group theory

birdtracks, Lie's, and exceptional groups

a blog

Predrag Cvitanović

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Chapter 2

Exotic symplectic holonomies

On 2016-10-24 Bernard Julia wrote to Predrag:

> I recently found a niche for your E_7 family [13]. Please have a look at Bryant [6] and check out table 2.1 in his review of Berger [1, 2] nonmetric holonomies. You will find among the exotic symplectics E_7, D_6, A_5 (where everybody agrees), and then your C_3, your $3\,A_1$ (as a possible solution) and your A_1.
>
> What do you make of this, except that the symplecticity is just right... ?
>
> I copy this to Pierre <pierre.ramond@gmail.com> who may appreciate.

My 1975 construction [11, 12, 14] of the E_7 row of the Magic Triangle was inspired by Brown's [3] observation that defining representation of E_7 has a symmetric quartic invariant (my construction, however, is a stand-alone derivation that owes nothing to Brown and Freudethal). In my *Negative dimensions and E_7 symmetry* [13] (as well as in Chapter 20. *E_7 family and its negative-dimensional cousins* of the birdtracks book [14]) I the negative-dimension relation through which E_7 emerges as a negative-dimensional relative of $SO(4)$. As nobody understood my diagrammatic notation, in 1980 I rewrote my (very compact in diagrammatic notation) derivation of the E_7 family in the more standard and lengthy index notation [13]. That also ended up blowing in the wind.

My reading Bryant [6] review paper *Recent advances in the theory of holonomy*, arXiv:math/9910059
[arXiv now includes blog links: wiki/Holonomy and mathoverflow]:

Exotic Holonomies. The full list of exotic holonomies (so called because the were omitted from the initial list of Berger [2]) was compiled by Chi, Merkulov, Schwachhöfer, and Bryant [4, 5, 10]; see, in particular, the tables in Merkulov and Schwachhöfer [16, 17], and table 2.1 here.

The first two examples in table 2.1, each with a torsion-free connection, were analyzed in ref. [2]. The study [4, 5] of the moduli space of rational curves on a complex surface with normal bundle $\mathcal{O}(3)$ turned up the next two entries in table 2.1 that were

IV. Exotic Symplectic Holonomies H	$\mathfrak{m}$	E_7 **row** algebra	rep.
		$\mathrm{U}(1)$	$\mathbb{R}^2$
$\mathrm{SL}(2,\mathbb{R})$ $\mathrm{SL}(2,\mathbb{C})$	$\mathbb{R}^4 \simeq \mathsf{S}^3(\mathbb{R}^2)$ $\mathbb{C}^4 \simeq \mathsf{S}^3(\mathbb{C}^2)$	A_1	$\mathbb{R}^4$
$\mathrm{SL}(2,\mathbb{R})\cdot\mathrm{SO}(p,q)$ [a] $\mathrm{SL}(2,\mathbb{C})\cdot\mathrm{SO}(n,\mathbb{C})$ [b] $\mathrm{Sp}(1)\cdot\mathrm{SO}(n,\mathbb{H})$ [c]	$\mathbb{R}^2 \otimes \mathbb{R}^{p+q}$ $\mathbb{C}^2 \otimes \mathbb{C}^n$ $\mathbb{H}^n$	$3\,\mathrm{A}_1$	$\mathbb{R}^8$
$\mathrm{Sp}(3,\mathbb{R})$ $\mathrm{Sp}(3,\mathbb{C})$	$\mathbb{R}^{14} \simeq \Lambda^3_0(\mathbb{R}^6)$ $\mathbb{C}^{14} \simeq \Lambda^3_0(\mathbb{C}^6)$	C_3	$\mathbb{R}^{14}$
$\mathrm{SL}(6,\mathbb{R})$ $\mathrm{SU}(1,5)$ $\mathrm{SU}(3,3)$ $\mathrm{SL}(6,\mathbb{C})$	$\mathbb{R}^{20} \simeq \Lambda^3(\mathbb{R}^6)$ $\mathbb{R}^{20} \simeq \Lambda^3(\mathbb{C}^6)^{\mathbb{R}}$ $\mathbb{R}^{20} \simeq \Lambda^3(\mathbb{C}^6)^{\mathbb{R}}$ $\mathbb{C}^{20} \simeq \Lambda^3(\mathbb{C}^6)$	A_5	$\mathbb{R}^{20}$
$\mathrm{Spin}(2,10)$ $\mathrm{Spin}(6,6)$ $\mathrm{Spin}(6,\mathbb{H})$ $\mathrm{Spin}(12,\mathbb{C})$	$\mathbb{R}^{32}$ $\mathbb{R}^{32}$ $\mathbb{R}^{32}$ $\mathbb{C}^{32}$	D_6	$\mathbb{R}^{32}$
E_7^5 E_7^7 $\mathrm{E}_7^{\mathbb{C}}$	$\mathbb{R}^{56}$ $\mathbb{R}^{56}$ $\mathbb{C}^{56}$	E_7	$\mathbb{R}^{56}$

Table 2.1: (left) Bryant Table IV [6]. The H are subgroups of $\mathrm{Sp}(S) \subset \mathrm{GL}(\mathfrak{m})$ for a nondegenerate skew symmetric bilinear form S on $\mathfrak{m}$, and the corresponding H-structures have an underlying symplectic structure. The restrictions are (a) $p+q \geq 3$ (for irreducibility), (b) $n \geq 3$ (for irreducibility), and (c) $n \geq 2$ (to be nonmetric). (right) The E_7 row of the Magic Triangle [11, 12, 14] is the family of all semi-simple Lie algebras [13] that admit an antisymmetric quadratic invariant $f_{\mu\nu} = -f_{\nu\mu}$ (i.e., they are subgroups of $\mathrm{Sp}(n), n$ even), together with a symmetric quartic invariant $d_{\mu\nu\rho\delta} = d_{\nu\mu\rho\delta} = d_{\mu\rho\nu\delta} = d_{\mu\nu\delta\rho}$.

omitted from Berger's nonmetric list [5]. The 'exotic' refers to any nonmetric subgroup $H \subset \mathrm{GL}(\mathfrak{m})$ that satisfies Berger's criteria but that does not appear on Berger's original nonmetric list. The construction [5] uncovered a number of unexpected identities. Chi, Merkulov, and Schwachhöfer [10] found other exotic symplectic examples, and noticed that the reconstruction technique generalized in the context of Poisson geometry (i.e., Poisson bracket, or a symplectic skew-symmetric invariant).

I am not able to decode any of the "exotic holonomies" literature that I have looked at. Questions to experts:

(quartic invariant) Is there is a quartic invariant someplace in the Poisson manifolds derivation of the exotic symplectic holonomies?

(ternary algebra) If not that, can you identify a class of ternary algebras (Freudenthal triple systems), see Faulkner [15] and Yamaguti and Asano [20].

($3\,A_1$) Is my 8-dimensional representation of semi-simple $3\,A_1$ consistent with your family of possible symmetries? I have no wiggle room - it must be $\mathbb{R}^8$. Would there be something special about that case?

($\mathrm{U}(1)$) Why don't you have $\mathrm{U}(1)$, $\mathbb{R}^2$ in your list? Too trivial for you?

(supersymmetry) Where is the list of supersymmetric partners [17] to table 2.1? Is $\mathrm{SO}(4) \simeq \mathrm{SO}(3) \oplus \mathrm{SO}(3) \simeq A_1 \oplus A_1$ in such list?

(exceptional magic) Would it make you happy to learn that your 'exotic' is a synonym for Cartan-Kylling's 'exceptional' [14]?

2.1 Holonomies blog

2016-12-03 Predrag Merkulov and Schwachhöfer [17] note " Another by-product result is the striking supersymmetry property of their real forms which occur as holonomies of torsion-free affine connections. With any symplectic Lie algebra $g \subset \mathrm{Sp}(V)$, dim $g = m$, dim $V = 2n$, one may associate naturally an $(m|2n)$-dimensional supermanifold "

Supersymmetry was actually the motivation for construction of my my E_7 family [13], but I have so far not been able to identify my $\mathrm{SO}(4)$ family in the above holonomies literature. It might be in this paper [17], for all I know...

2016-12-03 Predrag Robert Bryant is not only the current President of the American Mathematical Society. American Mathematical Society, he is also a NAS, and Landsberg's thesis adviser, etc. He explains the idea of "holonomy" by a basketball.

2016-12-07 Robert Bryant I haven't been thinking about these things for a very long time, so your questions probably would be better going to Chi, Merkulov, and/or Schwachhofer. I'll tell you what I know about your questions, but it's not much:

Quartic invariant: There are quartic polynomial invariants associated to these things in all cases.

For example, $\mathrm{Spin}(2,10)$ (up to a finite extension) acting on $\mathbb{R}^{32}$ is defined as the stabilizer of a certain quartic polynomial in 32 variables, and, of course, it also stabilizes a symplectic form on $\mathbb{R}^{32}$. Similarly E_7 (up to a finite extension) acting on $\mathbb{R}^{56}$ is defined by the quartic polynomial originally defined by Cartan.

The lower dimensions are classical:

For A_1 acting on $\mathbb{R}^4 = S^3(\mathbb{R}^2)$, which is the homogeneous cubics in two variables, the quartic is the classical discriminant of a cubic form in two variables.

For $3\,A_1$ acting on $\mathbb{R}^8 = \mathbb{R}^2 \otimes \mathbb{R}^2 \otimes \mathbb{R}^2$, the quartic invariant was discovered by Cayley in 1843 and is known as Cayley's hyperdeterminant.

In the case of A_5 acting on $\mathbb{R}^{20} = \Lambda^3(\mathbb{R}^6)$, there is a well-known quartic invariant polynomial, and A_5 is the stabilizer of that polynomial.

The C_3 case turns out to be the restriction of that quartic polynomial to the 14-dimensional invariant subspace of $\mathbb{R}^{20}$ that C_3 preserves.

You can find details on this material by searching for the term 'prehomogeneous vector spaces' in the literature.

Ternary algebra: I know nothing about this.

$3\,A_1$: I don't know of anything particularly special about this that makes it somehow nicer than the other cases. I'm not sure what your 'wriggle room' means. There are other real forms, besides the Cayley hyperdeterminant case (which is the natural choice for 'split form'); there are 4 distinct real forms altogether in the (real) 8-dimensional case.

$U(1)$: Yes, this case is 'trivial' because it actually preserves a quadratic form (whose square is, of course, the invariant quartic form), and so it has already been accounted for in the metric case.

Supersymmetry: I don't know anything about this.

Exceptional Magic: I think 'synonym' is too strong a word, here. Clearly, there is a connection of the exotic symplectic holonomies with the exceptional groups, but it's not one-to-one. You can find more about this in the 'Affine Holonomy' section (near the end) of the Wikipedia page https://en.wikipedia.org/wiki/Holonomy. There is also a paper by Cahen and Schwachhöfer [9] in which they discuss this at length and its relation with Poisson structures.

That's all that I know.

Yours,

Robert Bryant

2016-12-08 Predrag Recheck

Wybourne [18] *$SU_6 \times SU_{3c}$ scalars in E_7 irreps*

Butler, Haase and Wybourne [8] *Calculation of $3jm$ factors and the matrix elements of E_7 group generators*

Butler, Haase and Wybourne [7] *Calculation of $6j$ symbols for the exceptional group E_7*

Wybourne [19] *Enumeration of group invariant quartic polynomials in Higgs scalar fields*

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