group theory

birdtracks, Lie's, and exceptional groups

a blog

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1.7 Quantum marginal problem

The marginal distribution of a subset of a collection of random variables is the probability distribution of the variables contained in the subset. It gives the probabilities of various values of the variables in the subset without reference to the values of the other variables.

Marginal variables are those variables in the subset of variables being retained. These concepts are "marginal" because they can be found by summing values in a table along rows or columns, and writing the sum in the margins of the table.

Quantum marginal problem (QMP) : Find a pure quantum state $|\psi\rangle$ given some marginals ρ_ℓ.

2023-01-02 Erik Aurell The quantum marginal problem: is there a globally pure state which has a set of reduced density matrices (in Fraser's notation): Let there be

(a) a joint Hilbert space $J = X_1 \otimes X_2 \otimes X_k$,

(b) a family of index sets $S_1, S_2, \cdots S_m$ with cardinalities $k_1, k_2, ...k_m$ labelling Hilbert spaces $X_{S_i} = X_{i,j_1} \otimes \cdots \otimes X_{i,j_{k_1}}$, and

(c) for each Hilbert space X_{S_i} a density matrix ρ_i.

Question: does there exist a pure state $|\psi\rangle$ on J such that the partial trace of $|\psi\rangle\langle\psi|$ over the Hilbert space complementary to X_{S_i} equals ρ_i? That is, in standard notation, is there a $|\psi\rangle$ such that $\mathrm{Tr}_{\,J\setminus S_i}[|\psi\rangle\langle\psi|] = \rho_i$ for each i?

I have a paper with Pawel Horodecki [11] where we use some results in this field.

This problem was solved about 10 years ago in the case when the marginals are disjoint (those are the results we used with Horodecki) by Alexander Klyachko [72, 73] *Quantum marginal problem and representations of the symmetric group*, arXiv:quant-ph/0409113; *Quantum marginal problem and N-representability*, arXiv:quant-ph/0511102. In this case only the eigenvalues of marginals ρ_ℓ matter. In principle. Relies on algebraic geometry, not straightforward to apply. One defines sets of marginals of compatible states, then checks whether they contain any pure states. Testing for purity is a nonlinear problem. Trick: take a convex hull of copies of compatible states. Pure states have traces equal to 1; this reduces the problem to study of *symmetric* products of the copies / separable states.

Aside: Klyachko was born in 1946, in Vladivostok.

Disjoint marginals mean questions like if there exists a pure state over N particles with given 1-particle reduced density matrices. There is no general solution for overlapping marginals. Overlapping marginals means questions like if there exists a pure state over N particles with a set of given 2-particle reduced density matrices, where the connectivity graph of particles linked by these given 2-particle reduced density matrices is large.

Thomas C. Fraser [49] *A sufficient family of necessary inequalities for the compatibility of quantum marginals*, arXiv:2211.00685, gives a general (but abstract) solution to QMP. Here is Thomas' (a graduate student) short video on this work:

Thomas Fraser : *Simultaneous Spectral Estimation and the Quantum Marginal Problem* (2021)

I have yet to decide if the solution in this paper is practical in any way. It is however clear that it uses properties of symmetric tensors a lot. There is an appendix explaining Predrag's birdtrack notation, with calculations (which I did not try to do yet). Predrag's book is cited.

Maybe something new that we could think of together?

2023-01-02 Predrag Fraser [49] *Appendix E: Diagrammatics* is taken directly from my *Group Theory* [37]. So is most of 'tensor networks' literature, which never cites me but cites Penrose instead, so Fraser is being very kind. The new and key operation, not in my book, is eq. (E15) for operations on 'multi-partite Hilbert spaces', i.e., multiple copies of a tensor.

Appendix F: The $n = 1$ *Case* I can follow step-by-step, but do not understand the significance of the results.

2023-01-03 Erik .

Otfried Gühne : *The Quantum Marginal Problem* (2022)

is up to 32 min at least partly about other (but related) problems. Of the papers that he is refering to I have read

Yu, Simnacher, Wyderka, Nguyen and Gühne [127] *A complete hierarchy for the pure state marginal problem in quantum mechanics* (2021)

but only blanced at

Huber, Gühne and Siewert [59] *Absolutely maximally entangled states of seven qubits do not exist* (2017)

Huber, Eltschka, Siewert and Gühne [58] *Bounds on absolutely maximally entangled states from shadow inequalities, and the quantum MacWilliams identity*, (2018)

I know other work by his postdoc Nguyen, who coauthored a very good classical stat mech review;

Nguyen, Zecchina and Berg [89] *Inverse statistical problems: From the inverse Ising problem to data science* (2017), arXiv:1702.01522.

At 32 min Gühne starts on the problem as I see it. The (unpublished) Klyachko [73] paper he refers to is a classic in the field, though almost incomprehensible (at least to me). It concerns the problem where the marginals are non-overlapping. In that setting there is a much easier later paper for Gaussian states:

Eisert, Tyc, Rudolph and Sanders [43] *Gaussian quantum marginal problem* (2008), arXiv:quant-ph/0703225.

At around 43minutes or so comes the construction of Gühne which is in the same ballpark as Fraser's. It considers more and more copies of the system, and then conditions on those N copies of the system such that the quantum marginal problem is solvable by a technique called semi-definite programming, standard in the field. I never bothered to learn exactly what it is because (1) there are already many others who know it very well and use it and (2) I am not convinced it is in the end so practical to consider these problems of N copies of the original problem, where N eventually tends to infinity. I may of course be wrong on this point (2). After 44 minutes I think Gühne again looks at a special case, maybe not generally useful.

The last part of the talk is I think is not quite up-to-date. The case "AME(4,6)" was solved in Suhail Ahmad Rather, Adam Burchardt, Wojciech Bruzda, Grzegorz Rajchel-Mieldzioć, Arul Lakshminarayan, Karol Życzkowski [100] *Thirty-six entangled officers of Euler: Quantum solution to a classically impossible problem* (2022).

Życzkowski was one of the co-authors of the list of open problems Gühne refers to.

2023-01-03 Erik What is the quantum marginal problem good for?. In the first part (A) I take QMP to mean the specific techniques of Fraser and Gühne. I think this should have many correspondences to topics both Gábor and I worked upon in the past. Gábor may have other input as well. In the second part (B) I take QMP to mean the physical problem I looked at in the paper with Horodecki [11], and provide a non-technical formulation.

A. The honest answer is that I do not know. There is a literature already for classical probability problems which uses somewhat similar techniques (semi-definite programming) to prove existence / lack of existence / of solutions to combinatorial optimization problems. A set of lectures in that line is

Boaz Barak and David Steurer 2016 *sumofsquares.org/public* course,

with focus on the Sum of Squares (SOS) semidefinite programming hierarchy.

Scott Kirkpatrick was very enthusiastic about these methods for a while. He and I spent at least some months trying to find out if these methods could actually give something for the problems we know something about, random k-satisfiability problem (KSAT), etc. As far as we could understand, at most in a quite convoluted way. Of course, that experience does not prove that the same methods do not give anything for the quantum marginal problem. Semi-definite programming is widely used for other tasks in quantum information theory.

Now, the striking thing about Fraser is that it uses birdtrack techniques in the proofs. Not many people know these techniques. Even I probably know them better than your average physicist-in-the-street. If we want to do something, the first step could be to try to think of the next problem in difficulty following Fraser, and then try to do that. If that works then one should have a much more concrete view of the techniques as they apply (or not) to the quantum marginal problem. Perhaps a zoom in a week or so?

B. Hawking's "black hole information problem" can be stated as follows: suppose a pure state of matter collapses to a black hole and then radiates away completely in Hawking radiation. If quantum mechanics holds everywhere, the final state of Hawking radiation is also pure. However, the state of Hawking radiation as computed by Hawking is thermal for every mode. Hence, there is a quantum marginal problem: is there a pure state of all the Hawking particles such that the marginal for each mode is thermal?

Two remarks: (a) by "mode" one should probably think of photons of a certain frequency emitted in a certain time interval. The reason is that Hawking's calculation assumes a stationary black hole as background. Over times which are long compared to the black hole lifetime, the Hawking temperature changes. (b) thermal states of photons are Gaussian. Hence one can think of a more restricted problem: is there a Gaussian pure state of all the Hawking particles such that the marginal for each mode is thermal? It turns out that there is, by a fairly elementary use of inequalities found by Eisert and collaborators [43]. That's the technical content of my paper with Horodecki [11].

Now, the physical problem would be something like the following: there has to be some quantum correlations in the Hawking radiation for it to be in a pure state, given that the one-mode marginals are thermal. How much of such quantum correlations? What is the least amount? That there is some is shown by the following argument due to Don Page (see Quanta Magazine): suppose one Hawking particle is emitted. Then by momentum conservation the black hole must recoil in the opposite direction. Therefore the position in space from where the next Hawking particle is emitted is entangled with the direction in which the first particle is emitted. And so on. According to Page this is a weak effect, and not sufficient to render all Hawking radiation pure. If that is correct or not I do not know.

2023-01-07 Erik Alessio Serafini (University College London, UK) 20 February 2020 lectures (with online videos), see lecture notes (2020), Chapter *1. Gaussian States*, fill many holes in the elementary part of the presentation of Eisert*et al.* [43] and led to the following:

Consider Serafini covariance matrix (CM) of the most general Gaussian state written in diagonalized form, with eigenvalue ν_j for each of the n 2-dimensional blocks $\hat{x}_j$, $\hat{p}_j$, his eq. (1.61). This is the most general form of a covariance matrix of a Gaussian state. S is a symplectic transformation acting on the density matrix. Consider two cases:

A. ν_j as in eq. (1.61) and $S = 1$. This would be the (mixed) state of Hawking radiation in an information loss scenario. One can identify $\eta_j = \coth(\omega_j \beta_j \hbar/2)$, where ω_j is the frequency of mode j, and β_j is the inverse temperature of mode j; it is the standard factor for quantum harmonic oscillator baths.

B. ν_j in eq. (1.61) all equal to 1, and some non-unity S. This would be the (pure) state of Hawking radiation in an information return scenario, under the additional (simplifying) assumption that this total state is Gaussian.

 1.7 *Quantum marginal problem*

Suppose the diagonal elements of the covariance matrix are the same in case **A** and **B**. In case **A** the off-diagonal elements are zero, but in **B** they do not have to be. One can then ask two questions:

1. Is this possible (is there such an S)? This depends on the values of the η_j. It is the problem solved by Eisert*et al.* [43]
2. If it is possible, how small can the off-diagonal elements be in case **B**?

Naively one would guess very small, and smaller the larger the dimension. Unless I get this very wrong, real symmetric covariance matrix of size $2n \times 2n$ and the symplectic matrix S both have $2n^2 + n$ elements. There are only n diagonal elements to set to unity. The freedom in the other parts of S can be used to smear out the changes in the off-diagonal elements so that they are all small. Is this a guess justified or unjustified? Any input and/or idea?

Serafini has a book on *Quantum Continuous Variables: A Primer of Theoretical Methods*. "[...] addresses the theory of Gaussian states, operations, and dynamics in great depth and breadth, through a novel approach that embraces both the Hilbert space and phase descriptions."

2023-01-07 Predrag Serafini online lectures:

Continuous-variable Quantum Information 1

is totally pedagogical. He starts with the Heisenberg commutator

$$[\hat{x}_j, \hat{p}_k] = i\delta_{jk}\,\mathbb{1}$$

with $2n$ continuous variables $\hat{x}_j$, $\hat{p}_j$, and that's how we get into a symplectic setting. Heisenberg commutators make antisymmetric parts proportional to $\mathbb{1}$, so the interesting part of the bilinear Hamiltonian is the symmetric part. He defines symplectic transformations generated by symplectic generator, uses them to bring the covariance matrix of the most general Gaussian state to diagonalized form, with eigenvalue ν_j for each of the n 2-dimensional blocks $\hat{x}_j$, $\hat{p}_j$, his eq. (1.61).

Looks like we can work through all that.

Continuous-variable Quantum Information 2

Continuous-variable Quantum Information 3

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