

group theory  
birdtracks, Lie's, and exceptional groups  
**a blog**

Predrag Cvitanović

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## 1.7 Quantum marginal problem

The **marginal distribution** of a subset of a collection of random variables is the probability distribution of the variables contained in the subset. It gives the probabilities of various values of the variables in the subset without reference to the values of the other variables.

*Marginal variables* are those variables in the subset of variables being retained. These concepts are "marginal" because they can be found by summing values in a table along rows or columns, and writing the sum in the margins of the table.

*Quantum marginal problem (QMP)* : Find a pure quantum state  $|\psi\rangle$  given some marginals  $\rho_\ell$ .

**2023-01-02 Erik Aurell** The quantum marginal problem: is there a globally pure state which has a set of reduced density matrices (in Fraser's notation): Let there be

- (a) a joint Hilbert space  $J = X_1 \otimes X_2 \otimes \dots \otimes X_k$ ,
- (b) a family of index sets  $S_1, S_2, \dots, S_m$  with cardinalities  $k_1, k_2, \dots, k_m$  labelling Hilbert spaces  $X_{S_i} = X_{i,j_1} \otimes \dots \otimes X_{i,j_{k_1}}$ , and
- (c) for each Hilbert space  $X_{S_i}$  a density matrix  $\rho_i$ .

Question: does there exist a pure state  $|\psi\rangle$  on  $J$  such that the partial trace of  $|\psi\rangle\langle\psi|$  over the Hilbert space complementary to  $X_{S_i}$  equals  $\rho_i$ ? That is, in standard notation, is there a  $|\psi\rangle$  such that  $\text{Tr}_{J \setminus S_i}[|\psi\rangle\langle\psi|] = \rho_i$  for each  $i$ ?

I have a paper with Pawel Horodecki [11] where we use some results in this field.


This problem was solved about 10 years ago in the case when the marginals are disjoint (those are the results we used with Horodecki) by **Alexander Klyachko** [72, 73] *Quantum marginal problem and representations of the symmetric group*, [arXiv:quant-ph/0409113](https://arxiv.org/abs/quant-ph/0409113); *Quantum marginal problem and N-representability*, [arXiv:quant-ph/0511102](https://arxiv.org/abs/quant-ph/0511102). In this case only the eigenvalues of marginals  $\rho_\ell$  matter. In principle. Relies on algebraic geometry, not straightforward to apply. One defines sets of marginals of compatible states, then checks whether they contain any pure states. Testing for purity is a nonlinear problem. Trick: take a convex hull of copies of compatible states. Pure states have traces equal to 1; this reduces the problem to study of *symmetric* products of the copies / separable states.

Aside: **Klyachko** was born in 1946, in Vladivostok.

Disjoint marginals mean questions like if there exists a pure state over  $N$  particles with given 1-particle reduced density matrices. There is no general solution for overlapping marginals. Overlapping marginals means questions like if there exists a pure state over  $N$  particles with a set of given 2-particle reduced density matrices, where the connectivity graph of particles linked by these given 2-particle reduced density matrices is large.

Thomas C. Fraser [49] *A sufficient family of necessary inequalities for the compatibility of quantum marginals*, [arXiv:2211.00685](https://arxiv.org/abs/2211.00685), gives a general (but ab-

stract) solution to QMP. Here is Thomas' (a graduate student) short video on this work:

 Thomas Fraser : *Simultaneous Spectral Estimation and the Quantum Marginal Problem* (2021)


I have yet to decide if the solution in this paper is practical in any way. It is however clear that it uses properties of symmetric tensors a lot. There is an appendix explaining Predrag's birdtrack notation, with calculations (which I did not try to do yet). Predrag's book is cited.

Maybe something new that we could think of together?

**2023-01-02 Predrag** Fraser [49] *Appendix E: Diagrammatics* is taken directly from my *Group Theory* [37]. So is most of 'tensor networks' literature, which never cites me but cites Penrose instead, so Fraser is being very kind. The new and key operation, not in my book, is eq. (E15) for operations on 'multi-partite Hilbert spaces', i.e., multiple copies of a tensor.

*Appendix F: The  $n = 1$  Case* I can follow step-by-step, but do not understand the significance of the results.

**2023-01-03 Erik Aurell** .

 Otfried Gühne : *The Quantum Marginal Problem* (2022)

is up to 32 min at least partly about other (but related) problems. I have read Huber, Gühne and Siewert [59] *Absolutely maximally entangled states of seven qubits do not exist* (2017)

Huber, Eltschka, Siewert and Gühne [58] *Bounds on absolutely maximally entangled states from shadow inequalities, and the quantum MacWilliams identity*, (2018)

Yu, Simnacher, Wyderka, Nguyen and Gühne [127] *A complete hierarchy for the pure state marginal problem in quantum mechanics* (2021)

that he is referring to (though not in detail). I know other work by his postdoc Nguyen, who coauthored a very good classical stat mech review;

Nguyen, Zecchina and Berg [89] *Inverse statistical problems: From the inverse Ising problem to data science* (2017), [arXiv:1702.01522](https://arxiv.org/abs/1702.01522).

At 32 min Gühne starts on the problem as I see it. The (unpublished) Klyachko [73] paper he refers to is a classic in the field, though almost incomprehensible (at least to me). It concerns the problem where the marginals are non-overlapping. In that setting there is a much easier later paper for Gaussian states:

Eisert, Tyc, Rudolph and Sanders [43] *Gaussian quantum marginal problem* (2008), [arXiv:quant-ph/0703225](https://arxiv.org/abs/quant-ph/0703225).

At around 43minutes or so comes the construction of Gühne which is in the same ballpark as Fraser's. It considers more and more copies of the system, and then conditions on those N copies of the system such that the quantum marginal

problem is solvable by a technique called semi-definite programming, standard in the field. I never bothered to learn exactly what it is because (1) there are already many others who know it very well and use it and (2) I am not convinced it is in the end so practical to consider these problems of  $N$  copies of the original problem, where  $N$  eventually tends to infinity. I may of course be wrong on this point (2). After 44 minutes I think Gühne again looks at a special case, maybe not generally useful.

The last part of the talk is I think a bit inconclusive (or not quite up-to-date). The case "AME(4,6)" was solved in Suhail Ahmad Rather, Adam Burchardt, Wojciech Bruzda, Grzegorz Rajchel-Mieldzioć, Arul Lakshminarayan, Karol Życzkowski [100] *Thirty-six entangled officers of Euler: Quantum solution to a classically impossible problem* (2022).

I am not sure if there is not some almost-cheating behind this. Życzkowski was one of the co-authors of the list of open problems Gühne refers to. Maybe Życzkowski knew that he could solve that problem when he put together the list of open problems together with Pawel Horodecki. I do not know.

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