

is QED finite?

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New Trends in First Quantisation
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overview

- 1 what this is about
- 2 QED finiteness conjecture
- 3 bye bye, Feynman diagrams
- 4 worldline approach

4- and 5-loop contributions to the anomaly

in 2017 Laporta completed
the twenty-year project of computing analytically
the 891 4-loop electron magnetic moment diagrams¹

here we shall discuss **only the quenched set**, i.e., the
contributions of diagrams with no lepton loops

the 4- and 5-loop contributions to the anomaly $a = \frac{1}{2}(g - 2)$:

$$\begin{aligned}a^{(8)} &= -2.176866027739540077443259355895893938670 \\a^{(10)} &= 7.606(192) \dots \quad \text{Aoyama *et al.* 2018}^2 \\&= 6.782(113) \dots \quad \text{Volkov 2019}^3\end{aligned}$$

awesome, heroic achievements

¹Laporta17.

²AoKiNi18.

³Volkov19.

request #1

please always do look at the
quenched set separately

renormalons schnormalons, they will go gently into that good night :)

notation : electron-photon vertex Γ_μ

out-, in- electron momenta : $p_\pm = p \mp q/2$

evaluated on the mass shell $p_\pm^2 = m^2 = 1$

Dirac, Pauli form factors $F_1(q^2)$ and $F_2(q^2)$:

$$\bar{u}(p_+) \Gamma_\mu(p, q) u(p_-) = \bar{u}(p_+) \left\{ F_1(q^2) \gamma_\mu - \frac{F_2(q^2)}{2m} \sigma_{\mu\nu} q^\nu \right\} u(p_-),$$

spinors $\bar{u}(p_+)$ and $u(p_-)$ satisfy the Dirac equation

$$\bar{u}(p_+) \not{p}_+ = \bar{u}(p_+) m, \quad \not{p}_- u(p_-) = m u(p_-).$$

notation : renormalized vertex

$Z_1 = 1 + L$: vertex renormalization constant

Z_2 : electron wave function renormalization constant

Ward identity : $Z_1 = Z_2$.

renormalized vertex

by definition, the renormalized charge form factor $\tilde{F}_1(0) = 1$

The vertex renormalization constant L is given by the on-shell value of the unrenormalized charge form factor⁴

$$1 + L = F_1(0) = \frac{1}{4} \text{tr} [(\not{p} + 1) \not{p}^\nu \Gamma^\nu]_{q=0}$$

(the electron mass set to $m = 1$ throughout)

⁴BroSul67.

the anomalous magnetic moment of an electron

$$a = (g - 2)/2$$

is given by the static limit of the magnetic form factor

$a = \tilde{F}_2(0) = M/(1 + L)$, where⁵

$$M = \lim_{q \rightarrow 0} \frac{1}{4q^2} \text{tr} \left\{ \left[\gamma^\nu p^2 - (1 + q^2/2)p^\nu \right] (\not{p}_+ + 1) \Gamma_\nu (\not{p}_- + 1) \right\}$$

perturbative expansion for the magnetic moment anomaly

$$a = \frac{M(\alpha)}{1 + L(\alpha)} = \sum_{n=1}^{\infty} a^{(2n)} \left(\frac{\alpha}{\pi}\right)^n,$$

where $1 + L = F_1(0)$, $M = F_2(0)$ are computed from the unrenormalized proper vertex, given by the sum of all one-particle irreducible electron-electron-photon vertex diagrams with internal photons and electron mass counterterms. Expanding M and L we have

$$a^{(2)} = M^{(2)}$$

$$a^{(4)} = M^{(4)} - L^{(2)}M^{(2)}$$

$$a^{(6)} = M^{(6)} - L^{(2)}M^{(4)} - (L^{(4)} - (L^{(2)})^2)M^{(2)}$$

look at physical, mass-shell
observables

4- and 5-loop contributions to the anomaly

4-loop : 518 diagrams

5-loop : 6 354 diagrams

each of size $\approx \pm 10$, add them up:

$$a^{(2)} = +0.5$$

$$a^{(4)} = -0.33$$

$$a^{(6)} = +0.92$$

$$a^{(8)} = -2.18$$

$$a^{(10)} = +7.60 \quad (\text{not random graphs sum } \approx \pm 800 !!!)$$

Q : why are these numbers **so** insanely **small**?

Q : what is the **sign** of n th contribution?

gauge cancellations?

as a prelude, you might enjoy the Dunne and Schubert⁶
historical review of ideas about the QED
perturbation series

they note:

a point which remains poorly understood

“is the influence of gauge cancellations on the divergence
structure of a gauge theory.”

⁶DunSch06.

gauge invariance induced cancellations

If gauge invariance of QED guarantees that all UV and on-mass shell IR divergences cancel, could it be that it also enforces cancellations among the finite parts of contributions of different Feynman graphs?

gauge invariance

A gauge change generates a k^μ term in a photon propagator, and that affects a photon-electron vertex in a very simple way.

from $k = (\not{p} + \not{k} + m) - (\not{p} + m)$ it follows that

$$\frac{1}{\not{p} + \not{k} - m} \not{k} \frac{1}{\not{p} - m} = \frac{1}{\not{p} - m} - \frac{1}{\not{p} + \not{k} - m},$$

neighbouring photon insertions cancel, leading to

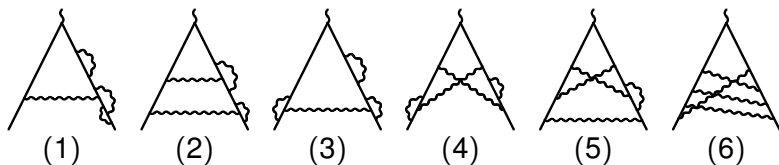
gauge invariant sets

gauge sets

A gauge set kmm' consists of all 1-particle irreducible vertex diagrams, with k photons crossing the external vertex (cross-photons) and m [m'] photons originating and terminating on the incoming [outgoing] electron leg (leg-photons)

$$a = \frac{1}{2}(g - 2) = \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} \sum_{m'=0}^{\infty} a_{kmm'} \left(\frac{\alpha}{\pi}\right)^{k+m+m'}.$$

representative 4-loop gauge set graphs



remaining diagrams : permute vertices, mirror diagrams

gauge set	kmm'	Laporta	approx
(1)	130	- 1.9710	- 2
(2)	220	- 0.1424	0
(3)	121	- 0.6219	- 1/2
(4)	211	1.0867	1
(5)	310	- 1.0405	- 1
(6)	400	0.5125	1/2

Laporta⁷ gauge-set contributions $a_{kmm'}^{(8)}$; my approximations

Signs are right, and the sets are close to multiples of 1/2

⁷Laporta17.

there are very few gauge sets

Order $2n$	Vertex graphs Γ_{2n}	Gauge sets G_{2n}	Anomaly $a(2n)$
2	1	1	1/2
4	6	2	0
6	50	4	1
8	518	6	0
10	6354	9	3/2
12	89 782	12	0
14	1 429 480	16	2

Comparison of the number of vertex diagrams without fermion loops, gauge sets, and the “gauge-set approximation”⁸ for the magnetic moment in $2n$ th order.

⁸Cvit77b.

Feynman's challenge, 12th Solvay Conference

Is there any method of computing the anomalous moment of the electron which, on first approximation, gives a fair approximation to the α term and a crude one to α^2 ; and when improved, increases the accuracy of the α^2 term, yielding a rough estimate to α^3 and beyond?⁹

⁹Feynman62.

the unreasonable smallness of gauge sets














When the diagrams are grouped into gauge sets, a surprising thing happens; while the finite part of each Feynman diagram is of order of 10 to 100, and each one is UV and IR divergent, for $n = 2, 3$ every gauge set adds up to approximately

$$\pm \frac{1}{2} \left(\frac{\alpha}{\pi} \right)^n ,$$

with the sign given by a simple empirical rule

$$a_{kmm'} = (-1)^{m+m'} \frac{1}{2}$$

1977 (slightly wrong) four-loop prediction

$2n$		anomaly
2	 $\frac{1}{2}$	5
4	 $-\frac{1}{2} (-65)$  $\frac{1}{2} (31)$	0 (-33)
6	 $\frac{1}{2} (56)$  $-\frac{1}{2} (-47)$  $\frac{1}{2} (43)$  $\frac{1}{2} (44)$	1 (93)
8	 $-\frac{1}{2}$  $\frac{1}{2}$  $-\frac{1}{2}$  $\frac{1}{2}$  $-\frac{1}{2}$  $\frac{1}{2}$	0 (?)

new "prediction" : $a^{(8)} = -2$, rather than 0.

2019 five-loop status

$2n$	(k, m, m')					anomaly
2	$(1, 0, 0)$ $\frac{1}{2}$					$\frac{1}{2}$
4	$(1, 1, 0)$ $-\frac{1}{2} (-.65)$	$(2, 0, 0)$ $\frac{1}{2} (.31)$				0 (-.33)
6	$(1, 2, 0)$ $\frac{1}{2} (.56)$	$(2, 1, 0)$ $-\frac{1}{2} (-.47)$	$(3, 0, 0)$ $\frac{1}{2} (.44)$			1 (.93)
8	$(1, 3, 0)$ $-\frac{1}{2} \cdot 4 (-1.97)$	$(2, 2, 0)$ $\frac{1}{2} \cdot 0 (-0.14)$	$(3, 1, 0)$ $-\frac{1}{2} \cdot 2 (-1.04)$	$(4, 0, 0)$ $\frac{1}{2} (.51)$		0 (-2.17)
10	$(1, 4, 0)$ $\frac{1}{2} \cdot 12 (6.2)$	$(2, 3, 0)$ $-\frac{1}{2} (-0.72)$	$(3, 2, 0)$ $\frac{1}{2} \cdot 0 (-0.40)$	$(4, 1, 0)$ $-\frac{1}{2} \cdot 2 (-1.02)$	$(5, 0, 0)$ $\frac{1}{2} \cdot 2 (1.09)$	$\frac{3}{2} \cdot 4 (6.78)$
	$(1, 3, 1)$ $\frac{1}{2} (0.90)$	$(2, 2, 1)$ $-\frac{1}{2} \cdot 4 (-2.16)$	$(3, 1, 1)$ $\frac{1}{2} \cdot 5 (2.62)$			
	$(1, 2, 2)$ $\frac{1}{2} (0.30)$					

gauge-set (k, m, m')

[naive ansatz $\pm \frac{1}{2}$] · [integer] \approx [(· · ·) Volkov 2019 numerical value]

an example of (slightly wrong) gauge-set approximation

With prediction $a_{kmm'} = (-1)^{m+m'}/2$, the “zeroth” order estimate of the electron magnetic moment anomaly is given by the “gauge-set approximation,” convergent and summable to all orders

$$a = \frac{1}{2}(g - 2) = \frac{1}{2} \frac{\alpha}{\pi} \frac{1}{\left(1 - \left(\frac{\alpha}{\pi}\right)^2\right)^2} + \text{“corrections”}.$$

gauge invariance matters

most colleagues believe that in 1952 Dyson¹⁰ had shown that the QED perturbation expansion is an asymptotic series (for a discussion, see Dunne and Schubert¹¹), in the sense that the n -th order contribution should be exploding combinatorially

$$\frac{1}{2}(g-2) \approx \cdots + n^n \left(\frac{\alpha}{\pi}\right)^n + \cdots,$$

contrast with my estimate

$$\frac{1}{2}(g-2) \approx \cdots + \frac{n}{2} \left(\frac{\alpha}{\pi}\right)^{2n} + \cdots.$$

hence “QED is finite” claim

¹⁰Dyson52.

¹¹DunSch06, HuTrSc17a.

request #4 : prove that quenched QED is finite

any bound on a gauge set,
exponential or slower, will do the
trick!

so far facts ; next, speculation with Edwards and Schubert

- 1 QED finiteness conjecture
- 2 bye bye, Feynman diagrams
- 3 worldline approach

bye bye, Feynman diagrams

it's been a good ride, but there are way too many of you

a fun fact

the idea of how to avoid Feynman diagrams can be traced to 1950 Feynman paper¹², though it took a long time for it to gain traction

by the time I explained the gauge set conjecture to him in 1975, Feynman had forgotten all about it

¹²Feynman50.

worldline path integral for the free scalar propagator

propagator for the Euclidean Klein-Gordon equation¹³ is

$$D_0(x, x') = \langle x | \frac{1}{-\square + m^2} | x' \rangle ,$$

exponentiate the denominator

$$D_0(x, x') = \int_0^\infty dT e^{-m^2 T} \langle x | e^{-T(-\square)} | x' \rangle ,$$

replace the the D -dimensional Laplacian by a path integral to obtain

$$D_0(x, x') = \int_0^\infty dT e^{-m^2 T} \int_{x(0)=x'}^{x(T)=x} \mathcal{D}x(\tau) e^{-\int_0^T d\tau \frac{1}{4} \dot{x}^2} ,$$

¹³Schubert12.

worldline formula for charged propagator

that emits and reabsorbs N photons as it propagates

Adding the QED interaction terms leads to the Feynman's worldline path integral representation¹⁴ of the charged scalar propagator in the presence of a background field $A(x)$,

$$D(x, x') = \int_0^\infty dT e^{-m^2 T} \int_{x(0)=x'}^{x(T)=x} \mathcal{D}x(\tau) e^{-S_0 - S_e - S_i},$$

where the suffix (0) indicates the free propagation

$$S_0 = \int_0^T d\tau \frac{1}{4} \dot{x}^2,$$

(e) is the interaction of the charged scalar with the external field

$$S_e = -ie \int_0^T d\tau \dot{x}^\mu A_\mu(x(\tau)),$$

¹⁴Feynman50.

worldline formula for N -photon propagator

and (i) are the virtual photons exchanged along the charged particle's trajectory

$$S_i = \frac{e^2}{2} \int_0^T d\tau_1 \int_0^T d\tau_2 \dot{x}_1^\mu D_{\mu\nu}(x_1 - x_2) \dot{x}_2^\nu,$$

where $D_{\mu\nu}$ is the x -space photon propagator.

The object of great interest to us is the internal virtual photons term

$$\int_{x(0)=x'}^{x(T)=x} \mathcal{D}x(\tau) e^{-S_i} = \int_{x(0)=x'}^{x(T)=x} \mathcal{D}x(\tau) e^{-\frac{e^2}{2} \int_0^T d\tau_1 \int_0^T d\tau_2 \dot{x}_1^\mu D_{\mu\nu}(x_1 - x_2) \dot{x}_2^\nu}$$

expanded perturbatively in α/π , this yields the usual $n!$ Feynman-parametric vertex diagrams

worldline formalism

however, the path integral is Gaussian in \dot{x}^μ , and if by integration by parts, \dot{x}^μ are eliminated in favor of x^μ , internal photons can be integrated over directly, prior to an expansion in $(\alpha/\pi)^n$, and one gets integrals in terms of

N -photon propagators

symmetrized sums over N photons

and not the usual Feynman graphs

each usual Feynman graph corresponds to one particular permutation of internal photon insertions, and from that comes the factorial growth in the number of graphs

worldline formalism

note : the worldline integrals are expressed in terms of N -photon propagators,
the central ingredient that defines the **gauge sets**

unlike the Feynman parameter integrals for individual vertex graphs, they are independent of the ordering of the momenta k_1, \dots, k_N ; the worldline formula contains all $\approx N!$ ways of attaching the N internal photons to the charged particle propagator

worldline representation combines
combinatorially many Feynman diagrams into

a **single** integral

worldline formalism

In QED the N -photon propagator formulation combines into one integral all Feynman graphs related by permutations of photon legs along fermion lines, that is, it yields a *single* integral for a gauge set kmm'

worldline formalism

worldline propagator

$N!$ Feynman graphs

$$\text{Worldline Propagator} = \text{Feynman Graph 1} + \text{Feynman Graph 2} + (\text{perms}) + \dots$$

master formula = sum of all symmetric insertions

"details" for spinor QED : see Edwards & Schubert

worldline formalism

$\Gamma^\mu(x_+, x_-)$ electron-external field-electron vertex

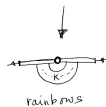
The diagram shows an equation between two Feynman diagrams. On the left is a circle with diagonal hatching, with an incoming arrow on the left and an outgoing arrow on the right, and a wavy line attached to the top. Below it is the text "1pI, renormalized". On the right is a summation over indices k, m, m' of a diagram. This diagram consists of a horizontal line with an incoming arrow on the left and an outgoing arrow on the right. A small circle is attached to the top of this line, with a wavy line extending upwards from it. Below the horizontal line are three loops labeled m , k , and m' . An arrow points from the text "magnetic moment vertex" to the small circle on the horizontal line. Below the loops is the text "worldline gauge sets".

$$1pI, \text{ renormalized} = \sum_{k, m, m'} \text{worldline gauge sets}$$

cute fact: not 1pI, but renormalized

worldline formalism

$$\begin{aligned}
 a^{(2)} &= a_{100} \quad [h\partial\tau \quad a_{010}, a_{001} = 0] & \# \\
 a^{(4)} &= a_{200} + a_{110} + a_{101} & 3 \\
 a^{(6)} &= a_{300} + a_{210} + a_{201} + a_{120} + a_{102} + a_{111} & 6 \\
 a^{(8)} &= a_{400} + a_{310} + a_{301} + a_{220} + a_{202} + a_{211} + a_{130} + a_{121} + a_{112} + a_{103} & 10 \\
 a^{(10)} &= a_{500} + a_{410} + a_{401} + a_{320} + a_{302} + a_{311} + a_{230} + a_{221} + a_{210} + a_{131} + a_{102} + a_{203} + a_{212} + a_{104} + a_{113} & 15
 \end{aligned}$$



worldline formalism

$$\text{Diagram 1} + \text{Diagram 2} = 0$$

Diagram 1: A horizontal line with an arrow pointing right. A vertical line segment extends downwards from the center of the line. A semi-circular arc is drawn below the horizontal line, centered on the vertical segment.

Diagram 2: A horizontal line with an arrow pointing right. A vertical line segment extends downwards from the center of the line. A semi-circular arc is drawn below the horizontal line, centered on the vertical segment. This diagram is identical to Diagram 1.

$$\{ \text{Diagram 3} + \text{Diagram 4} + \text{Diagram 5} \} + \text{Diagram 6} = 0$$

Diagram 3: A horizontal line with an arrow pointing right. A vertical line segment extends downwards from the center of the line. Two semi-circular arcs are drawn below the horizontal line, both centered on the vertical segment.

Diagram 4: A horizontal line with an arrow pointing right. A vertical line segment extends downwards from the center of the line. Two semi-circular arcs are drawn below the horizontal line, both centered on the vertical segment. This diagram is identical to Diagram 3.

Diagram 5: A horizontal line with an arrow pointing right. A vertical line segment extends downwards from the center of the line. Two semi-circular arcs are drawn below the horizontal line, both centered on the vertical segment. This diagram is identical to Diagram 3.

Diagram 6: A horizontal line with an arrow pointing right. A vertical line segment extends downwards from the center of the line. Two semi-circular arcs are drawn below the horizontal line, both centered on the vertical segment. This diagram is identical to Diagram 3.

worldline formalism

an example gauge set

$$a_{III} = \text{diagram}$$

$$= \left\{ \begin{array}{c} \text{diagram 1} \\ \text{diagram 2} \\ \text{diagram 3} \end{array} \right\} - \text{diagram 4} \left\{ \begin{array}{c} \text{diagram 5} \\ \text{diagram 6} \\ \text{diagram 7} \end{array} \right\}$$

worldline formalism

$$\begin{aligned}
 a_{111} = & \left\{ \begin{array}{l} \text{diagram 1} + \text{diagram 2} \\ \text{diagram 3} + \text{diagram 4} \end{array} \right\} M_{111} \\
 & \left\{ \begin{array}{l} \text{diagram 5} + \text{diagram 6} \\ \text{diagram 7} + \text{diagram 8} \end{array} \right\} \begin{array}{l} L_{010} M_{101} \\ M_{110} L_{001} \end{array} \\
 & + \left\{ \text{diagram 9} \right\} L_{010} M_{100} L_{00}
 \end{aligned}$$

The diagrams are worldline diagrams with vertices (circles) and external lines (horizontal lines). Red arrows labeled "Ward" indicate specific relationships or identities between the diagrams.

summary

- 1 a proof of the QED finiteness conjecture might be within reach
- 2 so might be methods for computing gauge invariant QFT sets without recourse to Feynman diagrams

you can download the current version of full notes here:

ChaosBook.org/~predrag/papers/finiteQED.pdf

The source code: GitHub.com/cvitanov/reducesymm/QFT