QFT and its discontents
a blog

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Contents

Chapter 1

Is QED finite?

For Emily, after she gets bored with "Baby Loves Quarks."

In 2017 Laporta [133, 134] completed the twenty-year project of computing analytically the individual contributions of 891 4-loop vertex diagrams contributing to the electron magnetic moment g. Vertex diagrams separate in 25 gauge sets (see figure **??**). The numerical contribution of each set is listed in table **??**. Adding only the quenched set V diagrams (diagrams with no lepton loops, see figures 1.5, 1.6 and 1.7), one finds for the 4- and 5-loop contributions to the anomaly $a[V] = \frac{1}{2}(g-2)\big|_V$:

$$
\begin{aligned}
a^{(8)}[V] &= -2.176866027739540077443259355895893938670 \\
&= -2.17\ldots \qquad \text{Aoyama } et\ al. \text{ 2012 [11]} \\
&= -2.569(237) \qquad \text{Kitano 2024 [122]} \\
&\approx 0 \qquad \text{Cvitanović 1977 [49]} \\
a^{(10)}[V] &= 6.782(113) \qquad \text{Volkov 2024 [194]} \\
&= 6.800(128) \qquad \text{Aoyama } et\ al. \text{ 2025 [10]} \\
&= 6.979(937) \qquad \text{Kitano 2024 [122]} \\
&\approx 3/2 \qquad \text{Cvitanović 1977 [49]}\,.
\end{aligned} \tag{1.1}
$$

There is a prediction dating back to 1977 for values of these terms: the predicted $a^{(8)}[V] \approx 0$ does not pan out, but the difference is small, considering that this is a sum of 518 vertex diagrams (or 47 self-energy diagrams) [119]. Likewise, the prediction for $a^{(10)}[V]$ is not too far off, considering that this is a sum of 6 354 vertex diagrams of table 1.1 (or 389 self-energy diagrams).

1.1 Electron magnetic moment

This section sets up the notation - the reader can safely skip it and start with sect. 1.2. For an introduction to the conventional magnetic moment calculation, see for example the Cvitanović online graduate QFT course, lectures 25 and 26 here.

An electron of mass m has a magnetic moment

$$\mu = \frac{e\hbar}{2mc}\,\frac{g}{2} \tag{1.2}$$

where g is the gyromagnetic ratio. In Dirac theory [63], the electron has $g = 2$.

Consider the electron-photon vertex Γ_μ of quantum electrodynamics, with $p_i = p - q/2$ and $p_o = p + q/2$ the momenta of incoming and outgoing electron lines, evaluated on the electron mass shell $p_i^2 = p_o^2 = m^2$. By Gordon decomposition the vertex can be written in terms of the Dirac and Pauli form factors $F_1(t)$ and $F_2(t)$:

$$\overline{u}(p_o)\Gamma_\mu(p,q)u(p_i) = \overline{u}(p_o)\left\{F_1(t)\,\gamma_\mu + \frac{F_2(t)}{2m}\,\sigma_{\mu\nu}q^\nu\right\}u(p_i)\,, \tag{1.3}$$

where $t = -q^2$, and the spinors $\overline{u}(p_i)$ and $u(p_i)$ satisfy the Dirac equation:

$$\overline{u}(p_o)\ \not{p}_o = m\,\overline{u}(p_o)\,, \qquad \not{p}_i\,u(p_i) = m\,u(p_i)\,.$$

We follow the notation of Bjorken and Drell [26] and Cvitanović and Kinoshita [58]. Z_1, Z_2, and Z_3, are the respectively the vertex, the electron wave function, and the photon wave function renormalization constants, and the electron mass will be set to $m = 1$ throughout. In what follows it is convenient to define $Z_1 = 1 + L$. For QED the charge conservation requires that the renormalized charge form factor satisfies $\tilde{F}_1(0) = 1$, which is guaranteed by the Ward identity [199] $Z_1 = Z_2$. The vertex renormalization constant L is given by the on-shell value of the unrenormalized charge form factor [34, 107] [1]

$$1 + L = F_1(0) = \frac{1}{4}\mathrm{tr}\,\left[(\not{p}+1)p^\nu\Gamma^\nu\right]_{q=0}\,, \tag{1.4}$$

and $a = (g-2)/2$, the anomalous magnetic moment of an electron is given by the static limit of the magnetic form factor $a = \tilde{F}_2(0) = M/(1+L)$, where [34]

$$M = \lim_{q\to 0}\frac{1}{4q^2}\mathrm{tr}\,\left\{\left[\gamma^\nu p^2 - (1+q^2/2)p^\nu\right](\not{p}_o+1)\Gamma_\nu(\not{p}_i+1)\right\}\,. \tag{1.5}$$

The perturbative expansions for the magnetic moment anomaly is defined as

$$a = \frac{M(\alpha_0)}{1+L(\alpha_0)} = \sum_{n=1}^{\infty} a_0^{(2n)}\left(\frac{\alpha_0}{\pi}\right)^n\,, \tag{1.6}$$

where $1 + L = F_1(0)$, $M = F_2(0)$ are computed from the unrenormalized proper vertex (1.3), given by the sum of all one-particle irreducible electron-electron-photon

[1] Predrag: 2018-11-26 Brodsky and Sullivan [34] is not an obvious reference - they just state the trace formulas, no derivation or attribution. Ishikawa, Nakazawa and Yasui [107] eq. (8) simply refers to "Cvitanović-Kinoshita procedure" [56, 57].

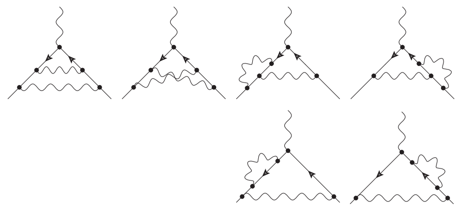


Figure 1.1: The two-loop vertex diagrams contributing to $a^{(4)}[V]$ magnetic moment anomaly. From Volkov [197].

vertex diagrams with internal photons, electron loops and electron mass counterterms. Expanding M and L we have

$$\begin{aligned} a_0^{(2)} &= M^{(2)} \\ a_0^{(4)} &= M^{(4)} - L^{(2)}M^{(2)} \\ a_0^{(6)} &= M^{(6)} - L^{(2)}M^{(4)} - (L^{(4)} - (L^{(2)})^2)M^{(2)} \end{aligned} \tag{1.7}$$

As shown in ref. [47], for the anomaly (1.6) expressed in terms of the unrenormalized coupling constant α_0, all $a_0^{(n)}$ are IR finite, for both QED and QCD. The UV finite expression for the anomaly (1.6) is obtained by the charge renormalization

$$\alpha = Z\alpha_0 \,, \qquad Z = \frac{Z_2}{Z_1} Z_3 \,, \tag{1.8}$$

where Z_1, Z_2, and Z_3 are computed as power series in the bare coupling constant α_0. For QED $Z_1 = Z_2$ by the Ward identity [199], and for QCD the Z_i's are related by the Taylor-Slavnov [181, 188] identities (1.8). The simple structure of (1.6) and (1.7) should simplify the worldline calculations of sect. 1.3.2.

The Dirac equation predicts that the magnetic moment of an electron of charge e and mass m is $\mu = e/2m$, i.e., in the absence of radiative corrections $\tilde{F}_1(0) = 1$ and $\tilde{F}_2(0) = 0$. In 1948 Schwinger [172] showed that in the leading, one-loop order in the fine structure constant α, the radiative corrections lead to the anomalous magnetic moment of form $\tilde{F}_2(0) = \alpha/2\pi + a^{(4)}\,(\alpha/\pi)^2 + \cdots$ (the result engraved on Schwinger's tombstone). These notes are about what to expect for the $(\alpha/\pi)^n$ term in this series.

1.2 Gauge sets

> Is there any method of computing the anomalous moment of the electron which, on first approximation, gives a fair approximation to the α term and a crude one to α^2; and when improved, increases the accuracy of the α^2 term, yielding a rough estimate to α^3 and beyond?
>
> — Feynman's challenge, 12th Solvay Conference [81]

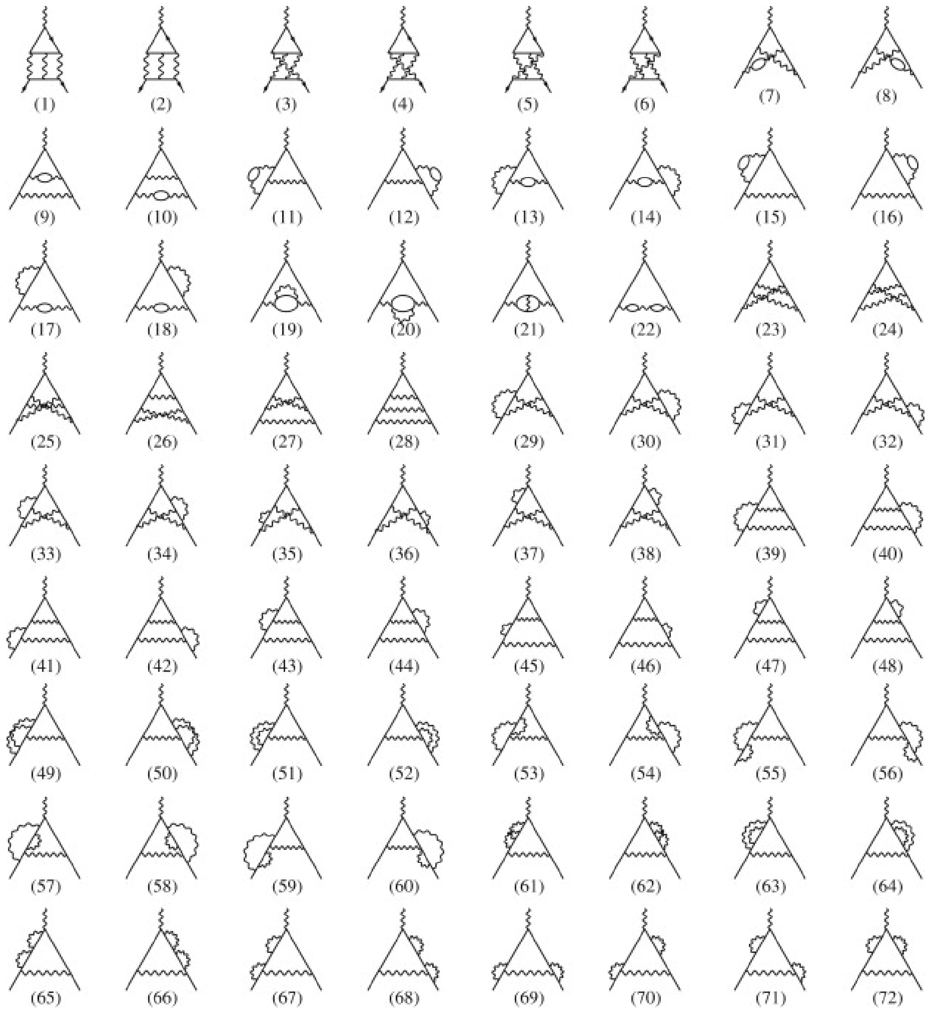


Figure 1.2: The three-loop vertex diagrams contributing to $a_1^{(6)}$ magnetic moment (from Jegerlehner and Nyffeler [109]). Lautrup *et al.* [139] were the first to note that subsets $(3, 0, 0) = \{23, 24, 25, 26, 27, 28\}$; $(2, 1, 0) = \{29, 31, 33, 35, 37, 39, 41, 43, 45, 47\}$ and its time-reversal $(2, 0, 1) = \{30, 32, 34, 36, 38, 40, 42, 44, 46, 48\}$; $(1, 2, 0) = \{49, 51, 53, 55, 57, 59, 61, 63, 65, 67\}$ and its time-reversal $(1, 0, 2) = \{50, 52, 54, 56, 58, 60, 62, 64, 66, 68\}$; and $(1, 1, 1) = \{69, 70, 71, 72\}$ are the minimal gauge sets, see figure 1.4.

Order $2n$	Vertex graphs Γ_{2n}	Gauge sets G_{2n}	Anomaly $a^{(2n)}$
2	1	1	1/2
4	6	2	0
6	50	4	1
8	518	6	0
10	6354	9	3/2
12	89 782	12	0
14	1 429 480	16	2

Table 1.1: Comparison of the number of quenched QED vertex diagrams (diagrams without fermion loops), gauge sets, and the gauge-set approximation (1.12) for the magnetic moment in $2n$th order. From ref. [49].

In 1972 Toichiro Kinoshita and Predrag Cvitanović had completed computing a large number of 3-loop anomalous magnetic moment Feynman diagrams and regularization counterterms [117], figure 1.2. The subsequent 4- and 5-loop numerical and analytic calculations were nothing short of heroic [12, 119, 133, 135, 193, 198]. The quantum field theory was used in the standard way [26], by expanding the magnetic moment into combinatorially many Feynman diagrams (see the numbers of vertex graphs in table 1.1). Each Feynman diagram corresponds to an integral in many dimensions, with oscillatory integrand with thousands of terms, each integral separately UV divergent, IR divergent, and unphysical, as its value depends on the definition of counterterms and the choice of gauge. The numerical values of these integrals typically range from ± 10 to ± 100. For example, the largest contributions of the 389 quenched self-energy diagrams listed in Aoyama *et al.* 2018 [14] are of order ± 20.

Adding up hundreds of such contributions, of wildly fluctuating values, yields (for the no-fermion loops subset V, in the notation of ref. [12])

$$a^{(6)}[V] \;=\; +(0.92 \pm 0.02)\left(\frac{\alpha}{\pi}\right)^3 .$$

But why "+" and not "-"? Why so small? Why does a sum of hundreds of diagrams and counterterms yield a number of order of unity, and not 10 or 100 or any other number?

If gauge invariance of QED guarantees that all UV and on-mass shell IR divergences cancel, could it be that it also enforces cancellations among the finite parts of contributions of different Feynman graphs?

1.2.1 QED vertex photons come in three "colors"

As first noted by Lautrup, Peterman and de Rafael [139], the renormalized on-mass shell QED vertex diagrams separate into a sum of minimal gauge-invariant subsets, each subset separately UV and IR finite. The only published proof of this elementary fact seems to be ref. [49]. The very reasonably priced ref. [51] might be worth a

 1.2 *Gauge sets*

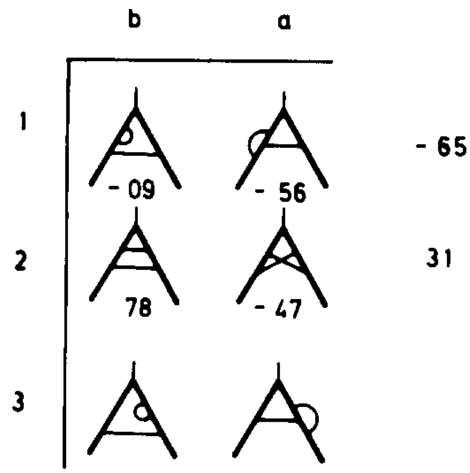


Figure 1.3: Rows: the fourth-order quenched QED Columns: external field insertions into the two self-energy sets. Rows: gauge sets (k, m, m') contributing to $a^{(4)}[V]$: (1) = $(1, 1, 0)$, (2) = $(2, 0, 0)$ and (3) = $(1, 0, 1)$. For diagrams related by time reversal (here (1) and (3)) the value listed under the first diagram of the pair is the total contribution of the pair. Contributions seem to be of order $\pm\frac{1}{3}\left(\frac{\alpha}{\pi}\right)^2$, and suggest that a set and its time-reversed partner should be counted separately. From ref. [49].

read, especially if one is interested in the non-abelian theories as well [47, 48, 50], see sect. 1.9.

A gauge change generates a k^μ term in a photon propagator, and that affects a tree electron vertex in a very simple way, $\not{k} = (\not{p} + \not{k} + m) - (\not{p} + m)$, or, diagrammatically [26, 51] (graphs drawn by H. Kißler [121]), [2]

$$\begin{aligned} \frac{1}{\not{p}+\not{k}-m}\not{k}\frac{1}{\not{p}-m} &= \frac{1}{\not{p}-m} - \frac{1}{\not{p}+\not{k}-m}, \\ &= \qquad - \qquad . \end{aligned} \tag{1.9}$$

To simplify matters, in what follows we shall consider only the no-fermion loop diagrams, or 'quenched-', or 'q-type' diagrams ('quenched', as this corresponds to the N_f-independent part of the vertex amplitude in QED with N_f flavors). The minimal gauge-invariant subsets without electron loops (see figure 1.2 diagrams $\{23-72\}$; figure 1.3; 1.4; 1.5; and 1.6) will be hereafter be referred to as *gauge sets*.

A quenched QED gauge set (k, m, m') consists of all 1-particle irreducible vertex diagrams without electron loops, with k photons crossing the external vertex (cross-photons) and $m[m']$ photons originating and terminating on the incoming [outgoing] electron leg (leg-photons), where $m \geq m'$. For asymmetric pairs of sets, with $m \neq m'$,

[2] Predrag: 2017-07-14 need to download, put axohelp.exe, the executable version of axohelp for MS-Windows into the directory where I can run it from, presumably Program Files/MiKTeX 2.9/miktex/bin/x64

the contribution to the anomaly $a_{kmm'}$ is, in the convention of ref. [49], the sum of the set and its mirror (time-reversed) image,

$$a[V] = \frac{1}{2}(g-2)\bigg|_V = \sum_{k=1}^{\infty}\sum_{m=0}^{\infty}\sum_{m'=0}^{m} a_{kmm'}\left(\frac{\alpha}{\pi}\right)^{k+m+m'} . \tag{1.10}$$

1.2.2 The unreasonable smallness of gauge sets

When the diagrams computed in ref. [58] are grouped into gauge sets, figure 1.3 to figure 1.6, a surprising thing happens; while the finite part of each Feynman diagram is of order of 10 to 100, every gauge set known at the time added up to approximately

$$\pm\frac{1}{2}\left(\frac{\alpha}{\pi}\right)^n ,$$

with the sign given by a simple empirical rule

$$a_{kmm'} = (-1)^{m+m'}\frac{1}{2} . \tag{1.11}$$

The sign rule is further corroborated by sets with photon self-energy insertions (but with the absolute size scaled down to $3-15\%$ of (1.11)). In figure 1.5 this rule is compared with the actual numbers, and the 1977 four-loop prediction is given [49]. With that prediction, the "zeroth" order estimate of the electron magnetic moment anomaly a is given by the "gauge-set approximation," convergent and summable to all orders

$$a = \frac{1}{2}(g-2) = \frac{1}{2}\frac{\alpha}{\pi}\frac{1}{\left(1-\left(\frac{\alpha}{\pi}\right)^2\right)^2} + \text{“corrections"} . \tag{1.12}$$

This is not how one usually thinks of perturbation theory. Most of our colleagues believe that in 1952 Dyson [71] had shown that the perturbation expansion is an asymptotic series (for a discussion, see Dunne and Schubert [70, 104]), in the sense that the n-th order contribution should be exploding combinatorially

$$\frac{1}{2}(g-2) \approx \cdots + n^n\left(\frac{\alpha}{\pi}\right)^n + \cdots ,$$

and not growing slowly like my estimate

$$\frac{1}{2}(g-2) \approx \cdots + \frac{n}{2}\left(\frac{\alpha}{\pi}\right)^{2n} + \cdots .$$

For me, the above is the most intriguing hint that something deeper than what we know today underlies quantum field theory, and the most suggestive lesson of our calculation.

1.2.3 Self-energy sets

There are two ways of grouping vertex diagrams, into *gauge sets*, and into *self-energy sets* (or the "externally gauge-invariant" sets). Every vertex diagram belongs both to a

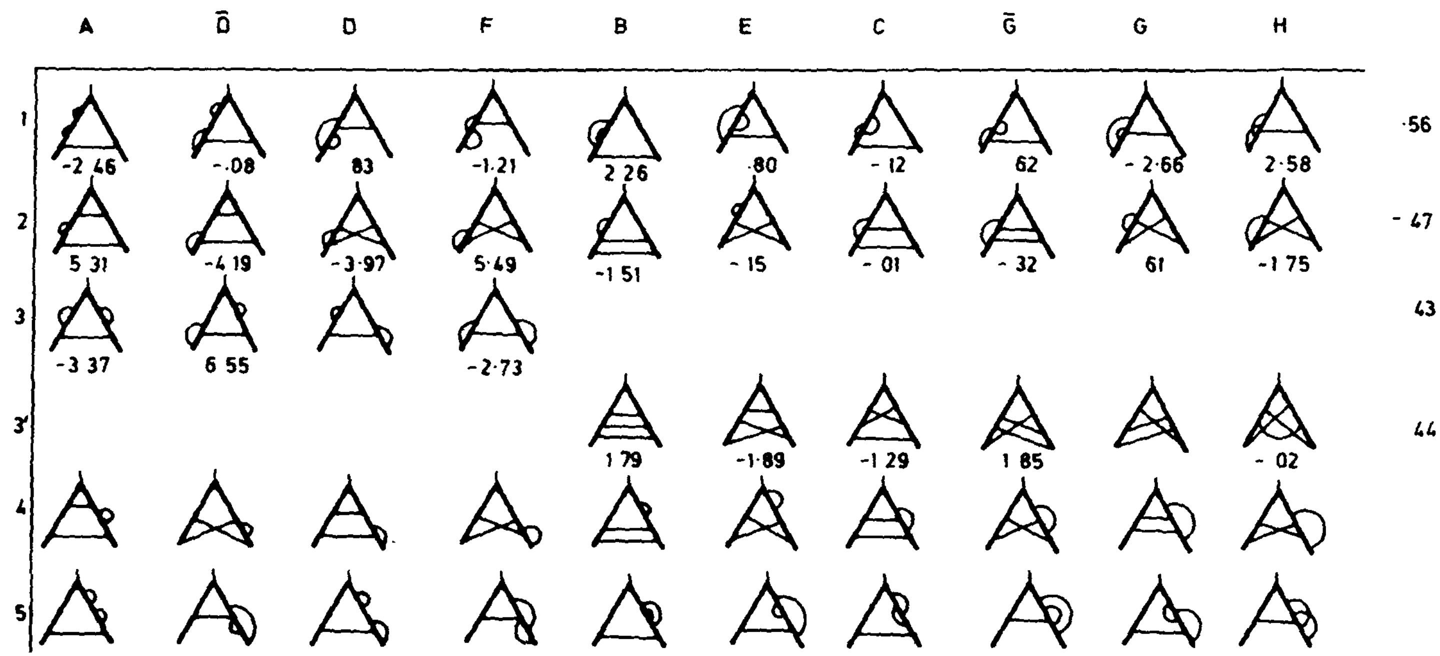


Figure 1.4: Every vertex diagram belongs both to a ‘gauge set’ and to a ‘self-energy set’. This table illustrates the two kinds of sets. The 3-loop gauge sets $km'm$ are arranged in the rows, and the self-energy sets (or the ‘externally gauge-invariant’ sets, vertex diagrams obtained by inserting an extra vertex into a self-energy diagram) in the columns, labeled as in Fig. 3 of ref. [58]. The values are finite parts in the $\ln\lambda$ IR cut-off approach, such as those listed in ref. [141]. For different IR separation methods (such as in ref. [58]) and different gauges, individual diagrams have different values. The gauge sets, however, are separately gauge invariant. The self-energy sets (whose number grows combinatorially with the order in perturbation theory) are not, only their sum is gauge invariant. From ref. [49].

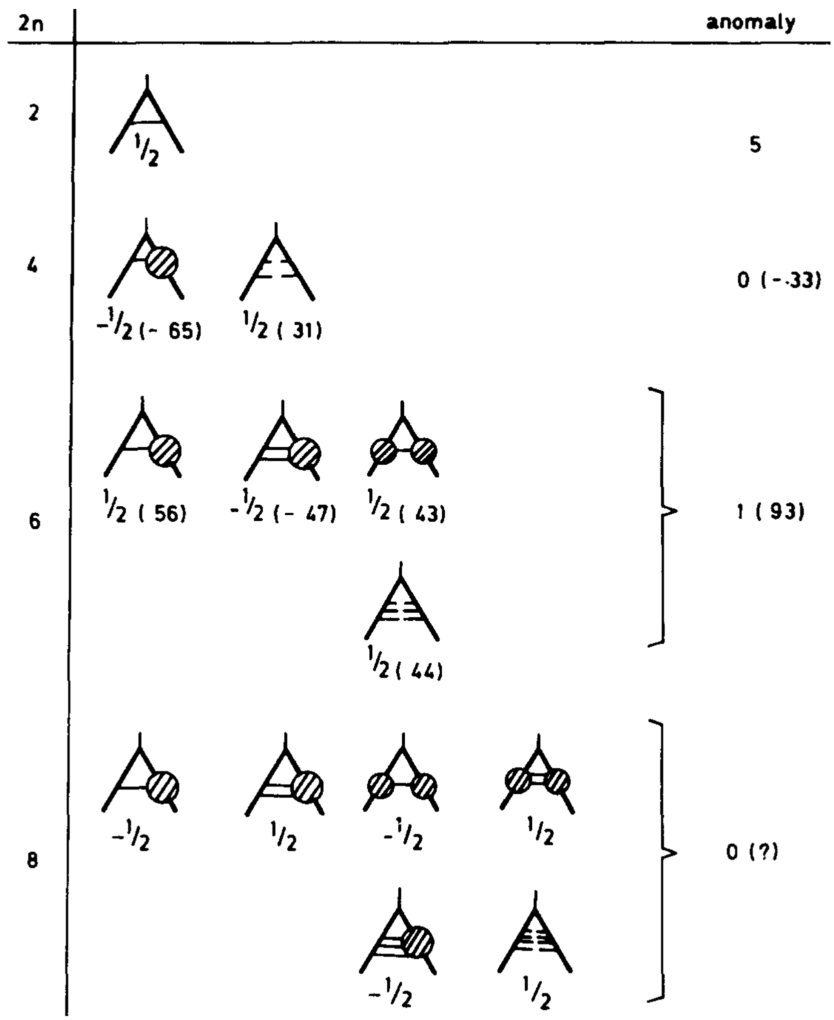
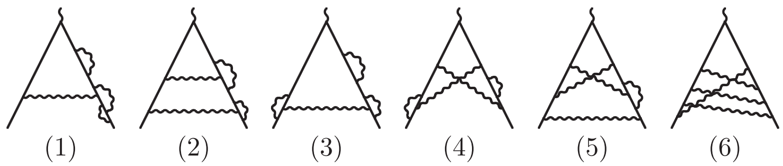


Figure 1.5: Comparison of the 1977 gauge-set approximation to the anomaly a and the actual numerical values of corresponding gauge sets, together with the 1977 eighth-order prediction of ref. [49]. For the updated listing, see table 1.2.

$2n$	(k,m,m')					anomaly
2	$(\mathbf{1},\mathbf{0},\mathbf{0})$ $\frac{1}{2}$					$\frac{1}{2}$
4	$(\mathbf{1},\mathbf{1},\mathbf{0})$ -$\frac{1}{2}$ (-.65)	$(\mathbf{2},\mathbf{0},\mathbf{0})$ $\frac{1}{2}$ (.31)				0 (-.33)
6	$(\mathbf{1},\mathbf{2},\mathbf{0})$ $\frac{1}{2}$ (.56)	$(\mathbf{2},\mathbf{1},\mathbf{0})$ -$\frac{1}{2}$ (-.47)	$(\mathbf{3},\mathbf{0},\mathbf{0})$ $\frac{1}{2}$ (.44)			1 (.93)
	$(\mathbf{1},\mathbf{1},\mathbf{1})$ $\frac{1}{2}$ (.43)					
8	$(\mathbf{1},\mathbf{3},\mathbf{0})$ -$\frac{1}{2}\cdot 4$ (-1.97)	$(\mathbf{2},\mathbf{2},\mathbf{0})$ $\frac{1}{2}\cdot 0$ (-0.14)	$(\mathbf{3},\mathbf{1},\mathbf{0})$ -$\frac{1}{2}\cdot 2$ (-1.04)	$(\mathbf{4},\mathbf{0},\mathbf{0})$ $\frac{1}{2}$ (.51)		0 (-2.17)
	$(\mathbf{1},\mathbf{2},\mathbf{1})$ -$\frac{1}{2}$ (-.62)	$(\mathbf{2},\mathbf{1},\mathbf{1})$ $\frac{1}{2}\cdot 2$ (1.08)				
10	$(\mathbf{1},\mathbf{4},\mathbf{0})$ $\frac{1}{2}\cdot 12$ (6.2)	$(\mathbf{2},\mathbf{3},\mathbf{0})$ -$\frac{1}{2}$ (-0.72)	$(\mathbf{3},\mathbf{2},\mathbf{0})$ $\frac{1}{2}\cdot 0$ (-0.40)	$(\mathbf{4},\mathbf{1},\mathbf{0})$ -$\frac{1}{2}\cdot 2$ (-1.02)	$(\mathbf{5},\mathbf{0},\mathbf{0})$ $\frac{1}{2}\cdot 2$ (1.09)	$\frac{3}{2}\cdot 4$ (6.78)
	$(\mathbf{1},\mathbf{3},\mathbf{1})$ $\frac{1}{2}$ (0.90)	$(\mathbf{2},\mathbf{2},\mathbf{1})$ -$\frac{1}{2}\cdot 4$ (-2.16)	$(\mathbf{3},\mathbf{1},\mathbf{1})$ $\frac{1}{2}\cdot 5$ (2.62)			
	$(\mathbf{1},\mathbf{2},\mathbf{2})$ $\frac{1}{2}$ (0.30)					

Table 1.2: Comparison of the $\pm\frac{1}{2}$ gauge-set (k, m, m') ansatz (1.11) with the actual numerical value of corresponding gauge set, stated in $(\cdots)$ bracket. Starting with 4-loops, the gauge-set ansatz (1.11) fails, but in suggestive ways. All gauge sets are surprisingly close to integer multiples of 1/2; the ones differing from multiple 1 are marked in red. The sign predictions are correct, except for the two anomalously small gauge sets $(2, 2, 0)$, and its "descendent" $(3, 2, 0)$.

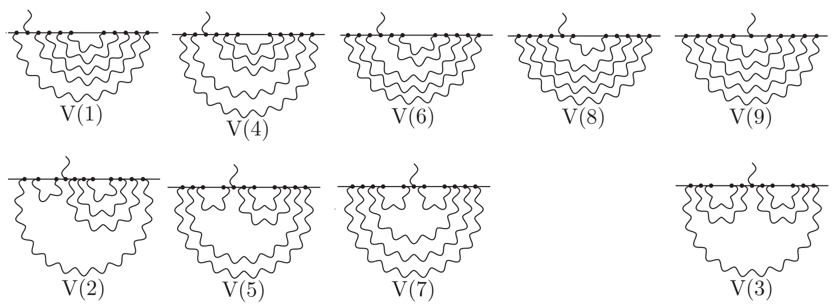


gauge set	(k, m, m)	value	ansatz	
(1)	(1,3,0)	- 1.9710	- 1/2	
(2)	(2,2,0)	-0.1424	1/2	(!)
(3)	(1,2,1)	- 0.6219	- 1/2	
(4)	(2,1,1)	1.0867	1/2	
(5)	(3,1,0)	- 1.0405	- 1/2	
(6)	(4,0,0)	0.5125	1/2	

Figure 1.6: Examples of 4-loop vertex diagrams belonging to Laporta [133] gauge sets (1) to (6). The remaining diagrams in the set can be obtained by permuting separately the vertices on the left and right side of the electron line, and considering also the mirror images of the diagrams. For all 25 gauge sets, see figure **??**. The table: Gauge-set contributions $a^{(8)}_{kmm'}$, see (1.10), as reported by Laporta [133] (for the full 25 gauge sets, see table **??**). The last column: 1977 Cvitanović predictions [49]. Signs are right, except for the set (2) = $(2, 2, 0)$, which is anomalously small, and the remaining sets are surprisingly close to multiples of 1/2. There might be factors of 2 having to do with symmetries, missing from the guesses of ref. [49], but I cannot see how that would work. Only (4) = $(2, 1, 1)$ and (6) = $(4, 0, 0)$ are symmetric, but (1) = $(1, 3, 0)$, (4) and (5) = $(3, 1, 0)$ seem to have an extra factor of 2 or 4.

gauge set and to a self-energy set, as illustrated by figure 1.4. Formulation of the $(g - 2)$ computation directly from self-energy graphs is due to Cvitanović and Kinoshita, see the "new formula" (6.22) in ref. [58]. Not only does this calculation use fewer Feynman graphs, but it enables calculation of the 3-loop electron magnetic moment by two independent methods (and in this way helps track down and eliminate errors in either calculation). As an important aside, Carroll [37, 38] gives the credit for self-energy sets only to the mass-operator formalism of Schwinger [173–175, 183], even though Carroll papers look closer to ref. [58] than to Schwinger and Sommerfield; ref. [58] is cited, and his derivation is also based on the Ward-Takahashi identity [187, 199]. The self-energy set formulation might be equivalent to Schwinger's, but it looks quite different in detail, and the authors were not aware of Schwinger mass-operator when they derived it.

The gauge sets are minimal, and separately gauge invariant (for a proof, see ref. [49]). The self-energy sets are not, only their sum is gauge invariant. Unlike gauge sets, whose number grows polynomially, the number of self-energy sets grows combinatorially - they save significant amount of computing for few-loops computations, but cannot be used to argue the finiteness of QED. That is the reason why Aoyama *et al.* [11, 12, 14, 15] calculations have nothing to say about the 1977 conjecture [49]: they do not compute individual vertex diagrams, but only the self-energy sets, and for



gauge set	(k, m, m')	value	ansatz
V(1)	(1,4,0)	6.17	1/2
V(4)	(2,3,0)	- 0.80	- 1/2
V(6)	(3,2,0)	- 0.41	1/2
V(8)	(4,1,0)	- 0.97	- 1/2
V(9)	(5,0,0)	1.08	1/2
V(2)	(1,3,1)	0.96	1/2
V(5)	(2,2,1)	- 2.16	- 1/2
V(7)	(3,1,1)	2.64	1/2
V(3)	(1,2,2)	0.33	1/2

Figure 1.7: 5-loop vertex diagrams quenched gauge sets. The 5-loop gauge-set contributions $a^{(10)}_{kmm'}$, as reported by Volkov [193, 194, 198]. The last column: 1977 Cvitanović predictions [49]. Signs are right, except for the set (6) = $(3, 2, 0)$, as is the case for similar 4-loop gauge set (2) = $(2, 2, 0)$ in figure 1.6. The remaining sets are close to multiples of 1/2, indicated in table 1.2.

them the set of all diagrams without a fermion loop (‘quenched-’ or ‘q-type’ diagrams) is a single ‘gauge set’ V. For example, for 5-loops the set V is a sum of 9 vertex gauge sets (where time-reversed pairs count as one set, see table 1.2), but Aoyama *et al.* [14] only give their sum (1.1).

1.2.4 Where do we go from here?

Gauge invariance is the bane of my life
— Predrag

Aoyama *et al.* 5-loop calculations already push the envelope of what is numerically attainable, they cannot switch from the self-energy diagrams formulation to the vertex diagrams formulation, it would mean (for the quenched set V) going from 389 self-energy graphs to 6 354 vertex diagrams. Stefano Laporta deserves a bit of well earned rest. So what is ahead?

As of spring 2025, Sergey A. Volkov is the only person who has carried through the requisite 5-loop calculations, see sect. 1.4.

Two approaches might be relevant to establishing bounds on, and perhaps even the direct computation of gauge sets (ignoring the $N = 2$ and $N = 4$ supersymmetric models): (1) *worldline formalism* pursued by Schubert and collaborators, see sect. 1.3, and (2) *Hopf algebraic approach* of Kreimer and collaborators, see sect. 1.5.

Very far out in the left field is the smooth conjugacy method of sect. 1.6 which would require a bit of real work to apply it to a field theory.

1.3 Worldline formalism

How and why Feynman in 1950 introduced 'worldline formalism' (initially for scalar QED, appendix to ref. [78], then for spinor QED, appendix to ref. [79]) is explained in Schubert 2001 report [168] (which also has an extensive bibliography up to 2001). In 1982 Affleck, Alvarez, and Manton [2] used the Feynman worldline path integral representation of the quenched effective action for scalar QED in the constant electric field.

For the remainder of this section we shall consider only the quenched QED, i.e., restrict our considerations only to sets of diagrams with no lepton loop insertions.

A formula for the charged scalar propagator to emit and reabsorb N photons as it propagates from x' to x can be derived as follows [5]. The free scalar propagator for the Euclidean Klein-Gordon equation [5, 169] is

$$D_0(x, x') = \langle x| \frac{1}{-\Box + m^2} |x'\rangle \,, \tag{1.13}$$

where $\Box$ is the D-dimensional Laplacian. Exponentiate the denominator following Schwinger,

$$D_0(x, x') = \int_0^\infty dT\, e^{-m^2 T} \langle x| e^{-T(-\Box)} |x'\rangle \,, \tag{1.14}$$

Replace the operator in the exponent by a path integral

$$D_0(x, x') = \int_0^\infty dT\, e^{-m^2 T} \int_{x(0)=x'}^{x(T)=x} \mathcal{D}x(\tau)\, e^{-\int_0^T d\tau \frac{1}{4} \dot{x}^2} \,, \tag{1.15}$$

where τ is a proper-time parameter (the fifth parameter [83]), and the dot denotes a derivative with respect to the proper time. This is the *worldline path integral* representation of the relativistic propagator of a scalar particle in Euclidean space-time. In the vacuum (no background field), it is easily evaluated by standard methods and leads to the usual space and momentum space free propagators, [3]

$$\int_{x(0)=x'}^{x(T)=x} \mathcal{D}x(\tau)\, e^{-\int_0^T d\tau \frac{1}{4} \dot{x}^2} = \frac{1}{(4\pi T)^{d/2}} \,. \tag{1.16}$$

[3] Predrag: 2017-06-17 Here a study of Sect. 6. *Worldline formalism* of Gelis and N. Tanji [92] might be helpful - it reexpresses the integral as an average over Wilson loops.

Adding the QED interaction terms leads to the Feynman's worldline path integral representation [78] of the charged scalar propagator of mass m in the presence of a background field $A(x)$,

$$D(x,x') = \int_0^\infty dT\, e^{-m^2T} \int_{x(0)=x'}^{x(T)=x} \mathcal{D}x(\tau)\, e^{-S_0-S_e-S_i}\,, \tag{1.17}$$

where the suffix (0) indicates the free propagation

$$S_0 = \int_0^T d\tau \frac{1}{4}\dot{x}^2\,, \tag{1.18}$$

(e) is the interaction of the charged scalar with the external field

$$S_e = -ie\int_0^T d\tau\, \dot{x}^\mu A_\mu(x(\tau))\,, \tag{1.19}$$

and (i) are the virtual photons exchanged along the charged particle's trajectory

$$S_i = \frac{e^2}{2}\int_0^T d\tau_1 \int_0^T d\tau_2\, \dot{x}_1^\mu\, D_{\mu\nu}(x_1-x_2)\, \dot{x}_2^\nu\,, \tag{1.20}$$

where $D_{\mu\nu}$ is the x-space photon propagator.

The formula (1.17) involves (i) a path integral over all the worldlines $x(\tau')$, i.e., closed paths in Euclidean space-time parameterized by the proper time $\tau' \in [0,\tau]$, and (ii) an ordinary integral over the length τ of these paths. The sum over all the worldlines accounts for the quantum fluctuations in space-time, and the prefactor $\exp(-m^2T)$ suppresses the very long worldlines that explore regions of space-time much larger than the Compton wavelength of the particles. The ultraviolet properties of the theory are encoded in the short worldlines limit $\tau \to 0$. The Euclidean space-time guarantees that both types of integrals are convergent.

Consider next the charged scalar field in external field, neglecting internal photon loops. By taking the constant external field $A(x)$ to be a sum of N plane waves, one obtains the rule for inserting N external photons:

$$\begin{aligned} D_{(N)}(x,x') \quad &= \quad (-\lambda)^N \int_0^\infty dT\, e^{-m^2T} \int_0^T d\tau_1 \cdots \int_0^T d\tau_N \\ &\times \int_{x(0)=y}^{x(T)=x} \mathcal{D}x\, e^{i\sum_{i=1}^N k_i\cdot x(\tau_i)} e^{-\int_0^T d\tau\, \frac{1}{4}\dot{x}^2}\,. \end{aligned} \tag{1.21}$$

For the spinor case, the magnetic moment will be given by the term linear in a constant external field $A(x)$, and in order to define gauge sets, one will have to distinguish the in- and out-electron lines.

The object of great interest to us is the quenched internal virtual photons term (1.20):

$$\int_{x(0)=x'}^{x(T)=x} \mathcal{D}x(\tau)\, e^{-S_i} = \int_{x(0)=x'}^{x(T)=x} \mathcal{D}x(\tau)\, e^{-\frac{e^2}{2}\int_0^T d\tau_1 \int_0^T d\tau_2\, \dot{x}_1^\mu\, D_{\mu\nu}(x_1-x_2)\, \dot{x}_2^\nu}\,. \tag{1.22}$$

(Fried and Gabellini [89] refer to this as the "linkage operator"). Expanded perturbatively in α/π, this yields the usual Feynman-parametric vertex diagrams. However, it is Gaussian in $\dot{x}^\mu$, and if by integration by parts, $\dot{x}^\mu$ are eliminated in favor of x^μ, internal photons can be integrated over directly, prior to an expansion in $(\alpha/\pi)^n$, and one gets integrals in terms of N*-photon propagators*, symmetrized sums over N photons, and not the usual Feynman graphs. Each usual Feynman graph corresponds to one particular permutation of internal photon insertions, and from that comes the factorial growth in the number of graphs.

These integrations by parts lead to the first and second proper-time derivatives of the Green's function, worked out in the literature (for example, in refs. [169, 186]), the details would take too much space to recap here. I find Bastianelli, Huet, Schubert, Thakur and Weber 2014 paper [21] quite inspirational. My notes on these papers are below, around page 41. Apologies, my notes are just a jumble, jottings taken as I try to understand this literature. They might be useful anyway, as pointers to the literature.

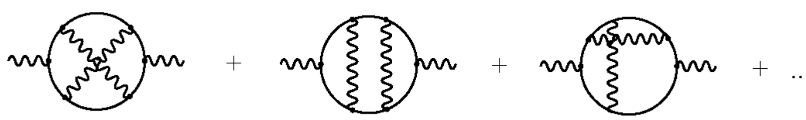


Figure 1.8: Quenched diagrams contributing to the three loop QED photon propagator. From ref. [21].

Thus, for the quenched scalar QED, the worldline integrals are expressed in terms of N-photon propagators, the central ingredient that defines the quenched gauge sets (1.10). Unlike the Feynman parameter integrals for individual vertex graphs, they are independent of the ordering of the momenta $k_1, \ldots, k_N$; the formula (1.22) contains all $\approx N!$ ways of attaching the N photons to the charged particle propagator. The formulation combines combinatorially many Feynman diagrams into a single integral. An example are the quenched contributions to the three-loop photon propagator shown in figure 1.8.

In QED the N-photon propagator formulation combines into one integral all Feynman graphs related by permutations of photon legs along fermion lines, that is, it should yield *one* integral for a gauge set $km'm$ defined in (1.10).

1.3.1 High-orders QED in worldline formalism

A non-perturbative formula for QED in a constant field, given for scalar QED in 1982 by Affleck, Alvarez, and Manton [2] is an example how the worldline formalism can yield high-order information on QED amplitudes. Huet, McKeon, and Schubert [103] continue this in their 2010 study of the l-electron loop, N-photon amplitudes in the limit of large photon numbers and low photon energies, this time for 1+1 dimensional scalar QED, in order to illustrate the large cancellations inside gauge invariant classes of graphs.

Affleck *et al.* [2] use the Feynman [78] 'worldline path integral' representation of the quenched effective action for scalar QED in the constant electric field, and calculate the amplitude in a stationary path approximation. The stationary trajectory so obtained is a circle with a field dependent radius, called "instanton" in this context. The worldline action on this trajectory yields the correct exponent, and the second variation determinant yields the correct prefactor. Using Borel analysis, they obtain non-perturbative information on the on-shell renormalized N-photon amplitudes at large N and low energies.

Parenthetically, independently and not by worldline formalism, but by dint of difficult calculations and much deep physics intuition, Lebedev and Ritus [140] have arrived at a nonperturbative mass shift interpretation for the spinor QED pair creation in constant electric field in 1984.

For the quenched spinor QED (fermion lines decorated by photon exchanges) closed-form expressions for general N require the worldline super- formalism [168], at the cost of introducing Fradkin 1966 [84] Grassmann path integral, or, alternatively, the second order formalism of Strassler [186].

The 2017 G. Torgrimsson, Schneider, Oertel and Schützhold [190], arXiv:1703.09203, sect. 3, uses the N-photon formalism to determine a saddle and the asymptotic form of two types of dynamically assisted Sauter-Schwinger effect.

The 2006 Dunne and Schubert [70] study of scalar and spinor QED N-photon amplitudes, in the quenched approximation (i.e., taking only the diagrams with one electron loop) led to "the following generalization of Cvitanović's conjecture: the perturbation series converges for all on-shell renormalized QED amplitudes at leading order in N_f. It must be emphasized that the on-shell renormalization is essential in all of the above." Unlike Cvitanović [49] purely numerical conjecture, theirs is a sophisticated argument, buttressed by Borel dispersion relations.

1.3.2 Electron magnetic moment in worldline formalism

Here we specialize the electron magnetic moment discussion of sect. 1.1 to the quenched subsector. Z_1, Z_2, and Z_3, are the respectively the vertex, the electron wave function, and the photon wave function renormalization constants. For quenched QED there are no fermion loops, there are no vacuum polarization contribution to the charge renormalization (1.8), $Z = Z_3 = 1$, so the bare coupling equals the physical coupling, $\alpha_0 = \alpha$. Furthermore, $Z_1 = Z_2$ by the Ward identity [199].

The anomalous magnetic moment of an electron $a = (g-2)/2$ is given by the static limit of the magnetic form factor $a = \tilde{F}_2(0) = M/(1+L)$ from (1.5), with perturbative expansion

$$a = \frac{M(\alpha)}{1+L(\alpha)} = \sum_{n=1}^{\infty} a^{(2n)} \left(\frac{\alpha}{\pi}\right)^n , \tag{1.23}$$

where $Z_1 = 1 + L = F_1(0)$, $M = F_2(0)$ are computed from the unrenormalized on-shell values of proper vertex (1.3), given by the sum of all one-particle irreducible (1pI) electron-electron-photon vertex diagrams with internal photon corrections (no electron loops). Expanding M and L we have

$$\begin{aligned} a_0^{(2)} &= M^{(2)} \\ a_0^{(4)} &= M^{(4)} - L^{(2)}M^{(2)} \\ a_0^{(6)} &= M^{(6)} - L^{(2)}M^{(4)} - (L^{(4)} - (L^{(2)})^2)M^{(2)} \end{aligned} \tag{1.24}$$

Each order in (1.23) is IR and UV finite, with the UV subdivergences are cancelled by $L^{(2m)}$ counterterms in (1.24).

A gauge set $km'm$ in expansion (1.10) consists of all 1-particle irreducible vertex diagrams without electron loops, with k photons crossing the external vertex (cross-photons) and $m[m']$ photons originating and terminating on the incoming [outgoing] electron leg (leg-photons). One can assume three different coupling, setting them all equal to α at the end of the calculation,

$$a = \sum_{m'=0}^{\infty} \left(\frac{\alpha'}{\pi}\right)^{m'} \sum_{k=1}^{\infty} \left(\frac{\alpha_v}{\pi}\right)^{k} \sum_{m=0}^{\infty} \left(\frac{\alpha}{\pi}\right)^{m} a_{km'm} \,. \tag{1.25}$$

The gauge set contributions are then

$$\begin{aligned} a^{(2)} &= a_{100} \\ a^{(4)} &= a_{200} + a_{110} + a_{101} \\ a^{(6)} &= a_{300} + a_{210} + a_{201} + a_{120} + a_{102} + a_{111} \\ a^{(8)} &= a_{400} + a_{310} + a_{301} + a_{220} + a_{202} + a_{211} + a_{130} + a_{103} + a_{121} + a_{112} \\ a^{(10)} &= a_{500} + a_{410} + a_{401} + a_{320} + a_{302} + a_{311} + a_{230} + a_{203} + a_{221} + a_{212} \\ &\quad + a_{140} + a_{104} + a_{131} + a_{113} + a_{122} \end{aligned} \tag{1.26}$$

Both $L^{(2m)}$ and $M^{(2m)}$ can be evaluated in terms of N-photon propagators.

1. To proceed, one needs something like a Bern-Kosower [24] type master formula for the electron line dressed with any number of photons, with a single constant external (arbitrarily weak) magnetic field insertion. For the magnetic moment calculation, the external vertex is distinguished by its $\sigma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]$ form (1.3), while all internal, virtual photon vertices are of the usual γ^μ form.

2. Please write down the worldline formula for the anomalous magnetic moment of the electron $a = \tilde{F}_2(0)$, corresponding to Dirac trace expression (1.5) for M.

3. As the external vertex transfers a (vanishing) momentum, the incoming and outgoing electron on-mass shell legs are distinct, and thus there are three kinds of N-photon propagators; k photons crossing the external vertex (cross-photons) and $m[m']$ photons originating and terminating on the incoming [outgoing] electron leg (leg-photons). One needs to prove in the worldline formalism that each $km'm$ integral (corresponding to a set of quenched set of 1-particle irreducible Feynman vertex diagrams without electron loops) is separately (i) a gauge set, and (ii) the minimal gauge invariant set.

4. Hopefully the distinction motivates the gauge set sign rule (1.11). Keep in mind, however, that this empirical rule is already violated by the gauge set $(2,2,0)$.

5. Please write down the worldline integral for one-loop anomaly $a_0^{(2)}$ in (1.7). The first thing to verify is that the worldline $(1,0,0)$ integral reproduces Schwinger's $\frac{1}{2}\left(\frac{\alpha}{\pi}\right)$ result [172], exactly. That is an exercise in converting the integral into Feynman-parametric form, already done several times for other amplitudes.

6. Please write down the worldline integral for 2-loop anomaly $a_0^{(4)} = M^{(4)} - L^{(2)}M^{(2)}$ in (1.7). Can $L^{(2)}M^{(2)}$ be absorbed into the integrand? If cancelations can be made pointwise, that would obviate a need for constructing UV (and IR?) counterterms.

7. For 2-loop anomaly there are only 2 quenched gauge sets $km'm$: $(2,0,0)$ and $(1,1,0)$, which equals $(1,0,1)$ by time reversal, see figure 1.3 and table 1.2. So, reformulate the 2-loop calculation as two worldline integrals, one for each gauge set. Most likely, want to do the gauge set $(2,0,0)$ first, as it seems to have simpler subdiagram structure (though not sure about that). Do not attempt (for now) to evaluate these analytically (though Broadhurst, Laporta, Kreimer, etc., would be interested to see whether some simplification occurs), main thing is to understand that the UV renormalization works, and that there are no intermediate IR divergences in this reformulation.

1.4 Volkov method

In ref. [195] Volkov explains that $A_1^{(2n)}$ is free from infrared divergences since they are removed by the on-shell renormalization. However, Volkov also states that there is no universal method in QED for canceling IR divergences in the Feynman graphs analogous to the R operation, and that the standard subtractive on-shell renormalization cannot remove IR divergences point-by-point in Feynman-parametric space, as it does for UV divergences. Moreover, it can generate additional IR-divergences.

That QED on-mass shell amplitudes are IR-free must be an old result; even I have several papers generalizing that to QCD [47, 48, 50, 59]. Tom Kinoshita and I solved the problem of point-by-point removal of IR divergences in Feynman-parametric space in my thesis [57], with a super-elegant formula (who needs forests?) for the UV and IR finite part of amplitude M_G,

$$\Delta M_G = \prod_{ij}(1 - I_{G/S_i})(1 - K_{G/S_j})M_G \,, \tag{1.27}$$

where the products are over all self-energy and vertex subdiagrams S_i and S_j. I have a bright memory of figuring out how to do it September 12, 1972 quiet evening in Ithaca, babysitting for a friend's toddler. But, as Volkov [195] and Aoyama *et al.* [13] explain, our approach was apparently not general enough to deal with the 4- and 5-loop contributions.

Volkov's algorithm is developed in *New method of computing the contributions of graphs without lepton loops to the electron anomalous magnetic moment in QED* [196].

It is based on the ideas used for proving UV-finiteness of renormalized Feynman amplitudes [7, 185]. He focuses on n-loop graphs with no lepton loops, or, in the notation of these notes, $a^{(2n)}[V]$. Volkov calculation groups Feynman graphs by self-energy graphs families because they have similar integrand structure. In contrast to refs. [12, 37, 38, 58] he does not evaluate these self-energy graphs directly; all his calculations are performed with vertex graphs, i.e., precisely what is needed to evaluate gauge sets of sect. 1.2. However, as illustrated in figure 1.5, each gauge-set vertex diagram belongs to a different self-energy diagram, so Volkov calculation will require a major reorganization of how integrands are generated, requiring months of recoding.

So far Volkov has evaluated the ladder graph and the fully crossed graph up to 5 loops. The cross graphs are of interest because they do not contain divergent subgraphs, so their contributions only depend on the gauge, but not on the choice of subtraction procedure.

While the contributions of individual vertex graphs (and self-energy sets [12]) are all over the place, all gauge sets are insanely small up to order 8, and it would be very sweet to see that this continues through order 10 (at least for the 5-loop graphs with no electron loops). My hunch is that starting with the gauge set $(5, 0, 0)$ of table 1.2 (5! vertex graphs, some of them symmetric pairs) would be the most rewarding. Stefano Laporta thinks it too hard, and suggests starting with the 5-loop relative $(1, 3, 1)$ (or $(1, 2, 2)$) of the 4-loop set $(1, 2, 1)$, which would entail less than 5! vertex graphs (I have not counted how many). As no high accuracy is needed, a numerical check of the QED finiteness conjecture would good enough if the gauge sets evaluated to two significant digits or so, but even that will need a lot of computer time.

1.5 Hopf algebraic approach

Hopf algebraic approach of Kreimer and collaborators [33, 121, 131, 132] is very appealing - it is just that I personally have no clue how to turn it into a direct $(g - 2)$ gauge set calculation. In the 2008 paper [132] Dirk Kreimer and Karen Yeats write:

> "One case where there is a natural interpretation is QED with a linear number of generators, namely
>
> $$X_1 = 1 + \sum_{k\geq 1} p(k)x^k \frac{X_1^{2k+1}}{(1 - X_2)^{2k}(1 - X_3)^{2k}}\,, \qquad (1.28)$$
>
> with X_2 and X_3 as before and with $p(k)$ linear, which corresponds to counting with Cvitanović's gauge invariant sectors [49]."

Even in this simple case I do not see how this counts the gauge sets. My generating function for G_{2n}, the number of gauge sets (eq. (7) in ref. [49]) is

$$\sum_{n=1}^{\infty} G_{2n} = \frac{X}{(1 + X)(1 - X)^3}\,. \qquad (1.29)$$

Broadhurst, Delbourgo and Kreimer [33] 1996 *Unknotting the polarized vacuum of quenched QED* unearthes much knot-theory magic, leading to cancelations of "transcedentals." While their particular conjecture did not work out in higher orders, the conceptual scheme might be another route to proving the QED is finite - if there is some finite knot-theory basis for expressing the value of every gauge set, and the gauge invariance induced cancelations are so strong to lead to the large cancelations of transcendentals (hyperlogarithms), then perhaps that gives bounds on the size of each gauge set which are slower than combinatorial. The number of different kinds of knots with n crossings is known to grow only exponentially, not faster.

Henry Kißler (on page 50) has a fresh idea for how to approach the finiteness conjecture, using the *Hepp bound*, see Panzer **2018-06-07** below.

Note that the Kißler and Kreimer [121] definition of a "gauge set" differs from (1.10) used here. They organize a gauge-dependent calculation into "gauge sets" of different parameter dependence.

My notes on these papers are below, starting on page 47.

1.6 Method of smooth conjugacies

> *If Feynman knew Poincaré: How to replace many diagrams by one*
>
> In quantum field theory the standard Feynman diagram methods become quickly unwieldy at higher orders. However, it is frequently observed that the sums of Feynman diagrams, each individually complicated, simplify miraculously to rather compact expressions.
>
> Here comes a possible reason why that can be traced back to Poincaré, and is perhaps not something that a field theorist would instinctively hark to as a method of computing perturbative corrections: make the dynamics linear ("free") by flattening out the vicinity of a path integral extremum by a smooth nonlinear coordinate transformation. The resulting perturbative expansion is more compact than the standard Feynman diagram perturbation theory.

The smooth conjugacy method sketched here would require some serious work to make it a workable quantum field theory scheme. The reader might prefer to skip straight to the worldline formalism sect. 1.3.

The periodic orbit theory is a classical, deterministic theory [53] that describes nonlinear systems in "chaotic" (for low-dimensional systems) or "turbulent" (for PDEs) regimes. The theory allows us to calculate long time averages in a chaotic system as expansions in terms of the periodic orbits (cycles) of the system. The simplest example (the deterministic analogue of the quantum evolution operator) is provided by the Perron-Frobenius operator

$$\mathcal{L}\rho(x') = \int dx\, \delta(f(x) - x')\rho(x) \tag{1.30}$$

for a *deterministic* map $f(x)$ which maps a density distribution $\rho(x)$ forward one integer step in time. The periodic orbit theory relates the spectrum of this operator and its weighted evolution operator generalizations to the periodic orbits via trace formulas, dynamical zeta functions and spectral determinants [53, 91].

For quantum mechanics the periodic orbit theory is exact on the semiclassical level [98], whereas the quintessentially quantum effects such as creeping, tunneling and diffraction have to be included as corrections. In particular, the higher order $\hbar$ corrections can be computed perturbatively by means of Feynman diagrammatic expansions [91]. We illustrate how this works by the parallel, but simpler example of *stochastic* dynamics. Cvitanović [52] *Chaotic Field Theory: A sketch* is a programmatic statement how this theory might connect to quantum field theory, and, by a way of motivation, an easy introduction into different approaches to incorporating stochastic corrections into classical dynamics.

What motivated the work [54, 55, 60] summarized in ref. [52] is the fact that the form of perturbative corrections for the stochastic problem is the same as for the quantum problem, and still the actual calculations are sufficiently simple that one can explore more orders in perturbation theory than would be possible for a full-fledged quantum theory. For the simple system studied, the result is a stochastic analog of the Gutzwiller trace formula with the "$\hbar$ corrections" computed to five orders beyond what has been attainable in the quantum-mechanical applications. Already a discrete time, 1-dimensional discrete Langevin equation [115, 136],

$$x_{n+1} = f(x_n) + \sigma\xi_n \,, \tag{1.31}$$

with ξ_n independent normalized random variables, suffices to reveal the structure of perturbative corrections. We treat a chaotic system with weak external noise by replacing the deterministic evolution δ-function kernel of Perron-Frobenius operator (1.30) by $\mathcal{L}_{FP}$, the Fokker-Planck kernel corresponding to (1.31), a peaked noise distribution function

$$\mathcal{L}_{FP}(x', x) = \delta_\sigma(f(x) - x') \,. \tag{1.32}$$

In the weak noise limit the kernel is sharply peaked, so it makes sense to expand it in terms of the Dirac delta function and its derivatives:

$$\delta_\sigma(y) = \sum_{m=0}^{\infty} \frac{a_m \sigma^m}{m!}\,\delta^{(m)}(y) = \delta(y) + a_2\frac{\sigma^2}{2}\delta^{(2)}(y) + a_3\frac{\sigma^3}{6}\delta^{(3)}(y) + \dots \,. \tag{1.33}$$

where

$$\delta^{(k)}(y) = \frac{\partial^k}{\partial y^k}\delta(y) \,,$$

and the coefficients a_m depend on the choice of the kernel. We have omitted the $\delta^{(1)}(y)$ term in the above because in our applications we shall impose the saddle-point condition, that is, we shift x by a constant to ensure that the noise peak corresponds to $y = 0$, so $\delta_\sigma^{'}(0) = 0$. For example, if $\delta_\sigma(y)$ is a Gaussian kernel, it can be expanded as

$$\begin{aligned} \delta_\sigma(y) &= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-y^2/2\sigma^2} = \sum_{n=0}^{\infty} \frac{\sigma^{2n}}{n!2^n}\delta^{(2n)}(y) \\ &= \delta(y) + \frac{\sigma^2}{2}\delta^{(2)}(y) + \frac{\sigma^4}{8}\delta^{(4)}(y) + \cdots \,. \end{aligned} \tag{1.34}$$

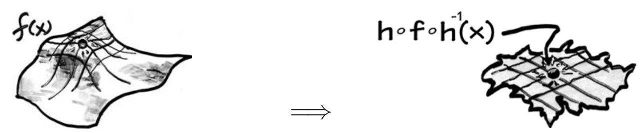
Analogies between noise and quantum mechanics can be explored by casting stochastic dynamics into path integral form (a stochastic Wiener integral). The periodic orbit

 1.6 *Method of smooth conjugacies*

theory is a nonperturbative, “WKB” method for approximating such integrals, which can then be improved by systematic perturbative corrections. In the weak noise case the standard perturbation theory is an expansion in terms of Feynman diagrams. For semiclassical quantum mechanics of a classically chaotic system such calculation was first carried out by Gaspard [91]. The stochastic version, implemented by Dettmann *et al.* [54], reveals features not so readily apparent in the quantum calculation. Perhaps some of these could be of interest to Kreimer and collaborators, sect. 1.5.

The Feynman diagram method becomes quickly unwieldy at higher orders.[4] However, in the Feynman diagram approach pursued in ref. [54], the authors observe that the sums of Feynman diagrams simplify miraculously to rather compact expressions.

Now the surprise; one can compute the same corrections faster and to a higher order in perturbation theory by integrating over the neighborhood of a given saddlepoint *exactly* by means of a nonlinear change of field variables. This elegant idea of flattening the neighborhood of a saddlepoint, introduced by Mainieri *et al.* [55], and referred to here as the *smooth conjugation method*, is perhaps an altogether new idea in field theory. The idea, that can be traced back to Poincaré [152], injects into field theory a method standard in the construction of normal forms for bifurcations [116]: perform a smooth nonlinear coordinate transformation $x = h(y)$, $f(x) = h(g(h^{-1}(x)))$ that flattens out the vicinity of a fixed point and makes the map *linear* in an open neighborhood, $f(x) \to g(y) = \mathbf{J} \cdot y$.



an arbitrary coordinatization intrinsic, flat coordinates

The resulting perturbative expansion turns out to be more compact than the standard Feynman diagram perturbation theory; whether it is better than the traditional loop expansions for computing field-theoretic saddlepoint correction remains to be seen.

What is new is that the problem is being solved locally, periodic orbit by periodic orbit, by translation to coordinates intrinsic to the periodic orbit.

This local rectification of a map can be implemented only for isolated non-degenerate fixed points (otherwise higher terms are required by the normal form expansion around the point), and only in finite neighborhoods, as the conjugating functions in general have finite radia of convergence.

In this approach the neighborhood of each saddlepoint is rectified by an appropriate nonlinear field transformation, with the focus shifted from the dynamics in the original field variables to the properties of the conjugacy transformation. The expressions thus obtained *correspond to sums* of Feynman diagrams, but are more compact.

We will try to explain this simplification in geometric terms that might be applicable to more general field theoretic problems. The idea is this: as the dynamics is nonlinear, why not search for a nonlinear field transformation $\phi = h(\tilde{\phi})$ (a smooth conjugacy) that

[4]The matrix method, introduced by Vattay *et al.* [60], based on Rugh’s [160] explicit matrix representation of the evolution operator will not be discussed here. If one is interested in evaluating numerically many orders of perturbation theory and many eigenvalues, this method is unsurpassed.

makes the intrinsic coordinates as simple as possible? Schematically –wrong in detail, but right in spirit– find a smooth conjugacy such that the action $S[\phi] = S_0[\phi] + S_I[\phi]$ in the partition function path integral becomes the free, quadratic action,

$$Z = e^W \;=\; \int [d\phi] e^{S[\phi]} \;=\; \int [d\tilde{\phi}] \frac{1}{|\det \partial h(\tilde{\phi})|^{\frac{1}{2}}} \, e^{\frac{1}{2}\tilde{\phi}^\top \frac{1}{\Delta}\tilde{\phi}} \,, \tag{1.35}$$

at the price of having the determinant of the conjugacy Jacobian show up as a weight.

Ref. [54] treats the problem of computing the spectrum of this operator by standard field-theoretic Feynman diagram expansions. Here we formulate the perturbative expansion in terms of smooth conjugacies and recursively evaluated derivatives. The procedure, which is relatively easy to automatize, enables us to go one order further in the perturbation theory, with much less computational effort than Feynman diagrammatic expansions would require.

[TO BE CONTINUED]

1.7 Lattice quenched QED

2025-04-08 Predrag .

Ryuichiro Kitano, Hiromasa Takaura, and Shoji Hashimoto [124] *Stochastic computation of* $g-2$ *in QED*, arXiv:2103.10106:

"We perform a numerical computation of the anomalous magnetic moment $g-2$ of the electron in QED by using the stochastic perturbation theory. Formulating QED on the lattice, we develop a method to calculate the coefficients of the perturbative series of $g-2$ without the use of the Feynman diagrams. We demonstrate the feasibility of the method by performing a computation up to the α^3 order and compare with the known results. This program provides us with a totally independent check of the results obtained by the Feynman diagrams, an example of the application of the numerical stochastic perturbation theory to physical quantities, for which the external states have to be taken on-shell. "

Reviews QED of $g-2$, but now on a Euclidean lattice. They use using numerical stochastic perturbation theory (NSPT).

In the stochastic quantization, one obtains the expectation values of operator products as fictitious time τ averages of the field products. The gauge field $A_\mu(n)$ is promoted to $A_\mu(n,\tau)$ governed by the Langevin equation

$$\frac{\partial A_\mu(n,\tau)}{\partial\tau} = -\frac{\delta S_{\text{lattice}}}{\delta A_\mu(n,\tau)} + \eta_\mu(n,\tau). \tag{1.36}$$

The Gaussian noise $\eta_\mu(n,\tau)$ satisfies:

$$\langle \eta_\mu(n,\tau)\rangle_\eta = 0, \quad \langle \eta_\mu(n,\tau)\eta_\nu(n',\tau')\rangle_\eta = 2\delta_{\mu\nu}\delta_{nn'}\delta(\tau-\tau'),$$

where $\langle\cdots\rangle_\eta$ denotes an ensemble average over the random noise. The ensemble average of field products converges to the correlation functions obtained by the path integral quantization:

$$\langle A_{\mu_1}(n_1,\tau)\cdots A_{\mu_k}(n_k,\tau)\rangle_\eta \to \langle A_{\mu_1}(n_1)\cdots A_{\mu_k}(n_k)\rangle,$$

as $\tau\to\infty$. This can be understood by the fact that the probability distribution of the path integral, e^{-S}, is the fixed point of the Fokker-Planck equation derived from the Langevin equation in (1.36). The ensemble average in the left-hand-side can be evaluated as the "time" average of the Langevin trajectories of the field product, i.e.,

$$\langle A_{\mu_1}(n_1)\cdots A_{\mu_k}(n_k)\rangle = \lim_{\Delta\tau\to\infty}\frac{1}{\Delta\tau}\int_{\tau_0}^{\tau_0+\Delta\tau} d\tau A_{\mu_1}(n_1,\tau)\cdots A_{\mu_k}(n_k,\tau).$$

Expand the field $A_\mu(n,\tau)$ as

$$A_\mu(n,\tau)=\sum_{p=0}^{\infty} e^p A_\mu^{(p)}(n,\tau), \tag{1.37}$$

and solve the Langevin equation for each $A_\mu^{(p)}$. The equation for gauge fields in each order p is given by

$$\frac{\partial A_\mu^{(p)}(n,\tau)}{\partial\tau} = -\left.\frac{\delta S_{\rm lattice}}{\delta A_\mu(n,\tau)}\right|_{(p)} + \eta_\mu(n,\tau)\delta_{p0},$$

where $|_{(p)}$ denotes the p-th order term. The noise is applied only for the lowest order, $A_\mu^{(0)}$. Higher order fields get fluctuated through interactions with lower order fields.

For on-shell fermions, $s(p)^2+m^2=0$ and $s(p+k)^2+m^2=0$, the vertex function can be expressed by form factors as

$$\begin{aligned}&-ie_{\rm P}\,\bar u(p)\Gamma_\mu(p,k)u(p+k) = \qquad (1.38)\\ &\quad -ie_{\rm P}\,\bar u(p)\left(F_1(\hat k^2)\gamma_\mu - F_2(\hat k^2)\frac{\sigma_{\mu\nu}\hat k_\nu}{2m}\right)u(p+k)+\mathcal{O}(a^2)\,.\end{aligned}$$

Ward-Takahashi identities take familiar form

$$\hat k_\mu G_\mu(p,k)=\frac{e\xi}{\hat k^2}\left(S(p+k)-S(p)\right)e^{-2\hat k^2/\Lambda_{\rm UV}^2}. \tag{1.39}$$

Here $S(p)$ is the full fermion propagator, and $G_\mu(p,k)$ is the vertex function.

2025-04-08 Predrag .

Ryuichiro Kitano and Hiromasa Takaura [123] *Quantum electrodynamics on the lattice and numerical perturbative computation of* $g-2$, arXiv:2210.05569

"We compute the electron g factor to the $O(\alpha^5)$ order on the lattice in quenched QED. We first study finite volume (FV) corrections in various infrared regularization methods to discuss which regularization is optimal for our purpose. We find that in QEDL the FV correction to the effective mass can have different parametric dependences depending on the size of Euclidean time t and match the 'naive on-shell result' only at the very large t region, $t \gg L$. We adopt finite photon mass regularization to suppress FV effects exponentially and also discuss our strategy for selecting simulation parameters and the order of extrapolations to efficiently obtain the g factor. We perform lattice simulation using small lattices to test the feasibility of our calculation strategy. This study can be regarded as an intermediate step toward giving the five-loop coefficient independently of preceding studies."

2025-04-08 Predrag .

Ryuichiro Kitano [122] *QED Five-Loop on the Lattice*, arXiv:2411.11554.

A method to evaluate $g-2$ on the lattice was proposed and developed in refs. [123, 124]. The no-lepton-loop part of the calculation corresponds to ignoring the fermion determinant in the path integral. In QED, ignoring the fermion determinant leads to the free theory of the gauge field and thus the path integral can be performed by generating Gaussian noises. Also, the absence of the lepton loop means that there is no renormalization of α, and thus the perturbative expansion can be formulated in terms of the bare parameter in the Dirac operator.

" [...] obtain the perturbative coefficients of $g-2$ in QED by using lattice simulations. The calculation is quite simple as we only have a single diagram to evaluate. By concentrating on the diagrams without a lepton loop, the formulation is further simplified as one can generate photon configurations by the free theory."

He concentrates on Feynman diagrams without lepton loops. The correlation functions are obtained as statistical averages of quantities evaluated under each configuration. The QED without the fermion action is a free theory. The Euclidean lattice action in the Feynman gauge can be written as

$$S_{\text{QED}} = \frac{1}{2}\sum_{n,\mu} \hat{A}_\mu(n)(-\nabla^2 + m_\gamma^2)\hat{A}_\mu(n). \tag{1.40}$$

We take the unit $a = 1$ lattice spacing. The photon field, $\hat{A}_\mu(n)$, is a ultraviolet (UV) regularized one defined by acting a differential operator,

$$\hat{A}_\mu(n) \equiv H\left(-\frac{\nabla^2}{\Lambda_{\text{UV}}^2}\right) A_\mu(n),$$

with a function $H(x)$ to satisfy $H(x) \to 0$ for $x \gg 1$ and $H(x) \to 1$ for $x \ll 1$. In this study, we take a sharp UV cut-off,

$$H(x) = \begin{cases} 0, & x \geq 1, \\ 1, & x < 1. \end{cases}$$

 1.7 *Lattice quenched QED*

In QED, this cut-off is gauge invariant.

For the calculations of the correlation functions, we use the naive Dirac operator on the lattice:

$$(D)^{\alpha\beta}_{nm} = m\delta_{nm}\delta_{\alpha\beta}+\frac{1}{2}\sum_{\mu}\left[(\gamma_\mu)_{\alpha\beta}e^{ieA_\mu(n)}\delta_{n+\mu,m} - (\gamma_\mu)_{\alpha\beta}e^{-ieA_\mu(n-\mu)}\delta_{n-\mu,m}\right].$$

There are unwanted doublers in the propagators obtained by inverting this Dirac operator. We will take care of those unwanted modes later. The parameter m is the fermion mass and e is the coupling constant. We formally expand every quantity in terms of e, and evaluate the statistical averages of each coefficient. In this way, at any stage of the calculation, we do not need a specific value of e. The coupling constant e is not a parameter in the simulation. This is different from lattice simulations of nonperturbative dynamics where the continuum limit is taken by tuning the coupling constant to a critical value. Since there is no lepton loop in our study, e will not be renormalized. This means that e can be identified as the physical coupling constant.

They measure fermion–fermion–current three-point functions to obtain the form factors, in the position space in the Euclidean temporal direction and in the momentum space in the spatial direction.

Each photon configuration, $A_\mu(n)$, is generated according to the free field theory. We evaluate the quantity inside the bracket of

$$G_\mu(t) = \left\langle \sum_{\mathbf{p}'} D^{-1}(t_{\text{sink}}, t; \mathbf{p}, \mathbf{p}')\gamma_\mu D^{-1}(t, t_{\text{src}}; \mathbf{p}' + \mathbf{k}, \mathbf{p} + \mathbf{k})\right\rangle, \quad (1.41)$$

and take the ensemble average, $\langle\cdots\rangle$. The locations t_{src}, t_{sink} and t are those of two fermions and the current operator, respectively. We fix t_{src} and t_{sink} at some particular locations, and view the correlation function as a function of t for later purpose.

He takes $L \to \infty$ and $T \to \infty$. Since we introduce the photon mass m_γ as the IR regulator, the finite volume effects are suppressed exponentially as $e^{-m_\gamma L}$ and $e^{-m_\gamma T}$. Therefore, he keeps the combinations of $m_\gamma L$ and $m_\gamma T$ large.

projections to the electric and the magnetic functions:

$$G_E(t) = \text{tr}\left[\frac{1+\gamma_4}{2}G_4(t)\right], \quad G_M(t) = i\sum_{i,j,k}\epsilon_{ijk}\text{tr}\left[\frac{1+\gamma_4}{2}\gamma_5\gamma_i G_j(t)\right]\mathbf{k}_k,$$

where i, j, k are indices for the spacial directions, x, y and z. The Euclidean temporal direction is labelled as 4. The trace is taken over the spinor indices. The Dirac representation of the gamma matrices is convenient here since the projection, $(1 + \gamma_4)/2$, simply picks up the upper two components of the source and sink spinors.

We perform the lattice simulations with five sets of lattice volumes, $L^3 \times T$ = $24^3 \times 48$, $28^3 \times 56$, $32^3 \times 64$, $48^3 \times 96$, and $64^3 \times 128$.

2025-04-14 Predrag Have a look at:

> J. Dimock *Quantum electrodynamics on the 3-torus -I*, arXiv:math-ph/0210020: " We study the ultraviolet problem for quantum electrodynamics on a three dimensional torus. We start with the lattice gauge theory on a toroidal lattice and seek to control the singularities as the lattice spacing is taken to zero. This is done by following the flow of a sequence of renormalization group transformations."

1.8 Summary

> Everyone makes mistakes—including Feynman
> — Toichiro Kinoshita [118]

As of 2024, Sergey A. Volkov has checked the QED finiteness conjecture numerically, by computing the 5-loops gauge sets [194].

Worldline formalism could be useful on a qualitative level, as a way of proving the finiteness of QED conjecture,

1. Develop a saddle point expansion for the N-photon propagator integrals, such that the leading term explains the apparent $\approx \pm 1/2$ (or a multiple thereof) size of each quenched gauge set. Affleck *et al.* [2] and G. Torgrimsson *et al.* [190] show the way.

2. Use that to establish bounds on gauge sets for large orders, prove finiteness of quenched QED. If that works, I trust electron loop insertions will be next, and thereafter renormalons [138], etc., will go gently into that good night.

and in a precise way, as a new computational tool:

1. Develop a worldline formulation of spinor QED in which each gauge set is given by a computable integral, in a way to be fleshed out in sect. 1.3.2.

2. Parenthetically, a reformulation of the self-energy diagrams magnetic moment calculation, sect. 1.2.3, would be an even greater computational time saver - all quenched diagrams contributions calculated at one go.

3. In either case, a worldline formulation might make it possible to evaluate orders beyond 5-loops, as the number of gauge sets grows only polynomially. A win-win.

4. A gauge set is by definition UV and IR finite. The worldline formalism quenched QED needs wave function counterterms, as in (1.6).

5. Things get interesting with reformulating the quenched 3-loop calculation as four worldline integrals / gauge sets, see figure 1.5 and table 1.2. In particular, the fermion line attachments of different kinds of N-photon propagators now get intertwined.

6. One electron-loop insertion into $(1,0,0)$ might be the easiest worldline integral to evaluate, but I find the quenched sets a higher priority.

7. One photon-photon scattering electron-loop insertion into $(2,0,0)$ might be the most tempting to evaluate, but I find the quenched sets a higher priority.

My main problem at the moment (well, there are many:) is that nobody seems to have written an explicit formula for the spinor QED anomalous magnetic moment in the worldline formalism.

1.9 QCD gauge sets - a blog

In 1981 Cvitanović *et al.* [59] constructed gauge invariant subsectors in QCD.

2016-12-10 Predrag Penante [151] 2016 On-shell methods for off-shell quantities in N=4 Super Yang-Mills: from scattering amplitudes to form factors and the dilatation operator has an up-to-date review of on-shell methods.

2016-12-26 Predrag Read Cruz-Santiago, Kotko and Staśto [45] 2015 *Scattering amplitudes in the light-front formalism*: "The idea is to divide the process into appropriate gauge invariant components. It turns out that the gauge invariant subsets are invariant under cyclic permutations of the external gluons. This decomposition was proposed in works of [58–61] for the tree level amplitudes. A thorough analysis of the relation between color structures and gauge invariance was done in ref. [59]. The color decomposition principle was extended beyond the tree level to loop amplitudes in [63]."

2016-12-26 Predrag Should also read Dixon [64] 1996 *Calculating scattering amplitudes efficiently*.

2017-05-26 Predrag The decomposition of scattering amplitudes into gauge invariant subsets of diagrams is studied by Boos and Ohl [19, 28]. Boos and Ohl [28] *Minimal gauge invariant classes of tree diagrams in gauge theories*, arXiv:hep-ph/9903357 (see arXiv:hep-ph/9911437 and arXiv:hep-ph/0307057 for more detail) is motivated by applications to Standard Model multi-particle diagrams, mostly at the tree level.

Perturbative calculations require an explicit breaking of gauge invariance for technical reasons and the cancellation of unphysical contributions is not manifest in intermediate stages of calculations. The contribution of a particular Feynman diagram to a scattering amplitude depends in the gauge fixing procedure and has no physical meaning. the identification of partial sums of Feynman diagrams that are gauge invariant by themselves is of great practical importance. Calculation a subset of diagrams that is not gauge invariant has no predictive power, because they depend on unphysical parameters introduced during the gauge fixing.

A *gauge invariance class* is a minimal subset of Feynman diagrams that is independent of the gauge parameter and satisfies the Slavnov-Taylor identities.

The set of diagrams connected by flavor and gauge flips they call *forest*, a set of diagrams connected by gauge flips the call *grove*. They shown that the groves are the minimal gauge invariance classes of tree Feynman diagrams. In unbroken gauge theories, the permutation symmetry of external gauge quantum numbers can be used to subdivide the scattering amplitude corresponding to a grove further into gauge invariant sub-amplitudes.

This (largely uncited) work seems to have no impact on the $(g-2)$ gauge sets discussed here.

2017-05-27 Predrag Reuschle and Weinzierl [155] *Decomposition of one-loop QCD amplitudes into primitive amplitudes based on shuffle relations* cite our ref. [59]. They say:

QCD calculations organise the computation of the one-loop amplitude as a sum over smaller pieces, called *primitive amplitudes*. The most important features of a primitive amplitude are gauge invariance and a fixed cyclic ordering of the external legs. Primitive amplitudes should not be confused with *partial amplitudes* (also referred to as a *dual amplitude* or a *color-ordered amplitude*), which are the kinematic coefficients of the independent colour structures. The first step in a discussion of perturbative Yang-Mills is the decoupling of color from kinematics,

$$A_{tot} = \sum c_J A_J \tag{1.42}$$

where A_{tot} represents the total amplitude for a scattering process, A_J are all the possible color structures, and A_J are partial amplitudes which depend only on the kinematical data (momenta and polarizations). Partial amplitudes are gauge invariant, but not necessarily cyclic ordered. Partial amplitudes are far simpler to calculate than the full amplitude. There exist linear relations among the partial amplitudes, called Kleiss-Kuijf relations, which reduce the number of linearly independent partial amplitudes to $(n-2)$! The leading contributions in an $1/N$-expansion (with N being the number of colours) are usually cyclic ordered, the sub-leading parts are in general not. The decomposition of the full one-loop amplitude into partial amplitudes is easily derived. However, it is less trivial to find a decomposition of the partial amplitudes into primitive amplitudes.

There are several possible choices for a basis in colour space. A convenient choice is the colour-flow basis [1].

2017-05-27 Predrag Schuster [170] *Color ordering in QCD*: " We derive color decompositions of arbitrary tree and one-loop QCD amplitudes into color-ordered objects called primitive amplitudes. "

2017-05-27 Predrag Zeppenfeld [202] *Diagonalization of color factors* (Georgia Tech has no access to this paper)

2017-05-27 Predrag Edison and Naculich [72] *Symmetric-group decomposition of SU(N) group-theory constraints on four-, five-, and six-point color-ordered amplitudes at all loop orders*: " Color-ordered amplitudes for the scattering of n particles in the adjoint representation of SU(N) gauge theory satisfy constraints that arise from group theory alone. These constraints break into subsets associated with irreducible representations of the symmetric group S_n, which allows them to be presented in a compact and natural way. "

2017-05-27 Predrag Kol and Shir [126, 127] *Color structures and permutations* has a useful overview of the literature in the introduction, and is a very interesting read overall.

We may permute (or re-label) the external legs in the expression for a color structure and thereby obtain another color structure. This means that the space of color structures is a representation of S_n, the group of permutations. A natural question is to characterize this representation including its character and its decomposition into irreducible representations (irreps).

The decomposition of color structures into irreps was suggested by Zeppenfeld [202].

The space of tree-level color structures TCS_n of dimension

$$\dim(TCS_n) = (n-2)! \tag{1.43}$$

is the vector space generated by all diagrams with n external legs and an oriented cubic vertex, which are connected and without loops (trees), where diagrams which differ by the Jacobi identity are to be identified.

The f-based and t-based color structures are related by celebrated Kleiss-Kuijf [125] relations (rederived in this paper).

The original problem, that of capturing symmetries of the partial amplitudes which originate with those of the color structures, is now formulated as the problem of obtaining the S_n character of the space of color structures. It turns out that (at least at tree level) this problem was fully solved in the mathematics literature by Getzler and M. M. Kapranov [93].

The free Lie algebra over some set A, denoted by $L(A)$, is the Lie algebra generated by A with no further relations apart for antisymmetry and the Jacobi identity which are mandated by definition.

Self duality under Young conjugation: for some n values TCS_n is self-dual under Young conjugation, namely under the interchange of rows and columns in the Young diagrams

2017-05-27 Predrag Getzler and Kapranov [93] *Modular operads*: " We develop a 'higher genus' analogue of operads, which we call modular operads, in which graphs replace trees in the definition. We study a functor F on the category of modular operads, the Feynman transform, which generalizes Kontsevich's graph complexes and also the bar construction for operads. We calculate the Euler characteristic of the Feynman transform, using the theory of symmetric functions: our formula is modelled on Wick's theorem. "

2017-05-27 Predrag Maltoni *et al.* [143] *Color-flow decomposition of QCD amplitudes*

The *color-flow* decomposition is based on treating the $SU(N)$ gluon field as an $N \times N$ matrix. (PC: I think that is what I actually do.) It has several nice features. First, a similar decomposition exists for all multiparton amplitudes, like the fundamental-representation decomposition. Second, the color-flow decomposition allows for a very efficient calculation of multiparton amplitudes. Third, it is a very natural way to decompose a QCD amplitude. As the name suggests, it is based on the flow of color, so the decomposition has a simple physical interpretation.

To calculate the amplitude, one orders the gluons clockwise, and draws color-flow lines, with color flowing counterclockwise, connecting adjacent gluons. One then deforms the color-flow lines in all possible ways to form the Feynman diagrams that contribute to this partial amplitude. The Feynman diagrams that contribute to a partial amplitude are planar. This is not due to an expansion in $1/N$; the partial amplitudes are exact. They note (see their Table 1) that the number of Feynman diagrams contributing to an n-gluon partial amplitude grows as $\approx 3 \cdot 8^n$. In contrast, the number of Feynman diagrams contributing to the full amplitude grows factorially, as $\approx (2n)!$.

2017-06-09 Predrag Henn *et al.* [101] *Four-loop photon quark form factor and cusp anomalous dimension in the large-N_c limit of QCD*, arXiv:1612.04389, is a thoroughly modern paper, a listing of different codes used to generate diagrams and evaluate integrals.

2017-06-16 Predrag Chang, Liu and Roberts [40] *Dressed-quark anomalous magnetic moments*

2017-06-16 Predrag Choudhury and Lahiri [43] *Anomalous chromomagnetic moment of quarks*

2017-06-16 Predrag Bermudez *et al.* [23] *Quark-gluon vertex: A perturbation theory primer and beyond*, arXiv:1702.04437: The on-shell limit enables us to compute anomalous chromomagnetic moment of quarks.

[...] we present some “physically" relevant results for the on-shell limit $p^2 = k^2 = m^2$ and $q^2 = 0$. The Dirac and Pauli form factors, $F_1(q^2)$ and $F_2(q^2)$, respectively, define the Gordon decomposition of the quark current as in (1.3). The anomalous chromomagnetic moment (ACM) of quarks can be identified as $F_2(q^2)$ for $q^2 \to 0$. The Abelian version of this decomposition with $C_F = 1$ and $C_A = 0$ is the electron-photon vertex of quantum electrodynamics. The great successes of the Dirac equation is the prediction of the magnetic moment of a charged fermion $\mu = eg/(2m)S$. The radiative corrections lead to [172]

$$\frac{e}{2m} \Rightarrow \left(1 + \frac{\alpha}{2\pi}\right)\frac{e}{2m}. \tag{1.44}$$

Note that the quark-gluon vertex differs from the electron-photon vertex already at one loop, by the contributions of an additional Feynman diagram, involving the triple-gluon vertex. In fact, apart from introducing additional color structure, this non-Abelian diagram introduces, at the one-loop level, a kinematical structure which is absent in the QED.

It is straightforward to see that for the soft gluon limit, $q^2 = 0$, the Abelian contribution for the ACM reduces to the non-Abelian counterpart of Schwinger’s result, $F_2^a(0) = -\alpha/12\pi$, already derived in ref. [40]. On the other hand, the corresponding non-Abelian contribution yields a divergence [43], for a non-zero quark mass, $m \neq 0$. We find this divergence to be logarithmic. For deep infrared gluon momenta it behaves as $F_2^b(q^2 \to 0) = C_b \ln\left(-q^2/m^2\right)$. Of course, perturbation theory in QCD is not the way to explore deep infrared region. All perturbative conclusions will be taken over by non-perturbative effects, overshadowing this divergence.

2017-06-16 Predrag Brambilla *et al.* [31] *QCD and strongly coupled gauge theories: challenges and perspectives*

2017-06-16 Predrag Simonov and Tjon [180] *The Feynman–Schwinger representation in QCD*:“ The proper time path integral representation is derived for Green’s functions in QCD. After an introductory analysis of perturbative properties, the total gluonic field is separated into a nonperturbative background and valence gluon part. For nonperturbative contributions the background perturbation theory is used systematically, yielding two types of expansions. As an application, we discuss the collinear singularities in the Feynman–Schwinger representation formalism. ”

2017-06-16 Predrag Kälin [114] *Cyclic Mario worlds – color-decomposition for one-loop QCD*, arXiv:1712.03539: “ a new color decomposition for QCD amplitudes at one-loop level [...] Starting from a minimal basis of planar primitive amplitudes we write down a color decomposition that is free of linear dependencies among appearing primitive amplitudes or color factors. [...] The standard SU(N) trace-based color decomposition at tree level does not take advantage of all linear dependencies of primitive amplitudes and color factors, as the sum goes over an overcomplete set of linear dependent primitive amplitudes and color factors. [...] we propose a compact color decomposition that eliminates linear dependencies of one-loop QCD amplitudes ”

He uses “Melia’s basis,” “Dyck words.”

2018-12-26 Tomeu Fiol, Jairo, Alan

- You are of course right that some of the color invariants that we evaluated are already computed in your “Birdtracks" book. We will include the reference in the upcoming revised version of our preprint.
- We did considered 1PI chord diagrams, but since we were chiefly interested in computing the logarithm of $< W >$, we didn’t find an immediate advantage in working with them, instead of with connected chord diagrams.
- Some months ago we realized that chord diagrams also appear in the evaluation of the fermion 2-point function in quenched QED. This led us to discover your paper [49] *Asymptotic estimates and gauge invariance*. However, we currently don’t see that the techniques we used in our paper can be of any use to address the conjecture you formulated there: while we are in principle computing quantities in a four dimensional non-Abelian gauge theory, the high degree of supersymmetry reduces the evaluation of the logarithm of $< W >$ to a problem in 0-dimensional QFT. A recent paper on quenched QED in 0 dimensions is

A recent paper on quenched QED in 0-dimensions is Borinsky [30] *Renormalized asymptotic enumeration of Feynman diagrams*, arXiv:1703.00840.

2019-06-27 Predrag I have not continued the discussion with Tomeu Fiol, on their Fiol, Martínez-Montoya and Fukelman [82] *Wilson loops in terms of color invariants*, but for my notes see sect. **??** *Non-Abelian exponentiation theorem*.

2019-07-23 Predrag Sidorov, Lashkevich and Solovtsova [177] *On the contribution of muon loops to the anomalous magnetic moment of the muon* might be hard to get, but I have a hard copy, if needed. The diagrams that they consider either have no lepton loops or contain only loops of the same leptons that are present in the external legs. The results are the same for electron or muon, so I'll refer only to electron from now on. They compute the anomaly $a^{(2n+2)}[v.p.]$ for single photon vertex with n electron loop insertions into the photon line analytically up $n = 7$ loops, and numerically up $n = 18$ (here $[v.p.]$ stands for "vacuum polarization).

From their analytic results it seems that the answer is always of the form

$$a^{(2n+2)}[v.p.] = c_1 + \sum_{i=2}^{n+1} c_i \zeta(i) \,, \tag{1.45}$$

and they observe some regularities in the signs of c_i. Up to $n = 6$ the value of $a^{(2n+2)}[v.p.]$ decreases, then it starts increasing. What is striking about their results is that while the individual terms in (1.45) are analytically unilluminating and rather large, their sum is very small. They quantify this by defining a "reduction factor," the ratio of $a^{(2n+2)}[v.p.]$ and the largest term in (1.45). That is very small, for example for $a^{(16)}[v.p.]$ the factor is of order 4×10^{-5}, and apparently decreasing with increasing n. To me that suggests that one should not use $\zeta(i)$ as the basis for such Feynman integrals, instead define close by (rational?) $b_i \simeq c_i$ such that

$$\hat{\zeta}(n+1) = b_1 + \sum_{i=2}^{n+1} b_i \zeta(i) = 0 \,, \tag{1.46}$$

or something in that spirit.

The analytic formula for 1-lepton loop 1PI photon polarization is taken from Aguilar, de Rafael and Greynat [3] *Muon anomaly from lepton vacuum polarization and the Mellin-Barnes representation*.

Baikov, Maier and Marquard [18] *The QED vacuum polarization function at four loops and the anomalous magnetic moment at five loops* is lots of computational series that I do not find helpful. The references, however, might be of interest.

Logashenko and Eidelma [142] *Anomalous magnetic moment of the muon* is a review, safely ignored here.

Strangely, they do not mention "renormalon," even though their explanation why $a^{(2n+2)}[v.p.]$ grow with n, their fig. 2 suggest that the large n growth could be estimated by a saddle-point method.

The renormalon divergence was discovered in 1970s, see Beneke [22] *Renormalons* and Lautrup [138] *On high order estimates in QED*. Lutrup notes that such n-lepton loop diagrams (each single diagrams a gauge set) give contributions to the anomaly whose amplitude grows like $n!$, and estimates the leading asymptotic behaviour.

Cavalcanti *et al.* [39] *Appearance and disappearance of thermal renormalons* haave an example where renormqaons matter and do not matter :)

1.10 Is QED finite? Notes

1.10.1 Quenched QED/QFT

1971-08-01 Lautrup, Peterman, and de Rafael 1972 *Recent developments in the comparison between theory and experiments in quantum electrodynamics* [139] list the 3-loop, no-electron loop "gauge invariant subclasses" (their Fig. 4.3).

The inventor of the "gauge invariant diagram sets" concept is Benny Lautrup.

1974-01-07 Samuel 1974 *Estimates of the eighth-order corrections to the anomalous magnetic moment of the muon* [162]:

"We speculate that in making radiative corrections to a class of graphs by inserting a single photon in all possible ways, one obtains a contribution which is roughly $-\frac{\alpha}{\pi}$ times the contribution of the class. This seems to be obeyed by the known contributions."

2013-11-24 Predrag As far as I can tell, terminology "of *quenched* type" was first introduced in this context by Marinari, Parisi and Rebbi [144] *Monte Carlo simulation of the massive Schwinger model*, in the context of lattice gauge theory. They write: "A first approximation to the effect of the gauge field on the fermion observables may be achieved by [...] neglecting the contributions from the fermionic vacuum polarization diagrams and, using a terminology developed in the theory of condensed matter, we shall call the expectation values thus obtained 'quenched'",

Parisi is reputable, so in the "quenched approximation" one neglects the fermionic vacuum polarization effects (i.e, the fermion loops) from the fermion determinant in the effective action. If it is good enough for Kinoshita, it is good enough for you.

"In the 'quenched approximation' the quark determinant is set equal to unity, i.e., neglecting the effect of virtual quark loops. In other words, this extreme approximation in terms of heavy quarks with a vanishing number of flavors assumes that gauge fields affect quarks while quarks have no dynamical effect on gauge fields."

A more general usage: "In Quantum Cosmology "quenching," amounts to quantizing a single scale factor thereby selecting a class of cosmological models, for instance, the Friedmann-Robertson-Walker space-time while neglecting the quantum fluctuations of the full metric."

But I still do not like it - it is mostly associated with Kogut, where it means something different (as in "... treating the gauge interaction in the quenched, planar (ladder) approximation"); search for quench here. Or here is what Brezin says:

"concept of quenching is well-known in the statistical mechanics of random media ; consider a system of particles, for instance, an electron gas, interacting with impurities. If these impurities are mobile, they will thermalize with the electron gas and the average physical quantities are obtained by a trace over the electron gas and the impurities degrees of freedom. However if the impurities are frozen, the 'quenched' case, the physical observables are obtained by calculating their value for fixed impurities and then averaging over these impurities. "

That is how I know it - nothing about fermion loops, just dirt physics...

From my point of view, the question is whether the sum of all corrections to (g-2) is a convergent series, or an asymptotic one. If on can prove the convergence for the quenched sector, I would expect each un-quenched sector (diagrams with one, two, ... lops) separately to be convergent, and their sum as well.

Here is something to amuse you: on amplituhedron. More serious: Lance Dixon on calculating amplitudes.

1.10.2 Is QED finite? A blog, continued

1950-12-22 Schwinger *On gauge invariance and vacuum polarization* [173] has 3700 citations. He writes: " This paper is based on the elementary remark that the extraction of gauge invariant results from a formally gauge invariant theory is ensured if one employs methods of solution that involve only gauge covariant quantities. We illustrate this statement in connection with the problem of vacuum polarization by a prescribed electromagnetic field. The vacuum current of a charged Dirac field, which can be expressed in terms of the Green's function of that field, implies an addition to the action integral of the electromagnetic field. Now these quantities can be related to the dynamical properties of a "particle" with space-time coordinates that depend upon a proper-time parameter. The proper-time equations of motion involve only electromagnetic field strengths, and provide a suitable gauge invariant basis for treating problems. Rigorous solutions of the equations of motion can be obtained for a constant field, and for a plane wave field. A renormalization of field strength and charge, applied to the modified lagrange function for constant fields, yields a finite, gauge invariant result which implies nonlinear properties for the electromagnetic field in the vacuum. [...] one can employ an expansion in powers of the potential vector. The latter automatically yields gauge invariant results, provided only that the proper-time integration is reserved to the last. This indicates that the significant aspect of the proper-time method is its isolation of divergences in integrals with respect to the proper-time parameter, which is independent of the coordinate system and of the gauge. The connection between the proper-time method and the technique of "invariant regularization" is discussed. Incidentally, the probability of actual pair creation is obtained from the imaginary part of the electromagnetic field action integral. Finally, as an application of the Green's function for a constant field, we construct the mass operator of an electron in a weak, homogeneous external field, and derive the additional spin magnetic moment of $\alpha/2\pi$ magnetons by means of a perturbation calculation in which proper-mass plays the customary role of energy. "

" A proper time wave equation, in conjunction with the second order Dirac operator, has been discussed by V. A. Fock [83] *Proper time in classical and quantum mechanics*. See also Nambu [149] (1950). "

 1.10 *Is QED finite? Notes*

In Appendix B Schwinger computes the anomalous spin magnetic moment $\alpha/2\pi$ produced by second-order electromagnetic mass effects.

1977-03-03 Drell and Pagels *Anomalous magnetic moment of the electron, muon, and nucleon* [65] attempt got the sign right, but was not successful in predicting the magnitude of the sixth-order magnetic moment; $0.15\left(\frac{\alpha}{\pi}\right)^3$ instead of $1.19\left(\frac{\alpha}{\pi}\right)^3$.

2013-12-08 Predrag to Piotr, Wanda and Andrea (Piotr Czerski <piotr.czerski@ifj.edu.pl>, wanda.alberico@to.infn.it, andrea.prunotto@gmail.com):

I'm no fan of Feynman diagrams (my rant is here), and I'm always looking for other ways to look at perturbative expansions. So just a little email - if you have a new angle [154] on subsets of diagrams which are gauge sets, I would be curious to learn how you look at that.

Just something to keep in mind :)

PS to Andrea: I realize you might rather forget this stuff (takes you a decade to write a paper?) but at least I got a ringtone out of you. The only problem is, I do not have a cell phone, so I do not know how to make it ring. At least I'm more technologically savvy than Peter Higgs.

2013-12-10 Andrea Sorry for late reply (well, we're used to longer gaps). Yes! I actually took 10 years to write this paper out of my master thesis, but I have some excuses: I did my PhD in Biochemistry (Zürich) and now I work on genetics (Lausanne). This summer my "old" professor Wanda found my work in a drawer and then contacted me, telling me that it would be a good idea to publish it.

About your request: I'm really interested in seeing if the rooted-map approach to Feynman diagrams can address the problem you've risen. But I have no idea what the "subsets of diagrams which are gauge invariant sets" are. I've checked a bit on the web but I'm sure you can give me better indications (the works I found were too technical: I need to know the basis of the problem). Can you send me some specific link at freshman level, in particular where I can see the geometry of these subclasses of diagrams?

2013-12-11 Predrag Googling is good, but it is faster to click on this link. The article defines the gauge sets.

2014-02-11 M. Borinsky *Feynman graph generation and calculations in the Hopf algebra of Feynman graphs* [29] "Programs for the computation of perturbative expansions of quantum field theory amplitudes are provided. feyngen can be used to generate Feynman graphs for Yang-Mills, QED and phi^k theories. feyncop implements the Hopf algebra of those Feynman graphs which incorporates the renormalization procedure necessary to calculate finite results in perturbation theory of the underlying quantum field theory. "

2016-02-08 Predrag Prunotto [153] *A Homological Approach to Feynman Diagrams in the Quantum Many-Body Theory*, and Prunotto, Alberico and Czerski [154] 2013 *Feynman Diagrams and Rooted Maps* has been submitted to the European

Physical Journal A as manuscript ID EPJA-103480, seems not to have been published anywhere by 2017. They write: “ The Rooted Maps Theory, a branch of the Theory of Homology, is shown to be a powerful tool for investigating the topological properties of Feynman diagrams, related to the single particle propagator in the quantum many-body systems. The numerical correspondence between the number of this class of Feynman diagrams as a function of perturbative order and the number of rooted maps as a function of the number of edges is studied. A graphical procedure to associate Feynman diagrams and rooted maps is then stated. Finally, starting from rooted maps principles, an original definition of the genus of a Feynman diagram, which totally differs from the usual one, is given. ”

2017-03-15 Predrag Dunne and Krasnansky [66] 2006 *“Background field integration-by-parts” and the connection between one-loop and two-loop Euler-Heisenberg effective actions*: “ We develop integration-by-parts rules for diagrams involving massive scalar propagators in a constant background electromagnetic field, and use these to show that there is a simple diagrammatic interpretation of mass renormalization in the two-loop scalar QED Euler-Heisenberg effective action for a general constant background field. This explains why the square of a one-loop term appears in the renormalized two-loop Euler-Heisenberg effective action, and dramatically simplifies the computation of the renormalized two-loop effective action for scalar QED, and generalizes a previous result obtained for self-dual background fields. ”

2017-05-23 Predrag M. G. Schmidt and C. Schubert [164] 1994 *Multiloop calculations in the string-inspired formalism: the single spinor-loop in QED*, arXiv:hep-th/9410100: They use the worldline path-integral Bern-Kosower formalism for to calculate the sum of all diagrams with one spinor loop and fixed numbers of external and internal photons. Of interest: in this formalism the three 2-loop photon polarization graphs, see figure 1.8, are a single integral, easier to evaluate than any of the three Feynman graphs. They also note an unexplained cancelation not only of poles, but also of “transcedentals.” A knot-theoretic explanation for the rationality of the quenched QED beta function is given in ref. [33].

2017-05-23 Predrag Nieuwenhuis and Tjon [150] 1996 *Nonperturbative study of generalized ladder graphs in a $\phi^2\chi$ theory*, arXiv:hep-ph/9606403:

2017-05-23 Predrag Christian Schubert [168] 2001 *Perturbative quantum field theory in the string-inspired formalism*, arXiv:hep-th/0101036:

The Feynman rules for (Euclidean) spinor QED in the second order formalism (see Morgan [148] 1995, Strassler [186] 1992, and references therein) are, up to statistics and degrees of freedom, the ones for scalar QED with the addition of a third vertex. The third vertex involves $\sigma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]$ and corresponds to the $\psi^\mu F_{\mu\nu}\psi^\nu$ – term in the worldline Lagrangian L_{spin}. For the details and for the non-abelian case see Morgan [148]. There also an algorithm is given, based on the Gordon identity, which transforms the sum of Feynman (momentum) integrals resulting from the first order rules into the ones generated by the second order rules.

2017-06-12 Predrag Morgan [148] *Second order fermions in gauge theories*, arXiv:hep-ph/9502230 seems to be the same discussion that I use in my QFT course to define the electron magnetic moment via $\sigma^{\mu\nu}$ (see lectures 25 and 26 here)

2017-06-12 Predrag Gies, Sanchez-Guillen, Vázquez [96] *Quantum effective actions from nonperturbative worldline dynamics*, arXiv:hep-th/0505275: " We demonstrate the feasibility of a nonperturbative analysis of quantum field theory in the worldline formalism with the help of an efficient numerical algorithm. In particular, we compute the effective action for a super-renormalizable field theory with cubic scalar interaction in four dimensions in quenched approximation (small- N f expansion) to all orders in the coupling. We observe that nonperturbative effects exert a strong influence on the infrared behavior, rendering the massless limit well defined in contrast to the perturbative expectation. "

2017-06-12 Predrag Giesand Hämmerling [94] *Geometry of spin-field coupling on the worldline*, arXiv:hep-th/0505072:

2017-03-15 Predrag Huet, McKeon, and Schubert [103] 2010 *Euler-Heisenberg Lagrangians and asymptotic analysis in 1+1 QED. Part I: Two-loop* (no GaTech online access, arXiv:1010.5315):" We continue an effort to obtain information on the QED perturbation series at high loop orders, and particularly on the issue of large cancellations inside gauge invariant classes of graphs, using the example of the l-loop N-photon amplitudes in the limit of large photon numbers and low photon energies. The high-order information on these amplitudes can be obtained from a nonperturbative formula, due to Affleck *et al.* [2], for the imaginary part of the QED effective Lagrangian in a constant field. The procedure uses Borel analysis and leads, under some plausible assumptions, to a number of nontrivial predictions already at the three-loop level. Their direct verification would require a calculation of this 'Euler-Heisenberg Lagrangian' at three-loops, which seems presently out of reach (though see Huet, Rausch de Traubenberg, and Schubert [105, 106] below). Motivated by previous work by Dunne and Krasnansky [66] on Euler-Heisenberg Lagrangians in various dimensions, in the present work we initiate a new line of attack on this problem by deriving and proving the analogous predictions in the simpler setting of 1+1 dimensional QED. In the first part of this series, we obtain a generalization of the formula of Affleck *et al.* [2] to this case, and show that, for both scalar and spinor QED, it correctly predicts the leading asymptotic behaviour of the weak field expansion coefficients of the two loop Euler-Heisenberg Lagrangians.

"The present work continues an effort [68–70, 145] to study the multiloop behaviour of the QED N-photon amplitudes using the QED effective Lagrangian, and in particular to prove or disprove Cvitanović's conjecture for these amplitudes."

2017-08-04 Predrag Valluri, Jentschura and Lamm [191] *The study of the Euler-Heisenberg Lagrangian and some of its applications*

2019-04-28 Predrag Huet, Rausch de Traubenberg and Schubert [106] *Dihedral invariant polynomials in the effective Lagrangian of QED* present a group-theoretical technique to calculate weak field expansion for a particular Feynman diagram using invariant polynomial basis for the dihedral group D_4. They show results obtained for the first coefficients of the three loop effective Lagrangian of 1+1 scalar QED in an external constant field. Their results suggest that a closed form involving rational numbers and values of the Riemann zeta function might exist for these coefficients.

The calculation is motivated by their 'exponential conjecture' eq. (1) which relates the all-order QED Euler-Heisenberg Lagrangian to the 1-loop Lagrangian. They say that the conjecture is "closely linked to Cvitanović's conjecture [49] on the convergence of the quenched series of QED.

They already know that QED self-dual fields provide substantial computational simplifications [67–69]. They note that the crossed-photons, Schwinger-parametrized diagram B in their fig. 3 has D_4 discrete symmetry.

Then they use something I do not remember seing before, but is sure very familiar in form: *Molien's generating function* (1.47) for the numbers of elements in the integrity basis for a given finite group.

Needless to say, there is a ton of references on Molien's function. Some random ones: my one-time budy Hanany and Mekareeya [99] *Counting gauge invariant operators in SQCD with classical gauge groups*, arXiv:1311.0746 uses the *plethystic exponential*, the *plethystic logarithm* (the two are related by the Möbius function), and the Molien-Weyl formula, with another ton of references to everything. In the series expansion of the plethystic logarithm the first terms with plus signs give the basic generators, while the first terms with the minus signs give the constraints between these basic generators.

arXiv:1311.0746

arXiv:1110.4891 focus is on breaking symmetries by means of invariant polynomials, for example $\mathrm{SO}(3) \to S_4$. (Heisenberg group shows up in their eq. (23), though they do not call it that.) Their "Mini group theory compendium" where they state a bunch of group-theoretic notions, might be useful. Their Figure 5 is mindboggling and instructive, a partial subgroup tree of $\mathrm{SU}(3)$. Even that is complicated. They generalize the Molien's generating function (1.47) to the generating function for counting covariant tensors, and work out S_4 in detail.

The spherical harmonics of $\mathrm{SU}(3)$ are known as the *complex spherical harmonics*. They construction explicitly the polynomial bases for (p, q) representations.

According to Holger Schellwat, *A gentle introduction to a beautiful theorem of Molien*, arXiv:1701.04692, Molien's Theorem (1897)) is a power series generating function formula for counting the number n_k of *linearly* independent polynomials (dimensions of integrity bases) of given degree which are invariant under the action of a finite group.

In general these polynomials are not algebraically independent, and Huet *et al.* [106] explain nicely the relations between primitive invariants, secondary invariants and syzygies that relate them. There also exist *semi-invariant* polynomials on whom the group acts by multiplying them by a group-element dependent overall complex phase, their Definition 2.5.

In calculations they use SINGULAR [62], an algebra system for polynomial computations (click here). They work out the D_4 integrity basis in detail and eventually succeed in analytically evaluating the Schwinger-parametrized diagram B, a nontrival calculation, but apparently very difficult without their application of group theory. They speculate that other diagrams with a high degree of symmetry could be computed this way, but do not have any further examples.

My remark is: every quenched QED wordline diagram has a finite group of discrete symmetries under relabeling photon insertions. Should we use this before starting any worldline integral evaluation (essential not to split the calculation into sum of individual Feynman diagrams).

Tangent 1: "Heisenberg group" seems to be closely related to cyclic group.

Tangent 2: (click here) proves that[5]
If A is a skew-symmetric matrix, then $I+A$ is nonsingular and $(I-A)(I+A)^{-1}$ is orthogonal.

As an appetizer, Schellwat displays this stunning formula:

$$\Phi(t) = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det\left(\mathbf{1} - tg\right)} = \sum_{k=0}^{\infty} n_k t^k \,, \tag{1.47}$$

where g is a shorthand for $D^\lambda(g)$, an n-dimensional irrep λ of G. One can immediately see elements of linear algebra, representation theory, and enumerative combinatorics in it, all linked together. From then on, going gets rougher, as this is a mathematics paper. He proves it on p. 15, using (roughly - will have to ponder, it looks like a relation between characters and determinants)

$$\mathrm{tr}\, D(g^d) = [t^d] : \frac{1}{\det\left(\mathbf{1} - tg\right)} \,, \tag{1.48}$$

where notation $[t^d]$: stands for "the coefficient of t^d term in the Taylor expansion of the function."

2017-05-24 Predrag Bastianelli, Huet, Schubert, Thakur and Weber [21] 2014 *Integral representations combining ladders and crossed-ladders* write:

This property is particularly interesting in view of the fact that it is just this type of summation which in QED often leads to extensive cancellations, and to final results which are substantially simpler than intermediate ones (see, e.g., ref. [33, 49]). More recently, similar cancellations have been found also for graviton amplitudes (see, e.g., ref. [17]). Although this property of the worldline formalism is well-known, and has been occasionally exploited [20, 157, 158,

[5]Predrag: 2019-04-28 move to Group Theory notes; Molien has to go to ChaosBook slicing chapter, eventually.

164] (see also ref. [88]) a systematic study of its implications is presently still lacking.

The first classes of Green's functions is the x-space propagator for one scalar interacting with the second one through the exchange of N given momenta.

This object, to be called "N-propagator", is given by a set of $N!$ simple tree-level graphs, is in the worldline formalism combined into a single integral.

The second class are the similarly looking x-space $N + 2$ - point functions defined by a line connecting the points x and y and N further points $z_1, \ldots, z_N$ connecting to this line in an arbitrary order.

An advantage of the worldline representation over the usual Feynman parameterization is the automatic inclusion of all possible ways of crossing the "rungs" of the ladders. They obtain such representations in explicit form both in x-space and in momentum space.

The inclusion of the crossed ladder graphs is essential for the consistency of the one-body limit where one of the constituents becomes infinitely heavy, and for maintaining gauge invariance.

They concentrate on the case of infinite N, *i.e.*, the sum over *all* ladder *and* crossed ladder graphs.

As their main application, they consider the case of two massive scalars interacting through the exchange of a massless scalar, obtain an the case of a massless exchanged particle (along the "rungs" of the ladders).

Applying asymptotic estimates and a saddle-point approximation to the N-rung ladder plus crossed ladder diagrams, they derive a semi-analytic approximation formula for the lowest bound state mass in this model.

They use the worldline formalism to derive integral representations for the N-propagators and the N-ladders - in scalar field theory, and give a compact expression combining the $N!$ Feynman diagrams contributing to the amplitude. They give these representations in both x and (off-shell) momentum space. Being off-shell, can be used as building blocks for more complex amplitudes. They derive a compact expression for the sum of all ladder graphs with N rungs, including all possible crossings of the rungs.

Nieuwenhuis and Tjon [150] 1996 have numerically evaluated the path integrals of the worldline representation for the same scalar model field theory, thus including all ladder *and* crossed ladder graphs.

2017-05-23 Predrag Huet, Rausch de Traubenberg, and Schubert [105] 2017 *Multi-loop Euler-Heisenberg Lagrangians, Schwinger pair creation, and the photon S-matrix*: "Schwinger pair creation in a constant electric field, may possibly provide a window to high loop orders; simple non-perturbative closed-form expressions have been conjectured for the pair creation rate in the weak field limit, for scalar QED in 1982 by Affleck, Alvarez, and Manton [2], and for spinor QED by Lebedev and Ritus [140] in 1984. Using Borel analysis, these can be used to obtain non-perturbative information on the on-shell renormalized N-photon amplitudes at large N and low energy."

"there is something quite implausible about it: a summation to all loop orders has produced the perfectly analytic factor $e^{\alpha\pi}$! This is certainly contrary to standard QED wisdom"

Preliminary results of a calculation of the three-loop Euler-Heisenberg Lagrangian in two dimensions indicate that the exponentiation conjecture by Affleck *et al.* and Lebedev/Ritus probably fails in $D = 2$.

Dunne and Schubert conjectured in 2005 that the QED $N-$photon amplitudes in the quenched (one electron loop) approximation are convergent in perturbation theory [70]. In this article they say; "Later they learned that Cvitanović in 1977 had already made the analogous conjecture for $(g-2)$ [49]."

2018-12-02 Predrag Gould, Mangles, Rajantie, Rose and Xie [97] *Observing thermal Schwinger pair production*, arXiv:1812.04089: "

"

2017-06-14 Predrag Academician Ritus [156] writes: " The requirement $e^2/\hbar c = 1$ leads to unique values of the point-like charge and its fine structure constant, $e_0 = \pm\sqrt{\hbar c}$, $\alpha_0 = 1/4\pi$. Arguments are adduced in favor of the conclusion that this value of the fine structure constant is the bare, nonrenormalized value. "

In 1951 Ritus was assigned to what was known at the time as the First Main Directorate of the USSR Council of Ministers, later rechristened the Ministry of Medium Machine Building (Sredmash) – a powerful state body placed above any other in the name of implementing the Soviet Government sponsored program of thermonuclear weapons design. [...] The legend of his infallibility when conducting complicated and cumbersome computations, just started to take root; its protagonist did nothing that would sully this reputation, neither then nor later. Science is unaware of any errors ever made by Ritus! (from ref. [27].)

2017-05-23 Predrag Das, Frenkel and Schubert [61] 2013 *Infrared divergences, mass shell singularities and gauge dependence of the dynamical fermion mass*; arXiv:1212.2057:

2017-05-23 Predrag Ahmadiniaz, Bashir and Schubert [5] 2016 *Multiphoton amplitudes and generalized Landau-Khalatnikov-Fradkin transformation in scalar QED*, arXiv:1511.05087:

$D_{\mu\nu}$ is the x-space photon propagator. In D dimensions and arbitrary covariant gauge

$$D_{\mu\nu}(x) = \frac{1}{4\pi^{\frac{D}{2}}}\Big\{\frac{1+\xi}{2}\Gamma\Big(\frac{D}{2}-1\Big)\frac{\delta_{\mu\nu}}{(x^2)^{\frac{D}{2}-1}}+(1-\xi)\Gamma\Big(\frac{D}{2}\Big)\frac{x_\mu x_\nu}{(x^2)^{\frac{D}{2}}}\Big\}\,. \quad (1.49)$$

Calculation methods:

1. The analytic or "string-inspired" approach, based on the use of worldline Green's functions: all path integrals are brought into Gaussian form; this requires some expansion and truncation. They are then calculated by Gaussian integration.
2. The semi-classical approximation, based on a stationary trajectory ("worldline instanton").

We will focus on the closed-loop case in the following, since it turns out to be simpler than the propagator one. Nevertheless, it should be emphasized that everything that we will do in the following for the effective action can also be done for the propagator.

Some reasonable gymnastics leads to the "Bern-Kosower master formula" [24, 25, 186]

2017-05-23 Predrag Strassler [186] *Field theory without Feynman diagrams: One-loop effective actions*, arXiv:hep-ph/9205205;

2017-05-23 Predrag Ahmad *et al.* [4] 2017 *Master formulas for the dressed scalar propagator in a constant field*, arXiv:1612.02944

2007-01-31 Kurusch Ebrahimi-Fard Here are the links I mentioned:

Anatomy of a gauge theory by Dirk Kreimer, arXiv:hep-th/0509135

Renormalization of gauge fields: A Hopf algebra approach by Walter D. van Suijlekom, arXiv:hep-th/0610137

The Hopf algebra of Feynman graphs in QED by Walter D. van Suijlekom, arXiv:hep-th/0602126

This is Jean-Yves Thibon's web-page, a very good combinatorialist!

2016-11-15 Kevin Hartnett Strange Numbers Found in Particle Collisions

2017-05-23 Predrag Should I talk to Spencer Bloch?

2017-05-23 Predrag Broadhurst, Delbourgo and Kreimer [33] 1996 *Unknotting the polarized vacuum of quenched QED* has lots of magic leading to cancelations of "transcedentals." They say: "Complete cancellation of transcendentals from the beta function, at every order, is to be expected only in quenched QED and quenched SED, where subdivergences cancel between bare diagrams."

Online collection of papers on this topic.

2013-10-23 Warren D. Smith D. J. Broadhurst and D. Kreimer: *Association of multiple zeta values with positive knots via Feynman diagrams up to 9 loops*, Physics Letters B 393 (1997) 403-412, arXiv:hep-th/9609128.

Furthermore, the number of different kinds of knots with N crossings, KnotCount(N), is known asymptotically to be bounded between two simple-exponentials,

$$A^N < \text{KnotCount}(N) < B^N$$

where $B \leq 13.5$ according to
D.J.A. Welsh,*On the number of knots and links, Sets, graphs and numbers* (Budapest, 1991), 713– 718, Colloq. Math. Soc. Janos Bolyai, 60, North-Holland, Amsterdam, 1992,
while $A \geq 2.68$ according to
C.Ernst & D.W.Sumners *The growth of the number of prime knots*, Proc Cambridge Philo Soc 102 (1987) 303-315.

So, that's the funny thing. My attempt to further-destroy Cvitanović, just led to an estimate involving simple exponential growth and NOT superexponential (e.g. factorial style). This is right on the boundary for convergence questions, i.e.

$$\sum A_N x^N$$

has a finite, nonzero radius of convergence if $|A_N|$ grows exponentially. So maybe there remains some hope for some form of Cvitanović conjecture.

The Welsh upper bound also works for links. Which means: if you believe this could rescue Cvitanović's quenched-QED convergence conjecture, that would also presumably mean you believe full unquenched QED series have finite nonzero radius of convergence.

2013-11-25 David Broadhurst <David.Broadhurst@open.ac.uk>

Dirk and kind of gave up when it turned out that a pair of counterterms at 7 loops, unidentified in 1996, have weight 11, whereas our intuition about the knots 10_139 and 10_154 has suggested weight 10.

Maybe there is some sort of connection, but I know not what.

2017-05-23 Predrag Dirk Kreimer and Karen Yeats [132] 2008 *Recursion and growth estimates in renormalizable quantum field theory*

Our method is very different in spirit from the constructive approach or the functional integral approach. It relies on a Hopf algebraic decomposition of terms in the perturbative expansion into primitive constituents, not unlike the decomposition of a ζ function into Euler factors.

Our construction of a basis of primitives with a given Mellin transform resolves overlapping divergences, thanks to the Hochschild cohomology of the relevant Hopf algebras [130].

We next assume there to be $p(k)$ primitives at k loops where p is a polynomial.

Yeats [201] 2017 *A Combinatorial Perspective on Quantum Field Theory*. I have put a copy here.

2016-08-20 Predrag Kißler [120] *Hopf-algebraic renormalization of QED in the linear covariant gauge*: "The possibility of a finite electron self-energy by fixing a generalized linear covariant gauge is discussed. An analysis of subdivergences leads to the conclusion that such a gauge only exists in quenched QED."

2017-06-16 Henry Kißler The term "finite electron self-energy" does not refer to the convergence of the perturbation series, but to an order-by-order cancellation of divergences. The idea was to gauge away all divergences in the self-energy order-by-order as used by Broadhurst [32] in "Four-loop Dyson-Schwinger-Johnson anatomy" for quenched QED.

2017-06-16 Predrag Broadhurst [32] *Four-loop Dyson-Schwinger-Johnson anatomy*: "Dyson–Schwinger equations are used to evaluate the 4-loop anomalous dimensions of quenched QED in terms of finite, scheme-independent, 3-loop integrals. The 4-loop beta function has 24 unambiguous terms. The rational, $\zeta(3)$ and $\zeta(5)$ parts of the other 22 miraculously sum to zero. Vertex anomalous dimensions have 40 terms, with no dramatic cancellations. Our methods come from work by the late Kenneth Johnson, done more than 30 years ago. They are entirely free of the subtractions and infrared rearrangements of later methods."

2016-08-20 Predrag Kißler and Kreimer [121] 2016 *Diagrammatic cancellations and the gauge dependence of QED*, arXiv:1607.05729: " The perturbative expansion given in terms of Feynman graphs might be rearranged in terms of meta graphs or subsectors with a maximum number of cancellations implemented. "

"The summation over all insertions requires considering *connected* rather then one-particle irreducible Feynman graphs."

In his (unpublished) thesis Henry defines the above "self-energy set" as an "equivalence class" obtained by all insertions of an external electron-photon vertex into the external electron line propagator of a self-energy graph. On mass-shell the summation over this class is gauge invariant by the Ward-Takahashi identity [187, 199].

This gets more interesting when he considers insertion of internal photon propagator "gaugeons" (a Lautrup name?). Then, for example (his eq. (2.75)) the sum of all connected quenched 2-loop graphs is equivalent to the one-loop self energy. Perhaps iteration of such insertions into Schwinger 1-loop anomaly might offer a proof of invariance of gauge sets...

" The four-photon interaction is called light-by-light scattering and supposed to be finite by renormalizability –in other words there is no interaction vertex in Quantum Electrodynamics that could serve to absorb the divergences of the four-photon type. Unfortunately, the author is not aware of a general argument beyond the first-loop order that proves this statement. In arXiv:1406.1618 and arXiv:1703.01094, a basis of Lorentz tensors was construct for the class four-photon graphs; it consists of 138 Lorentz tensors and the challenge is to show that each coefficient becomes finite when all graphs at a certain loop order are considered. "

He shows that the sum over all one-particle irreducible four-photon graphs is transversal (it vanishes whenever one of its external legs is contracted with the associated momentum). This global property restricts the possible occurrence of divergences to 43 Lorentz tensors (see arXiv:1505.06336 for an alternative basis of Lorentz tensors which includes the anti-symmetry tensor), whose coefficients remain to be proven finite.

They discuss how the QED tree-level cancellation identity implies cancellation between Feynman graphs of different topologies and determines the gauge dependence. They parameterize the momentum part of Landau gauge, then start by keeping only a liner term in graphs (i.e., insert only one Landau propagator, rest Feynman). They focus on the electron propagator in the massless limit of Quantum Electrodynamics. Not sure it is useful to us...
2017-06-16 Henry Your definition of gauge sets differs from the one we use in *Diagr. cancellations and the gauge dependence* [121]. The anomalous mag. moment is gauge independent due to the on-shell electrons, so studying the gauge parameter terms is not important, but I find it interesting to compare both definitions of gauge sets, maybe one can improve the other.

Read also Kreimer [131] 2000 *Knots and Feynman diagrams*.

2017-06-16 Henry Kißler Here an alternative idea to approach the finiteness conjecture: There is a bound for the value of a scalar Feynman diagram due to Eric Panzer, which he called the *Hepp bound*. This bound is derived using the 1974 Cvitanović and Kinoshita [56] Feynman-parametric representation. A natural question is: does this bound generalize to a sum of Feynman graphs when the sum goes over something as your gauge sets. Of cause, the gamma matrices in the numerator make things more complicated, but imposing on-shell conditions and choosing an appropriate gauge might simplify this task. Unfortunately, there is little in the literature about the Hepp bound [167]; the only thing I am aware of is a short section in the thesis of Iain Crump [44].
2017-06-16 Predrag I do not see how this would work: The Hepp invariant $H(G)$ is defined for a given graph G. Suppose we use instead of a Feynman diagram the $km'm$ multi-photons gauge set diagram $\tilde{G}$ (for examples, see figure 1.5 and table 1.2). In worldline formalism one has a proper time parametrization intermingled with Feynman-parametric bits. Any clue how the weights in the Hepp invariant $H(\tilde{G})$ would be defined? Just for a scalar theory, let us forget QED for the time being, along the lines of **2017-05-24 Predrag** entry above, on Bastianelli *et al.* [21]?

2017-06-19 Predrag Iain Crump [44] thesis *Graph Invariants with Connections to the Feynman Period in ϕ^4 Theory* (he follows Yeats [201]):

The *Feynman period* is a simplified version of the Feynman integral. The period is of special interest, as it maintains much of the important number theoretic information from the Feynman integral. It is also of structural interest, as it is known to be preserved by a number of graph theoretic operations.

2017-05-23 Predrag Badger, Bjerrum-Bohr and Vanhove [17] 2009 *Simplicity in the structure of QED and gravity amplitudes*.

2017-05-23 Predrag Rosenfelder and Schreiber [157, 158] 1996, arXiv:nucl-th/9504002, arXiv:nucl-th/9504005:

2017-06-02 Predrag Rosenfelder and Schreiber [159] 2004 *An Abraham-Lorentz-like equation for the electron from the worldline variational approach to QED*:

They discuss the of a spin-1/2 electron dressed by an arbitrary number of photons in the quenched approximation to QED. The approach is patterned after Feynman's celebrated variational treatment of the polaron problem [80], which was first applied by Mano, Progr. Theor. Phys. 14, 435 (1955) [8] to a relativistic scalar field theory and rediscovered and expanded by them in a series of papers [157, 158]. Its main features are the description of relativistic particles by worldlines [168] parametrized by the proper time, an exact functional integration over the photons and a variational approximation of the resulting effective action by a retarded quadratic trial action. In recent work we have extended this approach to more realistic theories, in particular to quenched QED [6] (the divergence structure and renormalization, a compact expression for the anomalous mass dimension of the electron). Here they calculate the finite contributions.

The variational formulation of worldline QED leads to an equation which is similar to Abraham, Lorentz and Dirac description of the electron and its self-interaction with the radiation field. The approach contains (almost) all the ingredients of the relativistic field theory of electrons and photons, in particular its divergence structure. This has been demonstrated by deriving an approximate nonperturbative expression for the anomalous mass dimension of the electron.

2017-05-23 Predrag K. Barro-Bergflödt, R. Rosenfelder and M. Stingl [20] 2006 *Variational worldline approximation for the relativistic two-body bound state in a scalar model*, arXiv:hep-ph/0601220.

2017-05-23 Predrag Fried and Gabellini [87] 2009 *Analytic, nonperturbative, almost exact QED: The two-point functions*. The remarkable (but speculative) result of this paper is that in a convenient gauge, the (unphysical) electron propagator renormalization is a multiplicative, non-perturbative *finite* factor bounded between 0 and 1:

$$Z_2 = \exp\left[-2\gamma\left(\left(\frac{\pi}{2}\right)^2 + \ln^2\left(\frac{\Lambda^2}{\mu^2}\right)\right)\right] , \tag{1.50}$$

where $\gamma = e^2/4\pi^2$ is the fine structure constant (referred to by the vulgar multitudes as α), $\mu \to 0$ is the infrared cutoff, and $\Lambda \to \infty$ is the UV cutoff. One would still need to compute the vertex renormalization Z_1 to get a gauge and renomalization method invariant result. Fried seem to only cite Schwinger, Fradkin and himself, so the similarity of this to 1982 Affleck, Alvarez, and Manton [2] nonperturbative result $e^{\alpha\pi}$ is not remarked upon.

Fried and Gabellini [88] 2012 *On the Summation of Feynman Graphs*, arXiv:1004.2202.

Fried and Gabellini [89] 2013 *QED vacuum loops and vacuum energy*

2017-03-15 Predrag I have tried reading Fried [86] 2014 *Modern Functional Quantum Field Theory: Summing Feynman Graphs*: "a simple, analytic, functional approach to non-perturbative QFT, using a functional representation of Fradkin to explicitly calculate relevant portions of the Schwinger Generating Functional (GF). In QED, this corresponds to

summing all Feynman graphs representing virtual photon exchange

between charged particles. It is then possible to see, analytically, the cancellation of an infinite number of perturbative, UV logarithmic divergences, leading to an approximate but most reasonable statement of finite charge renormalization. A similar treatment of QCD, with the addition of a long-overlooked but simple rearrangement of the Schwinger GF which displays Manifest Gauge Invariance, is then able to produce a simple, analytic derivation of quark-binding potentials without any approximation of infinite quark masses. A crucial improvement of previous QCD theory takes into account the experimental fact that asymptotic quarks are always found in bound state."

This book can be read online via GaTech library link here or here.

Even though I am a grandchild of Schwinger (via Tung Mow Yan), and have written/drawn a book where Schwinger's functional formalism is explained to everywoman, I still find the functional formalism of Schwinger and Fradkin [84] hard to follow. I believe the results are essentially the same as wordline formalism developed by Schubert *et al.*.

2017-06-15 Predrag I've now reread much of the relevant literature known to me. There might be much more - people who do things related to my 1977 paper [49] never alert me to their papers, presumably because the reports of my death have been greatly exaggerated.

2017-06-16 Predrag Jia and Pennington [110] *How gauge covariance of the fermion and boson propagators in QED constrain the effective fermion-boson vertex*

Jia and Pennington [111] *Gauge covariance of the fermion Schwinger–Dyson equation in QED*

Jia and Pennington [112] *Landau-Khalatnikov-Fradkin transformation for the fermion propagator in QED in arbitrary dimensions*

2017-06-16 Christian Melnikov, Vainshtein and Voloshin [146] *Remarks on the effect of bound states and threshold in g-2*: " The appearance of positronium poles in a photon propagator in QED formally requires a summation of an infinite series of terms in perturbation theory. [...] we show how these nonperturbative contributions disappear, using the case of the electron anomalous magnetic moment as an example. [...] it never happens that a summation of infinite classes of Feynman diagrams enhanced at any threshold generates additional effects beyond perturbation theory. The misunderstanding of this fact appears to be quite common. "

The same conclusion is reached by Eides [73] *Recent ideas on the calculation of lepton anomalous magnetic moments* and Fael and Passera [76] *Positronium contribution to the electron $g-2$.*

2017-06-16 Predrag Herzog and Ruijl [102] *The R*-operation for Feynman graphs with generic numerators*: “The R*-operation by Chetyrkin, Tkachov, and Smirnov is a generalisation of the BPHZ R-operation, which subtracts both ultraviolet and infrared divergences of euclidean Feynman graphs with non-exceptional external momenta. It can be used to compute the divergent parts of such Feynman graphs from products of simpler Feynman graphs of lower loops. In this paper we extend the R*-operation to Feynman graphs with arbitrary numerators, including tensors. We also provide a novel way of defining infrared counterterms which closely resembles the definition of its ultraviolet counterpart. We further express both infrared and ultraviolet counterterms in terms of scaleless vacuum graphs with a logarithmic degree of divergence. By exploiting symmetries, integrand and integral relations, which the counterterms of scaleless vacuum graphs satisfy, we can vastly reduce their number and complexity.”

Ruijl, Ueda, Vermaseren and Vogt [161] Four-loop QCD propagators and vertices with one vanishing external momentum: “ We have computed the self-energies and a set of three-particle vertex functions for massless QCD at the four-loop level. The vertex functions are evaluated at points where one of the momenta vanishes. Analytical results are obtained for a generic gauge group and with the full gauge dependence, which was made possible by extensive use of the Forcer program for massless four-loop propagator integrals. The bare results in dimensional regularization are provided in terms of master integrals and rational coefficients; the latter are exact in any space-time dimension.”

Chetyrkin and Tkachov [42] *Infrared R-operation and ultraviolet counterterms in the MS-scheme*, together withWe Chetyrkin and Smirnov [41] *R*-Operation corrected*

Smirnov and Chetyrkin [182] *R* operation in the minimal subtraction scheme*

Johnson and Zumino [113] *Gauge dependence of the wave-function renormalization constant in Quantum Electrodynamics*: “ [...] point out the existence of an exact and simple relation between the electron Green’s function renormalization constants in the general class of manifestly covariant gauges. ”

Korthals Altes and De Rafael [128] 1976 *Infrared structure of non-abelian gauge theories: An instructive calculation*

Korthals Altes and De Rafael [129] 1977 *Infrared structure of non-abelian gauge theories: Comments on perturbation theory calculations*

Frenkel *et al.* [85] 1976 *Infra-red behaviour in non-abelian gauge theories*

2017-06-27 Predrag Søndergaard, Palla, Vattay, and Voros [184] *Asymptotics of high order noise corrections*: “We consider an evolution operator for a discrete Langevin equation with a strongly hyperbolic classical dynamics and noise with finite moments. Using a perturbative expansion of the evolution operator we calculate high order corrections to its trace in the case of a quartic map and Gaussian noise. The asymptotic behaviour is investigated and is found to be independent up to a multiplicative constant of the distribution of noise.”

2017-06-16 Predrag Read also Simonov and Tjon 1996 article (PC: cannot find any such) which applies the the so-called Fock–Feynman–Schwinger representation (FSR), based on the Fock–Schwinger [173] proper time and Feynman path integral formalism, to QED.

Simonov [178] *Relativistic path integral and relativistic Hamiltonians in QCD and QED* writes: " The proper-time 4D path integral is used as a starting point to derive the new explicit parametric form of the quark-antiquark Green's function in gluonic and QED fields entering as a common Wilson loop. The subsequent vacuum averaging of the latter allows us to derive the instantaneous Hamiltonian. The explicit form and solutions are given in the case of the $q\bar{q}$ mesons in magnetic field. "

Simonov [179] *QED spectra in the path integral formalism* writes: " Relativistic Hamiltonians, derived from the path integrals, are known to provide a simple and useful formalism for hadron spectroscopy in QCD. The accuracy of this approach is tested using the QED systems, and the calculated spectrum is shown to reproduce exactly that of the Dirac hydrogen atom. [...] The calculated positronium spectrum, including spin-dependent terms, coincides with the standard QED perturbation theory to the considered order $O(\alpha^4)$. "

2017-08-03 Predrag Schwartz [171] Chapter33 *Effective actions and Schwinger proper time* looks very pedagogical.

2017-08-04 Predrag Mielniczuk *et al.* [147] *The anomalous magnetic moment of a photon propagating in a magnetic field*

2017-08-18 Predrag Antonov [8] *World-line formalism: Non-perturbative applications*

2017-09-17 Predrag Schäfer and I. Huet and H. Gies [163] *Worldline numerics for energy-momentum tensors in Casimir geometries*: "We develop the worldline formalism for computations of composite operators such as the fluctuation induced energy-momentum tensor. As an example, we use a fluctuating real scalar field subject to Dirichlet boundary conditions. The resulting worldline representation can be evaluated by worldline Monte-Carlo methods in continuous spacetime. The method generalizes straightforwardly to arbitrary Casimir geometries and general background potentials."

2017-09-17 Predrag A closer reading of sect. 6. *Worldline formalism* of Gelis and N. Tanji [92] *Schwinger mechanism revisited* might be helpful - it goes to the barycenter to reexpress the integral as an average over Wilson loops.

In sect. 6.4. *Lattice worldline formalism* they describe a formulation of the worldline formalism the (Euclidean) space–time discretized on a cubic lattice [113–116].

Schmidt and Stamatescu [165, 166] *Matter determinants in background fields using random walk world line loops on the lattice*, using Schwinger's propert time formalism, pointed out that the fermion and boson determinant on the lattice can be viewed as a gas of closed loops which can be simulated numerically via a random walk.

2017-09-17 Predrag Seiler and Stamatescu [176] *A note on the loop formula for the fermionic determinant* notes moved to siminos/spatiotemp/chapter/stabBlog.tex .

2017-09-17 Predrag Fry [90] *Nonperturbative quantization of the electroweak model's electrodynamic sector*: " Consider the Euclidean functional integral representation of any physical process in the electroweak model. Integrating out the fermion degrees of freedom introduces 24 fermion determinants. Suppose the functional integral over the Maxwell field is attempted first. This paper is concerned with the large amplitude behavior of the Maxwell effective measure. We examine the large amplitude variation of a single QED fermion determinant. To facilitate this the Schwinger proper time representation of this determinant is decomposed into a sum of three terms. The advantage of this is that the separate terms can be nonperturbatively estimated for a measurable class of large amplitude random fields in four dimensions. It is found that the QED fermion determinant grows to fast in the absence. Including zero mode supporting background potentials can result in a decaying exponential growth of the fermion determinant. This is prima facie evidence that Maxwellian zero modes are necessary for the nonperturbative quantization of QED "

Gies and Langfeld [95] *Loops and loop clouds — a numerical approach to the worldline formalism in QED* point out that the fermion (and boson) determinant on the lattice can be viewed as a gas of closed loops which can be simulated numerically via a random walk: " A numerical technique for calculating effective actions of electromagnetic backgrounds is proposed, which is based on the string-inspired worldline formalism. As examples, we consider scalar electrodynamics in three and four dimensions to one-loop order. Beyond the constant-magnetic-field case, we analyze a step-function-like magnetic field exhibiting a nonlocal and nonperturbative phenomenon: "magnetic-field diffusion". "

Epelbaum, Gelis and Wu [75] *From lattice Quantum Electrodynamics to the distribution of the algebraic areas enclosed by random walks on* $\mathbb{Z}^2$.

Epelbaum, Gelis and Wu [74] *Lattice worldline representation of correlators in a background field*: " We use a discrete worldline representation in order to study the continuum limit of the one-loop expectation value of dimension two and four local operators in a background field. We illustrate this technique in the case of a scalar field coupled to a non-Abelian background gauge field. The first two coefficients of the expansion in powers of the lattice spacing can be expressed as sums over random walks on a d-dimensional cubic lattice. Using combinatorial identities for the distribution of the areas of closed random walks on a lattice, these coefficients can be turned into simple integrals. Our results are valid for an anisotropic lattice, with arbitrary lattice spacings in each direction. "

Laufer and Orland [137] *Metric of Yang-Mills orbit space on the lattice*: " We find coordinates, the metric tensor, the inverse metric tensor and the Laplace-Beltrami operator for the orbit space of Hamiltonian SU(2) gauge theory on a finite, rectangular lattice, with open boundary conditions. This is done using a complete axial gauge fixing. ”

Vilela Mendes [192] *A consistent measure for lattice Yang–Mills* (no GaTech access to the journal): “ The construction of a consistent measure for Yang–Mills is a precondition for an accurate formulation of nonperturbative approaches to QCD, both analytical and numerical. Using projective limits as subsets of Cartesian products of homomorphisms from a lattice to the structure group, a consistent interaction measure and an infinite-dimensional calculus have been constructed for a theory of non-Abelian generalized connections on a hypercubic lattice. Here, after reviewing and clarifying past work, new results are obtained for the mass gap when the structure group is compact. ”

2017-09-17 Predrag a letter

to: Erhard Seiler <ehs@mpp.mpg.de>,
Ion-Olimpiu Stamatescu <I.O.Stamatescu@thphys.uni-heidelberg.de>

Dear Erhard and Ion-Olimpiu

I have read your paper [176] *A note on the loop formula for the fermionic determinant* (well, not read it sufficiently deeply) with great interest.

2017-09-19 Schubert With g-2 we have not made a lot of headway yet, either, but my students Cesar and Misha have started chipping away at it, and with James we are making some progress in using parity and integration by parts to simplify the integrands.

2017-09-19 Schubert Arkani-Hamed, Huang and Huang [16] *Scattering amplitudes for all masses and spins*, arXiv:1709.04891 contains an elegant calculation of the one-loop g-2 which shares with our approach the property that it lumps together the irreducible and the reducible diagrams.

2017-09-19 Predrag Arkani-Hamed, N. and Huang, T.-C. and Huang [16] *Scattering amplitudes for all masses and spins*: “ We introduce a formalism for describing four-dimensional scattering amplitudes for particles of any mass and spin. This naturally extends the familiar spinor-helicity formalism for massless particles to one where these variables carry an extra SU(2) little group index for massive particles, with the amplitudes for spin S particles transforming as symmetric rank 2S tensors. We systematically characterise all possible three particle amplitudes compatible with Poincare symmetry. [...]

We illustrate a number of applications of the formalism at one-loop, giving few-line computations of the electron (g-2) as well as the beta function and rational terms in QCD. “Off-shell" observables like correlation functions and form-factors can be thought of as scattering amplitudes with external "probe" particles of general mass and spin, so all these objects–amplitudes, form factors and correlators, can be studied from a common on-shell perspective. ”

If one gets through the first 55 pages, than the calculation of magnetic moment (7.3) is very simple :)

2024-04-08 Predrag Nima Arkani-Hamed, Keisuke Harigaya *Naturalness and the muon magnetic moment*, arXiv:2106.01373.

N. Arkani-Hamed, H. Frost, G. Salvatori, P-G. Plamondon, H. Thomas *All loop scattering as a counting problem* arXiv:2309.15913.

2017-12-17 Christian Schubert We have not forgotten about g-2 here, my student Misha is working on the LBL diagram, and we have also made progress in decomposing the pure fermion-line contributions, the ones that interest you most, into spin and orbit parts. My big hope is that not all will equally contribute for large N.

Last week James and I were in Bologna where we had an opportunity to talk to Remiddi. He seems not terribly interested in the convergence issue, but in any case he said he would find nothing odd if that happened to be true, since the issue of cancellations is so little understood.

Laporta has obtained a job at Padua University. More than well-deserved.

2018-01-08 Christian Schubert We are planning to have a workshop devoted to worldline methods in Germany at the Mainz Center of Theoretical Physics sometime in 2019, most likely in February. Would you be interested in principle to attend?

2018-06-02 Predrag Jegerlehner17 [108] *The Anomalous Magnetic Moment of the Muon* is a very detailed review of everything known about anomalous magnetic moments.

Todorov lecture [189] *From Euler's play with infinite series to the anomalous magnetic moment* is a fun overview of the history of the (g-2) calculations.

2018-06-02 Predrag Aoyama and T. Kinoshita and M. Nio [14] *Revised and improved value of the QED tenth-order electron anomalous magnetic moment*: " we have carried out a new numerical evaluation of the 389 integrals of Set V, which represent 6 354 Feynman vertex diagrams without lepton loops. We found that one of the integrals, called $X024$, was given a wrong value in the previous calculation due to an incorrect assignment of integration variables. The correction of this error causes a shift of -1.26 to the Set V contribution, and hence to the tenth-order universal (mass-independent) term 7.606 (192) $(\alpha/\pi)^5$ as the best estimate of the Set V contribution. " I have entered the correction into (1.1).

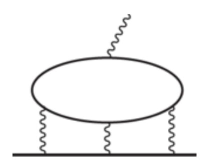
2018-11-27 Predrag Aoyama, Kinoshita and Nio [15] *Theory of the anomalous magnetic moment of the electron*. A review of the current status of the theory of the electron anomalous magnetic moment, including the contribution of Feynman diagrams of QED up to the tenth-order perturbation theory.

2020-06-15 Predrag Aoyama *et al.* [9] *The anomalous magnetic moment of the muon in the Standard Model*: We review the present status of the Standard Model calculation of the anomalous magnetic moment of the muon. This is performed in a perturbative expansion in the fine-structure constant and is broken down into pure QED, electroweak, and hadronic contributions. The pure QED contribution is by far the largest and has been evaluated up to and including $O(\alpha^5)$ with negligible numerical uncertainty. [...] the prospects to further reduce the theoretical uncertainty in the near future make this quantity one of the most promising places to look for evidence of new physics.

2025-04-08 Predrag Aoyama, Hayakawa, Hirayama and Nio [10] *Verification of the tenth-order QED contribution to the anomalous magnetic moment of the electron from diagrams without fermion loops*, (2025).

A discrepancy of approximately 5σ exists between the two previous results for the tenth-order QED contribution to the anomalous magnetic moment of the electron, calculated from Feynman vertex diagrams without fermion loops. We decomposed this contribution into 389 parts based on a self-energy diagram representation, enabling a diagram-by-diagram numerical comparison of the two calculations. No significant discrepancies were found for individual diagrams. However, the numerical differences of the 98 diagrams sharing a common structure were not randomly distributed. The accumulation of these differences resulted in the 5σ discrepancy. A recalculation with increased statistics in the Monte Carlo integration was performed for these 98 diagrams, with a revised result of 6.800±0.128, thereby resolving the discrepancy.

2019-06-04 Predrag Makiko Nio gave a very nice Northwestern physics colloquium on *Higher order QED corrections to the electron g-2: Numerical approach*. "Numerical approach" because Laporta was supposed to cover the "analytical approach," but he had to cancel for health reasons. She emphasised something that I found useful: the history of (the few) errors in the calculation. I was relieved to learn that my and Kinoshita's formula (1.27) for the UV and IR finite part of amplitude ΔM_G is correct, that the error Kinoshita made and Nio corrected was in the application of the self-energy (Ward-Takhashi) formula for $(g-2)$ to the internal 4-photon electron loops of type:



Kinoshita correctly included external photon vertex insertions and $\partial/\partial q$ derivatives into the electron line, but omitted the electron loop derivatives.

2019-06-04 Gerry Gabrielse The effects due to the finite radius of a Penning trap in which the magnetic moment is measured are discussed in the 75-page Lowell S. Brown and Gerald Gabrielse 1986 paper [35] *Geonium theory: Physics of a single electron or ion in a Penning trap*. At the end of section VII they say "We conclude that the radiative level shifts in geonium are certainly negligible." The also discuss the effects of cylindrical cavity Green's function, find them negligible as well, and much, much more. One deserves a Ph.D. just for being able to get through this paper.

But, in the light of the increased experimental accuracies since 1986 some these claims might have to be reexamined.

As far as the QED $(g-2)$ calculation is concerned, I imagine one would need to recompute the Schwinger $\alpha/2\pi$ with the photon (and electron?) Green's function for a finite cylindrical container. This is a much harder calculation, as instead of infinite Lorentz invariant free space, one has to use the translation-symmetry broken photon Green's function (photon propagator), with Dirichlet boundary conditions, something like their frequency domain cylindrical Green's function eq. (8.25), expanded in terms of radial Bessel functions. All the magic of usual Feynman integrals is gone; a job for a very good mathematical physicist. They probably do not raise any such any more...

2019-06-05 Predrag As an amusing curiosity / coincidence, today in our online plumbers' meeting we discussed Burak's current work on "Bohmian" walking droplets [36], an analysis of the ongoing IST Austria experiments. The experiment is done in a shallow cylindrical dish, 8cm in diameter, conical bottom with 1^o (!) slope. The equations are 2D shallow wave approximation PDEs for the surface, coupled to a point-like bouncing droplet, see sect. III of the above reference. Currently Burak plans to discretize his calculation on a grid, though eventually he might try to use a radial Bessel function eigenmodes for his cell. DNS fluid dynamicists do not use them, as they do not diagonalize Laplacians (not sure - this statement looks wrong to me) and there are no fast numerical Bessel transforms.

One person who could do Gabrielse calculation is der menschliche Panzer Andreas Wirzba [200]. He will not listen to me, but Gabrielse might persuade him.

2021-01-21 Predrag to Stephen Wolfram (to be moved to my blog)

Hardly needs repeating, but: you are totally amazing. I do not have 1/100th of your memory, if that much. Here are a few things that I possibly remember somewhat accurately... Again, a wonderful portrait. Now, all of us in our Discreet Charm of Bourgeoisie apartment know about Cadbury bars, but how did Tiny know that?

To the best of my recollection, I had never used SCHOONSCHIP. I had, in 1971 or so, written one of the ur-algebraic symbol manipulation programs, my own, described in my first physics paper [46], of no interest whatsoever today (not sure I have a copy of the paper anywhere, and certainly not the code):

Predrag Cvitanović
Computer generation of integrands for Feynman parametric integrals
Cornell preprint CLNS-234 (June, 1973) and Proc. 3rd Coll. on Advanced Comp. Meth. in Theoretical Physics (Marseille, 1973)

Our electron magnetic moment calculation was implemented by entering integrands on IBM punch cards, flying on a 4-person plane to Brookhaven with box-fulls of cards, and feeding batch jobs into the CDC computer there.

We had no computer, but Cornell Synchrotron had several state-of-the-art DEC PDP-10s, so advanced that they had keyboards and screen terminals attached. I was allowed to use them after 1AM, only to edit, but absolutely not compute. So I wrote an symbolic algebra language for evaluation of Dirac gamma-matrix traces entirely as macros in TECO, the DEC text editor. Every time they checked on me, I was in editor mode, so they never caught me.

Then Kinoshita went to CERN for a summer, and broke their computing budget - it was the first large physics calculation, as expensive as an experiment. I think my TECO code produced some errors, so he switched to SCHOONSCHIP and reentered all our integrands into it, without me, the amazing man that he is. I (Tom believes he) invented a wholly independent way of computing sets of integrals corresponding to self-energy insertions, and that enabled us to catch a single faulty Jacobian in a single integral, so our calculation had no errors.

I have a ton of things written about that and this in my diaries, which actually are in the next room. If it becomes of any interest to anyone, ever.

For many years there was a large metal cabinet in a Newwman Hall hallway, containing our entire calculation as punch cards - it might still be there.

Only Tom knows. I suffer big father-son guilt there, because I used to call him up every few years, but then he exploded a car tire in his home's garage, and went totally deaf, so I do not call him, and I never write letters (to anyone). But I assume he's as frisky as always. We have a running thing going, because I have crank theory that perturbation series for g-2 (and all on-shell physical observables) is not an asymptotic series which he doesn't think much of. So 3 years ago I returned to it, and even found a guru in Morelia, Michoacán, Mexico who might be able to prove that - put maybe 1/2 year into it again, failed again, with to show. But psychologically it must be that I wanted to show my (academic) father that I can do it. Psychology - I do not recognize any form of paternalistic authority - I only listen to women - and still...

The smart graduate student ("hrra-hrra T-uh-u-ft") I had always liked a lot. His adviser -it seems- I found grumpy, so he only features on the back cover of my

ChaosBook.org/FieldTheory

as "El Noble Cigaro", channeling Pauli, but I do not remember why I did not think all that highly of him.

I still have to look at your Project book - just drove back to Atlanta from Chicago and still have to look at the (paper) mail accumulated over the 306 days away. I better just press [Send] on this one :)

2022-09-27 Gabrielse Fan, Myers, Sukra and Gabrielse [77] *Measurement of the electron magnetic moment* (2022)):

The electron magnetic moment in Bohr magnetons,

$$-\mu/\mu_B = g/2 = 1.00115965218059(13)[0.13ppt]\,,$$

is consistent with a 2008 measurement [100] and is 2.2 times more precise. The most precisely measured property of an elementary particle agrees with the most precise prediction of the Standard Model (SM) to 1 part in 10^12, the most precise confrontation of all theory and experiment. The SM test will improve further when discrepant measurements of the fine structure constant α are resolved, since the prediction is a function of α. The magnetic moment measurement and SM theory together predict

$$\alpha - 1 = 137.035999166(15)[0.11ppb]\,.$$

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Chapter 2

Learning worldline QFT

2.1 Quantum Field Theory refresher

The goal. Take a narrow path, learn enough Quantum Field Theory (QFT) (and Green's functions and such, but no more) to be able to digest sect. 1.3 *Worldline formalism* and check James sect. 1.3.2 *Electron magnetic moment in worldline formalism* calculations.

What is a gauge theory? According to [6] there are two types of theories that can be called 'gauge theories', the Yang-Mills theories and constrained Hamiltonian theories.

Summary In order to get correct predictions from non-Abelian field theories, which are susceptible to large number of gauge copies, we need to choose a representative of each gauge orbit.

2.2 Morelia Worldline Formalism course

2017-07-03 Predrag First Morelia discussion (all errors are mine):

> Christian is inclined to compute (g-2) starting with their two-field spinor QED Bern-Kosower formula. In that formulation all photons are born equal; one of them is kept as the external field (the seagull vertex, or the k^μ coefficient) is the magnetic moment $\sigma_{\mu\nu}$), and the rest are contracted pairwise in all possible ways. In the quenched case, this yields one gauge set, the self-energy set of sect. 1.2.3.
>
> James would like to start with their electron propagator in constant external field, and keep the term linear in the external field. I vastly prefer that, because it should be possible to distinguisn in- and out-legs, and the three kinds of N-photon propagators that yield the minimal gauge sets.
>
> I probably need to got through the proof of gauge invariance with them.

2017-07-04 Predrag Morelia Schubertiad day 1: the lecture written up in Schubert [16] 2012 *Lectures on the worldline formalism*, sects. 1.4 *Gaussian integrals* and 1.5 *The N-photon amplitude*.

Christian was right. One has to start with scalar QED one-loop effective action to understand the Bern-Kosowar type master formulas. That yields a loop with any number of photons attached, each photon vertex carrying a 1D proper time Green's function. This could be computed by usual math methods techniques for computing Green's functions, but they find it useful for reasons that will be understood later to compute it as a sum of Fourier modes. The marginal modes (4 space-time translations) are fixed in the Gauss way, by shifting the origin to loops center of mass (i.e., different symmetry reduction for each loop).

We then separate the integration over x_0, thus reducing the path integral to an integral over the relative coordinate q:

$$x^\mu(\tau) = x_0^\mu(\tau) + q^\mu(\tau), \tag{2.1}$$

with the relative coordinate q periodic and satisfying constraint

$$\int_0^T d\tau\, q^\mu(\tau) = 0\,. \tag{2.2}$$

In the symmetry-reduce$q$-space the zero-mode integral then yields the energy-momentum conservation δ function. The 1D Laplacian $M = -d^2/d\tau^2$ has positive eigenvalues (the usual k^2 Fourier modes), (do the exercise!)

$$\det M = (4T)^D\,, \tag{2.3}$$

and the bosonic Green's function of $-\frac{1}{2}\frac{d^2}{d\tau^2}$ in the symmetry-reduced space is

$$G_B^c(\tau,\tau') = 2\langle\tau|\left(\frac{d^2}{d\tau^2}\right)^{-1}|\tau'\rangle = |\tau-\tau'| - \frac{(\tau,\tau')^2}{T} - \frac{T}{6}\,. \tag{2.4}$$

The first derivative $\dot{G}$ has a sign function, and $\ddot{G}$ has a $\delta(\tau-\tau')$ (because of the translation invariance, $d/d\tau$ can always e taken to act on the left variable τ).

This results in a Bern-Kosower [7] type master formula

$$\begin{aligned}
&\Gamma_{\rm scal}[k_1,\varepsilon_1;\ldots;k_N,\varepsilon_N] = \\
&\quad (-ie)^N(2\pi)^D\delta(\sum k_i)\int_0^\infty \frac{dT}{T}(4\pi T)^{-\frac{D}{2}}e^{-m^2T}\prod_{i=1}^N\int_0^T d\tau_i \\
&\quad\times\exp\left\{\sum_{i,j=1}^N\left[\tfrac{1}{2}G_{Bij}k_i\cdot k_j - i\dot{G}_{Bij}\varepsilon_i\cdot k_j + \tfrac{1}{2}\ddot{G}_{Bij}\varepsilon_i\cdot\varepsilon_j\right]\right\}|_{\rm lin(\varepsilon_i)}
\end{aligned} \tag{2.5}$$

for the one-loop N-photon amplitude in scalar QED, with photon momenta k_i and polarization vectors ε_i. m denotes the mass, e the charge and T the total proper time of the scalar loop particle.

This method is not applicable to open fermion lines.

A scary realisation. This is a special case of our general, nonlinear, arbitrary order interaction vertex smooth conjugacy calculation of sect. 1.6. In other words, our calculation is not just for scalar theory in vacuum, it is for any nonconstant background, and any order of interaction, i.e., inter alia general relativity. As we start from a saddlepoint (classical periodic solution) that is not translation invariant, we do not have to worry about fixing the marginal modes - there are none.

I fear having to explain the smooth conjugacy method to civilians.

2017-07-05 Predrag Morelia Schubertiad day 2: the $N = 2$ photon legs case is worked out in Schubert [16] 2012 lectures sect. 1.6 *The vacuum polarization*.

Even though the result is strictly zero (by Furry theorem, or by time reversal of odd number of $\dot{G}$ functions), $N = 3$ is useful to start understanding how integrations by parts work.

For N photon legs, see sect. 2.5 *Integration-by-parts and the replacement rule* and Ahmadiniaz, Schubert and Villanueva [4] *String-inspired representations of photon/gluon amplitudes*, arXiv:1211.1821: "The Bern-Kosower rules provide an efficient way for obtaining parameter integral representations of the one-loop N-photon/gluon amplitudes involving a scalar, spinor or gluon loop, starting from a master formula and using a certain integration-by-parts ("IBP") procedure. Strassler observed that this algorithm also relates to gauge invariance, since it leads to the absorption of polarization vectors into field strength tensors. Here we present a systematic IBP algorithm that works for arbitrary N and leads to an integrand that is not only suitable for the application of the Bern-Kosower rules but also optimized with respect to gauge invariance. In the photon case this means manifest transversality at the integrand level, in the gluon case that a form factor decomposition of the amplitude into transversal and longitudinal parts is generated naturally by the IBP, without the necessity to consider the nonabelian Ward identities. Our algorithm is valid off-shell, and provides an extremely efficient way of calculating the one-loop one-particle-irreducible off-shell Green's functions ("vertices") in QCD. In the abelian case, we study the systematics of the IBP also for the practically important case of the one-loop N-photon amplitudes in a constant field."

2017-07-06 Predrag Morelia Schubertiad day 3:

Idrish Huet explained to me how the numerical Monte-Carlos of worldline path integrals work.

2017-07-05 Christian says there is a relevant new arXiv from some Vietnamese authors, but I couldn't find it.

2017-07-07 Predrag Morelia Schubertiad day 4:

2017-07-09 5:22 am Predrag had a panic attack that we'll never get started on (g-2). Forgot all about going to Patzquaro, spent entire Sunday writing up the new sect. 1.1 *Electron magnetic moment* and sect. 1.3.2 *Electron magnetic moment in worldline formalism*.

2017-07-10 Predrag Morelia Schubertiad day 5: handwritten notes only.

2017-07-11 Predrag Morelia Schubertiad day 6:

Christian assigned homework: reformulate the magnetic moment vertex operator (1.3), projections (1.4) and (1.5) in configuration coordinates.

2017-07-12 Predrag Morelia Edwardsiad day 7: **Scalar QED open lines**. Following Ahmadiniaz, Bashir and Schubert [1] 2016 *Multiphoton amplitudes and generalized Landau-Khalatnikov-Fradkin transformation in scalar QED*, arXiv:1511.05087, and Ahmadiniaz, Bastianelli and Corradini [2] *Dressed scalar propagator in a non-Abelian background from the worldline formalism*, arXiv:1508.05144.

Worldline is sum of N! photon insertions, spinor indices and gauge transformations at the endpoints. In 1950 Feynman [11] gave the scalar QED worldline integral (1.17). Note dT for the open line, as opposed to dT/T for the closed loop: due to einbein gauge fixing that leads to different Fadeev-Popov for the loop than for the line. The worldine Green's function is now (compare to (2.4))

$$\begin{aligned}\Delta(\tau_1,\tau_2) &= \frac{1}{2}|\tau_1-\tau_2| - \frac{1}{2}(\tau_1-\tau_2) + \frac{\tau_1\tau_2}{T} \\ &= \frac{1}{2}\left(G_B(\tau_1,\tau_2) - G_B(\tau_1,0) - G_B(0,\tau_2) + G_B(0,0)\right) .\end{aligned} \tag{2.6}$$

2017-07-13 Predrag Morelia Edwardsiad day 8: handwritten notes only.

2017-07-14 Predrag Morelia Edwardsiad day 9:

Wrote down the master formula for $S^{xx'}_{(N)}$ and its kernel $K^{xx'}_{(N)}$. Rewrote it in momentum space. Verified that $N=0$ generates the free propagator. Checked the $N=1$ vertex. The hardest was computing the fermion self-energy, in terms of 2 dimensional regularization hypergeometric functions. Currently can compare to Davydychev [10] only numerically.

Read also Davydychev [9] *Geometrical methods in loop calculations and the three-point function*

2018-06-10 Predrag Les Houches Edwardsiad day 10:

James has an open-line Dirichlet Green's function that should suffice to compute (g-2). Predrag is sceptical, thinks that Dirichlet Green's function is a bad choice, as it breaks the translational invariance. Would prefer some periodic formulation, where a periodic space box is taken off to infinity, than put on the mass-shell either by some absorptive cut (yielding (1.5)) or amputated and renormalized by $\sqrt{Z_2}$ as in LSZ formulation.

I thought that in the quenched QED worldline formulation, the one-particle reducible graphs renormalize (g-2) contributions (my talk `reducesymm/presentations/LesHouch18/finiteQED.tex`) but my N-photon formulation of the QED vertex overcounts counter-terms, or, as Magnea says in his handwritten lecture 2, "one-particle reducible graphs should be included only once."

2.3 Learning wordline formalism notes

2018-01-25 Predrag Having gone through the first part of Peskin and Schroeder [14] would be a plus.

2018-01-08 Christian Schubert that's great that you want to learn the worldline formalism. The problem is that on the part that interests you most, the open fermion line, we still have not written up anything intelligible. But James is in the process of writing up his part of the lectures that we gave for you, which is going to be incorporated into the lecture notes that I have on the web with Olindo Corradini. So maybe for the time being you might want to work through the lectures as they are, and hopefully by the time he is through with this, James' fermion line part may already be in shape. Any questions, we are available.

2018-01-09 James Edwards Christian's suggests that you begin with the Olindo Corradini and Christian Schubert and notes [8], arXiv:1512.08694. 2012 notes by Schubert [16], (click here) may be also helpful (Predrag: I think the 2012 notes are included in the entirety into the Corradini and Schubert notes [8]. However, you might find parts of Corradini 2012 lectures useful, (click here)).

In the meantime, I will write up my notes on open fermion lines and share them with you as soon as possible.

2018-01-16 James Would you like to get involved in some ongoing calculations pertaining to g-2? Our worldline representation of the fermion propagator provides a new technique for attacking this problem and there has already been some progress in calculating some of the ingredients for the 3 loop contribution, but we would also like to look at 1+ loop rainbow diagrams.

To achieve this, we would study the open fermion line with a low energy photon attached along with a certain number of virtual photon loops. Now we know how to deal with the fermion line we have a good idea of how to calculate these quantities in a particularly efficient way (that automatically picks out the term linear in k with the right gamma matrix structure to allow for extraction of the structure constant we seek). We expect to be able to streamline the calculation and maybe even earn some advantages in the limit of large numbers of virtual photons. In principle, this can be extended to the case of a constant electromagnetic background too.

We can begin working by checking the one loop contribution as a warm-up (I have notes on this already) before looking at the 2-loop rainbow. This requires familiarity with the scalar propagator in the worldline approach before looking at open spinor lines and the coupling to external photons. The latter remain as my job to write up our notes from Predrag's visit, and I will strive to do this asap.

2018-02-13 Predrag I find discrete lattice problems very useful in understanding concepts in QFT. Study ChaosBook.org/FieldTheory/postscript.html Chapter 1 Lattice field theory. You will learn that Laplacian generates all walks on a lattice (making it easier to understand path integrals), that a Green's function is the propagator (sum over all walks), that the mysterious $e^{iET/\hbar}$ and $e^{ip\cdot x/\hbar}$'s of Quantum Mechanics are roots of unity (a consequence of time and space translation invariance), and how that enables you to compute the propagators / Green's functions.

All of the above can be then continued in the small lattice spacing limit into things you accept on faith when you read QFT textbook. But here it is just matrices and vectors, and you can check every step.

Do not use much time on the discrete lattice formalism, as we shall not use it in the current project.

2018-02-18 Predrag To see whether you have actually understood Green's function on the level needed for our term's goals, see whether you can follow worldline Green's function calculations following eq. (1.22), eq. (2.6) above, eqs. (2.35) and (2.41) in Corradini and Schubert notes [8].

2018-03-28 James At this stage, I don't think you need to worry about the gravitational case in section 1.4 and it's true that initially we are more concerned with Abelian QED (that said, I would recommend that you do take a look at section 2.5.1 when you have a chance, just because it gives the simplest extension of the worldline formalism to the non-Abelian case). These days we have some more powerful techniques to deal with colour degrees of freedom, but it's good to see the simpler presentation in that section).

I understand that you were referring to the material on open lines in the worldline approach, which is of course relevant to the g-2 calculation. I have almost written up my notes from Predrag's visit and hope to send them to you. I would encourage you to go through them critically in case any typos have crept in.

If you're eager to get started, I would suggest that you review the path integral calculation of quantum mechanical kernels (i.e. matrix elements of the form $K(x, y; T) = \langle y| \exp[-iTH]|x\rangle$ including electromagnetic interactions that will be used repeatedly in the open line calculations. For example, sects. 2.18 to 2.20 of Kleinert [12] path integral textbook has full details. [1]

2018-03-31 Predrag Skip section 2.5.1 (extension of the worldline formalism to the non-Abelian case) in Corradini and Schubert notes [8], we are very short on time. Focus on QED.

Please write up your derivation of the Green's functions (1.35) and (2.15), then (1.41), here, in this text, in LaTeX, with all signs and prefactors correct. Going through sect. 2.4 *The vacuum polarization* might be good - that connects these mysterious worldline Green's functions to the standard Feynman integrals like those you see in Peskin and Schroeder.

[1] Predrag: 2018-04-16 I have a copy of Kleinert [12]

2018-04-16 Predrag Go through Stone and Goldbart [17], *Mathematics for Physics: A Guided Tour for Graduate Students*, Chapter 5 *Green's Functions*.

It is all about 1-dimensional Green's functions - understanding material related to kind of function you need for worldline formalism should be helpfull to you.

Stone and Goldbart [17] is an advanced summary where you will find almost everything one needs to know. More pedestrian and perhaps easier to read is Arfken and Weber [5].

I like Mathews and Walker [13], based on lectures by Richard Feynman at Cornell University. You can download it from here. However, I'm not sure it will help you understand the problem at hand.

The above examples are the usual way students are taught Green's functions. I personally find deriving them as continuum limits of lattice formulations (click here) the most insightful and easiest to understand. My derivation is for periodic boundary conditions. You will have to also understand other kinds of boundary conditions that arise in your project.

2018-07-12 Added the Edwards reference Ahmadiniaz *et al.* [3] *One-particle reducible contribution to the one-loop spinor propagator in a constant field*

Added the Edwards / Cvitanović June 2018 whiteboard notes, *reducesymm/guopeng/*:

Mrenorm.pdf The perturbative expansions for the magnetic moment anomaly (1.6).

M4.pdf

GammaVertex.pdf

WardIds.pdf

worldLprop.pdf

M4.pdf

aSums.pdf The next five files are gauge sets $a_{km'm}$ defined in (1.10):

a111.pdf

a111Gset.pdf

a111old.pdf

a120.pdf

a210.pdf

1loopPhi3.pdf is an outline of Edwards 1-loop calculation.

There are two ways to turn the line into a loop with a virtual photon:

1. Start with the line with $N \geqslant 2$ photons and sew a pair of photons together; this is the Ahmadiniaz, Bashir and Schubert [1] approach to doing one-loop vertex corrections for scalar QED (compare to (2.6)).

2. Incorporate the virtual photon loop at the level of the worldline action, which is the approach advocated by Edwards in the calculations followed here. Note that the boundary conditions for the open line in both cases [1] are Dirichlet; the difference lies in the modification to the kinetic operator.

Possibly Davydychev [9] is also relevant.

2018-07-13 working through Edwards' *1loopPhi3.pdf* calculation sketch:

The note starts out with the N "photon" (actually, scalar) insertions propagator for a charged scalar field in external field (2.5). As the external scalar particles have no polarisation, the vertex operator is $V_{\text{scal}}[k] = \int_0^T d\tau\, e^{ik\cdot x(\tau)}$ in this case. Therefore, the master formula for amplitude for scalars is related to that of photons (2.5) by setting all $\varepsilon = 0$. [2]

$$\begin{aligned}
N \text{ photon ins. fig} &= \int_0^\infty dT\, e^{-m^2 T} \\
&= (ie)^N \int_0^\infty dT\, e^{-m^2 T} \int_0^T d\tau_1 \cdots \int_0^T d\tau_N \\
&\quad \times \int_{x(0)=y}^{x(T)=x} \mathcal{D}x\, e^{i\sum_{i=1}^N k_i\cdot x(\tau_i)} e^{-\int_0^T d\tau\, \frac{1}{4}\dot{x}^2} . \qquad (2.7)
\end{aligned}$$

The expression $\#/[(x_a - x_b)^2]^{D/2-1}$ is obtained by change of variable $u = 1/T$, and evaluating the Gaussian integral

$$\int_0^\infty \frac{dT}{(4\pi T)^{D/2}} e^{-(x_a-x_b)^2/4T} .$$

The Ingredient #1:

[3] I wonder if there should be a T in the denominator of the derivative term, because I can't find the extra T in the eq. (2.11) of "Spinning particles in the QM and QFT" [8]. [4] It is probable that in ref. [8] the authors had made a rescaling of the interval $[0, T]$ to the unit interval, which induces a $1/T$ for the derivative term; please recheck, and comment out these two footnotes once you have corrected the text. I calculate this by factorizing the Laplacian,

$$\begin{aligned}
\text{Det}\left(-\frac{1}{4}\frac{\partial^2}{\partial\tau^2} + \frac{B_{ab}}{4\bar{T}}\right) &= \text{Det}\left[-\frac{1}{4}\frac{\partial^2}{\partial\tau^2}\left(1 - (\frac{\partial^2}{\partial\tau^2})^{-1}\frac{B_{ab}}{\bar{T}}\right)\right] \qquad (2.8)\\
&= \text{Det}\left(-\frac{1}{4}\frac{\partial^2}{\partial\tau^2}\right) \text{Det}\left(1 - (\frac{\partial^2}{\partial\tau^2})^{-1}\frac{B_{ab}}{\bar{T}}\right) .
\end{aligned}$$

where $\text{Det}\left(-\frac{1}{4}\frac{\partial^2}{\partial\tau^2}\right) = (4\pi\bar{T})^{-\frac{D}{2}}$ is evaluated in (2.3). [5] As for the Det $\left(1 - (\frac{\partial^2}{\partial\tau^2})^{-1}\frac{B_{ab}}{T}\right)$ term, it would appear that [6] This is not correct! See eq. (8.9) in ref. [15]: you are

[2] Predrag: 2018-08-21 draw the figure
[3] Guopeng: 2018-07-29
[4] James: 2018-07-29
[5] Predrag: 2018-08-04] If you gone through exercise of evaluating (2.3), write it up.
[6] James: 2018-07-29, 2018-08-21

missing some delta functions. Schubert 1999 literature review [15] *Perturbative Quantum Field Theory in the String-Inspired Formalism*, arXiv:hep-th/0101036, is a first base for just about any question on the early development of worldline techniques. There you will find section 8 of great use...

$$-(\frac{\partial^2}{\partial\tau^2})^{-1}\frac{B_{ab}}{T}=\frac{G_{Bab}}{T}\,,$$

so that the determinant could be $(1+\frac{G_{Bab}}{T})^{-\frac{D}{2}}$, where G_{Bab} is the loop Green's function. [7] The result you need is $\Big(B_{ab}(\frac{d^2}{dt^2})^{-1}B_{ab}\Big)(\tau,\tau')=-G_{Bab}B_{ab}(\tau,\tau')$ which is eq. (8.9) in ref. [15]. To prove this, insert two complete sets of states:

$$\langle\tau|B_{ab}(\frac{d^2}{dt^2})^{-1}B_{ab}|\tau'\rangle=\int_0^T ds\int_0^T ds'\langle\tau|B_{ab}|s\rangle\langle s|(\frac{d^2}{dt^2})^{-1}|s'\rangle\langle s'|B_{ab}|\tau'\rangle\,,$$

write the matrix elements as functions of τ and s, τ' and s' and compute the s and s' integrals. As

$$B_{ab}(\tau_1,\tau_2)=[\delta(\tau_a-\tau_1)-\delta(\tau_b-\tau_1)][\delta(\tau_a-\tau_2)-\delta(\tau_b-\tau_2)]\,,$$

$-(\frac{\partial^2}{\partial\tau_1^2})^{-1}\frac{B_{ab}}{T}$ is proportional to $\tau-\frac{1}{2}(\tau_a+\tau_b)$, since

$$\begin{aligned}\frac{\partial^2}{\partial\tau_1^2}|\tau_1-\tau_2| &= 2\delta(\tau_1-\tau_2)\\ |\tau_a-\tau|-|\tau_b-\tau| &= 2\tau-(\tau_a+\tau_b)\,.\end{aligned}\tag{2.9}$$

[8] As discussed in ref. [15], the incorporation of a photon propagator between two points on a one loop diagram (that turns it into a two loop diagram) can be incorporated into the worldline action at the expense of changing the kinetic term – see eqs. (8.2) - (8.4) of ref. [15] :-). Finding the Green's function and the functional determinant of the new kinetic operator then ensures that the effect of the internal photon is automatically included into all further worldline calcultions. So the one-loop calculation turns into a two loop calculation upon application of the new Green's function. The same happens below for the open line: the incorporation of a virtual photon between two points on the line turns a tree level process into a one loop process. The effect of this photon can be absorbed into the worldline calculations by using $\Delta_{(1)}$ instead of the open line's Δ. **The Ingredient #2**:

$$\Delta_{(1)}(\tau_1,\tau_2|\tau_a,\tau_b):=\langle\tau_1|\frac{1}{-\frac{1}{4T}\frac{\partial^2}{\partial\tau^2}+\frac{B_{ab}}{4T}}|\tau_2\rangle\,.\tag{2.10}$$

For $\langle\tau_1|\frac{B_{ab}}{4T}|\tau_2\rangle$, the operator $\hat{B}_{ab}$ has matrix elements

$$B_{ab}(\tau_1,\tau_2):=\langle\tau_1|\hat{B}_{ab}|\tau_2\rangle=[\delta(\tau_a-\tau_1)-\delta(\tau_b-\tau_1]\,[\delta(\tau_a-\tau_2)-\delta(\tau_b-\tau_2)]\,.$$

[7] James: 2018-08-21
[8] James: 2018-08-21

[9]Be careful here about definitions of operators and their functional form (matrix elements) with respect to a given basis. Your calculation below doesn't really look right as it stands.

$$\begin{aligned}
& B_{ab}(\tau_1, \tau_2) \\
= \quad & < \tau_1|[\delta(\tau_a - \tau_1) - \delta(\tau_b - \tau_1)][\delta(\tau_a - \tau_2) - \delta(\tau_b - \tau_2)]|\tau_2 > \\
= \quad & \int_0^\infty d\tau \; < \tau_1|[\delta(\tau_a - \tau_1) - \delta(\tau_b - \tau_1)][\delta(\tau_a - \tau_2) - \delta(\tau_b - \tau_2)]|\tau >< \tau|\tau_2 > \\
= \quad & \delta(\tau_1 - \tau_2)([\delta(\tau_a - \tau_1) - \delta(\tau_b - \tau_1)][\delta(\tau_a - \tau_2) - \delta(\tau_b - \tau_2)] \qquad (2.11)
\end{aligned}$$

since $< \tau_1|\tau >= \delta(\tau_1 - \tau)$, $< \tau|\tau_2 >= \delta(\tau - \tau_2)$ and

$$\int_0^\infty d\tau \, \delta(\tau_1 - \tau)\delta(\tau - \tau_2) = \delta(\tau_1 - \tau_2) \, ,$$

but this calculation doesn't match the result in the *1loopPhi3.pdf* note.

However, note that when we calculate $\mathrm{Det}\left(1 - (\frac{\partial^2}{\partial\tau^2})^{-1}\frac{B_{ab}}{T}\right)$, we assume that $-(\frac{\partial^2}{\partial\tau^2})^{-1}\frac{B_{ab}}{T} = \frac{G_{Bab}}{T}$, [10] I don't know how to prove this. [11]This is not the correct result – see my comment above. so the matrix is diagonal matrix, that's why we have

$$\mathrm{Det}\left(1 - (\frac{\partial^2}{\partial\tau^2})^{-1}\frac{B_{ab}}{T}\right) = (1 + \frac{G_{Bab}}{T})^{-\frac{D}{2}}$$

[12]The result requires application of the $\ln\det = \mathrm{tr}\,\ln$ formula, and a subsequent expansion of the logarithm. This is a main result of the new work that I must write up for you. but in terms of the result in James calculation,

$$< \tau_1|B_{ab}(\tau_1, \tau_2)|\tau_2 >= \frac{(\Delta_{1a} - \Delta_{1b})(\Delta_{a2} - \Delta_{b2})}{\bar{T} + G_Bab}$$

[13] This is not what I'm saying, because you have in fact to calculate (see your (2.12)) the matrix elements to all orders in the expansion of the $\left(1 + (\frac{d^2}{dt^2})^{-1}\frac{B_{ab}}{T}\right)^{-1}(\frac{d^2}{dt^2})^{-1}$. Again, I must write this up and send it to you. All worldline Green's functions are exact. As commented above, the final result represents an all orders summation of the matrix elements in the expansion of $\langle\tau_1|[1+(-\frac{1}{4T}\frac{\partial^2}{\partial\tau^2})^{-1}\frac{B_{ab}}{4T}]^{-1}|\tau_2\rangle$. The error is in the claim $\langle\tau_1| - \frac{B_{ab}}{4T}|\tau_2\rangle = \frac{(\Delta_{1a}-\Delta_{1b})(\Delta_{a2}-\Delta_{b2})}{\bar{T}+G_{Bab}}$.

when $\tau_1 \neq \tau_2$, the matrix element should be zero if it is diagonal, so the matrix shouldn't be diagonal, then how is the determinant to be calculated?

[9]James: 2018-08-21
[10]Guopeng: 2018-07-29
[11]James: 2018-08-21
[12]James: 2018-08-21
[13]James: 2018-08-21

2018-07-30 Predrag Best to make intermediate calculation questions into footnotes, which you then comment out as you correct (or expand upon) the steps in the calculation as you fix them, so the resulting blog is the clean, correct calculation.

2018-08-03 In *1loopPhi3.pdf* note the $\Delta_{(1)}$ is defined as in (2.10). If I extract $-\frac{1}{4T}\frac{\partial^2}{\partial\tau^2}$, and expand the latter term, [14] I obtain

$$\Delta_{(1)}(\tau_1,\tau_2|\tau_a,\tau_b) = \langle\tau_1|[-\frac{1}{4T}\frac{\partial^2}{\partial\tau^2}]^{-1}[1+(-\frac{1}{4T}\frac{\partial^2}{\partial\tau^2})^{-1}\frac{B_{ab}}{4\bar{T}}]^{-1}|\tau_2\rangle$$
$$\simeq \langle\tau_1|[-\frac{1}{4T}\frac{\partial^2}{\partial\tau^2}]^{-1}[1-(-\frac{1}{4T}\frac{\partial^2}{\partial\tau^2})^{-1}\frac{B_{ab}}{4\bar{T}}]|\tau_2\rangle$$
$$= \langle\tau_1|[(-\frac{1}{4T}\frac{\partial^2}{\partial\tau^2})^{-1}-\frac{B_{ab}}{4\bar{T}}]|\tau_2\rangle\,. \tag{2.12}$$

Using

$$\langle\tau_1|(-\frac{1}{4T}\frac{\partial^2}{\partial\tau^2})^{-1}|\tau_2\rangle = \Delta(\tau_1,\tau_2)$$

I get

$$\langle\tau_1|-\frac{B_{ab}}{4\bar{T}}|\tau_2\rangle = \frac{(\Delta_{1a}-\Delta_{1b})(\Delta_{a2}-\Delta_{b2})}{\bar{T}+G_{Bab}}\,,$$

so it appears again that we need to know how to prove the latter equation. The reason why I expand the latter term is that I assume the deviation from the classical path is small. But I could be wrong.

Currently I am looking through Ahmadiniaz, Bashir and Schubert [1] and preparing my TOEFL exam:)

2018-08-14 A small typo found. In tree-level note, charge in the right side of eq. (1.97) in the second term is missing.

Some thoughts or questions about worldline formalism. When we derive the master formula for scalar QED, we assume that the electromagnetic field is composed of N photons, so we can use this formula to describe a scalar particle(like a proton or electron) interact with N photons. So when $N=0$, it describe a free scalar particle. When$N=1$, it describe a scalar particle emit one photon or absorb one photon, in this case, I wander whether it satisfy the Special Relativity. For an electron emit a photon, if we observe this process in the center of mass frame with respect to ingoing electron, then it will violate the energy conservation law. For $N=2$, a scalar particle emit(absorb) a photon and then absorb(emit) a photon at somewhere else, if the two photon is two different photons(I mean the source is different), then this describe Compton scattering process. If the two photon is the same one(for instance, emitted first then absorbed by the same electron), this describe self-energy process, in this case, we take the replacement

$$\epsilon_\mu\epsilon_\nu \longrightarrow \int\frac{d^Dk}{(2\pi)^D}\frac{\eta_{\mu\nu}}{k^2}$$

[14]Predrag: 2018-08-04] We are not allowed any such “small deviation” approximations, as Edwards' Green's functions are exact.

in master formula. This is obtained by “sewing" two photon vertices together.

In *1loopPhi3.pdf* calculation: The extra term behind the master formula

$$\int_0^T d\tau_a \int_0^T d\tau_b \frac{\sharp}{[(x(\tau_a)-x(\tau_b))^2]^{D/2-1}}$$

This act like the photon vector operator, thus we add this term.

2018-08-24 Ingredient1:

Following eq. (8.11) of ref. [15], Det $\left(-\frac{1}{4}\frac{\partial^2}{\partial\tau^2}+\frac{B_{ab}}{4\bar{T}}\right)$ is calculated application of the $\ln\det\ = \mathrm{tr}\ \ln$ formula as follows:

$$\begin{aligned}\log\frac{\mathrm{Det}\left(-\frac{1}{4}\frac{\partial^2}{\partial\tau^2}+\frac{B_{ab}}{4\bar{T}}\right)}{\mathrm{Det}\left(-\frac{1}{4}\frac{\partial^2}{\partial\tau^2}\right)} &= \log\mathrm{Det}\left(-\frac{1}{4}\frac{\partial^2}{\partial\tau^2}+\frac{B_{ab}}{4\bar{T}}\right)-\log\mathrm{Det}\left(-\frac{1}{4}\frac{\partial^2}{\partial\tau^2}\right) \\ &= \mathrm{tr}\ \log\left(-\frac{1}{4}\frac{\partial^2}{\partial\tau^2}+\frac{B_{ab}}{4\bar{T}}\right)-\mathrm{tr}\ \log\left(-\frac{1}{4}\frac{\partial^2}{\partial\tau^2}\right)\end{aligned} \tag{2.13}$$

and

$$\log\left(-\frac{1}{4}\frac{\partial^2}{\partial\tau^2}+\frac{B_{ab}}{4\bar{T}}\right)=\log\left((-\frac{1}{4}\frac{\partial^2}{\partial\tau^2})[1-(-\frac{1}{4}\frac{\partial^2}{\partial\tau^2})^{-1}\frac{B_{ab}}{4\bar{T}}]\right)$$

$$=\log\left(-\frac{1}{4}\frac{\partial^2}{\partial\tau^2}\right)+\log\left(1-(-\frac{1}{4}\frac{\partial^2}{\partial\tau^2})^{-1}\frac{B_{ab}}{4\bar{T}})\right)$$

So

$$\log\frac{\mathrm{Det}\left(-\frac{1}{4}\frac{\partial^2}{\partial\tau^2}+\frac{B_{ab}}{4\bar{T}}\right)}{\mathrm{Det}\left(-\frac{1}{4}\frac{\partial^2}{\partial\tau^2}\right)}=tr\log\left(1-(-\frac{1}{4}\frac{\partial^2}{\partial\tau^2})^{-1}\frac{B_{ab}}{4\bar{T}})\right)$$

After expanding to Taylor series and noting that $\left(B_{ab}(\frac{d^2}{dt^2})^{-1}B_{ab}\right)(\tau,\tau')=-G_{Bab}B_{ab}(\tau,\tau')$, I obtain

$$\log\left(1-(-\frac{1}{4}\frac{\partial^2}{\partial\tau^2})^{-1}\frac{B_{ab}}{4\bar{T}})\right)$$

$$=-(-\frac{1}{4}\frac{\partial^2}{\partial\tau^2})^{-1}\frac{B_{ab}}{4\bar{T}}-\frac{1}{2}(-\frac{1}{4}\frac{\partial^2}{\partial\tau^2})^{-1}\frac{B_{ab}}{4\bar{T}}(-\frac{1}{4}\frac{\partial^2}{\partial\tau^2})^{-1}\frac{B_{ab}}{4\bar{T}}$$

$$-\frac{1}{3}(-\frac{1}{4}\frac{\partial^2}{\partial\tau^2})^{-1}\frac{B_{ab}}{4\bar{T}}(-\frac{1}{4}\frac{\partial^2}{\partial\tau^2})^{-1}\frac{B_{ab}}{4\bar{T}}(-\frac{1}{4}\frac{\partial^2}{\partial\tau^2})^{-1}\frac{B_{ab}}{4\bar{T}}+\dots\dots\dots$$

$$=-\left((-\frac{1}{4}\frac{\partial^2}{\partial\tau^2})^{-1}\frac{B_{ab}}{4\bar{T}}-\frac{1}{2}(-\frac{1}{4}\frac{\partial^2}{\partial\tau^2})^{-1}\frac{B_{ab}}{4\bar{T}}\frac{G_{Bab}}{\bar{T}}+\dots\dots\dots\right)$$

$$=-(-\frac{1}{4}\frac{\partial^2}{\partial\tau^2})^{-1}\frac{B_{ab}}{4\bar{T}}\left(1-\frac{1}{2}\frac{G_{Bab}}{\bar{T}}+\frac{1}{3}\frac{G_{Bab}}{\bar{T}^2}+\dots\dots\dots\right)$$

 2.3 *Learning wordline formalism notes*

Using $\mathrm{tr}\, A = \int d\tau \langle\tau|A|\tau\rangle$ [15] Not sure whether this is the correct definition of a trace.

$$\mathrm{tr}\ \log\left(1-(-\frac{1}{4}\frac{\partial^2}{\partial\tau^2})^{-1}\frac{B_{ab}}{4\bar{T}})\right) = -\int d\tau \langle\tau|(-\frac{1}{4}\frac{\partial^2}{\partial\tau^2})^{-1}\frac{B_{ab}}{4\bar{T}}|\tau\rangle\,(\ldots\ldots\ldots)$$

and inserting a completeness series $\int d\tau' |\tau'\rangle\langle\tau'|$,

$$\begin{aligned}
&\int d\tau \int d\tau' \langle\tau|(-\frac{1}{4}\frac{\partial^2}{\partial\tau^2})^{-1}|\tau'\rangle\langle\tau'|\frac{B_{ab}}{4\bar{T}}|\tau\rangle \\
= &\int d\tau \int d\tau' \frac{[\delta(\tau_a-\tau)-\delta(\tau_b-\tau)][\delta(\tau_a-\tau')-\delta(\tau_b-\tau')]}{2\bar{T}}\Delta(\tau,\tau') \\
= &-\frac{\Delta(\tau_a,\tau_a)+\Delta(\tau_b,\tau_b)}{2\bar{T}} \neq \frac{G_{Bab}}{2\bar{T}}\,. \qquad (2.14)
\end{aligned}$$

The result we need should be $-\frac{G_{Bab}}{\bar{T}}$, so the summation could be $\log(1+\frac{G_{Bab}}{\bar{T}})$. Considering that the dimension is D, then we could get the right answer. But here I meet a problem.

Ingredient 2:

$$\begin{aligned}
\Delta^{(1)}(\tau_1,\tau_2|\tau_a,\tau_b) \;:=\; &\langle\tau_1|\left(-\frac{1}{4}\frac{\partial^2}{\partial\tau^2}+\frac{B_{ab}}{4\bar{T}}\right)^{-1}|\tau_2\rangle \\
= &\langle\tau_1|\left((-\frac{1}{4}\frac{\partial^2}{\partial\tau^2})(1+(-\frac{1}{4}\frac{\partial^2}{\partial\tau^2})^{-1}\frac{B_{ab}}{4\bar{T}})\right)^{-1}|\tau_2\rangle \\
= &\langle\tau_1|\Big((-\frac{1}{4}\frac{\partial^2}{\partial\tau^2})^{-1}-(-\frac{1}{4}\frac{\partial^2}{\partial\tau^2})^{-1}\frac{B_{ab}}{4\bar{T}}(-\frac{1}{4}\frac{\partial^2}{\partial\tau^2})^{-1} \\
&+\cdots\cdots)\,|\tau_2\rangle\,. \qquad (2.15)
\end{aligned}$$

Noting that the high orders can be expressed by the second term, for instance,

$$(-\frac{1}{4}\frac{\partial^2}{\partial\tau^2})^{-1}\frac{B_{ab}}{4\bar{T}}(-\frac{1}{4}\frac{\partial^2}{\partial\tau^2})^{-1}\frac{B_{ab}}{4\bar{T}}(-\frac{1}{4}\frac{\partial^2}{\partial\tau^2})^{-1} = -(-\frac{1}{4}\frac{\partial^2}{\partial\tau^2})^{-1}\frac{B_{ab}}{4\bar{T}}(-\frac{1}{4}\frac{\partial^2}{\partial\tau^2})^{-1}\frac{G_{Bab}}{\bar{T}}$$

the first term gives the free Green's function

$$\Delta(\tau_1,\tau_2) = \langle\tau_1|(-\frac{1}{4}\frac{\partial^2}{\partial\tau^2})^{-1}|\tau_2\rangle\,. \qquad (2.16)$$

As for the second term, we can insert two completeness series to obtain

$$\begin{aligned}
&\langle\tau_1|(-\frac{1}{4}\frac{\partial^2}{\partial\tau^2})^{-1}\frac{B_{ab}}{4\bar{T}}(-\frac{1}{4}\frac{\partial^2}{\partial\tau^2})^{-1}|\tau_2\rangle \\
= &\int d\tau \int d\tau' \langle\tau_1|(-\frac{1}{4}\frac{\partial^2}{\partial\tau^2})^{-1}|\tau\rangle\langle\tau|\frac{B_{ab}}{4\bar{T}}|\tau'\rangle\langle\tau'|(-\frac{1}{4}\frac{\partial^2}{\partial\tau^2})^{-1}|\tau_2\rangle \\
= &\int d\tau \int d\tau' \Delta(\tau_1,\tau)\frac{[\delta(\tau_a-\tau)-\delta(\tau_b-\tau)][\delta(\tau_a-\tau')-\delta(\tau_b-\tau')]}{4\bar{T}}\Delta(\tau',\tau_2) \\
= &\frac{[\Delta_{a1}-\Delta_{b1}][\Delta_{a2}-\Delta_{b2}]}{\bar{T}}\,. \qquad (2.17)
\end{aligned}$$

[15] Guopeng: 2018-08-25

Take the common factor out. Left in the brackets is $(1 + G_{Bab}/\bar{T})^{-1}$, so

$$\begin{aligned}\Delta^{(1)}(\tau_1, \tau_2|\tau_a, \tau_b) &:= \langle\tau_1|\left(-\frac{1}{4}\frac{\partial^2}{\partial\tau^2} + \frac{B_{ab}}{4\bar{T}}\right)^{-1}|\tau_2\rangle \\ &= \Delta_{12} + \frac{[\Delta_{a1} - \Delta_{b1}]\,[\Delta_{a2} - \Delta_{b2}]}{G_{Bab} + \bar{T}}\,. \qquad (2.18)\end{aligned}$$

2.4 New Trends in First Quantisation - Bad Honnef 2025

2025-04-14 Predrag World Quantum Day

2025-04-15 Predrag .

Rafael Porto: Impressive, very involved gravitational waves perturbative calculations

Duff 70's Damour et al 90's Classical electrodynamics in tersm of interparticle potentials

Fiorenzo Bastianelli "First quantisation and one-loop divergences in quantum gravity", a very clear exposition.

2025-04-17 Predrag .

Jan Plefka *High-precision gravitational wave physics from worldline quantum field theory*: very impressive worldline gravity perturbative expansions.

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Chapter 3

Daily QFT blog

3.1 QFT books, courses

2013-07-15, 2018-08-27 Predrag I've collected a bunch of QFT e-books, saved them in ChaosBook.org/library:

Grigorenko 2006 (click here)

Abarbanel 2013 (click here)

Altland and Simons [1] *Condensed Matter Field Theory*: see Chapter 1 *Collective Excitations: From Particles to Fields* (click here).

Check out also

Simons, Lecture I: *Collective Excitations: From Particles to Fields Free Scalar Field Theory: Phonons*

Simons *Quantum Condensed Matter Field Theory*

T. Banks [8], *Modern Quantum Field Theory: A Concise Introduction* download from here, review here :
(click here)

Chaichian and A. Demichev [20] *Path Integrals in Physics Volume I Stochastic Processes and Quantum Mechanics*: (click here)

Chaichian and A. Demichev [21] *Path Integrals in Physics Volume II Quantum Field Theory, Statistical Physics and other Modern Applications*: (click here)

Piers Coleman [25] *Introduction to Many-Body Physics*: (click here) + (click here)

CoKaWa 2004 (click here)

DasFerbel 2003 (click here)

Farmelo [47] *The Strangest Man: The Hidden Life of Paul Dirac, Mystic of the Atom*: (click here)

Hao 1989 (click here)

 3.1 QFT books, courses

Nichkawde 2013 (click here)

Das 2006 (click here)

Elvang and Huang [41, 42] *Scattering Amplitudes*, arXiv:1308.1697

Milton 2001 (click here)

NagashimaI 2010 (click here)

Elbaz [40]: (click here)

Sadovskii [95] *Quantum Field Theory*: (click here)

Sadovskii [94] *Diagrammatics: Lectures on Selected Problems in Condensed Matter Theory* (click here)

SeoSan 2012 (click here)

Szekeres [103] *A Course in Modern Mathematical Physics: Groups, Hilbert Space and Differential Geometry*: (click here)

Zee 2003 (click here)

An online references list

Howard Haber list of sources

2024-02-13 Predrag Very impressed by David Tong's course materials / online books. Also like the HTML layout (in this gitHub folder), might use something like that?

2014-08-13 Predrag In my notes it says that Dyson himself told me to read *Drawing theories apart: the dispersion of Feynman diagrams in postwar physics*, by Kaiser [65]. So we better read it - I will put a pilfered eBook copy Kaiser09.pdf into ChaosBook.org/library.

2024-02-18 Predrag Now that I have read further into Kevin Costello's textbook (click here), I have to acknowledge, sadly, that Kevin is not my brother by another mother. The book does not work for our QFT course. The lectures 1 and 2 linked below I still recommend, but by lecture 3, Costello takes us in direction that I should suffer in silence, without dragging you there. It's for *gasp!* mathematicians.

The free early version of the book (click here) has identical contents and numbers of pages, so I think it is good enough. I like following the nLab branched tree exposition of his approach.

I might use his sect. 2. *Functional integrals in quantum field theory* as the summary of the general field theory part of the course.

In my humble opinion, the original version is still the best:

Who's on first?

2024-02-18 Predrag Our course follows the French school. Francois David's course is the real deal.

Quantum Field Theory 2 (2023/2024 course)
Lecture 2 (November 15, 2023) derives Wick expansion.

Lecture 3 (November 17, 2023) derives Dyson-Schwinger eqs.
Lecture 4 (November 20, 2023) effective quantum action; 1-loop corrections

Quantum Field Theory 2 (2019/2020 course)
Lecture 4 ϕ^4 Feynman rules

2024-02-13 Predrag Sunil Mukhi course *Quantum Field Theory II* is very good.

For example, I watched

Lecture 13 - Evaluation of non-Gaussian path integrals is a very clean discussion of the semiclassical approximation and nonlinear corrections in the path-integral formulation of QM.

I'll alert you as I find lectures that fit into our course. Or you alert me.

2017-03-15 Predrag Read Setlur [98] *Dynamics of Classical and Quantum Fields: An Introduction*. Starts with
Geometrical meaning of Legendre transformation in classical mechanics;
Dynamical symmetries in the context of Noether's theorem;
The derivation of the stress energy tensor of the electromagnetic field; the expression for strain energy in elastic bodies, and the Navier Stokes equation; Functional integration is interpreted as a limit of a sequence of ordinary integrations, ... The rest is less obvious.

In principle, this book can be read online via `library.gatech.edu`, but how?

2018-01-31 Predrag Check out M. J. D. Hamilton [55] *Mathematical Gauge Theory*: " This book explains the mathematical background behind the Standard Model, translating ideas from physics into a mathematical language and vice versa. The first part of the book covers the mathematical theory of Lie groups and Lie algebras, fibre bundles, connections, curvature and spinors. The second part then gives a detailed exposition of how these concepts are applied in physics, concerning topics such as the Lagrangians of gauge and matter fields, spontaneous symmetry breaking, the Higgs boson and mass generation of gauge bosons and fermions. [...] Only a basic knowledge of differentiable manifolds and special relativity is required, summarized in the appendix. "

3.2 Daily QFT blog

2010-03-04 Predrag Kevin Mitchell is here, says we should study Littlejohn and M. Reinsch [75]: "Gauge fields in the separation of rotations and internal motions in the n-body problem." I will put it into ChaosBook.org/library. Read 3-body problem sections. See p. 14 for a discussion of the three-body coordinates and p. 25 for a discussion of the three-body section (gauge).

2011-07-27 PC What follows is casting eye far ahead - to the role of gauge invariance in Quantum Field Theories. Following articles seem of interest as follow-ups on Cvitanović [29], *Group theory for Feynman diagrams in non-Abelian gauge theories*:

Khellat [67] strikes me as dubious...

Martens [80] writes: "We calculate the two-loop matching corrections for the gauge couplings at the Grand Unification scale in a general framework that aims at making as few assumptions on the underlying Grand Unified Theory (GUT) as possible. In this paper we present an intermediate result that is general enough to be applied to the Georgi-Glashow SU(5) as a "toy model". The numerical effects in this theory are found to be larger than the current experimental uncertainty on αs . Furthermore, we give many technical details regarding renormalization procedure, tadpole terms, gauge fixing and the treatment of group theory factors, which is useful preparative work for the extension of the calculation to supersymmetric GUTs. "

Tye and Zhang [104] write: " Bern, Carrasco and Johansson have conjectured dual identities inside the gluon tree scattering amplitudes. We use the properties of the heterotic string and open string tree scattering amplitudes to refine and derive these dual identities. These identities can be carried over to loop amplitudes using the unitarity method. Furthermore, given the M-gluon (as well as gluon-gluino) tree amplitudes, M-graviton (as well as graviton-gravitino) tree scattering amplitudes can be written down immediately, avoiding the derivation of Feynman rules and the evaluation of Feynman diagrams for graviton scattering amplitudes

Eto *et al.* [46] write: "We construct the general vortex solution in the color-flavor-locked vacuum of a non-Abelian gauge theory, where the gauge group is taken to be the product of an arbitrary simple group and U(1). Use of the holomorphic invariants allows us to extend the moduli-matrix method and to determine the vortex moduli space in all cases. Our approach provides a new framework for studying solitons of non-Abelian varieties with various possible applications in physics."

and there is much much more...; will continue some other time.

2011-11-03 PC Today is that time. I'm sitting in Intractability Workshop: *Counting, Inference and Optimization on Graphs* with a bunch of high-level computer nerds, and I almost afraid to say what I'll say next (plumbers avoid physicists that say such things): In constructing our atlas of inertial manifold of turbulent pipe flow, we fix the $SO(2) \times O(2)$ phase separately on each local chart. The freedom of doing that is called "local gauge invariance" (blame Hermann Weyl for the ugly word) and in the limit of ∞ period cycles, cycle points are dense and their local charts are infinitesimal, so this is really local gauge invariance. In the world of computer science they use this freedom profitably, to reduce the number of terms they use in their computations. That suggests that there might be a (variational?) principle that selects an optimal choice of (relative) template phases (i.e., gauge transformations that connect a chart to the next chart).

Nerds call this 'reparametrization' - it supposedly speeds up calculations. Have not really seen that in quantum field theory, with exception of light cone gauges and their relatively recent applications by the Witten cult.

Literature: ref. [22, 23] and stuff on this site (if you can understand any of it).

Feel free to ignore this remark. It's future research.

2012-05-16 Parameswaran Nair vpnair@optonline.net writes on saddle solutions of Yang-Mills:

I attributed the conjecture to Hitchin; it was actually due to Atiyah and Jones. "The only finite action solutions of the YM equations are instantons, either self-dual or antiself-dual." This was the conjecture for which the refs provide counter examples.

Here is the paper by my student Schiff [96], who writes: " Following a proposal of Burzlaff (Phys.Rev.D 24 (1981) 546), we find solutions of the classical equations of motion of an abelian Higgs model on hyperbolic space, and thereby obtain a series of non-self-dual classical solutions of four-dimensional SU(3) gauge theory. The lowest value of the action for these solutions is roughly 3.3 times the standard instanton action. "

" In physics, despite the fact that the non-self-dual solutions correspond to saddle points, and not minima, of the Yang-Mills functional, to do a correct semiclassical approximation by a saddle-point evaluation of the path integral, it is certainly necessary to include a contribution due to nonself-dual solutions, and if it should be the case that there is a non-self-dual solution with action lower than the instanton action (this question is currently open, and of substantial importance), then such a contribution would even dominate. Unfortunately, it is questionable whether the semiclassical approximation can give a reliable picture of quantized gauge theories; it has been argued that in four-dimensional gauge theory small quantum fluctuations around classical solutions cannot be responsible for confinement, unlike in certain lower-dimensional theories. But it may still be possible to extract some physics from the semiclassical approach. A first step in such a direction would be to obtain a good understanding of the full set of non-self-dual solutions and their properties. [...] We pursue an old idea, due to Burzlaft [10], for obtaining a non-self-dual, "cylindrically symmetric" solution for gauge group SU(3). If we write $\mathbb{R}^4 = \mathbb{R} \times \mathbb{R}^3$, and identify some SU(2) [or SO(3)] subgroup of SU(3), with generators that we will denote T', then we can look at the set of SU(3) gauge potentials which are invariant under the action of the group generated by the sum of the T"s and the generators of rotations on the IR factor of E (we choose the T"s and the IR rotation generators to satisfy the same commutation relations). We call such potentials "cylindrically symmetric" (in analogy to the standard notion of cylindrical symmetry in IR, which involves writing R =RXIR and requiring rotational symmetry on the E factor). Such potentials will be specified by a number of functions of two variables: the coordinate on the IR factor of E (which we will denote x), and the radial coordinate of the E factor (which we will denote y). Clearly the equations of motion for such cylindrically symmetric potentials (if they are consistent) will reduce to equations on the space I (x,y):y 0]. „

The earlier work is by Sibner, Sibner, Uhlenbeck [99]. They write " The Yang-Mills functional for connections on principle SU(2) bundles over S 4 is studied. Critical points of the functional satisfy a system of second-order partial differential equations, the Yang-Mills equations. If, in particular, the critical point is a minimum, it satisfies a first-order system, the self-dual or anti-self-dual equations. Here, we exhibit an infinite number of finite-action non-minimal unstable critical points. They are obtained by constructing a topologically nontrivial loop of connections to which min-max theory is applied. The construction exploits the fundamental relationship between certain invariant instantons on S 4 and magnetic monopoles on H 3. This result settles a question in gauge field theory that has been open for many years. "

Bor [12] writes " We prove the existence of a new family of non-self-dual finite-energy solutions to the Yang-Mills equations on Euclidean four-space, with SU(2) as a gauge group. The approach is that of "equivariant geometry:" attention is restricted to a special class of fields, those that satisfy a certain kind of rotational symmetry, for which it is proved that (1) a solution to the Yang-Mills equations exists among them; and (2) no solution to the self-duality equations exists among them. The first assertion is proved by an application of the direct method of the calculus of variations (existence and regularity of minimizers), and the second assertion by studying the symmetry properties of the linearized self-duality equations. The same technique yields a new family of non-self-dual solutions on the complex projective plane. "

2012-04-19 Daniel "graveyard of obvious ideas" rings a little aggressive, no? "if you are a master of quantum-mechanics or QFT symmetries and their linear irreducible representations, [32] you may leave your baggage at the door" rings a little aggressive, too.
Predrag: Get's edgy. In "master of their linear irreducible representations" I make fun of myself. Let the referee object to that?

[2012-06-14 Predrag] Grin and bear it. Pulling my Senior discount card here.

2013-04-16 Predrag There seems to be whole literature on classical Yang-Mills (CYM). In *Entropy production in classical Yang-Mills theory from Glasma initial conditions* Hideaki Iida, Teiji Kunihiro, Berndt Müller, Akira Ohnishi, Andreas Schäfer, and Toru T. Takahashi, arXiv:1304.1807, write:

Pure Yang-Mills theory in temporal gauge with the Hamiltonian in the noncompact (A, E) scheme on a cubic spatial lattice. The initial condition satisfies Gauss' law; check its validity as well as Energy conservation carefully at every time step. Define distance (6), (7) that is gauge invariant under residual (time independent) gauge transformations.

now in CB

They call the stability matrix 'Hessian', and its eigenvalues at time slice the 'local Lyapunov exponents (LLEs)' [72]: LLE plays the role of a "temporally local" Lyapunov exponent, which specifies the departure rate of two trajectories in a short time period. Then they say this (?): "For a system where stable and unstable modes couple with each other as in the present case, an LLE does not generally agree with the Lyapunov exponent in a long time period." "Ref. [72] introduced another kind of Lyapunov exponent called the intermediate Lyapunov exponent (ILE), which is an "averaged Lyapunov exponent" for an intermediate time period; i.e., a time period which is sufficiently small compared to the thermalization time but large enough to sample a significant fraction of phase space. By its definition (13) it is the set of stability exponents for a finite time Jacobian matrix.

"Two comments are in order, here: A Lyapunov exponent [PC: not the Lyapunov exponent, they mean the stability exponent] can be (real) positive, negative, zero or purely imaginary. Liouville's theorem tells us that the determinant of the time evolution matrix U is unity, implying that the sum of all positive and negative ILEs is zero. The KS entropy is given as a sum of positive Lyapunov exponents. The second comment concerns gauge invariance of the Lyapunov exponents. In the Appendix we show that LLE and ILE are indeed gauge invariant under time-independent gauge transformations in the temporal gauge."

2013-11-27 Predrag One way to reduce symmetry seems obvious; **average over the group orbit**. That reduces the dimension of the state space by the dimension of the group orbit; in the reduced state space each group orbit is replaced by a point, it's average value. It is a natural construct in the theory of linear representations of groups, very important, where it is called a 'character' of the representation; I use it in my trace formula for systems with continuous symmetries. But it is a trace of a linear evolution operator.

Still, I do not seem to know how to do this for (1) nonlinear systems, (2) QFT gauge fixing. Barth and Christensen [9] is an example where this is done - perhaps a way too complicated example... (here are my notes on it, which I no longer understand myself :)

2014-01-11 Predrag I always wonder whether we should be reducing symmetries by averaging over group orbits (method of characters, used in my derivation of the spectral determinant in presence of continuous symmetries). Churchill, Kummer and Rod [24] write in *On averaging, reduction, and symmetry in Hamiltonian systems*: The existence of periodic orbits for Hamiltonian systems at low positive energies can be deduced from the existence of nondegenerate critical points of an averaged Hamiltonian on an associated "reduced space." The paper exploits discrete symmetries, including reversing diffeomorphisms, that occur in a given system. The symmetries are used to locate the periodic orbits in the averaged Hamiltonian, and thence in the original Hamiltonian when the periodic orbits are continued under perturbations admitting the same symmetries."

2014-01-11 Predrag Kummer [71], *On the construction of the reduced phase space of a Hamiltonian system with symmetry* writes: " Weinstein [...] uses this correspondence between connections and lifts in his construction of Sternberg's phase space for a particle in a Yang-Mills field. "

l. A. Weinstein, A universal phase space for particles in Yang-Mills fields, Lett. Math. Phys. 2 (1978), 417-420.

2. S. STERNBERG, Minimal coupling and the symplectic mechanics of a classical particle in the presence of a Yang-Mills field, Proc. Nat. Acad. Sci. 74 (1977),5253-5254.
3. S. STERNBERG, On the role of field theories in our physical conception of geometry in Differential geometric methods in mathematical physics II, Springer Lecture Notes in Mathematics, 676 (1977), 1-80.

2014-07-30 Predrag Paul Hoyer recent talk at Light Cone 2014 is too packed to be useful, but maybe has some pointers to recent interesting non-perturbative QCD results.

2014-08-01 Burak Sydney Coleman [26] gave a lecture on semi-classical formulations of QFT in 1975. He has several semi-classical treatment methods with names “Zeroth-Order Weak-Coupling Expansion", “Coherent-State Variation", “First-Order Weak-Coupling Expansion", “Bohr-Sommerfeld Quantization", and “DHN Formula". I didn’t yet understand what any of these means, I’m just starting to read it, but if we can find a way of adapting these methods to the numerical calculations that might be a good starting point.

2014-08-01 Predrag Coleman was a great expositor - on a level of a Nobelist. Would be cute if we found something in his lectures we can use today. “DHN Formula" (I knew all three - the first two authors are currently dead) should be the Gutzwiller formula for QFT. Paolo Muratore-Ginanneschi, a former student in our Copenhagen group, wrote something on that [86] that I still have not studied in depth. But we maybe should.

2014-08-01 Burak Coleman refers to the pedagogical overview by Rajaraman [92] for the derivation of “DHN Formula” and I think this will be a good starting point for me. So far, keywords are familiar, in the introduction, he talks about periodic orbits and their stability and how hard it is to find them for realistic cases. A lot of work apparently in this period has been done using Sine-Gordon equation because they had analytical solutions. I looked for articles which cite [92] but didn’t see anything that relies on numerical solutions of field equations.

2014-11-05 Predrag Do not know if this is something for us, but worth having a look at:

Luca Salasnich, *Discrete bright solitons in Bose-Einstein condensates and dimensional reduction in quantum field theory*, arXiv:1411.0160: “ We review the derivation of an effective one-dimensional (1D) discrete nonpolynomial Schrödinger equation from the continuous 3D Gross-Pitaevskii equation with transverse harmonic confinement and axial periodic potential. Then we study the bright solitons obtained from this discrete nonpolynomial equation showing that they give rise to the collapse of the condensate above a critical attractive strength. We also investigate the dimensional reduction of a bosonic quantum field theory, deriving an effective 1D nonpolynomial Heisenberg equation from the 3D Heisenberg equation of the continuous bosonic field operator under the action of transverse harmonic confinement. Moreover, by taking into account the presence of an axial periodic potential we find a generalized Bose-Hubbard model which reduces to the familiar 1D Bose-Hubbard Hamiltonian only if a strong inequality is satisfied. Remarkably, in the absence of axial periodic potential our 1D non-polynomial Heisenberg equation gives the generalized Lieb-Liniger theory we obtained some years ago."

2014-11-23 Predrag Zvonkin [111] (click here) writes in *Matrix integrals and map enumeration: An accessible introduction*: "Physicists working in two-dimensional quantum gravity invented a new method of map enumeration based on computation of Gaussian integrals over the space of Hermitian matrices. This paper explains the basic facts of the method and provides an accessible introduction to the subject."

2015-01-09 Predrag Tudor Dimofte gave a talk in Math on *Geometric representation theory, symplectic duality, and 3d supersymmetric gauge theory*

Abstract: Recently, a "symplectic duality" between D-modules on certain pairs of algebraic symplectic manifolds was discovered, generalizing classic work of Beilinson-Ginzburg-Soergel in geometric representation theory. I will discuss how such dual spaces (some known and some new) arise naturally in supersymmetric gauge theory in three dimensions.

Tudor is a mathematical physicist at the IAS, School of Physics, Princeton.

I went to the talk, and - wow! You would think I know something about a gauge theory but is is like it was in Lithuanian: I understood individual words, and the alphabet seemed to be Latin - there were things that looked like letter G or letter H and what we call quotient M/G is apparently called 'resolution'. The foundational paper is Braden, Licata, Proudfoot and Webster [16], and its followups on "Quantizations of conical symplectic resolutions II: category O and symplectic duality". Good luck reading these...

and of course, it was emphatically N=4 and not N=2, so now I'm at peace

:)

2015-02-04 Predrag Stephan Stetina <stetina@hep.itp.tuwien.ac.at> thesis on arXiv:1502.00122 uses my Field Theory, and says that 2PI graphs are no sweat (for me they were). Wrote to him:

You seem to have proven Feynman wrong :) That's no mean achievement. Congrats!

Me and my friends have been studying turbulence in fluid dynamics as a warmup for doing the same in Yang-Mills. If you see some interesting turbulence in relativistic fluid dynamics, we are always willing to have a look at things more field-theoretical.

2015-02-09 Stephan Stetina If you are referring to Feynman comments on your book, I definitely disagree with them - I found your book on field theory more than helpful!

It is very difficult to study (quantum) turbulence within our approach - however it would be very interesting to do so! The original idea was to derive the two-fluid hydrodynamics of superfluids from an underlying field theory. To be able to obtain analytical results, we had to apply some rather drastic simplifications:

We assumed the superfluid condensate to be uniform and homogeneous (which translate in a homogeneous superflow in the hydro picture). Further more we used imaginary time formalism which strictly limits us to study systems in equilibrium. It would most likely be very challenging (in particular numerically) to introduce a condensate with arbitrary space and time dependence. In the current calculations, a probable onset of turbulence manifests itself as "something going wrong" - for example above certain velocities of the superfluid it is no longer possible to construct a stable and homogeneous superfluid phase. Another example is the appearance of the "two-stream instability" which can also be detected in our approach (see for instance arXiv:1312.5993). I am not sure yet how much this approach has in common with the one you have cited.

2015-08-20 Predrag Strauss, Horwitz, Levitan, and Yahalom [102] *Quantum field theory of classically unstable Hamiltonian dynamics* might be a good starting point to learn about dynamical systems for which the motions can be described in terms of geodesics on a manifold. They say : " ... ordinary potential models can be cast into this form by means of a conformal map. The geodesic deviation equation of Jacobi, constructed with a second covariant derivative, is unitarily equivalent to that of a parametric harmonic oscillator, and we study the second quantization of this oscillator. The excitations of the Fock space modes correspond to the emission and absorption of quanta into the dynamical medium, thus associating unstable behavior of the dynamical system with calculable fluctuations in an ensemble with possible thermodynamic consequences. "

2016-01-08 Predrag Bogomolny wants us to study Englert and Schwinger [43–45]. Why? Ref. [45] *Semiclassical atom* seems to be reinventing Gutzwiller, without citing him. These papers are not cited much either, a pair of Nobel prizes notwithstanding :) Rohwedder and Englert [93] continue with *Semiclassical quantization in momentum space*. There is something called the Englert-Schwinger equation used in graphite studies [78]. Ullmo *et al.* [105] cite it: *Semiclassical density functional theory: Strutinsky energy corrections in quantum dots*

2016-04-06 Predrag Went to hear Sung-Jin Oh talk about

> "... wave equations that arise from geometric considerations. Prime examples include the wave map equation and the Yang-Mills equation on the Minkowski space. On one hand, these are fundamental field theories arising in physics; on the other hand, they may be thought of as the hyperbolic analogues of the harmonic map and the elliptic Yang-Mills equations, which are interesting geometric PDEs on their own. I will discuss the recent progress on the problem of global regularity and asymptotic behavior of solutions to these PDEs."

This kind of work might offer a path to computing non-trivial "exact coherent states" (non-perturbative classical solutions) of Yang-Mills. Experimentally I used Evernote on my phone to take notes and photos of the white board - 160406SungJinOh.pdf in this repository - maybe it helps if one wants to get started reading the literature, though it is going to be hard going.

Both in the Abelian case (what they call "Maxwell-Klein-Gordon" for a scalar charged particle and "Maxwell-Dirac" for spin 1/2), and in the non-Abelian case (what they call "Yang-Mills") they cheerfully set the particle mass to $m = 0$, which is a killer for us.

I tried to briefly explain to Sung-Jin Oh two things of possible interest to people solving the classical Maxwell-Klein-Gordon and Yang-Mills PDEs:

1. finiteness conjecture: perhaps they can find saddle-points (non-perturbative classical solutions of QED) that give the gauge sets as the starting step in perturbative calculations, rather than computing Feynman-diagrammatic corrections to trivial vacuum.

   My explanation probably did more harm than good, as I liberally mixed in QFT, and kept confusing the quantum and the classical problem - it is for a reason that PDE people do not know what expressions like "on-mass-shell amplitudes" and "Ward identities" mean.

2. I ran Sung-Jin Oh quickly through ChaosBook.org/tutorials to make him aware that for turbulent nonlinear PDEs one has to work in the ∞-dimensional state space, rather than look at the solutions only in the $(d+1)$-dimensional configuration space.

   He would have been more impressed if we could find such solutions for Eulerian flows, and give him a criterion which solutions are important, but I have no idea how to find smooth solutions for Euler (no viscosity Laplacian to help us there...).

All in all, I still have no idea for what kind of 'exact coherent states' to compute for Yang-Mills.

2016-05-26 Predrag William Graham Hoover and Kenichiro Aoki *Order and Chaos in the One-Dimensional ϕ^4 Model : N-Dependence and the Second Law of Thermodynamics*, arXiv:1605.07721 write: " We revisit the equilibrium one-dimensional ϕ^4 model from the dynamical systems point of view. We find an infinite number of periodic orbits which are computationally stable while at the same time exhibiting positive Lyapunov exponents. We formulate a standard initial condition for the investigation of the microcanonical chaotic number dependence of the model. We speculate on the uniqueness of the model's chaotic sea and on the connection of such collections of deterministic and time-reversible states to the Second Law of Thermodynamics.

2016-10-12 Predrag Read Nguyen [89] *The perturbative approach to path integrals: A succinct mathematical treatment*: " We study finite-dimensional integrals in a way that elucidates the mathematical meaning behind the formal manipulations of path integrals occurring in quantum field theory. This involves a proper understanding of how Wick's theorem allows one to evaluate integrals perturbatively, i.e., as a series expansion in a formal parameter irrespective of convergence properties. We establish invariance properties of such a Wick expansion under coordinate changes and the action of a Lie group of symmetries, and we use this to study essential features of path integral manipulations, including coordinate changes, Ward identities, Schwinger-Dyson equations, Faddeev-Popov gauge-fixing, and eliminating fields by their equation of motion. We also discuss the asymptotic nature of the Wick expansion and the implications this has for defining path integrals perturbatively and nonperturbatively. "

2016-10-28 Predrag Read Hegg and Phillips [59] *Strongly coupled fixed point in φ^4 theory*: " We show explicitly how a fixed point can be constructed in scalar $g\varphi^4$ theory from the solutions to a nonlinear eigenvalue problem. The fixed point is unstable and characterized by $\nu = 2/d$ (correlation length exponent), $\eta = 1/2 - d/8$ (anomalous dimension). For $d = 2$, these exponents reproduce to those of the Ising model which can be understood from the codimension of the critical point. The testable prediction of this fixed point is that the specific heat exponent vanishes. 2d critical Mott systems are well described by this new fixed point. "

2016-12-10 Predrag Possibly useful in a QFT course:

Weinzierl [107] *Tales of 1001 Gluons*.

2015-09-15, 2017-02-13 Predrag Dashen, Hasslacher and Neveu [36–38], *Nonperturbative methods and extended-hadron models in field theory. I. Semiclassical functional methods*, are reputed to be the first people to use WKB methods in field theory.

Juan-Diego Urbina is a big fan of the 3rd paper [38]: " a more modest approach by finding classical solutions of finite energy and bounded spatial extent. [...] We exhibit a four-dimensional model involving non-Abelian Yang-Mills fields. "

Juan-Diego:

Ablowitz, Faddeev and Korepin have solitons for nonlinear Schrodinger on discrete lattice, with quartic term written as $|\psi_i|^2 \frac{1}{2}(\psi_{i-1} + \psi_{i+1})$. Even time can be taken discrete. Nohl took a one solition solution, treated as a periodic orbit, got exact energy.

Soliton is an integrable, 4-parameter 4-dimensional manifold of solutions, in the infinite-dimensional space. All other action-angle pairs equal zero.

2016-12-26 Predrag Read Borinsky [13] *Renormalized asymptotic enumeration of Feynman diagrams*. It is a follow-up to Cvitanović, Lautrup and Pearson [35].

and Borinsky [14] PhD Thesis *Graphs in perturbation theory: Algebraic structure and asymptotics*

2017-05-26 Predrag Martin and Kearney [81] *An exactly solvable self-convolutive recurrence* fancy math reproduces (among much else) also counting of Cvitanović, Lautrup and Pearson [35] and Cvitanović [30]. The study the sequences $S(\alpha_1, \alpha_2, \alpha_3)$ of self-convolutive recurrences, derive a closed-form solutions as a Mellin transforms. The representation is useful for study the asymptotics via Laplace's method. Their counting problem is the number of connected, or indecomposable, permutations, which naturally leads them to QFT diagram counting; For example, the number of nonisomorphic connected Feynman diagrams of order $2(n + 1)$ arising in a simplified model of quantum electrodynamics (QED) [35]; the number of 'vertex graphs' of order 2n arising in the QED perturbation series for the electron magnetic moment [30]; the number of Feynman diagrams with exact propagators [30]; and the number of Feynman diagrams with proper self-energies arising in QED [35]. Their integral representation is apparently new.

Kugler [70] *Counting Feynman diagrams via many-body relations*, arXiv:1808.01759

Kaneko [66] *Enumeration of N-rooted maps using quantum field theory*: " Information about the number of Feynman graphs for a given physical process in a given field theory is especially useful for confirming the result of a Feynman graph generator used in an automatic system of perturbative calculations. A method of counting the number of Feynman graphs with weight of symmetry factor was established based on zero-dimensional field theory, and was used in scalar theories and QED. In this article this method is generalized to more complicated models by direct calculation of generating functions on a symbolic calculating system. This method is applied to QCD with and without counter terms, where many higher order are being calculated automatically. "

2017-05-29 Predrag Read Gopala Krishna, Labelle and Shramchenko [53] *Enumeration of N-rooted maps using quantum field theory*, arXiv:1705.05800:

A one-to-one correspondence is proved between the N-rooted ribbon graphs, or maps, with e edges and the (e-N+1)-loop Feynman diagrams of a certain quantum field theory. This result is used to obtain explicit expressions and relations for the generating functions of N-rooted maps and for the numbers of N-rooted maps with a given number of edges using the path integral approach.

Our main idea is to apply methods of quantum field theory to enumeration of graphs. We will refer to this theory as scalar quantum electrodynamics (scalar QED) to follow the notation of ref. [35], even though this is an abuse of language because our theory does not contain a spin one gauge field.

Gopala, Labelle and Shramchenko [54] *Enumeration of N-rooted maps using quantum field theory* prove, inter alia, the equality of the number of two-point Feynman diagrams in scalar QED [35] and the number of rooted maps.

2019-07-29 Predrag Vera [106] *Double-logarithms in $N = 8$ supergravity: impact parameter description & mapping to 1-rooted ribbon graphs*:

The numerical coefficients in this expression are present in other physical and mathematical problems. They appeared as early as 1976 in ref. [63] where different theorems for the enumeration of diagrams associated to many body theory were investigated. In 1978 Cvitanović *et al.* [35] made use of field theoretical functional methods to evaluate sums of combinatoric weights of Feynman diagrams.

2017-05-29 Predrag Read Pavlyukh and W. Hübner [91] *Analytic solution of Hedin's equations in zero dimensions*: " Feynman diagrams for the many-body perturbational theory are enumerated by solving the system of Hedin's equations in zero dimension. We extend the treatment of Molinari [83] and give a complete solution of the enumeration problem in terms of Whittaker functions. An important relation between the generating function of the electron propagator and anomalous dimension in quantum field theory of massless fermions and mesons in four dimensions (Yukawa theory) is found. The Hopf algebra of undecorated rooted trees yields the anomalous field dimension in terms of the solution of the same differential equation. Its relation to the mathematical problem of combinatorics of chord diagrams is discussed; asymptotic expansions of the counting numbers are obtained. "

2018-04-28 Predrag Castro and Roditi [17] *A combinatorial matrix approach for the generation of vacuum Feynman graphs multiplicities in ϕ^4 theory*. They cite Cvitanović, Lautrup and Pearson [35] and write: " [...] generate the set of all Feynman graphs and the respective multiplicities in a combinatoric way. These combinatorial matrices are explicitly related with the permutation group, which facilitates the construction of the vacuum Feynman graphs. Various insights in this combinatoric problem are proposed, which in principle provide an efficient way to compute the Feynman vacuum graphs and its multiplicities. "

Castro and Roditi [18] *A recursive enumeration of connected Feynman diagrams with an arbitrary number of external legs in the fermionic non-relativistic interacting gas*: " Enumeration of Feynman diagrams is an active research subject in quantum field [35] [2–3] and many-body theoretical research [83, 85, 91]. The combinatorial character of this process is contained in two equivalent formalisms: the functional and the field operator approaches. Although the enumeration of the Feynman diagrams is independent of the integrals that represent the physical processes, when we take the total contribution of certain classes of diagrams, the global structure of the generative combinatorics is relevant. (This can be seen, for instance, in recent results [19], where the symmetry factor—or multiplicity—of the related Feynman diagrams appear explicitly in the integrals.) "

2017-05-29 Predrag Read Molinari [83, 84] *Hedin's equations and enumeration of Feynman diagrams*

Molinari and N. Manini [85] *Enumeration of many-body skeleton diagrams* "Based on Hedin's equations for self-energy, polarization, propagator, effective potential, and vertex function, dressed (skeleton) Feynman diagrams are enumerated."

2022-08-15 Predrag Study

Zia, Redish and McKay [110] *Making sense of the Legendre transform* (2009)

Skarke [100] *Why is the Legendre transformation its own inverse?* (2013)

2018-03-31 Predrag Created the abstract for the Les Houches meeting, see `predrag/lectures/LesHouch18`. The talk is in this repo, `reducesymm/presentations/LesHouch18`
The worldline formula for the quenched QED form factors is currently wrong; essentially it puts Z_2 renormalization on each electron leg, rather than $\sqrt{Z_2}$.

2018-06-05 Predrag Notes on *Summer school on structures in local quantum field theory* Les Houches — June 4-15, 2018

Gerald Dunne *Resurgence and Perturbative/Non-Perturbative Relations* is pretty amazing - exact expression for trace formula in terms of WKB saddles and the non-perturbative corrections, based on 2-torus duality relations, see here for lecture notes (I have also written down some notes). I think this works only for integrable models. Told him to have a look at Wirzba [108] as a physically motivated chaotic problem whose analytic formulation is suited to Dunne's methods.

Jacob Bourjaily *Improving Integrands and Integrals for Amplitudes*: It's complicated. However, I did explain to him how to reduce all adjoint rep birdtracks (quotient Jacobi relation, treat the rest as a free algebra) to a basis set of treees + fully fully symmetric Casimir tensors; he should get back to me once he tries it.

2013-03-27 Predrag Do not understand this article: Jiménez-Lara and J. Llibre [64], *Periodic orbits and nonintegrability of generalized classical Yang–Mills Hamiltonian systems*.

Hu [62] *General initial value form of the semiclassical propagator*, write: " We show a general initial value form of the semiclassical propagator. Similar to cellular dynamics, this formulation involves only the nearby orbits approximation: the evolution of nearby orbits is approximated by linearized dynamics. This phase space smearing formulation keeps the accuracy of the original Van Vleck-Gutzwiller propagator. As an illustration, we present a simple initial value form of the semiclassical propagator. It is nonsingular everywhere and is efficient for numeric implementation. "

2018-06-07 Predrag Lorenzo Magnea (an intellectual, with opinions on many things, masquerading as a phenomenologist, five layers removed from LHC experimentalists) *Eikonal Correlators and form factors in perturbation theory*, and Franz Herzog *Geometric IR subtraction in real radiation*, arXiv:1804.07949, are our best shot for developing approximations to N-photon propagators to all orders. The idea is to claim that the N-photon propagator in the "rainbow" gauge sets a_{N00} is concentrated on (Schwinger) backbone that hops over the magnetic momentum vertex with a large momentum q, with all photons 'soft' respective to the backbone, with momenta $q - k_j$, $|k_j| \ll 1$. Then the same discussion as for IR contributions should apply, with only one γ^μ per N-photon propagator mass-shell vertex surviving in the IR limit, the rest reducing to scalar, gauge invariant and universal M_{soft}, commuting vertices, and exponentiating. The limit is explained in Magnea *Advanced Lectures on the Infrared structure of Perturbative QCD* handwritten lecture 2, sect. 2 *The soft approximation*, and the all order summation is given in lecture 3.

Study also Herzog [61] *Zimmermann's forest formula, infrared divergences and the QCD beta function*.

The trick then would be to formulate next-to-leading order (NLO) and next-to-next-to-leading order (NNLO) corrections systematically.

Henry Kißler *The t'Hooft–Veltman gauge*: Finally understood the point of Henry's thesis. The t'Hooft–Veltman gauge condition is quadratic in A^μ, leading to 4-vertices, and QED ghost sector with structure reminiscent of a non-Abelian QCD. By Ward identities it all adds up to zero, but individual graphs are non-trivial.

Erik Panzer *The Hepp bound for Feynman periods*. "Feynman period" is this cult's jargon for the value of a Feynman integral, a quantity hard to evaluate except for the known zoo of integrals evaluated so far, catalogued by Oliver Schnetz [97]. The "Hepp bound" or "Hepp invariant," however, is a *rational* number computed efficiently for all graphs. Some of the striking properties of this bound: it correlates very well with the actual Feynman period (the empirical ratio of the two fluctuates within 1%, a numerical fact not explained yet) and it respects all known identities among periods (I have some handwritten notes on this talk).

From gauge-sets point of view, analytic formulas for (g-2) seem useless, as they are sets of large exact numbers (see eq. (29) in Schnetz [97]; arXiv:1711.05118), that then sum up to a much smaller number.

I had extensive discussion with Erik, and gauge sets are now in his "to-do" list. While Schnetz and Panzer had focused on ϕ^4, and I have this deep prejudice that gauge theories are profoundly different (because I assume that the gauge invariance induces cancelations among finite parts), perhaps they are not so different. Let's think of gauge symmetry as we think of other continuous symmetries in theory of dynamical systems. Then quotienting the symmetry leads us to gauge sets, which have no symmetry, but each has $n!$ Feynman diagram contributions - a "ϕ^4" for each gauge set. Erik tells me that someone has proven that Borel-resummed ϕ^4 has a finite radius of convergence, so it still could work out that the quenched QED has a finite radius of convergence.

2018-06-08 Predrag Spencer Bloch *The algebraic geometry of the kite graph amplitude* assumed more pre-knowledge than what I posses. Lake I've spent last 20 years chatting to Spencer, and he's telling me what's new in the last 6 weeks, without defining a single thing.

Berghoff

2018-06-11 Predrag Dirk Kreimer *Complex graphs and graph complexes*

Schlotterer

Marcel Golz *Parametric QED* [49]. The Schwinger parametric Feynman integrals for gauge theories quickly become prohibitively complicated due to the very involved numerator polynomials. He reported on a simplification via combinatorics for QED; the integrand is only a very small sum of scalar integrands with Dodgson polynomial (introduced by Francis Brown) numerators. Dodgson polynomial (related to cycle polynomials) satisfy Dodgson identities. Marcel is interested in simplifications such as organization by gauge sets, so I gave him access to this blog.

2020-04-19 Predrag Golz bachelor thesis *Evaluation techniques for Feynman diagrams* states clearly what various generalizations of Riemann zeta function are: multiple zeta values, classical polylogarithm (a multiple polylogarithm in one variable is called hyperlogarithm), and the multiple polylogarithm.

Golz, Panzer and Schnetz [52] *Graphical functions in parametric space* arXiv:1509.07296

Golz [49] *New graph polynomials in parametric QED Feynman integrals* arXiv:1703.05134.

Golz [48] *Contraction of Dirac matrices via chord diagrams*, arXiv:1710.05164.

Golz [51] *Parametric quantum electrodynamics* (2018)

Golz [50] *Dodgson polynomial identities*, arXiv:1810.06220:

Dodgson polynomials appear in Schwinger parametric Feynman integrals and are closely related to the well known Kirchhoff (or first Symanzik) polynomial. In this article a new combinatorial interpretation and a generalisation of Dodgson polynomials are provided. This leads to two new identities that relate large sums of products of Dodgson polynomials to a much simpler expression involving powers of the Kirchhoff polynomial. These identities can be applied to the parametric integrand for quantum electrodynamics, simplifying it significantly. This makes QED Feynman integrals more accessible for both direct parametric integration via computer algebra and more abstract algebro-geometric methods.

Predrag 2023-11-05 Check out -while teaching- Kim, Cho and Lee [68] *The art of Schwinger and Feynman parametrizations* (2023). They refer to my parametric paper [33] only indirectly, via ref. [5] (2019).

I read this one for my PhD thesis writeup of parametric Feynman integrals [33]: Nambu [88] *Parametric representations of general Green's functions* (1957).

2018-06-11 Predrag Bogner

Brödel

Jaclyn Bell is a charismatic young Liverpoolian (from working class background), theoretical particle physics PhD advised by a Northern Irish childhood friend and colleague of Georgia Tech's Brian Kennedy, who did a stint on the BBC astronauts show as potential astronaut, and now runs a UK STEM education organization, with outreach to children in UK's poorer naighborhoods. If I got it right....

2018-06-12 Predrag Wulkenhaar (Meintz) had a breakthrough assist from Erik Panzer during this workshop, which seems to establish that a non-commutative matrix ϕ^4 model is integrable. (My mother's ϕ^4 QFT model remains definitely non-integrable.) I have handwritten notes on his talk.

Alexander Hock *Noncommutative 3-colour scalar quantum feld theory model in 2D* is a closely related model with the same structure: " We introduce the 3-colour noncommutative quantum field theory model in two dimensions. For this model we prove a generalised Ward-Takahashi identity, which is special to coloured noncommutative QFT models. It reduces to the usual Ward- Takahashi identity in a particular case. The Ward-Takahashi identity is used to simplify the Schwinger-Dyson equations for the 2-point function and the N -point function. The absence of any renormalisation conditions in the large (N , V)-limit in 2D leads to a recursive integral equation for the 2-point function, which we solve perturbatively to sixth order in the coupling constant. One important fact is the appearance of polylogarithms in the perturbative solution, which is generated by the closed integral equation. With the knowlegde of the experts here in Les Houches I hope to improve my results to higher order or even find an exact solution for the 2-point function. "

Masha Vlasenko *Motivic Gamma Functions*, see "what is a motivic gamma function?". Not a chance that I would understand anything.

2018-06-13 Predrag Burgos-Gil is a lovely Spanish-speaking refugee from Barcelona living in Madrid.

Ralph Kaufmann *Feynman categories and applications: geometry, number theory and physics*

David Prinz *Einstein-Maxwell-Dirac theory* is the canonical generalization of spinor electrodynamics to curved spacetimes of general relativity. He explains the underlying geometry of the theory and its Lagrange density, with gauge fixing, ghost terms, Feynman rules and tree-level interactions. The obstructions to multiplicative renormalization are overcome by a generalization of Furry's theorem.

2018-06-13 Predrag Blum *et al.* [11] *Calculation of the hadronic vacuum polarization contribution to the muon anomalous magnetic moment*: A first-principles lattice QCD+QED calculation at physical pion mass of the leading-order hadronic vacuum polarization contribution to the muon anomalous magnetic moment. The total contribution of up, down, strange, and charm quarks including QED and strong isospin breaking effects is $a_\mu^{HVP\,LO} = 715.4(18.7) \times 10^{-10}$. By supplementing lattice data for very short and long distances with R-ratio data, we significantly improve the precision to $a_\mu^{HVP\,LO} = 692.5(2.7) \times 10^{-10}$. This is the currently most precise determination of $a_\mu^{HVP\,LO}$.

Borsanyi *et al.* [15] *Hadronic vacuum polarization contribution to the anomalous magnetic moments of leptons from first principles*: compute the leading, strong-interaction contribution to the anomalous magnetic moment of the electron, muon, and tau using lattice quantum chromodynamics (QCD) simulations.

Calculations include the effects of u, d, s, and c quarks and are performed directly at the physical values of the quark masses and in volumes of linear extent larger than 6 fm. All connected and disconnected Wick contractions are calculated. They obtain $a_e^{LO-HVP} = 189.3(2.6)(5.6) \times 10^{-14}$, $a_\mu^{LO-HVP} = 711.1(7.5)(17.4) \times 10^{-10}$, and $a_\tau^{LO-HVP} = (0.8)(3.2 \times 10^{-8}$, where the first error is statistical and the second is systematic.

"Today, a_e is one of the most precisely measured [56] and computed [2–5] quantities in nature, with a total uncertainty below 1 ppb. Theory and experiment agree and the measurement can be used to make the most precise determination [3] of the fine-structure constant α".

2018-06-17 Colin Parker The best measurement of α that I know is Parker *et al.* [90] *Measurement of the fine-structure constant as a test of the Standard Model*. It also provides a good list of the other methods. Atoms, and not fancy emergent condensed matter phenomena, are actually the most certain measurements. Besides QHE and $g-2$, there are two atomic methods. One of the atomic methods is fine structure spectroscopy of helium, which is competitive with QHE. This measures a spin-orbit interaction, which is compared with detailed atomic physics calculations, so in that sense it is similar to $g-2$, except it's not as mny Feynman diagrams and more standard perturbation theory for atoms (which is not to say that it necessarily converges either, I don't know). The other method is to come directly from the Rydberg constant and the electron rest mass, via the electron to cesium mass ratio, and an absolute measurement of the cesium atomic mass using momentum transfer with photons. That's what's reported in the linked paper and it's the best that I know of: 0.20 ppb. It's in a 2.5 sigma tension with $g-2$ for what it's worth, but not in a way that makes the muon $g-2$ measurements make a ton of sense either.

2018-06-17 Colin I guess it's not totally surprising to see a bunch of terms cancel, the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots \tag{3.1}$$

(or any conditionally convergent series) will have a large difference between the absolute sum and the signed sum if you take a lot of terms.

2018-06-18 Predrag Can you cook up an example more dramatic than (3.1)? The corresponding $g-2$ series has only the first 5 terms computed so far, with the sum of magnitudes of contributions to the 5th term some 10^5 times larger than the result itself (in contrast to the term 1/5 in (3.1)). Not only are these huge cancelations, but they are occurring separately within each gauge set.

I only *conjecture* that the $g-2 = a(\alpha/\pi)$ is a convergent series in α/π. As long as I cannot prove, my convergence claim is a speculation, not mathematics.

2014-07-18 Predrag Mariño [79] lectures on non-perturbative effects are perhaps of interest: "a review of non-perturbative instanton effects in quantum theories, with a focus on large N gauge theories and matrix models. I first consider the structure of these effects in the case of ordinary differential equations, which provide a model for more complicated theories, and I introduce in a pedagogical way some technology from resurgent analysis, like trans-series and the resurgent version of the Stokes phenomenon. After reviewing instanton effects in quantum mechanics and quantum field theory, I address general aspects of large N instantons, and then present a detailed review of non-perturbative effects in matrix models."

2019-06-01 Predrag I find Córdova, Heidenreich, Popolitov and Shakirov [27] *Orbifolds and exact solutions of strongly-coupled matrix models* very surprising. The introduction is worth reading. They compute analytically the matrix model (QFT in zero dimensions) partition function for trace potential

$$S[X] = \mathrm{tr}\,(X^r)\,, \qquad \text{integer } r \leq 2\,. \tag{3.2}$$

Their "non-perturbative ambiguity" in the case of the $N = 1$ cubic matrix model seem to amount to the Stokes phenomenon, i.e., choice of integration contour for the Airy function.

Unlike the weak coupling expansions, the strong coupling expansion of

$$Z = \frac{1}{2\pi} \int dx e^{-\frac{1}{2g^2}x^2 - x^4}\,, \tag{3.3}$$

is convergent, not an asymptotic series.

There is a negative dimensions type duality $N \to -N$, their eq. (3.27). The loop equations, their eq. (2.10), are also interesting - they seem to essentially be the Dyson-Schwinger equations and Ward identities in my book's [31] formulation of QFT.

2019-10-05 Predrag Kreimer's masters student Harlan [57] *Moduli Spaces of Gluon Graphs* uses my *Field Theory* [31] to show there are no 5-vertices in gauge theories.

2020-06-04 Predrag Read Lunev [76] *Pure bosonic world-line path integral representation for fermionic determinants, non-Abelian Stokes theorem, and quasiclassical approximation in QCD*: " Simple bosonic path integral representation for path ordered exponent is derived. This representation is used, at first, to obtain new variant of non-Abelian Stokes theorem. Then new pure bosonic world-line path integral representations for fermionic determinant and Green functions are presented. Finally, applying stationary phase method, we get quasiclassical equations of motion in QCD. "

2020-09-19 Predrag Lifson, Reuschle and Sjödahl [74] *Introducing the chirality-flow formalism*: " At the algebra level, the Lorentz group consists of two copies of the (complexified) $\mathrm{SU}(2)$ algebra [...] we introduce the chirality-flow formalism for massless tree-level QED and QCD, [with] scattering amplitudes written down in terms of Lorentz-invariant spinor inner products, similar to how the color structure can be described in terms of a color flow.

[...] a flow picture, similar to the color-flow picture in QCD. Recalling the QCD Fierz identity, we analogously write the spinor Fierz identity in a flow form. [...] the photon exchange [is] recast into a flow-like picture [resulting in] massless tree-level QED Feynman rules. [...] chirality-flow formalism gives a transparent and intuitive way of understanding the Lorentz inner products appearing in scattering amplitudes, similar to how color structure can be thought about in terms of a color flow. [...] a simplification of the spinor-helicity method, where Dirac spinors are replaced by their left- and right-chiral components, and Dirac matrices by Pauli matrices. [...] we remove the Pauli matrices as well, and instead express all internal structure with Kronecker delta functions. ”

For details, see: Lifson, Reuschle and Sjödahl *The chirality-flow formalism* arXiv:2003.05877: “ We take a fresh look at Feynman diagrams in the spinor-helicity formalism. Focusing on tree-level massless QED and QCD, we develop a new and conceptually simple graphical method for their calculation. In this pictorial method, which we dub the chirality-flow formalism, Feynman diagrams are directly represented in terms of chirality-flow lines corresponding to spinor inner products, without the need to resort to intermediate algebraic manipulations. ”

2021-02-15 Predrag I find Kieran Finn’s *Quantizing the Eisenhart Lift*:

“The classical Eisenhart lift is a method by which the dynamics of a classical system subject to a potential can be recreated by means of a free system evolving in a higher-dimensional curved manifold, known as the lifted manifold”

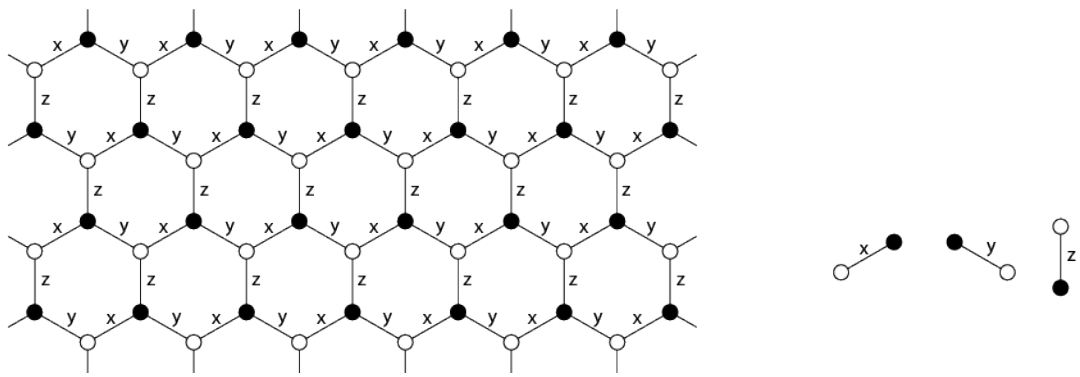
too good to be true. Maybe it’s a good start to semiclassics? Worth a more serious look... The video should be here.

2023-01-04 Predrag Have a look at Zee [109] *Quantum Field Theory, as Simply as Possible* (2022).

2023-05-08 Predrag Berglund and Klose [10] *Perturbation theory for the ϕ^4_3 measure, revisited with Hopf algebras* (2022).

Scanning through the paper it bit like scanning through my CV, so many overlaps. Though I’ve never work on any field theory in 3 dimensions. Currently our favorite ϕ^4 is any dimension, in practice in 1 and 2 dimension, and we prefer the anti-harmonic, ‘inverted pendula’ form of the action CL18 eq. (53) in the current draft of our ‘deterministic field theory’ paper we work at Klein-Gordon mass-square values where perturbation theory makes no sense (ϕ^4 dominates over ϕ^2). Here the solutions of Euler-Lagrange equations (classical periodic and relative periodic orbits) form a fractal set. Do not know what that means - any suggestion? - but it is convenient for us, as the symbolic dynamics is complete - every symbol sequence.

A bit of propaganda is here: (2000) paper Sect. 6. Smooth conjugacies, and the calculations are in the (1999) paper Sect. 5. Smooth conjugacies, on what I would today call a temporal 1-dimensional lattice.



(a) (b)

Figure 3.1: Three types of links in the honeycomb lattice.

2023-10-20 Predrag 2 Bader I do not know where to put this thread. Very broadly, I have 3 sets of blogs: on 'chaos and turbulence', on 'quantum field theory', and on 'group theory'. This topic seems to be the intersection of all three:).

For time being, I'll put blog about this in
GitHub.com/cvitanov/reducesymm/tree/master/QFT/blog.tex,
the end of Chapter 8 *Daily QFT blog*.

To get your own working copy,
git https://GitHub.com/cvitanov/reducesymm,

cd QFT; pdflatex blog; biber blog

If you want to add your own working notes as you work though this paper and the related theory, let me know and I'll add you to the GitHub repo collaborators.

2023-10-20 Predrag Kitaev reading club with Bader H. Aldossari
<baldossari3@gatech.edu>. Meetings Zoom 973 4569 4798 'Tigers rule'.

Alexei Kitaev [69]
Anyons in an exactly solved model and beyond, arXiv:cond-mat/0506438.

So far we have discussed 3 broad topics: crystalographic **lattices**; **group theory**; **perturbation theory**.

2023-10-21 Predrag On **lattices**:

Being lazy, everything I do is on the simplest of lattices, the square (or hyper-cubic) lattice. Currently my favorite exposition is Han's and my CL18 current draft of our 'deterministic field theory' paper. I would love to discuss it with you.

But if this exposition is too condensed, try ChaosBook Appendix A24 *Deterministic diffusion*. Things to understand are:

dicretized fields on lattices (I use only bosons; you have a pair of complex fields, i.e., a 'spinor' per lattice site, a 2-dimensional irrep of internal, lattice site field su(2) symmetry)

lattice derivatives, lattice Laplacians,

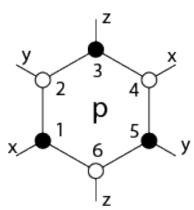
Periodic lattices, reciprocal lattices, see CL18 sect. 4. Bravais lattices.

eigenmodes of translation generators (shift operators) used to diagonalise (by discrete Fourier transforms) non-local operators, such as Laplacians, and invert them;

Green's functions CL18 eq. (61) , eq. (174) and CL18 sect. 8.3. *Green's function of two-dimensional spatiotemporal cat*;

Lattice propagators CL18 sect. 3.2. *Bravais lattice stability*. My favorite derivation of the propagator QMlectures eq. (1.21) is QMlectures sect. 1.1 *Wanderings of a drunken snail*.

2023-10-21 Predrag Consider a **honeycomb lattice** of figure 3.1 (a). Under the translation group, its primitive cell can be taken as a hexagon tile (lattice plaquette),



,

with dangling links labeled as in figure 3.1 (b).

If you take as primitive vectors the two unit lattice translations along 60^o and 120^o directions, then their sum translate the lattice by one tile vertically, and their difference generates one-tile horizontal translation.

That might not be in your taste, as these primitive vectors are orthogonal to primitive cell links. So view the lattice as consisting of two equivalent simple sublattices, "even" and "odd" (empty, full circles in the figure).

Shifts along "x-links", "y-links", and "z-links" directions, figure 3.1 (b), independent translations along x and y directions, z translation given by the sum -x-link - y-translation, translate the 2 triangular sublattices into each other. The lattice is invariant under pairs of successive different shifts. For example, successive shifts, first along x-links", than along "y-links translate the lattice by one lattice unit in the z direction

The point group of either sublattice is D_3, 3 rotations, and 3 reflections. A sensible theory should also be invariant under interchanging the two sublattices. Under all these discrete symmetries, the honeycomb is built from Kitaev's 'unit cell' that contains one vertex of each kind.

The Hamiltonian

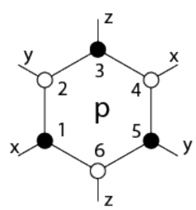
$$H = -J_x \sum_{x\text{-links}} \sigma_j^x \sigma_k^x - J_y \sum_{y\text{-links}} \sigma_j^y \sigma_k^y - J_z \sum_{z\text{-links}} \sigma_j^z \sigma_k^z , \tag{3.4}$$

where J_x, J_y, J_z are parameters, is a sparse, almost diagonal (almost = the nearest neighbors) operator (an infinite matrix acting on the infinite honeycomb lattice) of a block diagonal form: it acts on lattice site 2-component 1/2-spin field with a 2-dimensional local matrix (Pauli matrices, the identity). Were $J_x = J_y = J_z$, one could write it as a Laplacian (see CL18 eq. (2.49)), with zero Klein-Gordon mass 'potential'. Bader, can you do this exercise? I do not remember doing it.

The individual operators in the Hamiltonian:

$$K_{jk} = \begin{cases} \sigma_j^x \sigma_k^x, & \text{if } (j,k) \text{ is an } x\text{-link;} \\ \sigma_j^x \sigma_k^y, & \text{if } (j,k) \text{ is an } y\text{-link;} \\ \sigma_j^x \sigma_k^z, & \text{if } (j,k) \text{ is an } z\text{-link.} \end{cases} \tag{3.5}$$

Remarkably, *all* operators K_{jk} commute with lattice plaquette (i.e., hexagon) operators W_p:



$$W_p = \sigma_1^x \sigma_2^y \sigma_3^z \sigma_4^x \sigma_5^y \sigma_6^z = K_{12} K_{23} K_{34} K_{45} K_{56} K_{61}. \tag{3.6}$$

It's probably not remarkable. As explained above, any non-repeating pair of shifts along x-, y-, z-links is a unit lattice translation. The three consecutive such in the chain (3.6) add up to a zero translation, so anything will commmute with it, including operators K_{jk}, and different operators $W_{p'}$.

2023-10-17 Predrag On **group theory**:

Kitaev06 eq. (20) follows from the definition of a tensor, birdtracks.eu eq. (3.26): a finite 'rotation' of an n-legged tensor rotates each leg (a 'leg' is a free, not contracted index).

$e^{-iT\cdot\theta} M e^{iT\cdot\theta}$ is just how a Hermitian matrix M (2-index tensor, one index up, one down) transforms, Heisenberg version of the Schrödinger equation being a special case. As time evolution is a 1-dimensional abelian group, you don't even see an adjoint index. Basically, Heisenberg is a statement of finite time evolution, Schrödinger is the infinitesimal time step = differential equation version.

I find the infinitesimal statement of tensor invariance, birdtracks.eu eq. (4.36) and (4.33), easier to grasp.

If M is built from N generators in the N-dimensional adjoint representations 2 indices shared with M, but the 3rd index in the adjoint rep (like 3 Pauli matrices, the generators of su(2)), then various commutators give structure constants c_{ijk}. They are generators of adjoint representation rotations, hence the rotation matrix on the right hand side of Kitaev06 eq. (20).

Examples:
eq. (10.10), eq. (11.4) and thereabouts, eq. (12.7) and thereabouts, but much of it is special to so(4) being a semisimple su(2)×su(2), misleading about the general case. My notes are *not* a textbook on group theory, only pointers to the literature. To learn it my way, you need to go through videos listed at the beginning of each chapter, read the literature if suggested.

2023-10-21 Predrag I do spin representations in birdtracks.eu sect. 11.1, but that is probably too tough (for anyone:). If you can get through spinsters, birdtracks.eu sect. 14.1, my hat off to you:) My birdtracks are good if you have to do real calculations (like perturbative corrections). On a superficial level, standard literature is good enough.

2023-10-21 Predrag On **perturbation expansions**:

Currently my favorite bosonic field theory is scalar ϕ^4 any dimension, in practice in 1 and 2 dimensions.

We work at Klein-Gordon mass-square values where perturbation theory in powers of the coupling strength of the nonlinear potential makes no sense. Our potential term ϕ^4 dominates over the 'kinetic', Laplacian term. This is known as the anti-integrable limit CL18 eq. (73) in the current draft of our 'deterministic field theory' paper.

Kitaev06 Hamiltonian (3.4) only has a 'kinetic', Laplacian term. It has no on-site potential or spinor mass - it's a massless theory.

So Kitaev06, in 5.1 *Perturbation theory study* , breaks the honeycomb D_6 symmetry by singling out the vertical z-links in figure 3.1 (b), and combines the x- and y-links into generators of horizontal translations, his 'potential' V, resulting in a $D_2 \times D_2$ 'non-relativisitc' rectangular latice; lattice 'constants' or 'coupling strengths' or 'speed of light' are different in the horizontal and vertical directions:

Kitaev06 Hamiltonian is $H = H_0 + V$, where H_0 is the main part and V is the perturbation:

$$H_0 = -J_z \sum_{z\text{-links}} \sigma_j^z \sigma_k^z, \qquad V = -J_x \sum_{x\text{-links}} \sigma_j^x \sigma_k^x - J_y \sum_{y\text{-links}} \sigma_j^y \sigma_k^y$$

The important thing is that H_0 generates vertical translations, V horizontal translation, so they commute. The propagator can be expanded as a geometrical series in terms of the small parameter $J_x = J_y$,

$$\frac{1}{H_0 + V} = \frac{1}{H_0} \sum_{n=0}^{\infty} \left(H_0^{-1} V\right)^n , \tag{3.7}$$

and computes the first four terms. It's a calculation, as there are various powers of Pauli matrices in various terms. Odd terms vanish, because you need even powers of V to get to lattice translations.

As $\mathrm{D}_2 \times \mathrm{D}_2$ includes symmetries under reflections across x and z axes, I believe that one could have written the theory in a Dirac propagator form, linear in 'momentum', from the start, but working that out might merit a stand-alone publication.

Kitaev06 first sets to the anti-integrable limit $J_z \to \infty$, or $J_x = J_y = 0$. In that case this is a 1-dimensional latice, with D_2 symmetry. Its ground state is highly degenerate: each two spins connected by a z-link are aligned (spinup spinup or spindown spindown), but their common direction is not fixed. The ground state energy is $E_0 = -NJ_z$, where N is the number of unit cells, i.e., half the number of spins.

I stop here - do not know whether this is helpful to Bader, until he finds it helpful.

Come back when you get to the next hurdle.

2024-10-03 Bader never answered :) He now works with Zhu-Xi Luo.

2024-10-03 Predrag One hour of learning about Zhu-Xi Luo's work (her papers on arXiv).

Goal: density-density correlations on 2-d lattices

Her current research is inspired by her first publication, Zhu-Xi Luo, Yu-Ting Hu and Yong-Shi Wu [77] *On quantum entanglement in topological phases on a torus* (2016); arXiv:1603.01777. For me it is a hard paper, it would take much work to unpack.

They concentrate on the string-net ground state wave functions. The string-net model is described in Appendix A.

a unitary fusion category

Zhu-Xi started by explaining exact $Z^2 \times Z^2$ models.

string-net

in $\beta \to \infty$ zero temperature limit

Jacob Bourjaily 2023-10-23 (Penn State U.).

Adventures in Perturbation Theory (download video): "What are the mathematical/functional forms of predictions made using perturbative QFT? Beyond one loop, very little about this question has been known until quite recently, in part due to the appearance of non-polylogarithmic functions (often, if not always, involving Calabi-Yau geometries). I review the role and importance of such geometric structures, implications for the use of differential equations to "perform" loop integration, and describe how unitarity can be used construct (dramatically) improved bases of so-called 'master integrals' in diverse applications."

Nima Arkani-Hamed 2023-09-22 Rutgers seminar

Scattering amplitudes as a counting problem and the combinatorial geometry of fundamental physics,

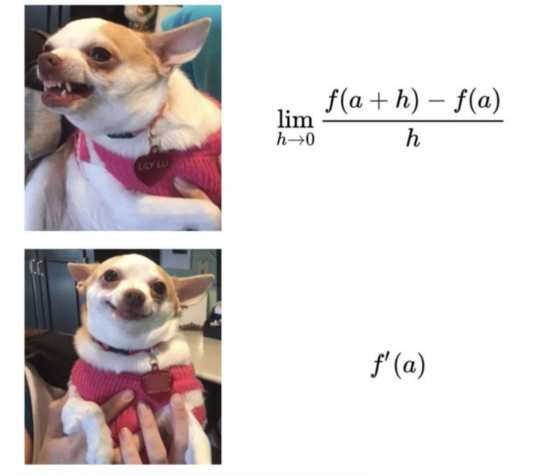


Figure 3.2: Are Cvitanović's discrete derivatives causing you discomfort?. Then consult your doctor as whether Costello's continuous spacetime QFT lectures might be just the right pill for you.

> " [...] a new understanding of scattering amplitudes based on combinatorial ideas in the kinematic space of the scattering data. [...] the simplest theory of colored scalar particles with cubic interactions, at all loop orders and to all orders in the 't Hooft expansion. [...] formula for loop-integrated amplitudes, with no sum over Feynman diagrams [...] determined by a counting problem to any order of the topological expansion. [...] into the physics of pions and Yang-Mills theory. "

See also: Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov and Trnka *Scattering Amplitudes and the Positive Grassmannian*, arXiv:1212.5605, and (probably the same?)

Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov and Trnka [6] *Grassmannian Geometry of Scattering Amplitudes* (2016).

Trnka lectures (2018).

Badger, Henn, Plefka and Zoia [7] *Scattering Amplitudes in Quantum Field Theory* (2024), (click here).

2024-06-17 Henriette Elvang Amplitudes 2024 Summer School lectures:

Scattering Amplitudes and Effective Field Theory

Study also *Amplitudes 2024 Conference* *videos*

2024-02-13 Predrag Motivating and deriving statistical and quantum mechanics in the disretized spacetime setting might appeal to a condensed or soft-matter physicist, but it's not for everyone: atomic, optical, nuclear or general relativity folk think of spacetime only as continuum, and of derivatives only in the Leibniz calculus lattice spacing $h \to 0$ limit. So maybe Costello lectures would be more in your taste.

Kevin Costello, Krembil William Rowan Hamilton Chair, strikes me a bit like a doppelganger of my younger self (minus the Chair).

Lecture 1

of his 2018 *Renormalization and Effective Field Theory* course is in the style of my ChaosBook.org/FieldTheorypdf.html, but I like my book better.

In our 'kinetic' vs 'potential' perspective, he starts out in the limit were the pure Laplacian dominates - fields very so fast as to overpower the potential, parametrized by μ^2, including the 2-point mass ϕ^2 term, her treated as a small, perturbative correction.

When Costello says "propagator", he only takes the inverse of the Laplacian, not a massive boson propagator with $\mu^2 > 0$ that is my preference. I see why - the inverse of the Laplacian in d dimension is simple

$$P(y, y') = \frac{1}{|y - y'|^{d-2}} \tag{3.8}$$

with $d = 2$ case equal $\ln |y - y'|$ (shouldn't the exponent be $(d - 2)/2$)? So not the massive Boson propagator which is a Bessel function.

As his focus will be effective field theory where high-energy "ultraviolet" divergences due to the singularity of an inverse Laplacian are renormalized away, separating this as the 'kinetic' term is right. In our semi-classical, deterministic field theory we are in the opposite, "anti-integrable' limit, where the μ^2 parameter potential dominates.

He does, however, introduce the massive propagator in

Lecture 2

by the heat kernel method (i.e., the continuum version of my random walks ending in death), so our approaches are very similar.

What we call Euclidean space he refers to as the 'Riemannian signature', as opposed to the physical, Lorentzian signature.

The things go off the rails starting with Lecture 3. In Lecture 5, (massless!?) scalar ϕ^4 over $\mathbb{R}^4$ enters - he'll presumably prove it exists. I'm out of here.

I feel he might be a kindred soul because he starts his Costello [28] *Renormalization and Effective Field Theory* (2011) monograph as this:

Most axiomatic formulations of quantum field theory in the literature start from the Hamiltonian formulation of field theory. [...] I believe that the Lagrangian formulation of quantum field theory, using Feynman's sum over histories, is more fundamental. The axiomatic framework developed in this book is based on the Lagrangian formalism, and on the ideas of low-energy effective field theory developed by Kadanoff (Kad66), Wilson (Wil71), Polchinski (Pol84) and others. [...] The only assumptions I am making about the nature of quantum field theory are the following:

1. The action principle: physics at every energy scale is described by a Lagrangian, according to Feynman's sum-over-histories philosophy.
2. Locality: in the limit as energy scales go to infinity, interactions between fields occur at points.

[...] The coefficients of the action of this local renormalization group flow on any particular theory are the functions of that theory. I include explicit calculations of the function of some simple theories, including the ϕ^4 theory on $\mathbb{R}^4$. [...] I then classify all possible renormalizable scalar field theories, and find the expected answer. For example, the only renormalizable scalar field theory in four dimensions, invariant under isometries and under the transformation $\phi^4 \to -\phi^4$, is the ϕ^4 theory. [...] Thus, in the approach to quantum field theory presented here, to prove renormalizability of a particular theory, one simply has to calculate the appropriate cohomology groups. No manipulation of Feynman graphs is required.

I should emphasize that time evolution does not play a role in this picture: quantum field theory on d-dimensional space-time is related to statistical field theory on d-dimensional space. We must assume, however, that the statistical system is in equilibrium. [...] The only difference between this picture and the quantum field theory formulation is that we have replaced $\hbar$ by T.

If we consider the limiting case, when the temperature T in our statistical system is zero, then the system is "frozen" in some extremum of the action functional $S()$. In the dictionary between quantum field theory and statistical mechanics, the zero temperature limit corresponds to classical field theory. In classical field theory, the system is frozen at a solution to the classical equations of motion. Throughout this book, I will work perturbatively. In the vocabulary of statistical field theory, this means that we will take the temperature parameter T to be infinitesimally small, and treat everything as a formal power series in T. Since T is very small, the system will be given by a small excitation of an extremum of the action functional. In the language of quantum field theory, working perturbatively means we treat $\hbar$ as a formal parameter. This means we are considering small quantum fluctuations of a given solution to the classical equations of motions.

[...] There are many ways to define "low energy". I will start by giving a definition which is conceptually very simple, but difficult to work with. In this definition, the low energy fields are those functions on our manifold M which are sums of low-energy eigenvectors of the Laplacian. [...] I will use a definition of effective field theory based on length rather than energy. The great advantage of this definition is that it relates better to the concept of locality.

(**2024-02-14 Predrag** at this time, I do not understand his *renormalization group equation* on p. 7).

2024-09-07 Predrag de Doncker, Ishikawa, Kato and Yuasa [39] *Analytic and numerical approaches for depictive 3-loop integrals using sector decomposition*, (2024).

The sector decomposition (SD) method [60], arXiv:0803.417, is well-established for calculating loop integrals. In this paper, we use a simple form of the SD method. [...] From the basic formula derived by this method, it is possible to obtain a specific expression of the coefficients that represent the ultraviolet (UV) divergence analytically. [...] We deal with the UV singularity by dimensional regularization.

The paper is based on the Nakanishi–Cvitanović–Kinoshita formalism [33, 34, 87]. After many clever parametric integral variable transformations they split the integrals in Laurent series in $1/\epsilon$, and arrive at

"Equation (18) is our basic formula to calculate the loop integral analytically and numerically."

with analytical formulas for prefactors as functions of m^2, $\ln m^2$, and nasty polylogs. They encounter thresholds at $s = 4\,m^2$, etc.., $s =$(external momentum squared) that I am not familiar with. Their results are in good agreement with Laporta [73].

[...] deep gratitude to Dr. T. Kaneko for showing us how to handle the SD method.

2024-09-07 Predrag A helpfull overview:

Harlander and Martine [58] *The development of computational methods for Feynman diagrams*, (2024)

2024-09-07 Predrag Martin [82] *Evaluation of three-loop self-energy master integrals with four or five propagators* (2023) is probably not useful to whatever is it that I'm seaarching for.

I obtain identities satisfied by the three-loop self-energy master integrals with four or five propagators with generic masses, including the derivatives with respect to each of the squared masses and the external momentum invariant. These identities are then recast in terms of the corresponding renormalized master integrals, enabling numerical evaluation of them by the differential equations approach.

By applying IBP relations repeatedly, one can discover identities between different loop integrals with common topological features, allowing one to eliminate many of them in favor of a finite [101] number of master integrals.

[...] the differential equations satisfied by the three-loop self- energy master integrals with four and five propagators are here found explicitly, enabling their numerical computation.

[...] [...] [...] [...] [...]

2018-06-16 Predrag Remember to send Henry Kißler a hard copy of my "Field Theory" book [31].

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