group theory

birdtracks, Lie's, and exceptional groups

a blog

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December 11, 2016

Contents

Chapter 1

Birdtracks

Enter here notes of general group-theoretic interest, perhaps for inclusion into revisions of birdtracks.eu. The notes are in
GitHub.com/cvitanov/reducesymm. If you download it
```
> cd dasgroup/
> pdflatex blog
```
For anything technical, please do not email me, but let me give you permissions to edit this GitHub repository. Then you can edit directly into the GitHub version, and let me know by email to
`dasgroup@mail.gatech.edu`
when you have `git push`ed something new to the server.

1.1 Notes on Alcock-Zeilinger and Weigert

2016-12-08 Predrag to Heribert: My "finiteness conjecture" is based on the observation that if internal photons are collected into gauge invariant sets, each set contributes a small, finite amount to what (off-mass shell) is usually assumed to be an asymptotic series. A gauge invariant set contributing to $(m + m' + k)$th order consists of m photon "strands" attached to the incoming electron, m' photon "strands" attached to the outgoing electron, and k photon "strands" crossing the external photon vertex. I do not have a direct method for evaluating a gauge set; instead, it takes a few years and a PhD thesis to evaluate these sets.

These photon "strands" have infrared divergences in individual diagrams, but as one is evaluating the magnetic moment, their sums do not.

Do you envision using Wilson lines formulation possibly accounting for clouds of soft photons crossing a QED vertex? Is there a direct calculation one could do without perturbatively expanding the Wilson lines to the usual individual multi-photon Feynman diagrams?

2016-12-02 Predrag My notes on Alcock-Zeilinger and Weigert are in GitHub.com/cvitanov/reducesymm/dasgroup/.

2016-11-30 Predrag Your Sect. 4.2 Proof of Theorem 3 *(generalized propagation rules)* seems not to need Young tableaux to work. A streamlined derivation might be to prove it for an individual transposition, the assemble whatever operator you need from transpositions?

2016-12-02 Predrag You might consider folding your Mathematica codes into `FormTracer` [10] see sect. 1.4, entry of **[2016-12-10]**.

2016-11-30 Predrag Skype session with Heribert and Judy, about their 3 birdtracking preprints. Got through the first one [2].

1.1.1 Physics motivation

(read up on Larry McLarren propaganda)

Applications of these tools in a QCD context where factorization invariably involves color singlet projections of Wilson line correlators, see several fields with possible applications:

Marquet and Weigert [22] *New observables to test the Color Glass Condensate beyond the large-N_c limit*

Weigert [33] *Non-global jet evolution at finite N_c*

Falcioni *et al.* [11] *Multiple gluon exchange webs*

Bomhof *et al.* [5] *The construction of gauge-links in arbitrary hard processes*

Since $\mathsf{SU}(N)$ is the gauge group of QCD, Young projection operators come into play through the theory of invariants, which relates the irreducible representations of $\mathsf{SU}(N)$ over $V^{\otimes n}$ to the Young tableaux of size n [13, 32]. The lack of Hermiticity of Young projection operators disqualifies them from the application to QCD calculations: for applications the operators need to be Hermitian (hence refs. [1, 16]) and all singlets are accounted for (hence ref. [4]).

Functional evolution evolution equation for QCD cross sections in high energy limit (Bjorken x less than 10^{-2}), as you push up energy make more and more soft gluons, making the system highly nonlinear. Parton model picture breaks down. BFKL pomeron equation is in Bjorken x, but distributions go exponentially large; Weigert contributed to formulating the nonlinear version.

Color Glass Condensate (within the standard model, only QCD does it):

The Balitsky-JIMWLK (Jalilian-Marian-Iancu-McLerran-Weigert-Leonidov-Kovner) is a tool to calculate the energy dependence of QCD observables at high energies. Gluon distribution in a proton as a function of impact parameter and rapidity can be described by the functional Langevin version of the JIMWLK renormalization group equation.

The meson production cross-sections contain four point correlators whose evolution follows from the JIMWLK framework. The four point correlators are here computed beyond the large-Nc limit.

N_c limit breaks gauge invariance, Weigert restores it minimally on the level of Wilson lines.

Needed for Wilson lines, in jet-like situations (scattering experiment jet observables) need to get all color singlets, SU(n).

Only thing that can happen are color rotations (that's where Wilson lines, driven by the soft gluons, come in), JIMWLK gives effective field theory for expectation values of these Wilson lines, globally colorless states.

Came from correlators of Wilson lines, needed to get all color singlets for $\mathsf{SU}(n)$.

The theory of invariants, relates the irreducible representations of $\mathsf{SU}(N)$ over $V^{\otimes n}$ to the Young tableaux of size n, see [13, 32] and other standard textbooks.

1.1.2 Simplification rules for birdtrack operators

Notes on Alcock-Zeilinger and Weigert [2].

They credit Young [35] for introducing Young projection operators, and refer to Tung [32] as the standard reference for them.

2 Notation, conventions and known results

The direction of the arrow on the index lines of a birdtrack encodes whether the line acts on the vector space V (arrow pointing from right to left) or its dual V^* (arrow pointing from left to right) [9]. In general birdtracks represent primitive invariants of $\mathsf{SU}(N)$ over a mixed algebra $V^{\otimes m} \otimes (V^*)^{\otimes n}$, where V^* is the dual vector space of V. Here only birdtracks acting on a space $V^{\otimes m}$ are considered (never on the dual). As all arrows go from right to left, they can be dropped.

The permutations of S_n are the *primitive invariants* [9] (of $\mathsf{SU}(N)$ over $V^{\otimes n}$). The real subalgebra of $\mathrm{Lin}\,(V^{\otimes n})$ that is spanned by these primitive invariants is denoted $\mathsf{API}\,(\mathsf{SU}(N), V^{\otimes n}) \subset \mathrm{Lin}\,(V^{\otimes n})$. API stands for "algebra of primitive invariants." One distinguishes

Semi-standard irregular tableaux Each number appears *at most once* within a tableau.

Young tableaux The boxes are top- and left-aligned. The numbers in the boxes to increase within each row from left to right and within each column from top to bottom.

Amputated tableaux The *column-amputated tableau* is obtained by removing all columns which do not overlap with the given row. The *row-amputated tableau* is obtained by removing all rows which do not overlap with the given column.

$A \subset B$ denotes that a *Hermitian* projection operator A projects onto a subspace completely contained in the image of a projection operator B, i.e., $A \subset B$ if and only if

$$A \cdot B = B \cdot A = A\,. \tag{1.1}$$

This simplification rule breaks down for the standard Young projection operators whenever they are not Hermitian.

The main result of this paper are the two kinds of simplification rules (cancellation or propagation) for birdtrack operators O comprised of symmetrizers and antisymmetrizers.

3 Cancellation rules

(1) Cancelation rules : Rules to determine whether certain symmetrizers or antisymmetrizers within an operator O are redundant, and thus can be *cancelled* from an operator. They can make a long expression significantly shorter, and thus easier to work with.

The two main cancellation rules are the *cancellation of wedged Young projection operators*, and the *cancellation of wedged ancestor-operators*.

3.1 Cancellation of wedged Young projection operators

Theorem 1. Outside S and A, inside a Young projection operator

The example that starts with Eq. (17), goes to the top of the page 10 motivates the general algorithm to remove inner symmetrizers.

The points 1. 2. and 3. are general, not just for this particular tableaux, hence:

Corollary 1. *Cancellation of wedged ancestor-operators*: can always get rid of an interior Young projection operator.

3.2 Cancellation of factors between bracketing sets

Cancelation rules : move sets of symmetrizers or antisymmetrizers through certain parts of the operator.

horizontal permutations of $\tilde{\Theta}$: $\mathbf{h}_{\tilde{\Theta}}$ is the subset of all permutations in S_n that only operate within the rows of $\tilde{\Theta}$; i.e. that do not swap numbers across rows.
vertical permutations of $\tilde{\Theta}$: $\mathbf{v}_{\tilde{\Theta}}$ is the subset of permutations in S_n that only operate within the columns of $\tilde{\Theta}$.

(I am too lazy to work through Tung's Lemma IV.5)

Corollary 2. *Cancellation of parts of the operator* One can always get rid of an inner Young projection operator. They lack explicit formula for the constants; so make sure non-zero, at the end evaluate the overall constant by other means (projection operator conditions).

Outer $\mathbf{A}_\Theta$ and $\mathbf{S}_\Theta$ belong to the same Young projection operator, see Eq. (23): there exists a (possibly vanishing) constant λ such that

$$\mathbf{S}_\Theta \; M \; \mathbf{A}_\Theta = \lambda \cdot Y_\Theta \,. \tag{1.2}$$

The rest of the section ensures that the constant is non-zero. It's quite of bit of work, I skipped it (unless they want me to work through some of it).

Dimensional zeroes

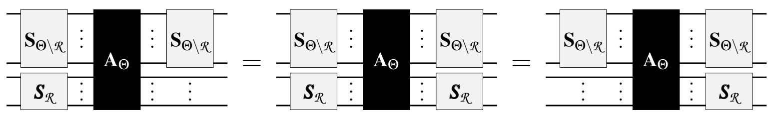
If any of the antisymmetrizers exceed the length N one has a *dimensional zero*. So one needs to assume N is high enough. It should work out once the calculation is done - every polynomial will have zeros for N to small for a given tableaux.

4 Propagation rules

rules when things commute

(2) Propagation rules : when it is possible to commute (*propagate*) a symmetrizer through an antisymmetrizer (or vice versa)? Then the cancellation rules might be applied, or features of a particular operator O, such as its Hermiticity can be made explicit. The answer:

$O =$



$= O^{\dagger}$.

It something I had used in the birdtracks.eu book in inchoate manner - they make it into a precise algorithm.

Example Eq. (45) is a bit tough

Example Eq. (48) through Eq. (50) is easy.

rest of the section is “Q. when you can commute?”

A. If can get rectangular tableaux, then it commutes

In particular, works also for semi-standard irregular tableaux,

Theorem 3 *(generalized propagation rules)*

Eq. (57) tells it

4.1 Proof of Theorem 2 (generalized propagation rules)

Proof is long and painful - I did not go through it. Should I?

5 Conclusion

Keppeler and Sjödahl [16] were the first to offer a simple method to construct Hermitian operators: their iteration is easy to understand, and the proofs of hermiticity are simple proofs. However, in practice, the algorithm is inefficient - the expression balloon quickly.

The methods of this paper are also recursive, but with the recursion cut down drastically. The gain is illustrated by Fig. 5.2 in the paper, here reproduced as figure 1.1.

1.1.3 Compact Hermitian Young projection operators

Notes on Alcock-Zeilinger and Weigert [1].

Young projection operators are (1) idempotent, (2) orthogonal and (3) complete. But, as the symmetrizers and antisymmetrizers comprising a given Young tableau do not necessarily commute, Young projection operators are in general not Hermitian.

Keppeler and Sjödahl [16] were first to construct Hermitian versions of Young projection operators in the birdtrack formalism, by an iterative algorithm. However, the KS-operators soon become unwieldy and thus impractical to work.

The construction algorithm presented here, based on the simplification rules of ref. [1], leads to drastically more compact and explicitly Hermitian expressions for the projection operators than the KS-algorithm [16]; an example is given in figure 1.1.

1.1 Notes on Alcock-Zeilinger and Weigert

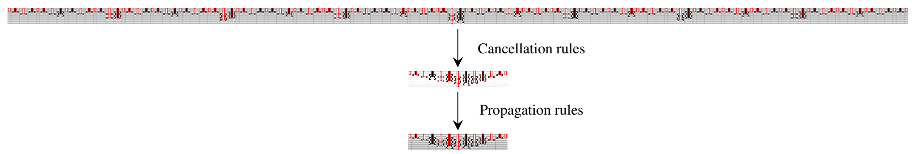


Figure 1.1: (top) A Hermitian birdtrack obtained by the iterative KS-algorithm [16]. Blow it up on the screen to see the details. (middle) The much shorter version obtained by application of the cancellation rules. (bottom) The explicitly symmetric (Hermitian) version achieved via the propagation rules.

Repeated here are most of the Keppeler and Sjödahl [16] *Hermitian Young projection operators*. Keppeler and Sjödahl used iterative methods, see ref. [1] bottom p. 18

Eq. (53) not obvious it is symmetric - [2] gives simplification rules, dramatic simplification, see Fig. 5.2

3.3 KS Construction principle for Hermitian Young projection operators

provides a direct route to bottom Fig. 5.2, paper proves that it really works

Young projection operators not being Hermitian has strange consequences. Eq. (12) not true, but for the Hermitian ones it is true.

1.1.4 Transition operators

Notes on Alcock-Zeilinger and Weigert [3].

The simplification rules of ref. [1] allow here a construction of transition operators between (Hermitian) Young projection operators corresponding to equivalent irreducible representations of $\mathsf{SU}(N)$, and an orthogonal basis for the algebra of invariants on $V^{\otimes m}$.

completes the picture, the full algebra of invariants

3 Young projection and transition operators

gives the counting argument that the number of primitive invariants equals the sum of diagonal operators and transition operators.

5.2 A full orthogonal basis for algebra of primitive invariants

write Clebsch, as in Eq. (71)

Eq. (73) transition operator between equivalent representations

they are unitary if restricted on the representations (top p 21)

together with the hermitian, they give you the full unitary basis

In Eq. (55a) algebra is decomposed into subalgebras, Eq. (55b) is as simple as can be.

Eq. (63) is non-Hermitian version

see and compare Fig. 2 (non Hermitian) and Fig. 3: (hermitian) same as birdtracks.eu, but without the transition operators.

Dimension of the algebra goes factorially, so algorithm works up to 8 or 9 (all algebra in Mathematica, up to 8 on the laptop).

1.1.5 Singlets

Notes on J. Alcock-Zeilinger and H. Weigert [4].

This paper says that these projection operators give you all singlets.

The orthogonal basis of ref. [3] is used to form a basis for the singlet states necessary to determine all color neutral Wilson line correlators. This has applications in QCD, such as ref. [22] and

Lappi *et al.* [21] *JIMWLK evolution of the odderon*

1.2 Notes on Keppeler and Sjödahl

2014-07-20 PC More birdtracking - a construction of orthogonal (Hermitian) projection operators:

Stefan Keppeler and Malin Sjödahl [16] *Hermitian Young projection operators*

Sjödahl [26, 27] *Tools for calculations in color space*, Malin.Sjodahl@thep.lu.se

and Keppeler's student Thorén [31].

1.3 Notes on Tai PhD thesis

Matthew Tai's 2014 PhD thesis [29, 30] *Family algebras and the isotypic components of $g \bigotimes g$* (PhD adviser Alexandre A. Kirillov [17, 18], of Institute for Information Transmission Problems, Russian Academy of Sciences) appears to supersede the Casimir and many other discussions of birdtracks.eu. My 2014-10-17 letter to Tai, mtai@math.upenn.edu:

Dear Matthew

Rumors of my death are exaggerated, so I always wonder about why nobody tells me anything about advances related to my work? Here you are, my best birdtracks student, and we have not even been introduced?

Anyway, I've started writing down some notes on your thesis in GitHub,
GitHub.com/cvitanov/reducesymm,

```
> cd dasgroup
> pdflatex blog
```

read Sect. *Notes on Tai PhD thesis*. For anything technical, please do not email me, but edit directly into the GitHub version, and let me know by email to
`dasgroup@mail.gatech.edu`
when you have `git push`ed something new to the server. Here are a few notes, from the first superficial reading. We can meet to discuss face to face anything any time on Skype or Google Hangouts.

1. Should I write in birdtracks.eu that chapter ? is superseded by your thesis?

2. With an eye on revising birdtracks.eu: which sections of the thesis in particular I should I study?

3. Do you have some clever way of generating your diagrams? Mine were all drawn by hand, using xfig. Do you want to contribute any of the scripts/programs to birdtracks.eu 'extras'?

4. why no link to birdtracks.eu?

5. ending lines with white dots rather than symmetrizers on external lines is clever. (but I would not know how to do that if there are internal symmetrizers and or several symmetrizer in the same diagram)

6. any errors, typos, etc. in birdtracks.eu I should fix?

7. I wonder where I got the 'Pfaffian' from (in your discussion of $D_r/SO(2k)$). I have no recollection - you happen to know a good reference? I should add Pfaffian to the index.

8. 'The degrees of the primitive Casimir operators' or 'exponents' are the (Betti numbers-1). Compare my *Table 7.1 Betti numbers for the simple Lie groups* with Tai *Table 10.1 Exponents for the exceptional Lie algebras*. "The name 'exponents' comes from the exponents of the hyperplane arrangement corresponding to the simple reflection planes of the Weyl Group of the Lie algebra. The exponents can also be considered topologically [...] also have representation-theoretic interpretations"

9. Can you contribute your thesis `*.bib` to birdtracks.eu?

10. for G_2, should I check Pieter Mostert unpublished paper?

11. for F_4, I should check 'Albert algebra' (related to Albert of birdtracks.eu ref. [70] C. W. Curtis [8] ...?)

12. My 'defining rep', 'fundamental 1-box Young tableaux representation' or 'defining n-dimensional rep' is 'reference representation' or 'standard representation'.

13.

14. typos

    p. 23 Clebsche vertices

1.4 Birdtracks blog

2000-02-01 PC This paper says my methods are not good enough: van Ritbergen, Schellekens and Vermaseren [25] *Group theory factors for Feynman diagrams*.

2000-09-09 Malin Sjödahl I have encountered a group theoretical problem, and I'm hoping that you might know of a solution to the problem (if a solution has been presented).

In QCD the external particles carry (anti-) quark and gluon indices that have to be summed, as we don't observe individual colors. The relevant color factor for the interference between two amplitudes M1 and M2 is thus

$$\sum q1...qn, qbar1...qbarn, g1...gm M1^{q1...qn,qbar1...qbarn,g1...gm} M2{*}^{q1...qn,qbar1...qbarn,g1...gm}.$$

This can be seen as a scalar product between two vectors M1 and M2; it is not hard to argue that the definition of a scalar product is fulfilled. This means that any amplitude can be written as a linear combination of basis states, and it would be nice to know of orthogonal bases for an arbitrary number of quarks and gluons. (I need to have a basis to do resummation, people doing loop calculations would benefit from having a basis when there are many external partons.)

If there are only quarks (and anti-quarks - an incoming anti-quark can be changed to an outgoing quark) such a basis can be constructed from the Young tableau projectors, for example as in figure 9.1 in your book.

The remaining problem is thus how to deal with the gluons, and my question is if you are aware of a systematic treatment of gluon indices to construct orthogonal states. Can one "recombine" the quark-indices of the Young tableaux to create orthogonal states? Or, do you know of an alternative strategy?

2013-06-20 PC to Malin /draft of Sep 10, 2010/
I'm - in manner of everybody now days - horribly behind, so when and if I answer is uncertain. But on the face of it the answer appears in this excellent book, which - if you are too poor to afford a coffee and a croissant in the Frankfurt airport - can also be downloaded for a click.

Try studying it, and if it looks like the answer is hidden there, I might be able to help - it's [draft of the letter stops here]

/continued June 20, 2013/
OMG - I have not forgotten, you have been on my guilt list for a long time, but hopefully time heals all wounds... Do you still want me to ponder your question, or is it all resolved, sealed, delivered, and published by now?

I do not know if it is of relevance to you, but we have a serious error in Appendix B, which I have not corrected yet in the book (Tony Kennedy's fix is the length of the book itself). There are also errata beyond the ones noted on the website that I have not listed yet...

I have started thinking again about how we fix gauges (I think now that using covariant gauges was a bad idea), but returning to QCD to implement my slicing feels so far beyond my reach... If there is something new and interesting happening in non-perturbative QCD, please do alert me :)

apologetically yours Predrag

2012-05-12 Stefan Keppeler <stefan.keppeler@gmail.com> Dear Predrag, over the last year I became a great fan of your birdtracking. Together with Malin Sjödahl I'm in the middle of writing up how to get decompositions like table 9.4 in your book for n-fold tensor products of the adjoint rep.

I think I found some typos in section 9.14, also within the rabbit-mouse birdtrack, of all equations ;-) I marked them in orange in the attached pdf.

I think the arrow in P_7 in table 9.4 should point in the other direction (or the sign in front of the second term be changed from minus to plus).

2016-12-10 PC Cyrol, A. K. and Mitter, M. and Strodthoff [10] *FormTracer - A Mathematica Tracing Package Using FORM* reviews the current software offerings. *FormTracer* includes different group tracing algorithms that are implemented in FORM [20, 25]. The most general algorithm is provided by the FORM color package [25] and allows to take traces of arbitrary simple compact Lie groups. Furthermore we include explicit tracing algorithms for the fundamental representation in SU(N) , SO(N) and Sp(N), adapted from routines 4 published with the color package [25] that use the Cvitanović algorithm [9] with additional support for partial traces. Finally, we include dedicated tracing algorithms for the fundamental representations in SU(2) and SU(3) that support partial traces, explicit numerical indices as well as transposed group generators. The use of explicit numerical indices requires to work in explicit representations. For SU(2) and SU(3) we choose generators proportional to Pauli and Gell-Mann matrices, respectively. Note that the fundamental SU(N) tracing algorithm also supports partial traces but does not guarantee the same degree of simplification as the specific SU(2) and SU(3) routines.

2013-02-22 PC Fomin and Pylyavskyy [12] *Tensor diagrams and cluster algebras*, arXiv:1210.1888, is a major orgy in birdtracking. Should study it some day.

2014-07-20 PC More birdtracking:

Gu and Jockers [15] *A note on colored HOMFLY polynomials for hyperbolic knots from WZW models*

Kol and Shir [19] *Color structures and permutations*.

2014-12-02 PC More birdtracking:

Geyer and Lazar [14] *Twist decomposition of nonlocal light-cone operators II: general tensors of 2nd rank*

Costa and Hansen [6] *Conformal correlators of mixed-symmetry tensors*

Rejon-Barrera and Robbins [24] *Scalar-vector bootstrap*

Costa *et al.* [7] *Projectors and seed conformal blocks for traceless mixed-symmetry tensors*

Pang, Rong and Su [23] ϕ^3 *theory with* F_4 *flavor symmetry in* $6-2\epsilon$ *dimensions: 3-loop renormalization and conformal bootstrap*

2016-12-08 Michael Stone m-stone5@uiuc.edu
It's quite interesting to read Young's original paper [34] on this issue. See the discussion after Theorem III on pa 263. This shows how to fix up the projectors, but the result is not pretty. Essentially the same discussion appears in the section of Young projectors in D E Littlewood's "University Algebra," which is where I was first alerted to the problem.

2016-12-08 Predrag Liu and Zerf [28] *Irreducible tensor basis and general Fierz relations for Bhabha scattering like amplitudes* has Fierz-e birdtracking for $SOn3, 1$.

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