue become a complex eigenvalue pair in Cartesian coordinates?
- $TW_2$ : contracting eigenvalues have very small imaginary part, so the presumably just rocket toward the relative equilibrium, not much spiraling there. At least the unstable eigenvalue seems slow compared to all other eigenvalues.
- $TW_1$ : Does the unstable eigenvalue become a complex eigenvalue pair in Cartesian coordinates?

**2012-04-12 Predrag:** to the C Gang: this is not [how one writes a paper](#).

**2012-04-15 Predrag to Keith:** It's too late for a new complex Lorenz equations attractor image. I have asked Evangelos to join us - we will use his `CLEperpReqb` in Fig. ??(c), and he will help Lei with the two-mode model simulations in the coming weeks.

**2012-04-17 Predrag to Chaos Gang:** Start writing up whatever you are doing in the C Gang term project report in `siminos/cgang/`. The gang is allowed to reuse any equations and or figures from the paper I wrote<sup>16</sup> but not a single sentence penned by me or there will be copyright violation suits. Cannot steal from Wikipedia either (if you find any website where we collectively can learn something about slicing, alert me instantly).

**2012-04-17 Predrag to Lei:** Evangelos (who did the CLE modeling that we used in the paper) is interested in whether the two-mode model exhibits more interesting 2-chart slice than the CLE, so he plans to join you in searching for interesting chaotic behavior. Do not feel guilty for not contributing to the paper - we divided the labor between completing the Rössler, CLE and working on two-mode in parallel, so you were doing your share.

**2012-04-20 Keith:** I also have Rössler return maps for a Poincaré section of our choice. Even though the interesting dynamics is/are contained in the near equilibrium chart, if you want this, I can supply, I believe I fixed errors.

**2012-04-20 Predrag to Keith:** Can you include it in your project report? Paper is too long and return maps are already in ChaosBook.org and Basu project. What would be great is to get forward Poincaré maps for two-mode model.

I would put a priority on the Dangelmayr two-mode model, as complex Lorenz equations seems too simple. Lei is MIA (last blog entry on two-mode was [2012-04-10]), you can perturb his relative equilibria and see what happens in Cartesian coordinates), so maybe you can just play with it yourself - it's no harder than complex Lorenz equations, maybe even easier, it is in 4 dimensions. If there are two distinct relative equilibria and chaos that hops between them, I hope there is a robust chart border separating them, with 2-chart atlas ridging it. That would be much more persuasive. Besides, the Dangelmayr two-mode model is known by many more people than complex Lorenz equations, would have more of an audience.

It's not guaranteed to work, I'm worried about the role that the invariant subspace plays - might still turn out not to be too interesting, even though it should have more relative equilibria than complex Lorenz equations.

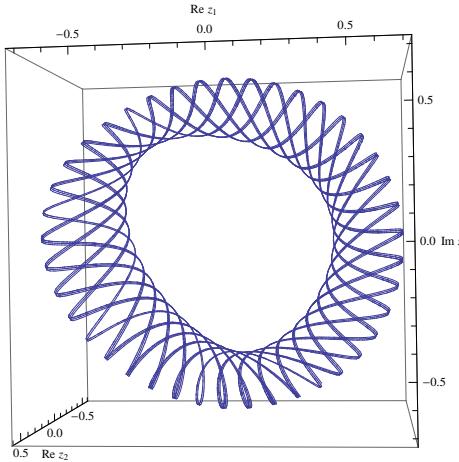


FIG. 15. Dangelmayr projected

**2012-04-20 Predrag to Lei:** Can you (and Evangelos?) write up two-mode your project in siminos/cgang/Lei/ , blog the progress here so gang is up-to-date?

**2012-04-20 Keith to Predrag:** Yes I was also going to take a look at the two-mode this weekend. I'll see what I can make out.

**2012-04-21 Bryce to Predrag:** Are we going to stick with Keith's figures and continue refining the paper (i.e. investigate the two-mode model?) or should I spend some time fine tuning my figures?

**2012-04-21 Predrag to Bryce:** I think Rössler figures are now pretty easy to understand (section borders and ridges are hopefully clear now), so I suggest playing with two-mode model instead.

**2012-04-21 Evangelos:** Our (20) seems like a special case of Porter and Knobloch<sup>30</sup> equation (12) but some brave young gangster has to derive the correspondence of parameters, so that we can properly cite them. Predrag's introduction of  $e_2$  seems to be the minimal modification required to break  $O(2)$  to  $SO(2)$ . Some exploration of (20) using siminos/cgang/Evangelos/dangelmayr\_so2\_int.nb shows we can have chaos, so we can stick to it. I leave it to Lei *et al.* to pick most interesting parameter values.

**2012-04-24 Lei:** Using Evangelos' code, I found the following set of parameters may be interesting.  $\mu_1 = -0.14, \mu_2 = 1.175, a_1 = -0.245, a_2 = -3.44, b_1 = 1.326, b_2 = -0.47, c_1 = 1, c_2 = -1, e_2 = 0.855$ . All the eigenvalues have positive real parts and both of the equilibrium points (in polar coordinates) have conjugate complex eigenvalues. So there is nice spirals around them. See Fig. 15 for projections 3-dim space.

**2012-04-25 Evangelos:** Lei, your parameter  $a_2$  had opposite sign in your text than what you use in your code. Also see [2012-04-25 Predrag] on the system not being chaotic.

**2012-04-24 Predrag to Lei:** • stability eigenvalues, eigenvectors of the equilibrium  $EQ_0$  at origin, at your parameter values - if it is stable, everything just might fall into it and die.

- stability eigenvalues, eigenvectors of  $TW_1$
- stability eigenvalues, eigenvectors of  $TW_2$
- $TW_1$ : unstable eigenvalue becomes a complex eigenvalue pair in Cartesian coordinates?
- $TW_2$ : Does the unstable eigenvalue become a complex eigenvalue pair in Cartesian coordinates?
- their plots in the Cartesian coordinates
- $\dot{\theta}$  to see how slow/fast are they.
- plots of small perturbations of the above equilibrium and relative equilibria in the Cartesian coordinates to see whether the dynamics looks chaotic

**2012-04-25 Lei to Predrag:** In polar coordinates

$$TW_1: (r_1, r_2, \psi) = (0.712184, 0.322201, -0.919559)$$

**Eigenvalues:**  $0.500667 \pm 1.87481i, 0.411071,$

**Eigenvectors:**  $\{\text{Re } \mathbf{e}^{(1)}, \text{Im } \mathbf{e}^{(1)}, \mathbf{e}^{(3)}\} =$   
 $(0.17281, 0.100144, 0.939794),$   
 $(-0.095779, 0.260235, 0),$   
 $(-0.560937, -0.380607, 0.735179)$

$TW_2$ :  $(r_1, r_2, \psi) = (1.24041, 0.357653, -0.238384)$

**Eigenvalues:**  $3.20719, 0.259474 \pm 2.44789i$

**Eigenvectors:**  $\{\mathbf{e}^{(1)}, \text{Re } \mathbf{e}^{(2)}, \text{Im } \mathbf{e}^{(2)}\} =$   
 $(-0.0393065, -0.102858, 0.993919),$   
 $(0.127876, 0.628119, -0.229194),$   
 $(-0.56813, 0, 0.462399)$

**2012-04-25 Predrag:** Fig. 15 is not chaotic - it simply says that one of your relative equilibria (plot them both in Fig. 15) is undergone a Hopf bifurcation into a stable limit relative periodic orbit that you have plotted. Keep changing parameters so this relative periodic orbit goes unstable and begets chaos.

**2012-04-25 Lei:** In cartesian coordinates, there seems to be no equilibrium points.

**2012-04-24 Keith to Lei:** Isn't the origin an equilibrium? And the periodic orbits shouldn't be a problem, since once we reduce they become equilibrium.

**2012-04-25 Lei:** All equilibria points in the reduced (set  $\psi = 2\theta_1 - \theta_2$ ) polar coordinates are actually periodic orbits in original coordinates.

**2012-04-25 Predrag:** The flow is  $\text{SO}(2)$ -equivariant, so there cannot be any equilibria and/or periodic orbits that do not belong to a flow-invariant subspace - this is the reason why we chose  $\text{SO}(2)$ -equivariant, not  $\text{O}(2)$ -equivariant system to study. These are not periodic orbits, these are relative equilibria. Please study the atlas paper and ChaosBook chapter on continuous symmetries. A relative equilibrium is a lazy drifter, a relative periodic orbit is a bold dancer.

We need  $\dot{\theta}$  for each of the two relative equilibria. Always list it when you describe a relative equilibrium.

$TW_2$  seem to have insanely unstable real eigenvalue, nothing probably comes close to it.

Is the complex eigenvalue pair of  $TW_2$  really mixing  $r_1$  and  $\psi$ ? Fascinating - like bricks and cats, one has dimension of length, the other is a dimensionless angle?

**2012-04-25 Evangelos:** Not sure its a good idea to compute stability of relative equilibria in polar coordinates, even though it is easier to locate relative equilibria in polar coordinates. I think one should do what we did in Ref. 46.

**2012-04-05 Predrag to two-mode gang:** before we get into the habit of calling it 'Dangelmayr system': His had  $\text{O}(2)$  symmetry, we study a related normal form with only  $\text{SO}(2)$ , hence I've been using the neutral but boring 'two-mode' appellation. As it is the custom name something after a person who has not done it, I have no serious objections to 'Dangelmayr system'.

**2012-04-25 Evangelos to Predrag:** It should most probably called two-mode system, see [2012-04-21 Evangelos]. Until some volunteer in the gang establishes the connection to two-mode system by computing the relation of their parameters to ours, I tentatively call it Dangelmayr system, since this is how you introduced it in (F6).

**2012-04-25 Predrag to two-mode gang:** continuing on the discussion above, of [2012-04-05 Lei to Predrag], a simple algebra question: you say there are 20 complex solutions for equilibria + relative equilibria, with relative equilibria overcounted as you allow  $\{\pm r_1, \pm r_2\}$ . Can you count how many distinct complex roots are there? That perhaps tells you what is the maximal number of templates this model could have. If we know the order of the polynomial, you can track the roots in the complex plane, see for what parameter values they go real. Evangelos might already know the answer, because if you rewrite the two-mode system in terms of Armbruster invariant polynomials, there is no  $\pm r_i$  problem.

**2012-04-05 Predrag to two-mode gang:** Really happy to see Fig. 11! Seeing chaos is a start, but not enough. Fig. 11 seems to be a result of a Hopf relative periodic orbit bifurcated off a relative equilibrium going chaotic (please, always plot all the relative equilibria). That's going to be not more interesting than the complex Lorenz flow. We need chaos that visits qualitatively different templates in the pedagogically simplest setting: a more complicated example, waiting to be sliced, are the equilibria + relative equilibria of Ref. 46. The only example

where 2-chart atlas of 2 sections (no continuous symmetry) was constructed is Ref. 47 (click [here](#)). You can see what templates should be by a glance at Fig. 1. The central template's chart (Fig. 8) worked out beautifully. I never managed to get across to Lan as to how to chart the second template neighborhood (Fig. 9, with a stunningly long periodic orbit), so we did not publish that return map (it might be in Lan's thesis). We were still using Fourier modes as coordinates then (stupid, stupid), and the section borders and ridges were yet undreamed of - that is the key contribution of the current C Gang paper.

**Evangelos to Lei:** In [2012-04-05 Lei to Predrag] you say you get a 10th order polynomial equation for relative equilibria. Could you please enter it here?

**2012-04-26 Lei to Keith:** Yes. Zero should be a equilibrium. I just omit it by manipulating algebraic equations...

**2012-04-26 Daniel:** Probably old hat at this point but I'm writing the equations for the two-mode system in case they ever come in handy. Here's Siminos's (20) in terms of 4 real variables, such that  $z_1 = x_1 + ix_2$  and  $z_2 = y_1 + iy_2$ . I did the algebra using Matlab's symbolic math toolbox, so these should be correct.

$$\begin{aligned}\dot{x}_1 &= \mu_1 x_1 + a_1 x_1^3 + b_1 x_1 y_1^2 + c_1 x_1 y_1 + a_1 x_1 x_2^2 + b_1 x_1 y_2^2 + c_1 x_2 y_2 \\ \dot{x}_2 &= \mu_1 x_2 + a_1 x_1^2 x_2 + c_1 x_1 y_2 + b_1 y_1^2 x_2 - c_1 y_1 x_2 + a_1 x_2^3 + b_1 x_2 y_2^2 \\ \dot{y}_1 &= \mu_2 y_1 + a_2 x_1^2 y_1 + c_2 x_1^2 + b_2 y_1^3 + a_2 y_1 x_2^2 + b_2 y_1 y_2^2 - c_2 x_2^2 + e_2 y_2 \\ \dot{y}_2 &= \mu_2 y_2 + a_2 x_1^2 y_2 + 2c_2 x_1 x_2 + b_2 y_1^2 y_2 - e_2 y_1 + a_2 x_2^2 y_2 + b_2 y_2^3\end{aligned}\quad (\text{F10})$$

$$A = \begin{pmatrix} \mu_1 + 3a_1 x_1^2 + b_1 y_1^2 & c_1 y_2 + 2a_1 x_1 x_2 & c_1 x_1 + 2b_1 x_1 y_1 & c_1 x_2 + 2b_1 x_1 y_2 \\ +c_1 y_1 + a_1 x_2^2 + b_1 y_2^2 & \mu_1 + a_1 x_1^2 + b_1 y_1^2 & 2b_1 y_1 x_2 - c_1 x_2 & c_1 x_1 + 2b_1 x_2 y_2 \\ c_1 y_2 + 2a_1 x_1 x_2 & -c_1 y_1 + 3a_1 x_2^2 + b_1 y_2^2 & \mu_2 + a_2 x_1^2 + 3b_2 y_1^2 & e_2 + 2b_2 y_1 y_2 \\ 2c_2 x_1 + 2a_2 x_1 y_1 & 2a_2 y_1 x_2 - 2c_2 x_2 & +a_2 x_2^2 + b_2 y_2^2 & \mu_2 + a_2 x_1^2 + b_2 y_1^2 \\ 2c_2 x_2 + 2a_2 x_1 y_2 & 2c_2 x_1 + 2a_2 x_2 y_2 & 2b_2 y_1 y_2 - e_2 & +a_2 x_2^2 + 3b_2 y_2^2 \end{pmatrix}. \quad (\text{F11})$$

**2012-04-26 Daniel to Evangelos:** Just tried to figure out how the parameters in our two-mode system (20) are related to the paramters in two-mode SO(2)-symmetric equations and found that they do not map directly (or perhaps you made a mistake copying your results into the blog). P&K's equation for  $\dot{z}_2$  has a linear term in  $z_2$  and no linear term in  $z_1$ . I don't have access to the original Dangelmayr paper, so I can't check that one, but Armbruster et al.<sup>56</sup> also have this term. Our  $\dot{z}_2$  has a linear term in  $z_1$  but no linear term in  $z_2$ . I guess if you are seeing chaos in our system, it might be fine, but for a larger audience we might want the P&K or AG&H systems. Let me know if you've made a mistake and I'll recalculate the various equations in 4D and polar coordinates.

**2012-04-26 Evangelos to Daniel:** That was a typo in (20), sorry, fixed now. There is no error in my Mathematica files, which use the correct Dangelmayr equations (which are also the starting point for P&K) and the parameters used in the figures are also correct. But it would be a good idea if someone else checked to see what the attractors look like.

**2012-04-26 Evangelos:** Added Dangelmayr paper<sup>31</sup> in CNS zotero library.

**2012-04-26 Predrag to Evangelos:** Zotero says "The attached file could not be located ..."

**2012-04-27 Evangelos:** Maybe we run out of space. I'll check tommorow, meanwhile I'll email it to you.

**2012-04-26 Predrag:** Thanks! I put it [here](#). This version is better than Evangelos', as I have made it text-searchable and a bit smaller than the journal download.

Dangelmayr paper<sup>31</sup> is a serious, detailed paper on normal forms for bifurcations, and quite interesting. I like his concise statement of invariance and equivariance. Invariant polynomials in 2 complex dimensions are so seductive, wish they were good also for 10's or 100's of dimensions...

Which student is going to (gasp!) read it, and (gasp! gasp!) blog about it?

Dangelmayr considers a system constructed from two arbitrary Fourier modes, a representation of the symmetry group O(2) which is defined by the operations of rotation and reflection,

$$(z_1, z_2) \rightarrow (e^{im\theta} z_1, e^{in\theta} z_2) \quad (\text{F12a})$$

$$(z_1, z_2) \rightarrow (\bar{z}_1, \bar{z}_2), \quad (\text{F12b})$$

respectively, acting on complex vectors  $(z_1, z_2) \in \mathbb{C}^2 \simeq \mathbb{R}^4$ . Here,  $m, n$  are positive integers where  $(m, n)$  satisfies either  $2 \leq m < n$  with  $(m, n)$  relatively prime or  $m = 1 < n$ , and the bar denotes complex conjugation. He cites an unpublished preprint by Buzano and Russo for establishing that there are three basic functions that are invariant under (F12),

$$u = z_1 \bar{z}_1, \quad v = z_2 \bar{z}_2, \quad w = z_1^n \bar{z}_2^m + \bar{z}_1^m z_2^n. \quad (\text{F13})$$

and that the basic equivariants are

$$\{z_1, \bar{z}_1^{n-1} z_2^m\}, \quad \{z_2, z_1^n \bar{z}_2^{m-1}\}. \quad (\text{F14})$$

Buzano and Russo published papers are tedious, we do not need to cite them - all about buckling rods, citing only Dangelmayr is fair enough. Any invariant function can be represented as  $f(u, v, w)$  for some  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ . A general smooth mapping  $\mathbb{C}^2 \rightarrow \mathbb{C}^2$  can be constructed in terms of the basic equivariants by multiplying each of the vectors in (F14) by a general invariant function and then adding the resulting terms. In this way we obtain the following form for a general smooth vector field which is equivariant under (F12):

$$\begin{aligned} \dot{z}_1 &= p_1 z_1 + q_1 \bar{z}_1^{n-1} z_2^m \\ \dot{z}_2 &= p_2 z_2 + q_2 z_1^n \bar{z}_2^{m-1}, \end{aligned} \quad (\text{F15})$$

where  $p_j, q_j$  are smooth invariant functions, that is, *real functions* of  $u, v, w$  and any external parameters. The cases in which  $m = 1$  and  $m > 1$  give rise to rather distinct behaviour. Also, the case where  $m = 1$  must be further distinguished according to whether  $n = 2$  or  $n > 2$ . Dangelmayr discusses  $(m, n) = (1, 2)$  and  $m > 2$  cases.

We consider here only the simplest normal form of the  $(m, n) = (1, 2)$  case, and go from  $O(2)$  to  $SO(2)$  by taking some parameters complex, and thus breaking the reflection invariance,  $\dot{\bar{z}}_j \neq \dot{\bar{f}}_j$ . As Dangelmayr consider only the real case, calling our system a special case of two-mode is fair enough.

**2012-04-21 Evangelos:** For future reference I note that the connection of the constants in (20) with  $e_2 = 0$  to the constants in equation (2.3) of Ref. 31 with  $n = 2, m = 1$  is  $\mu_1 = \nu\epsilon\alpha, a_1 = -\nu\epsilon, b_1 = -\nu\epsilon\rho, c_1 = -\nu\mu, \mu_2 = \epsilon\beta, a_2 = -\epsilon\kappa, b_2 = -\epsilon\epsilon', c_2 = \mu\mu'$ .

**2012-04-27 Predrag:**  $SO(2)$  is a subgroup of  $O(2)$ , without the reflection group element (F12b). As a subgroup, it has to satisfy invariance constraints in addition to those of  $O(2)$ , with one more invariant polynomial in its basis, called  $q$  in what follows. The stand-alone write up is now in Section III.

**2012-04-27 Predrag:** This could be more elegant - I made only  $(\mu_2 - i e_2)$  imaginary, and I am not sure why I am breaking the reflection symmetry of  $m = 2$  mode (that also breaks it in the  $m = 2$  flow-invariant subspace), but that's what we have done so far. Finding equilibria using these equations should be easy - they are like the polar equations but more elegant, extracting  $\psi$  is silly -  $w, q$  tell you you want equations for  $(\cos \psi, \sin \psi)$  instead.

**2012-04-28 Evangelos:** Indeed, using (30) should be the cleanest way to compute relative equilibria, as equations are given in a clear polynomial form. To find relative equilibria in  $u, v, w, q$  basis one would have to solve  $\dot{u} = \dot{v} = \dot{w} = \dot{q} = 0$  in (30) together with the syzygy (27) [ $w^2 + q^2 - 4u^2v = 0$ ], right? In other words, (27) is a constraint in allowable solutions which dynamics knows about, but when solving algebraic equations for (relative) equilibria, don't we have to impose it?

**2012-04-27 Predrag to Chaos Gang:** That sinking feeling...

OK, I learned my lesson: **you** do not write. The game plan is to abuse a defenseless assistant professor who must publish. She will write your articles for you in grad school, and then you'll be on the street again.

But, do you read? In your project you are referencing several impressive, well written two-mode articles, where some of the problems you are encountering are discussed and solved. Ethos of the profession is that if one references a publication, one has read it, and the key ones one has studied in depth. OK, not all of us have read *Principia* (I did it for the high school senior project, that was before we had smart phones, and even then I did not understand anything), but we do not cite it either. And it gets fuzzy in collaboration, as not all contributors contribute to everything. But then at least one of the coauthors can explain what a given cited paper says.

**2012-04-26 Daniel:** Uploaded code to `/siminos/cgang/Daniel/Matlab/`. Just type `EOMComplexTo4D` at the Matlab prompt and it will calculate the system in 4D Cartesian, 4D polar, and 3D polar coordinates. The  $z_j$  equations can be modified within the code if need be.

For future reference, our two-mode system (F6) is a special case ( $\omega_1 = 0$ ) of the system of Porter and Knobloch<sup>30</sup>. Our parameters are related to their parameters by (P&K  $\leftrightarrow$  ChaosGang two-mode):

$$\begin{aligned} \mu_1 &\leftrightarrow \mu_1, \epsilon\omega_1 \leftrightarrow 0, \tilde{\alpha} \leftrightarrow c_1, \tilde{d}_{11} \leftrightarrow a_1, \tilde{d}_{12} \leftrightarrow b_1, \mu_2 \leftrightarrow \mu_2, \\ \epsilon\omega_2 &\leftrightarrow -e_2, \beta \leftrightarrow c_2, \tilde{d}_{21} \leftrightarrow a_2, \text{ and } \tilde{d}_{22} \leftrightarrow b_2. \end{aligned}$$

**2012-04-27 Evangelos:** Great, it wasn't so terrible after all. So now we can call this system Porter-Knobloch flow.

**2012-04-27 Evangelos to Daniel and the gang:** I think since Porter and Knobloch use complex parameters, our model is a special case of theirs with  $\omega_1 = \alpha_i = \beta_i = c_{11} = c_{12} = c_{21} = c_{22} = 0$  and correspondence of parameters:

$$\mu_1 \leftrightarrow \mu_1, \alpha_r \leftrightarrow c_1, d_{11} \leftrightarrow a_1, d_{12} \leftrightarrow b_1, \mu_2 \leftrightarrow \mu_2, \epsilon\omega_2 \leftrightarrow -e_2, \beta_r \leftrightarrow c_2, d_{21} \leftrightarrow a_2, \text{ and } d_{22} \leftrightarrow b_2.$$

**2012-04-27 Evangelos:** If we let  $\omega_1 \neq 0$  we might generate some interesting dynamics (by adding an extra frequency we might gain an extra relative equilibrium).

**2012-04-26 Daniel to Chaos Gang:** Sorry to keep raining on the party, but I believe (F8) have another typo in them. If you take (F6) and set them to zero, you end up with a  $b_1 r_2^2$  term in the equation for  $\dot{r}_1$  rather than  $b_1 r_1 r_2$  term. This is confirmed by my `EOMComplexTo4D.m` code using (20) as a starting point. If you are working on the two-mode relative equilibria, you probably want to stop and double check this before moving on but I'm pretty sure I'm right. **To Lei:** Did you derive (F20) from (F6) or from (F8)? If it was the latter, you probably want to double check it.

**2012-04-27 Evangelos:** Daniel is correct, I have the same term in my notes.

**2012-04-27 Keith:** I have developed a program that finds zeros of the equations we have (it disregards imaginary solutions). About half of them actually come up as relative equilibria, and these are nice and the trajectories that I start on these stay one these; the other solutions are not doing this at all. I am checking the conditions we have about the flow being in the same direction as the tangent and only some of them meet the requirements. I am wondering if using the squared versions of the equations is introducing more solutions than there truly are. The other thought I am having is if this squaring could be introducing problems with the phases being offset by  $\pi$  or others.

**2012-04-27 Keith:** I spent some time really thinking about this, and I just tested it. You can't just use the squared equations, (F19), that we have, you need to also go back and verify it solves the original equation (F8), otherwise you come up with strange solutions that almost act as relative equilibria but they are not. I have tried to see if adding phase factors will adjust this, but it appears not. Does anyone have an insight as to why this is? It is probably something simple.

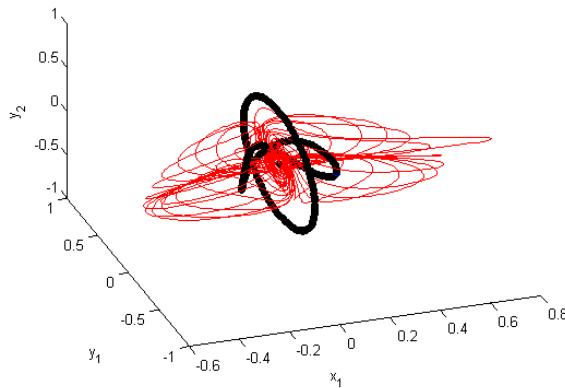


FIG. 16. Possibly what we are looking for in the two-mode system. Red labels the trajectory, black labels the relative equilibrium.

**2012-04-27 Keith:** So I toyed around with a bunch of things and kept twisting parameters, and eventually got this Fig. 16. I can give you the eigenvalues/vectors and relative equilibrium. I also have the parameters and I twisted them, not a computer. I want to plot it in reduced state space, but I need to go eat first. I just want to get this out there before I do that. I think this has the elements we are looking for. The trajectories are spiralling around one of the relative equilibria, and if you look close enough they are spiralling around the equilibrium at the origin. I plotted this in 4-d space, and in hind sight should have done the polar coordinates, but for now I think this image captures what I have.

**2012-04-27 Keith:** There is one problem, and its in a couple of the papers, but if you start on  $r_1 = 0$  you stay on it and tend to approach the relative equilibrium in that space, but this is OK, I think.

**2012-04-27 Predrag:** It's no problem, mon ... just a situation. It's a flow invariant subspace.

**2012-04-27 Bryce:** I'm in the process symmetry reducing the flow myself but in the meantime I wanted to verify a couple of things. Under what rotation is the Dangelmeyer system invariant? That is to say which entries of generator (assuming one is working in  $\{x_1, y_1, x_2, y_2\}$  coords.) are zero/one? Keith: which parameters did you use to produce Fig. 16?

**2012-04-27 Keith to Bryce:** The parameters were bad. They don't have the trajectory visiting both equilibrium, it just looks that way. If you still want them, Ill give them though. Generator:

$$T = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{pmatrix}. \quad (\text{F16})$$

The minus sign may be switched, its the same as the complex Lorenz equations. But this should be enough to get you started.

**2012-04-27 Predrag:** The infinitesimal transformations generator (F16) is correct.

**2012-04-27 Predrag to Daniel:** As you did check the invariance by showing that the Jacobian derivative,

$$t_a(v) - A(a) t_a(a) = 0, \quad (\text{F17})$$

vanishes, maybe you make it into problems set - solution like you did for my original proto-two-mode, it would be nice to have that explicitly in the two-mode project?

**2012-04-26 Predrag:** On factoring polynomials (this is somewhere in the blog of a few weeks ago, but perhaps now falling on more receptive ears): two-mode system has 2 invariant subspaces,  $r_1 = r_2 = 0$  and  $r_1 = 0, r_2 \neq 0$ , so you already know two roots of your polynomial; the  $u = v = 0$ ; and  $u = 0, \mu_2 + b_2 v = 0$ . Still do not know what the order of this polynomial really is, but whatever polynomial you get, you should *divide it* by the roots you already know, that brings the order of the polynomial down.

**2012-04-27 Daniel:** I know that the Chaos Gang is zipping ahead into uncharted territory, but here are some previously undocumented results for meager relative equilibria and the easy relative equilibria.

- By using Matlab to symbolically solve (F10), I found that the only equilibrium is the origin, independent of parameter choices. [2012-03-31 Predrag] Most solutions are ‘mixed modes’. However, [2012-04-06 Predrag] there is an equilibrium for  $r_1 = r_2 = 0$ . There is a invariant subspace  $r_1 = 0, r_2 > 0$  which you can solve analytically - it may cause us some trouble. If what you end up solving is a polynomial in  $r_1$ , you want to divide it with all these known roots, see whether anything of interest is still left. [2012-04-25 Predrag] Obviously there is one equilibrium, very important one, for all  $z_i = 0$ . Eigenvectors could be complex, due to the  $\text{SO}(2)$  equivariance of perturbations off the origin. One eigenvector pair must belong to the  $m = 2$  invariant subspace, the other should be pure  $m = 1$ , as at the origin linear theory of irreducible representations applies (reread ChaosBook.org Sect. 10.3 *Stability*). The eigenvalues are  $\lambda_1 = \mu_1$  with multiplicity 2 and  $\lambda_2 = \mu_2 \pm i\epsilon_2$ . The eigenvectors for  $\lambda_1$  are  $(1, 0, 0, 0)$  and  $(0, 1, 0, 0)$  in the  $(x_1, x_2, y_1, y_2)$  basis. The complex eigenvectors for  $\lambda_2$  are  $(0, 0, \mp i, 1)$ . [2012-04-25 Predrag] One always replaces complex eigenvector pair by  $\{\text{Re } \mathbf{e}^{(j)}, \text{Im } \mathbf{e}^{(j)}\}$ : that defines the real plane in which spiral-out takes place. The eigenvectors for  $\lambda_2$  thus are  $(0, 0, 1, 0)$  and  $(0, 0, 0, 1)$ . I have uploaded the code that generated these as /siminos/cgang/Daniel/Matlab/TwoModeOriginStability.m

- [2012-04-27 Predrag] As we want hyperbolicity, pick  $\mu_j$  of opposite signs. In order to avoid the complex Lorenz equations near visits to the invariant subspace (there it was the  $z$  axis, here it is the  $m = 2$  subspace) pick contraction within the subspace,  $\mu_2 < 0$ , but make it overall repelling by making sure that  $\mu_1 > -\mu_2 > 0$ . No parameters that you have reported chaos for so far satisfy that.
  - I also started going after relative equilibria. The first (and obvious) is in the  $r_1 = 0, r_2 > 0$  invariant subspace. Starting from (F6), I got that there are relative equilibria at  $r_1 = 0, r_2 = \pm\sqrt{-\mu_2/b_2}$ , and  $\psi = \sin^{-1} \left( \pm \frac{e_2}{2c_1} \sqrt{\frac{-b_2}{\mu_2}} \right)$ . Given the additional constraints that  $r_2$  must be real and positive, I conclude that the relative equilibria reduce to a single one at  $(r_1, r_2, \psi) = \left( 0, \sqrt{-\mu_2/b_2}, \sin^{-1} \left( \frac{e_2}{2c_1} \sqrt{\frac{-b_2}{\mu_2}} \right) \right)$ , which is only physical if  $\mu_2/b_2 < 0$  and  $\left| \frac{e_2}{2c_1} \sqrt{\frac{-b_2}{\mu_2}} \right| \leq 1$ .
- [2012-04-27 Predrag] please compute its stability. As it is within the repelling  $m = 2$  invariant subspace, it's existence and stability probably does not matter much, but if you can make it unstable, do so.

**2012-04-25, 2012-04-27 Predrag:** Thanks! moved the earlier entries that nobody read into your text.

**2012-04-25 Bryce:** Nothing fancy here but I've written code to symmetry reduce (F10). I am currently choosing a random slice (no thinking yet!) just to see if it worked and the results were quite surprising Fig. 17.

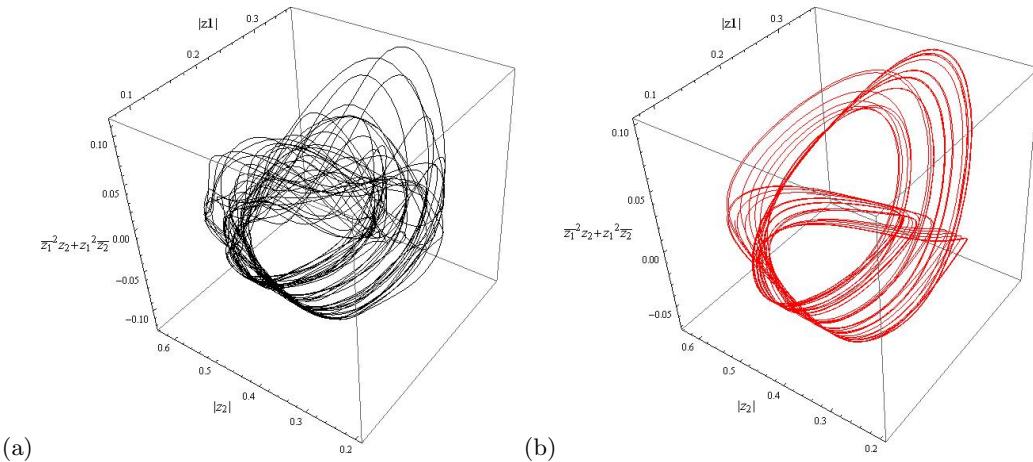


FIG. 17. Projection of Dangelmayr system (F10) “attractor” for (not a good choice) parameters (E1). (a) Projection onto original state space:  $(|z_1|, |z_2|, z_1^2 z_2 + z_1 z_2^2)$ . (b) Symmetry reduced trajectory using the following template for the slice  $(0.220299, 0.784274, -1.03596, 2.46552)$ .

**2012-04-27 Keith:** I am still writing my own project up, I have just been posting in here because everyone was working on the 2-mode, is that alright? I haven't been posting there because I focused on this for a while, and haven't updated that one. I do not know how much I need to input in this one therefore, or if me going off and studying the return map stuff is alright.

**2012-04-28 Predrag to Keith:** Of course, the return map stuff is important.

**2012-04-28 Lei to Predrag:** Can I get access to Dangelmayr's paper? It seems I don't know the password.

**2012-04-28 Predrag:** cnsuser cnsweb

**2012-04-27 Predrag:** Regarding (20) in terms of 4 real variables (F10)- unfortunately for complex Lorenz equations one writes  $z_1 = x_1 + ix_2$  and  $z_2 = y_1 + iy_2$  (so rotations are in ‘x’ and ‘y’ planes - this mixes them up). Might be confusing in the time to come.

**2012-04-28 Daniel:** Well, it's sort of trivially easy fix. I just thought I'd keep 1's together and 2's together since that's the way they appear in eqs. (F6) and (20). Want me to change them to  $z_1 = x_1 + ix_2$  and  $z_2 = y_1 + iy_2$ ? I would take me 5 minutes. [2012-04-29 Daniel] Changed all 4D Cartesian representations of  $z_1$  and  $z_2$  in blog and my Matlab codes to be consistent with the substitution used in DasBuch for the complex Lorenz equations system as outlined by Predrag in the previous post.

**2012-04-28 Predrag:** Low priority, but if only 5 minutes, do it sometime.

**2012-04-28 Predrag:** A stand-alone write up of invariant polynomials formulation of two-mode is now in Section III.  
Edit and improve it there, in the project proper.

**04-28-2012 Daniel:** Starting from the following definitions: <sup>105</sup>

$$\begin{aligned} u &= z_1 \bar{z}_1, \quad v = z_2 \bar{z}_2 \\ w &= (z_1^2 \bar{z}_2 + \bar{z}_1^2 z_2)/2 = \operatorname{Re}(z_1^2 \bar{z}_2) = u|v|^{1/2} \cos \psi \\ q &= (z_1^2 \bar{z}_2 - \bar{z}_1^2 z_2)/2i = \operatorname{Im}(z_1^2 \bar{z}_2) = u|v|^{1/2} \sin \psi, \end{aligned} \quad (\text{F18})$$

Related by the syzygy  $w^2 + q^2 - u^2 v = 0$ . <sup>107</sup> Starting from these definitions, I get the following for  $\dot{u}$  and  $\dot{v}$ :

$$\begin{aligned} \dot{u} &= 2\mu_1 u + 2a_1 u^2 + 2b_1 u v + 2c_1 w \\ \dot{v} &= 2\mu_2 u + 2a_2 u v + 2b_2 v^2 + 2c_2 w \\ \dot{w} &= (4\mu_1 + 2\mu_2) w + (4a_1 + 2a_2) u w + (4b_1 + 2b_2) v w \\ &\quad + 4c_1 u v + 2c_2 u^2 - 2e_2 q \\ \dot{q} &= (4\mu_1 + 2\mu_2) q + (4a_1 + 2a_2) u q + (4b_1 + 2b_2) v q + 2e_2 w \end{aligned}$$

Again... these are using the (F18) definitions of  $w$  and  $q$ , not the ones that Predrag has specified in Section III.  
Use at your own risk.

**2012-04-28 Predrag:** Please use (26) which follows Dangelmayr rather than (F18) - in that case (F19) and (30).  
Apologies for causing the confusion.

**2012-04-28 Keith to Predrag:** The  $r_1 = 0$  invariant space should remain a circle because all points on its group orbit already meet the condition for being in the slice, which yes, I meant the template point is:  $\hat{a}' = [1, 0, 0, 0]$ .  
The order of coordinates I am using is  $[x_1, x_2, y_1, y_2]$  where  $x_1$  and  $x_2$  are the first mode and  $y_1$  and  $y_2$  are the 2nd mode as you seem to suggest above.

**2012-04-28 Keith to Predrag:** Yes, the  $r_1 = 0$  follows Daniel's rule, though I am certain this was mentioned in the Armbruster paper. Also they seem to suggest using the  $z_i = x_i + iy_i$ . I have switched my notation for what you have suggested.

**2012-04-28 Predrag:** I'm used to complex Lorenz equations convention, where  $x_i$  and  $y_i$  are two rotation planes, but Armbruster  $z_i = x_i + iy_i$  is how we always write complex numbers. That should trump the newfangled complex Lorenz equations convention. You guys decide which one.

**2012-04-29 Predrag:** [with 2012-04-29 Daniel's correction] Eqs. (36) are what Lei had as of [2012-04-26] (Keith and Predrag had checked some of the algebra):

$$\begin{aligned} c_2^2 u^2 (\mu_1 + a_1 u + b_1 v)^2 - c_1^2 v^2 (\mu_2 + a_2 u + b_2 v)^2 &= 0 \\ \frac{1}{c_1^2} (\mu_1 + a_1 u + b_1 v)^2 + \frac{v^2 e_2^2}{(c_2 u + 2c_1 v)^2} &= v. \end{aligned} \quad (\text{F19})$$

Factoring the first equation one gets

$$\frac{u}{c_1} (\mu_1 + a_1 u + b_1 v) \pm \frac{v}{c_2} (\mu_2 + a_2 u + b_2 v) = 0, \quad (\text{F20})$$

but we know from (35) that only '-' sign is correct. <sup>109 111</sup> Adding, and using (34) I get

$$w^2 - 4u^2 v + \frac{4e_2^2 u^2 v^2}{(c_2 u + 2c_1 v)^2} = 0, \quad (\text{F21})$$

<sup>106</sup> PC: I agree this would be more natural, but that is not the  $w$  Dangelmayr used so we are stuck with unnatural factors of 2.  
Grin and bear it.

<sup>108</sup> DB 2012-04-28: Looks like Predrag changed these but I want to put these down for now.

<sup>110</sup> PC: 2012-04-28 The invariant polynomials get this right and

probably might cure the wrong roots problems that [2012-04-27 Keith] has noted

<sup>112</sup> DB 2012-04-30: This can also be shown by just solving the first two equations in (F8) for  $\cos \psi$  and setting them equal to each other. Gets the sign right too, no roots problem.

the same as the second equation of (35), so all is well (thanks Daniel for fix that pesky factor of 2!).

As we already know  $[0, 0, 0, 0]$  and  $[0, v, 0, 0]$  roots, we should divide them out, and that is what we have done. Finding roots of bivariate polynomials is not easy. As fate would have it, you can learn all about right now:

**Sunday, April 29, 2012** — 5:00pm — Klaus 1116

by Jason Cantarella (U Georgia), Southeast Geometry Seminar General Audience Lecture:

*The Square Peg Theorems or What does it mean to solve simultaneous equations?*

It's now been 101 years since Otto Toeplitz asked the simple question: Does every closed curve in the plane with no self-intersections contain four points which form the vertices of an inscribed square? This talk will be an introduction for a general audience to the various mathematical ideas which prove and generalize parts of the square peg theorem, complete with some good pictures and animations. The key idea for undergraduates is that the simple act of solving a system of two equations in two unknowns (something that everyone who took high-school math has probably done) is actually the jumping off point for a wonderful journey into modern mathematics.

At this point Google brain **kicks in**:

**Bezout's Theorem** Let  $f(x, y) = g(x, y) = 0$  be a system of two polynomial equations in two unknowns. If it has only finitely many common complex zeros  $(x, y) \in \mathbb{C}^2$ , then the number of those zeros is at most  $\deg(f) \cdot \deg(g)$ .

The degree of each term in a polynomial in two variables is the sum of the exponents, and the degree of the polynomial equals the largest such sum. For our first equation  $\deg(f) = 2$ , and for the second  $\deg(f) = 6$ . So we are up to maximum 12 roots; as we know two already, we are down to 6 roots. There is no  $(u, v) \rightarrow (-u, -v)$  symmetry, so we will have to weed out the negative  $u$  or  $v$  - would be nice to prove something about the number of positive roots (the roots in the open positive quadrant  $(\mathbb{R}_+)^2$ ) ahead of computing them. For a single variable polynomial one has:

**Descartes' Rule of Signs** The number of positive real roots of a polynomial  $g(x)$  is bounded above by the number of sign alternations of its consecutive coefficients.

For bivariate polynomials one has to count 'alternating mixed cells'. This is an active research area of mathematics, so best to give up while we are still ahead.

**2012-04-28 Bryce:** The simplest root I was able to find (aside from the obvious ones) was the following:

$$\left\{ u \rightarrow \frac{b_2 \mu_1 - b_1 \mu_2}{a_2 b_1 - a_1 b_2}, v \rightarrow -\frac{a_2 \mu_1 - a_1 \mu_2}{a_2 b_1 - a_1 b_2}, w \rightarrow 0, q \rightarrow \frac{2u}{e_2} (2c_1 v + c_2 u). \right\} \quad (\text{F22})$$

Haven't messed around with parameters in (F22) though...

**2012-05-08 Daniel:** Double checked eq. (F22) and found that it was wrong (factors of 2 and  $u$  and  $v$  were switched in the term in parentheses in the equation for  $q$ ). Verified that my corrected solution was an equilibrium of the two-mode system using Matlab. Uploaded relevant code as `/siminos/cgang/Daniel/Matlab/W0EquilibriumCheck.m`. [2012-05-08 Predrag] the error was introduced by me, in process of deMathematicizing Bryce's formula. Thanks Daniel for fix that pesky factor of 2! Should be able to do verify it by hand now, using (F23).

**2012-04-29 Predrag to Bryce:** I simplified your solution, please recheck my algebra; do not see it in (35) as yet. As  $w = 2u|v|^{1/2} \cos \psi$ , vanishing  $w$  implies that  $\psi = \pi/2$ . Cannot see the symmetry that would impose it. Wonder why would that be a solution? Perhaps because of the funky  $O(2)$  symmetry breaking  $e_2$  term. As  $q$  is proportional to  $e_2$ , there are no solutions with  $q = 0$ . However, (35) does simplify for  $w = 0$

$$\begin{aligned} c_2 u (\mu_1 + a_1 u + b_1 v) &= c_1 v (\mu_2 + a_2 u + b_2 v) \\ (c_2 u + 2c_1 v)^2 &= e_2^2 v, \end{aligned} \quad (\text{F23})$$

Both equations have  $\deg(f) = 2$ , so at most 4 roots, but that includes the two we already know. The  $v > 0$  case is a bit weird - a constraint on coefficients?  $0 = \mu_2 + b_2 v$ ,  $4c_1^2 v = e_2^2$ . Must have made an error somewhere...

**2012-08-10 Predrag:** The relative equilibrium discovered by Bryce on 2012-04-28 ([F22](#))

$$\begin{aligned} u &= \frac{b_2\mu_1 - b_1\mu_2}{a_2b_1 - a_1b_2}, \\ v &= -\frac{a_2\mu_1 - a_1\mu_2}{a_2b_1 - a_1b_2}, \\ w &= 0, \\ q &= \frac{2u}{e_2} (2c_1v + c_2u). \end{aligned} \quad (\text{F24})$$

Does it satisfy syzygy  $w^2 + q^2 - 4u^2v = 0 \rightarrow q^2 = 4u^2v$ ? It says:

$$\frac{4}{e_2^2} (2c_1v + c_2u)^2 = v, \quad (\text{F25})$$

but all constants are independent, so how could it ever be a general solution? Do not know why we let it survive so long.

**2012-04-29 Predrag:** A cry in wilderness. OK, now that I've worked out your polynomials, can you, now that you are owed by my secret skills (OK - do not tell anybody - I looked at the articles that you cite), can you now for once [hear](#) me?

I'll be honored if you blow me off and then do the right thing, but remember Zsa Zsa's paperweight. Don't just screw around with zillion parameters. ChaosBook tells you to

1. determine the equilibria (in the invariant polynomial basis), then
2. study their stability,
3. move parameters so that equilibria are nicely hyperbolic, i.e., on the whole they repel more than they contract.

This way they will kick the trajectory around, not suck it in as happened to Keith's Hot Pants or Lei's Fig. 15. You want the repelling directions to win. If  $\{\lambda_e\}$  is the set of expanding eigenvalues, and  $\{\lambda_c\}$  the set of contracting ones, I believe you want as few expanding directions as possible (otherwise it is hard to find the contracting directions toward the hyperbolic point, and you want  $\sum \lambda_e > -\sum \lambda_c > 0$ , but not by very much, otherwise you never get close to the equilibrium).

In particular, for the equilibrium at the origin, pick  $\mu_j$  of opposite signs. In order to avoid the complex Lorenz equations-type near visits to the invariant subspace (there it was the  $z$  axis, here it is the  $m = 2$  subspace) pick contraction within the subspace,  $\mu_2 < 0$ , but make it overall repelling by making sure that  $\mu_1 > -\mu_2 > 0$ . [No parameters that you have reported so far in your experiments satisfy that.](#)

**2012-04-29 Keith:** This was what I was aiming to do for the previous stuff, but for some reason I could only get an attractor centered around a single template, and not two. Or else as I posted, I would get a limit cycle or trajectories that converged just towards one of the equilibria. I will continue to toy with the parameters, but am mostly focused on finishing up the return map stuff.

**2012-04-29 Keith to Predrag:** How does one divide out roots to a multinomial? I know how to do it for a single variable polynomial, but not a multi-variable one. You can't use the same formalisms because that doesn't make sense (ie I can't divide by:  $(r_1 - r_0^{(1)})(r_2 - r_0^{(2)})$  where  $r_0^{(1)}$  and  $r_0^{(2)}$  are the roots because this would imply that  $r_1 = r_0^{(1)}$  is always a root, and that may not be true  $\forall r_2$ ). I suspect one has to divide by a multinomial itself, but I do not know a way to get this.

**2012-04-29 Predrag:** Clipping from above [2012-04-29 Predrag]: "I do not know how to divide bivariate polynomials. Finding roots of bivariate polynomials is not easy." It seems to be front-line research topic in algebraic varieties that you can learn about at 5pm today. Urgh.

**2012-04-29 Daniel to Gang: [IMPORTANT!](#)** Updated notation in 2mode "paper", blog, and MY matlab codes from  $z_1 = x_1 + iy_1$  and  $z_2 = x_2 + iy_2$  to  $z_1 = x_1 + ix_2$  and  $z_2 = y_1 + iy_2$  to reflect notation used in analysis of Complex Lorenz in DasBuch as per Predrag's request. Although we've sort of moved on to  $(u, v, w, q)$ , I suggest that everybody take notice and start using this notation if you need to use 4D cartesian coordinates for anything. In fact, if you have used the notation  $z_1 = x_1 + iy_1$  and  $z_2 = x_2 + iy_2$  in your codes, take 15 minutes and update it (simple find and replace works wonders here. First, rename  $y_1$  to something else, say  $k$ , then rename  $x_2$  to  $y_1$ , and finally replace  $k$  with  $x_2$ .) so it doesn't come back and bite us later.

**2012-04-29 Predrag:** Went to talk, told Jason Cantarella about our bivariate polynomials. As we are interested in the positive  $u, v > 0$  quadrant only, he suggest constructing vector fields from our two polynomials, and checking the flow on  $u = 0$  and  $v = 0$  axes, and some enclosing curve; then use Hopf index theorem to count the minimal number of enclosed zeros.

**2012-04-29 Predrag:** Factored out the pair of  $u = 0$  roots and got our relative equilibrium equations down to (36), of degree  $\deg(f) \cdot \deg(g) = 2 \cdot 4$  and we are down to 7 coefficients, of which one is positive and one is negative. Please check my algebra, confirm here when done. [DB 2012-05-01] Checked this last night and it was fine.

**2012-04-29 Predrag to Bryce and Lei:** Just to make command and control structure clear: you are writing the two-mode project report. Keith is writing a separate one on return maps, and Daniel will continue being your adviser; he will join you later in a write-up if we use it for a publication or include it into ChaosBook. To view just the project (without this blog) toggle the `draftfalse` switch in `siminos/cgang/setup2modes.tex`.

**2012-04-29 Daniel to Gang:** I have been double checking the various equations that result in going from  $(z_1, z_2)$  to  $(u, v, w, q)$  using Matlab's symbolic toolbox and have found a few mistakes. A couple of the coefficients were wrong. I have corrected them in the text. Please look at my corrections in Section III and see `/siminos/cgang/Daniel/Matlab/PKComplexToInvariant.m` for details. If you have been using any of these equations for your computations, please check that you are using correct ones.

**2012-04-29 Predrag to Daniel:** Thanks! this has fixed (35) but I still have a pesky factor of 2 in (F19)...

**2012-04-30 Daniel citing Evangelos:** Triple checked (30). Claims I am wrong. Corrected various terms.

**2012-04-30 Daniel to Evangelos:** I don't see how you get enough  $v$ 's to get a  $v^2$  in  $\dot{u}$  in (30). The highest power of  $v$  in  $\dot{z}_1$  is one and there are none in  $z_1$ . That  $b_1$  term is of a higher order than you can get by multiplying  $z_1$  (order 1/2 in  $u$ ) and  $\dot{z}_1$  (order 3/2 in  $u, v$  and products thereof). Also remember that  $w$ 's and  $q$ 's have a  $2 * u * (v^5)$  out front. Maybe you forgot the sqrt? I think most of the new terms you put in suffer from this same problem (they're too high order). Anyway, don't have Mathematica at home so I'll double check your code when I get to Tech, but I'm pretty sure you've made a mistake somewhere.

**2012-04-30 Evangelos to Daniel:** Well, I got enough  $v$ 's by writing the initial equations incorrectly! Sorry for the confusion. Strange thing is I checked three times and did no see the error. Getting too old...

**2012-04-30 Predrag:** Unitary transformations preserve the magnitude of a complex vector  $\|a\| = \|g(\theta)a\|$ , so  $g^\dagger g = 1$ . Expanding the norm to leading order in  $\theta \langle (1 + \theta T)(1 + \theta T) \rangle$  shows that the generators are antihermitian,  $T^\dagger = -T$ , and from that it follows by complex conjugation of the dot product  $a^\dagger T a$  that  $a^\dagger T a = 0$  for any vector  $a \in \mathbb{R}^d$ .

**2012-04-30 Predrag:** <sup>113</sup>To evaluate the slice condition for unitary groups (like our U(1)-equivariant two-mode system)

$$\frac{\partial}{\partial \theta} \|a - g\hat{a}'\|^2 = 0, \quad (\text{F26})$$

start by expanding

$$(a - g\hat{a}')^\dagger (a - g\hat{a}') = \|a\|^2 - \hat{a}'^\dagger g^\dagger a - a^\dagger g\hat{a}' + \|\hat{a}'\|^2$$

so

$$\begin{aligned} \frac{\partial}{\partial \theta} (a - g\hat{a}')^\dagger (a - g\hat{a}') &= -\frac{\partial}{\partial \theta} (\hat{a}'^\dagger g^\dagger a + a^\dagger g\hat{a}') = -2 \operatorname{Re} \left( a^\dagger \frac{\partial g}{\partial \theta} \hat{a}' \right) \\ &= -2 \operatorname{Re} (a^\dagger g T \hat{a}'). \end{aligned} \quad (\text{F27})$$

Substitute  $a = g\hat{a}$ . <sup>115</sup> This yields the *slice condition* for unitary groups:

$$\operatorname{Re} (\hat{a}^\dagger T \hat{a}') = 0 \quad (\text{F28})$$

<sup>114</sup> 2CB

<sup>116</sup> PC: This is OK for abelian groups, as along the way I assumed  $gT = Tg$ , but we have to redo it more carefully for nonabelian

groups (like  $SU(n)$ ). I think the answer is the 'Cartan derivative at unity', but we'll cross that ridge when we come to it. For example, we slice spiral chaos induced by the Euclidean  $E(2)$  symmetry.

We will need it, if the C Gang prefers to work with a state space in  $\mathbb{C}^2$  rather than  $\mathbb{R}^4$ . My problem is that I cannot visualize  $\mathbb{C}^2$ , hence I am stuck in the material world.

**2012-05-01 Lei:** Discussed with Bryce and trying to finish our term paper by summarizing all we have about PK system. I'm really confused about the LaTeX templates used in the paper, so I wrote my own templates and put it in `cgang/2-mode-Bryce-Lei`. Other sections just copied from the previous paper, rewrite a little using my own words. Just begins, keep moving.

**2012-05-01 Daniel:** Following up on Predrag's cry in the wilderness of 2012-04-29, we need the trace of the stability matrix,  $\text{tr } A = \sum \lambda_e > -\sum \lambda_c$ , to be greater than zero. For future reference, the trace of the stability matrix for a generic point is given by

$$\text{tr } A = 2(\mu_1 + \mu_2) + (4a_1 + 2a_2)r_1^2 + (2b_1 + 4b_2)r_2^2 \quad (\text{F29})$$

This agrees with my earlier result of 2012-04-27 for the origin, but may allow us to put further constraints on the parameter choices, especially since  $u, v \geq 0$ .

**2012-04-29 Daniel:** Here's a couple of other results that may be useful for constraining parameter choices or looking for relative equilibria. Setting the equation for  $\dot{\psi}$  in the polar form of the two-mode system to zero or starting from the equation for  $q$  in (34) and the fact that  $q = 2u|v|^{1/2} \sin \psi$ , one can solve for  $\sin \psi$ . Since  $-1 \leq \sin \psi \leq 1$ , one can show that any solution must satisfy the following two conditions in  $(r_1, r_2)$  space:

$$\frac{c_2}{2c_1}r_1^2 + \left(r_2 \pm \frac{e_2}{4c_1}\right)^2 \geq \left(\frac{e_2}{4c_1}\right)^2 \quad (\text{F30})$$

The + condition is satisfied for any  $r_1, r_2 \geq 0$ , but the - condition is an additional constraint on what  $r_1$  and  $r_2$  (or  $u$  and  $v$ ) can be. Might prove useful if we have to brute force search for roots.

If one starts with the  $A_1$  equation for  $w$  in (34) and  $w = 2u|v|^{1/2} \cos \psi$  and uses the constraint that  $\cos \psi \leq 1$ , you can work out that

$$\frac{a_1}{b_1}r_1^2 + \left(r_2 + \frac{c_1}{2b_1}\right)^2 \geq \frac{c_1^2 - 4\mu_1 b_1}{4b_1^2} \quad (\text{F31})$$

This result and the conditions from the range of  $\sin$  (F30) are nice but not great because they tell us that the  $r$ 's must lie *outside* of particular ellipses (i.e., where the root are not, but not where the roots *are*). However, it seems (unless I messed up the algebra, so **somebody please double check this and report back here**) that the final condition  $\cos \psi \geq -1$  gives the more powerful constraint

$$\frac{a_1}{b_1}r_1^2 + \left(r_2 - \frac{c_1}{2b_1}\right)^2 \leq \frac{c_1^2 - 4\mu_1 b_1}{4b_1^2} \quad (\text{F32})$$

This constraint tells us that valid  $r$ 's need to be *inside* of a certain ellipse, which really cuts down on the available state space! Furthermore, the lefthand side of the inequality is non-negative, so the righthand side must also be non-negative.

**2012-05-04 Sarah:** **Here's a first stab** at an interactive Rössler web page. It works on IE, but only somewhat on Firefox.

I have planes and conics waiting to be added. I'd like to try and setup a model with buttons to turn vectors, sections, etc on and off. Still have some bugs to work out. Let me know how it looks to you.

**2012-05-06 Predrag to Sarah:** It's great! Let me know when you are ready for public version, and I'll link it to [ChaosBook.org](#). Turning vectors, sections, etc. on and off is probably necessary, as right now it is pretty slow when I try to turn the Rössler attractor around.

**2012-05-16 Predrag to Daniel:** Regarding [2012-04-27 Predrag], [2012-04-28 Daniel], and [2012-04-29 Daniel] above: Have you really changed to  $z_1 = x_1 + ix_2$  and  $z_2 = y_1 + iy_2$ ? Left-hand-side of (21) and (F10) looks like the old version;  $\mu_i$  factors do not match up. Please recheck. [2012-06-11 Daniel] Somehow missed this request. Double-checked them and it turns out that I forgot to update the left-hand sides of (21) and (F10). I have now corrected them. For details of the derivation see `/siminos/cgang/Daniel/Matlab/EOMComplexTo4.m`.

**2012-07-25 Evangelos:** I have just finished writing a plasma physics paper and I will have some time to work on the two-mode system. I have no choice anyway, since I have to give a 15 min talk in Dynamics Days Europe about slicing and I'd better have something new to say.

Are you still interested in / have time for this? I would suggest that we split the tasks in some clear way so that we do not duplicate effort (except for verification).

**2012-07-25 Daniel:** I would definitely be interested in helping you out. I don't have a TON of time (since I DO need to graduate at some point), but I should be able to do some work in my spare time. I agree that splitting up the tasks (and perhaps defining a goal and outlining what the plan of attack will be) may help make things more efficient.

**2012-07-26 Predrag:** The current draft *is* an outline - improve it in the style that helps you see the goal clearly.

**2012-07-26 Keith:** I have waited a day to think about this before responding. I don't want to be irresponsible since I have been extremely busy this summer; however, the temptation to see this to completion as well as seeing if we can produce a return map is too great. Having said that, as long as you are ok with me not being as involved over the next week to two weeks (things should calm down by the end of next week, but I say two to be safe), I would love to be able to help out mostly out of curiosity to see if I can generate my own return map as I was looking forward to do for this 2-mode problem. If this is problematic, then I can await the published results. If I am holding it down as well, then feel free to drop me as well.

**2012-07-26 Predrag to Keith:** OK, your job is generate return maps. It will take a few weeks of experimentation with parameters before Chaos Gang settles on a particular system to study. If you feel you cannot do it, let us know.

I have dropped Bryce and Lei from the two-mode paper, as they have not responded whether they are still in the gang or not.

**2012-07-30 Evangelos:** I hope it won't take a few weeks to settle on parameters as I only have two weeks of "quality" time before Dynamics Days (Aug. 4-24 I am in Greece and usually it's very hard to work there). However, Predrag might be right - I've spent the whole day Friday trying to find a system with two unstable relative equilibria and no attracting limit cycle and could not do it.

**2012-07-30 Evangelos:** 2-modes system is driving nuts - I can't find interesting behavior. Color-coded sign of  $\text{tr } A$  in `dangelmayr_so2.int.nb` in hope it might help.

**2012-07-30 Evangelos:** Starting from (29) I compute the following stability matrix for the two-mode system in invariant variables:

$$A = \begin{pmatrix} 2\mu_1 + 4a_1 u + 2b_1 v & 2b_1 u & c_1 & 0 \\ 2a_2 v & 2\mu_2 + 2a_2 u + 4b_2 v & c_2 & 0 \\ 4c_2 u + 4c_1 v + (2a_1 + a_2)w & 4c_1 u + (2b_1 + b_2)w & 2\mu_1 + \mu_2 + (2a_1 + a_2)u + (2b_1 + b_2)v & 2e_1 - e_2 \\ (2a_1 + a_2)q & (2b_1 + b_2)q & -2e_1 + e_2 & 2\mu_1 + \mu_2 + (2a_1 + a_2)u + (2b_1 + b_2)v \end{pmatrix}. \quad (\text{F33})$$

The trace of this matrix is

$$\text{tr } A = 2(3\mu_1 + 2\mu_2) + 4(2a_1 + a_2)u + 6(b_1 + b_2)v, \quad (\text{F34})$$

not in agreement with (F29),

$$\text{tr } A = 2(\mu_1 + \mu_2) + 2(2a_1 + a_2)u + 2(b_1 + b_2)v$$

Daniel, are you using a different notation? Or am I doing something wrong? Once cross-checked/corrected, I'll move (F33) and (F34) to the main text.

**2012-07-30 Daniel to Evangelos:** Hmm... To get (F29), I started from (20), plugged in  $z_1 = x_1 + ix_2$  and  $z_2 = y_1 + iy_2$ , to get (21), and then calculated the jacobian (F11) and its trace. This was done using `/cgang/Daniel/Matlab/EOMComplexTo4.m`. I then, replaced  $x_1^2 + x_2^2$  by  $u$  and  $y_1^2 + y_2^2$  by  $v$  to get (F29).

If I start from (20), go to the  $[u, v, w, q]$  basis to get (30), then take the jacobian and trace, I get the same result (F33) and (F34) that you do. This was calculated using `/cgang/Daniel/Matlab/PKComplexToInvariant.m`

I guess the traces of the stability matrices for the two-mode system written in the  $[x_1, x_2, y_1, y_2]$  basis should be expected to be different from the stability matrix for the system written in the  $[u, v, w, q]$  basis, right?

**2012-08-01 Predrag:** Traces of stability matrices should be the same? In the symmetry-reduced state space there is one eigenvalue less, but its value is zero.

**2012-07-31 Evangelos to Daniel:** I think that  $A$  (and its trace) have to be computed in invariant coordinates of some form. For instance, a relative equilibrium is a periodic orbit in original space and if we just pick a point on it and compute  $A$ , it tells us nothing about stability, right?

**2012-08-01 Predrag:** Stability matrix  $A$  is explicitly coordinate dependent. However, Floquet multipliers of invariant sets (equilibria, relative equilibria, periodic orbits, relative periodic orbits, ...) are smooth-conjugacy invariants, so they, their sums (traces) and their products (determinants) should not change under (locally differentiable) coordinate changes. We should write this down somewhere more precisely, but going from equivariant coordinates to invariant coordinates + syzygies keeps all multipliers the same.

In case at hand, while  $A = A(a(\tau))$  depends on the point  $a(\tau)$  in the relative equilibrium orbit,  $\text{tr } A(a(\tau)) = \text{tr } A$  should not. If I am correct, for a relative equilibrium the eigenvalues  $\{\lambda_j\}$  of  $A(a(\tau))$  are constants independent of  $a(\tau)$ , and the Floquet multipliers for one traversal of the relative equilibrium orbit are  $\Lambda_j = \exp(T\lambda_j)$ , where  $T = 2\pi/c$ , and  $c$  is the phase velocity.

**2012-08-01 Daniel:** If what Predrag says is true and the traces should be the equivalent independent of whether we use equivariant coordinates or invariant coordinates, then I have either made a mistake in calculating one or the other. It is kind of worrisome that these do not agree for the stability of the origin. I've double checked my calculation, but I don't see anything wrong with it. Is it possible that they are not the same and that their difference (or something like that) lets us extract a multiplier or something. Or maybe equating the two gives some kind of information. I just don't see what I did wrong in my computation.

**2012-07-31 Evangelos to Daniel:** One should go to a reduced space where the relative equilibrium is an equilibrium and compute its stability. Since the dimensions of the original and reduced spaces are not the same (note there is a syzygy), we don't have the nice property of invariance of stability eigenvalues under smooth conjugacies.

What confuses me still, is whether one should use the syzygy explicitly. Should we first eliminate one variable using the syzygy, then compute the Jacobian as a  $[3 \times 3]$  matrix? Maybe I'll try this here.

**2012-08-01 Predrag:** That's an open question that Evangelos never wrote up in his thesis, and would be nice to have a clean statement about how it works. Infinitesimal variations are constrained to the  $d - N$ -dimensional hypersurface defined by the syzygies - should not be necessary to substitute them explicitly. I assume this is answered somewhere in Gilmore and Letellier<sup>9</sup> (read it [here](#)) or if not, elsewhere in the immense literature that uses invariant polynomials. Most likely, the answer is the same as in constraining the Newton searches to section a slice (for relative periodic orbits); add every syzygy condition by a Lagrange multiplier.

Or perhaps [ChaosBook.org](#) 4.5.1 *Stability of Poincaré return maps* - there the constraint  $U(a + \delta a) = 0$  is your ticket.

**2012-08-01 Daniel:** I seem to remember the syzygy simply coming out as an equation when I was trying to solve for equilibria. I think after massaging  $g(u, v)$  in (35).

**2012-07-30 Daniel to Evangelos and Predrag:** What would be the features of a "good" set of parameters? Obviously we need the term that breaks the reflection symmetry, but other than that? What's our metric for chaos? Do we want to be calculating Lyapunov exponents or frequency spectra of time series and looking for broadband noise?

**2012-07-31 Evangelos:** Predrag could you answer this? I've added a second  $e_1$  term in hopes of getting more interesting behavior but have not played with it nor rederived relative equilibria conditions yet. The problem with any obvious way to detect chaos (such as Lyapunov exponents) is that it does not provide a feeling on whether there are qualitatively different regions in phase space, which we could pick as our templates. Maybe the right thing to do is to scan the parameter space plotting the largest Lyapunov exponent and at the end looking in details cases which seem promising. The problem is the large number of parameters involved.

**2012-08-01 Predrag:** <sup>117</sup>Until advent of period doubling experimentalist used their expensive multi-channel analyzers to extract broad band spectra from chaotic dynamics. I thought we had killed that monstrosity once for all -

<sup>118</sup> 2CB

as chaotic motions (in particular, the unstable periodic orbits) have no fixed time scale, Fourier spectra provide no information about chaotic dynamics, they are good only for pulling out stable periodic motions. Chaotic dynamics can only be untangled in the state space, and it's 'Fourier' transform are the trace formulas of the periodic orbit theory. That's extra price: requires thinking, and the way the world is moving now (there is a moronic Op-Ed in New York Times arguing that algebra should be eliminated from American K12, replaced by 'Citizen Statistics'), we might never get there.

Most of the stuff physicists write about nonlinear problems is thoughtless crap, the prime exhibit being 'Lyapunov exponents'. They are not well defined (unless we are talking about 'covariant Lyapunov vectors'), and they are uncomputable, as a generic dynamical system is not nicely 'ergodic', 'axiom A' system. We never know whether a physical system of interest is a 'strange attractor' or a transient. With probability 1 one it undecidable either one or the other. There is a very good book on internet that politely tries to make the point, and explains that the

" Chaotic dynamics is generated by the interplay of locally unstable motions, and the interweaving of their global stable and unstable manifolds. These features are robust and accessible in systems as noisy as slices of rat brains. Poincare, the first to understand deterministic chaos, already said as much (modulo rat brains). Once this topology is understood, a powerful theory yields the observable consequences of chaotic dynamics, [...]"

" While mathematically correct, the definition of chaos as 'positive Lyapunov + positive entropy' is useless in practice, as a measurement of these quantities is intrinsically asymptotic and beyond reach for systems observed in nature. More powerful is Poincaré's vision of chaos as the interplay of local instability (unstable periodic orbits) and global mixing (intertwining of their stable and unstable manifolds). In a chaotic system any open ball of initial conditions, no matter how small, will in finite time overlap with any other finite region and in this sense spread over the extent of the entire asymptotically accessible state space. Once this is grasped, the focus of theory shifts from attempting to predict individual trajectories (which is impossible) to a description of the geometry of the space of possible outcomes, and evaluation of averages over this space. "

" the non-wandering set of  $f$  is the key to understanding the long-time behavior of a dynamical system; all calculations undertaken here will be carried out on non-wandering sets. "

So, I feel that what one needs to do is to (1) identify the simplest invariant solutions, (2) cut out the stable ones, (2) see whether stable / unstable manifolds of the unstable ones intersect and fold. That should reveal Smale horseshoes embedded in the non-wandering set, regardless of whether the non-wandering set is a 'strange attractor' or a 'strange repeller'.

Now, we nevertheless often just run finite time Lyapunov exponent estimates as function of parameters, as that requires no thinking. As Evangelos points out the problem is large parameter space - so I think we first have to show that there are at least two unstable invariant structures whose unstable manifolds communicate before we start fishing for a 'strange attractor'.

**2012-08-01 Daniel:** I realize that perhaps FTLE's and power spectra and not the smartest tools, but perhaps we can use them to have the computer try a bunch of parameter combinations and come up with some interesting candidates that we can then look at, no?

**2012-07-31 Evangelos:** Added  $e_1$  in (20). However in invariant polynomials (29) only the difference  $2e_1 - e_2$  appears, so it seems we don't get anything new by introducing it. I keep it for the time being just to see how it affects dynamics in original space and the projection from original to reduced space.

**2012-08-01 Predrag:** We have done this somewhere in a more general setting, I forget where. You can break the U(1) symmetry only once, so these imaginary parts of parameters always show up summed up, I believe.

**2012-07-31 Evangelos:** Compute the stability matrix from (31):

$$A = \begin{pmatrix} 2\mu_1 + 4a_1u + 2b_1v & 2b_1u & c_1 \\ 2a_2v & 2\mu_2 + 2a_2u + 4b_2v & c_2 \\ 4c_2u + 4c_1v + (2a_1 + a_2)w + \frac{4(2e_1 - e_2)uv}{\sqrt{4u^2v - w^2}} & 4c_1u + (2b_1 + b_2)w + \frac{2(2e_1 - e_2)u^2}{\sqrt{4u^2v - w^2}} & 2\mu_1 + \mu_2 + (2a_1 + a_2)u + (2b_1 + b_2)v - \frac{(2e_1 - e_2)w}{\sqrt{4u^2v - w^2}} \end{pmatrix}. \quad (\text{F35})$$

The trace reads

$$\text{tr } A = 4\mu_1 + 3\mu_2 + 3(2a_1 + a_2)u + (4b_1 + 5b_2)v + \frac{(-2e_1 + e_2)w}{\sqrt{4u^2v - w^2}} \quad (\text{F36})$$

This is interesting: we now have a dependence on  $w$  and on the symmetry breaking parameters  $e_1$  and  $e_2$ . I'll have to compare (F35) with (F33) numerically, see which one correctly encapsulates dynamics [F35] should be the correct one since it takes into account the functional dependence of all variables before taking derivatives].

**2012-08-02 Predrag:** In a way of motivation for a need for multiple charts, I have added Fig. 8 from the small-cell Kuramoto-Sivashinsky system studied in Ref. 47.

**2012-08-02 Predrag:** A wild idea that requires no thinking, just running simulations. Ashley and Sebastian have now independently shown that replacing Fourier space Euclidean norm  $\delta_{jk}$  by the Sobolev  $H^{-1}$  norm  $\delta_{jk}/k^2$  flattens out the group orbits and decreases the number of linear charts needed to cover the ergodic attractor (see Figs. 18 and 19).

How about going the opposite way and using the Sobolev  $H^1$  norm  $\delta_{jk} k^2$  for the two-mode flow, in order to make the attractor more curved rather than less? Then you start with a chaotic looking simulation, then pick a state  $\hat{a}'^{(1)} = a(\tau_1)$  if you like to construct the first slice hyperplane. Monitor the cos of the angle between the two tangent vectors:

$$\cos \psi = \frac{\langle t(\hat{a}(\tau)) | t' \rangle}{\|t(\hat{a}(\tau))\| \|t'\|}. \quad (\text{F37})$$

When that hits zero, you have hit the chart border. Pick that point as  $\hat{a}'^{(2)} = \hat{a}(\tau_2)$ , backtrack to the ridge between the two slice hyperplanes, restart there by moving in the second slice. If we are lucky, we shall need (a) at least two charts, (b) no more than two charts.

Take it for a spin!

**2012-08-02 Daniel (in ignorance):** So I guess the Fourier space Euclidian norm is just the sum of the squares of the Fourier coefficients, right? Then this  $H^1$  norm weights the short-wavelength components. I think I can visualize this for a 1D scalar function and for a 2D scalar field. But what does this do for a 5D vector? I can't really see it. Also, how is  $H^1$  defined in 5D full state space? I guess the  $k^2$  (by the way, the  $k$  in  $k^2$  is a different  $k$  than the one in  $\delta_{jk}$ , right?) corresponds to a second spatial derivative of some sort. Once I understand this, I'll code it up. I believe that this forum post has the relevant information, but I don't completely understand what the summations and integrals are over.

**2012-08-02 Predrag:** Dunno what is 5D vector? If you are referring to two-mode (21), it is 4D.  $k$  above is the wave number, i.e., in the representation (F38) the Sobolev  $H^1$  norm  $\delta_{jk} k^2$  metric tensor is

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \quad (\text{F38})$$

For example:

$$\langle a|y \rangle = x_1 x'_1 + x_2 x'_2 + 4(y_1 y'_1 + y_2 y'_2). \quad (\text{F39})$$

The above `math.stackexchange.com` is not helpful. Jean-Luc Thiffeault<sup>62</sup> reviews Sobolev norms in Sect. 3 of [arXiv:1105.1101](#).

**2012-08-03 Daniel: Doh!** Yes, by 5D I meant two-mode got my dimensionality confused with complex Lorenz. Also forgot that  $z_1$  and  $z_2$  are actually  $k = 1, 2$  spatial modes, so the definition of the Sobolev norm is sort of obvious. My bad! That being said, when we calculate the angle between tangent vector, do we use Sobolev norm everywhere in our calculations of the dynamics in the slice (for example, in calculating  $\dot{\phi}$  in the equations of motion constrained to a slice hyperplane) or just to check the cosine of the angle between  $t(\hat{a}(\tau))$  and  $t'$ ?

**2012-08-03 Predrag:** Group tangent  $t(\hat{a}(\tau))$  is always computed on the reduced state space trajectory  $\hat{a}(\tau)$  in the slice hyperplane fixed by  $t'$  (not the full state space trajectory), and the angle (F37) is computed using the Sobolev  $H^1$  norm (F39). You just run an ergodic trajectory, keep plotting  $\cos \psi$ , stop whenever it gets small (let's say  $\approx 0.1$ ) but before it hits the chart border. If that happens, we are in the 2-chart business. Might even happen for Euclidean norm, but I think Sobolev  $H^1$  that emphasizes the  $k = 2$  mode is more likely to kinkier group orbits. This can be automatized, so you can start different runs for parameter values that seem to give strange attractors.

Do'h...

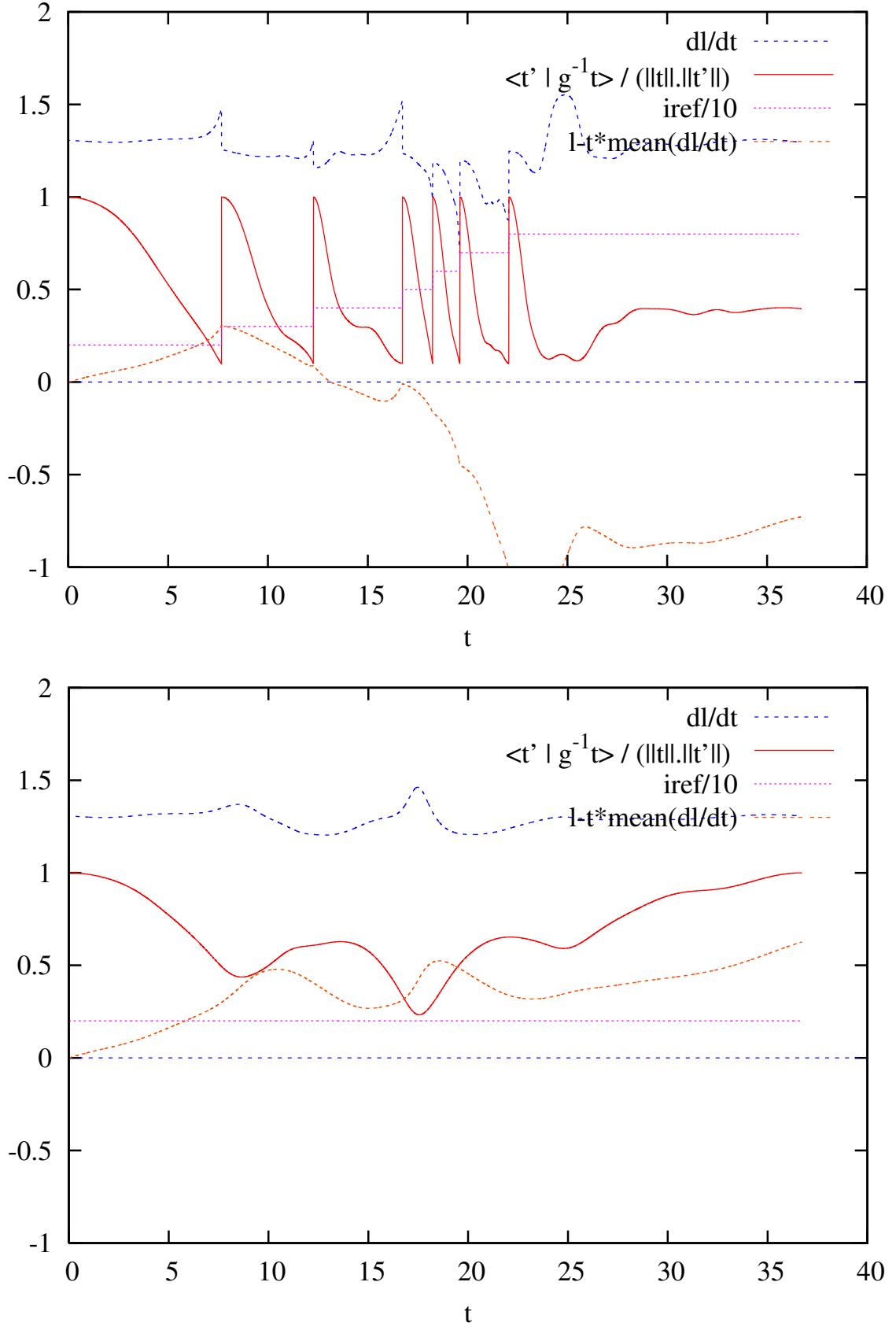


FIG. 18. Chart border condition for the pipe flow relative periodic orbit  $RPO_{36.72}$  discovered in Ref. 17: (a) [2012-05-22 Ashley] Tracking  $RPO_{36.72}$  starting with  $\hat{a}' = a(\tau = 0)$  and adding new templates on the fly, whenever  $\cos \psi < 0.1$ . (b) [2012-07-27 Ashley] Tracking  $RPO_{36.72}$  starting with  $\hat{a}' = a(\tau = 0)$ . Just as in frame (a), except here using the Sobolev  $H^{-1}$  norm,  $\cos \psi$  does not drop below 0.1.

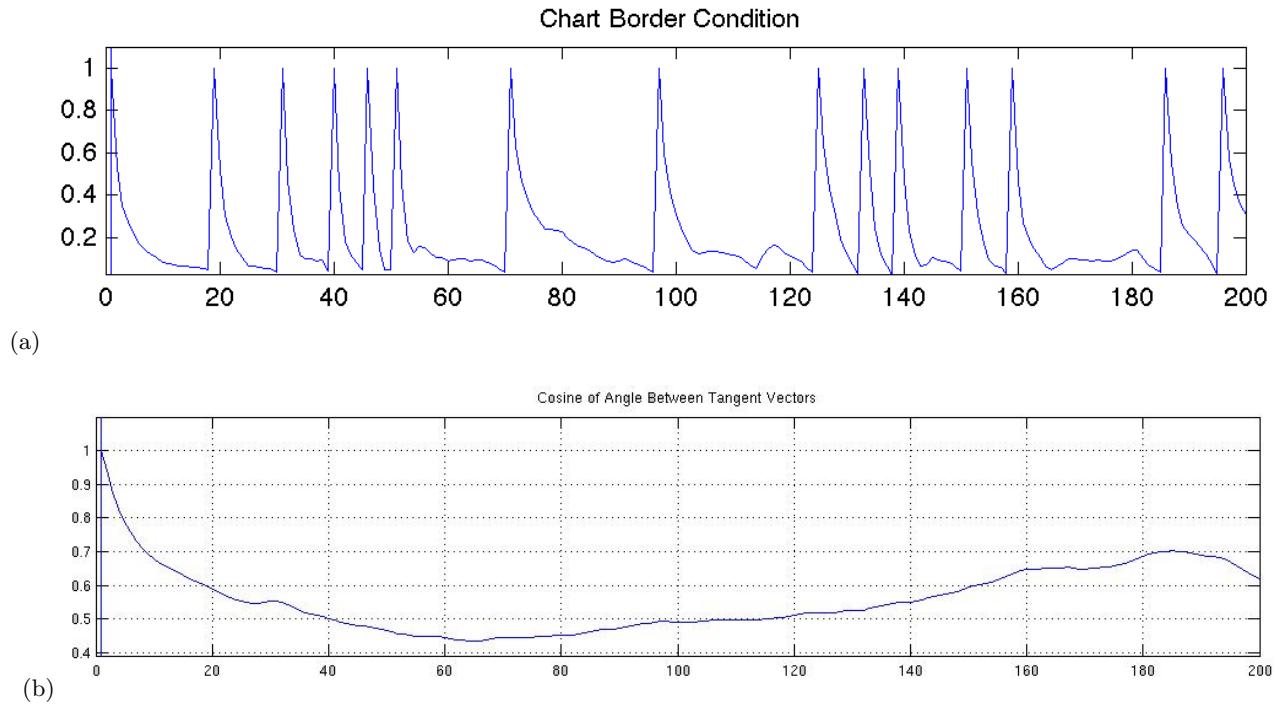


FIG. 19. Chart border condition for the baroclinic instability<sup>61</sup>. The current state is used as the new template each time the cos of the angle between the two tangent vectors (F37) is smaller than 0.05. (a) [2012-07-19 Sebastian] Using the Euclidean norm. (b) [2012-07-27 Sebastian] Using the Sobolev  $H^{-1}$  norm.

**2012-08-06 Edgar Knobloch:** I do not like so much to work in the invariant coordinates, as in them much information about symmetries is lost.

In trying to understand chaotic regions, I too would start with the simplest things, equilibria, relative equilibria and their stabilities.

There is theorem due to Pascal Chossat<sup>51,63</sup> that in systems of hydrodynamic type (2D Navier-Stokes) close to this codimension 2 bifurcation  $c_1 c_2 < 0$  (download the article from [here](#)). This is also the case for which one gets structurally stable heteroclinic cycles if symmetry is O(2) (i.e., when  $e_1 = e_2 = 0$ ).

Caroline Nore, Laurette Tuckerman *et al.*<sup>64</sup> have a nice derivation of (20), for the French washing machine. Different  $e_j$  if different angular velocities for the top and the bottom. That only the difference  $2e_1 - e_2$  appears is obvious from the form of polar coordinates (28).

**2012-08-06 Predrag to Evangelos:** Can you again remove all  $e_1$ , explain why only the difference  $2e_1 - e_2$  appears in the text proper? ES: It's in my todo list.

**2012-08-06 Edgar Knobloch:** There is a generic method for computing how many relative equilibria there are for polynomial equations such as (36). For several examples see Callahan and Knobloch<sup>65,66</sup>.

The general  $A(a)$  in (F11) can only be block-diagonalized into  $[3 \times 3] \oplus [1 \times 1]$ , the 1-dimensional subspace eigenvector pointing in the marginal, rotation direction.

**2012-08-06 Predrag:** Presumably that is what the infinitesimal equivariance condition (2) means - would be nice to make it explicit. It would be useful for computing stability of relative equilibria, less so for the relative periodic orbits, I think.

**2012-08-07 Keith:** playing catch up and reading articles on fitting curves (if I find anything good or easy, Ill let you know) in space to try to tackle the return map in an easier way.

**2012-08-08 Daniel to Keith:** Do you have a map already? If so, could you post it? I think it should be easy to fit with splines or cubic interpolants, both of which are built-in Matlab (and probably Mathematica) functions.

**2012-08-08 Keith to Daniel:** Not yet, I am more working on this new method for a few days. I have created return maps for other systems, but these relied on using a summed up distance along the curvilinear manifold and

required a step that would order the points in space in a meaningful way. Worst comes to it, we could always use this, but I am trying to use a method I found for finding curvilinear distances along the unstable manifold with parametric equations for *unordered points* in space.

**2012-05-07 Predrag to Chaos Gang:** It's not over until it is over.

## Appendix G: Burak's two-mode

Do edit this git file (created 2013-08-10), it is a new addition to the original svn version (which for now stays untouched, the mother version is still the svn one).

### 1. Burak's two-mode blog

**2012-05-07 Predrag to Chaos Gang:** It's not over until it is over.

**2013-07-25 Predrag:** - instead of computing complex Lorenz flow one more time, how about giving a try to two-mode SO(2)-equivariant flow, defined in  
`reducesymm/cgang/2modes.tex`

(you can get it by a click [here](#), provided you had already pdflatex-ed 2modes.tex). Have a look at it, and then meet with Daniel Borrero, 3. floor Schatz lab, who can walk you through what we had already done and learned (all in the 2modes.pdf blog). You can play with it for -let's say- two weeks, see whether you can find an interesting strange attractor worth slicing. If that does not work out, we'll give up, and go to Kuramoto-Sivashinsky instead, which is much more important for our overall goals...

**2013-08-06 Predrag:** A more precise statement of what we are trying to achieve with this model is in `reducesymm/cgang/2modes.tex`:

"For the 4-dimensional model at hand are using the invariant polynomials  $\{u, v, w, q\}$  dynamics only to develop intuition, but to illustrate the general method of slices, everything has to do be done in  $\mathcal{M} = \{x_1, x_2, y_1, y_2\}$  and slice  $\hat{\mathcal{M}}$  as well. You can see that even for the simplest conceivable SO(2) 4-dimensional flow one has to think about how to construct the invariant polynomials basis, and it is hard to imagine how anyone could do that for very high-dimensional flows."

If we can find a nice strange attractor with comparable amplitudes in the two modes, and show how to slice it, that would be the simplest example of power of slicing, as the symmetry-reduced dynamics is 3-dimensional and something a human can look at, turn around as 3-dimensional Mathematica or Matlab figure.

If in addition it turns out that my favorite Sobolev norm (read `reducesymm/blog/norms.tex`) reduces the number of local slice hyperplanes needed to cover the attractor, I'll be doubly happy.

**2013-08-05 Burak:** I read the two-mode blog and [Chapter 4 - Local stability](#), confirmed most of the findings in blog, naively experimented on the parameters of the system in  $x_i, y_i$  basis tried to find equilibria, got nothing, then talked to Daniel, and re-read the blog and come up with a Monte-Carlo (kind of) algorithm hoping that it could get me a strange attractor. So far, I only got periodic orbits. Fig. 20 is a typical one.

**2013-08-06 Predrag:** Got worried that there were no updates for 11 days - how about if agree on a schedule, let's say git pushes [every Monday and Friday](#)?

<sup>119</sup>The periodic orbits (actually, relative equilibria) that you find are presumably *limit cycles*. In ChaosBook I define 'limit cycles' as periodic orbits which are strictly exponentially attracting forward in time. Parenthetically, in her thesis De Witte<sup>67,68</sup> defines a 'limit cycle' as an "isolated periodic orbit" thusly:

"A cycle of a continuous-time dynamical system, in a neighbourhood of which there are no other cycles, is called a limit cycle."

That presumably has advantage of being true for both directions of time, but I do not think we need to get that finicky...

**2013-08-05 Burak:** Here is what I did:

- Generate pseudo-random set of parameters ensuring  $\mu_1 > -\mu_2 > 0$ ,  $c_1 = 1$  and  $c_2 = -1$  as suggested in [2012-04-29 Predrag] and [2012-08-06 Edgar Knobloch]

<sup>120</sup> 2CB

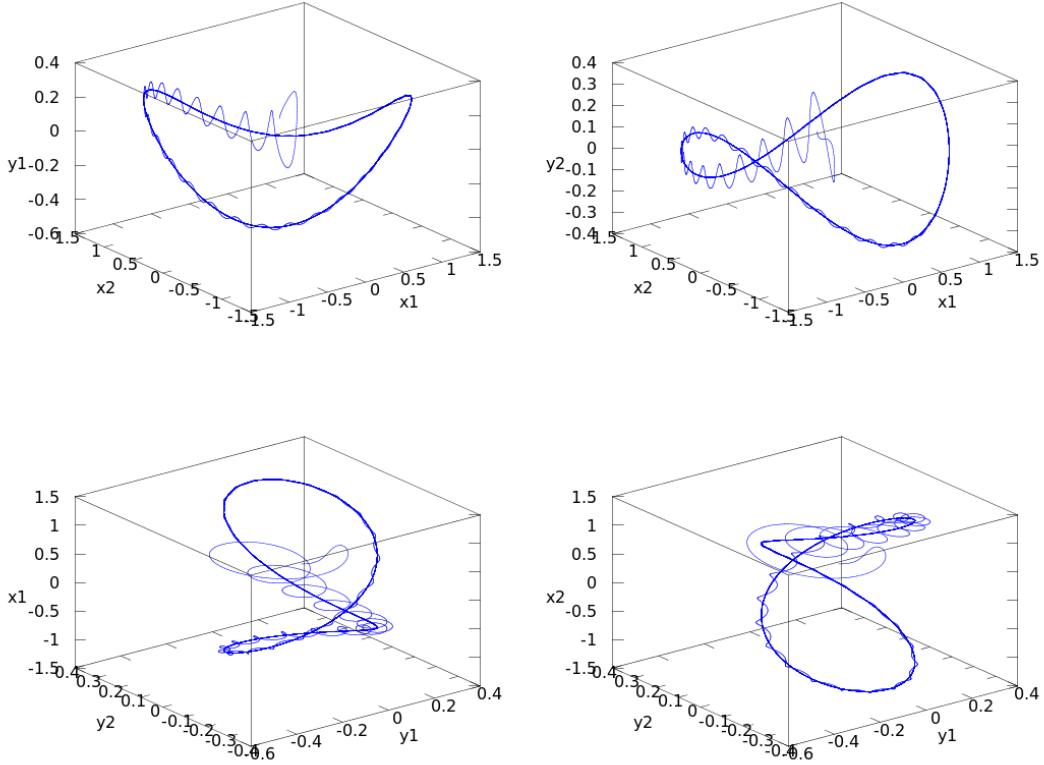


FIG. 20. A typical attractive relative equilibrium of two-mode flow.

- Numerically compute roots of (36) in  $u > 0, v > 0$  region starting from pseudo-random pair of points  $(u, v)$ , to find an equilibrium in invariant polynomial basis.

$$\begin{aligned} \tilde{f}(\tilde{u}, \tilde{v}) &= \tilde{u} A_1 - \tilde{v} A_2 = 0, & \deg(f) &= 2 \\ \tilde{g}(\tilde{u}, \tilde{v}) &= (A_1^2 - c_1 \tilde{v}) (\tilde{u} + 2 \tilde{v})^2 + e_2^2 \tilde{v}^2 = 0, & \deg(g) &= 4 \\ \text{where } A_1 &= \mu_1 + \tilde{a}_1 \tilde{u} + \tilde{b}_1 \tilde{v}, \\ A_2 &= \mu_2 + \tilde{a}_2 \tilde{u} + \tilde{b}_2 \tilde{v}. \end{aligned} \quad (G1)$$

- Calculate corresponding w and q and check if the syzygy holds (it does).
- Calculate eigenvalues of the stability matrix at this point.
- If the stability matrix has at least one eigenvalue with positive real part (repulsive), at least one eigenvalue with negative real part (attractive) and a complex pair of eigenvalues with non-zero imaginary part (spiral); keep the parameters and the equilibrium point.
- Numerically calculate the points  $x_i, y_i$  corresponding to the equilibrium in invariant polynomial basis, using following relations:

$$\begin{aligned} u &= x_1^2 + x_2^2, \\ v &= y_1^2 + y_2^2, \\ w &= 2x_1^2 y_1 + 4x_1 x_2 y_2 - 2x_2^2 y_1, \\ q &= -2x_1^2 y_2 + 4x_1 x_2 y_1 + 2x_2^2 y_2. \end{aligned} \quad (G2)$$

- Integrate the Porter - Knobloch velocity function to see time evolution in the full state space.

So far, I got divergent solutions and periodic orbits using parameters that I found this way. My questions:

- If I check the eigenvalues of the stability matrix for full state space, I get 3 of the eigenvalues almost same with the ones I get for the invariant polynomial basis, and one eigenvalue 0 (usually something less than  $10^{-4}$ ). This gives me the feeling of I am doing things correct, however, I want to make more sense out of this. Is there a clear discussion about how these eigenvalues remain unchanged under coordinate transformations (I saw the discussion about traces in the blog, I confirmed the result that traces of stability matrices in  $u, v, w, q$  basis and  $\mathcal{M} = \{x_1, x_2, y_1, y_2\}$  basis are not the same at the origin.).
- Is what I did reasonable at all? Is there any obvious wrong logic?
- Would you suggest any other restrictive criteria to pick a “good” set of parameters, in addition to the ones I force on eigenvalues of the stability matrix? I thought, maybe I should take parameters for which the positive and negative real-part eigenvalues are of the same order.
- Is an equilibrium in invariant polynomial basis ( $u, v, w, q$ ) a relative equilibrium in the full state space  $\mathcal{M} = \{x_1, x_2, y_1, y_2\}$  basis? ([2013-08-13 Predrag] correct, it is.) If not, what sense I should make out of the fact that the relations (G2) do not provide a unique point  $\mathcal{M} = \{x_1, x_2, y_1, y_2\}$  for given  $(u, v, w, q)$ . ([2013-08-13 Predrag] You are on a group orbit in  $\mathcal{M}$ , to find out where requires the full reconstruction equations.)

After writing these questions and some more reading, I realized that I did not include anything to eliminate stable limit cycles. I am now starting to read [Chapter 5 - Cycle stability](#) and then I will try to implement a way of picking equilibria other than attracting limit cycles.

**2013-08-06 Predrag:** As we were not successful in finding an interesting strange attractor, probably best not to be influenced by my (mostly misguided) intuition; keep experimenting, and keep checking it with Daniel, who remembers what we had tried last time around. As to our goals, see the “more precise statement” above.

My only remark for now is that Fig. 20 is a relative equilibrium of two-mode flow, meaning that the group orbit and time orbit coincide, it is not a “periodic orbit”. If you are *on the relative equilibrium* you should get one of the full state space Floquet multipliers equal to 1 to machine precision. The reason is why the Floquet exponent is only  $\approx 10^{-4}$  is that you are converging to the relative equilibrium forward in time, and that is only exponential; once you have Newton codes for [finding periodic orbits](#) running, the convergence will be super-exponential.

**2013-08-08 Burak:** Does Fig. 21 look like a strange attractor? It wanders around a relative equilibrium but I'm not sure if it is a periodic orbit. I tried to slice it but my slicing code is buggy. I picked a template point on the relative equilibrium shown with red curve on Fig. 21, the result is Fig. 24. Fig. 24 is a longer run, and it looks more like a periodic orbit when I run it longer. Is there an easy way of telling whether it is a periodic orbit or not?

**2013-08-13 Predrag:** I do not know what you mean by ‘periodic orbit’. The whole point of this model is that it has no periodic orbits, only relative invariant solutions, so one must slice to get any periodic orbits at all?

**2013-08-14 Burak:** By ‘periodic orbit’, I mean an orbit that repeats itself. Fig. 21 is a simulation of two-mode flow from  $t = 0$  to 100. When I simulated the flow with the same parameters from  $t=0$  to 400 I got Fig. 24 and if I run it longer I get a very similar looking picture. That's what I was guessing that to be a periodic orbit.

**2013-08-14 Predrag:** There is no ‘periodic orbit’ in these figures, in sense of your definitions of Section ???. I think you found an attractive relative periodic orbit (attractive torus is the full state space) which after symmetry reduction becomes an attractive periodic orbit, AKA a limit cycle. If you plot it in  $\{u, v, w, q\}$ , it will be a limit cycle. (Your ‘Symmetry reduced’ frames in Fig. 24 look very wrong.)

One way to proceed would be to change parameters in such a way that this Hopf cycle goes unstable. If it does it through 2nd Hopf bifurcation, that is a start of a generic transition to chaos via mode locking and then period doublings.

**2013-08-14 Burak:** If you look closely to the upper-left plot in Fig. 21, there is a piece of the blue curve which looks like the red curve squeezed and turned upside-down (**Predrag** I see it). The starting point of that simulation corresponds to a point on that piece of the blue curve. The points on this small piece maps to an equilibrium in the invariant polynomials basis, so that's why I was calling that an unstable relative equilibrium (**Predrag** agreed). After realizing that the actual dynamics (blue curve) looks like it is making a ‘wurst’ around a group orbit of the system, I decided to check whether if it is around some relative equilibrium of the system or not.

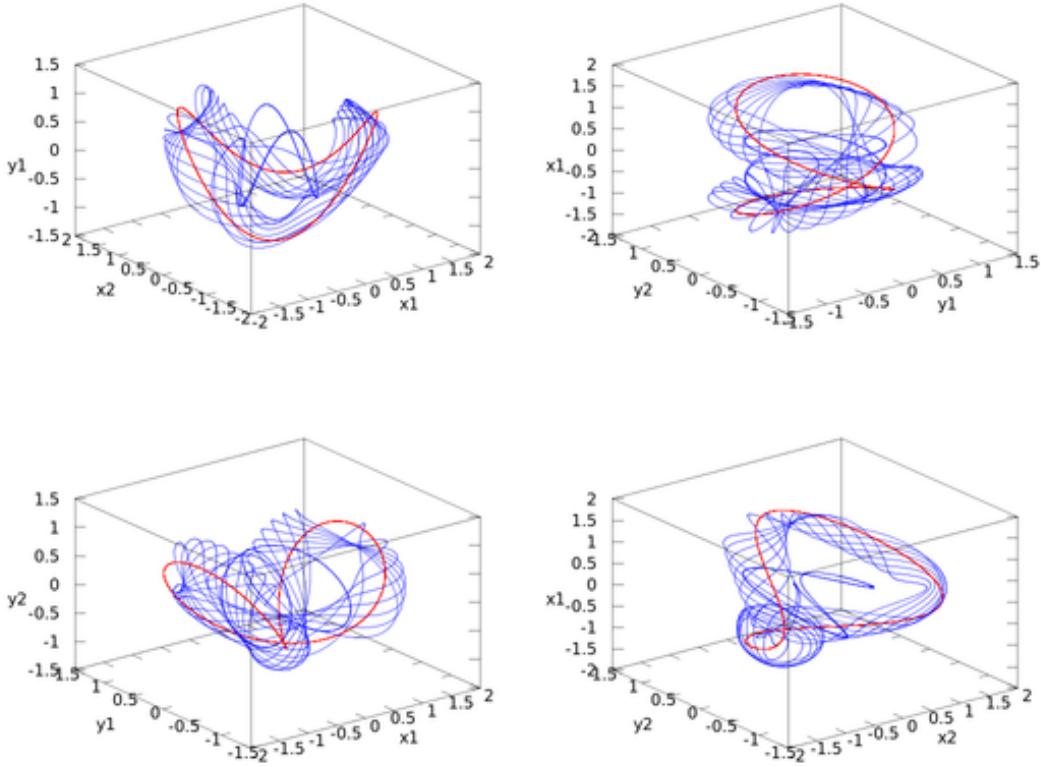


FIG. 21. 3D projections of trajectories of two-mode flow in full state space for parameters:  $\mu_1 = 1.23436$ ,  $a_1 = -0.32304$ ,  $b_1 = -1.07444$ ,  $c_1 = 1.00000$ ,  $\mu_2 = -0.23149$ ,  $a_2 = 0.44110$ ,  $b_2 = -0.42287$ ,  $c_2 = -1.00000$ ,  $e_2 = 0.67556$ . Blue curve is a trajectory starting close to an unstable relative equilibrium, the blue curve in the middle of the top left frame, which converges to a stable relative periodic orbit, better seen in Fig. 24. This relative periodic orbit originates from a Hopf bifurcation of a relative equilibrium, here the coexisting attractive relative equilibrium plotted as red curve.

For that reason, I started from other points corresponding to the relative equilibria of the system, integrated and plotted on top of the blue curve; one of these curves is the red curve here. Red and blue curves are time evolutions starting from two different relative equilibria of the system. While red curve is a stable relative equilibrium, blue curve starts from an unstable relative equilibrium and then starts to rotate and shift around the stable relative equilibrium shown red.

**2013-08-08 Burak:** According to my simulations, an attracting equilibrium in the invariant polynomials basis corresponds to a stable relative equilibrium in the full state space. ([[2013-08-13 Predrag](#)] correct.) Eliminating these parameter values gives more interesting dynamics.

**2013-08-13 Predrag:** Fig. 21 does not look like a strange attractor. You really want to plot these things in  $\{u, v, w, q\}$  representation first, and if something looks chaotic, look at it in a Poincaré section; there it is much easier to see whether there is a stretch & fold unstable manifold with fractal structure.

**2013-08-14 Burak:** I am going to try to compute Poincaré sections and return maps tomorrow. I have to go back a little bit since my existing Poincaré section codes for ChaosBook exercises are extremely sloppy.

**2013-08-20 Predrag:** You probably want to implement the Hénon trick, ChaosBook section [3.2 Computing a Poincaré section](#).

**2013-08-19 Burak:** Took longer than I thought. I have the two-modeflow in invariant polynomials basis (Fig. 22), Poincaré section and radial return map (Fig. 23). Convergence to a periodic orbit is clearly seen on these plots,

points with  $r = 6.8$  mapped on themselves.

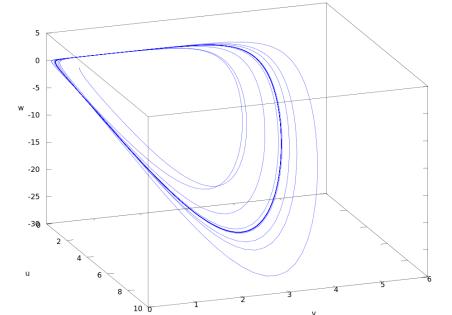


FIG. 22. A projection of two-mode dynamics in the invariant polynomials basis  $\{u, v, w, q\}$ . Parameters:  $\mu_1 = 1.23436$ ,  $a_1 = -0.32304$ ,  $b_1 = -1.07444$ ,  $c_1 = 1.00000$ ,  $\mu_2 = -0.23149$ ,  $a_2 = 0.44110$ ,  $b_2 = -0.42287$ ,  $c_2 = -1.00000$ ,  $e_2 = 0.67556$ .

**2013-08-20 Predrag:** Looks right - you are converging to a limit cycle with a negative least contracting Floquet multiplier. For limit cycles Poincaré sections are boring, they will be useful only one you are looking at strange attractors. (I have, experimentally, commented your commit on GitHub, but that looks useless. Did GitHub alert you to the comment?) Have you computed Floquet multipliers of this limit cycle? They should give you the convergence rate exactly.

Now see whether you can pick a convenient parameter that moves one or a pair of Floquet multipliers through 1 and on to chaos.

As explained in [ChaosBook](#), 1-dimensional Poincaré return maps, such as of the radial distance from  $w$  axis in Fig. 23 are misleading.

**2013-08-21:** I got notification for GitHub comment. I tried to calculate the Floquet multipliers but failed really badly at calculating the Jacobian. I thought that I could estimate the Jacobian (in the sense that it is defined in the ChaosBook, not like in Strogatz's book, I think his definition of Jacobian is the Stability Matrix of the ChaosBook) by multiplying the Jacobians for small time steps in a time ordered manner, but result was horrible. I think the numerical errors build up in an intolerable way. Can you refer me some paper/thesis/book that describes the correct way of computing this? I'm assuming that someone did this before and I'd rather not to re-discover it.

**2013-08-08 Burak:** I think this one (Fig. 25) is chaotic.

**2013-08-10 Predrag:** I think you should cheat and find chaos first in the invariant polynomials basis  $\{u, v, w, q\}$  - that is already symmetry reduced. After it looks chaotic in the invariant polynomials basis, plot the same trajectory in the equivariant  $\mathcal{M} = \{x_1, x_2, y_1, y_2\}$  coordinates. That should look messy. After that construct a slice  $\hat{\mathcal{M}}$ . Examples are Fig. 11 and Fig. 12.

That might sound masochistic (why not slice from the start?), but we are only learning how to slice, and it is easier when you already have a symmetry-reduced representation. For very high-dimensional flows we will not have the luxury of an invariant polynomials basis.

**2013-08-12 Burak:** I got the parameters that I used in Fig. 25 by generating random parameters, discarding if there are attractive equilibria or the time evolution is convergent or divergent in invariant polynomials basis. Unfortunately, this one was a periodic orbit in the invariant polynomials basis.

I added another criteria in my parameter generating code on Saturday to eliminate periodic orbits and discovered a bug in it later today. I spent most of today on varying parameters one by one and trying to see if those variations breaks that periodicity. I didn't get anything interesting yet.

I will run another 'new parameter set finder' tonight with the working periodic orbit elimination.

**2013-08-13 Predrag:** One way to diagnose chaos is to pick a stable solution (like Fig. 20) and follow it as it undergoes a Hopf bifurcation into a stable limit cycle. Then one keeps changing parameters until this periodic orbit goes unstable and begets chaos, through period doublings and beyond. To do this, you need to be able to compute Floquet multipliers of your invariant solutions. For example, does the unstable relative periodic orbit in Fig. 21 have complex pair of multipliers (underwent a Hopf bifurcation that turned it unstable) or two real multipliers (perhaps on the way to period doubling sequence?)

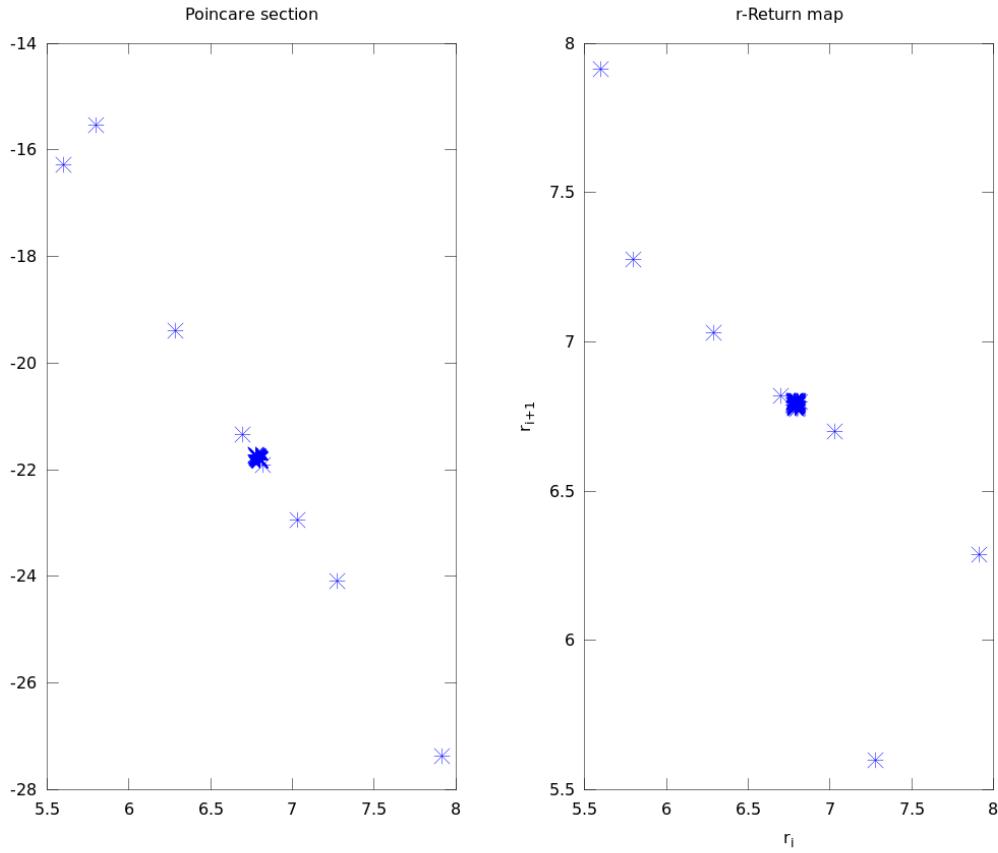


FIG. 23. Left: Poincaré section for the two-mode flow shown in Fig. 22. Section hyperplane passes through  $\hat{a} = (u = 10, v = 6, w = 0, q = 0)$  and the plane normal is  $\hat{n} = (-6, 10, 0, 0)$  ( $\hat{a}$  rotated 90 degrees about  $w$  axis). Right: Poincaré return map of radial distance from  $w$  axis.

**2013-08-13 Predrag:** I have asked a Geophysical Fluid Dynamics (Woods Hole) Fellow Yuuki Yasuda (Earth and Planetary Science, U. Tokyo, yyuuki@eps.s.u-tokyo.ac.jp) to learn how to use **AUTO**, for the same reason - to follow bifurcations of initially invariant solutions, see how they go into chaotic behavior. It is well written code which I think is only good in small dimensions, so we have not used it for our high-dimensional hydrodynamics calculations. Yuuki will report to me today how it is working; I'll report whether it might be useful to you.

**2013-08-13 Predrag:** I keep getting confused about whether you are plotting a relative equilibrium or a periodic orbit, and as we will need this anyhow, please define equilibrium, relative equilibrium, periodic orbit, and relative periodic orbit in Section ??, just to be sure we are on the same page. (I keep using macros for their names, because depending on the publication and audience, a ‘relative equilibrium’ might be called a ‘rotating wave’, etc.)

**A truncation of PDE representations:** The two-mode system is the simplest, coarsest example of a truncation of a Fourier representation of a PDEe. Consider, for example, the 1D Kuramoto-Sivashinsky equation.<sup>121</sup> The time evolution of the ‘flame front velocity’  $u = u(x, t)$  on a periodic domain  $u(x, t) = u(x + L, t)$  is given by

$$u_t = F(u) = -\frac{1}{2}(u^2)_x - u_{xx} - u_{xxxx}, \quad x \in [-L/2, L/2]. \quad (\text{G3})$$

Here  $t \geq 0$  is the time, and  $x$  is the spatial coordinate. The subscripts  $x$  and  $t$  denote partial derivatives with respect to  $x$  and  $t$ . In what follows we shall state results of all calculations either in units of the ‘dimensionless system size’

<sup>122</sup> PC: 2013-08-13 Predrag copied this from Ref. 46, to set up the conventions for the Kuramoto-Sivashinsky equation calculations.

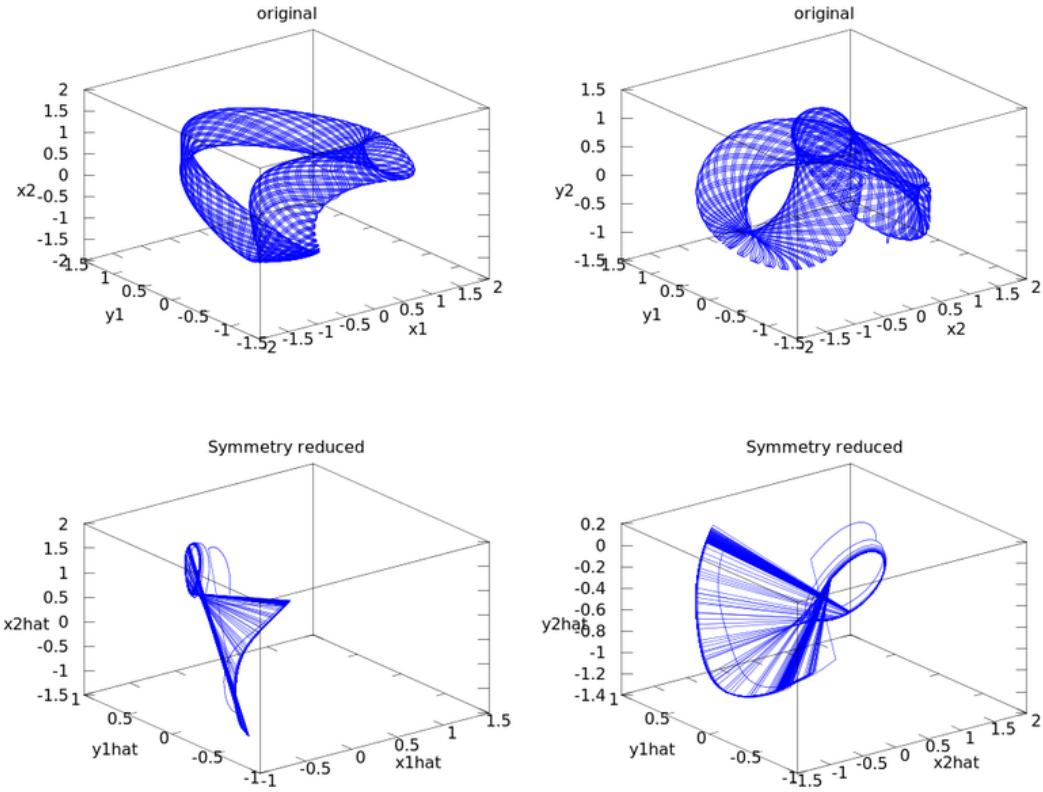


FIG. 24. Projections of two-mode dynamics in full state space and symmetry reduced space. Parameters:  $\mu_1 = 1.23436$ ,  $a_1 = -0.32304$ ,  $b_1 = -1.07444$ ,  $c_1 = 1.00000$ ,  $\mu_2 = -0.23149$ ,  $a_2 = 0.44110$ ,  $b_2 = -0.42287$ ,  $c_2 = -1.00000$ ,  $e_2 = 0.67556$

$\tilde{L}$ , or the system size  $L = 2\pi\tilde{L}$ . All numerical results presented in this report are for  $\tilde{L} = 22/2\pi \simeq 3.5014$ . Spatial periodicity  $u(x, t) = u(x + L, t)$  makes it convenient to work in the Fourier space,

$$u(x, t) = \sum_{k=-\infty}^{+\infty} a_k(t) e^{ikx/\tilde{L}}, \quad (\text{G4})$$

with the 1-dimensional PDE (G3) replaced by an infinite set of ODEs for the complex Fourier coefficients  $a_k(t)$ :

$$\dot{a}_k = v_k(a) = (q_k^2 - q_k^4) a_k - i \frac{q_k}{2} \sum_{m=-\infty}^{+\infty} a_m a_{k-m}, \quad (\text{G5})$$

where  $q_k = k/\tilde{L}$ . Since  $u(x, t)$  is real,  $a_k = a_{-k}^*$ , and we can replace the sum by a  $k > 0$  sum. In the two-mode system we keep only the  $k \in \{\pm 1, \pm 2\}$  terms. This is very wrong as an approximation to the Kuramoto-Sivashinsky equation, but –in the spirit of the Lorentz equations– OK for hoping to learn something about the qualitative dynamics of this class of PDEs.

**2013-08-13 Predrag:** A problem with two-mode system is too many parameters (seven! - see (37)). How about reducing the number of parameters by demanding that our two-mode system is the  $k \in \{\pm 1, \pm 2\}$  truncation of Kuramoto-Sivashinsky? There is only one parameter left (the system size  $L$ ), so that is probably too radical – maybe it will yield no interesting dynamics at all, but let something physical like this guide you in reducing the number of parameters.

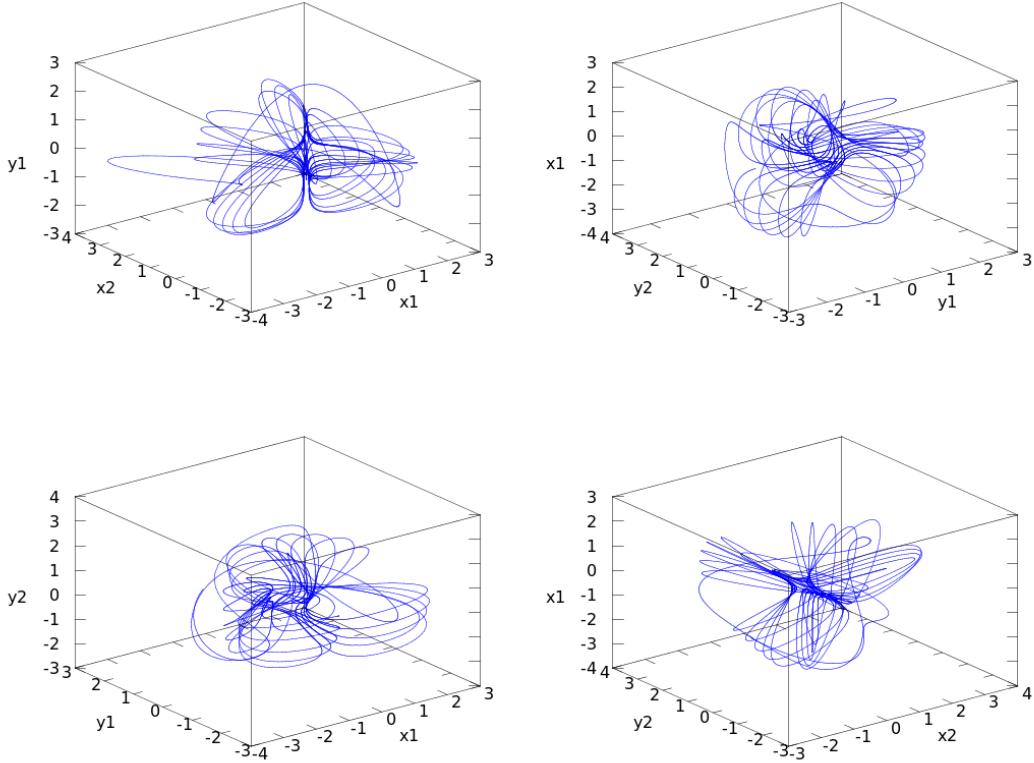


FIG. 25. 3D projections of trajectories of two-mode flow in full state space for parameters:  $\mu_1 = 1.768907$ ,  $a_1 = 0.406357$ ,  $b_1 = -1.660768$ ,  $c_1 = 1.00000$ ,  $\mu_2 = -0.675565$ ,  $a_2 = 0.083130$ ,  $b_2 = -0.047035$ ,  $c_2 = -1.00000$ ,  $e_2 = -0.455152$

The physical setting is that a dissipative turbulent system has a finite number of Fourier modes nonlinearly coupled and of comparable amplitudes, while high modes are very strongly suppressed by dissipative terms like  $q_k^4$  in (G5). (Xiong Ding can explain the ‘physical dimension’ to you). So you will be interested in chaotic solutions for which the two modes  $u = z_1 \bar{z}_1$ ,  $v = z_2 \bar{z}_2$  in (30) are of comparable magnitude.

One good exercise is to go through Ruslan’s 15.7.1 2009-08-26 Epicycles: 2-Fourier modes in the [main blog](#). In this model he looks at a trivial dynamics of two uncoupled modes. I do not agree with Ruslan pessimistic conclusion - search for ‘epicycles’ within the blog for more optimistic angles.

**2013-08-30 Burak:** I computed dynamics of two mode truncation of 1D Kuramoto-Sivashinsky equation and found nothing interesting. Solutions either diverges or converges to the origin. Mathematica files where I derived the velocity functions and stability matrices, along with the Matlab files for numerical work are in [/blog/burak/2modeKuramotoShivashinski](#).

**2013-08-21 Burak:** I varied the parameters I used in Fig. 22 and Fig. 23 and the fixed point in the Poincaré section (attracting limit cycle in the flow) got bifurcated to a two cycle. After playing a little bit more, I got Fig. 26. In Fig. 26, every trajectory goes near the origin, make a small turn there and start go away and make another turn at a different location far away from the origin and go back. This behavior is better seen in the Poincaré return map on the right hand side of Fig. 27. On the return map, different points with larger radial distances are mapped to the neighborhood of the radius 5, and points with radii close to 5 are mapped to different points (without following a regular pattern.). This, I think, satisfies the ‘extreme dependence on initial state’ condition of chaos.

After liking what I saw in the invariant polynomials basis, I went ahead and integrated the system in the full

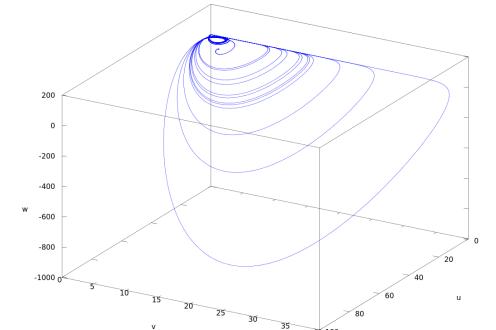


FIG. 26. A projection of two-mode dynamics in the invariant polynomials basis  $\{u, v, w, q\}$ . Parameters:  $\mu_1 = 1.23436$ ,  $a_1 = -0.32304$ ,  $b_1 = -1.07444$ ,  $c_1 = 1.00000$ ,  $\mu_2 = -0.23149$ ,  $a_2 = 0.44110$ ,  $b_2 = -0.42287$ ,  $c_2 = -1.00000$ ,  $e_2 = 1.8$ .

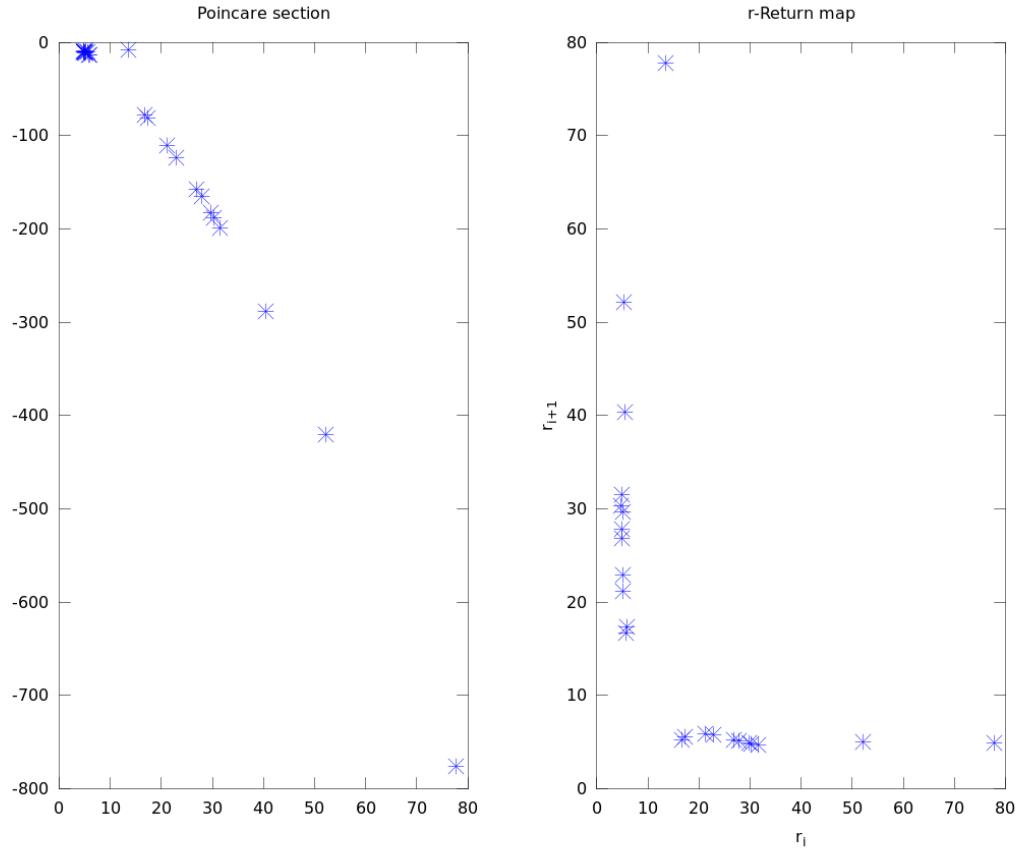


FIG. 27. Left: Poincaré section for the two-mode flow shown in Fig. 26. Section hyperplane passes through  $\{u, v, w, q\} = \{10, 4, 0, 0\}$  and the plane normal is  $\hat{n} = (-4, 10, 0, 0)$  ( $\{u, v, w, q\}$  point rotated 90 degrees about  $w$  axis). Right: Poincaré return map of radial distance from  $w$  axis.

state space and try to slice it using method of moving frames. One projection of the full state space flow is in Fig. 28 and the Fig. 29 is how the symmetry-reduced state space looks like. I tried a few template points (I tried to find equilibria and use them, didn't look good at all) and got the prettiest picture with the template  $\hat{a}' = (1, 1, 0, 0)$ .

**2013-08-22 Predrag:** This looks promising. Comparing Fig. 26 and Fig. 29 you can see one of the reasons why I do not like symmetry reduction by slices to invariant polynomials basis, even in a few dimensions, where syzygies are few: invariant polynomials scrunch everything into the origin. If you and Daniel settle on this one as an

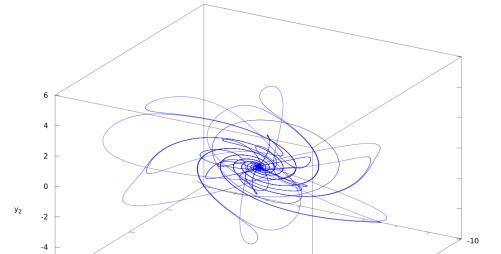


FIG. 28. A projection of two-mode dynamics in the full state space  $\{x_1, x_2, y_1, y_2\}$ . Parameters:  $\mu_1 = 1.23436$ ,  $a_1 = -0.32304$ ,  $b_1 = -1.07444$ ,  $c_1 = 1.00000$ ,  $\mu_2 = -0.23149$ ,  $a_2 = 0.44110$ ,  $b_2 = -0.42287$ ,  $c_2 = -1.00000$ ,  $e_2 = 1.8$ .

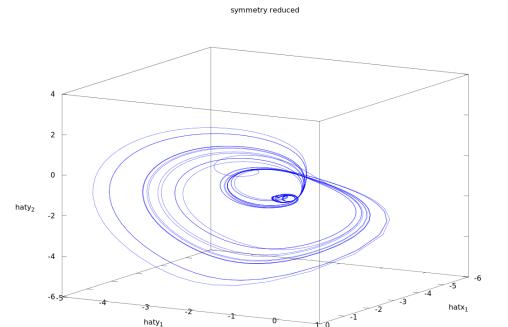


FIG. 29. A projection of two-mode dynamics of Fig. 28 after symmetry reduction using the method of moving frames (post processing). Slice template:  $\hat{a}' = (1, 1, 0, 0)$ .

example (I would prefer an example where you need at least two slice hyper-planes - presumably when the two modes are of comparable strength; and when some parameters are eliminated by requiring that the two-mode system is close to the Kuramoto-Sivashinsky 2-mode truncation), than you should do Poincaré sections more carefully, using points ant not '\*', constructing the unstable manifolds by continuing the Floquet vectors of the initial (now unstable) Hopf cycle, and constructing the return map from them, as discussed in [Chapter 12 Stretch, fold, prune](#).

**2013-08-21 Daniel:** Ok... so while you guys are off doing all the thinking, I brute forced a set of parameters that appear to give chaotic dynamics. Basically, I used the same Monte Carlo-ish technique that I used to find "good" templates for slicing Complex Lorenz flowwhere I let the computer vary parameters by a little bit and see if they give better results. If so they are used a new starting point and the calculation is repeated; if not they are discarded. To measure "success" I calculated finite time Lyapunov exponents for the two-mode system in the invariant polynomials basis and required that the leading one be positive (the more positive the better), the second one be zero (guaranteed by autonomous dynamics), and the third one be negative (the more negative the better). The algorithm prints them out in order, so the fourth one is automatically more negative than the third. The code I used is in `/cgang/Daniel/Matlab/ChaoticExponentsCalculation` and should have enough comments for intelligibility. The algorithm used is based on Wolf et al.<sup>45</sup>. I let the computer run this for a day or so and got the following set of parameters:

$$\begin{aligned} \mu_1 &= -2.8023, \mu_2 = 0.6128, a_1 = -0.9146, a_2 = -2.6636 \\ b_1 &= -0.6141, b_2 = -0.0144, c_1 = -4.1122, c_2 = 1.8854, e_2 = 2.1769. \end{aligned} \quad (\text{G6})$$

**2013-08-23 Predrag to Daniel:** Got your rotatable Matlab file - it was probably OK yesterday, I just did something wrong then....

It's both a boring looking attractor (this is is really in a single slice hyperplane coordinates; not the invariant polynomials basis?), and a little bit too singular - maybe two charts needed?

**2012-08-25 Daniel to Predrag:** I might have had the axis labels wrong in the Matlab fig that I sent you. Basically in invariant coordinates it looks like Fig. 30 and in Cartesian coordinates it looks like Fig. 31.

**2013-08-21 Daniel:** Calculating the Lyapunov exponents for the two-mode system using the parameters from (G6) gives

$$\{\lambda_i\} = \{\lambda_i\} = \{0.09, 0.00, -9.14, -17.77\}. \quad (\text{G7})$$

**2013-08-23 Predrag to Daniel:** are these the Lyapunov exponents, or the real parts of stability exponents? I guess it is Lyapunov exponents. Ask Xiong to compute stability exponents for this system - while finite time stability and Lyapunov exponents differ, I do not know whether they really converge to each other for long times. You are really able to compute  $\lambda_4 = -17.77$  to even one digit, let alone 4 digits?

**2013-08-25 Daniel to Predrag:** <sup>123</sup> Well Wolf *et al.* <sup>45</sup> call them Lyapunov exponents in the paper where the algorithm that I used is described. I don't know that I would necessarily trust the -17.77 to any precision but it is what the code spits out. It does spit out the REALLY negative Lyapunov exponent for the Lorenz system (-22.46) that is reported in the literature <sup>45,69</sup>, as well as the -9.77 that is reported for Rössler to within 0.1, so while maybe I wouldn't swear that the fractional part is correct, I think the -17 is probably reliable.

**2013-08-25 Predrag:** That's a copout. I gave you the link (*Lyapunov exponents*) because the literature is totally confused, and describes two different things by the same name. I believe that covariant vectors and the corresponding multipliers are the correct object, Lyapunov exponents are not the right thing. If you read the description of their algorithm you will know what they are computing, so do check that. My collaborators were never able to compute very negative stability and/or Floquet exponents accurately (if at all), but the algorithm of Ginelli *et al.* <sup>70</sup> that Xiong should implement is supposed to compute *all* to machine precision. So once you have some exponents you think are worth checking, ask Xiong to check his algorithm against yours. Being a classical graduate student, Xiong has so far refused to work on anything less than the 32-dimensional Kuramoto-Sivashinsky system.

**2013-08-25 Daniel to Predrag:** Ok... so looked at the Wolf et al. paper more carefully. I'm pretty sure it's Lyapunov exponents. I know they are a relic of the 80's and not endorsed by Das Buch, but I was only using them to try and find parameters that would give us chaotic behavior. I just know that the leading exponent is positive, the second one is within 0.001 of zero, and the third and fourth ones are negative. I'll stop by to discuss tomorrow just to make sure.

**2013-08-21 Daniel:** Coincidentally, the first three are almost exactly the Lyapunov exponents for the Rössler system ( $\lambda_1 = 0.09, \lambda_2 = 0.00$ , and  $\lambda_3 = -9.77$ ). Fig. 30 shows what it ends up looking like in the invariant polynomials basis. If you run it longer it just fills out the attractor. Fig. 31 shows it in the full state space. If you run it longer it just fills out the attractor.

**2013-08-23 Burak:** I integrated the system with Daniel's parameters and my simulation converged to equilibrium  $(0, 425.56, 0, 0)$  in the invariant polynomials basis (see Fig. 32 (a)). I found all eigenvalues of the stability matrix at this point to be negative with zero imaginary parts. As I expected, this resulted in a relative equilibrium in the full state space (see Fig. 32 (b)).

I also sliced this solution with the same template that I used for my parameters and got Fig. 33 for the time that the solution looked chaotic (t from 0 to 12), it then makes a jump (see Fig. ?? (a) for evolution from t=14 to t=17) as the solution in invariant polynomials basis does, and then converges to the relative equilibrium (see Fig. ?? (b) for evolution from t=15.5 to t=20) by going through a spiral. what figure does fig:BBsymmredPKDanielPars14t20 refer to? If it was erroneous, rewrite this post; also all other figures that have turned to ?'s now. Whenever you comment out / remove / rename a labeled environment, search for the commented-out label and fix all references to it. I'm blogging this because I found it interesting to see a "spiral in" type of convergence on the symmetry reduced manifold while all eigenvalues of the stability matrix of the equilibrium in the invariant space have zero imaginary parts.

**2013-08-23 Predrag:** Having equilibrium at  $(0, 425.56, 0, 0)$  in the invariant polynomials basis must mean that it sits in an invariant subspace (there is much discussion of a such subspace earlier in the blog, check whether it is in this subspace). Then it will have eigenvectors within the subspace, and others that point out - all of that should be significant. 'Spiral in' I find unexpected, maybe there is an unstable equilibrium you are missing?

<sup>124</sup> 2CB

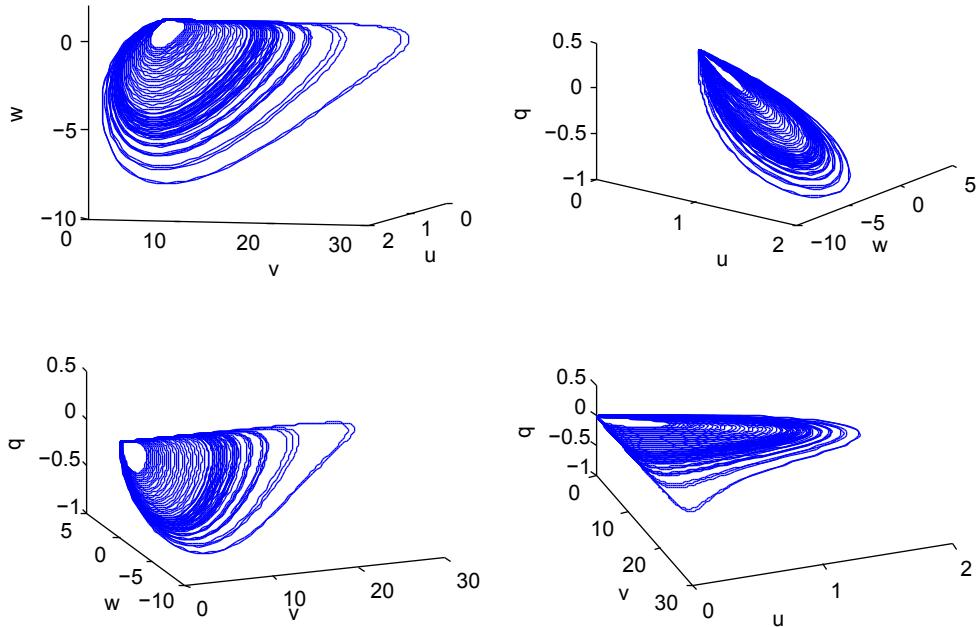


FIG. 30. Two-mode system in invariant polynomials basis using parameters (G6).

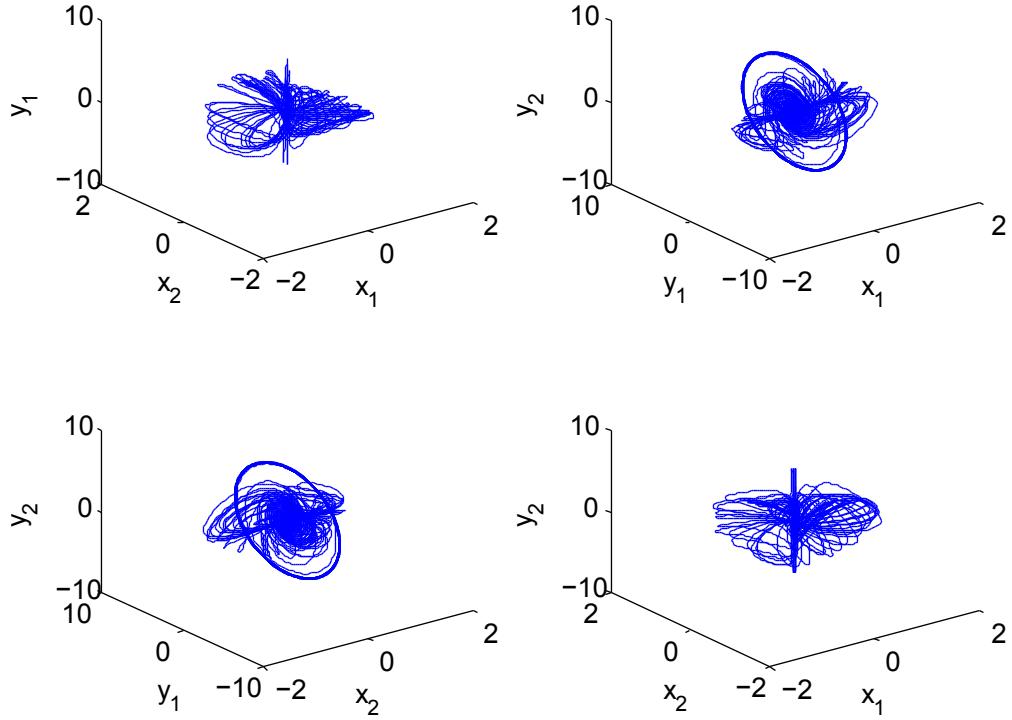
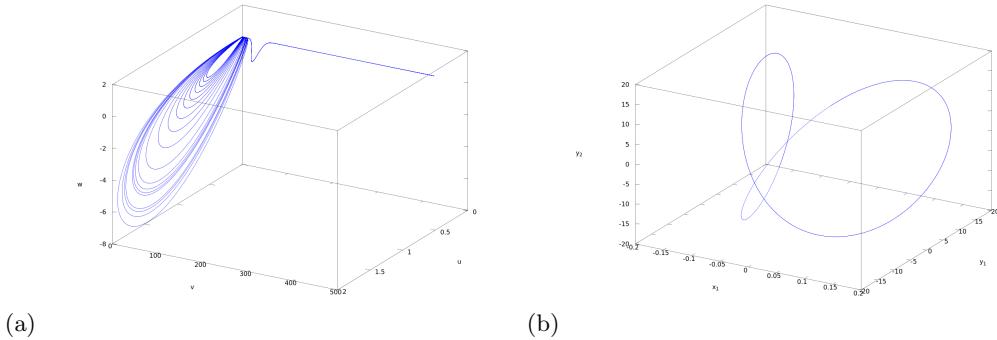
**2013-08-23 Burak:** The inconsistent look of the equilibria was due to a numerical error. I used different time steps for simulations in invariant basis ( $\Delta t = 0.001$ ) and full state space ( $\Delta t = 0.01$ ). When I checked in the invariant polynomials basis, I found that when I use the smaller time step, simulation converges to a relative equilibrium (zero velocity) but if I use the bigger time step, it ends up in a periodic orbit. I calculated the final point of the full state space simulation and found a point corresponding to the cycle in the invariant polynomial space. ([2013-08-24 Predrag] Do you mean: “I calculated the relative equilibrium in the full state space and found the corresponding equilibrium in the invariant polynomials basis”?) ([2013-08-25 Burak] Yes.) My point is, there is no direct correspondence between the invariant polynomials basis plot that I posted (Fig. 32 (a)) and full state space plot (Fig. 32 (b)). Since symmetry reduction plots are generated from the full state space data, they are also not so accurate. I am now simulating the full state space with higher resolution, I already found that the full state space system converges to an relative equilibrium in the  $(y_1, y_2)$  plane and  $x_1 = x_2 = 0$  all the time. This is expected from the transformation relations (G2) and all points on this relative equilibrium are mapped to the invariant polynomials basis equilibrium  $(0, 425.56, 0, 0)$ . Sliced results will be ready tomorrow morning.

**2013-08-23 Predrag:** Fig. 33 is cute. Basically, you have to keep checking whether your trajectory has crossed the border of your single slice hyperplane. If so, two-hyperplane (with a ridge in-between) cover of the slice is needed.

**2013-08-23 Daniel:** Just ran a couple of hundred simulations using the parameters from (G6) with random initial conditions taken from a Gaussian ball centered at  $(u, v, w, q) = (0, 0, 0, 0)$  with radius of about 8. Each simulation was run for 1000 time units. I found that the trajectories ended up at Burak's equilibrium about 70% of the time, so it's basin of attraction is definitely significant. Then again, 30% of trajectories fall into the chaotic attractor and stay there for long times, so it's basin of attraction is not tiny either. It also doesn't seem like the chaotic dynamics are transient. I can integrate out to 20000 time units and still be in the attractor if I start in its basin of attraction (e.g.,  $(u, v, w, q) = (1, 1, 1, 1)$ ). Maybe we can tweak the parameters to make Burak's equilibrium unstable or hyperbolic? Looking at the symbolic expressions for the eigenvalues, I don't see a particularly obvious way to do this. The expressions are a mess!

**2013-08-23 Predrag:** I do not mind coexisting attractors - that's what we have for the pipe and the plane Couette flows anyway.

**2013-08-23 Burak:** I tried to calculate the dynamics within the slice and then reconstruct today. It worked perfectly on 1 slice for Daniel's parameters. I am committing the Matlab (Octave) functions and scripts that I used at [/blog/burak/PorterKnobloch](http://blog/burak/PorterKnobloch) If you want to test them, first run `generator.m`, this will generate `generator.mat`

FIG. 31. Two-mode system in the full state space  $(x_1, x_2, y_1, y_2)$  using parameter values (G6).FIG. 32. (a) A projection of two-mode dynamics in the invariant polynomials basis  $\{u, v, w, q\}$ , parameters given in (G6). (b) Relative equilibrium in the full state space corresponding to the equilibrium in (a).

with the group generator matrix in it. You can then integrate the system by running `PKintegrate.m` and then you can reduce the symmetry by running `PKComputeMovingFrames.m` and you can do everything starting from the reduced space by running `PKComputeDynamicsWithinSlice.m`. Now to compare the results from both approaches you can run `PKPlotDynamicsWithinSlice.m`. You should see exactly the same dynamics on the slice and in the full state space calculated with different techniques. I am not posting these results but basically I get the exact same plot with Fig. 33 by calculating the dynamics entirely on the slice.

I was not this lucky with my parameters. When I tried to simulate the dynamics on the slice with the same initial point, I got something that started similar to the slice from the moving frames, and then settled on a limit cycle, results of two methods are side by side on Fig. 34 for comparison. I also plotted the variation of the group parameters,  $\phi(t)$  (computed within the slice) and  $\phi(\tau)$  computed using moving frames, and the square of the Euclidian norm of the difference between state vectors,  $|x_r(t) - x(t)|^2$ , calculated both by reconstruction from the slice and the direct integration in the full state space (Fig. 35). Any ideas why is this the case?

**2013-08-24 Burak:** For amusement, I rotated the symmetry reduced dynamics such that one of the coordinates was

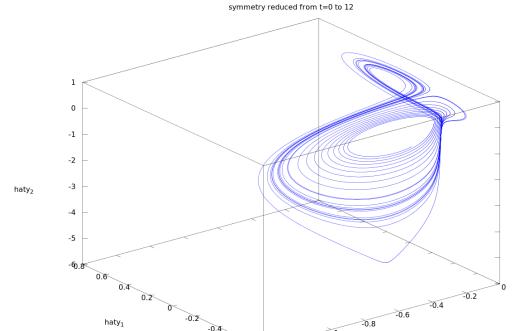


FIG. 33. A projection of two-mode dynamics of Fig. 32 in a single slice hyperplane using the method of moving frames (post processing). A single template:  $\hat{a}' = (1, 1, 0, 0)$ .

always 0 hence got the entire dynamics confined in 3 dimensions so that we can see it without any projections. I used Daniel's parameters to produce Fig. 36. You can use `PKPlotMovingFrames.m` after computing the moving frames by following the steps in [2013-08-23 Burak] to produce this plot.

**2013-08-25 Daniel:** There is actually a less contrived way to look at the dynamics in 3D. Just write down  $(x_1, x_2, y_1, y_2)$  as  $(r_1 \cos \theta_1, r_1 \sin \theta_1, r_2 \cos \theta_2, r_2 \sin \theta_2)$ . Then the equations of motion reduce to  $(r_1, r_2, \theta)$  where  $\theta = 2\theta_1 - \theta_2$ .

See `/cgang/Daniel/Matlab/EOMComplexTo4.m` for details.

**2013-08-24 Predrag:** Mhm... “a rotated frame such that all points have  $\hat{a}_2 = 0$ ” in Fig. 36 sounds like my Chaos-Book chapter written many years ago, and that I have to rewrite from scratch, almost... Things have advanced since. Can you guys use the notation of the c-gang paper<sup>16</sup>? It’s not a ‘rotated frame’ but a slice hyperplane, and that is defined by writing down the group tangent vector  $t'$  evaluated at the template  $\hat{a}'$ . That is almost never a hyperplane corresponding to setting a full state space coordinate to zero, the flow is nonlinear, and a good template is intrinsic to the flow, not to the coordinate system used.

**2013-08-25 Burak:** Sorry for wrong terminology. It is not a rotated coordinate system. I applied all points of the reduced flow the group operation, such that the  $\hat{a}'_2 = 0$  where  $\hat{a}' = g(\phi)\hat{a}$ . I was interpreting this as a visualization of the slice manifold.

**2013-08-24 Predrag:** Sorry, at the first parsing, I have no interpretation of Fig. 36. Daniel - can you have the first go?

**2013-08-24 Burak:** I varied Daniel's parameters a little bit and I think found a better behaving set. I computed the trace of the stability matrix at the equilibrium and tried to make it less negative. To do that, I varied  $b_1$  and got better looking flows for  $b_1 \rightarrow b_1/n$ . Fig. 37 shows the flow in the invariant basis and the Fig. 38 is the corresponding Poincaré section for the parameter set given in (G8). Some fine tuning on  $b_1$  may give a better chaotic attractor but I have to leave soon so I am blogging prematurely. There might also be a bug in my Poincaré section code because numbers does not make sense to me but I believe the picture is qualitatively correct.

$$\begin{aligned} \mu_1 &= -2.8023, \mu_2 = 0.6128, a_1 = -0.9146, a_2 = -2.6636 \\ b_1 &= -0.20470, b_2 = -0.0144, c_1 = -4.1122, c_2 = 1.8854, e_2 = 2.1769. \end{aligned} \quad (\text{G8})$$

**2013-08-24 Predrag to Daniel:** Looks like this might result in a publication. At what point do you think we should invite the original cgang contributors, offer them a chance to join this? In my case, I believe that a co-authorship should require a substantial contribution, minor ones go into the acknowledgments.

**2013-08-25 Daniel to Predrag:** Well, I think only Keith expressed interest originally. Anyway, I think if we are going to get them involved with this, it might as well be sooner rather than later so that they can make a significant contribution (and don't have too much to catch up on, which might disincentivize them to actually do anything).

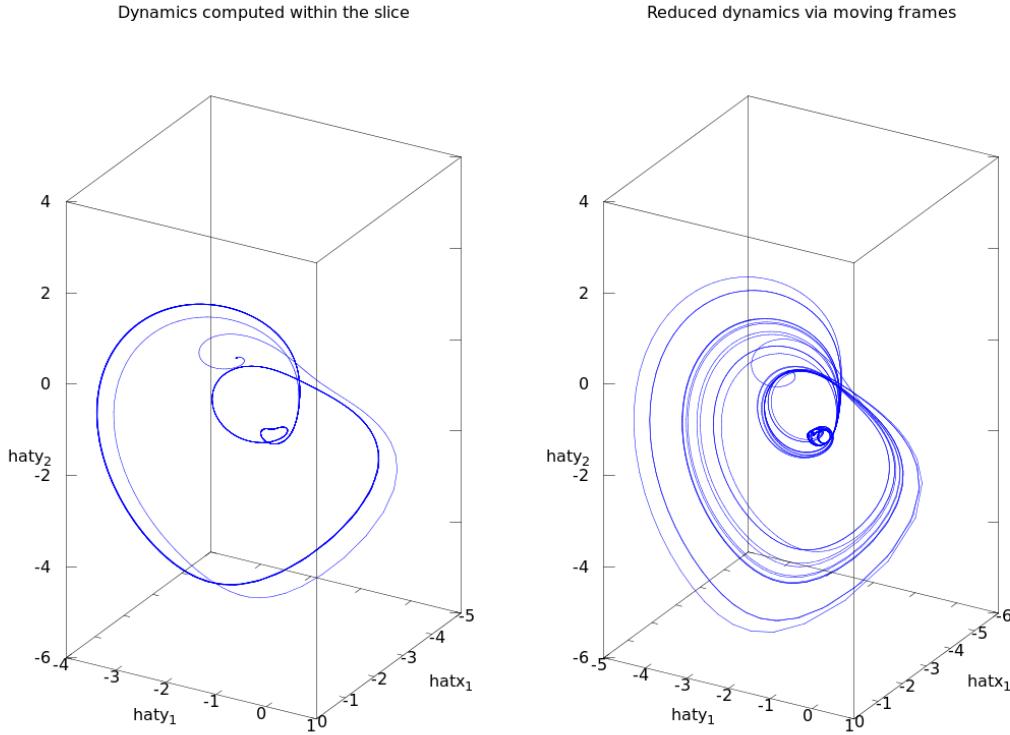


FIG. 34. Left: Dynamics of two-mode system with the parameter set used in Fig. 28 computed within the slice with the template  $\hat{a}' = (1, 1, 0, 0)$ . Right: Symmetry reduced dynamics of two-mode system with the parameter set used in Fig. 28 computed using Moving Frames Method. **ES:** Is the trajectory in both panels the same? Is the integration time longer in the second case? I don't understand why there is difference between the two figures if the template is the same. **BB:** I believe this is due to a numerical error resulting from the trajectory passing close to the chart border. I did not check this one specifically, but I observed a similar situation today and found out that this was the case. Please see my discussion in [2013-09-14 Burak].

**2013-08-25 Predrag:** Can you contact him? If you search through the blog, Bryce seems to have contributed, even though he was MIA when we wrote the ‘atlas’ article; maybe someone else too who we have forgotten by now. Tell them both that they can be coauthors if they contribute to the research content AND actively write and edit the article.

**2013-08-31-2013 Daniel to C-gang:** Contacted Keith, Bryce, and Lei (I’m assuming Siminos is already in the loop). Only heard back from Bryce, who said he is interested. Told him to coordinate with Burak since they are both taking Predrag’s QFT class.

**2013-08-24 Predrag:** The general drift if where you are going looks promising. Remember, you always must check whether the single slice hyperplane reduced dynamics crosses the chart border. If it does, pick a 2. template (that captures dynamics that lies beyond), and check that dynamics crosses the two-slice hyper-planes ridge before it hits the chart border. And so on. Looking forward to drawing the 3-slabs drawing for the 3 slices case.

To make sure you understand this (Daniel does), email now appointment time to Ashley ashleypwillis@gmail.com and Kimberley for Monday, and explain to them how this works. It is very important to do it as soon as possible, as Ashley is trying to implement it for pipes right now and does not understand it (nobody reads papers<sup>16</sup> nowadays - the situation is hopeless :) (but good). And he is here only for 10 days more.

You might also get Kimberly into this git. So far she has not learned to use svn either :)

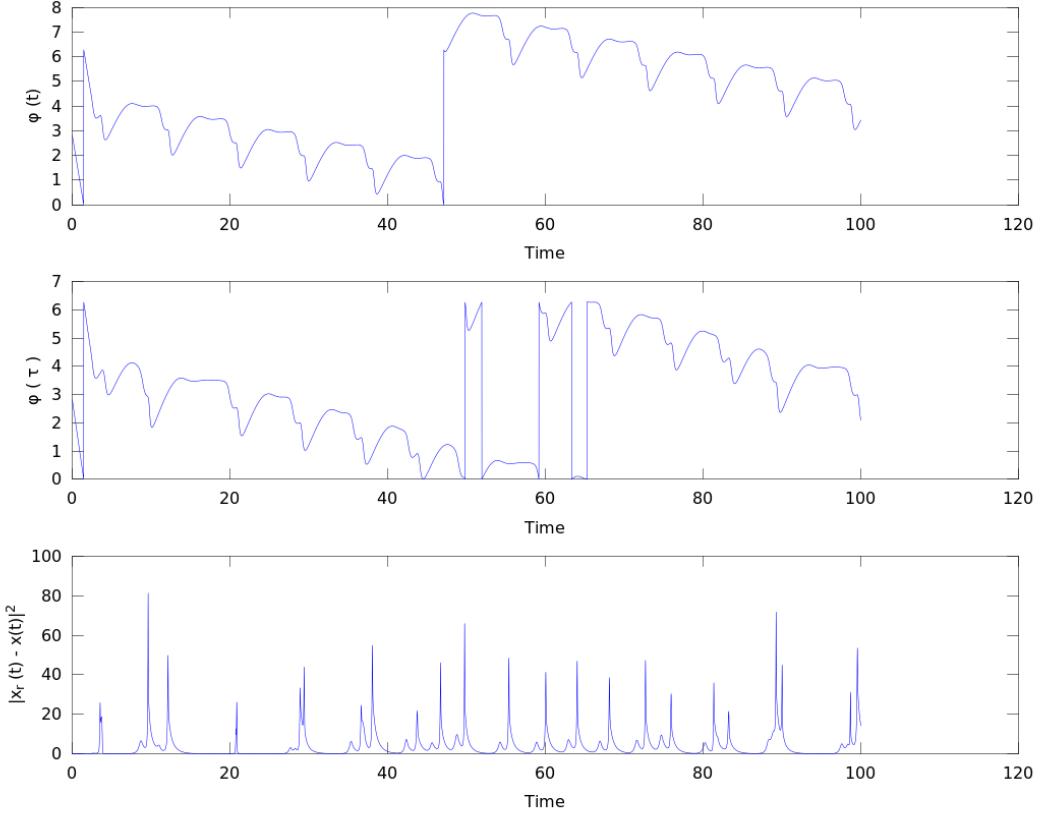


FIG. 35. [Top] Group parameter versus time obtained from the computation of dynamics within the slice. [Middle] Group parameter versus time obtained from the moving frames calculations. [Bottom] Square of the Euclidian norm of the difference between state vectors calculated both ways. The same parameters as in Fig. 28.

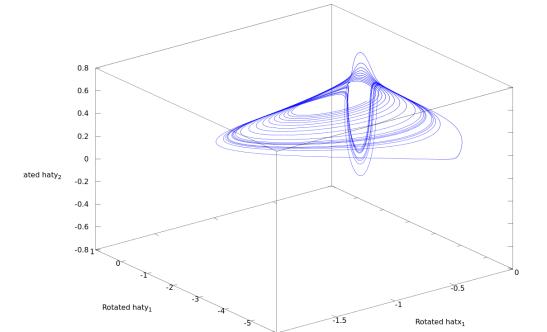


FIG. 36. Symmetry reduced two-mode dynamics of Fig. 32 on the 3D slice manifold  $\hat{\mathcal{M}}$ . See Fig. 41.

**2013-08-25 Burak:** I am sending Kimb the invitation to the Google updates group and the quick start guide (for Debian-based Linux Distros. I am also blogging it in `strategy.tex`) you can give her collaborator access once she gets herself a GitHub account.

**2013-08-25 Burak:** Using the parameters in (G8), except that in this numerical search, I mistakenly used  $b_2 = -0.00144$  (see [2013-09-02 Burak] for the discussion), I scanned the invariant coordinates  $u$  and  $v$  from 0 to 1000 (increasing each by steps of 10) and looked for a equilibrium starting a Newton-Raphson search from each  $(u, v)$  combination. Code is here:

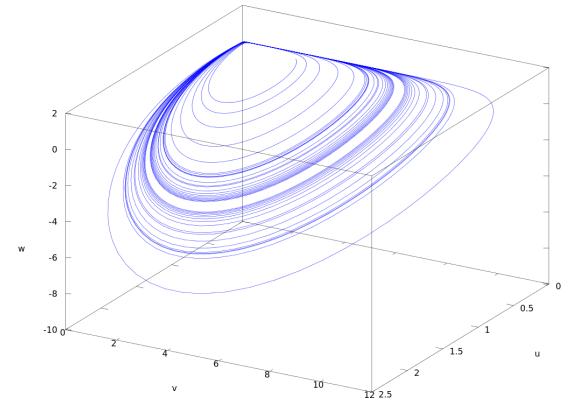


FIG. 37. Two-mode dynamics in the invariant space with parameters (G8)

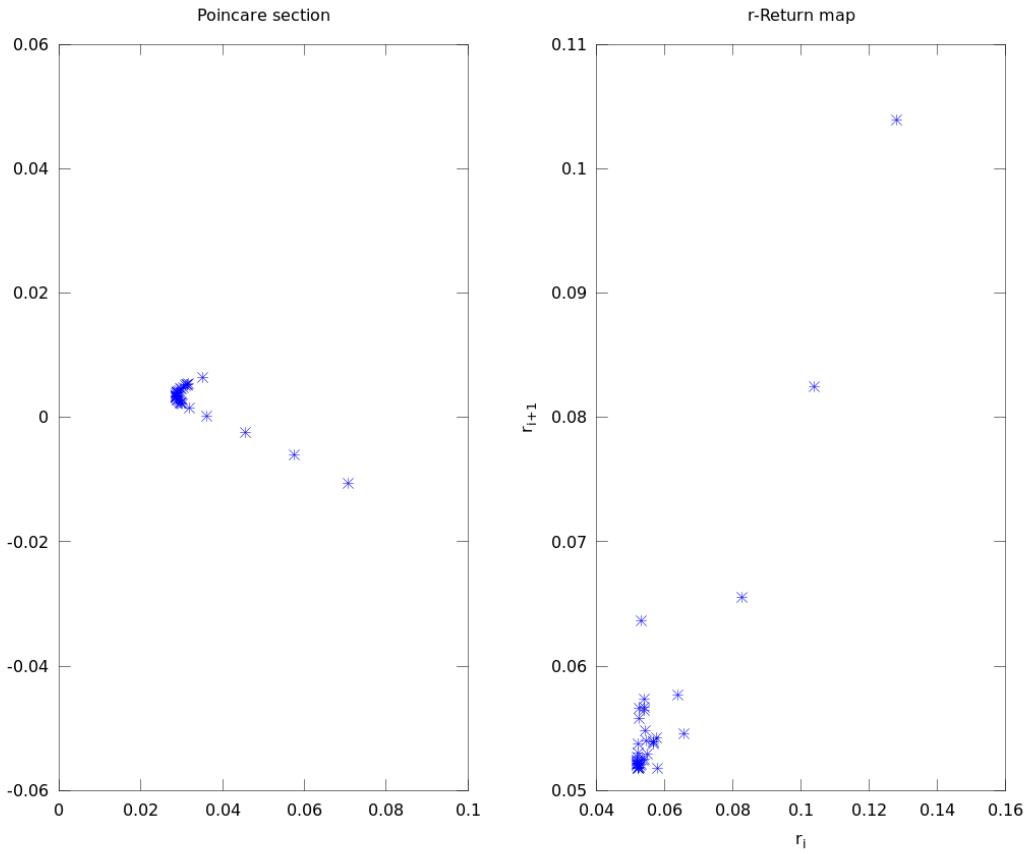


FIG. 38. Left: Poincaré section for the two-mode flow shown in Fig. 37. Section hyperplane passes through  $\hat{x} = (u = 2.5, v = 12, w = 0, q = 0)$  and the plane normal is  $\hat{n} = (-12, 2.5, 0, 0)$  ( $\hat{x}$  rotated 90 degrees about  $w$  axis). Right: Poincaré return map of radial distance from  $w$  axis.

/blog/burak/PorterKnobloch/rootfinding/pkroots.m. The equilibria that I found are

$$\begin{aligned}
 (u, v, w, q) &= (0, 0, 0, 0) \\
 &= (0.12496, 0.62510, -0.18504, -0.06933) \\
 &= (0.21517, 373.61171, -8.31743, -0.11355) \\
 &= (0.02618, 373.61171, -1.01417, -0.01386) \\
 &= (0.13684, 374.34003, -5.29472, -0.07231) \\
 &= (0.00269, 375.58562, -0.10412, -0.00142) \\
 &= (0.37194, 372.15177, -14.34919, -0.19577) \\
 &= (0, 425.56, 0, 0).
 \end{aligned} \tag{G9}$$

(for the last one, see [2013-08-26 Burak] below)

**2013-08-25 Predrag:** Wow!

**2013-08-25 Burak:** Corresponding stability matrix eigenvalues in respective order:

$$\begin{aligned}
 \lambda_i &= (1.22560, -4.99180 \pm 2.17690i, -5.60460) \\
 &= (0.02993 \pm 1.97539i, -5.82124, -11.62009) \\
 &= (7.99278, -10.73945, -159.45337, -318.12740, ) \\
 &= (2.74751, -3.87388, -159.32547, -318.65092) \\
 &= (6.44915, -8.52430, -159.40058, -318.34481) \\
 &= (0.69456, -1.61956, -159.30945, -318.71575) \\
 &= (10.22285, -14.31342, -159.55807, -317.69045).
 \end{aligned} \tag{G10}$$

Each found equilibrium has at least one positive eigenvalue.

**2013-08-25 Predrag:** I find it easier to survey  $\{\lambda_j\}$  if one lists them by decreasing real parts  $\lambda^{(j)} = \mu^{(j)} \pm i\omega^{(j)}$ . The most worrisome is  $0.02993 \pm i.197539$ ; why is it so close to being marginal? You also need the eigenvectors  $e^{(j)}$  to make sense of how these fit together. You use the unstable ones to start tracing (short) segments of unstable manifolds, to see where they go; some examples are in Ref. 46 and [ChaosBook.org/tutorials](#).

**2013-08-25 Burak:** I also computed the dynamics within the reduced space and generated the exact same trajectory that I got from the direct integration in the full state space using the parameters in (G8). You can reproduce this result by following the directions in [2013-08-23 Burak].

**2013-08-25 Daniel:** Hmm... according to our results from ca. 2012-04-28 which are around (36) in the manuscript, we should have 8 equilibria: the origin,  $(0, -\mu_2/b_2, 0, 0)$ , and six others that we haven't been able to solve for analytically at the time. Did no  $(0, -\mu_2/b_2, 0, 0)$  equilibrium pop out this time? I think that HAS to be there, independent of parameters. Just in case, I checked all the equilibria listed by Burak in (G9) and they all seem to satisfy the syzygy (27) (at least to a couple of decimal places since Burak only listed the values are only listed to 5 decimal places).

**2013-08-26 Burak:** I checked and confirmed that  $(0, -\mu_2/b_2 = 425.56, 0, 0)$  is indeed an equilibrium and it is stable. I also started simulations from the each equilibria I posted above at (G9) they all ended up going towards the strange attractor. I think I'll be able to get  $(0, -\mu_2/b_2, 0, 0)$  also hyperbolic or unstable, now that we have an analytic expression for where the root is, it will be fairly straightforward.

**2013-08-26 Daniel to Burak:** Remember that  $u$  and  $v$  are constrained to be positive since they are  $x_1^2 + x_2^2$  and  $y_1^2 + y_2^2$  in disguise. I might be possible to just restrict  $-\mu_2/b_2$  to be a negative number so the equilibrium is in a part of phase space that we don't care about.

**2013-08-30 Daniel:** After discussing with Predrag, I tried playing around with the parameters a little bit to see if I could get rid of some of them and still maintain chaotic dynamics. I discovered that it is possible to get reduce the number of parameters to 4 (or maybe 5, no sure if setting  $e_2 = 1$ , counts as removing a parameter) by basically either zeroing out parameters or setting to 1 or -1. Here's a set that gives some nice chaos:

$$\begin{aligned}
 \mu_1 &= -2.8023, \mu_2 = 1, a_1 = -1, a_2 = -2.6636 \\
 b_1 &= 0, b_2 = 0, c_1 = -4.1122, c_2 = 1.8854, e_2 = 1.
 \end{aligned} \tag{G11}$$

The attractor looks pretty much the same as what we have been seeing with the parameter sets from (G8) and (G6). Notice that two terms are wiped out completely ( $b_1$  and  $b_2$  are zeroed out so it simplifies things a little bit. Furthermore, I think it sends the extra attractor  $(0, -\mu_2/b_2, 0, 0)$  out to infinity, so the strange attractor might be the only one. Burak, maybe you can run your root finder with this set of parameters and see what happens to the other roots?

The Lyapunov exponent calculation using these parameters gives

$$\{\lambda_i\} = \{0.117, 0.00, -5.7, -11.2\}, \quad (\text{G12})$$

to be contrasted with

$$\{0.09, 0.00, -9.14, -17.77\}$$

from (G7).

As Predrag has stressed earlier this probably isn't the optimal thing to calculate, but at least tells us that there is a positive Lyapunov exponent.

**2013-08-30 Burak:** I searched for roots using Daniel's new simplified set of parameters starting from different initial guesses as I did for the previous case. I only found  $(0.19346, 0.54822, -0.28187, -0.05119)$  other than the origin with the stability matrix eigenvalues  $(-11.01364, 0.05064 \pm i2.45364, -5.51033)$ . I also simulated the system with these parameters and calculated the Poincaré section and I confirm chaotic behavior.

**2013-08-31 Predrag:** What is striking about the equilibria in (G9) are the  $v \approx 375$  values, while the attractor  $v$  values are of order 10. These equilibria are unlikely to play any significant role in shaping the strange attractor. Can you find values of parameters that send these off to infinity? If so, can that might reduce the order of the polynomial, and perhaps reduce the number of parameters to a few?

The logic of the paper would then be:

- We would like to illustrate how the interplay of different physical modes complicates group orbits, and necessitates slicing by a set of chart.
- $N = 2$  truncation of Kuramoto-Sivashinsky is too radical to exhibit any interesting dynamics, so we adopt the well studied  $\text{SO}(2)$  symmetric bifurcation normal form two-mode system as our starting model. Unfortunately it has 7 parameters, none with physical interpretation when dynamics is far from the bifurcation.
- However, we note that most equilibria tend to lie very far from the strange attractor of interest. We chose parameters such that these equilibria are sent off to infinity, thus reducing the number of parameters to ???. Some of these we can set to simple values without losing the qualitative dynamics of interest, and thus our model is ??-parameter sub-model of the two-mode model.

**2013-08-31 Daniel to Burak:** For some reason, I haven't been able to get your rootfinding code to work. It must be some Matlab/Octave translation thing and I'm not really sure how it works anyway. As Predrag discussed above, it would be nice to have an excuse for setting some of the parameters to zero, for example, that doing so sends some of the roots off to infinity. I have a theory that the magnitude of the solutions that you found for your modified set of parameters is large because they go like  $1/a$  small parameter. The only small parameters in your parameter set are  $b_1$  and  $b_2$ . Setting  $b_2$  to zero sends  $(0, -\mu_2/b_2, 0, 0)$  to infinity, so might be possible that  $b_1$  controls the other ones. Could you start from (G8), set  $b_2$  to zero, check if this affects the roots. If not, make  $b_1$  a little bit smaller and see if they move further away from the origin. If they do, make it even smaller and see if they go further out. This might be a way to justify zeroing out  $b_1$ .

**2013-09-02 Burak:** To have a better understanding of the behavior of the fixed points, I started from (37), scaled variables and parameters back to the ones without tilda, and solved  $f(u, v) = 0$  for  $u$ . Since this polynomial is of degree 2, I got two possible solutions for  $u$ :

$$u_{root,1}(v) = -\frac{B}{2A} \mp \frac{\sqrt{\Delta}}{2A}$$

where  $\Delta = B^2 - 4AC$ ,  $A = a_1c_2$ ,  $B = -va_2c_1 + vb_1c_2 + c_2\mu_1$ ,

$$C = -v^2b_2c_1 - vc_1\mu_2. \quad (\text{G13})$$

Since  $v$  and  $u$  are always greater than 0, only positive root in (G13) has a meaning for the two-mode system, however, which one is positive depends on the parameters and the value of  $v$ . I substituted these roots into the unscaled  $g(u, v)$  from (37) and got the following equations for  $v$

$$F_1(v) = g(u_{root,1}(v), v) = 0 \quad (\text{G14a})$$

$$F_2(v) = g(u_{root,2}(v), v) = 0 \quad (\text{G14b})$$

Now, we have two polynomials of one variable roots of which are the roots of (37). By looking at the behavior of these polynomials, I realized a few things. First: There was a typo in my numerical calculation of the fixed points for the parameters given by (G8). In the numerical calculation, I entered  $b_2 = -0.00144$  instead of  $-0.0144$  and this resulted in finding a fixed point around  $v = 375.368$  with a positive corresponding  $u = 0.0261764$ . If I had done that calculation correctly with parameters (G8), I wouldn't find any of the roots with  $v = 375.368$ , because with the correct value of  $b_2 = -0.0144$ , this root of  $F_1(v) = 0$  is shifted to the  $v = 392.35$  where the corresponding  $u$  is  $u = -1.82583$  and since a negative  $u$  value is not allowed, the fixed point disappears. This method of looking for the fixed points confirmed my previous numerical findings. For the parameters given by (G11) variation of  $F_1(v)$  with  $v$  is shown in Fig. 39. I am committing the Mathematica file that I used to

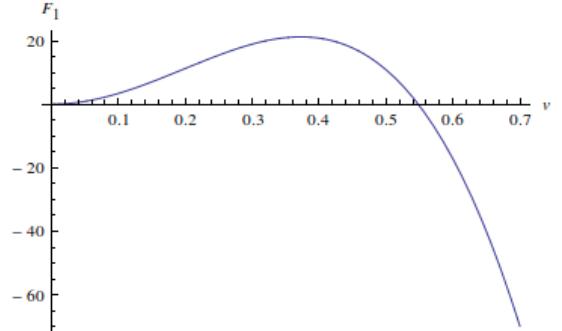


FIG. 39. Variation of  $F_1(v)$  given by (G17) evaluated with the parameters (G11). Note that the curve intersects the  $v$  axis at  $v = 0.548221$  which is the only fixed point of this system other than the origin. Numerical evaluation shows that this curve monotonically decreases, hence there is no other fixed point.

compute these polynomials at [/blog/burak/PorterKnobloch/Mathematica/PKfgeneral.nb](#).

**2013-09-05 Burak:** I tried to develop a way of finding two templates that can capture the dynamics completely and failed. Here is the explanation of what I did: For a chosen template point  $\hat{a}'$ , chart border  $\hat{a}^*$  is defined by the following equations:

$$\begin{aligned} \langle \hat{a}^* | t' \rangle &= 0, \\ \langle t(\hat{a}^*) | t' \rangle &= 0. \end{aligned} \quad (\text{G15})$$

By solving these equations symbolically, we can write the chart border  $\hat{a}^*$ , as a vector of two independent components and two components that are uniquely determined by the independent ones consistent with the fact that the chart border defines a  $(d-2)$  dimensional manifold. The result is

(see [/blog/burak/PorterKnobloch/Mathematica/PKchartborder.nb](#) for the computation):

$$\begin{aligned} \hat{a}^* = \left\{ \hat{a}_1^*, \hat{a}_2^*, \frac{-\hat{a}_1^*(\hat{a}_1'\hat{a}_3' + 2\hat{a}_2'\hat{a}_4') - \hat{a}_2^*(\hat{a}_2'\hat{a}_3' - 2\hat{a}_1'\hat{a}_4')}{4(\hat{a}'_3^2 + \hat{a}'_4^2)}, \right. \\ \left. \frac{-\hat{a}_1^*(-2\hat{a}_2'\hat{a}_3' + \hat{a}_1'\hat{a}_4') - \hat{a}_2^*(2\hat{a}_1'\hat{a}_3' + \hat{a}_2'\hat{a}_4')}{4(\hat{a}'_3^2 + \hat{a}'_4^2)} \right\}. \end{aligned} \quad (\text{G16})$$

After having an expression for the chart border, I tried to come up with a condition of a good choice of two templates  $\hat{a}'_1$  and  $\hat{a}'_2$ . According to my argument (I'm not 100% sure that this is correct, please check. I suspect that it is too restrictive.), if the following inequalities are satisfied,  $\hat{a}'_1$  and  $\hat{a}'_2$  provide a two chart atlas that can capture dynamics avoiding the chart borders:

$$\langle \hat{a}'^{(1)} - \hat{a}'^{(2)} | t'^{(2)} \rangle \langle \hat{a}^{*(1)} - \hat{a}'^{(2)} | t'^{(2)} \rangle < 0 \quad (\text{G17a})$$

$$\langle \hat{a}'^{(2)} - \hat{a}'^{(1)} | t'^{(1)} \rangle \langle \hat{a}^{*(2)} - \hat{a}'^{(1)} | t'^{(1)} \rangle < 0 \quad (\text{G17b})$$

I tried to sketch how I got these inequalities on Fig. 40. Difference vectors shown in red color are the ones which are dotted with the template tangent in (G17a). My claim here is that the dot products of these vectors,

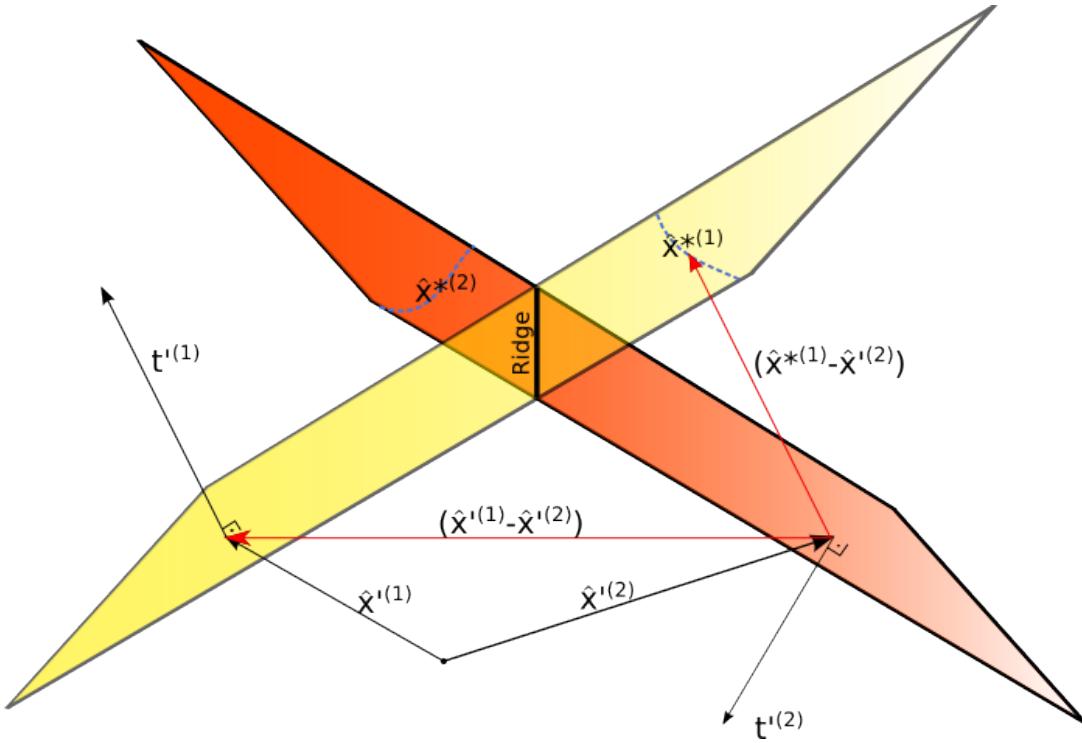


FIG. 40. Sketch of a “good” two-chart atlas where the respective borders of both charts are on the side of the ridge that the templates are not.

$(\hat{a}'^{(1)} - \hat{a}'^{(2)})$  and  $(\hat{a}^{*(1)} - \hat{a}^{*(2)})$ , with the template tangent  $t'^{(2)}$  should have opposite signs if they are on the different sides of the ridge. (G17b) is the same condition for the first chart. After getting equations (G16), (G17a) and (G17b), I numerically picked a first template point  $\hat{a}'^{(1)}$  on the relative equilibrium of the two-mode system with parameters (G11), then applied group transformation on this point and picked the result as a second template candidate (I thought this would be a good start since this way it is possible to transform the normal vector in a systematic manner, as a function of  $\phi$ , rather than pulling a random candidate):

$$\hat{a}'^{(2)} = g(\phi)\hat{a}'^{(1)}. \quad (\text{G18})$$

With the candidate template (G18), I evaluated (G17a) and (G17b) and checked their signs. I assigned a range of parameters to the free parameters in (G16) and evaluated (G17a) and (G17b) at each combination. If algorithm finds that at least one the inequalities are not satisfied, it varies group operation parameter and repeats the process. My intention was to keep the second template point for which the inequalities were simultaneously satisfied, however, this never happened. Then, instead of picking second template by transforming first one, I started generating random candidates and checked if the inequalities satisfied, again did not get anything worked.

Questions to everybody: Does the reasoning behind equations (G17a) and (G17b) make sense to you? Do you see anything too restrictive in the procedure I described?

**2013-09-14 Burak:** I computed two-mode dynamics with Daniel’s simplified parameters (G11) for a long time interval,  $t = (0, 1000)$ , starting close to the relative equilibrium. [2013-09-14 Predrag] Which one? I reduced it onto one slice defined by the template  $\hat{a}' = (1, 1, 0, 0)$  using moving frames. I also computed the dynamics directly starting on the slice hyperplane; plots of both are on Fig. 41. As you can see, they are not exactly the same, Fig. 42 shows when they start to be different and how different they are.

**2013-09-14 Predrag:** Are the coordinates of Fig. 41 the coordinates of the slice hyperplane, i.e., are they Gram-Schmidt orthogonal to the template group tangent  $t' = T\hat{a}' = T(1, 1, 0, 0) = (1, -1, 0, 0)$ ? Plot all relative equilibria; their unstable manifolds might be giving the shape to your strange attractor. Can you also draw the 2D slice border plane in these coordinates, so one can see that this slice is a half of a 3D hyperplane? You

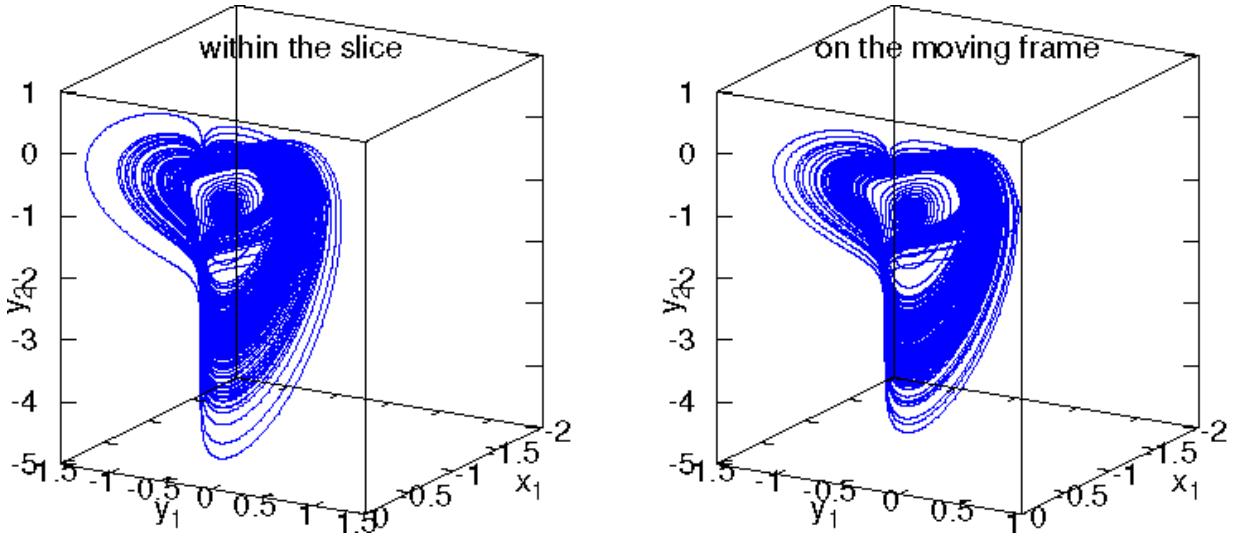


FIG. 41. Left: Dynamics of two-mode system with the parameter set (G11) integrated [within the 3D slice](#) hyperplane normal to the group tangent  $t' = (1, -1, 0, 0)$ . The 2D slice border plane is in these coordinates the  $\{0, y_1, y_2\}$  plane normal to the 4-vector  $(1, 1, 0, 0)$ . As the slice border coincides with the  $m = 2$  flow-invariant subspace, the symmetry-reduced flow cannot cross the slice border. Right: Symmetry reduced dynamics of two-mode system with the parameter set (G11) computed using Moving Frames Method.

probably want to rotate your 3D Matlab projection, so the slice border is the back wall, not the front one. As it contains the pure  $m = 2$  Fourier mode flow-invariant subspace, the relative equilibrium in this subspace might be important in shaping the dynamics.

**2013-09-14 Burak:** As you can see from Fig. 42, full  $a$  trajectory computed by two methods are same (to the machine precision) for the first 200 time steps and then they start to differ around  $t=250$ . If we look at the point on the trajectory for this time, we see that the trajectory passes through the region given by  $x_{1,2} \approx 10^{-5}$  which is quite close to the chart border  $x_{1,2} = 0$  as a result, reduced state space evolution equations comes close to the divergence (but does not diverge) and starts to generate numerical errors. I used the same fixed timestep for both calculations and this result shows that this is not a good idea. One needs to use an adaptive time-stepping algorithm, that would ensure a certain precision, especially for the flow within the slice. I'm 99% sure that single slice will work for two-mode system after integrating a little bit more care.

The reason that I'm sure that single slice would work for two-mode system has to do with its invariant subspace. In the invariant polynomial basis, the invariant subspace is  $(u, v, w, q) = (0, v, 0, 0)$  which corresponds to  $(x_1, x_2, y_1, y_2) = (0, 0, y_1, y_2)$  in the full state space which is the chart border. Hence the dynamics that would start outside the chart border will never get into the chart border in the two-mode system. In other words, in two-mode system, the invariant subspace is the subspace where the dynamics is confined into the pure  $m = 2$  Fourier mode. Since the two-mode system is the most general system with 2 Fourier modes, this urges me to check whether a similar statement can be made in systems with higher Fourier modes. Namely, if we can show that all invariant subspaces of an  $m$ -mode  $SO(2)$  symmetric system will exclude  $m = 1$  region, we can say that an  $m$ -mode  $SO(2)$  symmetric system can always be symmetry reduced using a single slice given by  $(1, 1, 0, 0, 0, \dots)$  without any trouble. I'll be working on this for a while before fixing integration routines which is a rather more trivial task.

**2013-09-14 Predrag:** It's OK, Fig. 42 is as it should be. You have probably frequently passed close to  $x_{1,2} \approx 10^{-5}$  before  $t = 250$ , doubt that that is the cause of the loss of accuracy; the trajectory has positive stability exponents. Flow within the slice is an integration independent of the moving frame post-processing, with independent computational errors, so for unstable flow they should agree only up to the Lyapunov time. Relative periodic orbits are great, because once you start computing them, they should agree to whatever accuracy you want to compute them, an so will their Floquet multipliers.

**2013-09-14 Predrag:** A very minor gripe: your edits are all line per paragraph. It's OK in wrapping mode in editors, but it is impossible to read on GitHub.com site, because they are shown as single lines, extending

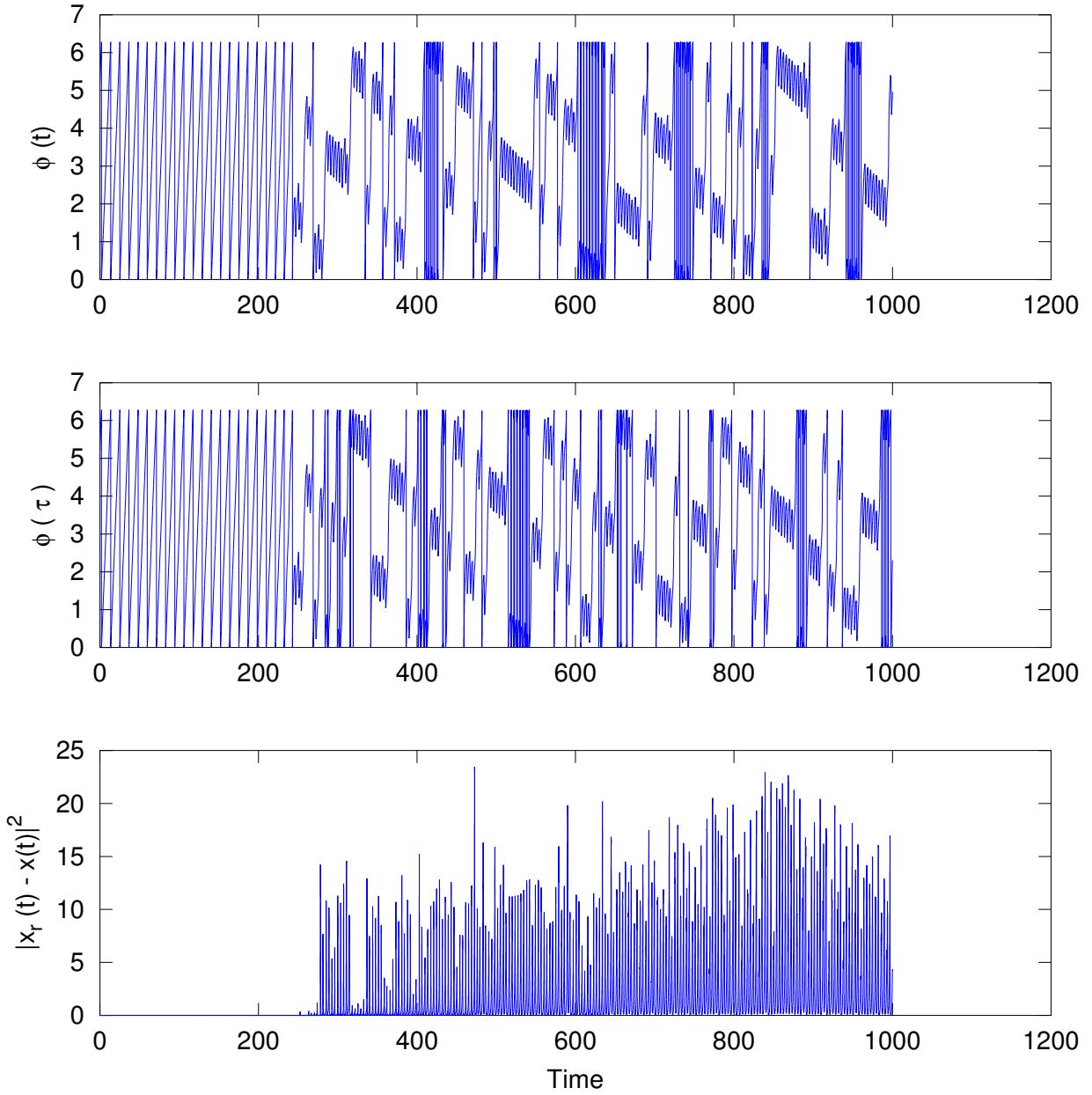


FIG. 42. [Top] Group parameter versus time obtained from the computation of dynamics within the slice. [Middle] Group parameter versus time obtained from the moving frames calculations. [Bottom] Square of the Euclidian norm of the difference between state vectors calculated both ways. Parameters are listed in (G11).

forever off the screen. If your editor can format paragraphs, maybe use that? If it is inconvenient, ignore this, it does not matter.

**2013-09-18 Daniel:** So I guess I'm a little late to the party but I adapted my old code for slicing complex Lorenz flow to slice the two-mode system in Cartesian  $(x_1, x_2, y_1, y_2)$  coordinates using  $(1, 1, 0, 0)$  as a template and basically get the same result as Burak did 2013-09-14. I can confirm that the trajectory never appears to cross a chart boundary (i.e.,  $\phi$  stays bounded and never appears to blow up). I also plotted the results projected onto a Gramm-Schmitt basis orthogonal to the group tangent  $(1, -1, 0, 0)$  such that  $v_1 = (\sqrt{2}/2, \sqrt{2}/2, 0, 0)$ ,  $v_2 = (0, 0, 1, 0)$ , and  $v_3 = (0, 0, 0, 1)$ . This does not seem to make the picture significantly different than what Burak got, so I won't repost figure in the blog. I have saved rotateable Matlab figs in /cgang/Daniel/Matlab/Slicing/SingleSlice(1-1-0-0).fig and /cgang/Daniel/Matlab/Slicing/SingleSliceInSliceCoordinates.fig if anybody wants to take a look. The chart

border basically sits on the  $x_1$  or  $v_1 = 0$ . planes I've included the equilibrium at the origin as red dot, but still need to put in the other equilibrium that Burak found numerically. Burak, could you provide this in  $(x_1, x_2, y_1, y_2)$  coordinates? My bet is that it sits in the middle of the lobe that doesn't have the origin in it...

**2013-09-19 Daniel:** Ok... so I added Burak's numerical equilibrium from 2013-09-02 to the figures. To do this, I started with Burak's  $v = 0.548221$ , plugged into G13 to find the root where  $u > 0$  and got  $u = 0.1934602$ . I then converted this to Cartesian coordinates  $(x_1, x_2, y_1, y_2) = (\sqrt{u}, 0, \sqrt{v}, 0)$  and rotated this into the slice. I plotted these as black dots in my Matlab figs (file locations in my post from 2013-09-18). These points seems to be outside the attractor. I also tried  $(x_1, x_2, y_1, y_2) = (0, \sqrt{u}, \sqrt{v}, 0)$  and rotated THAT into the slice. These are plotted in the Matlab figs as magenta points. This actually seemed to work better in the sense that it plops an equilibrium right down in the middle of the lobed structure, which looks more correct to me. I am kind of confused as to why I get two different points depending on which point I choose. I guess it has to do with fixing the relative phase of the two modes? Or maybe my postprocess rotation into the slice code is buggy. Burak, could you check to see if your code gives different points for the two choices of  $(x_1, x_2, y_1, y_2)$ ?

**2013-09-19 Burak:** I don't think finding  $u$  and  $v$  for the roots is enough for mapping it correctly to the full state space, you have to find  $w$  and  $q$  as well and then find a numerical solution to (G2). My starting point for simulations is one of these points  $(x_1, x_2, y_1, y_2) = (0.12712, -0.42107, 0.68001, 0, 29294)$  corresponding to the equilibrium in the invariant polynomial basis  $(u, v, w, q) = (0.19346, 0.54822, 0.28187, 0.05119)$  (this is the equilibrium in the invariant polynomial basis that I reported in [2013-08-30 Burak]). Trajectory, for a while, traces the group orbit of this point until it goes away from it and becomes chaotic. In my simulations, If you look at the time interval during which the trajectory is on the relative equilibrium in the reduced space, you see that all these points on the group orbit are mapped onto a single point. As the system evolves in time, trajectory spirals out from this point and this region corresponds to the small spiral in the center of the attractor in Fig. 41. I coded stuff using which I can plot these processes separately but my netbook's graphics processors has a problem , so I'll be able to blog these hopefully tonight, when I get home.

By the way, I am unable to look at .fig files since I don't have a Matlab installation and I have no way of legally getting one since I don't have access to a Georgia Tech computer or money to waste on it. I also have endless sympathy for open source things and that's another reason why I don't use Matlab. If you can commit your plots as pdf files, without blogging them if you think it's unnecessary, I can look at them.

**2013-09-19 Predrag:** If you need to run Matlab, you can run it on CNS linux network, or on the MS windows box in the computer cubicle 5th floor Howey W508E.

**2013-09-19 Daniel:** Fixed figure to include the correct relative equilibrium in the slice projected coordinates. Here's a pdf for poor Burak who cannot see Matlab figs Fig. 43. Rotate-able version in /cgang/Daniel/Matlab/Slicing/SingleSliceInSliceCoordinates.fig.

**2013-09-20:** Thanks! Fig. 43 is very similar to what I got, I'm not blogging another version of this, I'm moving onto Poincaré sections on slice right now, results will hopefully be here tomorrow.

**2013-09-19 Predrag:** What about spiralling away from the other relative equilibrium  $NO_2$ ? in Fig. 43?

**2013-09-19 Predrag:** This is really looking pretty. Next,

- a Poincaré section, or more likely two
- parametrize it/them using curvilinear length along the unstable manifolds of the two relative equilibria - I assume one of them is in the invariant subspace? See ChaosBook.
- construct return map, forward maps.
- describe symbolic dynamics, find a set of relative periodic orbits.
- complete the current draft the paper, publish.

**2013-09-19 Predrag:** Please go to Adam Kamor and learn how to use his unstable manifold Poincaré sections code - looks very smart (Adam Fox approved), then put a copy of the code into our git repository /code/. It should help you get the return maps for the 2-mode problem, and it should become very useful once we symmetry reduce KS flow (and NS! :)

**2013-09-22 Burak:** I started computing Poincaré sections on the Gram-Schmidt coordinates. As a first template, I chose the equilibrium  $\hat{x} = (-0.4398, -0.7285, 0.1323)$  (black dot on the middle of the green spiral

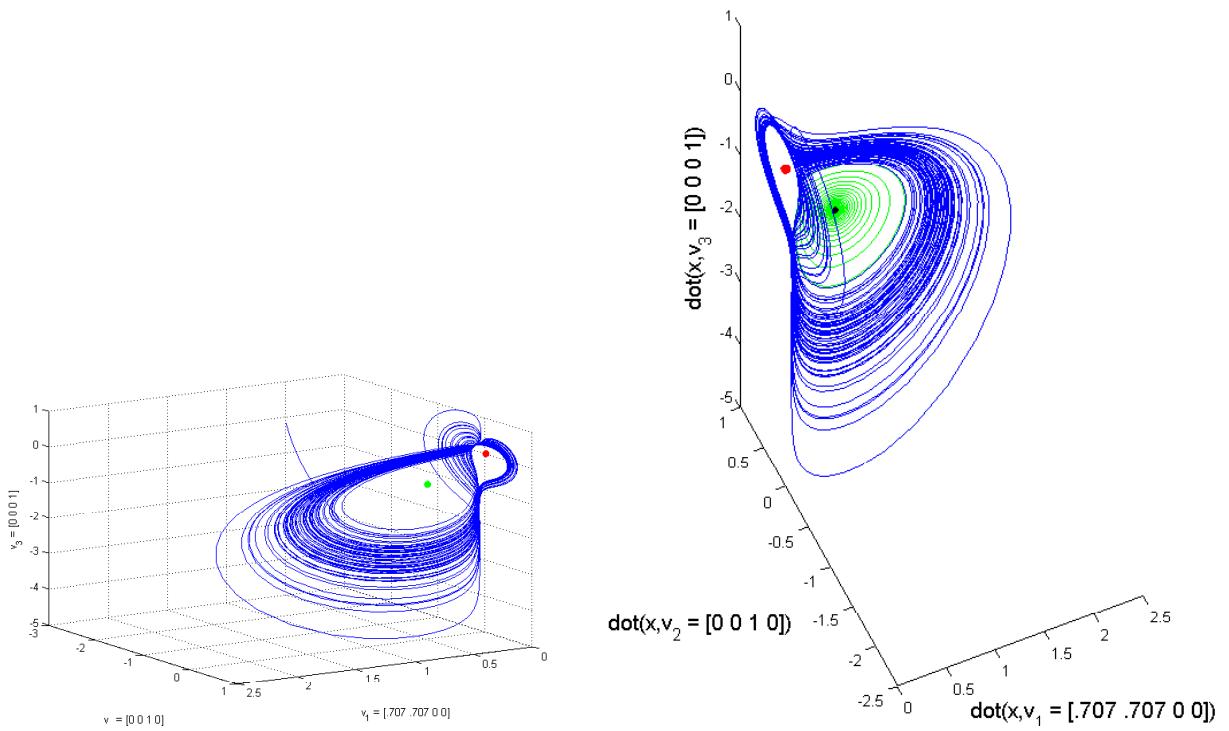


FIG. 43. Left: The invariant subspace which is also the slice hyperplane border contains the unstable relative equilibrium  $NO2?$  marked red, which shapes the ‘border’ part of the strange attractor (EXPLAIN). Right: Dynamics of two-mode system with the parameter set (G11) reduced to a slice hyperplane of the template  $(x_1, x_2, y_1, y_2) = (1, 1, 0, 0)$  and projected onto a Gramm-Schmitt basis orthogonal to the template group tangent  $(x_1, x_2, y_1, y_2) = (1, -1, 0, 0)$ . Relative equilibrium at the origin is shown in red. Points started near the origin fall into the attractor rather quickly. Relative equilibrium  $NO1?$  is shown in black with a trajectory escaping from it along its unstable manifold and into the attractor shown in green. Once the system leaves this green region it does not come back. Trajectories started away from this region never enter it.

on Fig. 44) and chose the normal vector in such a way that the Poincaré section includes the vertical axis:  $\hat{n} = (0.85607, -0.51686, 0.00000)$ . Points on this Poincaré section is on Fig. 45.

Fig. 44 is equivalent to Fig. 43 I am posting this just to have a reference for the Poincaré sections that I am going to present in this blog, since I used my own basis on those.

**2013-09-22 Predrag:** I would chose the two reduced state space coordinates  $\{\hat{a}_1, \hat{a}_2, \hat{a}_3\}$  to span the chart border/invariant subspace plane, the 3. one normal to the two (maybe you do that already). I’m not seeing the Poincaré section (Siminos can point you to the mathematica codes he used to get shading effects), but I would guess the optimal one would go through the other relative equilibrium and cut transversaly the flow coming closest to the observer; that will probably give you a very nice return map.

**2013-09-22 Burak:** I then generalized my code for arbitrary plane normals (using Euler angles) and observed that the Poincaré section looks qualitatively same if I rotate the plane around the vector:  $(-0.4398, -0.7285, 0)$ . I can explain this much better if I can draw these planes along with the flow, I tried but could not yet succeeded in this. Do you have a favourite software for these kind of plots? (transparent surfaces etc.)

I’m not sure if I can define a useful curvilinear distance using which I can construct the return map. To get a better understanding, I will stop coding for a while and read the relevant ChaosBook chapters.

**2013-09-22 Predrag:** I propose we use the same name for invariant solutions when plotted in the full  $\{x_1, x_2, y_1, y_2\}$  state space and the  $\{\hat{a}_1, \hat{a}_2, \hat{a}_3\}$  reduced state space. For example,  $\{x_1, x_2, y_1, y_2\} = (0, 0, 0, 0) \rightarrow \{\hat{a}_1, \hat{a}_2, \hat{a}_3\} = (0, 0, 0)$  is an equilibrium (the only one in this problem, coinciding with the  $(0, 0, 0)$  flow invariant subspace - I guess every equilibrium is by definition a 0-dimensional flow invariant subspace, and relative equilibrium a 1-dimensional flow invariant subspace). Any full state space relative equilibrium and relative periodic orbit should still be called that when plotted as equilibrium and periodic orbit in the reduced state space. Invariant

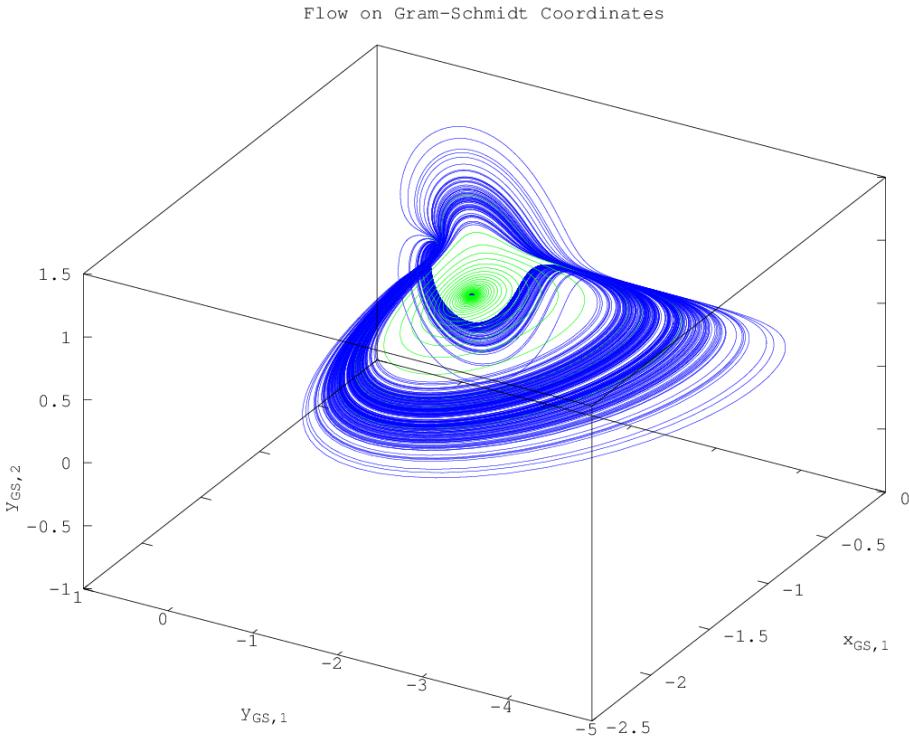


FIG. 44. Symmetry reduced two-mode flow on the Gram-Schmidt coordinates.

polynomials ( $u, v, w, q$ ) are only a crutch; except for the section in which we discuss them, all other relative equilibrium and periodic points should be listed in  $\{x_1, x_2, y_1, y_2\}$  and in  $\{\hat{a}_1, \hat{a}_2, \hat{a}_3\}$  coordinates.

**2013-09-22 Predrag:** Can you mark the red dot (the unstable invariant subspace equilibrium at the origin) in Fig. 44? If I understand you, your single Poincaré section goes through both important equilibria/relative equilibria (if there are other nearby relative equilibria, mark them too, the distant ones do not matter). You might need a pair of Poincaré sections, each going through one of the (reduced state space) equilibria and its stable eigenvector, and then rotated in some convenient manner, perhaps so it goes through the other equilibrium.

**2013-09-22 Predrag:** The list (G10) is missing the stability eigenvalues for the relative equilibrium within the invariant subspace. You also want to distinguish the eigen-directions which stay within the subspace from those who go out of it.

**2013-09-22 Predrag:** to Bryce and Evangelos: It looks like Burak is doing all the heavy pulling. While Daniel crosschecks the results and helps him with the final stretch and the figures, can you two complete the first draft of the article, Sects. I to VI, by October 1, 2013?

**2013-09-28 Burak:** Since there are a lot of experimental posts in this blog, I decided to write a compact summary to clarify which parameter set we are using, what are the roots and equilibria etc. This will help me to have a record of my own progress and I thought this would also help in writing the paper.

For the two-mode system defined by (30) in invariant polynomial basis, and (21) in 4D state space, we use the following set of parameters, for which the flow is chaotic, suggested by Daniel in [2013-08-30 Daniel]:

$$\begin{aligned} \mu_1 &= -2.8023, \mu_2 = 1, a_1 = -1, a_2 = -2.6636 \\ b_1 &= 0, b_2 = 0, c_1 = -4.1122, c_2 = 1.8854, e_2 = 1. \end{aligned} \quad (\text{G19})$$

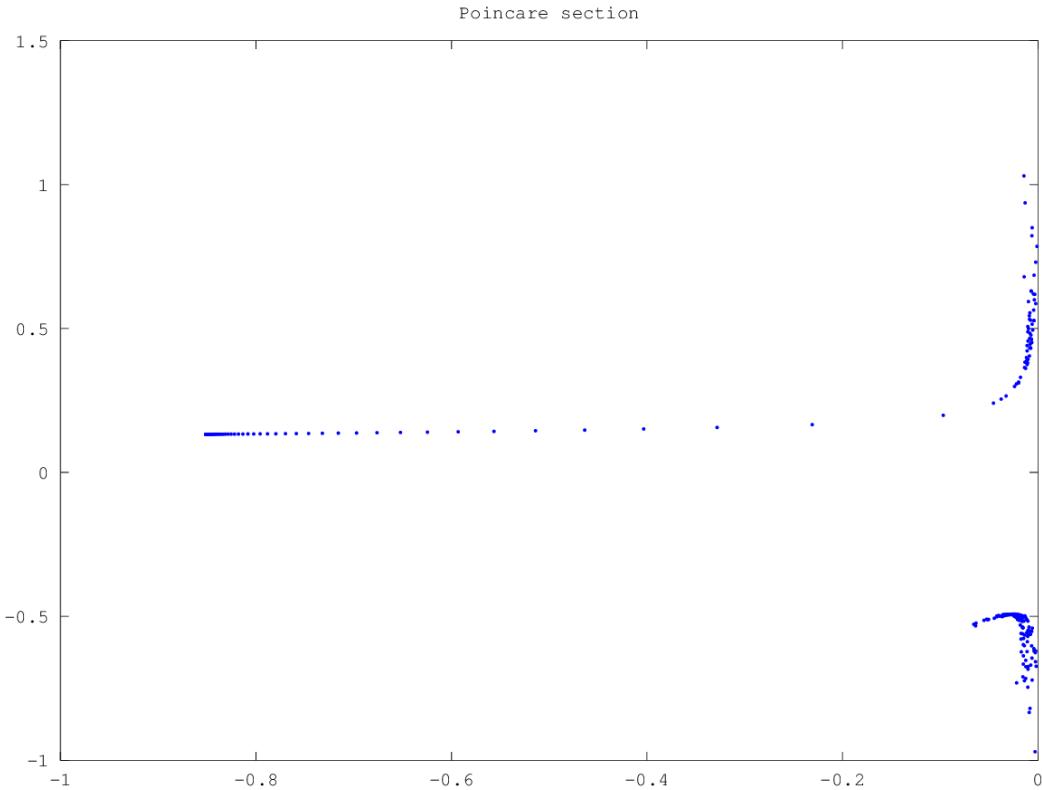


FIG. 45. Poincaré section plane in the Gram-Schmidt coordinates, defined by going through the relative equilibrium  $\{\hat{a}_1, \hat{a}_2, \hat{a}_3\} = (-0.4398, -0.7285, 0.1323)$  and being normal to  $\hat{n} = (0.85607, -0.51686, 0.00000)$ .

With these parameters, the system has 2 roots in the invariant polynomial basis:

$$(u, v, w, q)_{EQ} = (0, 0, 0, 0) \\ (u, v, w, q)_{TW} = (0.19346, 0.54822, -0.28187, -0.05119). \quad (G20)$$

I am  $(100 - \epsilon)\%$  sure that these are only roots. See [2013-08-25 Burak], [2013-08-30 Burak] and [2013-09-02 Burak] for the details of finding them. The bivariate polynomials of (36) simplify significantly with the choice of parameters (G19), and numerical solutions becomes easier to deal with.

Root at the origin of the invariant polynomial basis corresponds to the only equilibrium of the system in the full state space, and the second root corresponds to a relative equilibrium. The details (Jacobians, eigenvalues & eigenvectors) about these roots will come soon.

We apply the method of moving frames to reduce the  $SO(2)$  symmetry of the system using a single template defined by  $\hat{a}' = (1, 1, 0, 0)$  for which the chart border, given by  $x_1 = x_2 = 0$ , is an invariant subspace of the two-mode system. Hence, it is safe. <sup>125</sup>

<sup>127</sup>

<sup>126</sup> ES 2013-09-29: This is true, but at the same time such slices are of little value for KS and, I think, fluids. For KS, inclusion of a third mode will break invariance of the  $k = 1$  subspace and one would have to worry about crossing it. What would be the scope of this paper then? What have we really shown? It is a nice study of the PK system, but can we claim progress in terms of general slicing?

<sup>128</sup> BB 2013-09-30: If you choose your template as, say,  $\hat{a}' = (1, 0, 0, 0)$  all solutions with completely imaginary first Fourier mode will be on the chart border. As you transform this tem-

plate with the group action for different parameters, you will get other linear relations between  $x_1$  and  $x_2$  for the chart border. Only for  $\hat{a}' = (x, x, 0, 0, 0, \dots)$  (see main blog for the proof), chart border becomes  $x_1 = x_2 = 0$ . (FALSCH. SEE siminos/blog [2013-11-18 Burak] FOR CORRECTION.) By an “improper domain” I mean this: Let’s say you have a square wave with the frequency  $\omega_0$ , if you Fourier expand this function with the harmonics of  $\omega_0/2$ , you will get the first Fourier mode to be 0, and second mode to be non zero.

In order to visualise the system in 3D, we define<sup>129</sup> the Gram-Schmidt basis as  $e_{i,GS} = g(\pi/4)e_i$ , where  $e_{x1,GS} = (0.70711, -0.70711, 0)$  is parallel to  $t'$  hence the reduced flow has no component in that direction<sup>131</sup>. Three Gram-Schmidt coordinates generated this way are:

$$\begin{aligned} e_{x,GS} &= (0.70711, 0.70711, 0, 0) \\ e_{y1,GS} &= (0, 0, 0, -1) \\ e_{y2,GS} &= (0, 0, 1, 0) \end{aligned} \quad (\text{G21})$$

<sup>133</sup>

Here, the choice of  $e_{y1,GS}$  and  $e_{y2,GS}$  might seem unnecessary since it would work if they were kept the way they were before. The reason I defined them this way is to have a consistent procedure of going from one basis to another using orthogonal transformations. Namely, whenever I want another set of basis to project things onto, I apply the same orthogonal transformations to all cartesian basis in the previous coordinate system, and use the final transformed vectors as the new basis. For example, when I need a basis to draw things on a particular Poincaré section, I will apply the SO(3) rotations to the usual cartesian basis in such a way that transformed  $e'_x$  will be parallel to the Poincaré section normal, and use the transformed  $e'_y$  and  $e'_z$  as the basis for the Poincaré section. This way, I will have a systematic procedure for going back and forth. <sup>135</sup>

The symmetry reduced two-mode flow on the Gram-Schmidt coordinates, the equilibrium, the relative equilibrium and their respective unstable directions are shown in Fig. 46 (I'm sorry for posting almost the same figure for a third time, but the reason that I'm doing this is to avoid further ambiguities, I am hoping to stick with the definitions of today's post.). In the following, I will explain how I got the unstable directions in the figure (It has some intuitive steps that I'm not sure if they are correct) starting with the origin. Stability eigenvalues and eigenvectors (Eigenvalues and eigenvectors of the matrix  $A_{ij} = \partial V_i / \partial x_j$ ) of the linearized flow at the origin of the full state space are

$$\begin{aligned} \lambda_1 &= 1 + 1i, v_1 = (0, 0, 0.70711, 0.70711i) \\ \lambda_2 &= 1 - 1i, v_2 = (0, 0, 0.70711, -0.70711i) \\ \lambda_3 &= -2.8023, v_3 = (1, 0, 0, 0) \\ \lambda_4 &= -2.8023, v_4 = (0, 1, 0, 0) \end{aligned} \quad (\text{G22})$$

I defined the unstable direction as the sum of the eigenvectors with expanding eigenvalues (eigenvalues with positive real parts), weighted by exponentials of their respective eigenvalues (in the “Jacobians” language, this corresponds to multiplying the Jacobian eigenvector with its respective eigenvalue):

$$\begin{aligned} v_{u,0} &= e^{\lambda_1} v_1 + e^{\lambda_2} v_2 \\ &= (0, 0, 2.07705, -3.23481) \end{aligned} \quad (\text{G23})$$

This is the direction towards which a small perturbation near the origin will expand (right?)<sup>137</sup>. This vector is already on our slice ( $\langle t' | v_{u,0} \rangle = 0$ ). After projecting onto Gram-Schmidt coordinates and normalizing, I obtained the black vector of the Fig. 46:

$$\hat{v}_{u,0,GS} = (0, 0.84147, 0.54030) \quad (\text{G24})$$

<sup>130</sup> ES 2013-09-29: I thought that GS is a standart procedure to orthonormalize a given set of vectors. I don't see why you can choose to define this basis any way you want. Here I refer to the (n-1) basis vectors which are all orthogonal to the slice tangent as Gram-Schmidt basis and describing one way of choosing such coordinates so that everyone knows exactly what coordinates that I'm talking about. Motivation of explaining this choice is just because the choice is not unique.

<sup>132</sup> ES 2013-09-29: Then why do you want to use it? I am not using it.

<sup>134</sup> ES 2013-09-29: Is it really  $e_{x,GS}$  or is it  $e_{x1,GS}$ ? Is there a missing or extra minus sign? Since the reduced flow has no component in  $x_{1,GS}$  direction, I just took  $x_{2,GS}$  and dropped the

index.

<sup>136</sup> ES 2013-09-29: I still don't understand your motivation for this. Any vector you can take it onto the slice in a well defined manner, so I don't see the problem you are trying to solve with this choice of  $e_{y1,GS}$  and  $e_{y2,GS}$ .

<sup>138</sup> ES 2013-09-29: Here, you take a direction on the unstable eigenspace and call it unstable direction. It is as confusing as it could get, please revise. This is just an intermediate step to have more guidance while choosing Poincaré sections. I did not know if there was another name for it, or it was completely wrong to think this way. If it is useless, you can ignore it, if there is a motivation for a revision, then I can do it.

### Flow on Gram-Schmidt Coordinates

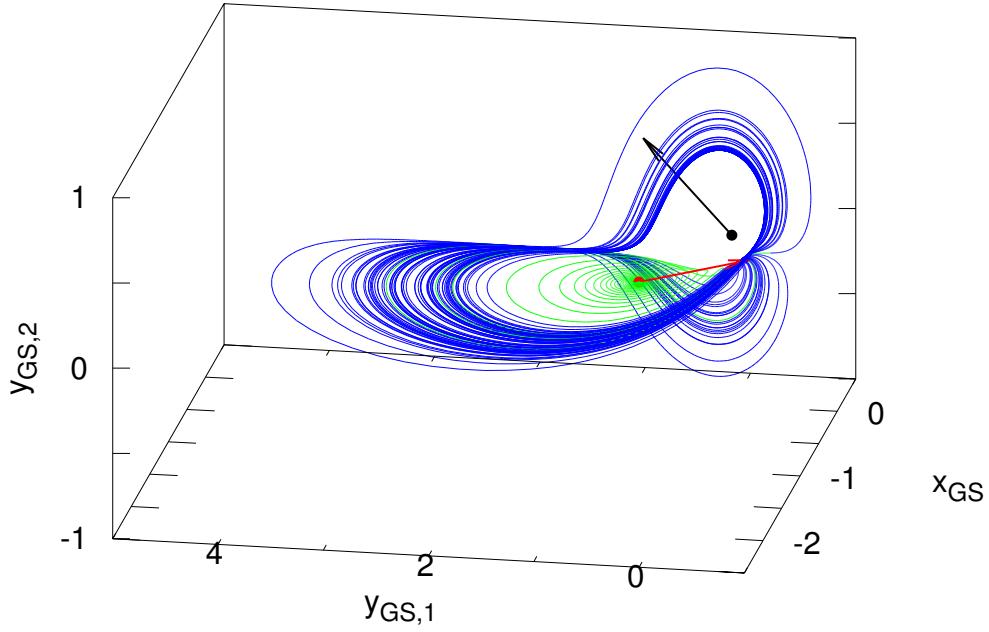


FIG. 46. Symmetry reduced two-mode flow on the Gram-Schmidt coordinates of (G21) Initial point of the flow is the relative equilibrium of the two-mode system. This point and its unstable direction are shown respectively with the red dot and the red arrow on the plot. Trajectory spirals out (green) from this point and then starts to trace the strange attractor (blue). The equilibrium at the origin and its corresponding unstable direction is shown with the black dot and the black arrow.

I set up the variational equation for the full state space to compute the Jacobian of the relative equilibrium. Starting from  $a_0 = (-0.3110, -0.3110, -0.1323, -0.7285)$ , I integrated  $\dot{J} = A(a)J, J_0 = \mathbf{1}$  along with the system itself for a time  $T$ , until the trajectory arrives at the initial point. Eigenvalues ( $\Lambda_i$ ) and eigenvectors ( $e_i$ ) of  $J^T$  are:

$$\begin{aligned}
 \Lambda_1 &= 0.99996, \quad e_1 = (-0.2013, 0.2013, -0.9432, 0.1714) \\
 \Lambda_2 &= 0, \quad e_2 = (0.5524, -0.7885, 0.2523, -0.0974) \\
 \Lambda_3 &= -1.7910 + 0.1170i, \\
 e_3 &= (-0.0922 - 0.4096i, -0.0567 - 0.4141i, -0.0794 + 0.0449i, -0.8005) \\
 \Lambda_4 &= -1.7911 - 0.1170i, \\
 e_4 &= (-0.0922 + 0.4096i, -0.0567 + 0.4141i, -0.0794 - 0.0449i, -0.8005)
 \end{aligned} \tag{G25}$$

<sup>139</sup> Here, the first eigenvalue is very close to unity and the corresponding eigenvector is in the direction of the  $t(a_0) = Ta_0$  which is an expected result, hence I think I computed the Jacobian correctly. Similar to what I

<sup>140</sup> ES 2013-09-29: There is a discussion on stability of relative equilibria in [2012-07-30 Evangelos] up to the second comment labeled [2012-08-01 Predrag]. Please try to compute stability of your relative equilibrium by considering it as an equilibrium in reduced space (i.e. do it in invariant variables) and then compare with what you get. Also, please try to verify Predrag's suggestion that you can compute the Floquet multipliers using the eigenvalues on a single point on the relative equilibrium and

the period. What I am doing here is calculating the Floquet vectors and multipliers along a relative equilibrium in the full state space. If this will be useful in choosing Poincaré sections, then I don't see a motivation for redoing calculations, I will go back to your suggestion if I cannot manage to get a useful Poincaré section.

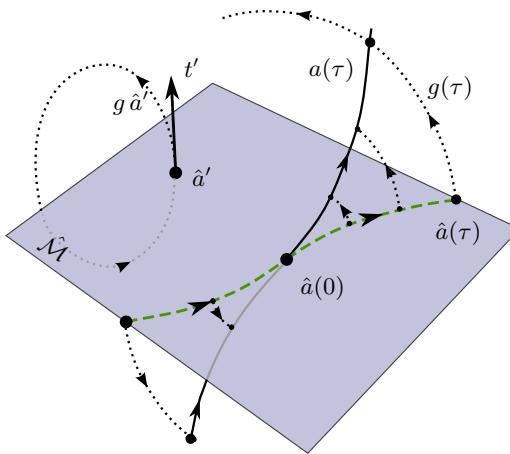


FIG. 47. Slice hyperplane  $\hat{\mathcal{M}}$  is a hyperplane passing through the template point  $\hat{a}'$ , and normal to the group tangent  $t'$  at  $\hat{a}'$ . It intersects all group orbits (indicated by dotted lines here) in an open neighborhood of  $\hat{a}'$ . The full state space trajectory  $a(\tau)$  and the reduced state space trajectory  $\hat{a}(\tau)$  belong to the same group orbit  $\mathcal{M}_{a(\tau)}$  and are equivalent up to a group rotation  $g(\theta)$  (from [ChaosBook.org](#)).

did for the origin, I calculated the weighted sum of eigenvectors with expanding eigenvalues to get the unstable direction:

$$\begin{aligned} v_{u,TW} &= \Lambda_3 e_3 + \Lambda_4 v_4 \\ &= (0.42604, 0.30014, 0.27399, 2.86727). \end{aligned} \quad (\text{G26})$$

I then subtracted the component of this vector in the direction of the group tangent to isolate its projection at the slice hyperplane ([This step intuitively makes sense to me but I'm not 100% sure about it](#)):

$$\begin{aligned} \hat{v}_{u,TW} &= v_{u,TW} - \langle t(a_0) | v_{u,TW} \rangle t(a_0) \\ &= (0.36309, 0.36309, 0.27399, 2.86727). \end{aligned} \quad (\text{G27})$$

Finally I projected this vector onto the Gram-Schmidt basis (G21) and normalized to get

$$\hat{v}_{u,TW,GS} = (0.175506, -0.980014, 0.093648). \quad (\text{G28})$$

Eq. (G28) is shown with the red arrow on the Fig. 46.

Computing Jacobians and drawing vectors took a longer time than I expected as I had to go back and redo things that before I was sloppy at. I am now going back to the Poincaré sections on which the unstable directions will lie.

**2013-09-22 Predrag:** At the moment, nothing substantial to add, only some bookkeeping suggestions concerning the pretty Fig. 46. Once the figure is almost publication ready -like this one seems to be-

- Record in `reducesymm/figSrc/00ReadMe.txt` the program that has generated it, perhaps save the program in the appropriate `reducesymm/figSrc/...` subdirectory.
- If possible, save the figure in TeX or LaTeX format; an example is Fig. 47. It looks ugly, but it is a great time saver in the editing process, as you can insert macros for LaTeX labels which do tend to change a lot in the article editing process.
- Do not include labels like ‘Flow on Gram-Schmidt Coordinates’.
- Trim all white edges around figures like BBunstablevectors.pdf

**2013-10-01 Burak:** I generated individual Poincaré sections including for the equilibrium and relative equilibrium and their respective “unstable directions” (defined in my previous post, probably not so helpful) shown in Fig. 46.

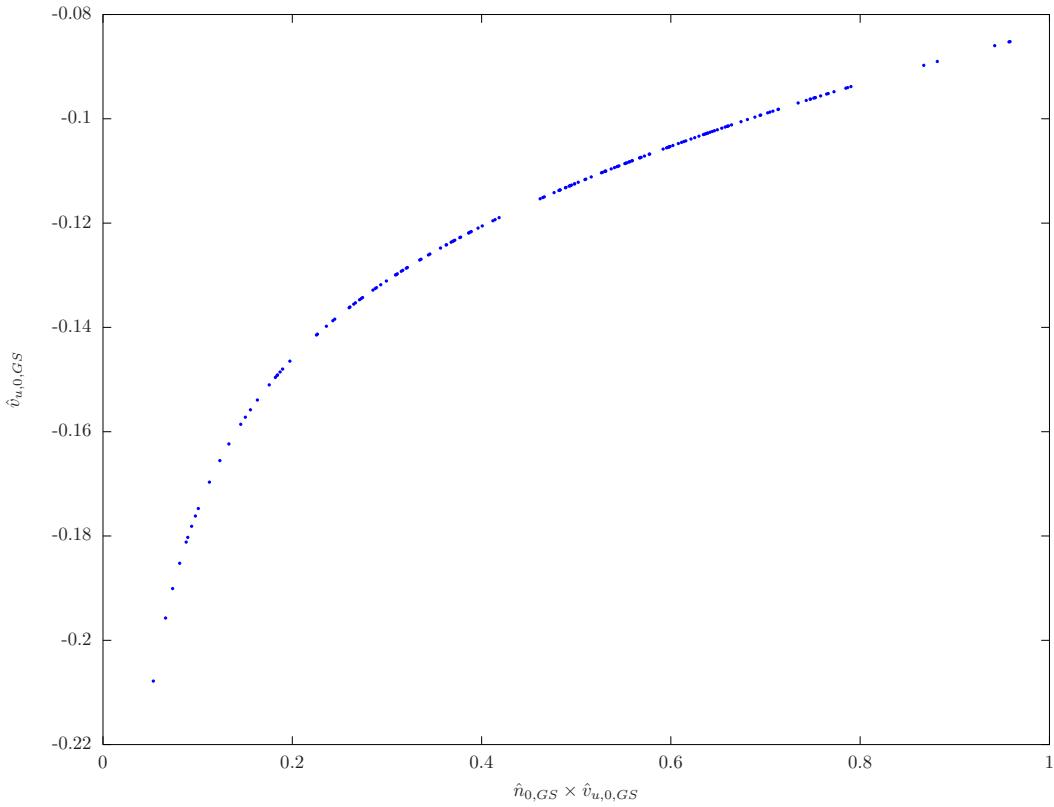


FIG. 48. Poincaré section plane which includes the equilibrium  $EQ_{NameMe}$  and is perpendicular to the normal vector given in (G29).

Poincaré section plane Fig. 48 including the origin is perpendicular to

$$\hat{n}_{0,GS} = (0, -0.54030, 0.84147) \quad (\text{G29})$$

Poincaré section plane Fig. 76 (a) including the relative equilibrium is perpendicular to

$$\hat{n}_{1,GS} = (0.977790, 0.184593, 0.099259) \quad (\text{G30})$$

**2013-10-02 Burak:** Here (Fig. 49 and Fig. 49) are the return maps of the Poincaré sections that I posted yesterday morning. On both return maps,  $s_n$  are arclengths along the respective Poincaré section curves, starting from the equilibrium/relative equilibrium. Hopefully, I will blog relative periodic orbits tomorrow.

**2013-10-03 Predrag:** Cool - looks like return maps for the Lorentz flow. You can already start nailing down your relative periodic orbits using the  $\{0, 1\}$  symbolic dynamics of Fig. ?? for initial Newton guesses for relative periodic orbits in the full 4-dimensional state space. Plotting the periodic points of the same relative periodic orbits in Fig. 49 will help you elucidate the relation between the two sections - it looks like Fig. 76 (a), suffices to explain everything.

**2013-10-03 Predrag:** It will look even prettier if you - instead of plotting the strange attractor - plot the unstable manifold ala Adom Kamor. That will give you the return map(s) also outside the strange attractor. As the attractor is exceedingly thin (estimate that similarly to what is done in ChaosBook for the Rössler), your initial guesses based on this 1-dimensional return map will be very accurate.

**2013-10-03 Predrag:** Regarding return maps: please plot squares as squares.

**2013-10-03 Predrag:** Low priority: it would be nice to plot the Poincaré section planes in Fig. 44, like it is done for the symmetry-reduced Lorentz flow in ChaosBook.

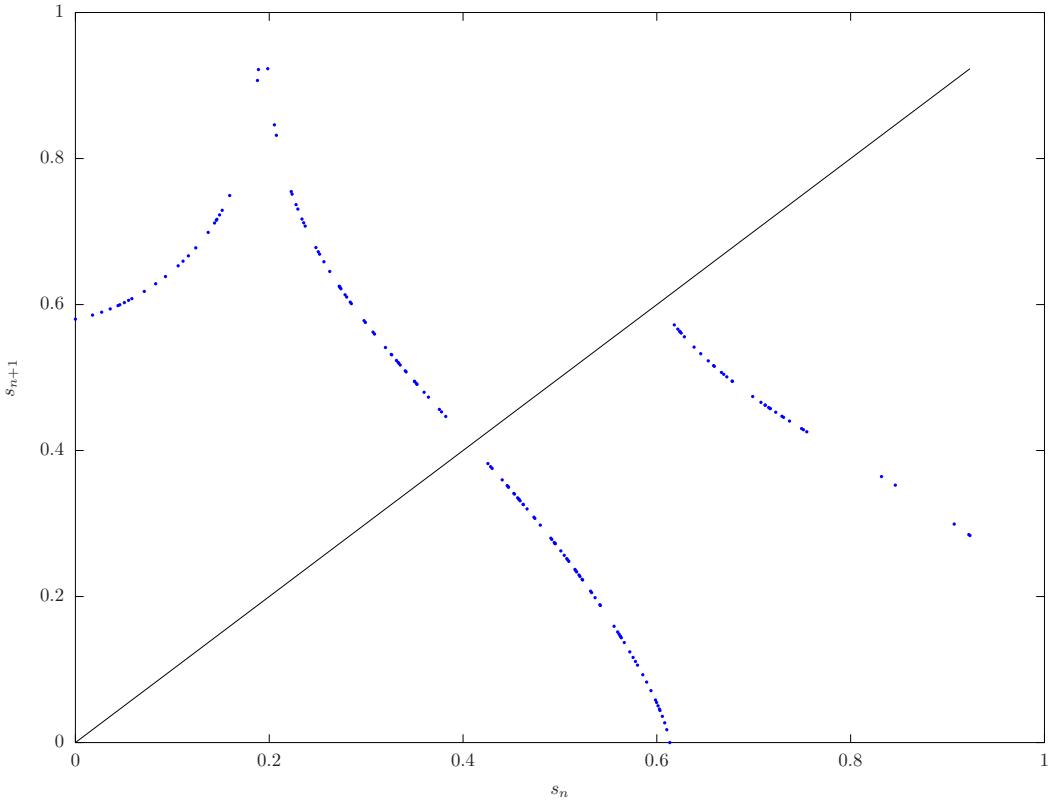


FIG. 49. Poincaré return map corresponding to the section in Fig. 48.

**2013-10-03 Predrag:** Re. figure source files such as PSECT1.tex:

- protests about lots of things it is missing?
- “GNUPLOT: LaTeX picture with Postscript” - does not it have an pdf option? Or I’m misreading the file...
- I recommend adding a comment line like one I added in LaTeX source to Fig. 76 (a), indicating the source for the figure - comes in handy later when editing, changing labels etc

**2013-10-03 Daniel:** So I’ve been out of the loop... what still needs to be done? Should I be trying to replicate Burak’s work to see if I get the same thing? Is there something that should be done in parallel? Should I try to do the whole Poincaré section, return map, periodic orbit solver song and dance in invariant coordinates to see if we get the same thing? Is there something more productive? Also, Burak, could you upload a file with the relative equilibrium that you calculated specified to machine precision (or at least more than 3-4 decimal places)? Might be useful to have it written down in the  $(x_1, x_2, y_1, y_2)$  basis.

**2013-10-07 Predrag:** Burak’s and Daniels outline of Das Artikel is in Fig. 50

**2013-10-07 Burak:** A short post about relative periodic orbits:

After interpolating with cubic splines, the return map Fig. 76 (b) looks like Fig. 77. The intersection corresponds to a relative periodic orbit which, in reduced space, looks like Fig. 51. I’m not blogging the details, numbers etc. since they are subject to change now. I couldn’t yet figure out how to trim the white spaces around the figures, sorry, I will fix that soon.

**2013-10-07 Burak:** After setting  $b_{1,2} = 0$ , the bivariate polynomials become:

$$\begin{aligned} f(u, v) &= -v(1 + ua_2)c_1 + uc_2(ua_1 + \mu_1) \\ g(u, v) &= v^2c_1^2e_2^2 + (2vc_1 + uc_2)^2(-vc_1^2 + (-u + \mu_1)^2) \end{aligned} \quad (\text{G31})$$

- Slicing
  - Motivation  $\mathbb{P} \Rightarrow \mathbb{P}$
  - Eqs & Chart Borders
  
- Development of the system
  - Invariants & equivariants (Ref: Dangelmaier, Peter Knoblauch)
  - General eq.s in invariant polynomials & full ssp
  - Roots, bivariate polynomials,  $b=0$ , other roots, final set of parameters, BBC, model (?)
  
- Single slice  $\Rightarrow$  Sliced dynamics
  - $y \xrightarrow{P_{\text{set}} \mapsto P_{\text{main}}} \mathbb{P}'$ 's
  - inv.  $\mathbb{P}$ -space  $\equiv$  chart borders
  - $\Phi$
- Two Slice Atlas (?)

FIG. 50. Burak's and Daniels outline of Das Artikel.

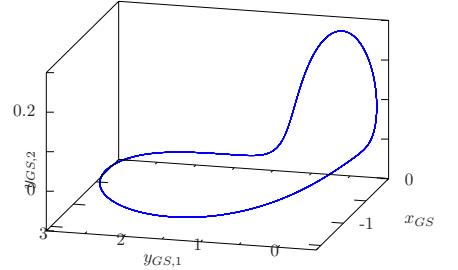


FIG. 51. A relative periodic orbit in the reduced space.

$f(u, v) = 0$  can be easily solved for  $v$ :

$$v(u) = \frac{uc_2(ua_1 + \mu_1)}{(1 + ua_2)c_1} \quad (\text{G32})$$

Plugging this into  $g(u, v)$  and simplifying gives:

$$g(u) = u^2 c_2^2 \left( \frac{e_2^2 (ua_1 + \mu_1)^2}{(1 + ua_2)^2} + \left( 1 + \frac{2(ua_1 + \mu_1)}{1 + ua_2} \right)^2 \left( (u - \mu_1)^2 - \frac{uc_1 c_2 (ua_1 + \mu_1)}{1 + ua_2} \right) \right) \quad (\text{G33})$$

Two-fold root at the origin can immediately be divided out. If one sets  $a_1 = -1$ , which is the case with Daniel's parameters,  $(u - \mu_1)$  also becomes a common multiplier and hence, can be divided out. After doing that and some more manipulation we get

$$g_{\text{red}}(u) = \frac{e_2^2 (u - \mu_1)(1 + ua_2) + (uc_1 c_2 + (u - \mu_1)(1 + ua_2))(1 + ua_2 - 2(u - \mu_1))^2}{(1 + ua_2)^3}. \quad (\text{G34})$$

Now the roots of the two-mode system is determined by the numerator of the (G34), which is a quartic polynomial in  $u$ , hence, exactly solvable. Observing that parameters  $c_1, 2$  only appears as  $c_1 c_2$  in this equation, I tried to further simplify the parameters by setting  $c_2 \rightarrow 1$  and  $c_1 \rightarrow c_1 c_2$ . This scaling preserves chaos. After this, parameters which are set to 1s and 0s are:

$$\mu_2 = 1, a_1 = -1, b_1 = 0, b_2 = 0, c_2 = 1, e_2 = 1 \quad (\text{G35})$$

There are three parameters left to determine. I rounded the remaining parameters to 2 decimals and got the following set:

$$\begin{aligned} \mu_1 &= -2.8, a_2 = -2.66, c_1 = -7.75 \\ \mu_2 &= 1, a_1 = -1, b_1 = 0, b_2 = 0, c_2 = 1, e_2 = 1 \end{aligned} \quad (\text{G36})$$

The flow with these parameters is nicely chaotic, I actually got a radial Poincaré return map in the invariant polynomial basis qualitatively very similar to the one I got in the reduced state space with the previous parameters. There are analytical expressions for the roots of quartic polynomials, however, they are too long and I could not make a sense out of them. With (G36) Numerical roots of the quartic polynomial are:

$$\begin{aligned} u &= -5.52172 \\ &= -0.991847 - 0.14571i \\ &= -0.991847 + 0.14571i \\ &= 0.19357 \end{aligned} \quad (\text{G37})$$

With the divided out roots (2 at  $u = 0$  and 1 at  $u = \mu_1$ ) and the one at the infinity ( $v = -\mu_2/b_2$ ), we have all the roots now (This may be wrong. There might be another root that is sent to infinity by setting  $b_1 = 0$ , since the original bivariate polynomials are of degree 2 and 4 and the root at  $v = -\mu_2/b_2$  is already divided out. I cannot immediately see this, however, it is not important once the choice of 0s and 1s (G35) is made, after that one can get all the roots analytically.). Only the positive real root listed above, along with the root at the origin, directly influences the flow in the full state space. I'm not able to comment on the possible intermittency right now.

**2013-10-07 Predrag:** Why does the front page of the paper have only the title and authors?

**2013-10-07 Burak:** It's something weird with the Chaos style file and the fact that we have been compiling as preprint instead of reprint. If you toggle this it looks fine. Do we have a plan for where we want to submit this?

**2013-10-10 Predrag:** Currently I'm thinking of Chaos, as it is a sequel (should have been a prequel) to Ref. 16, but we could send it someplace else, if you wish.

**2013-10-10 Predrag:** Please always plot square as a square, not as a rectangle, as currently in Fig. 49, Fig. 76 (b) and Fig. 77.

**2013-10-11 Predrag:** We still have another case to investigate, the Dangelmayr normal form system,<sup>31</sup> with the parameters  $\{e_1, e_2\} = \{0, 0\}$  and the O(2) symmetry. We need it, as this is the symmetry of Kuramoto-Sivashinsky as well. The system (20) then has also a discrete symmetry (complex conjugation), and a 2-dimensional invariant subspace spanned by  $\{\text{Im } z_1, \text{Im } z_2\} = \{x_2, y_2\}$ .<sup>141</sup> That should lead to ‘preperiodic orbits’ discussed in Ref. 46. We should probably quotient the reflection symmetry, work in a fundamental domain.

**2013-10-12 Burak:** I worked/experimented on  $e_{1,2} = 0$  case and could not find a chaotic flow. For now, I have only one thing to say about what won't work:

It is impossible to get anything chaotic by taking (G36) and setting  $e_2$  to 0. Velocity function for the invariant coordinate  $q$ , after substituting 0s and 1s (G35) is

$$\dot{q} = qu(-2 + a_2) + q(1 + 2\mu_1) \quad (\text{G38})$$

Since  $u$  always takes positive values and the parentheses in this equation are both negative for the parameter choice (G36), the flow always approaches to  $u = 0$ . Once this happens, two-mode flow becomes effectively 2 dimensional since two of the invariant coordinates determine the third one through the syzygy. I experimented with other parameters but could not find anything interesting yet.

<sup>142</sup> PC: please recheck

**2013-10-15 Burak:** I have been trying to find a parameter set with  $e_2 = 0$  for which the flow is chaotic. So far, I could not find chaos, but I found something that might be promising. After a lot of analytical trials and getting nothing useful, I went back to my older parameters and set  $e_2 \rightarrow 0$  and varied other parameters and finally got:

$$\begin{aligned} \mu_1 &= 2, a_1 = 0.41, b_1 = -1.7, a_2 = 0.16, b_2 = -0.05 \\ c_1 &= 1, \mu_2 = -1, c_2 = -1, e_2 = 0 \end{aligned} \quad (\text{G39})$$

Flow in the invariant polynomial basis looks like Fig. 52. It converges to a limit cycle. I'm blogging this because the limit cycle might be destabilized by tweaking the parameters (G39).

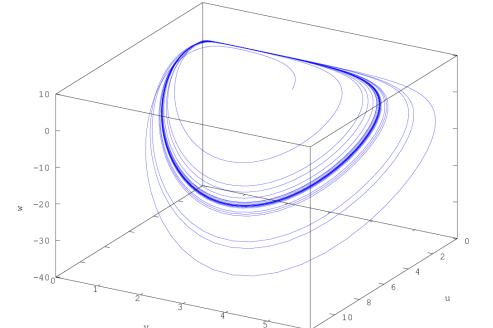


FIG. 52. Flow with parameters (G39) on the invariant polynomial basis.

I also computed the flow in the full state space (Fig. 53) and symmetry reduced using the magic slice (Fig. 54) without quotienting the reflection symmetry (don't know how to do it yet). Both post processing and integration on the slice methods give the same results up to small numerical errors.

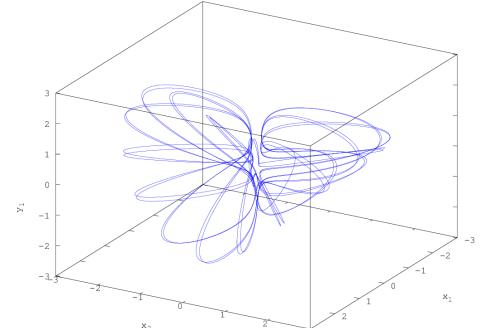


FIG. 53. Projection of the flow with parameters (G39) in the full state space.

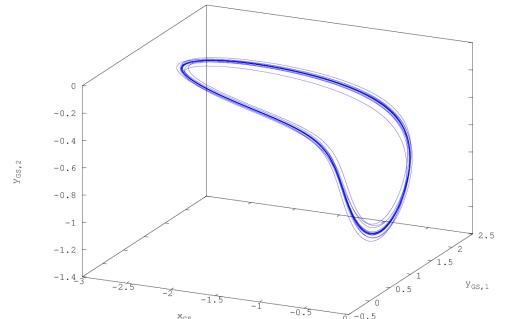


FIG. 54. Symmetry reduced flow with parameters (G39) on the Gram-Schmidt basis.

**2013-10-15 Burak:** A new set of parameters for (21), O(2)-equivariant case:

$$\begin{aligned} a_1 &= 0.47 \\ \mu_1 &= 1, b_1 = -1, c_1 = 1, \mu_2 = -1, a_2 = 0, b_2 = 0, c_2 = -1, e_2 = 0 \end{aligned} \quad (\text{G40})$$

This, I think, is the simplest looking set of parameters we have found so far. Three of the original nine parameters are set to zero and five of them are set to either 1 or -1. For smaller values of  $a_1$ , The flow in the invariant space converges to a limit cycle as in Fig. 52, and for the larger values it approaches to an equilibrium (might be a relative equilibrium in the equivariant state space). With  $a_1 = 0.47$ , flow in the invariant space looks like Fig. 55.

I also computed the full state space flow (it's a mess) and symmetry reduced it without quotienting the reflection symmetry. I'm posting the reduced flow on the Gram-Schmidt coordinates of (G21) for a short time interval in Fig. 56. It's looks a bit messy if I plot it for the entire flow ( $t_{final} = 500$ ). Topologically, it's like a torus bent in a shape similar to that of Fig. 54

Please double-check the parameters and the existence of chaos. If this looks promising, please make suggestions about the next steps, I'm guessing the first one to be quotienting the reflection symmetry. It is possible to get all the equilibria in the invariant space, because the parameters are rather simple.

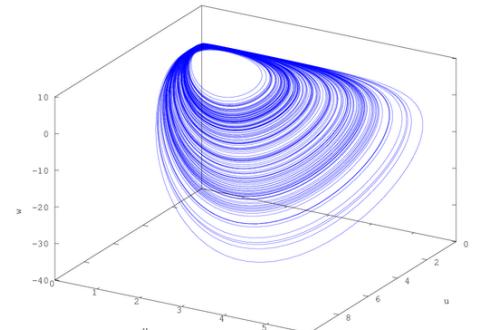


FIG. 55. Flow with parameters (G40) on the invariant polynomial basis.

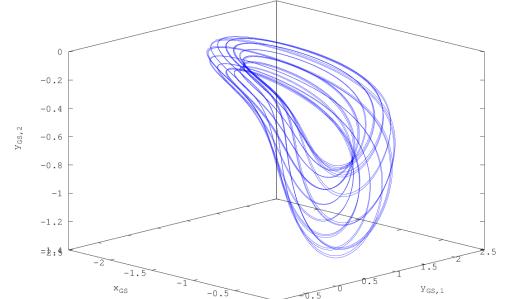


FIG. 56. Symmetry reduced flow with parameters (G40) on the Gram-Schmidt basis.

**2013-10-21 Burak:** I am starting moving the code from Octave/Matlab to NumPy, completed slicing, Poincaré sections are the next. The previous code was using slow integrators and solvers written by me. This one mostly uses built in NumPy functions, which are much faster. I did this transformation mainly to speed up things to make unstable manifold algorithm feasible to run. Also, NumPy has powerful plotting tools so I will be able to plot Poincaré section planes soon.

I'm committing the code, a README file and bash scripts to run the programs at: [blog/burak/PorterKnobloch/python](http://blog/burak/PorterKnobloch/python). I prepared small scripts to install the required packages for Linux users, so if you want to re-run something, it should be really easy to do it on Linux. That is the way of generating 3D interactive figures for now, I will try to find a web based way of sharing rotatable figures.

This transition is temporarily slowing my progress down but it will ultimately be speed it up.

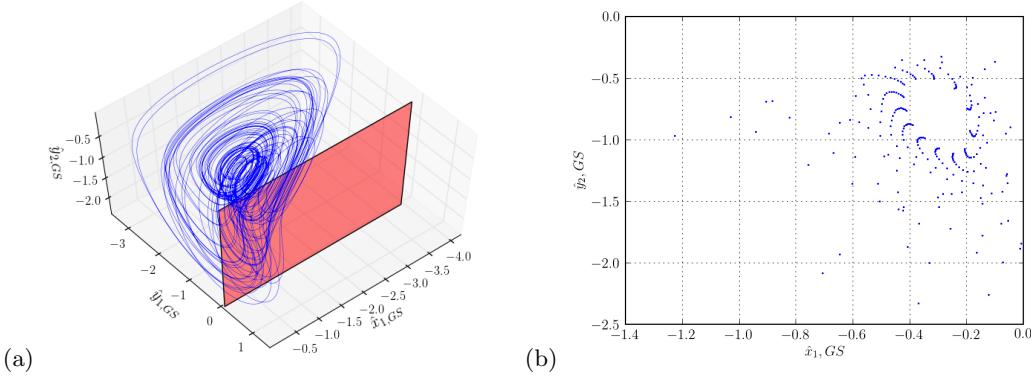


FIG. 57. (a) Reduced flow (blue) on the Gram-Schmidt basis and the Poincaré section hyperplane. (b) Corresponding Poincaré section. Parameters (G40)

**2013-10-23 Predrag:** Some thoughts on the days' Poincaré section Fig. 57(b): it looks like a very slow transient to cycle of discrete time 13. Would be impossible to see that without Poincaré section... Invariant polynomial representation seems useless. You want to mark relative equilibria and their eigenvectors, then have the section go through the important relative equilibrium, traverse the complex eigenvector pair plane.

To reveal the attractor, you want to first run it for a long time, then start plotting. Possible genesis; there was once a stable Hopf limit cycle bifurcated off an relative periodic orbit gone unstable. A shorter period stable Hopf limit cycle bifurcates off next; they mode-lock with discrete time ratio  $m/13$ ,  $m$  relative prime to 13, i.e.,  $m = 1, 2, \dots, 12$ . Not sure about this at all, but your system is sure a rare beast :)

**2013-10-23 Burak:** The  $O(2)$  invariant system with the parameters (G40) has seven roots on the invariant polynomials:

$$\begin{aligned}
 (u, v, w) &= (0, 0, 0) \\
 &= (16.6667, 8.33333, -16.6667)(\text{double}) \\
 &= (-1.12714, 4.16894, -8.33787) \\
 &= (0.270366, 0.239869, -0.479739) \\
 &= (-1.38372, 1.26085, -2.5217) \\
 &= (1.09745, 0.793116, -1.58623).
 \end{aligned} \tag{G41}$$

None of these but the origin has a real correspondence in the full  $a$  (This result badly needs to be double checked). I let Mathematica try to find the real  $(x_1, x_2, y_1, y_2)$  that solves (G2) also did an independent Newton search, none of which found a solution. For now, I believe, this system has no relative equilibrium.

Having no interesting relative equilibrium, I then decided on trying a Poincaré section including the origin and its unstable direction which, on Gram-Schmidt basis simply is  $\hat{a}_{1,GS}$  direction. This Poincaré section is shown in Fig. 57. In Fig. 57, initial condition is a point very close to the origin. I then started from the final point of this simulation and run another 1000 time units and computed the Poincaré section again (Fig. 58). I think this 13-cycle on the Poincaré section is the bifurcation of the limit cycle of Fig. 54, however, since the attractor is not thin, I'm not able to make an arc-length return map as I did for the previous case. I'm thinking whether it would be possible to do a 'local symmetry reduction' since all the points on the Poincaré section look like they have a symmetry of an ellipse, if that symmetry can be reduced then it may look like it's converging to a single point. While writing these, I did a much longer run, which made the squid convergence more apparent Fig. 59, however, as you can see there are more legs in this picture, which makes me think that this might be converging to a single limit cycle, but very slowly.

**2013-10-24 Burak:** After playing with  $a_1$  and getting different sorts of cycles on the Poincaré section of Fig. 57 and Fig. 58, I decided to do it systematically. I took 1000 equally spaced values between 0.45 and 0.49 for  $a_1$  and integrated the flow for 1000 time units, computed the Poincaré section for the last 500 time units of the simulation and plotted the values on the Poincaré section individually against the varying parameter. Resulting figure: Fig. 60. Looking at this, I think the chaotic behavior is somewhere around  $a_1 = 0.475$ .

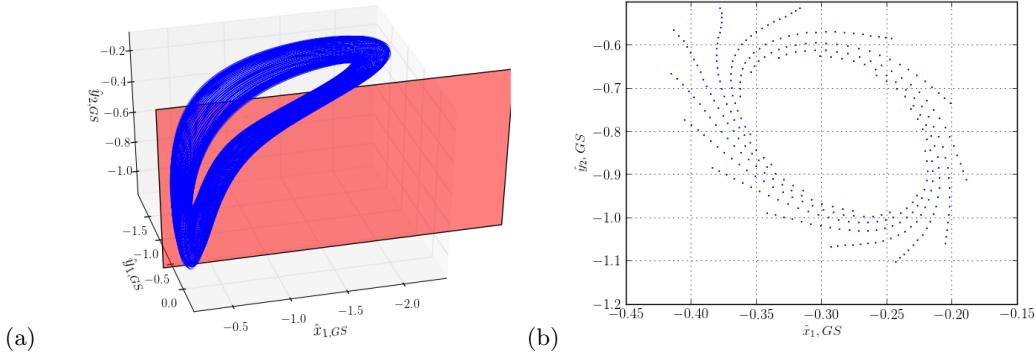


FIG. 58. (a) Reduced flow (blue) on the Gram-Schmidt basis and the Poincaré section plane. (b) Corresponding Poincaré section. Parameters (G40).

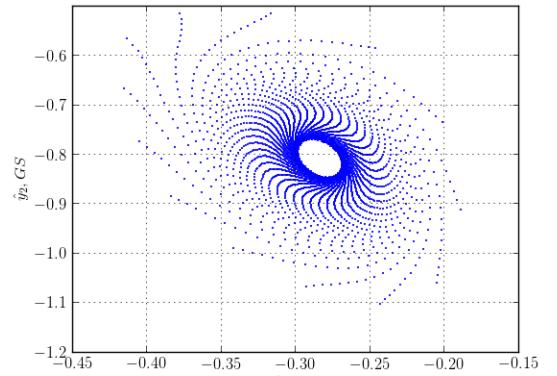


FIG. 59. Convergence to an apparent 13-cycle on the Poincaré section of Fig. 57.

**2013-10-25 Predrag:** Poincaré section Fig. 57 (b) seems to be converging to a 13-cycle, but it does not seem to do so in the other, longer time sections? It seems that the dynamics is converging to a torus of neutral stability, leading to excruciatingly slow convergence to it?

**2013-10-28 Burak:** The region of the bifurcation diagram Fig. 60 which I thought was chaotic, unfortunately (or maybe not) was not chaotic, in fact, if I counted correctly, it is a 68-cycle. I am arriving the 'not chaotic' conclusion by observing that long runs of the system is confined on the surface of a torus in the symmetry reduced space, hence the dynamics is 2 dimensional so it cannot exhibit chaos. I also checked variation of  $x$  with time and saw recurrence with period approximately 472 time units. Nevertheless, I think, this is still interesting dynamics and it demonstrates the importance of the symmetry reduction. For  $a_1 = 0.4775$ , Poincaré section is shown in Fig. 61

**2013-10-28 Burak:** I reduced the reflection symmetry as follows: I first computed the reduced dynamics within the slice, then found the points on the trajectory with  $y_2(t^*) < 0$ , and then mapped these points as

$$\begin{aligned} x_2(t^*) &\rightarrow -x_2(t^*), \\ y_2(t^*) &\rightarrow -y_2(t^*). \end{aligned} \quad (\text{G42})$$

This operation corresponds to complex conjugation of the initial mode amplitudes. Result of this operation is in Fig. 62. I didn't find this plot particularly informative but if you would like to rotate and zoom in to this plot, please run

`bash 02integrateonslice.sh`

in the folder:

`blog/burak/PorterKnobloch/python/`

**2013-10-29 Predrag:** Comparing with Fig. 56, 57 (a) and 58 (a) I see your point - it does not look good when orbit is discontinuously injected in a counter-moving direction. Usually orbits are self-dual under reflection (in which

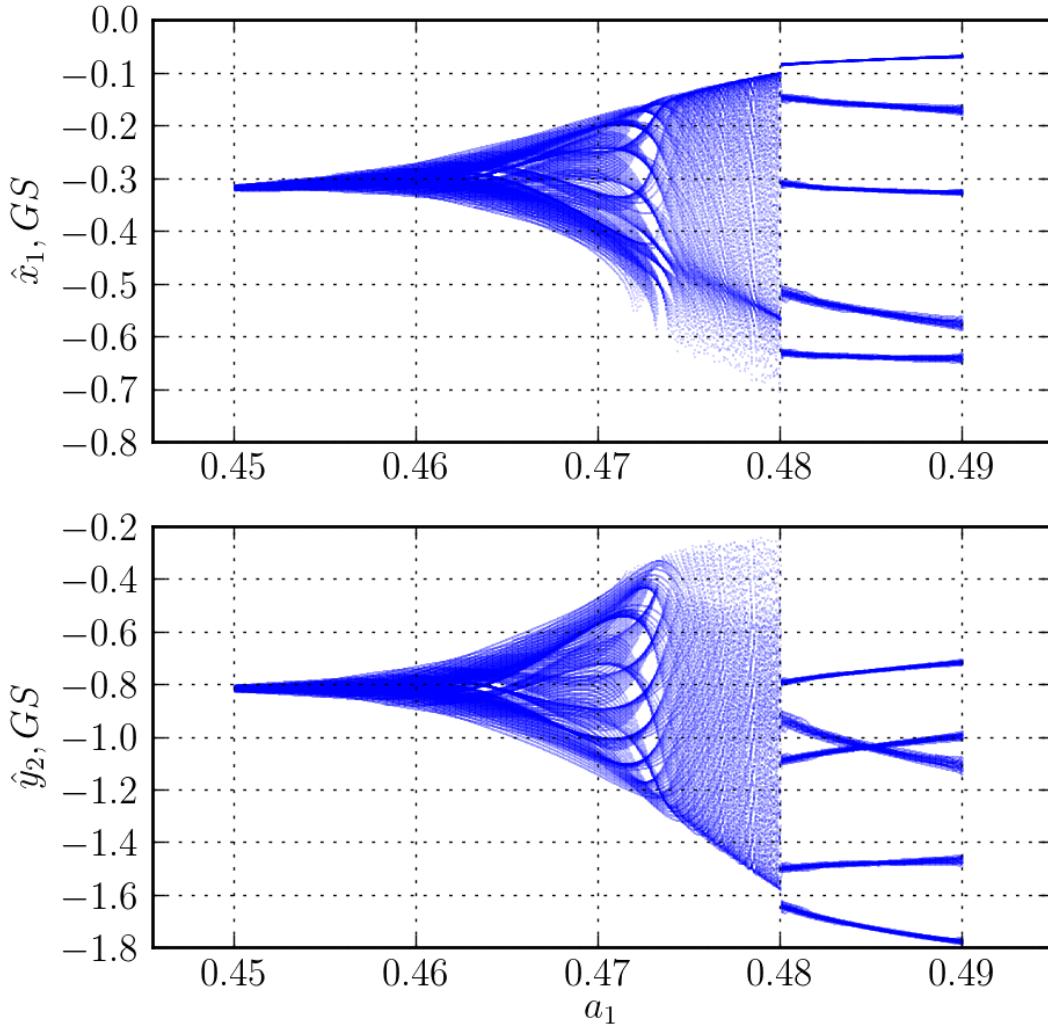


FIG. 60. Bifurcation diagram for the Poincaré section (see Fig. 57 (b), 58 (b), and 59 for the section evaluated at  $a_1 = 0.47$ ) of the  $O(2)$  invariant flow with parameters (G40),  $a_1$  varying.

case you would find this more insightful) or come as dual pairs; where is the (G42)-reflected dual of your orbit in the above figures? There might be something about  $O(n)$  vs.  $SO(n)$  that we are misunderstanding...

**2013-10-29 Predrag:** Regarding Poincaré section of Fig. 57 - I seem to remember that we argued that Poincaré sections should be discrete symmetry invariant (? how would one guarantee transversality of section crossings then? have to reread ChaosBook.org suggestions plus Ref. 16 myself :). That would put the section plane into  $(x_2, y_2) = (0, 0)$ . In any case, it might be useful to mark those points in Fig. 62 as red dots, to see where the reflection happens.

**2013-10-28 Burak:** Figure 5 in Porter-Knobloch paper<sup>30</sup> shows a chaotic flow with an attractor topology very similar to what we got in Fig. 46. There, they are using some kind of a symmetry invariant basis, which, I suspect may be equivalent to the ‘magic slice’. I thought it would be a good start to simulate the system with their parameters, however, there is no direct mapping of their parameters to ours since we don’t have the most general 2-mode system here. I’ll work on this correspondence in more detail.

**2013-10-28 Burak:** As discussed in [2012-04-29 Daniel to Gang], our current mapping from two complex modes to

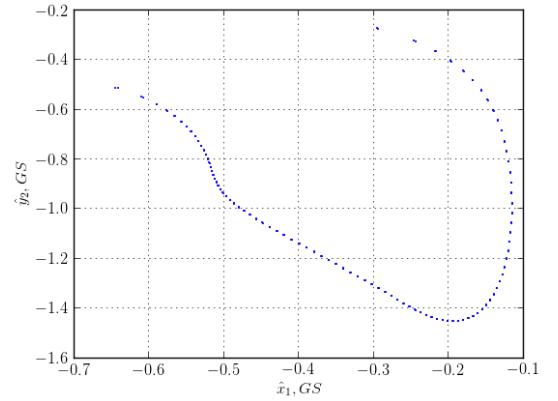


FIG. 61. A relative periodic orbit with a very long period ( $T \approx 472$ ) on the Poincaré section of Fig. 57 in the reduced state space. Parameters (G40), with  $a_1 = 0.4775$ .

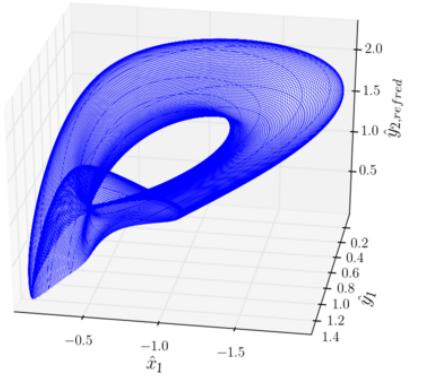


FIG. 62. Dynamics in the reduced state space with reflection symmetry reduced to mapping solutions to the half-space with  $y_2 > 0$  as described in (G42). Compare with Fig. 56, 57 (a) and 58 (a). Parameters (G40), with  $a_1 = 0.4775$ .

the real valued 4D state space is as follows:

$$\begin{aligned} z_1 &= x_1 + ix_2, \\ z_2 &= y_1 + iy_2. \end{aligned} \tag{G43}$$

While this is similar to the mapping that you used for the complex Lorenz equations, I think  $z_j = x_j + iy_j$  is more informative since it is straightforwardly generalizable to the more than 2 mode cases with  $x_j$  and  $y_j$  respectively corresponding to the real an imaginary parts of the  $j^{th}$  mode. I also checked the previous two mode literature Ref. 31 and Ref. 30 do not have a mapping to real variables like (G43). If it makes sense to you, I am suggesting using  $z_j = x_j + iy_j$  for  $\mathbb{C}^2$  to  $\mathbb{R}^4$  mapping, I can do all the changes in Das Artikel and the codes in approximately an hour.

**2013-10-29 Predrag:** I bow my head in shame - “follow the precedent” creed makes no sense here, complex Lorenz equations has two  $m = 1$  representations, while here we care about different  $m$ ’s. If you have the strength, (re)introduce the sensible notation, please.

**[2013-10-30 Burak]** Implemented the redefinitions, in the main `2modes.tex` file only.

**2013-10-03 Evangelos to Burak:** I hate it to sound like Kassandra, but I do not see where ‘magic’ in your slice comes from. Using the non-magic template  $\hat{a}' = \{0, \sqrt{2}, 0, 0, 0\}$  for the two-mode system gives a figure equivalent (see Fig. 63) to what you get if you use  $\hat{a}' = \{1, 1, 0, 0\}$ . To me this is no surprise: since your flow is rotationally symmetric, any two templates that can be rotated into each other will give you equivalent results. Note that the equation for the slice could be written as  $\text{Re } z_1 = 0$ , and that we might as well have used template  $\hat{a}' = \{0, 1, 0, 0\}$  since this also leads to the same condition. Singularities occur here only if  $z_1 = 0$ , which corresponds to the invariant subspace of the 2modes flow, and thus we are safe.

**2013-11-04 Burak to Evangelos:** I agree that templates  $\hat{a}' = \{1, 1, 0, 0\}$  and  $\hat{a}' = \{0, 1, 0, 0\}$  are geometrically equivalent to each other and they would yield similar results using method of moving frames, however, I would expect you to be in trouble if you integrate within the slice of  $\hat{a}' = \{0, 1, 0, 0\}$ .

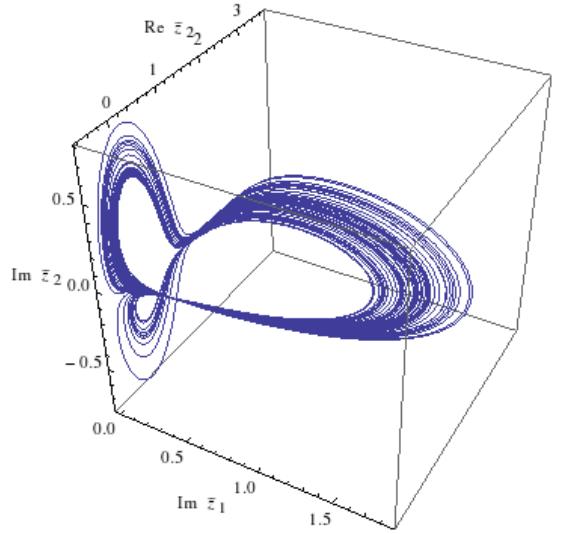


FIG. 63. Symmetry reduced two-mode system for  $\mu_1 = -2.8023, \mu_2 = 1, a_1 = -1, a_2 = -2.6636, b_1 = 0, b_2 = 0, c_1 = -4.1122, c_2 = 1.8854, e_1 = 0, e_2 = 1$ . Using moving frames method with  $\hat{a}' = \{0, \sqrt{2}, 0, 0\}$ . Compare with Fig. 46 (E. Siminos 2013-10-03).

**2013-11-03 Predrag:** In Ref. 71 the claim is made that the ‘method of comoving frames’

$$\dot{a}_\perp = v_\perp(a) = v(a) - \frac{t(a) \cdot v(a)}{t(a)^2} t(a) \quad (\text{G44})$$

‘calms’ the flow. It is very easy for you to implement it for the full state space flow of Fig. 53, and see what you get. My expectation is the flow will not be very different, as the method of comoving frames is not a symmetry reduction scheme, but merely a different way to explore the same group orbit of the flow in the full state space. Another line of the argument:

In the presence of an  $N$ -continuous parameter symmetry, each state space point  $a$  owns  $(N+1)$  tangent vectors: one  $v(a)$  along the time flow  $a(\tau)$  and the  $N$  group tangents  $t_1(a), t_2(a), \dots, t_N(a)$  along infinitesimal symmetry shifts, tangent to the  $N$ -dimensional group orbit  $\mathcal{M}_a$ . A group tangent and the state space flow velocity at an instant  $a = a(\tau)$  of a typical turbulent state are not likely to be close to parallel. For fluid flows the state space is high dimensional, and if the angles  $\theta_j(\tau) = \theta_j(a(\tau))$  between them

$$\cos \theta_j(a) = \frac{v(a) \cdot t_j(a)}{\|v(a)\| \|t_j(a)\|} \quad (\text{G45})$$

are probabilistically distributed, two such vectors in high dimensional state space are mostly nearly orthogonal, and there is no ‘calming’ effect in the comoving frame.

Maybe a time series plot of (G45) would also be informative...

**2013-11-05 Burak:** I implemented in method of comoving frames (G44) by integrating (G44) for  $O(2)$ -equivariant case and plotted trajectories for a relative periodic orbit in Fig. 64.

[2013-11-05 Predrag] (*RPO<sub>01</sub> repeated how many periods?*)

**2013-11-11 Burak:** After trying to improve the numerical precision of the relative equilibrium in Fig. ?? and Fig. ?? I realized that it actually is not a relative equilibrium, I'll try to explain why: I thought the second of the roots of invariant polynomials that I listed in (??) would have a correspondence in the equivariant state space provided by  $u, v > 0$ , however, the condition is more strict than that. Let's write down the equations that connects the state variables to the invariant polynomials:

$$\begin{aligned} u &= x_1^2 + y_1^2, \\ v &= x_2^2 + y_2^2, \\ w &= 2x_1^2x_2 - 2y_1^2x_2 + 4x_1y_1y_2, \\ q &= -2x_1^2y_2 + 2y_1^2y_2 + 4x_1x_2y_1 \end{aligned} \quad (\text{G46})$$

Now, since these equations are invariant under  $SO(2)$  operations, if there is a solution  $x = (x_1, y_1, x_2, y_2)$  to them for a particular  $(u, v, w, q)$  then, all the vectors  $g(\theta)x, \theta \in [0, 2\pi]$  are also solutions. Thus, as a particular solution,

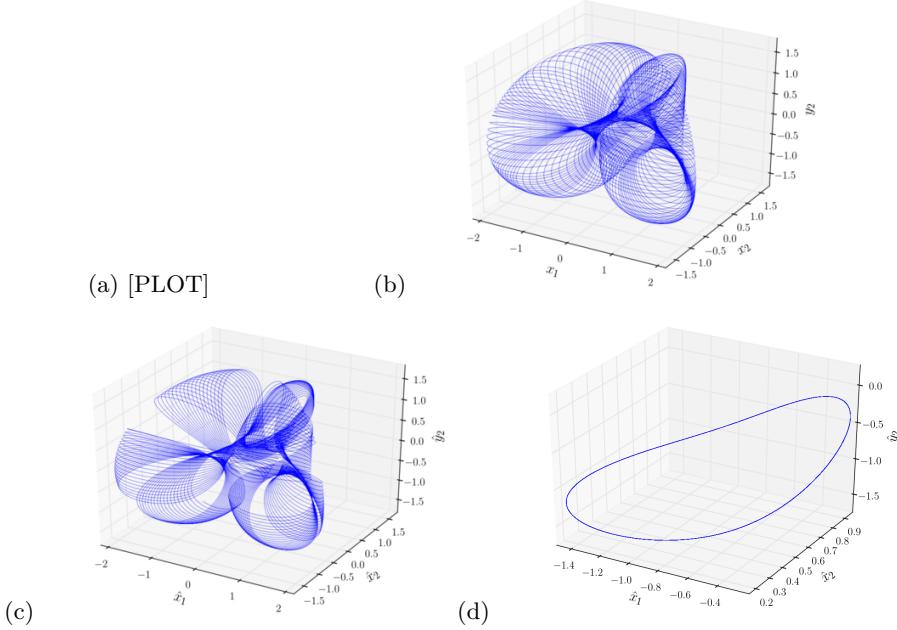


FIG. 64. (a) A single traversal of relative periodic orbit  $\overline{01}$ , period  $T_{01} = ???$ , shift  $\ell_{01} = ???$  (the 2-cycle of the Poincaré return map such as Fig. 77). This is a 3D projection of the full 4D state space  $O(2)$ -equivariant two-mode flow (21). (b) Relative periodic orbit  $\overline{01}$  integrated for ?? periods  $T_{01}$ . It explores ergodically the group orbit of the relative periodic orbit, a 2D invariant torus embedded in 4D. Other projections on any three coordinates  $\{x_1, y_1, x_2, y_2\}$  are qualitatively similar. (b) The same trajectory computed integrating the comoving frame transformed equations (G44). As the method of comoving frames is not a symmetry-reduction method, the trajectory explores ergodically the same invariant torus, but with a different shift  $\ell_{01}^{cm} = ???$  per period. (c) In the slice this trajectory retraces the corresponding periodic orbit. Parameters: (G40) with  $a_1 = 0.45$ .

we can choose the one within the slice hyperplane of  $\hat{a}' = (1, 1, 0, 0)$ , namely  $x = (x_1, \hat{x}_1, \hat{y}_1, y_2)$ . Plugging this candidate solution into (G46), we get:

$$\begin{aligned} u &= x_1^2 + \hat{x}_1^2, \\ v &= \hat{x}_1^2 + y_2^2, \\ w &= 4x_1^2 y_2, \\ q &= 4x_1^2 \hat{x}_1 \end{aligned} \tag{G47}$$

From the first relation we get  $u = 2x_1^2$ , plugging this into the last two, we get  $x_2 = q/2u$ ,  $y_2 = w/2u$ , finally plugging these into the second equation we obtain a condition for an equilibrium in the invariant polynomials to have a corresponding relative equilibrium in the equivariant state space:

$$4u^2v = w^2 + q^2 \tag{G48}$$

The second root in the list does not satisfy (G48), however, LHS of equation is 0.023101 while RHS is 0.022620. The state space point  $x_0 = (0.31056, 0.31056, 0.04191, 0.38981)$  which I thought was a solution to (G47) was in fact not a solution, but a minimum and the red curves in Fig. 3(a,b) were not an unstable relative equilibrium but trajectories effected by some sort of an intermittency.

This, I think, is interesting, but it also proves that my previous Jacobian calculations in [2013-09-28 Burak] were not accurate. If I compute the Jacobian as if this orbit is periodic and look for the eigenvalues, I still get something close to 1 on the group tangent direction, however, the accuracy is low to 3 or 4 decimal points, hence, I'm not sure whether the expanding eigendirections of the Jacobian are meaningful or not.

**2013-11-11 Burak:** CORRECTION: What I found in my previous post is the syzygy. So you can disregard it.

**2013-11-17 Burak:** I spent my weekend on learning a 3D visualization software called [Paraview](#) (this is very nice and open source) which uses open source scientific visualization libraries of [Visualization Toolkit](#). At the end,

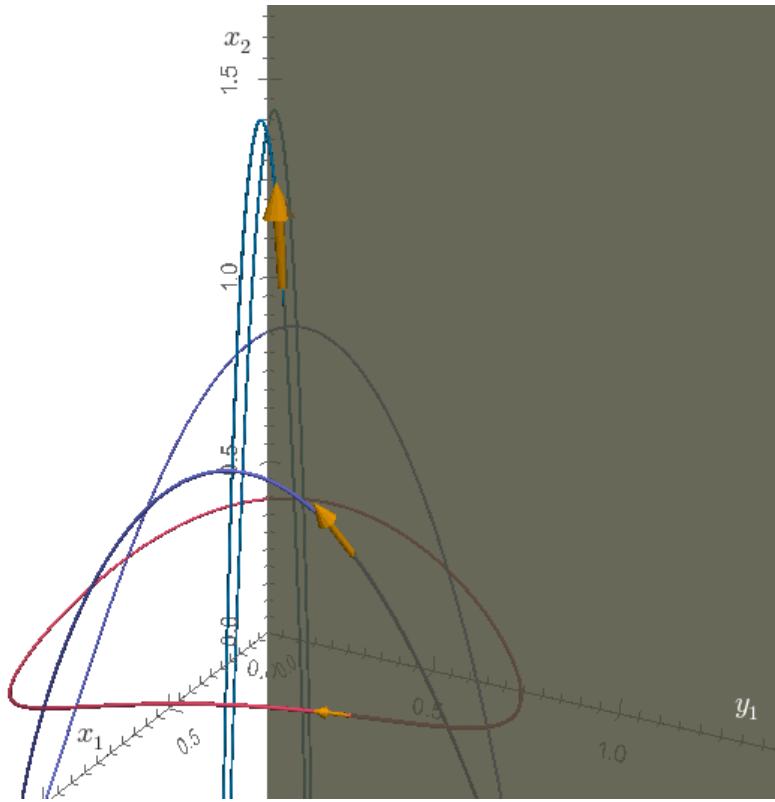


FIG. 65. Fancy 3D visualization of the magic slice, group orbits and group tangents.

I generated Fig. 65, where you can see a 2d projection of the magic slice, 3d projections of group orbits and tangent vectors. This is open to improvement, ideas to make it look better are welcome, you can see on Paraview website that the program is much more than a 3D plotter.

If you want to play with it, you can download a binary of Paraview from its website, it runs stand-alone without a problem on my Ubuntu 12.04-64bit, and then load the state file  
[blog/burak/PorterKnobloch/paraview/magicslicehd.pvsm](http://blog/burak/PorterKnobloch/paraview/magicslicehd.pvsm)

**2013-11-17 Predrag:** Best you show me :)

**2013-11-17 Burak:** After realizing that I can do it, I wrote some scripts that converts solutions to vtk data files that can be readable by Paraview, and plotted curves as 3D objects. An example is Fig. 66 where I drew the symmetry equivariant and sliced flow on top of each other. I'm not sure if this may be helpful in explaining the method of slices, but I think it's beautiful, so I'm blogging it.

My vtkscripts are here:

[blog/burak/PorterKnobloch/paraview/vtkscripts/](http://blog/burak/PorterKnobloch/paraview/vtkscripts/)

**2013-11-18 Predrag:** My reading of Fig. 65 is that Ruslan Davidchack, Evangelos Siminos and Al Shapere are right. Unless I am missing some subtlety about the invisible second dimension of the magic slice, it looks like we are allowed to rotate it around the  $x_2$  axis, and at all angles of that rotation, all group orbits (as long as all Fourier coefficients are non-vanishing) are sliced only once. In particular, if the slice group tangent vector is normal to  $y_1$ , all group orbits intersect the (rotated) magic slice orthogonally. If that is true, the simplest of all guesses at a slice might work, after all... That would be good news. What am I missing?

**2013-11-18 Burak:** You are not missing anything. I went back and rechecked my algebra and found a mistake (a division by zero), so it should be rotatable, as geometry suggests. There is one point that I do not agree with Ruslan, I remember him suggesting at some point that integration within the slice shouldn't be used and one should always post process the data; I think that the opposite is the best idea since we want to resolve the dynamics within the slice so our integration steps must adapt to the velocity functions within the slice. I still never managed to make numerics work on 1st mode slices other than the magic slice, to which, right now I have no explanation to, but there should be one.

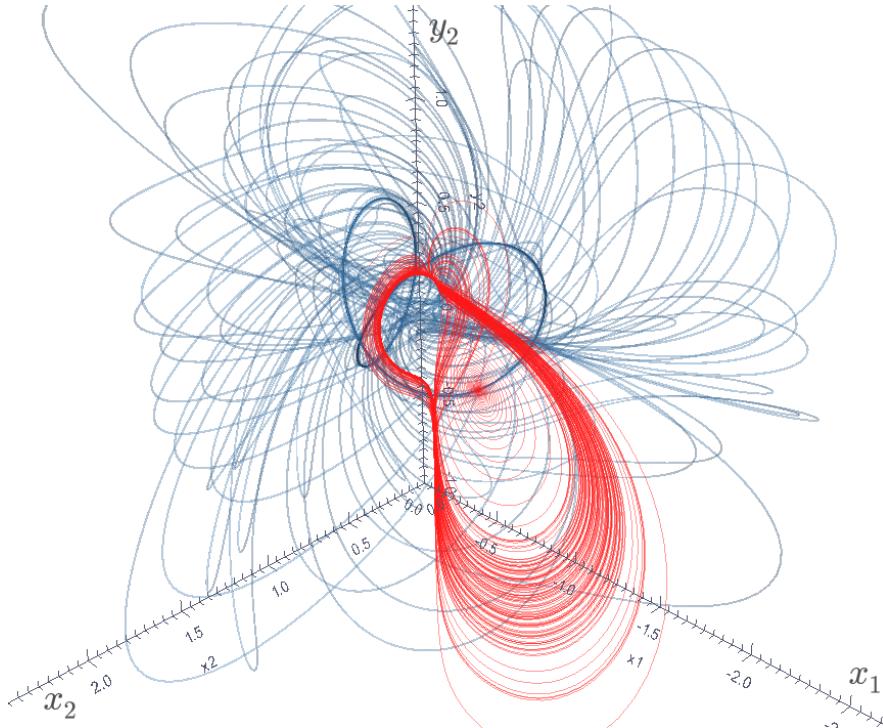


FIG. 66. A chaotic trajectory and the relative equilibrium  $TW_1$  of the full 4-dimensional  $\{x_1, y_1, x_2, y_2\}$  state space equivariant flow projected on three coordinates  $\{x_1, x_2, y_2\}$ , plotted together with the symmetry-reduced flow within the magic slice.

**2013-11-18 Burak:** Well I made them work, there were stupid mistakes before. Now everything make sense and is less interesting. I suggest we discard the magic slice and do the reduction onto  $\hat{a}' = (1, 0, 0, 0)$ . However, we should carefully study its applicability to Kuramoto-Sivashinsky and pipes, if it's not applicable, then go back to the multiple chart atlas.

**2013-11-19 Burak:** I made an argument before, saying that, in the  $O(2)$  equivariant case, if the invariant polynomial  $q$  vanishes in time, the flow becomes effectively two dimensional ( $w$  is determined by  $u$  and  $v$  through the syzygy (27)). After discussing with Predrag yesterday and today I decided to blog about it:

First of all, Danglemayr<sup>31</sup> never mentions  $q$  or a syzygy. Since we can produce the  $O(2)$  symmetric system by setting  $e_i \rightarrow 0$ , our invariant polynomials and the equations they satisfy should still be valid.

We can write down the velocity function for the invariant polynomials for the  $O(2)$  case by setting  $e_i \rightarrow 0$  in (30) as

$$\begin{aligned} \dot{u} &= 2\mu_1 u + 2a_1 u^2 + 2b_1 u v + c_1 w \\ \dot{v} &= 2\mu_2 v + 2a_2 u v + 2b_2 v^2 + c_2 w \\ \dot{w} &= (2\mu_1 + \mu_2) w + (2a_1 + a_2) u w + (2b_1 + b_2) v w + 4c_1 u v + 2c_2 u^2 \\ \dot{q} &= [2\mu_1 + \mu_2 + (2a_1 + a_2) u + (2b_1 + b_2) v] q. \end{aligned} \quad (\text{G49})$$

Key observation here is that once we set  $e_i \rightarrow 0$ ,  $q$  gets decoupled from the other equations and the final equation we got is actually linear in  $q$ , with a time-varying real coefficient. For a first order linear differential equation with real coefficients such the one we have here, only two things are possible: it either diverges to infinity or converges to zero, depending on the sign of coefficient being plus or minus. In our case, the coefficient varies in time and it can change its sign depending on the parameters, which, is indeed what I tried while looking for a chaotic attractor. However, after experimenting on these parameters for a while, and having a second look at the equations, I am thinking that  $q$  either diverges or converges to 0.

As an experiment, I made a bifurcation diagram of  $q$ , similar to Fig. 60 by varying  $a_1$ , evaluating (G49) for 1000 time units and recording the last 100 values of  $q$ , and got Fig. 67. As you can see most of the time it converges

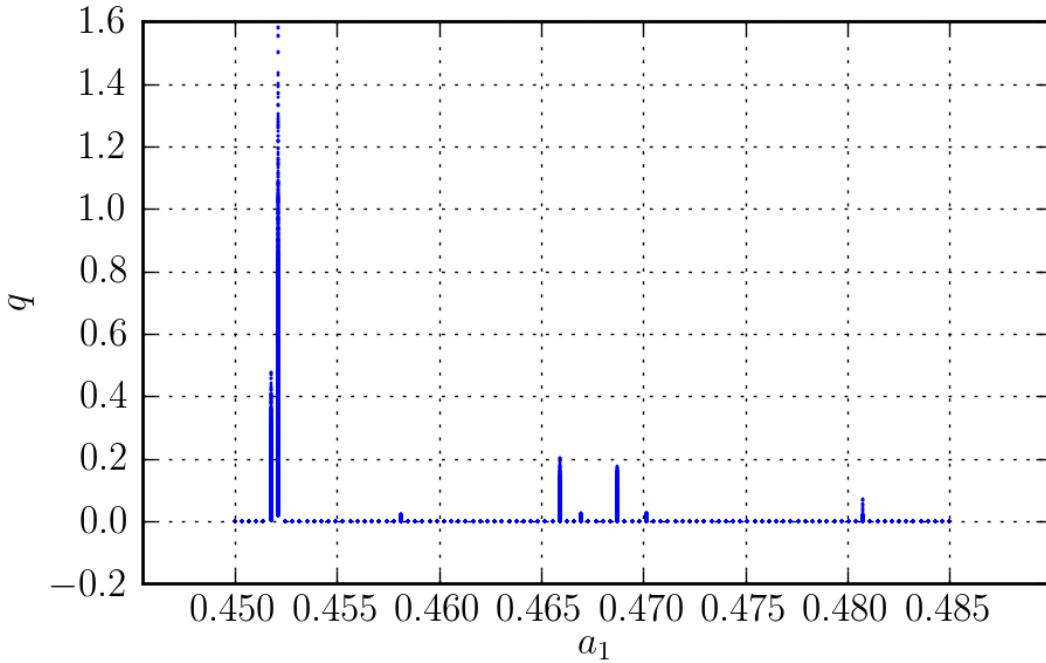


FIG. 67. Bifurcation diagram for  $q$  in  $O(2)$  invariant case obtained by integrating (G49) for 1000 time steps and keeping the last 100 of them with parameters (G40),  $a_1$  varying.

to 0 and for the ones with values other than zero, I thought, it could be converging slowly. To collect more evidence, I integrated (G49) with parameters (G40) setting  $a_1 = 0.452121212121$  the  $a_1$  value that corresponds to the largest peek in the bifurcation diagram Fig. 67 and plotted  $q$  versus time Fig. 68. As you can see from the figure, I proved myself wrong. In this case,  $q$  oscillates regularly as the orbit is a relative periodic orbit, but doesn't converge to anything.

For now, this discussion is inconclusive, however, I still think that we can come up with an argument related to linear being of  $\dot{q}$  and the dimensionality of the system in the invariant polynomial basis to state that observing chaos in the  $O(2)$  normal forms that we are dealing here is impossible.

**2013-12-08 Burak:** Found relative periodic orbits and their binary itineraries, blogging a preliminary result in table IV. While table IV has 50 relative periodic orbits, some of them computed multiple times, so the code needs some improvement. Also looking at these numbers, accuracy that I could get was around 4 decimal places, which is not so great. I used the build in Newton solver `fsolve` from `SciPy`, which is an implementation of `hybrd` and `hybrdj` functions from `MINPACK Fortran library` requiring relative accuracy of  $10^{-12}$  between iterations.

After a little more investigation, I found out that the relative periodic orbits with higher periods are not accurate at all. I developed the algorithm to search for relative periodic orbits on the Poincaré section this weekend, however, it ended up being less successful then looking at the reduced state space which is what I did last week. I'm blogging the figure of two relative periodic orbits here: Fig. 69. By going back to my previous algorithm, I probably will add a third one here.

Searching for cycles had been more challenging than I expected and now I'm really short on time. I am switching to the Dynamics Days poster right now.

**2013-12-10 Predrag:** You have to get a good multiple shooting or variational relative periodic orbit Newton routine running, you'll need it later. Here we expect all periodic points and cycle stabilities to machine precision.

You need to list only one periodic point per cycle, lexically ordered, as in ChaosBook Table 15.1. I have removed some of the repeats and extra cycle points. Current accuracy is sub par.

$\overline{101010} = \overline{10}^3$  is the third repeat of  $\overline{10}$ .

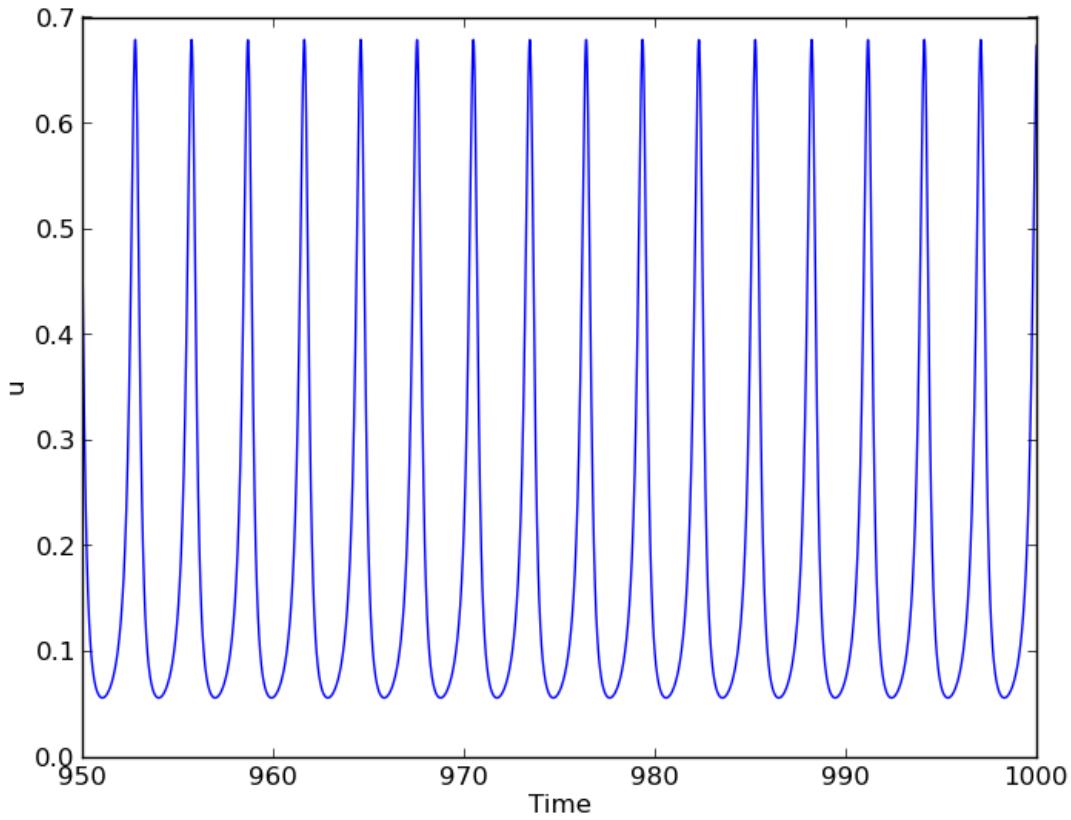


FIG. 68. Invariant polynomial  $q$  versus time, obtained by integrating (G49), using parameters (G40) with  $a_1 = 0.452121212121$ .

We also need a short list of periods, phases, Floquet multipliers and Floquet exponents for these cycles. I'll try to recycle some old table, but - curiously enough - neither Siminos nor Davidchack nor Froelich seem to have created such a table...

There will be pruning, as in 11.5 Kneading theory - should mention it. It is the kneading invariant obtained by the forward iteration of the critical (the highest) point of the return map.

**2014-01-19 Burak to Predrag:** I came to the point of applying the multiple shooting method to the Rossler system and before coding, I finished reading Chapter 13 - Fixed points, and how to get them and got confused. Last time when we talked at your office, you told me that when I set up a Newton search for a relative periodic orbit, I should add the symmetry parameters and time to the list of variables that I am looking for and run the search in the full state space. However, in 13.4, in summary, you say that when one looks for a particular orbit with a continuous symmetry, such as a periodic orbit symmetric under time translations, one has a marginal Floquet multiplier, hence  $(1 - J)$  which appears in the Newton set up becomes non-invertible. For this reason, one should always apply a constraint, such as a Poincaré section to the Newton solver to make it convergent. This was my initial idea and I looked for the periodic orbits within the slice and the Poincaré section in the two-mode system, then we decided that I should apply multiple shooting method. Did you have any particular reason for looking for the relative periodic orbits in the full state space?

**2014-01-20 Burak:** Here (Fig. 70) are two long periodic orbits of Rossler system that I found by multiple shooting method on the Poincaré section. I'm not committing the code yet since parts of it is assigned as a homework problem for the ChaosBook course. Tomorrow, I will start adapting the algorithm to two-mode system to find the relative periodic orbits with an acceptable precision.

**2014-01-31 Burak, 2014-02-03 Predrag:** <sup>143</sup>

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<sup>144</sup> 2CB

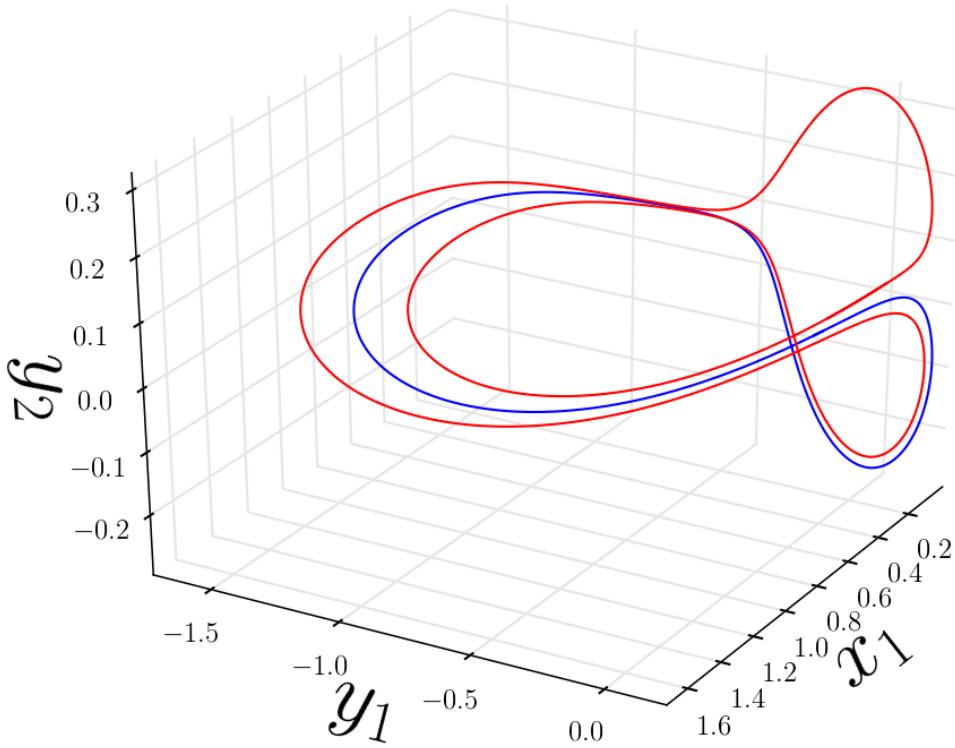


FIG. 69. Relative periodic orbits  $\bar{1}$  and  $\bar{0}\bar{1}$  for the two-mode system. See also Fig. 73.

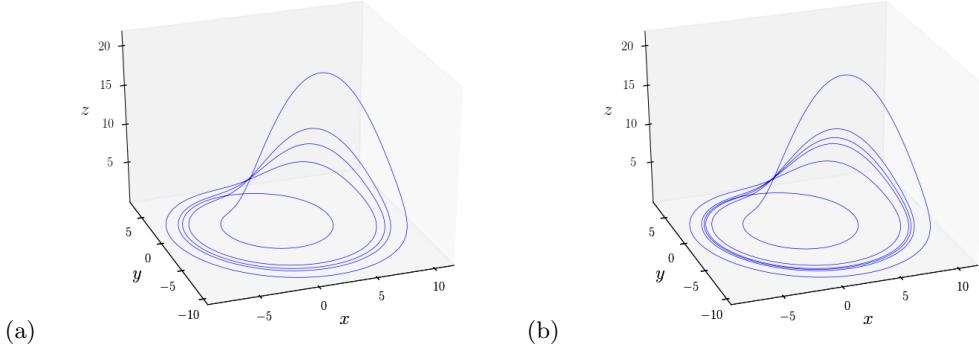


FIG. 70. (a) Periodic orbit 01111 of Rossler system. (b) Periodic orbit 011111 of Rossler system.

**Multiple shooting method for search for the relative periodic orbits:** Let us assume that we have a set of good guesses for a set of state space points, flight times and 1D symmetry group parameter increments  $\{a_i^{(0)}, \tau_i^{(0)}, \theta_i^{(0)}\}$  such that the points  $\{a_i^{(0)}\}$  lie close to the relative periodic orbit  $p$ ,

$$a_{i+1}^{(0)} \approx g(\theta_i^{(0)}) f^{\tau_i^{(0)}}(a_i^{(0)}) \quad \text{cyclic in } i = 1, \dots, n. \quad (\text{G50})$$

Here the period and the shift of the relative periodic orbit  $p$  are  $T_p \approx \sum \tau_i$ ,  $\theta_p \approx \sum \theta_i$ . We want to determine  $(\Delta a_i, \Delta \tau_i, \Delta \theta_i)$  corresponding to the exact relative periodic orbit,

$$a_{i+1} + \Delta a_{i+1} = g(\theta_i + \Delta \theta_i) f^{\tau_i + \Delta \tau_i}(a_i + \Delta a_i) \quad \text{cyclic in } i = 1, \dots, n. \quad (\text{G51})$$

Cycle point	$(x_{p,1}, y_{p,1}, x_{p,2}, y_{p,2})$
1	(0.4525310, 0.0000000, 0.0491730, 0.0323220)
01	(0.4516980, 0.0000000, 0.0202880, 0.0389600)
011	(0.4501950, 0.0000000, -0.0317750, 0.0497160)
010	(0.4519660, 0.0000000, 0.0295760, 0.0379080)
111	(0.4524410, 0.0000000, 0.0460270, 0.0312930)
0101	(0.4515050, 0.0000000, 0.0135930, 0.0402150)
1101	(0.4521150, 0.0000000, 0.0347420, 0.0369650)
1110	(0.4529440, 0.0000000, 0.0634840, 0.0302250)
11111	(0.4525310, 0.0000000, 0.0491590, 0.0323140)
01111	(0.4506150, 0.0000000, -0.0172370, 0.0462820)
01011	(0.4519510, 0.0000000, 0.0290460, 0.0374940)
011101	(0.4513060, 0.0000000, 0.0067200, 0.0431280)
010110	(0.4518470, 0.0000000, 0.0254650, 0.0374470)
101111	(0.4538490, 0.0000000, 0.0948170, 0.0214900)
0110101	(0.4504000, 0.0000000, -0.0246730, 0.0487990)
0110101	(0.4503810, 0.0000000, -0.0253450, 0.0473590)
0111101	(0.4508060, 0.0000000, -0.0106260, 0.0464390)
0111101	(0.4508250, 0.0000000, -0.0099580, 0.0465640)
0011111	(0.4509220, 0.0000000, -0.0065940, 0.0722500)
0111011	(0.4515410, 0.0000000, 0.0148390, 0.0417900)
0101111	(0.4519500, 0.0000000, 0.0290300, 0.0372790)
1101111	(0.4520600, 0.0000000, 0.0328350, 0.0385940)
1101010	(0.4521500, 0.0000000, 0.0359670, 0.0371310)
1101111	(0.4520040, 0.0000000, 0.0309010, 0.0393920)
1101010	(0.4521420, 0.0000000, 0.0356680, 0.0364640)
1101010	(0.4521460, 0.0000000, 0.0358010, 0.0367990)
1111010	(0.4523470, 0.0000000, 0.0427940, 0.0336690)
1111111	(0.4525200, 0.0000000, 0.0487720, 0.0328350)

TABLE IV. A periodic point per each short relative periodic orbit, chosen to be the lexically lowest by its binary itinerary. Two-mode system, parameter set (22).

To linear order in

$$(\Delta a_i^{(m+1)}, \Delta \tau_i^{(m+1)}, \Delta \theta_i^{(m+1)}) = (a_i^{(m+1)} - a_i^{(m)}, \tau_i^{(m+1)} - \tau_i^{(m)}, \theta_i^{(m+1)} - \theta_i^{(m)})$$

the improved Newton guess  $(a_i^{(m+1)}, \tau_i^{(m+1)}, \theta_i^{(m+1)})$  is obtained by minimizing the effect of perturbations along the spatial, time and phase directions,<sup>145 147</sup>

$$a'_{i+1} - g_{i+1} f^{\tau_i}(a_i) = g_{i+1} (t_{i+1} \Delta \theta_i + v_{i+1} \Delta \tau_i + J_{i+1} \Delta a_i), \quad (\text{G52})$$

where, for brevity,  $a_i^{(m+1)} = a_i^{(m)} + \Delta a_i^{(m)} = a'_i$ ,  $a_i^{(m)} = a_i$ ,  $g(\theta_i) = g_{i+1}$ ,  $v(a_i) = v_i$ ,  $J^{\tau_i}(a_i) = J_{i+1}$ ,  $t(a_i) = T a_i = t_i$ , etc.. For sufficiently good initial guesses, the improved values converge under Newton iterations to the exact values  $(\Delta a_i, \Delta \tau_i, \Delta \theta_i) = (\Delta a_i^{(\infty)}, \Delta \tau_i^{(\infty)}, \Delta \theta_i^{(\infty)})$  at a super-exponential rate.<sup>149</sup> In order to deal with the marginal multipliers along the time and group orbit directions, one needs to apply a pair of constraints, which eliminate variations along the marginal directions on the relative periodic orbits 2D torus (see Fig. 64). As our guesses  $\{a_i^{(m)}\}$  are computed in the global slice Fig. ?? and Poincaré section Fig. 5 (a), and read off the (nearly) unimodal return map Fig. 5 (b), one can impose the corresponding global slice and section constraints (D3) and (D4),

$$[\text{writetheglobalconstraintshere}], \quad (\text{G53})$$

<sup>146</sup> PC: 2014-02-02 I have changed Burak's (G52) by  $T f^{\tau_i}(a_i) \rightarrow T g(\theta_i) f^{\tau_i}(a_i)$ . The rest of the formulas need to be changed accordingly.

<sup>148</sup> PC: 2014-02-02 to Burak: please write up the detailed derivation, with what  $O(\Delta^2)$ 's are dropped as a problem / solution set for `cycles.tex`.

<sup>150</sup> PC: 2014-02-02 I am often running into the situation that the linear operators are better marked by the final rather than by the initial points of trajectory segments, as in  $J^{\tau_i}(a_i) = J_{i+1}$ . Implementing this requires a huge rewrite of the ChaosBook

for each return map cycle point  $a_i$  in Fig. 5 (b), compute the first return time and phase shift  $(a_i^{(m)}, \tau_i^{(m)}, \theta_i^{(m)})$ , and then restart with  $a_{i+1}$  in Fig. 5 (b).

However, imposition of the global slice condition induces high velocities along the phase reconstruction coordinate  $\dot{\theta}(\tau)$  close to the chart border. These high  $\dot{\theta}(\tau)$  are artifacts of the symmetry reduction methods, they are not present in the full state space flow. For that reason local transversality constraints - a local Poincaré section orthogonal to the flow, and a local slice orthogonal to the group orbit at each point along the orbit,

$$v_i \cdot \Delta x_i = 0, \quad t_i \cdot \Delta x_i = 0, \quad (\text{G54})$$

are natural and numerically preferable; in the limit of infinitely many infinitesimal time steps, the solutions that they yield are the relative periodic orbits in the comoving frame transformed equations (G44), see Fig. 64 (b), with the comoving frame  $\theta_p^{cm}$ . This  $\theta_p^{cm}$  might be the so much bandied around ‘geometrical phase.’. As the method of comoving frames is not a symmetry-reduction method, the trajectory explores ergodically the same invariant torus, but with a different shift per period. Once the relative periodic orbit is computed, its shift  $\theta_p$  in the ‘lab frame’ can be computed by imposing the fixed slice condition (G53) on any cycle point.

We can rewrite everything as one matrix equation:

$$\mathbf{A}\Delta = E, \quad (\text{G55})$$

where,

$$\mathbf{A} = \begin{pmatrix} g_2 J_2 & g_2 v_2 & Tg_2 f^{\tau_1}(a_1) & -\mathbf{1} & 0 & 0 & 0 & \cdots & 0 \\ v_1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ t' & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & g_3 J_3 & g_3 v_3 & Tg_3 f^{\tau_2}(a_2) & -\mathbf{1} & \cdots & 0 \\ 0 & 0 & 0 & v_2 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & t' & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ -\mathbf{1} & 0 & \cdots & 0 & 0 & 0 & g_1 J_1 & g_1 v_1 & Tg_1 f^{\tau_1}(a_1) \\ 0 & 0 & \cdots & 0 & 0 & 0 & v_n & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & t' & 0 & 0 \end{pmatrix} \quad (\text{G56})$$

$$\Delta = \begin{pmatrix} \Delta a_1 \\ \Delta \tau_1 \\ \Delta \theta_1 \\ \Delta a_2 \\ \Delta \tau_2 \\ \Delta \theta_2 \\ \vdots \\ \Delta a_n \\ \Delta \tau_n \\ \Delta \theta_n \end{pmatrix} \quad \text{and } E = \begin{pmatrix} a_2 - g_2 f^{\tau_1}(a_1) \\ 0 \\ 0 \\ a_3 - g_3 f^{\tau_2}(a_2) \\ 0 \\ 0 \\ \vdots \\ a_1 - g_1 f^{\tau_n}(a_n) \\ 0 \\ 0 \end{pmatrix} \quad (\text{G57})$$

We than solve (G55) for  $\Delta$  and update our initial guess by adding the vector of the computed  $\Delta$  values, and iterate.

I applied this algorithm to the relative periodic orbit candidates that I found from the return map to get the relative periodic orbits precisely, however, orbits, other than  $\bar{1}$  did not converge.

I also added an adaptive stepping algorithm which basically reduces the step length if a Newton step increases the error, after adding this, it search started to converge a non-zero minimum value. I plotted the element of  $E$  with largest absolute value which, for  $\bar{0}1$  looks like Fig. 71.

One thing, that I clearly see by looking at the numbers is that, while the constraint  $t' \cdot \Delta x_i$  keeps  $(\Delta x_i)_2 = 0$ , since,  $t' = (0, 1, 0, 0)$ , maximum error in  $E$  appears to be in the second element of each  $a_{i+1} - g(\theta_i) f^{\tau_i}(a_i)$ . This can either due to  $\tau_i$  or  $\theta_i$  variations, since  $(\Delta x_i)$  obeys the constraint and keeps  $y_1$  component of the  $a_i = 0$ .

There must be something I am missing, however, I could not find it yet. Any suggestions? Maybe a restriction on  $\tau_i$ 's or  $\theta_i$ 's?

This derivation is based on eq. (13.11) of [Fixed points and how to get them](#).

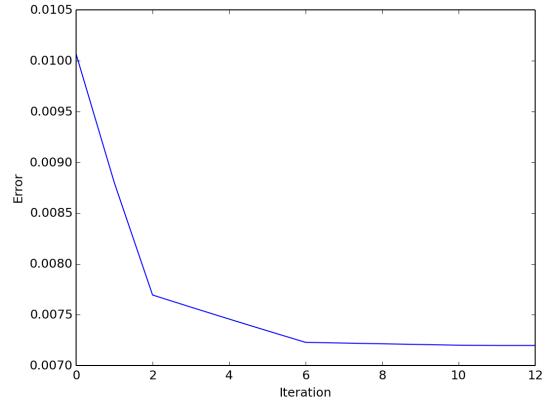


FIG. 71. Error vs Newton search steps for relative periodic orbit  $\overline{O1}$  of two-mode system.

**2014-02-02 Predrag:** An idea of how to represent the two-mode state space. The splaying of the strange attractor close to the slice border in Fig. ?? is an artifact of the Fourier modes representation. Think of a straight line, constant velocity trajectory in Cartesian  $(x_1(t), x_2(t))$  coordinates passing close to the origin (our slice border). In polar coordinates  $z_1 = |u|^{1/2} e^{i\theta_1}$ ,  $z_2 = |v|^{1/2} e^{i\theta_2}$ ,  $|u|(t) \rightarrow 0_+$ , and  $\dot{\theta}(t)$  whips through,  $\dot{\theta}(t)$  making a fast jump by  $\pm\pi$ , see complex Lorenz flow Fig. 72. So we should really chose more sensible coordinates which rescale  $\theta$  in such a way that when the trajectory passes by the singularity as a straight line, the new velocity is nearly constant vector as the trajectory passes the singularity.

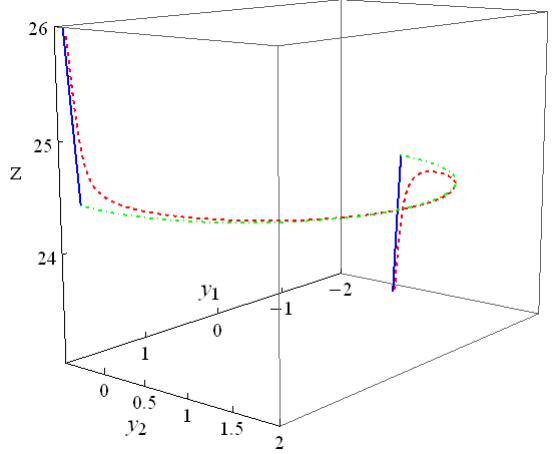


FIG. 72. The blue trajectory passes through the slice border of complex Lorenz flow. Notice how there is the gap in the trajectory, this corresponds to the jump cause by passing through the singularity. The red trajectory has initial point differing from the blue's by  $(0, 0, 0.1, 0, 0)$ , so it does not pass through a singularity. The green trajectory is the group orbit of  $x_{sing}$  between the two  $\theta$  that rotate  $v(x_{sing})$  in the slice. As expected, the red trajectory begins near the blue trajectory, closely follows the green trajectory after the singularity point, reaches the other side of the blue trajectory and begins to follow it again. Read Sect. 2.6 *Slice singularities* in [siminos/froehlich/blog.pdf](#).

**2014-02-02 Predrag:** Another idea of how to visualize the two-mode states. We are toying with the two-mode system as a model for truncations of 1D PDEs such as Kuramoto-Sivashinsky. For those the natural visualization is in the configuration space, i.e., as a periodic function  $u(x, t) = u(x + 2\pi, t)$  on the domain  $x \in [0, 2\pi]$ . You can think of  $(z_1, z_2)$  as the coefficients in the  $k = \{1, 2\}$  truncation of the Fourier expansion

$$u(x, t) = \sum_{k=-\infty}^{+\infty} z_k(t) e^{ikx/2\pi}. \quad (\text{G58})$$

How about doing the inverse Fourier transform, and plotting color coded amplitude  $u(x, t)$  in the  $[x, t]$  plane, as we do (for example) in Ref. 46 ([click here](#)). Chaotic dynamics is competition between 1-wiggle and 2-wiggle solutions, that might be more obvious in the configuration space. Relative equilibria and relative periodic orbits will drift with different phase velocities. One can perhaps study energy, dissipation rate, etc.? In other words, motivate the model by the physical problems we are really interested in.

**2014-02-02 Predrag:** I started rewriting the multiple shooting method for search for the relative periodic orbits, with intention of including it into ChaosBook. As an example? exercise? in an appendix? - in any case have to include the calculation for a relative equilibrium, possibly in the Lagrange multiplier formulation, if that is easier to understand...

I stopped, however, until Burak edits the formulas.

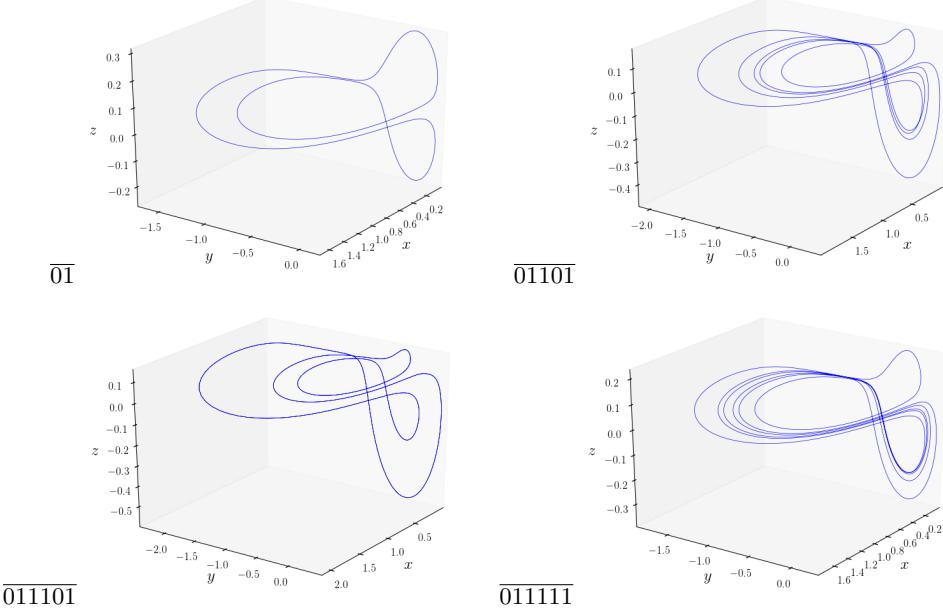


FIG. 73. Several relative periodic orbits of two-mode system plotted in the slice hyperplane. Cycle  $\overline{0}11111$  exhibits accumulation toward  $\bar{1}$  for long sequences of consecutive 1's. See also Fig. 10.

Itinerary	$(x_{p,1}, y_{p,1}, x_{p,2}, y_{p,2})$	Period
1	(0.4525719, 0.0, 0.0509257, 0.0335428)	3.6415120
01	(0.4517771, 0.0, 0.0202026, 0.0405222)	7.3459412
0111	(0.4514665, 0.0, 0.0108291, 0.0424373)	14.6795175
01101	(0.4503967, 0.0, -0.0170958, 0.0476009)	18.3874094
01111	(0.4505384, 0.0, -0.0171185, 0.0475980)	18.3874094
011111	(0.4512323, 0.0, 0.0037677, 0.0438172)	21.9923439
011101	(0.4575102, 0.0, -0.0327127, 0.0497776)	22.1591782

TABLE V. The shortest relative periodic orbits of the two-mode system. Parameter values (22).

**2014-02-03 Predrag:** You always want to plot  $\log_{10}(\text{error})$  in Fig. 71, to verify the you are getting the quadratic super-exponential convergence expected of Newton. What happened to odd iterations?

**2014-02-03 Predrag:** What bugs me about Newton is that one must pick a particular norm in (G54); that is very explicit in the Lagrangian cost function formulation of the boy scout version of ChaosBook. So in applying this to Kuramoto-Sivashinsky you should try different Sobolev norms. Would be grand if there were a norm-free iteration method...

**2014-02-03 Burak:** I think choosing the global slice for every point is a better idea for our approach, since our guesses are from that particular slice and if we force them to move in their local slice, our initial guesses may be not as good as they were before.

[**2014-02-03 Predrag**] Do not worry, the errors will drop like a ton of rocks, with no need to use short time steps close to close to the chart border.

**2014-02-03 Burak:** Here are four relative periodic orbits in the slice hyperplane: Fig. 73.

**2014-02-03 Predrag:** You have to explain and plot the critical point (top of the knifedge) in Fig. 5 (b). It is worked out for the Lorenz in ChaosBook.

**2014-02-07 Burak:** Relative periodic orbits, their symbolic dynamics and periods are listed in table V.

**2014-02-07 Burak:** We can define a new time variable as  $d\tau = x_1 dt$  and rewrite (8) for the first mode slice ( $\hat{a}' = (1, 0, 0, 0, \dots)$ ) as:

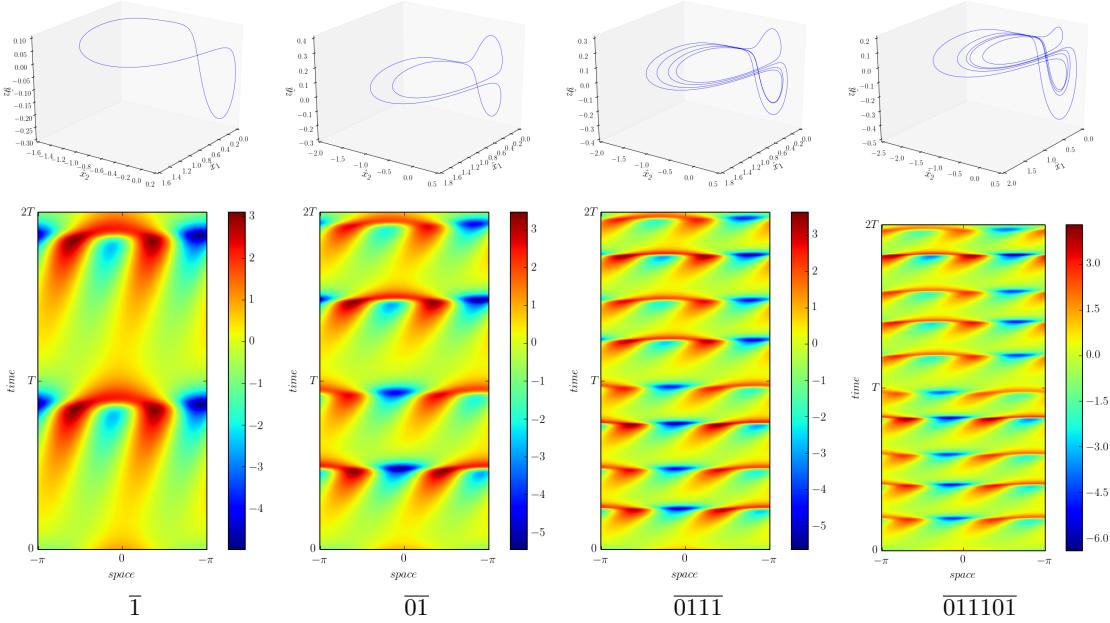


FIG. 74. Several relative periodic orbits of two-mode system plotted in the slice (top) and the configuration space (bottom).

$$\begin{aligned} \frac{d\hat{a}}{d\tau} &= \hat{a}_1 v(\hat{a}) - \frac{d\theta(\hat{a})}{d\tau} t(\hat{a}) \\ \frac{d\theta(\hat{a})}{d\tau} &= -v(\hat{a})_2. \end{aligned} \quad (\text{G59})$$

Fig. 75 shows two-mode dynamics obtained by integrating (G59).

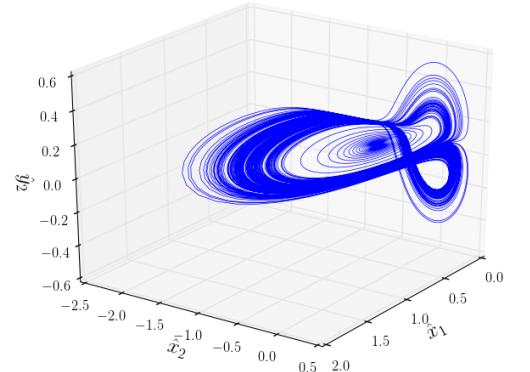


FIG. 75. Two-mode dynamics in the reduced state space obtained by integrating (G59).

**2014-02-07 Predrag:** The O(2) two-mode symmetry analysis now needs to be written up: The anti-symmetric invariant subspace takes out 1/2 of Fourier modes (see symmetry discussions of Ref. 46, 72, and 73, and probably many better papers out there that I'm not aware of) so two-mode it is 2-dimensional and boring :). Could have a spiral or something, though. Define it and nail it down. That is the space of solutions self-dual under complex conjugations. All other solutions came as pairs under the discrete ‘reflection’. You can plot them in ‘angle double’ polar coordinate as we do for the Lorenz attractor - perhaps you show that is also taking place in a 2-dimensional invariant subspace: you have 4 dimensions total, and 2 are taken by the anti-symmetric invariant subspace.

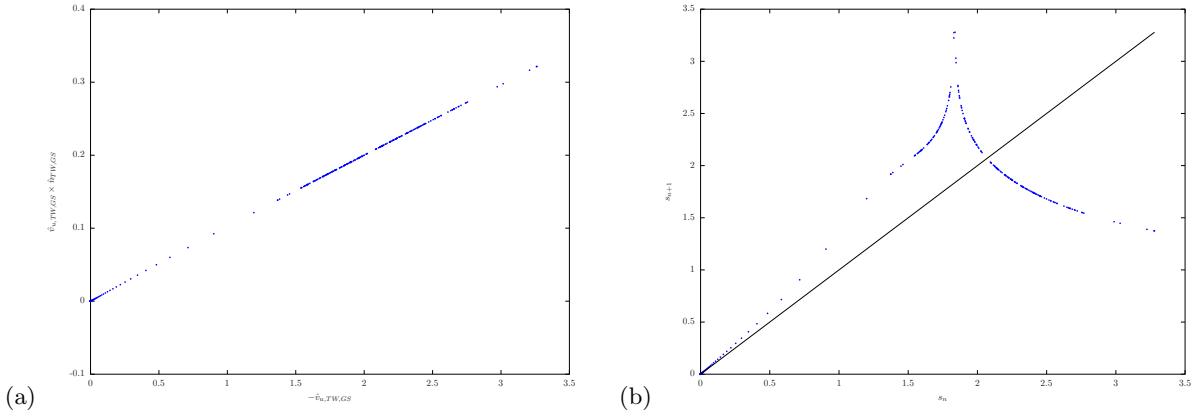


FIG. 76. (a) Poincaré section plane which includes the relative equilibrium  $TW_{NameMe}$  and is perpendicular to the normal vector given in (G30). Shown positions are relative to the section template (the relative equilibrium point in the slice). (b) Poincaré return map corresponding to the section in Fig. 76(a). The equilibrium  $\bar{0}$  is at  $(0, 0)$ , the relative equilibrium  $\bar{1}$  is the other fixed point on the diagonal, and the shortest cycle  $\bar{0}\bar{1}$  hops between the two branches (unmarked).

It's the mandatory problem set for the denizens of 3rd and 5th floor Howey, and that means YOU :) (Not to mention the Little Internet Book teachings-resistant [comrade Ding](#)). Also needs to be written up as the problem/solution pair for this, if not the next Tuesday.

**2014-04-02 Burak:** Porter-Knibloch polar coordinates are probably equivalent to the first mode slice. Here are their coordinates:

$$r_1 = |z_1|, (x, y) = r_2(\cos(\theta + \phi_1), \sin(\theta + \phi_1)), \text{ where, } \theta = \phi_2 - 2\phi_1 \quad (\text{G60})$$

the correspondance between this coordinates and the 1st mode slice would have been exact if the phases that appear in  $\sin$  and  $\cos$  were simply  $\theta$ , yet I can't see why there is an extra  $\phi_1$ , but this probably still is an equivalent representation upto a phase factor. They actually have comments on why if one simply picks  $r_2$  as an invariant coordinate, then the associated phase becomes singular, which is equivalent to our "border" concept, and say that they pick these coordinates since  $r_1 = 0$  is an invariant subspace (see the discussion between equations 13 and 14 in Ref. 30).

**2014-04-02 Burak:** OK. I think the explanation to the extra  $\phi_1$  factor is that it's a typo. Ref. 30 refers to a few papers, two of which are authored by Edgar Knibloch, about this transformation and neither of them has the extra  $\phi_1$  term. Also in the figure 5 of Ref. 30, where they show the reduced chaotic trajectory, the axis labels are wrong.  $x$  and  $r_1$  must change places. The result: This representation is equivalent to our 1st mode slice, they wrote this down for 1 : 3 resonance as well. In that case, relative phase becomes  $\theta = \phi_3 - 3\phi_1$ . Our first mode slice, in some sense, is a generalization of this idea.

**2014-05-07 Burak to Evangelos and Daniel:** I have just finished re-writing abstract and the first three sections of the two-mode article. Since it was re-written at many different times, it is quite possible that there are notational inconsistencies, please have a look at them, and try to make changes. Everything should be consistent with Ref. 18. I will focus on the Section IV, and reproduce the results that we would like to present with publication quality figures. If you can work on the first three sections in parallel, we can quickly finalize this paper. I separated the sections of the paper into different files: `abstract.tex`, `introduction.tex`, `contsymmt.tex`, `flow.tex`, `numerics.tex`, `conclusion.tex`, `acknowledgments.tex`, so that we can work in parallel without generating conflicts.

**2014-05-14 Predrag:** Regarding the (nearly) unimodal return map Fig. 5:

Fig. 5 axes number labels are too small.

Fig. 5 (a) is in conflict with Fig. 76(a) which looks like the section in Fig. ?? and I suspect is the right figure. Please plot the unstable manifold of the relative equilibrium  $TW_{NameMe}$  as a thin line, with dots sprinkled upon it. I would not plot the transient sequence starting close to  $TW_{NameMe}$ , we want to emphasize the recurrent strange set only.

Fig. 77 looks better than Fig. 5(b) In either case, please plot square as a square, not a rectangle.

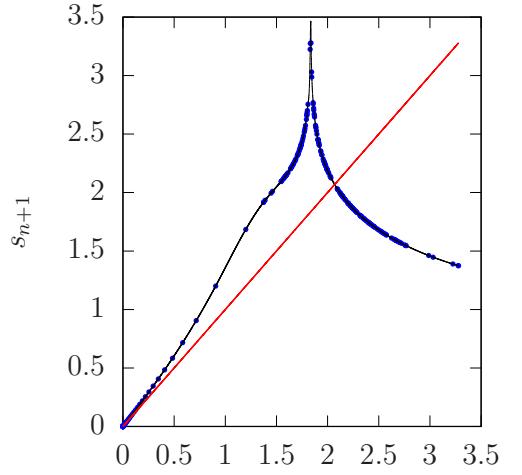


FIG. 77. Poincaré return map corresponding to the section in Fig. 76.

**2014-05-14 Predrag:** Regarding the symbolic dynamics. Here is proposal that I hope will also render the Lorentz attractor symbolic dynamics a limit of a sequence of finite grammars. There are two kinds of binary dynamics horseshoes: stretch & fold, as the Rössler, and ‘tear’, like Lorenz. In the latter case the 2D unstable manifold of the ‘upper’ equilibrium is forced to intersect with the invariant subspace, leading to the infinite-slop cusp in the return map. No dynamics actually visits the close neighborhood of the ‘lower’ equilibrium, so in conjugating it to a dike map, we can approximate the dynamics by taking points to the left of the cusp in Fig. 5 (b) with kneading sequences that correspond to the finite grammars. The truncation error can be estimated as the  $1/\Lambda_{last}$  where ‘last’ is the rightmost orbit in the finite-grammar approximation. This error will drop like a ton of rocks...

**2014-05-14 Predrag to Evangelos and Daniel:** As coauthors, please start working on this paper. Burak has done all the pulling since Sept 2013.

**2014-05-14 Evangelos:** If there is still problem with multiple shooting I could very easily adapt my routines from complex Lorenz flow to find all relative periodic orbits up to a given length. Let me know.

**2014-05-14 Daniel:** I'm kind of out of the loop. What should I work on? For now I will crunch through the manuscript for readability and check for consistency in notation. I'll also double check the symbolics for consistency. Is there anything more productive that I should be doing?

**2014-05-15 Burak to Evangelos:** Please read my [2014-05-07 Burak to Evangelos and Daniel] post and don't touch, for now, the Section IV. The version you see (before Evangelos' edits) were from six months ago and I'm currently working on it.

**2014-05-14 Daniel to Gang:** Went over the first part of section II. I changed it significantly so please go over it and make sure I didn't say anything crazy. Also, is there a reason we are using the tilde on  $a_p$ ? I don't really see it anywhere else and don't remember using it in the atlas paper. I have temporarily gotten rid of it as part of a series of two-mode paper specific macro redefinitions in `def2modes.tex` which I hope will make things line up with BudCvi14. Feel free to revert if there is a need for it or to change the global macro definitions. Where are those again, by the way?

**2014-05-16 Daniel to Predrag and Burak:** Please see comment about figure 1 in two-mode text. Have we given up on wursts?

**2014-05-19 Burak:** We have avoided mentioning sausages in Ref. 18, and I think we can do the same here. I'm open to discuss about this, but I think we can simply say that “symmetry reduction is a coordinate transformation such that every dynamically equivalent point is mapped to a single point in the reduced space.” This is a simple method and we are going to present relative periodic orbits and compute traces in our results, so I think it's better to keep the symmetry reduction discussion as simple as possible.

**2014-05-17 Daniel to Burak, Predrag, and Evangelos:** Below equation (15) we constrain the slice hyperplane to  $x_1 > 0$ . Should this constraint be on  $x_1$  or on  $\hat{x}_1$ ? I think it's the latter, so I changed it in the manuscript

but please confirm and revise accordingly. Also please check red text comments in the manuscript. Also what happens to the zeroeth mode from the expansion. It's symmetry invariant but still plays a role in the dynamics, no? How do we justify getting rid of it in the discussion? Also, have we firmly concluded that there is no chaos in  $O(2)$  equivariant two-mode-like systems? I found my old code with which I determined the parameters for  $SO(2)$ . Is it still of interest? Hard to tell from the blog entries above.

**2014-05-19 Burak:** Constraint is  $\hat{x}_1 \geq 0$  and we should discuss about the equality case. It's not clear to me.

**2014-05-20 Evangelos:** Equality in  $\hat{x}_1 \geq 0$  is needed because one would like to define the chart border as the hyperplane satisfying both the slice condition and the border condition.

**2014-05-19 Burak:** *Summary of the “no chaos in  $O(2)$ ” discussion:* Note that in (30), if you set  $e_1 = e_2 = 0$ , which is the case for  $O(2)$ , you dynamically decouple  $q$  from other three invariant polynomials. Moreover,  $\dot{q}$  becomes linear in  $q$ . Observing this, I guessed that since  $\dot{q}$  is linear in  $q$ , it will either vanish, or diverge depending on system parameters, however, remaining three invariant polynomials will still satisfy the syzygy, making the dynamics effectively two dimensional, thus, no chaos. This was case for many set of parameters, however, I also managed to create exceptions, see Fig. 67. As you can see for most of the parameter range,  $q$  indeed vanishes, however, it oscillates for some. This is not conclusive, and I think my guess should still be correct and non-zero values in Fig. 67 are transient; it just takes too long for them to actually vanish. We should make this clear, or may be consult a mathematician for a rigorous proof that  $q$  would vanish or diverge in the long term. Here is the summary of that question:

$$\dot{q} = \xi(t)q, \quad (\text{G61})$$

where  $\xi(t)$  is some combination of other state space variables and parameters, but does not depend on  $q$ . If we had a strange attractor,  $\xi(t)$  would take positive and negative values, bounded below and above. So my guess was,  $q$  would eventually vanish or diverge depending on the time-average of  $\xi(t)$  being negative or positive respectively. If we can make this rigorous, we can safely say that 1 : 2 resonance with mode interactions up to third order, cannot exhibit chaos unless the reflection symmetry is broken.

**2014-05-28 Predrag:** My worry is that our two-mode is a special case of Dangelmayr, which itself is a cubic truncation of the most general function Taylor expansion around an equilibrium. Two-mode truncation for Kuramoto-Sivashinsky (which is quadratic) was boring, that's why we went to Dangelmayr cubic normal form. My hunch is that if we added quartic term, there would be chaos also for the  $O(2)$  case, and it would not surprise me if you found  $O(2)$  chaos for a more general Pöter-Knobloch than our two-mode model.

**2014-05-19 Burak:** Here is one way to look at (G61): Assume we discretize the time as  $t_k = k\delta t$  and  $\xi_k = \xi(t_k) = \text{const.}$  for a small period. Then we advance from  $q_k$  to  $q_{k+1}$  by:

$$q_{k+1} = e^{\xi_k \delta t} q_k = e^{\xi_k \delta t} e^{\xi_{k-1} \delta t} \dots e^{\xi_0 \delta t} q_0 = e^{\sum_{j=0}^k \xi_j \delta t} q_0 \quad (\text{G62})$$

Now if we take the limit  $\delta t \rightarrow 0$ , for  $q(t)$ , we get:

$$q(t) = e^{\int_0^t d\tau \xi(\tau)} q(0). \quad (\text{G63})$$

(G63) is a valid solution to (G61) and you can check this by plugging it in (G61). Now the integral in the exponential in (G63), measures the area under  $\xi(\tau)$  for  $\tau \in [0, t]$ . Now, let us assume that the time average of  $\xi(t)$  for some large time interval  $T$  is  $\xi_{avg}$ , then the integral in the exponential for  $t = nT$  can be written as:

$$\int_0^t d\tau \xi(\tau) = n\xi_{avg} T, \quad (\text{G64})$$

thus, in the limit  $n \rightarrow \infty$ ,  $q$  will either vanish or diverge depending on the sign of  $\xi_{avg}$ . Now the question:

$\xi_{avg}$  will exist for periodic solutions, but will it exist when there is a chaotic attractor? It would, of course, cause a contradiction since what we are saying is if  $\xi_{avg}$  exists, dynamics cannot exhibit chaos. We can think of  $\xi_{avg}$  as a measurable that depends on state space variables, in that sense, its long term average should exist, right?

**2014-05-28 Predrag:** You do not need to derive this, you know from many other cases you have worked out in QM and other classes that if  $\xi(\hat{a}(t))$  is a scalar observable, (G63) is the solution of (G61), with no need for time ordering.

As long as dynamics is bounded, observable  $\xi(t) = \xi(\hat{a}(t))$  is bounded and so is  $\xi_{avg}$ . What it converges to depends on the initial  $p \in \hat{\mathcal{M}}$ ; on any periodic orbit it takes a particular value. I think we are allowed to state that on any chaotic set, convergence is guaranteed by the Birkhoff ergodic theorem.

Of course, we would be happier if we had a proof in  $\hat{a} \in \hat{\mathcal{M}}$ , rather than polynomial coordinates.

**2014-05-22 Daniel to Gang:** Started double checking equations in Section III using computer algebra and assuming that we are sticking with  $z_k = x_k + y_k$  notation. So far I have done (21), (27), (28), (29), (30), and (31). Intermediate equations that are built into paragraphs have not been checked yet. I am marking checked equations with comments with the double check date. So far I have not found any mistakes, so it appears that my transformation of the blog to  $z_k = x_k + y_k$  from a while back worked fine.

**2014-05-22 Daniel to Gang:** Continued double checking equations. Having issues verifying the equation for  $q$  in (34), but will continue working on it. I also realized that we have some notational issues between reporting points in the Cartesian basis and the invariant polynomial basis when we report specific points. Like what does  $(0, -\mu_2/b_2, 0, 0)$  mean? I suggest something like using parenthesis for  $(x_1, y_1, x_2, y_2)$  and square brackets for  $[u, v, w, q]$ . If you guys agree or have a better suggestion, I'll start going through and modifying the notation in the paper.

**2014-05-26 Burak:** I'm not sure if having two different notation for state space vectors, can we make sure that the representation that we are referring is clear from the context? We have  $a = (x_1, y_1, x_2, y_2)^T$  for real-valued equivariant state space variables, may be we can define invariant polynomial vectors with letter  $p$ , like,  $p = (u, v, w, q)^T$  and when we refer to the  $TW_1$  in different representations we can make it clear by denoting as  $a_{TW_1}$  or  $p_{TW_1}$ . How does this sound? I also had never been careful about the transpose signs when I wrote vector elements in parentheses, we probably need to add  $^T$ s to several places.

$p_{TW_1} = (0, -\mu_2/b_2, 0, 0)$  is the relative equilibrium in invariant polynomial basis because  $v = |z_2|^2 = -\mu_2/b_2$  is a relative equilibrium.

**2014-05-28 Predrag:** Burak's notation sounds good to me.

**2014-05-25 Evangelos:** The return map with  $\overline{010}$  in Fig. 5(d) is very interesting. I now understand your discussion above about pruning. A suggestion that might, or might not work, is to use the unstable manifold of cycle  $\overline{010}$  to construct a new return map. Since this cycle seems to lie on the “border” of the attractor, one might hope to get a return map with a different turning point, leading to symbolic dynamics that are easier to deal with.

**2014-05-25 Evangelos:** Regarding the cusp in Fig. 5(d), is it related to an heteroclinic connection from  $TW_1$  to the origin, or is it a heteroclinic connection to cycle? If I had to guess based on Fig. 5(a), I would say it's the former.

**2014-05-25 Predrag:** The heteroclinic connection is from the critical value  $f(s_C) \rightarrow s_{TW_1}$ , I believe.

There is no need to look at any periodic orbit unstable manifolds, as Burak is born under a lucky star. It just so happens (for the system parameter values as they are) that anything bracketed by the 3-cycle periodic points open interval  $(s_{001}, s_{100})$  follows the golden mean symbolic dynamics, with a single pruned block  $_00_0$ . Cycle  $\overline{001}$  is the only orbit that can have two consecutive 0's, for anything that starts to the right of  $s_{001}$  has to have itinerary  $.01\dots$ . I think that is clear from Fig. 5(d): our dike map is obtained by replacing all values  $f(s) > s_{001}$  by a flat plateau  $\bar{f}(s) = s_{001}$ . We are justified in doing this bit of surgery, because the slope of the map within this interval is very large, and consequently contributions of those cycles is vanishingly small. An added bonus - we cannot compute the cycles within this interval anyway, they are too unstable. ChaosBook has golden mean examples worked out, so now one can find all cycles of a given length, stick them into cycle expansions - on the level of symbolic dynamics shadowing is exact - and see how the escape rate, the Lyapunov exponent and the diffusion coefficient converge. Should be like a ton of rocks.

**2014-05-26 Daniel:** So speaking of lucky star parameter values... Do we want to confess how these were arrived at? I can write it up if necessary OR we can just say “empirically arrived at” or “by numerical experimentation”. Just let me know.

**2014-05-26 Burak:** I think “empirically arrived” is good enough, however, we may want to use your Lyapunov exponent subroutine to compare the Lyapunov exponent prediction from periodic orbit theory.

**2014-05-25 Burak:** I have one question: Do we have an algorithm for generating prime periodic orbit itineraries up to a certain length for unimodal maps? What I basically want to do is to generate prime periodic orbit symbol

sequences up to a certain length, compute their maximum topological coordinates and than compare it to the kneading value. I know that there is a table in Ref. 8 with all prime periodic orbits but I'd prefer to learn how to generate them if there is a smart way of doing it.

**2014-05-25 Predrag:** I've written codes for generating prime cycles number of times. Golden mean is especially easy, as you can turn it into complete binary (explained in ChaosBook) by defining  $a = 01$ ,  $b = 1$ , and remembering that  $n_p = 2n_a + n_b$  when you convert back into  $\{0, 1\}$  code. Problem is that most of my programs seem to be written in HP Basic or Pascal, so I cannot even look at them from my laptop:) I probably have printouts in my office, so might be easy to retype them in python. I've put some mathematica programs into DropBox Physics/Burak/, check whether any of them do the job (cannot open them on my laptop either). Ask Tingnan how he generates prime cycles for the Lorentz gas, he might have something workable (it's c++).

**2014-05-26 Evangelos:** There is some Mathematica code that I have written for complete binary symbolic dynamics in repository `vaggelis/mathematica/trunk/unimodal.nb`, so by Predrag's substitution above, it should also work for you. It uses Duval's algorithm (see references at appendix in my thesis) to generate the symbolic sequences. I suspect it is not the smartest way to do it, and mathematica implementation is slow, but it did the work up to length ten or eleven for CLE. There is even a routine that will do multiple shooting given an approximation to your return map and its derivative (e.g. an interpolating function object in Mathematica). There is a self-generated mathematica package called `unimodal.m`, which you can load from another notebook. The documentation is not excellent, so let me know if you have questions. An example usage is given in `vaggelis/testing/flows/CLEfinalTmp.nb`, but this notebook is a mesh.

**2014-05-26 Burak:** Thanks! It took me a little while to understand why a Lyndon word corresponds to a prime cycle, [Wikipedia article](#) had the following illuminating sentence:

Being the singularly smallest rotation implies that a Lyndon word differs from any of its non-trivial rotations, and is therefore aperiodic.

The fact that it's the smallest among its cyclic rotations implies that it cannot be composed from the repetition of a smaller piece. I'm noting it here, so that I can find it when I forget why this method worked. Sorry about the slow progress, combinatorics is certainly not one of my strengths.

Thank you Evangelos, pseudocode in your thesis will definitely help.

**2014-06-10 Burak:** Table VI shows two-mode relative periodic orbits up to length, their periods, phase shifts after one period and the Floquet multipliers. I used Xiong's periodic Schur decomposition code<sup>49</sup> to compute the Floquet multipliers and input n Jacobian's for n-cycles. As you can see that the marginal eigenvalues are not exactly one, which implies that the insanely contracting eigenvalue is probably also inaccurate, nevertheless, most expanding one should be fine, and that's what we need for the calculations.

**2014-06-10 Predrag:** <sup>151</sup>Shadowing cycle-by-cycle: Started construction of golden-mean curvatures in table VII. The table is obtained from the binary table in the [ChaosBook](#) by substitution  $0 \rightarrow 01$ . The curvatures should be small <sup>153</sup> (see, for example, Ref. 74, table 1), but maybe not all that small, thanks to the singularity that might unbalance the cancelations... I'm also unsure whether to include  $t_{001}$  or not. The table peters out above  $n > 8$ ; someplace I have counting for the golden mean prime cycles, but where?

**2014-06-18 Burak:** Before moving forward, I wanted to blog about what I did lately. After not seeing nice convergence with cycles upto topological length 8, I computed the ones with topological lengths upto 10. Now there are 36 of them, listed in table VI. To compute Floquet multipliers accurately, I used Xiong's periodic schur decomposition code with  $n \times 100$  subdivisions of the trajectory for an  $n$ -cycle, whole computation took approximately a day. After having these, I computed the dynamical zeta function and the spectral determinant. Procedure for dynamical zeta function was described in Ref. 8, but for the spectral determinant, I used something slightly different than what was in the book. So I'm writing it here for future use: Take the following definition of the spectral determinant with the order-tracking parameter  $z$  to be set to 1 at the end of calculations:

$$\det(s - \mathcal{A}) = \exp \left( - \sum_p \sum_{r=1}^{\infty} \frac{1}{r} \frac{e^{r(\beta A_p - s T_p)} z^{n_p r}}{|\det(\mathbf{1} - M_p^r)|} \right) = \prod_p \exp \left( - \sum_{r=1}^{\infty} \frac{1}{r} \frac{e^{r(\beta A_p - s T_p)} z^{n_p r}}{|\det(\mathbf{1} - M_p^r)|} \right). \quad (\text{G65})$$

<sup>152</sup> 2CB

<sup>154</sup> PC: 2014-06-10 include AACII table in the ChaosBook.org

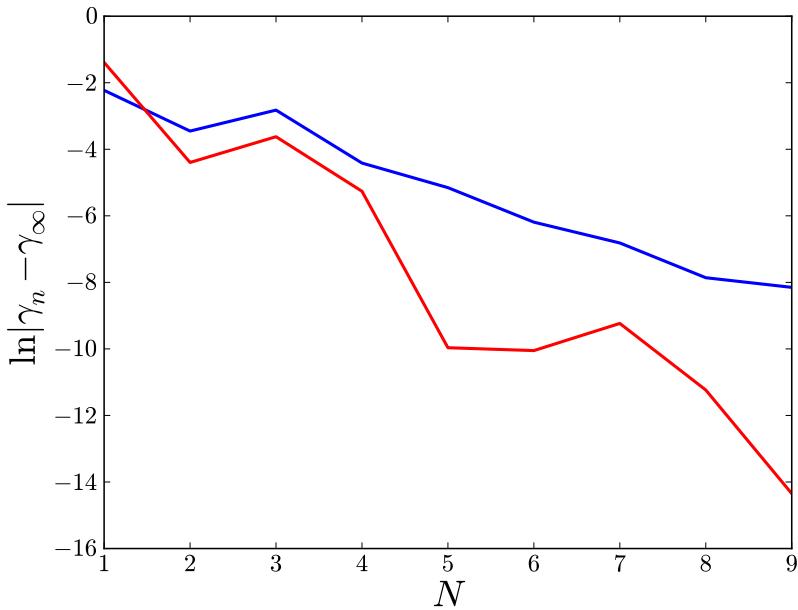


FIG. 78. Convergence of the escape rate ( $\gamma$ ) estimated from the dynamical zeta function (blue) and the spectral determinant (red).

Now for an  $N$ -th order approximation to this determinant, we compute the sum that appears inside the exponential to the upto  $N^{th}$  order in  $z$  for  $p^{th}$  cycle, as follows:

$$\begin{aligned} -\sum_{r=1}^{\infty} \frac{1}{r} \frac{e^{r(\beta A_p - s T_p)} z^{n_p r}}{|\det(1 - M_p^r)|} &\approx -\sum_{r=1}^{rn_p \leq N} \frac{1}{r} \frac{e^{r(\beta A_p - s T_p)} z^{n_p r}}{|\det(1 - M_p^r)|} \\ &= c_1(\beta, s) z^{n_p} + c_2(\beta, s) z^{2n_p} + \dots + c_R(\beta, s) z^{R n_p} \quad (\text{G66}) \\ \text{where, } R n_p &\leq N < (R+1)n_p \end{aligned}$$

For each cycle we expand the exponential of (G66) upto  $N - th$  order in  $z$ :

$$\det(s - \mathcal{A})^{(p)} = 1 + b_1(\beta, s)z^1 + b_2(\beta, s)z^2 + \dots + b_N(\beta, s)z^N. \quad (\text{G67})$$

Finally, we compute the spectral determinant iteratively, droping terms with order greater than  $N$  after each step, as follows:

$$\det(s - \mathcal{A})_{(i+1)} = \det(s - \mathcal{A})_{(i)} \det(s - \mathcal{A})^{(i+1)} \quad \text{where, } \det(s - \mathcal{A})_{(1)} = \det(s - \mathcal{A})^{(1)}. \quad (\text{G68})$$

Where we denoted the contributions from the individual cycles (G67) with superscripts and the refined spectral determinant after the contribution of the  $(i+1)^{th}$  cycle with subscript  $(i+1)$ . Let us denote the  $N^{th}$  order contribution to the spectral determinant after setting  $z = 1$ , following Ref. 8 as follows:

$$F_N(s, \beta) = 1 - \sum_{n=1}^N Q_n(s, \beta). \quad (\text{G69})$$

In order to compute the escape rate (hoping to find something close to 0), I looked for the leading zero of (G69) with  $\beta = 0$ , namely, solved  $F_N(s_0, 0) = 0$  close to the origin. Leading zeros of the spectral determinant, with approximations upto order 10 are listed in Table VIII . While the final numbers seem to be converging with 6-digit accuracy, something strange happens when  $N = 7$ . In order to visualize this, I plotted the errors of the  $N^{th}$  order estimates to the escape rate,  $\gamma = -s_0$  relative to its  $10^{th}$  order estimate in Fig. 78. As you can see from Fig. 78, while the estimates from spectral determinant converges with much higher accuracy than those from the dynamical zeta function, dynamical zeta function estimates seem to have a better convergence behavior.

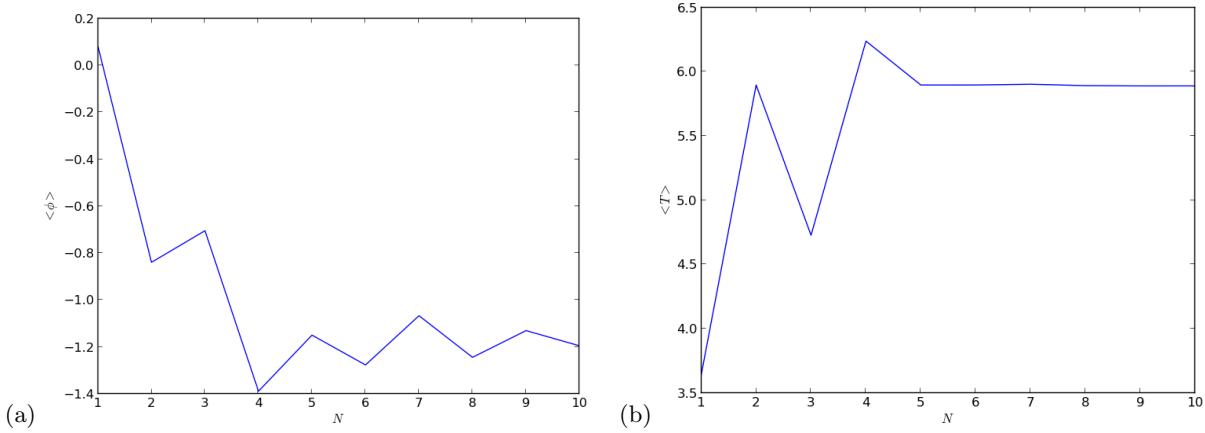


FIG. 79. (a) Cycle expansion estimate of the  $\langle \phi \rangle$  with respect to the expansion order. (b) Cycle expansion estimate of the  $\langle T \rangle$  with respect to the expansion order.

Jump in the error when  $N = 7$  (see the red curve in Fig. 78) is worrisome, and hopefully is due to something being wrong. Tomorrow I'll check the individual shadowing terms that could be the source of this error and also compute other observables such as the mean phase speed. I think I'm close to finishing the results for this paper, had been harder than I expected.

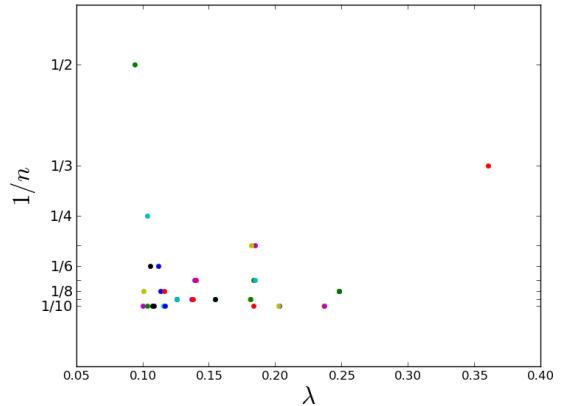


FIG. 80. Distribution of Floquet exponents with respect to  $1/n$  where  $n$  is the topological length of the cycles.

**2014-06-20 Burak:** Fig. 79 shows the mean phase drift  $\langle \phi \rangle$  per return time (a) and the mean return time  $\langle T \rangle$  (b). I didn't do any extrapolation on these yet, but their ratio is on the same order with average phase velocity from the numerical experiment. Return time converges with 3 digit accuracy, whereas the phase shift converges with 1 digit accuracy. I found a mistake in my implementation, and then fixed it, now I believe that my calculation is correct. Note that first order estimates are simply the period and the phase shift of  $\bar{I}$ .

Fig. 80 is a visualization of the distribution of Floquet exponents (see table VI) with respect to the cycle length. We see that as the symbol 1 repeats, Floquet exponents of the cycles converges to that of the  $\bar{I}$ . Another thing to note on this plot is the large Floquet exponent of the only length-3 cycle,  $\bar{0}1\bar{1}$ , which causes the shadowing effects kick in much slower than we expected.

**2014-06-23 Daniel:** At Burak's request I have used my Lyapunov exponent code based on Wolf et al.<sup>45</sup> to calculate the leading Lyapunov exponent for the two-mode system with  $\mu_1 = -2.8, \mu_2 = 1, a_1 = -1, a_2 = -2.66, b_1 = 0, b_2 = 0, c_1 = -7.75, c_2 = 1, e_1 = 0$ , and  $e_2 = 1$ . I did my calculation in invariant polynomial coordinates. Since this method requires an initial conditions to start the calculation, I started with 16 different initial conditions in the range  $u, v \in [0, 3]$  and  $\psi \in [0\pi]$ . I got a mean value of  $\lambda_1 = 0.1174$  with standard error of  $\pm 0.0023$  (95% confidence level), which seems to agree quite nicely with Burak's calculation of 0.118268 from cycle expansions. I'll run some more trials later tonight and report back with better statistics.

**2014-06-23 Burak:** Thanks! My averages from cycle expansions are in table IX. I computed escape rate, mean period, Lyapunov exponent, mean phase velocity and the diffusion coefficient. While I got 6 digit convergence

for the Lyapunov exponent, diffusion coefficient seems to converge only with single digit. Even and odd expansion orders seem to follow individual trends. I'm currently running the Newton solver to find relative periodic orbits with  $n = 12$  also improving accuracy of the Floquet multipliers. I think, however, this is about as good as it gets. I'm writing the formulas that I used to compute averages here so that Predrag can check and confirm that I didn't do anything criminal:

Mean period:

$$\langle T \rangle_N = \left. \frac{\partial F_N}{\partial s} \right|_{\beta=0, s=s_0}, \quad (\text{G70})$$

with,  $F_N$  given by (G69).

Lyapunov exponent and mean phase velocity computed by the same formula:

$$\langle a \rangle_N = -\frac{1}{\langle T \rangle_N} \left. \frac{\partial F_N}{\partial \beta} \right|_{\beta=0, s=s_0}, \quad (\text{G71})$$

with,  $A_p = \phi_p, \ln \Lambda_p$  respectively for the phase velocity and the mean phase velocity and the Lyapunov exponent. Variance in the phase shift:

$$\langle \sigma_\phi^2 \rangle_N = \langle (\phi - \langle \phi \rangle)^2 \rangle_N = -\left. \frac{\partial^2 F_N}{\partial \beta^2} \right|_{\beta=0, s=s_0}. \quad (\text{G72})$$

NOTE: This formula differs from the one in the ChaosBook (current stable version) formula (20.22) by a minus sign and I believe the minus sign should be there for two reasons: 1) I get negative variances if I don't have it. 2) In the spectral determinant (G65), observables measured over the cycles ( $A_p$ ) appear in the exponential with positive signs, so taking a second partial derivative with respect to  $\beta$  should not kill the minus sign that is in (G71).

Finally, the diffusion coefficient:

$$D_N = \frac{1}{2} \frac{\langle \sigma_\phi^2 \rangle_N}{\langle T \rangle_N} \quad (\text{G73})$$

**2014-06-23 Daniel:** Ran the same calculations as above but for 100 different initial conditions. I got an ensemble average of  $\lambda_1 = 0.1198 \pm 0.0008$ . I guess that formally doesn't overlap with Burak's result (if the 6 digit accuracy claim is correct), but then again I used 1985's state of the art. I don't know that anyone can claim that this method is good beyond a couple of decimal places. Also, the  $\pm 0.0008$  is a standard error at the 95% confidence level ( $1.96\sigma/\sqrt{N}$ ). The standard deviation is actually more like 0.004. Hope this is useful... back to the thesis cave...

**2014-06-24 Burak:** There is still room for investigation, but slightly under estimating the Lyapunov exponent makes sense to me because we cut off the sharpest, thus the most unstable, piece of the return map to have finite grammar and good convergence. This probably is trade-off of this approximation.

**2014-06-24 Evangelos:** I have some naive questions in order to make sure I understand what is going on. In order to find your cycles you approximate the return map by a dike map, with golden mean symbolic dynamics, right? Right. This way you consider very unstable cycles with sequence '00' as pruned. However, for the cusp map, not all of these cycles are pruned, right? If we try to locate and include some of these cycles, would convergence of your cycle expansions improve or not? For instance you have '001', but as far as I understand it is not included in cycle expansion. What would happen if you include it?

**2014-06-24 Burak:** I tried including  $\overline{001}$  when I had much less cycles and it worsened the convergence. The motivation of approximating the map with a dike map with golden mean symbolic dynamics was the fact that golden mean symbolic dynamics have finite grammar hence all the terms after  $t_1$  and  $t_{01}$  in the cycle expansion to the zeta function are curvature corrections. So we hoped for a super exponential convergence after the third term, however, it turns out this was an overly optimistic expectation because the curvature corrections are far from being exact due to the high Floquet exponent of  $\overline{011}$  (see Table VI) hence whenever it appears in a shadowing combination that term cancels only a small portion of the respective fundamental cycle contribution, so the higher order terms continue to matter.

**2014-06-24 Evangelos:** The main reason I am asking is that, if I understand people correctly, it is not straightforward to compute averages for maps that have a cusp. To my understanding, if you try to simply use ensemble averages over trajectories, convergence can be poor because rare visits in the “pruned” region can have large influence since the values of observables may be very different in this region. From that perspective, the introduction of the dike map for averaging is very-very interesting.

**2014-06-24 Burak:** The alternative approach is to find as many cycles as possible, and then compute all the pseudocycles, and discard the ones that are above some certain stability threshold. This is called “Stability ordering” and I haven’t tried it yet. It’s not all clear to me how to implement this for the spectral determinant, but I can start with the zeta function and try to keep the same terms. So the answer to this question is, we don’t know yet whether the finite grammar approach does a better job in terms of convergence compared to the alternative.

**2014-06-24 Evangelos:** My other naive question is, to interepret escape rate, should we consider the map to be a repeller, with the flat-top of the dike map being the “open” part of the map? Then, this would seem to tell us how often we have “rare events”.

**2014-06-24 Burak:** I believe this is correct. And this is the reason that the escape rate is not exactly zero, but a small number.

**2014-06-24 Evangelos:** In (G71), is it  $A_p = \phi_p$  or  $A_p = \dot{\phi}_p$  for the phase velocity?

**2014-06-24 Burak:** That is  $\phi_p$ , and it’s important. I got confused last week about it and asked Predrag why I cannot define  $\phi_p/T_p$  as a measurable, and got immeadiately informed that I did not understand the periodic orbit theory [ES: Apparently, neither did I :-)]. The reason that we cannot average over the “phase velocities of individual cycles” is the necessity that the measurable must be additive along the orbit, so that we can define it’s infinitesimal generator in the derivation of trace formulas. That’s why we first compute the average phase shift per cycle, and then divide it by average cycle period.

**2014-06-24 Evangelos:** Something that confuses me is the following: does your cycle expansion use the reduced dynamical system on the slice only? Is there a differentiation between invariant and equivariant observables? For instance,  $\lambda_1$ ,  $\dot{\phi}$ , etc. should all be SO(2) invariant, but would it make sense to ask what  $\langle x_1 \rangle$  is? Also, in order to get  $\langle \phi \rangle$ , one uses the reconstruction equation, so it seems that some reference to the original system is implied?

**2014-06-24 Burak:** I use reduced system, within the slice. In general case the continuous symmetry reduced trace formulas have group characters described in [the paper that nobody understands](#), however, as far as I understood these terms doesn’t matter for these set of measurables. Heuristically, I think of it as all the quantities that I’m computing here translates identically to their values in the full state space so I don’t need a “correction” that cares about symmetry (I might be dead wrong). For example, Lyapunov exponent should be coordinate independent, so it totally makes sense to compute it directly within the slice. Phase shifts of relative periodic orbits are also quantities that I can compute within the slice so I can define that as a measurable along the reduced orbit since it fulfills the additivity requirement. Also since I don’t have any discrete-symmetry copies of these cycles so I don’t need to worry about that too.

**2014-06-26 Evangelos:** I agree with this. Actually, the reason I have trouble to understand [the paper that nobody understands](#), is that the invariant measure should be symmetry invariant, so I do not understand why we would need more than computing cycle expansions on the slice.

**2014-06-24 Burak:** I computed the all accessible relative periodic orbits with lengths upto  $n = 8$  including some with ‘00’ sequence and computed averages with all of them ordering the spectral determinant with respect to the topological lengths. I still am not clear about how to order terms in the Spectral determinant according to their stabilities. Results are in table X. Convergence-wise, it goes slightly slower than the finite grammar approximation, which is something we would expect. Notice that the escape rate is smaller than that of the table IX, which again makes sense because now we are tessellating the entire attractor without skipping regions. Another thing to note is that the estimate of the Lyapunov exponent in this case is  $\lambda = 0.1196$  which is consistent with Daniel’s numerical estimate and my guess of underestimating the Lyapunov exponent in the finite grammar case due to the fact that we always skip most unstable cycles. To me, it seems like the finite grammar approximation earns us 1 digit more accuracy while bringing systematic errors. And again, spectral determinant that produce results in table X is ordered in  $z^n$ , not in the pseudocycle stabilities, so the convergence of the sequences in table X can potentially be improved.

**2014-06-26 Evangelos:** Do you mean that you now have all cycles permitted by symbolic dynamics up to length 8? Why is there a jump in the escape rate at length 8 in table X? If finite grammar approximation converges faster towards a skewed result, then it's not very useful. Stability expansion might be a good idea for this problem. I have written a routine that implements escape rate calculation with stability expansion, when I was back at Georgia Tech. If you could tell me how to read multipliers of cycles from a file, I could try it, to see if it is worth going towards that direction.

**2014-06-26 Burak:** I added two sqlite3 databases:

`reducesymm/burak/2modes/data/rpo.db` and `reducesymm/burak/2modes/data/rpoall.db` in the repository. Both of them have a table named rpos in them, the content of which you can see by:

```
select * from rpos
```

Columns of the table rpos goes like this:

```
rpono (int) | itinerary (str) | x1 (real) | y1 (real) | x2 (real) | y2 (real) | ...
period (real) | phase (real) | FloquetMults (str)
```

Floquet multipliers are strings like this:

```
'[-1.48372354e+00 1.00000001e+00 9.99999640e-01 -8.99728179e-10]'
```

I now it looks terrible, but I found it convenient to work like this on python because there is a build in function that converts the this type of strings directly to the numpy array.

For example if you run something like

```
for rpono = 1:127
itinerary, x1, y1, x2, y2, T, phi, Lambda = Query('select * from rpos where rpono = rpono')
```

You can get all the rpos.

The difference between `rpo.db` and `rpoall.db` is that the former has golden mean pruned rpos, whereas the latter has all periodic orbits upto topological length 12.

Are you going to use the dynamical zeta function or the spectral determinant? If you know how to stability order the spectral determinant can you explain? I'm really confused about that.

Here is a [stackoverflow page](#) about how to open sqlite dbs in Mathematica. I'm sure you can do it on Matlab too.

**2014-07-01 Evangelos:** Thanks for the database files. I tried the following in ipython:

```
import sqlite3
select * from rpos
```

I get the following complaint:

```
from rpos select *
^
SyntaxError: invalid syntax
```

I will try to export a table with all multipliers from python, then import it in mathematica. Did I not import some module, or did I skip a test?

I will use the dynamical zeta function. I do not know how to do it with the spectral determinant.

**2014-07-01 Burak:** I just wrote a little python script for you so that you can read the rpos into a python list, in general you need a cursor object to send queries to the db, so it goes like:

```

import sqlite3
conn = sqlite3.connect('data/rpo.db')
c = conn.cursor()
c.execute("SELECT * FROM rpos WHERE rpono = "+str(rpono))
a = c.fetchall()
conn.close()

```

but you don't need to worry about these. In `reducesymm/burak/2modes` if you run

```
ipython -i ReadRpoDb.py
```

you will have an interactive ipython session with a list named `rpos` whose columns are:

```
[TopologicalLength, Period, ExpandingFloquetMultiplier, PhaseShift]
```

hope this helps.

**2014-07-14 Evangelos:** Thanks, it helps!

**2014-06-24 Burak:** I updated table X and figures Fig. 5, Fig. ?? and Fig. ??, I'll start writing results section tonight. I commented out the second Fourier mode slice figure, and I think we can also discard the second mode slice in the text too, since it is not essential to this paper. Since we are presenting dynamical averages computed periodic orbit theory, I think we should change the introduction and abstract a little bit. I'm also thinking that we can move the bivariate polynomials to the appendix because that section currently looks like a detour from the main idea to me. What do you think?

**2014-06-24 Evangelos:** I agree with moving the bivariate polynomials to the appendix. You may also want to add your bifurcation figures in an appendix. People will find this information interesting.

Regarding second Fourier mode slice figure, I would think it's not a bad idea to keep it in the text. The reason is that it's better to show that what you have can be easily generalized.

**2014-07-02 Burak:** Finished first writing of the results section. There is a lot of room for improvement, I should probably add more results, extrapolations etc. but as an outline, I think this can work. However, I think we should re-write most of the introduction and abstract, emphasizing that this is a demonstration of the periodic orbit theory that can, in principle, be applied to Naiver-Stokes and such.

**Burak to Predrag:** I tried to write a brief review of the trace formulas and cycle expansions in order to motivate the finite grammar approximation, but I only cited ChaosBook in that part, should I cite the individual papers for each concept? If so, can you help me with that?

**Burak to Daniel:** I mention the numerical estimate of the Lyapunov exponent at the last paragraph before the conclusions, can you add the name of the method and the reference there?

**2014-07-06 Burak:** I rewrote the abstract, please read it critically and change it as you find appropriate. I'll reorganize the introductory sections a little bit, so that we can emphasize that this study is a prototype for what one should ultimately apply to turbulence. I wrote the abstract with this in mind, and I think we will have a better-motivated paper this way.

**2014-07-08 Burak:** I tried to distribute the slice references in the method of slices section, feel free to make changes. I think I'll go back to improving results now.

**2014-07-14 Evangelos:** I am probably missing something important here, or I have forgotten everything, so please be patient with me. I do not understand why you get different results for escape rate for  $N = 1$  in table X and table IX? At this level you only use the first cycle, so I would expect to get the same answer, irrespective of how you approximate the grammar.

I am asking because I have great doubts about my stability ordered evaluation of escape rate using the inverse  $\zeta$  function, so I was trying to see what I get using only one cycle. Using only the first cycle  $\bar{1}$ , I get the following estimate for the escape rate from the inverse  $\zeta$  function:

$$\gamma \simeq 1 - 1/|\Lambda_{\bar{1}}| \simeq 0.326 \quad (G74)$$

which does not depend on how we approximate the grammar. What do I miss?

**2014-07-14 Burak:** You are right,  $N = 1$  results in table X and table IX must be the same, and they are close to each other but differ at the fifth decimal place. It is completely my fault, and result of my implementation. 'Finite grammar' and 'all rpos' are two different databases and I probably used different multiple shooting subdivisions for each case, so Floquet multiplier of  $\bar{1}$  in two cases differ by a small amount, causing the apparent discrepancy.

**2014-07-15 Evangelos:** This could explain it, but when I look to first cycle Floquet multiplier, I get exactly the same number from both databases. Any other possible explanation? By the way, all the cycles have only one unstable eigendirection, right? [2014-07-18: Burak found a bug and fixed this.]

**2014-07-14 Burak:** The quantity that you are looking at in (G74) is not the escape rate, but it's the flow conservation rule that should be close to zero if periodic orbits cover the attractor well. You implicitly assume that the escape rate is '0' and plug that in the zeta function, and expect the zeta function to vanish for 0 escape rate, and write that conservation rule. However, for a particular order in expansion, escape rate is found by solving  $\zeta(\beta = 0, s_0) = 0$ , where  $s_0$  is the leading 0 of the dynamical zeta function; then the escape rate is  $\gamma = -s_0$ .

Here are numbers I get when I compute the flow conservation rule for the zeta function (finite grammar Falsch! all RPOs) in order upto 5 in the expansion:

$$\begin{aligned} \zeta(0, 0)^{-1} = & 0.326019994263891, -0.173842965806291, 0.126663646598951, \\ & -0.0681237161617973, 0.0293660939936478, \dots \end{aligned} \quad (\text{G75})$$

so our initial numbers are same. For the escape rate, I get the following:

$$\begin{aligned} \gamma = & 0.10834917226127472, -0.031018879366355036, 0.04607247616487041, \\ & -0.012530861909001791, 0.007462614819364516, \dots \end{aligned} \quad (\text{G76})$$

These numbers are not the same with those in table IX, but I believe it's normal because even at the first order in expansion, we approximate the denominator  $|1 - \Lambda|$  in the spectral determinant by  $|\Lambda|$  and this approximation is simply wrong for  $\bar{1}$ , since  $\Lambda_{\bar{1}} \approx -1.5$ .

**2014-07-15 Evangelos:** You are right, I assumed that escape rate should be zero and computed the flow conservation rule. I now see why I cannot compare with escape rate directly at  $N=1$ , thanks. (by the way, shouldn't it be  $\zeta(0, 0)^{-1}$  above? fixed it)

**2014-07-15 Evangelos:** Are you sure that the results above are for the finite grammar approximation? When I use 'all rpos', I get

$$\begin{aligned} \zeta(0, 0)^{-1} = & 0.326019994263891, -0.173842965806291, 0.126663646598951, \\ & -0.09265054161752, -0.16116463641532, \dots \end{aligned} \quad (\text{G77})$$

i.e., the same numbers up to  $N = 3$  (after you discard the first cycle).

When I use the 'finite grammar' database, I get:

$$\begin{aligned} \zeta(0, 0)^{-1} = & 0.326019994263891, -0.173842965806291, 0.144591718961083, \\ & -0.074722469255388, -0.143236564053192, \dots \end{aligned} \quad (\text{G78})$$

In this case, we differ already at  $N = 3$ , the opposite of what I would expect.

If I understand your database 'all rpos' correctly, the first missing rpos (compared to complete binary) appears at topological length four, right? Is this because multiple shooting did not converge, or do you expect that the missing cycles are really forbidden?

**2014-07-15 Burak:** You're right, I used 'all rpos' database for the numbers above, sorry for the confusion. The difference between two sets starts at the topological length  $n = 3$ , since in the finite grammar approximation, we discard  $\bar{001}$ .

**2014-07-14 Burak:** I made changes in the outline and added O(2) discussion in the text, and then something weird happened: `2modes.tex` stopped compiling and started producing errors:

```
! Output loop---100 consecutive dead cycles.
```

inconsistently. I cannot find what causes this, also if I turn flag on, it compiles, so there shouldn't be anything wrong with the O(2) section. If I comment out different part of text, it sometimes compiles sometimes does not, it might have something to do with the bibtex, but I could not find it. Also google didn't help. It might as well be that something weird going on with `light.physics`. It behaves awkwardly in the last few days (for some reason it started to log me out whenever I started chrome, although I didn't change any system settings). I just commented the O(2) subsection out, so that it compiles, but it's at the end of invariant polynomial subsection (III.1) in `flow.tex`. I'll try to make it work from home.

**2014-07-18 Burak to co-authors:** I re-implemented spectral determinants in Mathematica and found a bug in my previous code while doing so. After fixing it, everything converged. I updated results tables in the text. You can see that finite grammar approximation gives way much better convergence compared to the 'all relative periodic orbits' case. With this result, I think we are done with this paper and we only need to fix text and possibly figures. We might add a separate discussion section and we should expand conclusions. Please read the paper critically and fix the errors you see. If you find a problem with a figure, let me know, I probably need to fix some ticks, labels and such.

**2014-07-19 Evangelos:** Great! Now table X and table IX start differ at level 3, as they should! How about the calculation of the flow conservation sum rule? Do we now get the same results up to fifth order? I would be interested to check my stability ordered calculation, even if we do not need it anymore.

**2014-07-19 Evangelos:** The missing rpo '0011' which should have multiplier around 70, seems to be crucial for stability ordering, because not having it in the set implies one has to impose very low stability threshold. Would it be possible to find it? This would also make the comparison between table X and table IX more fair.

**2014-07-19 Evangelos:** I find the fast convergence of the finite grammar result very interesting (exciting, I would say). It shows that you can get a convergent average for a system that has rare events in a systematic way, and at the same time avoiding using orbits in the most unstable regime. In a sense, this is a systematic stability ordering method. The question is, would it be possible to also predict by how much we underestimate Lyapunov exponents, or overestimate escape rate, etc?

For the escape rate, the answer seems to be the ratio of the 'cut out' interval over the interval bounded by the three-cycles. The latter can be estimated from Fig. 5 to be around 1.3, but for the 'cut out' region we would have to look at Burak's data. The difference between the red and cyan lines is too small to read it off from the figure.

How could we get a similar estimate for the systematic error introduced by the finite grammar approximation in the Lyapunov exponent? Could we get an estimate based on the stability of the first excluded cycle?

**2014-07-19 Evangelos:** I am insanely busy preparing figures for a collaboration with experimentalists that cannot wait, but I will try to go back to editing the manuscript as soon as possible (and I hope to be able to go back to KS work soon).

**2014-07-22 Burak:** I went over the Section V and all the figures one more time, fixed the ticks, labels and such. There are still many parts that needs to be corrected/improved or re-written, but I think the outline won't change. So please work on it and make changes as necessary, ask me if anything is unclear. I'll stop working on this for a while, probably until I get some input from you, I hope we can send this off soon.

**2014-09-15 Burak:** I just finished revising this paper, and now it is 17 pages including appendices. I understand that you are not going to make any contributions to it in near future (although some proof reading would have been nice) as coauthors I at least expect you to read and make sure you understand and agree with the content of this paper. For example the continuous factorization of the classical trace is highly technical and it took me a serious amount of time to grasp that, please have a look at it and make sure that you are ok with that and the rest. I guess this is the ethical minimum for authoring an article.

**2014-09-15 Daniel:** Sorry it took so long, but it's been crazy busy here trying to figure out how to do the whole professor thing. I've been advising seniors on possible thesis projects and it's just been a giant time sink. Anyway, I'm back into at least editing mode.

**2014-11-15 Daniel:** So I modified the title yet again... The previous title was "Periodic orbit analysis of A system with continuous symmetry - a tutorial". I have changed it to "Periodic orbit analysis of systemS with continuous symmetry". Here's my rationale: First, it makes sense to advertise this as a paper that will help people looking at systems other than our two mode model, so the change from singular "a system" to "systems" makes sense to me. This is probably not controversial and I don't foresee you guys having issues with it. If you guys absolutely

hate “systems” rather than “a system”, I would have no objections to reverting it. The second part, which was removing “-a tutorial” from the title, might be a worthy of more discussion. I realize that Predrag just added it back. From the standpoint of the Director of CNS and a full professor this makes sense. His reputation is established and he’s not submitting CVs anywhere. So having a paper that evangelizes his approach to dealing with chaotic systems with a title that reflects this (i.e., “-a tutorial”) is fun and useful since it highlights the educational value of the paper. On the flipside, Burak and I are/will be submitting CVs to places that are looking for researchers. We don’t have a huge publication list. Most of these places will shun anything that has to do with (gasp) education! This means that any paper that has “-a tutorial” in the title will automatically get discounted as not a valuable scientific contribution and therefore not a useful feather in Burak and my hats. Without the “-a tutorial” they might actually have to read the paper to figure out whether it fits their definition of useful contribution, which they won’t and it’ll just get tacked onto the publication count. I think that even without the “-a tutorial” the paper will still attract those that are looking to apply periodic orbit analysis to systems with continuous symmetry. The ambiguity of the title in terms of “systems” (or “a system”) guarantees that, so Predrag’s goal of getting people to learn how to do it right is still satisfied. Anyway... those are my thoughts... I am definitely open to discuss this...

**2015-01-15 Burak:** Sorry that it took me a two months to respond, life is hard. Contrary to your assumption, I gave the ‘tutorial’ title to this paper with the exactly same intention of yours: to improve its impact. I disagree with you that the name ‘tutorial’ would lead people to ignore the paper, I believe the truth is the opposite. Take [this tutorial](#), for example, it’s cited 20k times, and it’s just a pedagogical review of a highly useful technique. Another experimental evidence which supports my point happened last week in dynamics days: [Ned Corron](#) looked at my poster and then the references, and asked whether this paper is published, and I told him that it’s on arXiv, and he said that he will check it out and he expressed his love for tutorials. So my point is, if the utility of a method is proven, than people tend to look for a pedagogical resource to actually learn the method and start to use it in their own research. I am hoping this paper to be the pedagogical resource, where people go to learn the first Fourier mode slice and cycle expansions for systems with continuous symmetries. But of course, for that to happen, we actually need to prove the strength of our method. Since you are away from us, let me give you a short summary on the current status in here: Our PRL on the first Fourier slice is very close to being published. Kimberly, Ashley and Predrag have found over 30 relative periodic orbits in the pipe flow using the first Fourier mode slice, and they are going to send these results as another PRL. Last month, I computed cycle averages for Kuramoto-Sivashinsky system and started writing another article on that calculation, which you can find it in [siminos/ksRecycled/tex/ksRecycled.tex](#). So the situation is very optimistic about the periodic orbit theory of turbulence. I hope you find my arguments convincing, we can continue discussing if you don’t.

**2015-1-9 Daniel:** Is the notation  $\mathcal{M}_{a(t)}$  for group orbit weird in Fig. 1? We use  $\mathcal{M}$  for state space, slices, and other manifolds... I guess technically, the group orbit is a manifold too.. hmm... anyway, something to think about.

**2014-01-15 Burak:** I believe Predrag replaced this expression with the group action on the state.

**2014-01-15 Burak:** To Predrag and Daniel: I think this paper is ready for journal submission. If you agree, let’s make a final reading of it and start the publication process. Since this is a rather long paper, we are likely to make major revisions on it through the publication process, and I think it would be nice to have third person opinions while making those.

**2014-01-16 Burak:** I did the painful  $\theta \rightarrow -\theta$  replacement everywhere, which I think made everything consistent. I went over other footnotes, and made minor changes to places that I found necessary.

Daniel, please have a final read (without scrutinizing :) ) and make edits as you see necessary. You have already gone through these 20 pages in great detail, so I think we’re in good shape, our message is clear. Predrag left the submission decision to us, so once I have your approval, I’ll produce the arXiv-v2 and start the submission process.

**2013-08-10 Predrag to Chaos Gang:** It’s not over until it is over.

Itinerary	$(x_{1,RPO}, y_{1,RPO}, x_{2,RPO}, y_{2,RPO})$	Period	Phase Shift	$\Lambda$	$\lambda$	$1/ \Lambda $
1	(0.43998334, 0.0, -1.49738474, 0.06662960)	3.64151210	0.08096967	-1.48372448	0.10834935	0.67397958
01	(0.43998241, 0.0, -1.68577388, 0.06639063)	7.34594139	-2.94647188	-2.00054924	0.09439522	0.49986273
011	(0.43998947, 0.0, -2.34810215, 0.06585569)	11.07958929	-2.50675931	54.16223330	0.36030071	0.01846305
0111	(0.44003834, 0.0, -1.76712578, 0.06630300)	14.67951785	-2.74691265	-4.55967094	0.10335833	0.21931407
01011	(0.43998513, 0.0, -1.62023898, 0.06646745)	18.39155389	-5.61529854	-30.00623420	0.18494387	0.03332641
01111	(0.43977723, 0.0, -2.10098129, 0.06601616)	18.38740977	-2.48213930	28.41881540	0.18202952	0.03518796
010111	(0.43999442, 0.0, -1.64621985, 0.06643622)	22.01106024	-5.69175989	10.27073510	0.10582401	0.09736401
011111	(0.43998417, 0.0, -1.83733973, 0.06623397)	21.99234432	-2.54998709	-11.74396350	0.11200895	0.08515013
0101011	(0.43990280, 0.0, -1.73119801, 0.06634107)	25.75817617	-8.47999896	113.88771800	0.18383340	0.00878058
0101111	(0.43999442, 0.0, -1.62417672, 0.06646262)	25.67681336	-5.51145860	-36.48795680	0.14008679	0.02740630
0110111	(0.44088525, 0.0, -2.22289861, 0.06592989)	25.75787503	-5.33628931	-116.42441700	0.18469079	0.00858926
0111111	(0.43997774, 0.0, -1.99986955, 0.06609251)	25.667555610	-2.37917849	35.35624260	0.13890980	0.02828355
01010111	(0.43998269, 0.0, -1.70559577, 0.06636855)	29.36551167	-8.63856554	-19.27058890	0.10075016	0.05189255
01011011	(0.44005044, 0.0, -1.61704862, 0.06647110)	29.46941737	-8.13146531	-1506.56257000	0.24831118	0.00066376
01011111	(0.43998217, 0.0, -1.63597560, 0.06644845)	29.30616862	-5.50220132	27.80123670	0.11346009	0.03596962
01101111	(0.44005062, 0.0, -2.33742440, 0.06586175)	29.46848392	-4.99188731	1486.68765000	0.24786840	0.00067264
01111111	(0.43999430, 0.0, -1.88149885, 0.06619307)	29.28887949	-2.36470458	-30.09311370	0.11623170	0.03323019
010101011	(0.44000230, 0.0, -1.66696833, 0.06641198)	33.09476023	-11.46667234	-166.20914700	0.15450322	0.00601652
010101111	(0.43996086, 0.0, -1.72766026, 0.06634465)	33.05136072	-8.41479347	92.81120120	0.13707658	0.01077456
010110111	(0.44007365, 0.0, -1.61754537, 0.06647040)	33.07862275	-8.24424279	399.15014200	0.18106369	0.00250532
010111011	(0.43990606, 0.0, -1.66510971, 0.06641448)	33.09338507	-8.32608749	166.21469300	0.15451065	0.00601632
010111111	(0.43998273, 0.0, -1.62697955, 0.06645927)	32.95790196	-5.37453849	-62.84938710	0.12563728	0.01591105
011011111	(0.44087344, 0.0, -2.27481442, 0.06589722)	33.07760203	-5.10213743	-403.94330800	0.18143016	0.00247559
011101111	(0.44004257, 0.0, -1.73844423, 0.06633287)	33.05137696	-5.26946723	-96.35442080	0.13821007	0.01037835
011111111	(0.43998392, 0.0, -1.95765574, 0.06612696)	32.94577133	-2.24146250	62.60039370	0.12556305	0.01597434
0101010111	(0.43999736, 0.0, -1.67590913, 0.06640175)	36.70762520	-11.58463895	39.77552470	0.10034024	0.02514109
0101011011	(0.43992459, 0.0, -1.73222144, 0.06633990)	36.83761797	-10.98833282	6088.34818000	0.23655525	0.00016425
0101011111	(0.43997720, 0.0, -1.71474111, 0.06635856)	36.66947945	-8.44364774	-51.76167420	0.10762765	0.01931931
0101101111	(0.44005148, 0.0, -1.61712758, 0.06647100)	36.75201081	-8.03575516	-1729.34670000	0.20285962	0.00057825
0101110111	(0.43998473, 0.0, -1.657533969, 0.06642298)	36.69492495	-8.43930577	-45.12565580	0.10381411	0.02216034
0101111011	(0.43999361, 0.0, -1.62052780, 0.06646707)	36.77924698	-8.09686122	-855.91668000	0.18358649	0.00116834
0101111111	(0.43998453, 0.0, -1.63247118, 0.06645264)	36.59342987	-5.32595338	69.13840240	0.11576150	0.01446374
0110110111	(0.43978957, 0.0, -2.34449530, 0.06585827)	36.83748357	-7.84575901	-6150.54388000	0.23683202	0.00016259
0110111111	(0.44064330, 0.0, -2.32543115, 0.06586732)	36.75083215	-4.89545111	1720.82316000	0.20273168	0.00058112
0111011111	(0.43997961, 0.0, -1.75349843, 0.06631728)	36.66788261	-5.29854939	54.17439520	0.10887480	0.01845890
0111111111	(0.43999020, 0.0, -1.90449494, 0.06617255)	36.57754043	-2.19024123	-72.70335370	0.11718632	0.01375452
01010101011	(0.43996709, 0.0, -1.69601735, 0.06637921)	40.44561988	-14.39294038	389.50254200	0.14747877	0.00256738
01010101111	(0.43999267, 0.0, -1.66823690, 0.06641056)	40.38317606	-11.38200421	-165.97030300	0.12658264	0.00602517
01010110111	(0.43990080, 0.0, -1.73151746, 0.06634074)	40.43854004	-11.20630155	-625.74706500	0.15922796	0.00159809
01010110111	(0.44003019, 0.0, -1.69692667, 0.06637797)	40.44491554	-11.25044458	-394.83423000	0.14781749	0.00253271
01010111111	(0.43998225, 0.0, -1.72463522, 0.06634783)	40.33145763	-8.29384012	137.29232200	0.12204152	0.00728373
01011010111	(0.44001105, 0.0, -1.61956523, 0.06646817)	40.40378832	-11.28560922	-377.31471800	0.14684464	0.00265031
01011011011	(0.44004398, 0.0, -1.61698546, 0.06647120)	40.54897301	-10.63840938	-81477.23790000	0.27887461	0.00001227
01011011111	(0.44006750, 0.0, -1.61736001, 0.06647065)	40.37592625	-8.04344046	1176.26465000	0.17510680	0.00085015
01011101111	(0.43998178, 0.0, -1.66387856, 0.06641563)	40.37951497	-8.24135150	168.58705700	0.12698152	0.00593165
01011110111	(0.43999066, 0.0, -1.62135755, 0.06646607)	40.37952614	-8.19479569	270.70814800	0.13870993	0.00369402
01011111011	(0.43996456, 0.0, -1.64278828, 0.06644040)	40.40095928	-8.14459922	382.75428600	0.14720921	0.00261264
01011111111	(0.43998708, 0.0, -1.62854249, 0.06645737)	40.23939599	-5.22352076	-124.72282700	0.11993455	0.00801778
01101101111	(0.43995834, 0.0, -2.34789364, 0.06585589)	40.54810071	-7.49870151	80470.81570000	0.27857408	0.00001243
01101110111	(0.44076723, 0.0, -2.24922917, 0.06591342)	40.43819622	-8.06486630	626.38020600	0.15925432	0.00159647
01101111111	(0.44072230, 0.0, -2.29515284, 0.06588520)	40.37474119	-4.90196362	-1184.45093000	0.17528371	0.00084427
01101111111	(0.43999751, 0.0, -1.74161976, 0.06632967)	40.33163974	-5.14811292	-143.02898700	0.12305593	0.00699159
01110101111	(0.44001018, 0.0, -2.07604806, 0.06603377)	40.37620587	-5.05219398	-279.96132400	0.13955376	0.00357192
01111111111	(0.43998357, 0.0, -1.93953645, 0.06614223)	40.22596328	-2.08977306	126.24217000	0.12027560	0.00792128

TABLE VI. relative periodic orbits of the two-mode system. Parameter values (22).

TABLE VII. The golden-mean curvature expansion up to length 8, listed in such a way that the sum of terms along the  $p$ th horizontal line is the curvature  $\hat{c}_p$  associated with a prime cycle  $p$ , or a combination of prime cycles such as the  $t_{100101} + t_{100110}$  pair.

$-t_1$
$-t_{01}$
$-t_{011} + t_{01}t_1$
$-t_{0111} + t_{011}t_1$
$-t_{01011} + t_{011}t_{01}$
$-t_{01111} + t_{0111}t_1$
$-t_{010111} + t_{01011}t_1 + t_{0111}t_{01} - t_1t_{011}t_{01}$
$-t_{011111} + t_{01111}t_1$
$-t_{0101011} + t_{01011}t_{01}$
$-t_{0101111} + t_{01111}t_{01} + t_{010111}t_1 - t_{01}t_{0111}t_1$
$-t_{0110111} + t_{0111}t_{011}$
$-t_{0111111} + t_{01111}t_1$
$-t_{01010111} + t_{010111}t_{01} + t_{0101011}t_1 - t_{01}t_{01011}t_1$
$-t_{01011011} + t_{01011}t_{011}$
$-t_{01011111} + t_{0101111}t_1 + t_{011111}t_{01} - t_{01}t_{01111}t_1$
$-t_{011010111} + t_{0110111}t_1 + t_{011111}t_{011} - t_1t_{01111}t_{011}$
$-t_{01111111} + t_{0111111}t_1$
$-t_{010101111} + t_{0101111}t_{01} + t_{01010111}t_1 - t_{01}t_{010111}t_1$
$-t_{0101011011} + t_{01011011}t_{01} + t_{0101011}t_{011} - t_{01}t_{01011}t_{011}$
$[...]$
$-t_{0101010111} + t_{01010111}t_{01} + t_{010101011}t_1 - t_{01}t_{0101011}t_1$
$-t_{01010101011} + t_{010101011}t_{01} + t_{01010101}t_{011} - t_{01}t_{010101}t_{011}$
$[...]$
$-t_{0101010101011} + t_{01010101011}t_{01}$
$-[...]$

$N$	$s_0$
1	-0.24982996
2	0.01159761
3	-0.02744631
4	0.00445552
5	-0.00068103
6	-0.00068490
7	-0.00063043
8	-0.00071487
9	-0.00072866
10	-0.00072807

TABLE VIII. Leading zero of the spectral determinant ( $\beta = 0$ ) computed using the finite grammar approximation

$N$	$\gamma$	$\langle T \rangle$	$\lambda$	$\langle \dot{\phi} \rangle$	$D$
1	0.249829963	3.6415122	0.10834917	0.0222352	-0.000000
2	-0.011597609	5.8967605	0.10302891	-0.1391709	0.143470
3	0.027446312	4.7271381	0.11849761	-0.1414933	0.168658
4	-0.004455525	6.2386572	0.10631066	-0.2141194	0.152201
5	0.000681027	5.8967424	0.11842700	-0.2120545	0.164757
6	0.000684898	5.8968762	0.11820050	-0.1986756	0.157124
7	0.000630426	5.9031596	0.11835159	-0.1997353	0.157345
8	0.000714870	5.8918832	0.11827581	-0.1982025	0.156001
9	0.000728657	5.8897511	0.11826873	-0.1982254	0.156091
10	0.000728070	5.8898549	0.11826788	-0.1982568	0.156217
11	0.000727891	5.8898903	0.11826778	-0.1982561	0.156218
12	0.000727889	5.8898908	0.11826780	-0.1982563	0.156220

TABLE IX. Cycle expansion estimates based on the golden mean approximation (69) to symbolic dynamics for the escape rate  $\gamma$ , average cycle period  $\langle T \rangle$ , Lyapunov exponent  $\lambda$ , average phase velocity  $\langle \dot{\phi} \rangle$  and the diffusion coefficient  $D$ , up to cycle length  $N$ .

$N$	$\gamma$	$\langle T \rangle$	$\lambda$	$\langle \dot{\phi} \rangle$	$D$
1	0.249829963	3.6415122	0.10834917	0.0222352	0.000000
2	-0.011597609	5.8967605	0.10302891	-0.1391709	0.143470
3	0.022614694	4.8899587	0.13055574	-0.1594782	0.190922
4	-0.006065601	6.2482261	0.11086469	-0.2191881	0.157668
5	0.000912644	5.7771642	0.11812034	-0.2128347	0.168337
6	0.000262099	5.8364534	0.11948918	-0.2007615	0.160662
7	0.000017707	5.8638210	0.12058951	-0.2021046	0.160364
8	0.000113284	5.8511045	0.12028459	-0.2006143	0.159233
9	0.000064082	5.8587350	0.12045664	-0.2006756	0.158234
10	0.000093124	5.8536181	0.12035185	-0.2007018	0.158811
11	0.000153085	5.8417694	0.12014700	-0.2004520	0.158255
12	0.000135887	5.8455331	0.12019940	-0.2005299	0.158465

TABLE X. Cycle expansion estimates of the escape rate  $\gamma$ , average cycle period  $\langle T \rangle$ , Lyapunov exponent  $\lambda$ , average phase velocity  $\langle \dot{\theta} \rangle$  and the diffusion coefficient  $D$  with respect to the expansion order  $N$ .