QFT and its discontents
a blog

Predrag Cvitanović

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Chapter 1

Is QED finite?

> Is there any method of computing the anomalous moment of the electron which, on first approximation, gives a fair approximation to the α term and a crude one to α^2; and when improved, increases the accuracy of the α^2 term, yielding a rough estimate to α^3 and beyond?
>
> — Feynman's challenge, 12th Solvay Conference [36]

2017-05-21 Predrag to

Sergey A. Volkov <volkoff_sergey@mail.ru>
Skobeltsyn Institute of Nuclear Physics,
Moscow State University, Moscow, 119234 Russia

Dear Sergey

I have just read your *New method of computing the contributions of graphs without lepton loops to the electron anomalous magnetic moment in QED* [78] (and the earlier ref. [79]) with great interest, and I can see that you are set to compute the 5-loop correction to the electron $(g-2)$. This is a very hard calculation, and approaching it strategically is a necessity.

May I suggests that you order your calculation by gauge sets of figure 1.5, as illustrated (for the 4-loop case) in Stefano Laporta's figure 1.6? My hunch is that the gauge set (6) = $(4,0,0)$ would be the most interesting, though Stefano thinks it too hard, and suggests starting with a 5-loop relative $(1,3,1)$ (or $(1,2,2)$?) of the set (3) = $(1,2,1)$ instead. While the contributions of individual vertex graphs (and self-energy sets [7]) are all over the place, all gauge-invariant sets are insanely small up to order 8, and it would be very sweet to see that this continues through order 10 (at least for the 5-loop graphs with no electron loops).

By the way, to check my conjecture one needs the gauge sets only to two significant digits or so, no high accuracy is needed.

best regards
Predrag

Order $2n$	Vertex graphs Γ_{2n}	Gauge sets G_{2n}	Anomaly $a^{(2n)}$
2	1	1	1/2
4	6	2	0
6	50	4	1
8	518	6	0
10	6354	9	3/2
12	89 782	12	0
14	1 429 480	16	2

Table 1.1: Comparison of the number of vertex diagrams without fermion loops, gauge sets, and the gauge-set approximation (1.3) for the magnetic moment in $2n$th order. From ref. [21].

As a motivation for these notes, the reader is referred to Dunne and Schubert 2005 paper [30] which starts with a nice historical review of ideas about the the perturbation series in QED. They note: "Despite of the many insights which have been gained along these lines, a point which remains poorly understood is the influence of gauge cancellations on the divergence structure of a gauge theory." In this spirit, we start here by explaining the numerics that motivates the QED finiteness conjecture, and then review the current approaches that offer a promise of establishing it. It is perhaps time to prove the conjecture, and use it to develop a different computational method for computing QFT amplitudes.

1.1 Gauge sets

In 1972 Toichiro Kinoshita and I had completed computing a large number of 3-loop anomalous magnetic moment Feynman diagrams and regularization counterterms [44], figure 1.1. The subsequent 4- and 5-loop numerical and analytic calculations are nothing short of heroic [7, 45, 53]. We had used the quantum field theory in the standard way, by expanding the magnetic moment into combinatorially many Feynman diagram (see the numbers of vertex graphs in table 1.1). Each Feynman diagrams corresponds to an integral in many dimensions with integrand with thousands of terms, each integral UV divergent, IR divergent, and unphysical, as its value depends on the definition of counterterms and the choice of gauge. Each integral, with its dreadfully oscillatory integrand, is evaluated by Monte Carlo methods in 10-20 dimensions with no hint of what the answer should be; in our case ± 10 to ± 100 was a typical range.

We added up hundreds of such contributions, of wildly fluctuating values, and obtained (for the no-fermion loops subset V, in the notation of ref. [7])

$$A_1^{(6)}[V] \;=\; +(0.92 \pm 0.02)\left(\frac{\alpha}{\pi}\right)^3 .$$

But why "+" and not "-"? Why so small? Why does a sum of hundreds of diagrams and counterterms yield a number of order of unity, and not 10 or 100 or any other number?

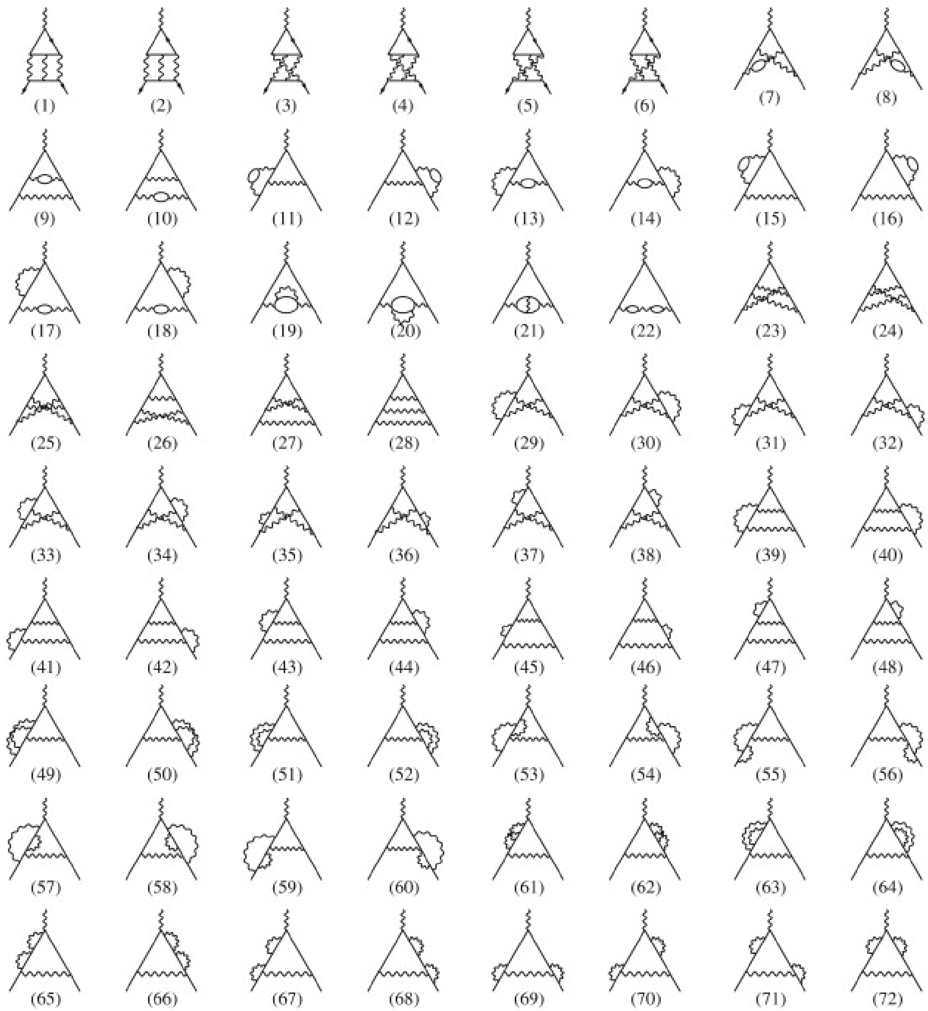


Figure 1.1: The three-loop vertex diagrams contributing to $A_1^{(6)}$ magnetic moment (from Jegerlehner and Nyffeler [43]). Lautrup *et al.* [55] were the first to note that subsets $(3,0,0) = \{23,24,25,26,28\}$; $(2,1,0) = \{29,31,33,35,37,39,41,43,45,47\}$ and its time-reversal $(2,0,1) = \{30,32,34,36,38,40,42,44,46,48\}$; $(1,2,0) = \{49,51,53,55,57,59,61,63,65,67\}$ and its time-reversal $(1,0,2) = \{50,52,54,56,58,60,62,64,66,68\}$; and $(1,1,1) = \{69,70,71,72\}$ are the minimal gauge sets, see figure 1.3.

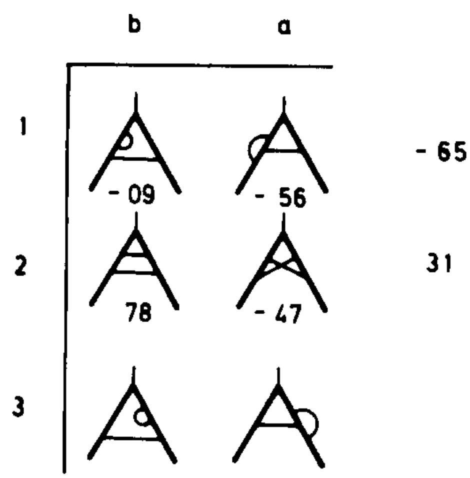


Figure 1.2: Rows: the fourth-order gauge sets (k, m, m'): (1) = $(1, 1, 0)$, (2) = $(2, 0, 0)$ and (3) = $(1, 0, 1)$. Columns: the two self-energy sets. For diagrams related by time reversal (here (1) and (3)) the value listed under the first diagram of the pair is the total contribution of the pair. Contributions seem to be of order $\pm\frac{1}{3}\left(\frac{\alpha}{\pi}\right)^2$, and suggest that a set and its time-reversed partner should be counted separately. From ref. [21].

If gauge invariance of QED guarantees that all UV and on-mass shell IR divergences cancel, could it be that it also enforces cancellations among the finite on-mass shell contributions?

As first noted by Lautrup, Peterman and de Rafael [55], the renormalized on-mass shell QED vertex diagrams separate into a sum of minimal gauge-invariant subsets, each subset separately UV and IR finite. To simplify matters, in what follows we shall consider only the no-fermion loop diagrams, or 'quenched-', or 'q-type' diagrams ('quenched', as this corresponds to the $O(N_f)$ part of the amplitude in QED with N_f flavors). The minimal gauge-invariant subsets without electron loops (for example, figure 1.1 diagrams $\{23-72\}$; figure 1.2; 1.3; 1.4; and 1.6) will be hereafter be referred to as *gauge sets*.

A gauge set (k, m, m') consists of all 1-particle irreducible vertex diagrams without electron loops, with k photons crossing the external vertex (cross-photons) and $m[m']$ photons originating and terminating on the incoming [outgoing] electron leg (leg-photons), where $m \geq m'$. For asymmetric pairs of sets, with $m \neq m'$, the contribution to the anomaly $a_{kmm'}$ is, in my definition, the sum of the set and its mirror (time-reversed) image,

$$a[V] = \frac{1}{2}(g-2)\bigg|_V = \sum_{k=1}^{\infty}\sum_{m=0}^{\infty}\sum_{m'=0}^{m} a_{kmm'}\left(\frac{\alpha}{\pi}\right)^{k+m+m'} . \qquad (1.1)$$

When the diagrams that we had computed [24] are grouped into gauge sets, figure 1.2 to figure 1.6, a surprising thing happens; while the finite part of each Feynman diagram is of order of 10 to 100, every known gauge set adds up to approximately

$$\pm\frac{1}{2}\left(\frac{\alpha}{\pi}\right)^n ,$$

with the sign given by a simple empirical rule

$$a_{kmm'} = (-1)^{m+m'}\frac{1}{2} . \tag{1.2}$$

The sign rule is further corroborated by sets with photon self-energy insertions (but with the absolute size scaled down to $3-15\%$ of (1.2)). In figure 1.4 I compared this rule with the actual numbers and made my 1977 four-loop prediction [21].

The "zeroth" order estimate of the electron magnetic moment anomaly a is now given by the "gauge-set approximation," convergent and summable to all orders

$$a = \frac{1}{2}(g-2) = \frac{1}{2}\frac{\alpha}{\pi}\frac{1}{\left(1-\left(\frac{\alpha}{\pi}\right)^2\right)^2} + \text{“corrections"} . \tag{1.3}$$

This is not how one usually thinks of perturbation theory. Most of our colleagues believe that in 1952 Dyson [31] had shown that the perturbation expansion is an asymptotic series (for a discussion, see Dunne and Schubert [30]), in the sense that the n-th order contribution should be exploding combinatorially

$$\frac{1}{2}(g-2) \approx \cdots + n^n\left(\frac{\alpha}{\pi}\right)^n + \cdots ,$$

and not growing slowly like my estimate

$$\frac{1}{2}(g-2) \approx \cdots + \frac{n}{2}\left(\frac{\alpha}{\pi}\right)^{2n} + \cdots .$$

For me, the above is the most intriguing hint that something deeper than what we know underlies quantum field theory, and the most suggestive lesson of our calculation.

In 1977 Laporta [53] has published the individual contribution of the 891 4-loop vertex diagrams contributing to the electron $(g-2)$ (evaluated up to 1100 digits of precision). The vertex diagrams separate in 25 gauge-invariant sets (figure 1.8). The numerical contribution of each set, truncated to 40 digits, is listed in the table 1.2. Adding only the diagrams without closed electron loops (see figure 1.6 and 1.5), one finds for the quenched set (in Aoyama *et al.* [6] nomenclature, the set of diagrams with no lepton loops):

$$\begin{aligned}
A_1^{(8)}[V] &= -2.176866027739540077443259355895893938670 \\
&= -2.17\ldots \qquad \text{Aoyama } et\ al.\ [6]\ (2012) \\
&\approx 0 \qquad\qquad \text{Cvitanović } [21]\ (1977) \\
A_1^{(10)}[V] &= 8.726(336)\ldots \qquad \text{Aoyama } et\ al.\ [7]\ (2015) \\
&\approx 3/2 \qquad\qquad \text{Cvitanović } [21]\ (1977) .
\end{aligned} \tag{1.4}$$

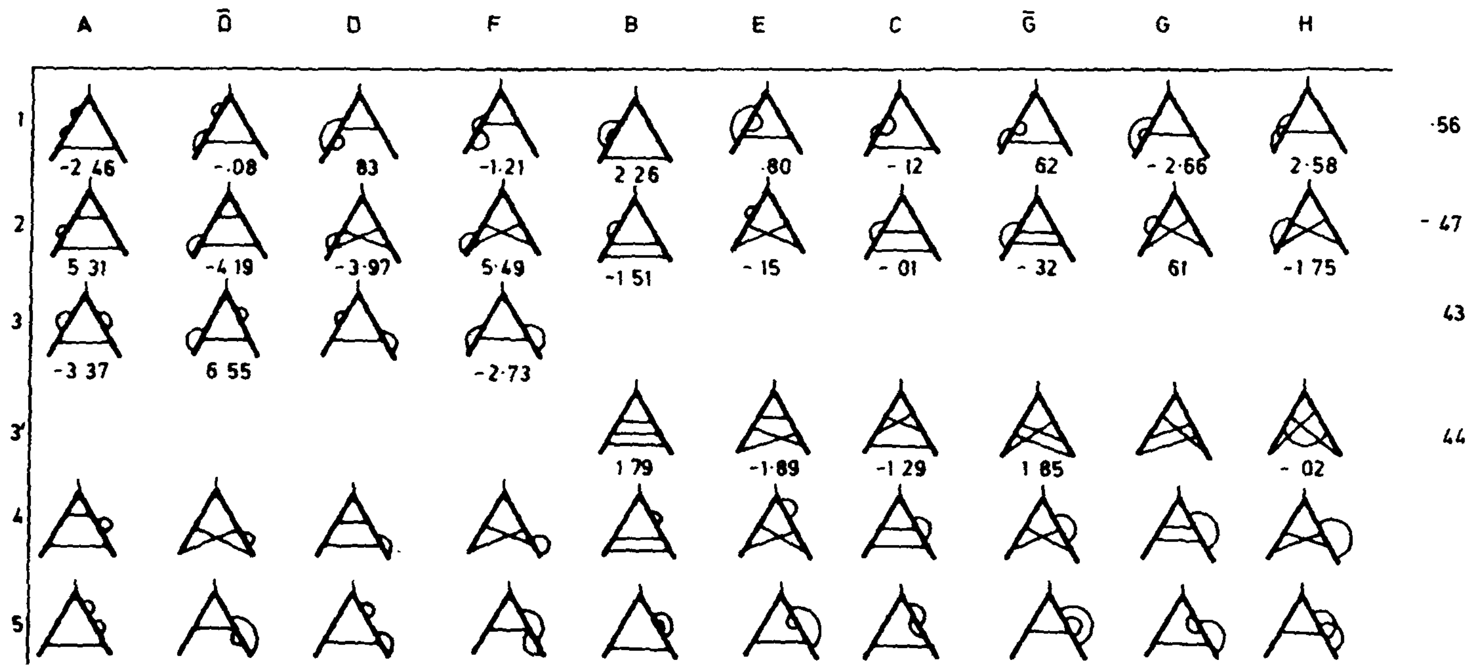


Figure 1.3: Every vertex diagram belongs both to a 'gauge set' and to a 'self-energy set'. This table illustrates the two kinds of sets. The sixth-order gauge sets (k, m, m') are arranged in the rows, and the self-energy sets (or the 'externally gauge-invariant' sets, vertex diagrams obtained by inserting an extra vertex into a self-energy diagram) in the columns, labeled as in Fig. 3 of ref. [24]. The values are finite parts in the $\ln \lambda$ IR cut-off approach, such as those listed in ref. [57]. For different IR separation methods (such as in ref. [24] and different gauges, individual diagrams have different values. The gauge sets, however, are separately gauge invariant. The self-energy sets (whose number grows combinatorially with the order in perturbation theory) are not, only their sum is gauge invariant. From ref. [21].

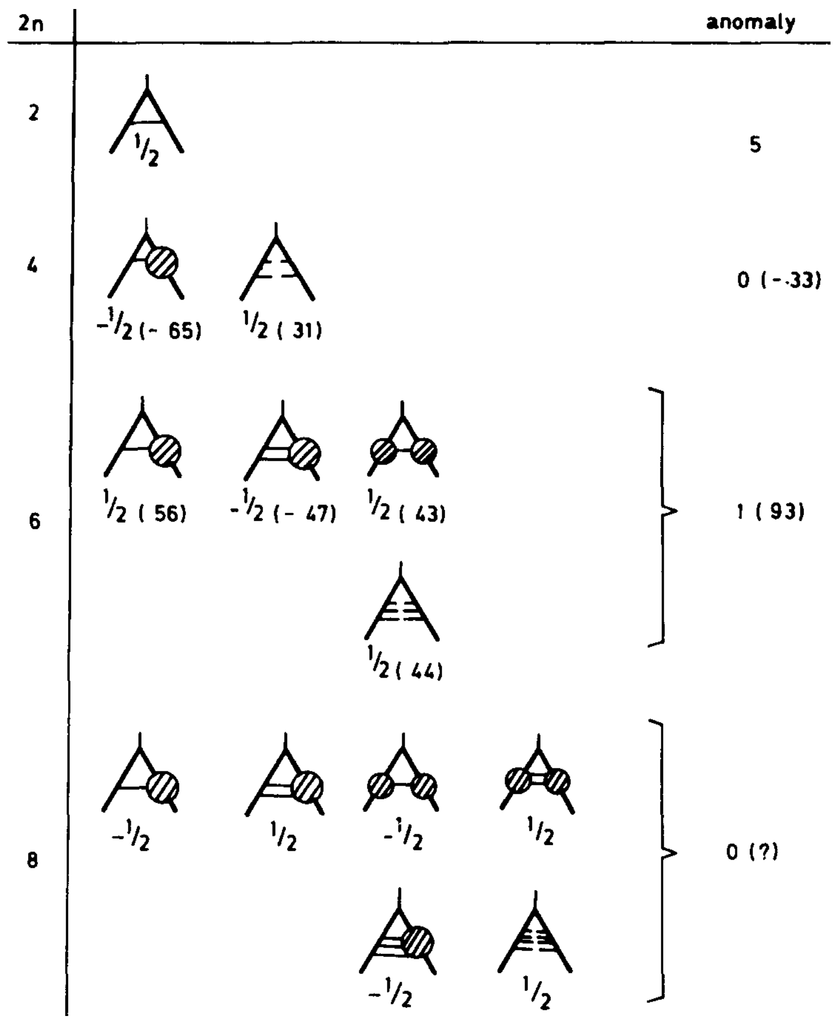
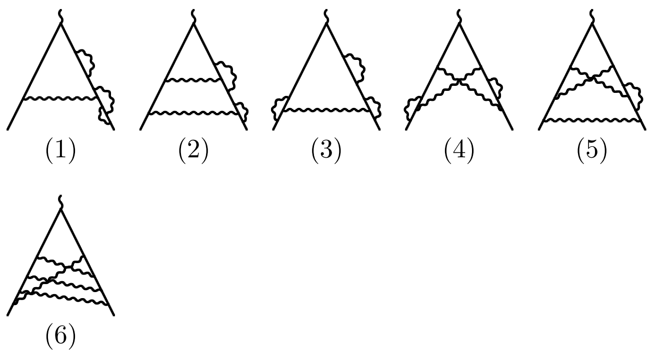


Figure 1.4: Comparison of the 1977 gauge-set approximation to the anomaly a and the actual numerical values of corresponding gauge sets, together with an eighth-order prediction, from ref. [21]. For the updated listing, see figure 1.5.

$2n$	(k,m,m')					anomaly
2	$(\mathbf{1},\mathbf{0},\mathbf{0})$ 1/2					$\frac{1}{2}$
4	$(\mathbf{1},\mathbf{1},\mathbf{0})$ -1/2 (-.65)	$(\mathbf{2},\mathbf{0},\mathbf{0})$ 1/2 (.31)				0 (-.33)
6	$(\mathbf{1},\mathbf{2},\mathbf{0})$ 1/2 (.56)	$(\mathbf{2},\mathbf{1},\mathbf{0})$ -1/2 (-.47)	$(\mathbf{3},\mathbf{0},\mathbf{0})$ 1/2 (.44)			1 (.93)
	$(\mathbf{1},\mathbf{1},\mathbf{1})$ 1/2 (.43)					
8	$(\mathbf{1},\mathbf{3},\mathbf{0})$ -1/2 (-1.97)	$(\mathbf{2},\mathbf{2},\mathbf{0})$ 1/2 (-.14)	$(\mathbf{3},\mathbf{1},\mathbf{0})$ -1/2 (-1.04)	$(\mathbf{4},\mathbf{0},\mathbf{0})$ 1/2 (.51)		0 (-2.17)
	$(\mathbf{1},\mathbf{2},\mathbf{1})$ -1/2 (-.62)	$(\mathbf{2},\mathbf{1},\mathbf{1})$ 1/2 (1.08)				
10	$(\mathbf{1},\mathbf{4},\mathbf{0})$ 1/2 (?)	$(\mathbf{2},\mathbf{3},\mathbf{0})$ -1/2 (-?)	$(\mathbf{3},\mathbf{2},\mathbf{0})$ 1/2 (?)	$(\mathbf{4},\mathbf{1},\mathbf{0})$ -1/2 (?)	$(\mathbf{5},\mathbf{0},\mathbf{0})$ 1/2 (?)	$\frac{3}{2}$ (8.72)
	$(\mathbf{1},\mathbf{3},\mathbf{1})$ 1/2 (?)	$(\mathbf{2},\mathbf{2},\mathbf{1})$ -1/2 (?)	$(\mathbf{3},\mathbf{1},\mathbf{1})$ 1/2 (?)			
	$(\mathbf{1},\mathbf{2},\mathbf{2})$ 1/2 (?)					

Figure 1.5: Updated figure 1.4 comparison of the gauge-set approximation (1.3) and the actual numerical values of corresponding gauge sets, together with the 5-loop prediction. Starting with 4-loops, the gauge-set approximation fails in detail, but still the signs are right, except for the anomalously small set $(2,2,0)$, and the remaining sets are surprisingly close to multiples of 1/2.



gauge set	(k, m, m')	value	prediction	
(1)	(1,3,0)	- 1.9710	- 1/2	
(2)	(2,2,0)	- 0.1424	1/2	(!)
(3)	(1,2,1)	- 0.6219	- 1/2	
(4)	(2,1,1)	1.0867	1/2	
(5)	(3,1,0)	- 1.0405	- 1/2	
(6)	(4,0,0)	0.5125	1/2	

Figure 1.6: (top panel) Examples of 4-loop vertex diagrams belonging to the gauge sets (1) to (6). The remaining diagrams in the set can be obtained by permuting separately the vertices on the left and right side of the electron line, and considering also the mirror images of the diagrams. From Laporta [53]; for all 25 gauge-invariant sets, see figure 1.8. (bottom panel) Gauge-set contributions $a^{(8)}_{kmm'}$, see (1.1), as reported by Laporta [53] (for the full 25 gauge-invariant sets, see table 1.2). The last column: 1977 Cvitanović predictions [21]. Signs are right, except for the set (2) = $(2, 2, 0)$, which is anomalously small, and the remaining sets are surprisingly close to multiples of 1/2. There might be factors of 2 having to do with symmetries, missing from the guesses of ref. [21], but I cannot see how that would work. Only (4) = $(2, 1, 1)$ and (6) = $(4, 0, 0)$ are symmetric, but (1) = $(1, 3, 0)$, (4) and (5) = $(3, 1, 0)$ seem to have an extra factor of 2 or 4.

While my 1977 prediction $A_1^{(8)}[V] \approx 0$ (instead of the correct $A_1^{(8)}[V] \approx -2$) does not pan out, the difference is small, considering that this is a sum of 518 vertex diagrams (or 47 self-energy diagrams) [45]. Likewise, my prediction for $A_1^{(10)}[V]$ is not too far off, considering this is a sum of 6354 independent vertex diagrams of table 1.1 (belonging to 389 self-energy sets).

1.1.1 Self-energy sets

There are two ways of grouping vertex diagrams, into *gauge sets* and into *self-energy sets* (or the "externally gauge-invariant" sets). Every vertex diagram belongs both to a gauge set and to a self-energy set, as illustrated by figure 1.3. Reformulation of the $(g-2)$ computation directly from self-energy graphs is due to Cvitanović and Kinoshita, see the "new formula" (6.22) in ref. [24], based on the Ward-Takahashi identity. Not only does the calculation use fewer Feynman graphs, but it was very important for us, as it enabled us to calculate the 3-loop electron magnetic moment by two wholly independent methods (and in this way we did actually identify and eliminate a subtle numerical error in one of the graphs). Parenthetically, Carroll papers [16, 17] puzzle me. He gives the credit to the mass-operator formalism of Schwinger [72–74], but no credit to us [24], even though his papers look closer to ours than to Schwinger and Sommerfield; he cites us, he also uses the Ward-Takahashi identity. Our formulation might be equivalent to Schwinger's, but it looks quite different in detail, and I was not aware of Schwinger mass-operator when we derived it.

The gauge sets are minimal, and separately gauge invariant (for a proof, see ref. [21]). The self-energy sets are not, only their sum is gauge invariant. Unlike gauge sets, whose number grows polynomially, the number of self-energy sets grows combinatorially - they save significant amount of computing for few-loops computations, but cannot be used to argue the finiteness of QED. That is the reason why Aoyama *et al.* [6, 7] calculations have nothing to say about my 1977 paper [21]: they do not compute individual vertex diagrams, but only the self-energy sets, and for them the set of all diagrams without a fermion loop ('quenched-' or 'q-type' diagrams) is a single 'gauge-invariant set' V. For example, for 5-loops the set V is a sum 9 vertex gauge sets (where time-reversed pairs count as one set, see figure 1.5), but Aoyama *et al.* [7] only give their sum (1.4).

1.2 Where do we go from here?

Aoyama *et al.* 5-loop calculations push the envelope of what is numerically attainable, they no motive to switch from self-energy diagrams formulation to the vertex diagrams formulation, it would mean (for the quenched set V) going from 389 self-energy graphs to 6354 vertex diagrams. Stefano Laporta deserves a bit of well earned rest. So what is ahead?

I've now searched through the literature, at least the literature familiar to me (people who do things related to my 1977 paper [21] never alert me to their papers, presumably because the news of my death are somewhat exaggerated). Sergey A. Volkov appears to be the only person set up do the requisite 5-loop calculations.

There seem to be 2 approaches that might be relevant to establishing bounds on, and direct computation of gauge sets (ignoring various $N=2$ and $N=4$ supersymmetric models): (1) *Hopf algebraic approach* of Kreimer and collaborators, and (2) *worldline formalism* pursued by Schubert and collaborators.

1.2.1 Volkov method

That QED on-mass shell amplitudes are IR-free must be an old result; even I have several papers generalizing that to QCD [19, 20, 22, 25]. Tom Kinoshita and I solved the problem of point-by-point removal of IR divergences in Feynman-parametric space in my thesis [23]. I have a bright memory of figuring out how to do it one quiet evening in Ithaca, babysitting for a friend's toddler. But our approach was apparently not general enough to deal with 4-loop and higher order corrections. In ref. [79] Volkov explains that $A_1^{(2n)}$ is free from infrared divergences since they are removed by the on-shell renormalization. However, Volkov also states that there is no universal method in QED for canceling IR divergences in the Feynman graphs analogous to the R operation, and that the standard subtractive on-shell renormalization cannot remove IR divergences point-by-point in Feynman-parametric space, as it does for UV divergences. Moreover, it can generate additional IR-divergences. Volkov's algorithm is developed in *New method of computing the contributions of graphs without lepton loops to the electron anomalous magnetic moment in QED* [78]. It is based on the ideas used for proving UV-finiteness of renormalized Feynman amplitudes [5, 75]. He focuses on n-loop graphs with no lepton loops, or, in Aoyama *et al.* [6] nomenclature, $A_1^{(2n)}[V]$.

While Volkov organizes Feynman graphs by self-energy graph families, in contrast to refs. [7, 16, 17, 24] he does not evaluate these self-energy graphs directly; all his calculations are performed with vertex graphs. Individual vertex graphs are precisely what is necessary for evaluating gauge sets [21].

So far Volkov has evaluated the ladder graph and the fully crossed graph up to 5 loops. The cross graphs are of interest because they do not contain divergent subgraphs, so their contributions only depend on the gauge, but not on the choice of subtraction procedure.

1.2.2 Hopf algebraic approach

Hopf algebraic approach of Kreimer and collaborators [15, 46, 51, 52] is very appealing - it is just that I personally have no clue how to turn it into a direct $(g-2)$ gauge set calculation. In the 2008 paper [52] Dirk Kreimer and Karen Yeats write: " One case where there is a natural interpretation is QED with a linear number of generators, namely

$$X_1 = 1 + \sum_{k\geq 1} p(k) x^k \frac{X_1^{2k+1}}{(1-X_2)^{2k}(1-X_3)^{2k}}\,, \tag{1.5}$$

with X_2 and X_3 as before and with $p(k)$ linear, which corresponds to counting with Cvitanović's gauge invariant sectors [21]. " but I do not see how this counts the gauge

 1.2 *Where do we go from here?*

sets. My generating function for G_{2n}, the number of gauge sets (eq. (7) in ref. [21]) is

$$\sum_{n=1}^{\infty} G_{2n} = \frac{X}{(1+X)(1-X)^3}\,. \tag{1.6}$$

1.2.3 Worldline formalism

How and why Feynman in 1950 introduced 'worldline formalism' (initially for scalar QED, appendix to ref. [33], then spinor QED, appendix to ref. [34]) is nicely explained in Schubert [69], which also has an extensive bibliography up to 2001. In 1982 Affleck, Alvarez, and Manton [1] used the Feynman worldline path integral representation of the quenched effective action for scalar QED in the constant electric field, and, independently, Lebedev and Ritus [56] did the same for the spinor QED in 1984.

A formula for the charged scalar propagator to absorb and emit N photons along the way of its propagation from x' to x can be derived as follows [3]. The free scalar propagator for the Euclidean Klein-Gordon equation [3, 70] is

$$D_0(x,x') = \langle x|\frac{1}{-\Box + m^2}|x'\rangle\,, \tag{1.7}$$

where $\Box$ is the D-dimensional Laplacian. Exponentiate the denominator following Schwinger,

$$D_0(x,x') = \int_0^\infty dT\, e^{-m^2T}\langle x|e^{-T(-\Box)}|x'\rangle\,, \tag{1.8}$$

Replace the operator in the exponent by a path integral

$$D_0(x,x') = \int_0^\infty dT\, e^{-m^2T}\int_{x(0)=x'}^{x(T)=x}\mathcal{D}x(\tau)\, e^{-\int_0^T d\tau \frac{1}{4}\dot{x}^2}\,, \tag{1.9}$$

where τ is a proper-time parameter. This is the worldline path integral representation of the relativistic propagator of a scalar particle in Euclidean spacetime. It is easily evaluated and leads to the usual space and momentum space free propagators. Adding the interaction terms leads to the Feynman's path integral representation [33] of the charged scalar propagator of mass m in the presence of a background field $A(x)$,

$$D(x,x') = \int_0^\infty dT\, e^{-m^2T}\int_{x(0)=x'}^{x(T)=x}\mathcal{D}x(\tau)\, e^{-S_0-S_e-S_i}\,, \tag{1.10}$$

where (0) is the free propagation

$$S_0 = \int_0^T d\tau \frac{1}{4}\dot{x}^2\,, \tag{1.11}$$

(e) is the interaction of the charged scalar with the external field

$$S_e = -ie\int_0^T d\tau\, \dot{x}^\mu A_\mu(x(\tau))\,, \tag{1.12}$$

and (i) are the virtual photons exchanged along the charged particle's trajectory

$$S_i = \frac{e^2}{2} \int_0^T d\tau_1 \int_0^T d\tau_2 \, \dot{x}_1^\mu \, D_{\mu\nu}(x_1 - x_2) \, \dot{x}_2^\nu \,, \tag{1.13}$$

where $D_{\mu\nu}$ is the x-space photon propagator.

Consider first the charged scalar field in external field, neglecting internal photon loops. By taking the constant external field $A(x)$ to be a sum of N plane waves, one obtains the rule for inserting N external photons:

$$\begin{aligned} D_{(N)}(x, x') &= (-\lambda)^N \int_0^\infty dT \, e^{-m^2 T} \int_0^T d\tau_1 \cdots \int_0^T d\tau_N \\ &\times \int_{x(0)=y}^{x(T)=x} \mathcal{D}x \, e^{i \sum_{i=1}^N k_i \cdot x(\tau_i)} e^{-\int_0^T d\tau \, \frac{1}{4} \dot{x}^2} \,. \end{aligned} \tag{1.14}$$

For the spinor case, the magnetic moment will be given by the term linear in a constant external field $A(x)$, and in order to define gauge sets, one will have to distinguish the in- and out-electron lines.

The object of great interest to us is the quenched internal virtual photons term (1.13):

$$\int_{x(0)=x'}^{x(T)=x} \mathcal{D}x(\tau) \, e^{-S_i} = \int_{x(0)=x'}^{x(T)=x} \mathcal{D}x(\tau) \, e^{-\frac{e^2}{2} \int_0^T d\tau_1 \int_0^T d\tau_2 \, \dot{x}_1^\mu \, D_{\mu\nu}(x_1 - x_2) \, \dot{x}_2^\nu} \,. \tag{1.15}$$

Expanded perturbatively in α/π, this yields the usual quenched, Feynman-parametrized vertex diagrams. However, it is Gaussian in $\dot{x}^\mu$, and if by integration by parts, $\dot{x}^\mu$ are eliminated in favor of x^μ, internal photons can be integrated over directly, prior to an expansion in $(\alpha/\pi)^n$, and one gets integrals in terms of N-photon propagators, and not the usual vertex graphs. Each usual vertex graph represents one permutation of internal photon insertions, and from that comes the factorial growth in the number of graphs.

These integrations by parts lead to the first and second proper-time derivatives of the Green's function are worked out in the literature (for example, in refs. [70, 76]), would take too much space here. I find Bastianelli, Huet, Schubert, Thakur and Weber 2014 paper [11] (the 2017-05-24 blog entry below) quite inspirational. My notes on these papers are below (page 19). Apologies, they are just a jumble, jottings taken as I try to understand this literature.

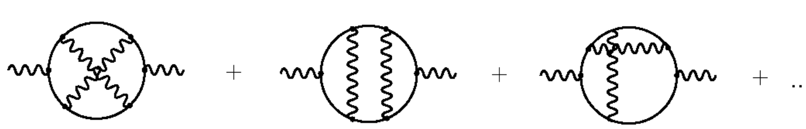


Figure 1.7: Quenched diagrams contributing to the three loop QED photon propagator. From ref. [11].

Thus, for the quenched scalar QED, the worldline integrals are expressed in terms of N-photon propagators, the ingredient that defines the quenched gauge sets (1.1).

Unlike the Feynman parameter integrals for individual vertex graphs, they are independent of the ordering of the momenta $k_1, \ldots, k_N$; the formula (1.15) contains all $\approx N!$ ways of attaching the N photons to the charged particle propagator. The formulation combines many Feynman diagrams into a single integral. An example are the quenched contributions to the three-loop photon propagator shown in figure 1.7.

In QED it combines into one integral all Feynman graphs related by permutations of photon legs along fermion lines, that is, it should yield *one* integral for a gauge set (k, m, m') defined in (1.1), provided one may distinguish the leg-photons from the cross-photons.

1.2.4 High-orders QED in worldline formalism

A non-perturbative formula for QED in a constant field, given for scalar QED in 1982 by Affleck, Alvarez, and Manton [1], and for spinor QED by Lebedev and Ritus [56] in 1984, is an example how the worldline formalism can yield high-order information on QED amplitudes. Huet, McKeon, and Schubert [41] continue this in their 2010 study of the l-loop N-photon amplitudes in the limit of large photon numbers and low photon energies in the setting of 1+1 dimensional scalar QED, in order to illustrate the large cancellations inside gauge invariant classes of graphs.

Affleck *et al.* [1] used the Feynman [33] 'worldline path integral' representation of the quenched effective action in scalar QED in the constant electric field, and calculate the amplitude in a stationary path approximation. The stationary trajectory so obtained is a circle with a field dependent radius, called "instanton" in this context. The worldline action on this trajectory yields the correct exponent, and the second variation determinant the correct prefactor. Using Borel analysis, they obtain non-perturbative information on the on-shell renormalized N-photon amplitudes at large N and low energy.

For the quenched spinor QED (fermion lines decorated by photon exchanges) closed-form expressions for general N require the worldline super- formalism [69], at the cost of introducing Fradkin 1966 [37] Grassmann path integral.

Dunne and Schubert [30] study N-photon amplitudes, in scalar and spinor QED, in the quenched approximation, i.e., taking only the diagrams with one electron loop and are led to "the following generalization of Cvitanović's conjecture: the perturbation series converges for all on-shell renormalized QED amplitudes at leading order in N_f. It must be emphasized that the on-shell renormalization is essential in all of the above." Unlike Cvitanović [21] purely numerical conjecture, theirs is a much more sophisticated argument, buttressed by Borel dispersion relations.

1.3 Summary

Worldline formalism could be useful in a crude way, as a way of proving the finiteness of QED conjecture, and in a precise way, as a new computational tool:

1. Develop a saddle point expansion for N-photon propagator such that the leading term explains the apparent $\approx \pm 1/2$ (or a multiple thereof) size of each quenched gauge set. Affleck *et al.* [1] show they way.

2. Use that to establish bounds on gauge sets for large orders, prove finiteness of quenched QED. If that works, I trust electron loop insertions will be next, and thereafter renormalons [54], etc., will go gently into that good night.
3. Develop a new worldline formulation of QED in which each gauge set is given by a computable integral. That might make it possible to evaluate orders beyond 5-loops, as the number of gauge sets grows only polynomially.
4. A gauge-invariant set is by definition UV and IR finite. Does that mean that its worldline formalism integrand is pointwise finite, no need for counterterms?

My main problem (well, there are many:) is that nobody seems to have written an explicit formula for the QED anomalous magnetic moment in the worldline formalism.

1.4 Is QED finite? A blog

1977-03-03 Predrag Drell and Pagel [28] *Anomalous magnetic moment of the electron, muon, and nucleon* attempt got the sign right, but was not successful in predicting the magnitude of the sixth-order magnetic moment; $0.15\left(\frac{\alpha}{\pi}\right)^3$ instead of $1.19\left(\frac{\alpha}{\pi}\right)^3$.

1971-08-01 Lautrup, Peterman, and de Rafael [55] 1972 *Recent developments in the comparison between theory and experiments in quantum electrodynamics* list the 3-loop, no-electron loop "gauge invariant subclasses" (their Fig. 4.3).

1974-01-07 Samuel [67] 1974 *Estimates of the eighth-order corrections to the anomalous magnetic moment of the muon*:

"We speculate that in making radiative corrections to a class of graphs by inserting a single photon in all possible ways, one obtains a contribution which is roughly $-\frac{\alpha}{\pi}$ times the contribution of the class. This seems to be obeyed by the known contributions."

2013-12-08 Predrag to Piotr, Wanda and Andrea (Piotr Czerski <piotr.czerski@ifj.edu.pl>, wanda.alberico@to.infn.it, andrea.prunotto@gmail.com):

I'm no fan of Feynman diagrams (my rant is here), and I'm always looking for other ways to look at perturbative expansions. So just a little email - if you have a new angle [62] on subsets of diagrams which are gauge invariant sets, I would be curious to learn how you look at that.

Just something to keep in mind :)

PS to Andrea: I realize you might rather forget this stuff (takes you a decade to write a paper?) but at least I got a ringtone out of you. The only problem is, I do not have a cell phone, so I do not know how to make it ring. At least I'm more technologically savvy than Peter Higgs.

2013-12-10 Andrea Sorry for late reply (well, we're used to longer gaps). Yes! I actually took 10 years to write this paper out of my master thesis, but I have some excuses: I did my PhD in Biochemistry (Zürich) and now I work on genetics (Lausanne). This summer my "old" professor Wanda found my work in a drawer and then contacted me, telling me that it would be a good idea to publish it.

About your request: I'm really interested in seeing if the rooted-map approach to Feynman diagrams can address the problem you've risen. But I have no idea what the "subsets of diagrams which are gauge invariant sets" are. I've checked a bit on the web but I'm sure you can give me better indications (the works I found were too technical: I need to know the basis of the problem). Can you send me some specific link at freshman level, in particular where I can see the geometry of these subclasses of diagrams?

2013-12-11 Predrag Googling is good, but it is faster to click on this link. The article defines the gauge invariant sets.

2016-02-08 Predrag Prunotto [61] *A Homological Approach to Feynman Diagrams in the Quantum Many-Body Theory*, and Prunotto, Alberico and Czerski [62] 2013 *Feynman Diagrams and Rooted Maps* has been submitted to the European Physical Journal A as manuscript ID EPJA-103480, seems not to have been published anywhere by 2017. They write: " The Rooted Maps Theory, a branch of the Theory of Homology, is shown to be a powerful tool for investigating the topological properties of Feynman diagrams, related to the single particle propagator in the quantum many-body systems. The numerical correspondence between the number of this class of Feynman diagrams as a function of perturbative order and the number of rooted maps as a function of the number of edges is studied. A graphical procedure to associate Feynman diagrams and rooted maps is then stated. Finally, starting from rooted maps principles, an original definition of the genus of a Feynman diagram, which totally differs from the usual one, is given. "

2016-12-10 Predrag Penante [60] 2016 On-shell methods for off-shell quantities in N=4 Super Yang-Mills: from scattering amplitudes to form factors and the dilatation operator has an up-to-date review of on-shell methods.

2016-12-26 Predrag Read Cruz-Santiago, Kotko and Staśto [18] 2015 *Scattering amplitudes in the light-front formalism*: "The idea is to divide the process into appropriate gauge invariant components. It turns out that the gauge invariant subsets are invariant under cyclic permutations of the external gluons. This decomposition was proposed in works of [58–61] for the tree level amplitudes. A thorough analysis of the relation between color structures and gauge invariance was done in ref. [25]. The color decomposition principle was extended beyond the tree level to loop amplitudes in [63]."

Should also read Dixon [27] 1996 *Calculating scattering amplitudes efficiently*.

2017-03-15 Predrag Dunne and Krasnansky [29] 2006 *"Background field integration-by-parts" and the connection between one-loop and two-loop Heisenberg-Euler effective actions*: " We develop integration-by-parts rules for diagrams involving massive scalar propagators in a constant background electromagnetic field, and use these to show that there is a simple diagrammatic interpretation of mass renormalization in the two-loop scalar QED Heisenberg-Euler effective action for a general constant background field. This explains why the square of a one-loop term appears in the renormalized two-loop Heisenberg-Euler effective action, and dramatically simplifies the computation of the renormalized two-loop effective action for scalar QED, and generalizes a previous result obtained for self-dual background fields. "

2017-05-23 Predrag M. G. Schmidt and C. Schubert [68] 1994 *Multiloop calculations in the string-inspired formalism: the single spinor-loop in QED*, arXiv:hep-th/9410100: They use the worldline path-integral Bern-Kosower formalism for to calculate the sum of all diagrams with one spinor loop and fixed numbers of external and internal photons. Of interest: in this formalism the three 2-loop photon polarization graphs, see figure 1.7, are a single integral, easier to evaluate than any of the three Feynman graphs. They also note an unexplained cancelation not only of poles, but also of "transcedentals." A knot-theoretic explanation for the rationality of the quenched QED beta function is given in ref. [15].

2017-05-23 Predrag Nieuwenhuis and Tjon [59] 1996 Nonperturbative study of generalized ladder graphs in a $\phi^2\chi$ theory arXiv:hep-ph/9606403:

2017-05-23 Predrag Christian Schubert [69] 2001 *Perturbative quantum field theory in the string-inspired formalism*, arXiv:hep-th/0101036:

2017-03-15 Predrag Huet, McKeon, and Schubert [41] 2010 *Euler-Heisenberg lagrangians and asymptotic analysis in 1+1 QED. Part I: Two-loop* (no GaTech online access, arXiv:1010.5315):" We continue an effort to obtain information on the QED perturbation series at high loop orders, and particularly on the issue of large cancellations inside gauge invariant classes of graphs, using the example of the l-loop N-photon amplitudes in the limit of large photon numbers and low photon energies. The high-order information on these amplitudes can be obtained from a nonperturbative formula, due to Affleck *et al.* [1], for the imaginary part of the QED effective lagrangian in a constant field. The procedure uses Borel analysis and leads, under some plausible assumptions, to a number of nontrivial predictions already at the three-loop level. Their direct verification would require a calculation of this 'Euler-Heisenberg lagrangian' at three-loops, which seems presently out of reach (though see Huet, de Traubenberg, and Schubert [42] below). Motivated by previous work by Dunne and Krasnansky [29] on Euler-Heisenberg lagrangians in various dimensions, in the present work we initiate a new line of attack on this problem by deriving and proving the analogous predictions in the simpler setting of 1+1 dimensional QED. In the first part of this series, we obtain a generalization of the formula of Affleck *et al.* [1] to this case, and show that, for both scalar and spinor QED, it correctly predicts the leading asymptotic behaviour of the weak field expansion coefficients of the two loop Euler-Heisenberg lagrangians.

2017-05-24 Predrag Bastianelli, Huet, Schubert, Thakur and Weber [11] 2014 *Integral representations combining ladders and crossed-ladders* write:

This property is particularly interesting in view of the fact that it is just this type of summation which in QED often leads to extensive cancellations, and to final results which are substantially simpler than intermediate ones (see, e.g., ref. [15, 21]). More recently, similar cancellations have been found also for graviton amplitudes (see, e.g., ref. [8]). Although this property of the worldline formalism is well-known, and has been occasionally exploited [10, 64, 65, 68] (see also ref. [39]) a systematic study of its implications is presently still lacking.

The first classes of Green's functions is the x-space propagator for one scalar interacting with the second one through the exchange of N given momenta.

This object, to be called "N-propagator", is given by a set of $N!$ simple tree-level graphs, is in the worldline formalism combined into a single integral.

The second class are the similarly looking x-space $N+2$ - point functions defined by a line connecting the points x and y and N further points $z_1, \ldots, z_N$ connecting to this line in an arbitrary order.

An advantage of the worldline representation over the usual Feynman parameterization is the automatic inclusion of all possible ways of crossing the "rungs" of the ladders. They obtain such representations in explicit form both in x-space and in momentum space.

The inclusion of the crossed ladder graphs is essential for the consistency of the one-body limit where one of the constituents becomes infinitely heavy, and for maintaining gauge invariance.

They concentrate on the case of infinite N, *i.e.*, the sum over *all* ladder *and* crossed ladder graphs.

As their main application, they consider the case of two massive scalars interacting through the exchange of a massless scalar, obtain an the case of a massless exchanged particle (along the "rungs" of the ladders).

Applying asymptotic estimates and a saddle-point approximation to the N-rung ladder plus crossed ladder diagrams, they derive a semi-analytic approximation formula for the lowest bound state mass in this model.

They use the worldline formalism to derive integral representations for the N-propagators and the N-ladders - in scalar field theory, and give a compact expression combining the $N!$ Feynman diagrams contributing to the amplitude. They give these representations in both x and (off-shell) momentum space. Being off-shell, can be used as building blocks for many more complex amplitudes. They derive a compact expression for the sum of all ladder graphs with N rungs, including all possible crossings of the rungs.

Nieuwenhuis and Tjon [59] 1996 have numerically evaluated the path integrals of the worldline representation for the same scalar model field theory that we are considering here, thus including all ladder *and* crossed ladder graphs.

2017-05-23 Predrag Huet, de Traubenberg, and Schubert [42] 2017 *Multiloop Euler-Heisenberg Lagrangians, Schwinger pair creation, and the photon S-matrix*: "Schwinger pair creation in a constant electric field, may possibly provide a window to high loop orders; simple non-perturbative closed-form expressions have been conjectured for the pair creation rate in the weak field limit, for scalar QED in 1982 by Affleck, Alvarez, and Manton [1], and for spinor QED by Lebedev and Ritus [56] in 1984. Using Borel analysis, these can be used to obtain non-perturbative information on the on-shell renormalized N-photon amplitudes at large N and low energy."

Preliminary results of a calculation of the three-loop Euler–Heisenberg Lagrangian in two dimensions indicate that the exponentiation conjecture by Affleck *et al.* and Lebedev/Ritus probably fails in $D = 2$.

Dunne and Schubert conjectured in 2005 that the QED $N-$photon amplitudes in the quenched (one electron loop) approximation are convergent in perturbation theory [30]. In this article they say; "Later they learned that Cvitanović in 1977 had already made the analogous conjecture for $(g-2)$ [21]."

2017-05-23 Predrag Das, Frenkel and Schubert [26] 2013 *Infrared divergences, mass shell singularities and gauge dependence of the dynamical fermion mass*; arXiv:1212.2057:

2017-05-23 Predrag Ahmadiniaz, Bashir and Schubert [3] 2016 *Multiphoton amplitudes and generalized Landau-Khalatnikov-Fradkin transformation in scalar QED*, arXiv:1511.05087:

$D_{\mu\nu}$ is the x-space photon propagator. In D dimensions and arbitrary covariant gauge

$$D_{\mu\nu}(x) = \frac{1}{4\pi^{\frac{D}{2}}}\Big\{\frac{1+\xi}{2}\Gamma\Big(\frac{D}{2}-1\Big)\frac{\delta_{\mu\nu}}{(x^2)^{\frac{D}{2}-1}}+(1-\xi)\Gamma\Big(\frac{D}{2}\Big)\frac{x_\mu x_\nu}{(x^2)^{\frac{D}{2}}}\Big\}\,. \quad (1.16)$$

Calculation methods:

1. The analytic or "string-inspired" approach, based on the use of worldline Green's functions: all path integrals are brought into Gaussian form; this requires some expansion and truncation. They are then calculated by Gaussian integration.
2. The semi-classical approximation, based on a stationary trajectory ("worldline instanton").

We will focus on the closed-loop case in the following, since it turns out to be simpler than the propagator one. Nevertheless, it should be emphasized that everything that we will do in the following for the effective action can also be done for the propagator.

Some reasonable gymnastics leads to the "Bern-Kosower master formula" [12, 13, 76], arXiv:hep-ph/9205205

2017-05-23 Predrag Strassler [76] *Field theory without Feynman diagrams: One-loop effective actions*

2017-05-23 Predrag Ahmad *et al.* [2] 2017 *Master formulas for the dressed scalar propagator in a constant field*, arXiv:1612.02944

2017-05-23 Predrag Broadhurst and R. Delbourgo and D. Kreimer [15] 1996 *Unknotting the polarized vacuum of quenched QED* has lots of magic leading to cancelations of "transcedentals." They say: "Complete cancellation of transcendentals from the beta function, at every order, is to be expected only in quenched QED and quenched SED, where subdivergences cancel between bare diagrams."

2017-05-23 Predrag Dirk Kreimer and Karen Yeats [52] 2008 *Recursion and growth estimates in renormalizable quantum field theory*

Our method is very different in spirit from the constructive approach or the functional integral approach. It relies on a Hopf algebraic decomposition of terms in the perturbative expansion into primitive constituents, not unlike the decomposition of a ζ function into Euler factors.

Our construction of a basis of primitives with a given Mellin transform resolves overlapping divergences, thanks to the Hochschild cohomology of the relevant Hopf algebras [50].

We next assume there to be $p(k)$ primitives at k loops where p is a polynomial.

Yeats [80] 2017 *A Combinatorial Perspective on Quantum Field Theory*. I have put a copy here.

2016-08-20 Predrag Kißler and Kreimer [46] 2016 Diagrammatic cancellations and the gauge dependence of QED: " The perturbative expansion given in terms of Feynman graphs might be rearranged in terms of meta graphs or subsectors with a maximum number of cancellations implemented. An early attempt to construct gauge invariant subsectors in QCD was given by Cvitanović *et al.* [25]. "

They discuss how the QED tree-level cancellation identity implies cancellation between Feynman graphs of different topologies and determines the gauge dependence. They parameterize the momentum part of Landau gauge, then start by keeping only a liner term in graphs (i.e., insert only one Landau propagator, rest Feynman). Not sure it is useful to us...

Read also Kreimer [51] 2000 *Knots and Feynman diagrams*.

2017-05-23 Predrag Badger, Bjerrum-Bohr and Vanhove [8] 2009 *Simplicity in the structure of QED and gravity amplitudes*

2017-05-23 Predrag Rosenfelder and Schreiber [64, 65] 1996, arXiv:nucl-th/9504002, arXiv:nucl-th/9504005:

2017-06-02 Predrag Rosenfelder and Schreiber [66] 2004 *An Abraham-Lorentz-like equation for the electron from the worldline variational approach to QED*:

They discuss the of a spin-1/2 electron dressed by an arbitrary number of photons in the quenched approximation to QED. The approach is patterned after Feynman's celebrated variational treatment of the polaron problem [35], which was first applied by Mano, Progr. Theor. Phys. 14, 435 (1955) [8] to a relativistic scalar field theory and rediscovered and expanded by them in a series of papers [64, 65]. Its main features are the description of relativistic particles by worldlines [69] parametrized by the proper time, an exact functional integration over the photons and a variational approximation of the resulting effective action by a retarded quadratic trial action. In recent work we have extended this approach to more realistic theories, in particular to quenched QED [4] (the divergence structure and renormalization, a compact expression for the anomalous mass dimension of the electron). Here they calculate the finite contributions.

The variational formulation of worldline QED leads to an equation which is similar to Abraham, Lorentz and Dirac description of the electron and its self-interaction with the radiation field. The approach contains (almost) all the ingredients of the relativistic field theory of electrons and photons, in particular its divergence structure. This has been demonstrated by deriving an approximate nonperturbative expression for the anomalous mass dimension of the electron.

2017-05-23 Predrag K. Barro-Bergflödt, R. Rosenfelder and M. Stingl [10] 2006 *Variational worldline approximation for the relativistic two-body bound state in a scalar model*, arXiv:hep-ph/0601220.

2017-03-15 Predrag Read Fried [38] 2014 *Modern Functional Quantum Field Theory: Summing Feynman Graphs*: a simple, analytic, functional approach to non-perturbative QFT, using a functional representation of Fradkin to explicitly calculate relevant portions of the Schwinger Generating Functional (GF). In QED, this corresponds to

summing all Feynman graphs representing virtual photon exchange

between charged particles. It is then possible to see, analytically, the cancellation of an infinite number of perturbative, UV logarithmic divergences, leading to an approximate but most reasonable statement of finite charge renormalization. A similar treatment of QCD, with the addition of a long-overlooked but simple rearrangement of the Schwinger GF which displays Manifest Gauge Invariance, is then able to produce a simple, analytic derivation of quark-binding potentials without any approximation of infinite quark masses. A crucial improvement of previous QCD theory takes into account the experimental fact that asymptotic quarks are always found in bound state.

This book can be read online via `library.gatech.edu`

2017-05-23 Predrag H. M. Fried and Y. Gabellini [39] 2012 *On the Summation of Feynman Graphs*, arXiv:1004.2202.

2017-05-26 Predrag The decomposition of scattering amplitudes into gauge invariant subsets of diagrams is studied by Boos and Ohl [9, 14]. Boos and Ohl [14] *Minimal gauge invariant classes of tree diagrams in gauge theories*, arXiv:hep-ph/9903357 (see arXiv:hep-ph/9911437 and arXiv:hep-ph/0307057 for more detail) is motivated by applications to Standard Model multi-particle diagrams, mostly at the tree level.

Perturbative calculations require an explicit breaking of gauge invariance for technical reasons and the cancellation of unphysical contributions is not manifest in intermediate stages of calculations. The contribution of a particular Feynman diagram to a scattering amplitude depends in the gauge fixing procedure and has no physical meaning. the identification of partial sums of Feynman diagrams that are gauge invariant by themselves is of great practical importance. Calculation a subset of diagrams that is not gauge invariant has no predictive power, because they depend on unphysical parameters introduced during the gauge fixing.

A *gauge invariance class* is a minimal subset of Feynman diagrams that is independent of the gauge parameter and satisfies the Slavnov-Taylor identities.

The set of diagrams connected by flavor and gauge flips they call *forest*, a set of diagrams connected by gauge flips the call *grove*. They shown that the groves are the minimal gauge invariance classes of tree Feynman diagrams. In unbroken gauge theories, the permutation symmetry of external gauge quantum numbers can be used to subdivide the scattering amplitude corresponding to a grove further into gauge invariant sub-amplitudes.

This (largely uncited) work seems to have no impact on the $(g-2)$ gauge sets discussed here.

2017-05-27 Predrag Reuschle and Weinzierl [63] *Decomposition of one-loop QCD amplitudes into primitive amplitudes based on shuffle relations* cite our ref. [25]. They say:

QCD calculations organise the computation of the one-loop amplitude as a sum over smaller pieces, called *primitive amplitudes*. The most important features of a primitive amplitude are gauge invariance and a fixed cyclic ordering of the external legs. Primitive amplitudes should not be confused with *partial amplitudes* (also referred to as a *dual amplitude* or a *color-ordered amplitude*), which are the kinematic coefficients of the independent colour structures. The first step in a discussion of perturbative Yang-Mills is the decoupling of color from kinematics,

$$A_{tot} = \sum c_J A_J \tag{1.17}$$

where A_{tot} represents the total amplitude for a scattering process, A_J are all the possible color structures, and A_J are partial amplitudes which depend only on the kinematical data (momenta and polarizations). Partial amplitudes are gauge invariant, but not necessarily cyclic ordered. Partial amplitudes are far simpler to calculate than the full amplitude. There exist linear relations among the partial amplitudes, called Kleiss-Kuijf relations, which reduce the number of linearly independent partial amplitudes to $(n-2)!$ The leading contributions in an $1/N$-expansion (with N being the number of colours) are usually cyclic ordered, the sub-leading parts are in general not. The decomposition of the full one-loop amplitude into partial amplitudes is easily derived. However, it is less trivial to find a decomposition of the partial amplitudes into primitive amplitudes.

There are several possible choices for a basis in colour space. A convenient choice is the colour-flow basis [77].

2017-05-27 Predrag Schuster [71] *Color ordering in QCD*: " We derive color decompositions of arbitrary tree and one-loop QCD amplitudes into color-ordered objects called primitive amplitudes. "

2017-05-27 Predrag Zeppenfeld [81] *Diagonalization of color factors* (Georgia Tech has no access to this paper)

2017-05-27 Predrag Edison and Naculich [32] *Symmetric-group decomposition of SU(N) group-theory constraints on four-, five-, and six-point color-ordered amplitudes at all loop orders*: " Color-ordered amplitudes for the scattering of n particles in the adjoint representation of SU(N) gauge theory satisfy constraints that arise from group theory alone. These constraints break into subsets associated with irreducible representations of the symmetric group S_n, which allows them to be presented in a compact and natural way. ”

2017-05-27 Predrag Kol and Shir [48, 49] *Color structures and permutations* has a useful overview of the literature in the introduction, and is a very interesting read overall.

We may permute (or re-label) the external legs in the expression for a color structure and thereby obtain another color structure. This means that the space of color structures is a representation of S_n, the group of permutations. A natural question is to characterize this representation including its character and its decomposition into irreducible representations (irreps).

The decomposition of color structures into irreps was suggested by Zeppenfeld [81].

The space of tree-level color structures TCS_n of dimension

$$\dim(TCS_n) = (n-2)! \tag{1.18}$$

is the vector space generated by all diagrams with n external legs and an oriented cubic vertex, which are connected and without loops (trees), where diagrams which differ by the Jacobi identity are to be identified.

The f-based and t-based color structures are related by celebrated Kleiss-Kuijf [47] relations (rederived in this paper).

The original problem, that of capturing symmetries of the partial amplitudes which originate with those of the color structures, is now formulated as the problem of obtaining the S_n character of the space of color structures. It turns out that (at least at tree level) this problem was fully solved in the mathematics literature by Getzler and M. M. Kapranov [40].

The free Lie algebra over some set A, denoted by $L(A)$, is the Lie algebra generated by A with no further relations apart for antisymmetry and the Jacobi identity which are mandated by definition.

Self duality under Young conjugation: for some n values TCS_n is self-dual under Young conjugation, namely under the interchange of rows and columns in the Young diagrams

2017-05-27 Predrag Getzler and Kapranov [40] *Modular operads*: “ We develop a ‘higher genus’ analogue of operads, which we call modular operads, in which graphs replace trees in the definition. We study a functor F on the category of modular operads, the Feynman transform, which generalizes Kontsevich’s graph complexes and also the bar construction for operads. We calculate the Euler characteristic of the Feynman transform, using the theory of symmetric functions: our formula is modelled on Wick’s theorem. ”

2017-05-27 Predrag Maltoni *et al.* [58] *Color-flow decomposition of QCD amplitudes*

The *color-flow* decomposition is based on treating the $SU(N)$ gluon field as an $N \times N$ matrix. (PC: I think that is what I actually do.) It has several nice features. First, a similar decomposition exists for all multiparton amplitudes, like the fundamental-representation decomposition. Second, the color-flow decomposition allows for a very efficient calculation of multiparton amplitudes. Third, it is a very natural way to decompose a QCD amplitude. As the name suggests, it is based on the flow of color, so the decomposition has a simple physical interpretation.

To calculate the amplitude, one orders the gluons clockwise, and draws color-flow lines, with color flowing counterclockwise, connecting adjacent gluons. One then deforms the color-flow lines in all possible ways to form the Feynman diagrams that contribute to this partial amplitude. The Feynman diagrams that contribute to a partial amplitude are planar. This is not due to an expansion in $1/N$; the partial amplitudes are exact. They note (see their Table 1) that the number of Feynman diagrams contributing to an n-gluon partial amplitude grows as $\approx 3 \cdot 8^n$. In contrast, the number of Feynman diagrams contributing to the full amplitude grows factorially, as $\approx (2n)!$.

2012-07-13, 2017-05-21 Aoyama *et al.* [6, 7], *Tenth-order electron anomalous magnetic moment: Contribution of diagrams without closed lepton loops* summarize the results of their numerical work on the complete determination of the 10th-order contribution to the anomalous magnetic moment.

Their quenched set V gives
$A_1^{(10)}[V] = 8.726(336)$, while I predict
$A_1^{(10)}[V] \approx 3/2$

2017-04-30 Predrag to Stefano Laporta <stefano.laporta@bo.infn.it>:

Thanks for the listings of gauge sets, and of course, for your whole amazing project. While in detail my 1977 guesses are wrong, the overall finiteness conjecture still looks good - it's crazy how small all these individual contributions are.

The reason I got into a collaboration with Kinoshita (nominally, Tung-Mow Yan was my adviser) is that he gave me the 3-loop ladder diagram (at 4-loops, it would belong to your gauge set (6) = $(4, 0, 0)$) to evaluate, and I derived a compact formula for the parametric integrand of n-loops ladder diagram. So maybe gauge sets of type (6) = $(4, 0, 0)$ are the easiest to evaluate?

Is evaluating gauge set $(5, 0, 0)$ with 5 loops be feasible? If that turns out to be +1/2, that would be sweet.

2017-05-05 Laporta to Predrag

To my knowledge, there are no ideas how to bound the sizes of the gauge sets.

In the set (6) = $(4, 0, 0)$ the ladder is easy to evaluate, but the others are not. The evaluation of the set $(5, 0, 0)$ with five loops would be a real challenge.

Among the sets (1)-(6), the easiest to evaluate is the set (3) = $(1, 2, 1)$.

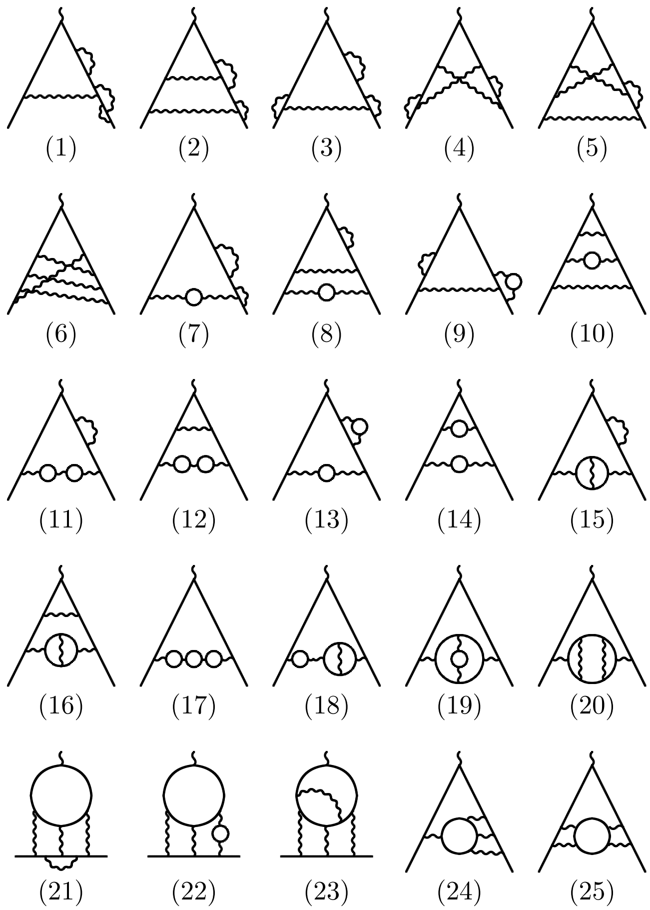


Figure 1.8: Examples of the $A_1^{(8)}$ 4-loop vertex diagrams belonging to the 25 gauge-invariant sets. The number indicates the gauge-invariant set to which the diagram belongs. Gauge sets (1) = $(1, 3, 0)$, (2) = $(2, 2, 0)$, (3) = $(1, 2, 1)$, (4) = $(2, 1, 1)$, (5) = $(3, 1, 0)$, (6) = $(4, 0, 0)$, In the case of the sets 1-16, 24 and 25, remaining diagrams in the set can be obtained by permuting separately the vertices on the left and right side of the main electron line, and considering also the mirror images of the diagrams. From Laporta [53].

1	(1,3,0)	- 1.971075616835818943645699655337264406980	- 1/2	
2	(2,2,0)	- 0.142487379799872157235945291684857370994	1/2	(!)
3	(1,2,1)	- 0.621921063535072522104091223479317643540	- 1/2	
4	(2,1,1)	1.086698394475818687601961404690600972373	1/2	
5	(3,1,0)	- 1.040542410012582012539438620994249955094	- 1/2	
6	(4,0,0)	0.512462047967986870479954030909194465565	1/2	
7		0.690448347591261501528101600354802517732		
8		- 0.056336090170533315910959439910250595939		
9		0.409217028479188586590553833614638435425		
10		0.374357934811899949081953855414943578759		
11		- 0.091305840068696773426479566945788826481		
12		0.017853686549808578110691748056565649168		
13		- 0.034179376078562729210191880996726218580		
14		0.006504148381814640990365761897425802288		
15		- 0.572471862194781916152750849945181037311		
16		0.151989599685819639625280516106513042070		
17		0.000876865858889990697913748939713726165		
18		0.015325282902013380844497471345160318673		
19		0.011130913987517388830956500920570148123		
20		0.049513202559526235110472234651204851710		
21		- 1.138822876459974505563154431181111707424		
22		0.598842072031421820464649513201747727836		
23		0.822284485811034346719894048799598422606		
24		- 0.872657392077131517978401982381415610384		
25		- 0.117949868787420797062780493486346339829		

Table 1.2: Contribution to $A_1^{(8)}$ of the 25 gauge-invariant sets of figure 1.8, as reported by Laporta [53].

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I've now searched through the literature. There seem to be 2 approaches that might be relevant to establishing bounds on, and actually computation of gauge-invariant sets (I am ignoring various $N = 2$ and $N = 4$ supersymmetric models), Hopf algebraic approach of Kreimer and collaborators, and the worldline formalism pursued by Schubert and collaborators. It operates with N-photon propagators, the ingredient that defines the quenched gauge-invariant sets for $(g - 2)$. The summary of this literature is in sect. 1.2.

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