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Deterministic diffusion analysis through periodic orbits

The starting observation is in [DANA]: in the analysis of lifts on the cylinder of Anosov diffeomorphisms he claims that a positive diffusion emerges from a balance between closed orbits on the cylinder and unstable periodic accelerator modes. In a sense this leads to the picture that diffusion is determined by the relative populations of closed orbits and accelerating orbits: in the asymptotic limit this distribution should converge to a Gaussian distribution characterized by the diffusion coefficient D.

Probably the simplest context in which the problem of deterministic diffusion arises is that of circle maps [DETDIFF]: in this case diffusion sets in (for suitably chosen parameter values) when one consider a lift of the circle maps, so that the motion is in principle unbounded.

Take for instance a map like that in Fig. 1

Fig. 1: Overshooting lift of a circle map

In this case periodic orbits of the map on the circle contain either closed orbits for the lift or running orbits, to which we can attach a label σ_p which gives the number of box jumped (which can be either positive or negative) after which the orbit closes on the torus.

By taking a gaussian distribution one has that the probability of having jumped σ steps in n time units will be given by;

1 $$p_n(\sigma) \;=\; \frac{1}{\sqrt{4\pi Dn}}\, e^{\left(-\sigma^2/4Dn\right)} \tag{1.1}$$

The main problem into consideration of periodic orbit expansion in this context is that computing averages for dynamical systems is generally feasible if one has to deal with multiplicative weights associated with the trajectory, and obviously the factors σ_p to which we are interested do not have this property. This can be circumvented by using exponentials, so turning additive weights into multiplicative ones.

Once we use (1.1) it is fairly simple to observe that

2 $$\langle e^{\alpha\sigma}\rangle_n \;=\; e^{nD\alpha^2} \tag{1.2}$$

to compute the average we can start from the $(0,1)$ box (so we have not to subtract any initial box–labelling factor) and we can write

3 $$\langle e^{\alpha\sigma}\rangle_n \;=\; \int_0^1 dx \int_{-\infty}^{+\infty} dy\, \delta\left(y - f^n(x)\right)\, e^{\alpha[y]} \tag{1.3}$$

This is easily seen to receive contributions (with different weights) by all periodic orbits of the map on the circle and can be recast in the form

4 $$\langle e^{\alpha\sigma}\rangle_n \;=\; \sum_{x\in Fix\tilde{f^n}} \frac{e^{\sigma_x\alpha}}{|1-\Lambda_x|} \tag{1.4}$$

Where the subscript individuates the map on the circle.

As usual one can introduce a formal parameter z and observe that $\alpha^2 D = -\log z_c$ where z_c is the point where

$$\sum_1^\infty z^n \sum_{x\in Fix\tilde{f^n}} \frac{e^{\sigma_x\alpha}}{|1-\Lambda_x|}$$

diverges.

D is then determined by looking at the first zero of the zeta function

5 $$Z(z) \;=\; \prod_{m=0}^{\infty}\prod_{\{p\}} \left(1 - \frac{z^{n_p} e^{\sigma_p}}{\left|\Lambda_p\right| \Lambda_p^m}\right) \tag{1.5}$$

and one has the relation $z_c(\alpha) = e^{-\alpha^2 D}$. which in particular means

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$$D = -\frac{1}{2}\frac{\partial^2}{\partial\alpha^2} z_c(\alpha)\big|_{\alpha=0} \tag{1.6}$$

• It was after looking at your paper that I realized that this way of finding D does not need involvement of gaussian distribution, so it should be fairly general. This has been very stupid of me.

When $\alpha = 0$ one recovers the usual escape rate zeta function, whose first zero is obviously one, as the map on the circle does not lead to any escape.

Let's now consider a map like the one in Fig. 1, with $a = 2$ (that is the maximum of the map on the unit box is 2) [DETDIFF], and consider $\zeta_0^{-1}(z)$ instead of the whole zeta function.

7
$$\zeta_0^{-1}(z) = \prod_{\{p\}} \left(1 - \frac{z^{n_p} e^{\alpha\sigma_p}}{|\Lambda_p|}\right) \tag{1.7}$$

We can write explicitly this function as we have an unrestricted grammar in seven symbols (see Fig. 2).

Fig. 2: The map of Fig. 1 when $a = 2$

We have so, as curvatures cancels exactly due to completeness and linearity

$$\zeta_0^{-1}(z) = 1 - t_1 - t_2 - t_3 - t_+ - t_\oplus - t_- - t_\ominus$$

that is

$$\zeta_0^{-1}(z) = 1 - \frac{3}{7}z - \frac{4}{7}z\cosh\alpha$$

so that $z_c(\alpha) = \frac{7}{(3+4\cosh\alpha)}$, and by (1.6) we get $D = 2/7$ which is known to be the correct result [DETDIFF].

We can easily generalize this result to every integer value of a: as a matter of fact we get a complete grammar for each of these cases and linearity then allows exact cancellation of all curvatures:

$$\zeta_0^{-1}(z,a) \;=\; 1 - \frac{3}{4a-1}z - \frac{4}{4a-1}z\sum_{j=1}^{a-1}\cosh(j\alpha)$$

from which we get $z_c(\alpha,a) = (4a-1)/(3+\sum_{j=1}^{a-1}4\cosh(j\alpha))$. Again by (1.6) we can get

$$D(a) \;=\; \frac{a(a-1)(2a-1)}{[3(4a-1)]}$$

which is known to be the exact result [DETDIFF], where we have employed $\sum_{k=1}^{n}k^2 = n(n+1)(2n+1)/6$.

Observe that though the complete zeta function factorizes into $Z_{CO}(z)\cdot Z_{AM}(z)$, where CO refers to closed orbits and AM to accelerator modes, this factorization would spoil completely the resummation of the zeta function, $Z_{CO}(z)$ in particular is the zeta function describing escaping of initial conditions from the unit box.

Notes

1. Also in this simpler context many things again may be considered: non–linear case, asymmetric maps and inclusion of drift, maybe anomalous diffusion..

2. What do you think?

References

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