QFT and its discontents
a blog

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Contents

Chapter 1

Is QED finite?

For Emily, after she gets bored with "Baby Loves Quarks."

In 2017 Laporta [104] completed the twenty-year project of computing analytically the individual contributions of 891 4-loop vertex diagrams contributing to the electron magnetic moment g. Vertex diagrams separate in 25 gauge-invariant sets (see figure 1.8). The numerical contribution of each set is listed in table 1.2. Adding only the quenched set V diagrams (diagrams with no lepton loops, see figure 1.4 and 1.6), one finds for the 4- and 5-loop contributions to the anomaly $a[V] = \frac{1}{2}(g-2)\big|_V$:

$$\begin{aligned}
a^{(8)}[V] &= -2.176866027739540077443259355895893938670 \\
&= -2.17\ldots \qquad \text{Aoyama } et\ al. \text{ 2012 [7]} \\
&\approx 0 \qquad \text{Cvitanović 1977 [39]} \\
a^{(10)}[V] &= 8.726(336)\ldots \quad \text{Aoyama } et\ al. \text{ 2015 [9]} \\
&\approx 3/2 \qquad \text{Cvitanović 1977 [39]}\,.
\end{aligned} \tag{1.1}$$

There is a prediction dating back to 1977 for values of these terms: the predicted $a^{(8)}[V] \approx 0$ does not pan out, but the difference is small, considering that this is a sum of 518 vertex diagrams (or 47 self-energy diagrams) [93]. Likewise, the prediction for $a^{(10)}[V]$ is not too far off, considering that this is a sum of 6354 vertex diagrams of table 1.1 (or 389 self-energy diagrams).

Why are these numbers so small? As a prelude to these notes, you might enjoy the introduction to Dunne and Schubert 2005 paper [56] which starts with a historical review of ideas about the QED perturbation series. They note: "[...] a point which remains poorly understood is the influence of gauge cancellations on the divergence structure of a gauge theory."

In this spirit, I review in sect. 1.2 the numerics that motivated the 1977 QED finiteness conjecture, and then review the approaches that offer a promise of establishing it. A proof of the conjecture might be within reach; so might be methods for computing gauge invariant QFT sets, without recourse to traditional Feynman diagrams. A fun fact is that the idea of how to avoid Feynman diagrams can also be traced to 1950

Feynman paper [61], though it took a long time for it to gain traction. By the time I explained the gauge set conjecture to him in 1975, Feynman had forgotten all about it.

Please download the current version of these notes: ChaosBook.org/∼predrag/papers/finiteQED.pdf.

The source code: GitHub.com/cvitanov/reducesymm/QFT

1.1 Electron magnetic moment

This section sets up the notation - the reader can safely skip it and start with sect. 1.2. For an introduction to the conventional magnetic moment calculation, see for example the Cvitanović online graduate QFT course, lectures 25 and 26 here.

Consider the electron-photon vertex Γ_μ of quantum electrodynamics, with $p_i = p - q/2$ and $p_o = p + q/2$ the momenta of incoming and outgoing electron lines, evaluated on the electron mass shell $p_i^2 = p_o^2 = m^2$. By Gordon decomposition the vertex can be written in terms of the Dirac and Pauli form factors $F_1(q^2)$ and $F_2(q^2)$:

$$\overline{u}(p_o)\Gamma_\mu(p,q)u(p_i) = \overline{u}(p_o)\left\{ F_1(q^2)\gamma_\mu - \frac{F_2(q^2)}{2m}\,\sigma_{\mu\nu}q^\nu \right\}u(p_i)\,, \tag{1.2}$$

where the spinors $\overline{u}(p_i)$ and $u(p_i)$ satisfy the Dirac equation:

$$\overline{u}(p_o)\ \not{p}_o = m\,\overline{u}(p_o)\,, \qquad \not{p}_i\,u(p_i) = m\,u(p_i)\,.$$

We follow the notation of Bjorken and Drell [19] and Cvitanović and Kinoshita [47]. Z_1, Z_2, and Z_3, are the respectively the vertex, the electron wave function, and the photon wave function renormalization constants, and the electron mass will be set to $m = 1$ throughout. In what follows it is convenient to define $Z_1 = 1 + L$. For QED the charge conservation requires that the renormalized charge form factor satisfies $\tilde{F}_1(0) = 1$, which is guaranteed by the Ward identity $Z_1 = Z_2$. The vertex renormalization constant L is given by the on-shell value of the unrenormalized charge form factor [28]

$$1 + L = F_1(0) = \frac{1}{4}\mathrm{tr}\,\left[(\not{p}+1)p^\nu\Gamma^\nu\right]_{q=0}\,, \tag{1.3}$$

and $a = (g-2)/2$, the anomalous magnetic moment of an electron is given by the static limit of the magnetic form factor $a = \tilde{F}_2(0) = M/(1+L)$, where [28]

$$M = \lim_{q\to 0}\frac{1}{4q^2}\mathrm{tr}\,\left\{\left[\gamma^\nu p^2 - (1+q^2/2)p^\nu\right](\not{p}_o+1)\Gamma_\nu(\not{p}_i+1)\right\}\,. \tag{1.4}$$

The perturbative expansions for the magnetic moment anomaly is defined as

$$a = \frac{M(\alpha_0)}{1+L(\alpha_0)} = \sum_{n=1}^{\infty} a_0^{(n)}\left(\frac{\alpha_0}{\pi}\right)^n\,, \tag{1.5}$$

where $1 + L = F_1(0)$, $M = F_2(0)$ are computed from the unrenormalized proper vertex (1.2), given by the sum of all one-particle irreducible electron-electron-photon vertex diagrams with internal photons, electron loops and electron mass counterterms. Expanding M and L we have

$$
\begin{aligned}
a_0^{(2)} &= M^{(2)} \\
a_0^{(4)} &= M^{(4)} - L^{(2)} M^{(2)} \\
a_0^{(6)} &= M^{(6)} - L^{(2)} M^{(4)} - (L^{(4)} - (L^{(2)})^2) M^{(2)}
\end{aligned}
\tag{1.6}
$$

As shown in ref. [37], for the anomaly (1.5) expressed in terms of the unrenormalized coupling constant α_0, all $a_0^{(n)}$ are IR finite, for both QED and QCD. The UV finite expression for the anomaly (1.5) is obtained by the charge renormalization

$$
\alpha = Z\alpha_0\,, \qquad Z = \frac{Z_2}{Z_1} Z_3\,,
\tag{1.7}
$$

where Z_1, Z_2, and Z_3 are computed as power series in the bare coupling constant α_0. For QED $Z_1 = Z_2$ by the Ward identity, and for QCD the Z_i's are related by the Taylor-Slavnov [140, 148] identities (1.7). The simple structure of (1.5) and (1.6) should simplify the worldline calculations of sect. 1.6.2.

The Dirac equation predicts that the magnetic moment of an electron of charge e and mass m is $\mu = e/2m$, i.e., in the absence of radiative corrections $\tilde{F}_1(0) = 1$ and $\tilde{F}_2(0) = 0$. In 1948 Schwinger [136] showed that in the leading, one-loop order in the fine structure constant α, the radiative corrections lead to the anomalous magnetic moment of form $\tilde{F}_2(0) = \alpha/2\pi + a^{(2)}\,(\alpha/\pi)^2 + \cdots$ (the result engraved on Schwinger's tombstone). These notes are about what to expect for the $(\alpha/\pi)^n$ term in this series.

1.2 Gauge sets

> Is there any method of computing the anomalous moment of the electron which, on first approximation, gives a fair approximation to the α term and a crude one to α^2; and when improved, increases the accuracy of the α^2 term, yielding a rough estimate to α^3 and beyond?
>
> — Feynman's challenge, 12th Solvay Conference [64]

In 1972 Toichiro Kinoshita and Predrag Cvitanović had completed computing a large number of 3-loop anomalous magnetic moment Feynman diagrams and regularization counterterms [92], figure 1.1. The subsequent 4- and 5-loop numerical and analytic calculations were nothing short of heroic [9, 93, 104, 105]. The quantum field theory was used in the standard way [19], by expanding the magnetic moment into combinatorially many Feynman diagrams (see the numbers of vertex graphs in table 1.1). Each Feynman diagram corresponds to an integral in many dimensions, with oscillatory integrand with thousands of terms, each integral separately UV divergent, IR divergent, and unphysical, as its value depends on the definition of counterterms and the choice of gauge. The numerical values of these integrals typically range from ± 10 to ± 100.

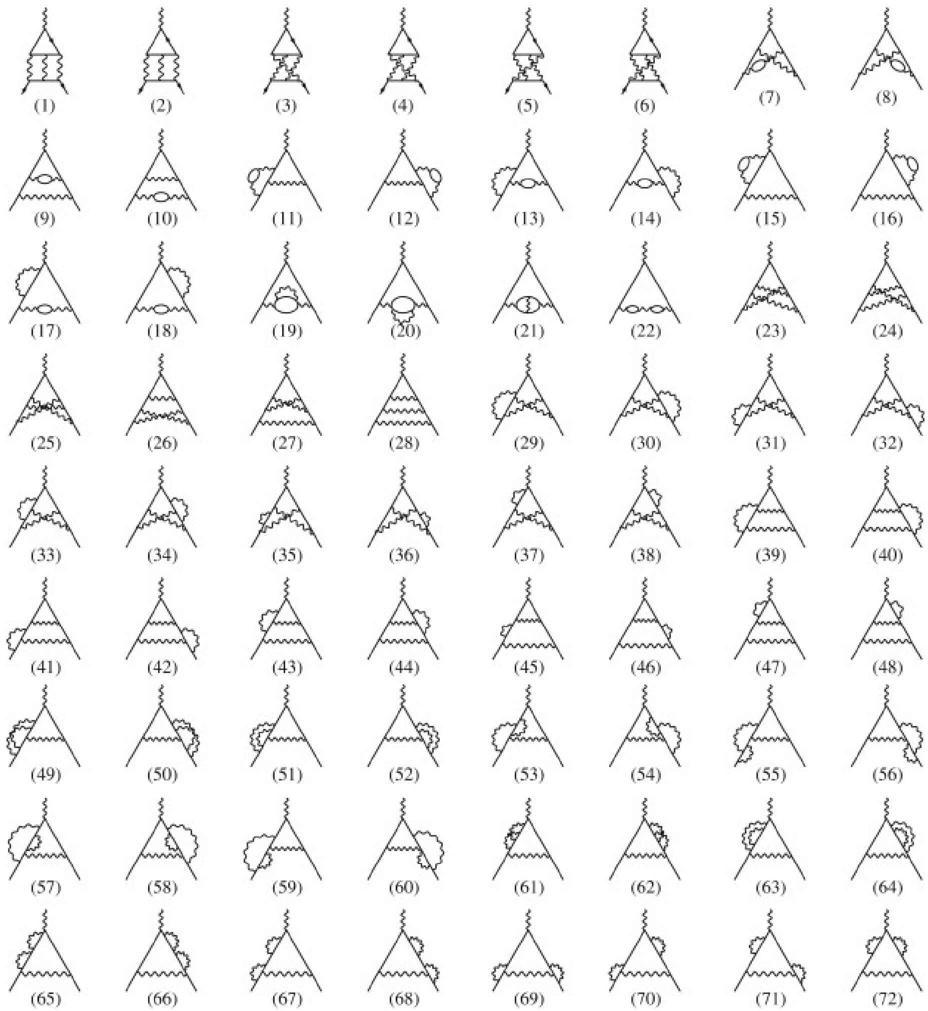


Figure 1.1: The three-loop vertex diagrams contributing to $A_1^{(6)}$ magnetic moment (from Jegerlehner and Nyffeler [81]). Lautrup *et al.* [108] were the first to note that subsets $(3,0,0) = \{23, 24, 25, 26, 28\}$; $(2,1,0) = \{29, 31, 33, 35, 37, 39, 41, 43, 45, 47\}$ and its time-reversal $(2,0,1) = \{30, 32, 34, 36, 38, 40, 42, 44, 46, 48\}$; $(1,2,0) = \{49, 51, 53, 55, 57, 59, 61, 63, 65, 67\}$ and its time-reversal $(1,0,2) = \{50, 52, 54, 56, 58, 60, 62, 64, 66, 68\}$; and $(1,1,1) = \{69, 70, 71, 72\}$ are the minimal gauge sets, see figure 1.3.

Order $2n$	Vertex graphs Γ_{2n}	Gauge sets G_{2n}	Anomaly $a^{(2n)}$
2	1	1	1/2
4	6	2	0
6	50	4	1
8	518	6	0
10	6354	9	3/2
12	89 782	12	0
14	1 429 480	16	2

Table 1.1: Comparison of the number of vertex diagrams without fermion loops, gauge sets, and the gauge-set approximation (1.10) for the magnetic moment in $2n$th order. From ref. [39].

Adding up hundreds of such contributions, of wildly fluctuating values, yields (for the no-fermion loops subset V, in the notation of ref. [9])

$$A_1^{(6)}[V] \;=\; +(0.92 \pm 0.02)\left(\frac{\alpha}{\pi}\right)^3 .$$

But why "+" and not "-"? Why so small? Why does a sum of hundreds of diagrams and counterterms yield a number of order of unity, and not 10 or 100 or any other number?

If gauge invariance of QED guarantees that all UV and on-mass shell IR divergences cancel, could it be that it also enforces cancellations among the finite parts of contributions of different Feynman graphs?

As first noted by Lautrup, Peterman and de Rafael [108], the renormalized on-mass shell QED vertex diagrams separate into a sum of minimal gauge-invariant subsets, each subset separately UV and IR finite. To simplify matters, in what follows we shall consider only the no-fermion loop diagrams, or 'quenched-', or 'q-type' diagrams ('quenched', as this corresponds to the N_f-independent part of the vertex amplitude in QED with N_f flavors). The minimal gauge-invariant subsets without electron loops (see figure 1.1 diagrams $\{23 - 72\}$; figure 1.2; 1.3; 1.4; and 1.6) will be hereafter be referred to as *gauge sets*.

A gauge set (k, m, m') consists of all 1-particle irreducible vertex diagrams without electron loops, with k photons crossing the external vertex (cross-photons) and $m[m']$ photons originating and terminating on the incoming [outgoing] electron leg (leg-photons), where $m \geq m'$. For asymmetric pairs of sets, with $m \neq m'$, the contribution to the anomaly $a_{kmm'}$ is, in the convention of ref. [39], the sum of the set and its mirror (time-reversed) image,

$$a[V] = \frac{1}{2}(g-2)\bigg|_V = \sum_{k=1}^{\infty}\sum_{m=0}^{\infty}\sum_{m'=0}^{m} a_{kmm'}\left(\frac{\alpha}{\pi}\right)^{k+m+m'} . \tag{1.8}$$

When the diagrams computed in ref. [47] are grouped into gauge sets, figure 1.2 to figure 1.6, a surprising thing happens; while the finite part of each Feynman diagram is

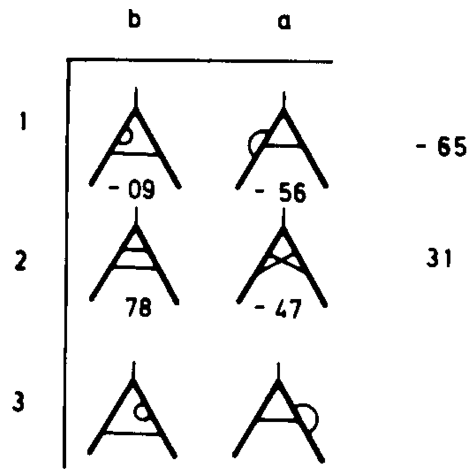


Figure 1.2: Rows: the fourth-order gauge sets (k, m, m'): (1) = $(1, 1, 0)$, (2) = $(2, 0, 0)$ and (3) = $(1, 0, 1)$. Columns: external field insertions into the two self-energy sets. For diagrams related by time reversal (here (1) and (3)) the value listed under the first diagram of the pair is the total contribution of the pair. Contributions seem to be of order $\pm\frac{1}{3}\left(\frac{\alpha}{\pi}\right)^2$, and suggest that a set and its time-reversed partner should be counted separately. From ref. [39].

of order of 10 to 100, every gauge set known at the time added up to approximately

$$\pm\frac{1}{2}\left(\frac{\alpha}{\pi}\right)^n ,$$

with the sign given by a simple empirical rule

$$a_{kmm'} = (-1)^{m+m'}\frac{1}{2} . \tag{1.9}$$

The sign rule is further corroborated by sets with photon self-energy insertions (but with the absolute size scaled down to $3 - 15\%$ of (1.9)). In figure 1.4 this rule is compared with the actual numbers, and the 1977 four-loop prediction is given [39]. With that prediction, the “zeroth” order estimate of the electron magnetic moment anomaly a is given by the “gauge-set approximation,” convergent and summable to all orders

$$a = \frac{1}{2}(g-2) = \frac{1}{2}\frac{\alpha}{\pi}\frac{1}{\left(1-\left(\frac{\alpha}{\pi}\right)^2\right)^2} + \text{“corrections"} . \tag{1.10}$$

This is not how one usually thinks of perturbation theory. Most of our colleagues believe that in 1952 Dyson [57] had shown that the perturbation expansion is an asymptotic series (for a discussion, see Dunne and Schubert [56, 79]), in the sense that the n-th order contribution should be exploding combinatorially

$$\frac{1}{2}(g-2) \approx \cdots + n^n\left(\frac{\alpha}{\pi}\right)^n + \cdots ,$$

and not growing slowly like my estimate

$$\frac{1}{2}(g-2) \approx \cdots + \frac{n}{2}\left(\frac{\alpha}{\pi}\right)^{2n} + \cdots .$$

For me, the above is the most intriguing hint that something deeper than what we know today underlies quantum field theory, and the most suggestive lesson of our calculation.

1.2.1 Self-energy sets

There are two ways of grouping vertex diagrams, into *gauge sets*, and into *self-energy sets* (or the "externally gauge-invariant" sets). Every vertex diagram belongs both to a gauge set and to a self-energy set, as illustrated by figure 1.3. Formulation of the $(g-2)$ computation directly from self-energy graphs is due to Cvitanović and Kinoshita, see the "new formula" (6.22) in ref. [47]. Not only does the calculation use fewer Feynman graphs, but it was very important for us, as it enabled us to calculate the 3-loop electron magnetic moment by two independent methods (and in this way we did actually identify and eliminate a subtle numerical error in one of the graphs). Parenthetically, Carroll papers [29, 30] puzzle me. He gives the credit for self-energy sets to the mass-operator formalism of Schwinger [137, 138, 142], but no credit to us [47], even though his papers look closer to ours than to Schwinger and Sommerfield; he cites us, his derivation is also based on the Ward-Takahashi identity. Our formulation might be equivalent to Schwinger's, but it looks quite different in detail, and I was not aware of Schwinger mass-operator when we derived it.

The gauge sets are minimal, and separately gauge invariant (for a proof, see ref. [39]). The self-energy sets are not, only their sum is gauge invariant. Unlike gauge sets, whose number grows polynomially, the number of self-energy sets grows combinatorially - they save significant amount of computing for few-loops computations, but cannot be used to argue the finiteness of QED. That is the reason why Aoyama *et al.* [7, 9] calculations have nothing to say about the 1977 conjecture [39]: they do not compute individual vertex diagrams, but only the self-energy sets, and for them the set of all diagrams without a fermion loop ('quenched-' or 'q-type' diagrams) is a single 'gauge-invariant set' V. For example, for 5-loops the set V is a sum of 9 vertex gauge sets (where time-reversed pairs count as one set, see figure 1.5), but Aoyama *et al.* [9] only give their sum (1.1).

1.2.2 Where do we go from here?

Gauge invariance is the bane of my life
— Predrag

Aoyama *et al.* 5-loop calculations already push the envelope of what is numerically attainable, they would be loath to switch from self-energy diagrams formulation to the vertex diagrams formulation, it would mean (for the quenched set V) going from 389 self-energy graphs to 6354 vertex diagrams. Stefano Laporta deserves a bit of well earned rest. So what is ahead? I've now reread much of the relevant literature known to me. There might be much more - people who do things related to my 1977 paper [39]

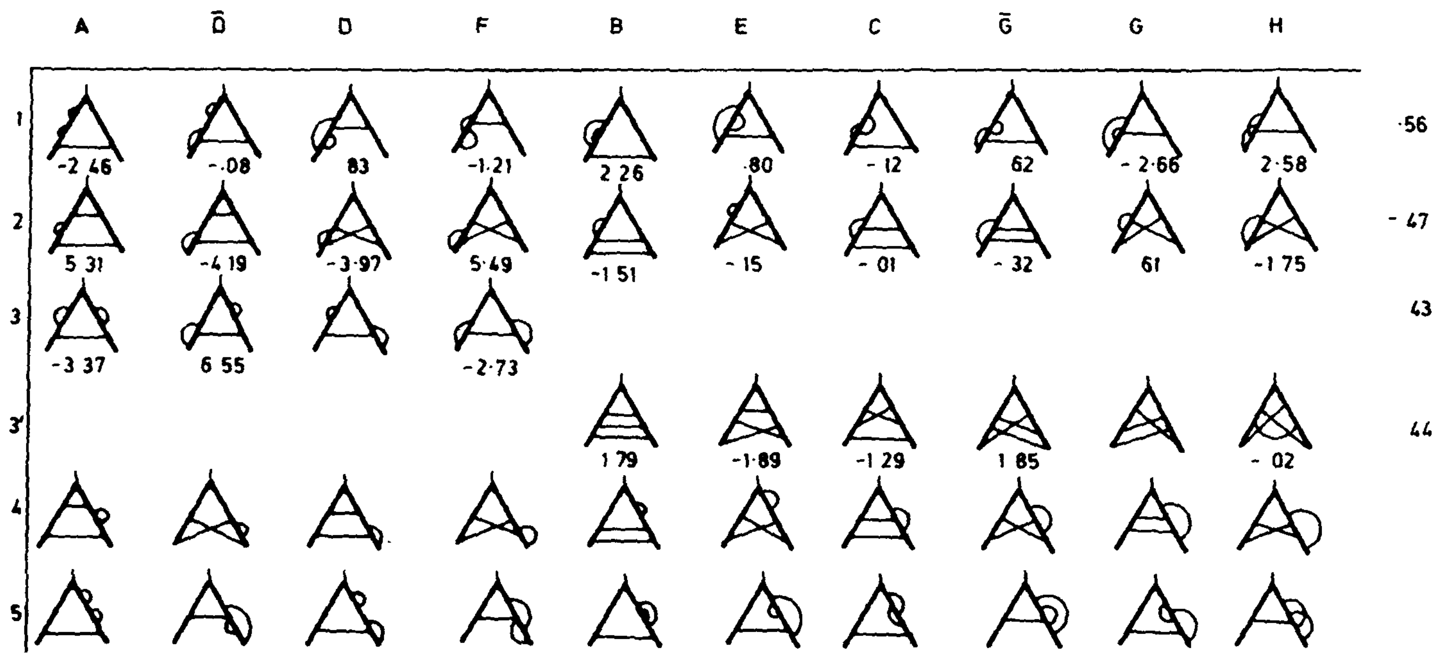


Figure 1.3: Every vertex diagram belongs both to a ‘gauge set’ and to a ‘self-energy set’. This table illustrates the two kinds of sets. The 3-loop gauge sets (k, m, m') are arranged in the rows, and the self-energy sets (or the ‘externally gauge-invariant’ sets, vertex diagrams obtained by inserting an extra vertex into a self-energy diagram) in the columns, labeled as in Fig. 3 of ref. [47]. The values are finite parts in the $\ln \lambda$ IR cut-off approach, such as those listed in ref. [110]. For different IR separation methods (such as in ref. [47]) and different gauges, individual diagrams have different values. The gauge sets, however, are separately gauge invariant. The self-energy sets (whose number grows combinatorially with the order in perturbation theory) are not, only their sum is gauge invariant. From ref. [39].

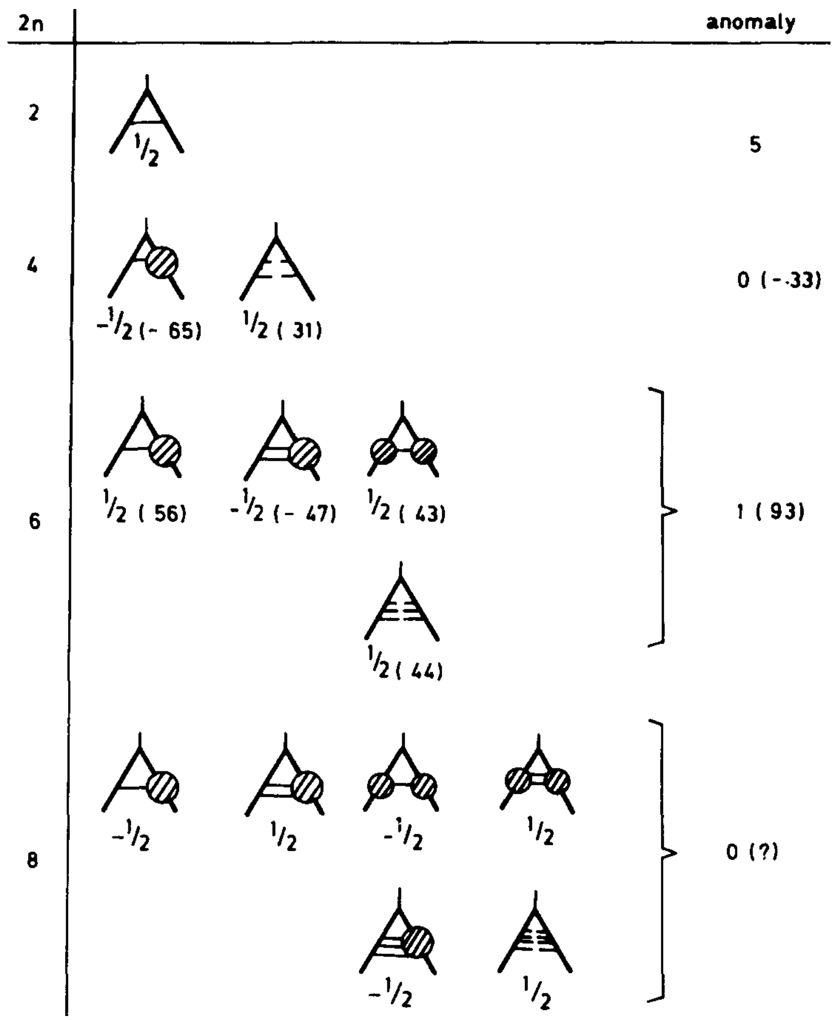
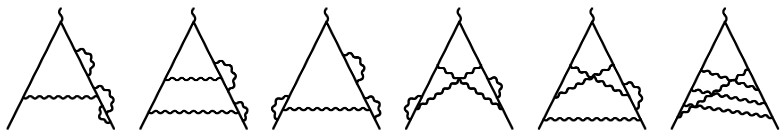


Figure 1.4: Comparison of the 1977 gauge-set approximation to the anomaly a and the actual numerical values of corresponding gauge sets, together with the 1977 eighth-order prediction of ref. [39]. For the updated listing, see figure 1.5.

$2n$	(k, m, m')					anomaly
2	$(\mathbf{1},\mathbf{0},\mathbf{0})$ 1/2					$\frac{1}{2}$
4	$(\mathbf{1},\mathbf{1},\mathbf{0})$ -1/2 (-.65)	$(\mathbf{2},\mathbf{0},\mathbf{0})$ 1/2 (.31)				0 (-.33)
6	$(\mathbf{1},\mathbf{2},\mathbf{0})$ 1/2 (.56)	$(\mathbf{2},\mathbf{1},\mathbf{0})$ -1/2 (-.47)	$(\mathbf{3},\mathbf{0},\mathbf{0})$ 1/2 (.44)			1 (.93)
	$(\mathbf{1},\mathbf{1},\mathbf{1})$ 1/2 (.43)					
8	$(\mathbf{1},\mathbf{3},\mathbf{0})$ -1/2 (-1.97)	$(\mathbf{2},\mathbf{2},\mathbf{0})$ 1/2 (-.14)	$(\mathbf{3},\mathbf{1},\mathbf{0})$ -1/2 (-1.04)	$(\mathbf{4},\mathbf{0},\mathbf{0})$ 1/2 (.51)		0 (-2.17)
	$(\mathbf{1},\mathbf{2},\mathbf{1})$ -1/2 (-.62)	$(\mathbf{2},\mathbf{1},\mathbf{1})$ 1/2 (1.08)				
10	$(\mathbf{1},\mathbf{4},\mathbf{0})$ 1/2 (?)	$(\mathbf{2},\mathbf{3},\mathbf{0})$ -1/2 (-?)	$(\mathbf{3},\mathbf{2},\mathbf{0})$ 1/2 (?)	$(\mathbf{4},\mathbf{1},\mathbf{0})$ -1/2 (?)	$(\mathbf{5},\mathbf{0},\mathbf{0})$ 1/2 (?)	$\frac{3}{2}$ (8.72)
	$(\mathbf{1},\mathbf{3},\mathbf{1})$ 1/2 (?)	$(\mathbf{2},\mathbf{2},\mathbf{1})$ -1/2 (?)	$(\mathbf{3},\mathbf{1},\mathbf{1})$ 1/2 (?)			
	$(\mathbf{1},\mathbf{2},\mathbf{2})$ 1/2 (?)					

Figure 1.5: Updated figure 1.4 comparison of the gauge-set approximation (1.10) and the actual numerical values of corresponding gauge sets, together with the 5-loop prediction. Starting with 4-loops, the gauge-set approximation fails in detail. Still, the signs are right, except for the anomalously small set $(2, 2, 0)$, and the remaining sets are surprisingly close to multiples of 1/2.



gauge set	(k, m, m')	value	prediction	
(1)	(1,3,0)	- 1.9710	- 1/2	
(2)	(2,2,0)	- 0.1424	1/2	(!)
(3)	(1,2,1)	- 0.6219	- 1/2	
(4)	(2,1,1)	1.0867	1/2	
(5)	(3,1,0)	- 1.0405	- 1/2	
(6)	(4,0,0)	0.5125	1/2	

Figure 1.6: (top) Examples of 4-loop vertex diagrams belonging to Laporta [104] gauge sets (1) to (6). The remaining diagrams in the set can be obtained by permuting separately the vertices on the left and right side of the electron line, and considering also the mirror images of the diagrams. For all 25 gauge-invariant sets, see figure 1.8. The table: Gauge-set contributions $a^{(8)}_{kmm'}$, see (1.8), as reported by Laporta [104] (for the full 25 gauge-invariant sets, see table 1.2). The last column: 1977 Cvitanović predictions [39]. Signs are right, except for the set (2) = $(2, 2, 0)$, which is anomalously small, and the remaining sets are surprisingly close to multiples of 1/2. There might be factors of 2 having to do with symmetries, missing from the guesses of ref. [39], but I cannot see how that would work. Only (4) = $(2, 1, 1)$ and (6) = $(4, 0, 0)$ are symmetric, but (1) = $(1, 3, 0)$, (4) and (5) = $(3, 1, 0)$ seem to have an extra factor of 2 or 4.

never alert me to their papers, presumably because the reports of my death have been greatly exaggerated.

At this time, Sergey A. Volkov appears to be the only person set up do the requisite 5-loop calculations, see sect. 1.3.

Two approaches might be relevant to establishing bounds on, and perhaps even the direct computation of gauge sets (ignoring the $N = 2$ and $N = 4$ supersymmetric models): (1) *Hopf algebraic approach* of Kreimer and collaborators, see sect. 1.4, and (2) *worldline formalism* pursued by Schubert and collaborators, see sect. 1.6.

Very far out in the left field is the smooth conjugacy method of sect. 1.5 which would require a bit of real work to apply it to a field theory.

1.3 Volkov method

In ref. [150] Volkov explains that $A_1^{(2n)}$ is free from infrared divergences since they are removed by the on-shell renormalization. However, Volkov also states that there is no universal method in QED for canceling IR divergences in the Feynman graphs analogous to the R operation, and that the standard subtractive on-shell renormalization cannot remove IR divergences point-by-point in Feynman-parametric space, as it does for UV divergences. Moreover, it can generate additional IR-divergences.

That QED on-mass shell amplitudes are IR-free must be an old result; even I have several papers generalizing that to QCD [37, 38, 40, 48]. Tom Kinoshita and I solved the problem of point-by-point removal of IR divergences in Feynman-parametric space in my thesis [46], with a super-elegant formula (who needs forests?) for the UV and IR finite part of amplitude M_G,

$$\Delta M_G = \prod_{ij} (1 - I_{G/S_i})(1 - K_{G/S_j}) M_G \,, \tag{1.11}$$

where the products are over all self-energy and vertex subdiagrams S_i and S_j. I have a bright memory of figuring out how to do it one quiet evening in Ithaca, babysitting for a friend's toddler. But, as Volkov [150] and Aoyama *et al.* [10] explain, our approach was apparently not general enough to deal with the 4- and 5-loop contributions.

Volkov's algorithm is developed in *New method of computing the contributions of graphs without lepton loops to the electron anomalous magnetic moment in QED* [151]. It is based on the ideas used for proving UV-finiteness of renormalized Feynman amplitudes [6, 144]. He focuses on n-loop graphs with no lepton loops, or, in the notation of these notes, $a^{(2n)}[V]$. Volkov calculation groups Feynman graphs by self-energy graphs families because they have similar integrand structure. In contrast to refs. [9, 29, 30, 47] he does not evaluate these self-energy graphs directly; all his calculations are performed with vertex graphs, i.e., precisely what is needed to evaluate gauge sets of sect. 1.2. However, as illustrated in figure 1.4, each gauge-set vertex diagram belongs to a different self-energy diagram, so Volkov calculation will require a major reorganization of how integrands are generated, requiring months of recoding.

So far Volkov has evaluated the ladder graph and the fully crossed graph up to 5 loops. The cross graphs are of interest because they do not contain divergent subgraphs, so their contributions only depend on the gauge, but not on the choice of subtraction procedure.

While the contributions of individual vertex graphs (and self-energy sets [9]) are all over the place, all gauge-invariant sets are insanely small up to order 8, and it would be very sweet to see that this continues through order 10 (at least for the 5-loop graphs with no electron loops). My hunch is that starting with the gauge set $(5, 0, 0)$ of figure 1.5 (5! vertex graphs, some of them symmetric pairs) would be the most rewarding. Stefano Laporta thinks it too hard, and suggests starting with the 5-loop relative $(1, 3, 1)$ (or $(1, 2, 2)$) of the 4-loop set $(1, 2, 1)$, which would entail less than 5! vertex graphs (I have not counted how many). As no high accuracy is needed, a numerical check of the QED finiteness conjecture would good enough if the gauge sets evaluated to two significant digits or so, but even that will need a lot of computer time.

1.4 Hopf algebraic approach

Hopf algebraic approach of Kreimer and collaborators [27, 95, 102, 103] is very appealing - it is just that I personally have no clue how to turn it into a direct $(g-2)$ gauge set calculation. In the 2008 paper [103] Dirk Kreimer and Karen Yeats write:

"One case where there is a natural interpretation is QED with a linear number of generators, namely

$$X_1 = 1 + \sum_{k\geq 1} p(k)x^k \frac{X_1^{2k+1}}{(1-X_2)^{2k}(1-X_3)^{2k}}\,, \tag{1.12}$$

with X_2 and X_3 as before and with $p(k)$ linear, which corresponds to counting with Cvitanović's gauge invariant sectors [39]."

Even in this simple case I do not see how this counts the gauge sets. My generating function for G_{2n}, the number of gauge sets (eq. (7) in ref. [39]) is

$$\sum_{n=1}^{\infty} G_{2n} = \frac{X}{(1+X)(1-X)^3}\,. \tag{1.13}$$

Broadhurst, Delbourgo and Kreimer [27] 1996 *Unknotting the polarized vacuum of quenched QED* unearthes much knot-theory magic, leading to cancelations of "transcedentals." That might be another route to proving the QED is finite - if there is some finite knot-theory basis for expressing the value of every gauge set, and the gauge invariance induced cancelations are so strong to lead to the large cancelations of transcendentals (hyperlogarithms), then perhaps that gives bounds on the size of each gauge set which are slower than combinatorial. The number of different kinds of knots with n crossings is known to grow only exponentially, not faster.

Henry Kißler (on page 48) has a fresh idea for how to approach the finiteness conjecture, using the *Hepp bound*.

Note that the Kißler and Kreimer [95] definition of a "gauge set" differs from (1.8) used here. They organize a gauge-dependent calculation into "gauge sets" of different parameter dependence.

My notes on these papers are below, starting on page 46.

1.5 Method of smooth conjugacies

If Feynman knew Poincaré: How to replace many diagrams by one

In quantum field theory the standard Feynman diagram methods become quickly unwieldy at higher orders. However, it is frequently observed that the sums of Feynman diagrams, each individually complicated, simplify miraculously to rather compact expressions.

Here comes a possible reason why that can be traced back to Poincaré, and is perhaps not something that a field theorist would instinctively hark to as a method of computing perturbative corrections: make the dynamics linear ("free") by flattening out the vicinity of a path integral extremum by a smooth nonlinear coordinate transformation. The resulting perturbative expansion is more compact than the standard Feynman diagram perturbation theory.

1.5 *Method of smooth conjugacies*

The smooth conjugacy method sketched here would require some serious work to make it a workable quantum field theory scheme. The reader might prefer to skip straight to the worldline formalism sect. 1.6.

The periodic orbit theory is a classical, deterministic theory [42] that describes nonlinear systems in "chaotic" (for low-dimensional systems) or "turbulent" (for PDEs) regimes. The theory allows us to calculate long time averages in a chaotic system as expansions in terms of the periodic orbits (cycles) of the system. The simplest example (the deterministic analogue of the quantum evolution operator) is provided by the Perron-Frobenius operator

$$\mathcal{L}\rho(x') = \int dx\, \delta(f(x) - x')\rho(x) \tag{1.14}$$

for a *deterministic* map $f(x)$ which maps a density distribution $\rho(x)$ forward one integer step in time. The periodic orbit theory relates the spectrum of this operator and its weighted evolution operator generalizations to the periodic orbits via trace formulas, dynamical zeta functions and spectral determinants [42, 71].

For quantum mechanics the periodic orbit theory is exact on the semiclassical level [75], whereas the quintessentially quantum effects such as creeping, tunneling and diffraction have to be included as corrections. In particular, the higher order $\hbar$ corrections can be computed perturbatively by means of Feynman diagrammatic expansions [71]. We illustrate how this works by the parallel, but simpler example of *stochastic* dynamics. Cvitanović [41] *Chaotic Field Theory: A sketch* is a programmatic statement how this theory might connect to quantum field theory, and, by a way of motivation, an easy introduction into different approaches to incorporating stochastic corrections into classical dynamics.

What motivated the work [43, 44, 49] summarized in ref. [41] is the fact that the form of perturbative corrections for the stochastic problem is the same as for the quantum problem, and still the actual calculations are sufficiently simple that one can explore more orders in perturbation theory than would be possible for a full-fledged quantum theory. For the simple system studied, the result is a stochastic analog of the Gutzwiller trace formula with the "$\hbar$ corrections" computed to five orders beyond what has been attainable in the quantum-mechanical applications. Already a discrete time, 1-dimensional discrete Langevin equation [86, 106],

$$x_{n+1} = f(x_n) + \sigma\xi_n\,, \tag{1.15}$$

with ξ_n independent normalized random variables, suffices to reveal the structure of perturbative corrections. We treat a chaotic system with weak external noise by replacing the deterministic evolution δ-function kernel of Perron-Frobenius operator (1.14) by $\mathcal{L}_{FP}$, the Fokker-Planck kernel corresponding to (1.15), a peaked noise distribution function

$$\mathcal{L}_{FP}(x', x) = \delta_\sigma(f(x) - x')\,. \tag{1.16}$$

In the weak noise limit the kernel is sharply peaked, so it makes sense to expand it in terms of the Dirac delta function and its derivatives:

$$\delta_\sigma(y) = \sum_{m=0}^{\infty} \frac{a_m\sigma^m}{m!}\,\delta^{(m)}(y) = \delta(y) + a_2\frac{\sigma^2}{2}\delta^{(2)}(y) + a_3\frac{\sigma^3}{6}\delta^{(3)}(y) + \dots\,. \tag{1.17}$$

where

$$\delta^{(k)}(y) = \frac{\partial^k}{\partial y^k}\delta(y)\,,$$

and the coefficients a_m depend on the choice of the kernel. We have omitted the $\delta^{(1)}(y)$ term in the above because in our applications we shall impose the saddle-point condition, that is, we shift x by a constant to ensure that the noise peak corresponds to $y = 0$, so $\delta_\sigma^{'}(0) = 0$. For example, if $\delta_\sigma(y)$ is a Gaussian kernel, it can be expanded as

$$\begin{aligned}\delta_\sigma(y) &= \frac{1}{\sqrt{2\pi\sigma^2}}e^{-y^2/2\sigma^2} = \sum_{n=0}^{\infty}\frac{\sigma^{2n}}{n!2^n}\delta^{(2n)}(y) \\ &= \delta(y) + \frac{\sigma^2}{2}\delta^{(2)}(y) + \frac{\sigma^4}{8}\delta^{(4)}(y) + \cdots\,. \qquad (1.18)\end{aligned}$$

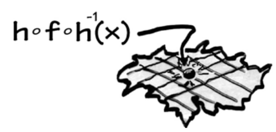
Analogies between noise and quantum mechanics can be explored by casting stochastic dynamics into path integral form (a stoch astic Wiener integral). The periodic orbit theory is a nonperturbative, “WKB” method for approximating such integrals, which can then be improved by systematic perturbative corrections. In the weak noise case the standard perturbation theory is an expansion in terms of Feynman diagrams. For semiclassical quantum mechanics of a classically chaotic system such calculation was first carried out by Gaspard [71]. The stochastic version, implemented by Dettmann *et al.* [43], reveals features not so readily apparent in the quantum calculation. Perhaps some of these could be of interest to Kreimer and collaborators, sect. 1.4.

The Feynman diagram method becomes quickly unwieldy at higher orders.[1] However, in the Feynman diagram approach pursued in ref. [43], the authors observe that the sums of Feynman diagrams simplify miraculously to rather compact expressions.

Now the surprise; one can compute the same corrections faster and to a higher order in perturbation theory by integrating over the neighborhood of a given saddlepoint *exactly* by means of a nonlinear change of field variables. This elegant idea of flattening the neighborhood of a saddlepoint, introduced by Mainieri *et al.* [44], and referred to here as the *smooth conjugation method*, is perhaps an altogether new idea in field theory. The idea, that can be traced back to Poincaré [118], injects into field theory a method standard in the construction of normal forms for bifurcations [87]: perform a smooth nonlinear coordinate transformation $x = h(y)$, $f(x) = h(g(h^{-1}(x)))$ that flattens out the vicinity of a fixed point and makes the map *linear* in an open neighborhood, $f(x) \to g(y) = \mathbf{J} \cdot y$.

an arbitrary coordinatization

$\Longrightarrow$



intrinsic, flat coordinates

The resulting perturbative expansion turns out to be more compact than the standard

[1] The matrix method, introduced by Vattay *et al.* [49], based on Rugh’s [128] explicit matrix representation of the evolution operator will not be discussed here. If one is interested in evaluating numerically many orders of perturbation theory and many eigenvalues, this method is unsurpassed.

Feynman diagram perturbation theory; whether it is better than the traditional loop expansions for computing field-theoretic saddlepoint correction remains to be seen.

What is new is that the problem is being solved locally, periodic orbit by periodic orbit, by translation to coordinates intrinsic to the periodic orbit.

This local rectification of a map can be implemented only for isolated non-degenerate fixed points (otherwise higher terms are required by the normal form expansion around the point), and only in finite neighborhoods, as the conjugating functions in general have finite radia of convergence.

In this approach the neighborhood of each saddlepoint is rectified by an appropriate nonlinear field transformation, with the focus shifted from the dynamics in the original field variables to the properties of the conjugacy transformation. The expressions thus obtained *correspond to sums* of Feynman diagrams, but are more compact.

We will try to explain this simplification in geometric terms that might be applicable to more general field theoretic problems. The idea is this: as the dynamics is nonlinear, why not search for a nonlinear field transformation $\phi = h(\tilde{\phi})$ (a smooth conjugacy) that makes the intrinsic coordinates as simple as possible? Schematically –wrong in detail, but right in spirit– find a smooth conjugacy such that the action $S[\phi] = S_0[\phi] + S_I[\phi]$ in the partition function path integral becomes the free, quadratic action,

$$Z = e^W \;=\; \int [d\phi] e^{S[\phi]} \;=\; \int [d\tilde{\phi}] \frac{1}{|\det \partial h(\tilde{\phi})|^{\frac{1}{2}}} \, e^{\tilde{S}_0[\tilde{\phi}]} \,, \tag{1.19}$$

at the price of having the determinant of the conjugacy Jacobian show up as a weight.

Ref. [43] treats the problem of computing the spectrum of this operator by standard field-theoretic Feynman diagram expansions. Here we formulate the perturbative expansion in terms of smooth conjugacies and recursively evaluated derivatives. The procedure, which is relatively easy to automatize, enables us to go one order further in the perturbation theory, with much less computational effort than Feynman diagrammatic expansions would require.

[TO BE CONTINUED]

1.6 Worldline formalism

How and why Feynman in 1950 introduced 'worldline formalism' (initially for scalar QED, appendix to ref. [61], then for spinor QED, appendix to ref. [62]) is explained in Schubert 2001 report [133] (which also has an extensive bibliography up to 2001). In 1982 Affleck, Alvarez, and Manton [1] used the Feynman worldline path integral representation of the quenched effective action for scalar QED in the constant electric field, and, independently, Lebedev and Ritus [109] did the same for the spinor QED in 1984.

A formula for the charged scalar propagator to emit and reabsorb N photons as it propagates from x' to x can be derived as follows [3]. The free scalar propagator for the Euclidean Klein-Gordon equation [3, 134] is

$$D_0(x, x') = \langle x| \frac{1}{-\Box + m^2} |x'\rangle \,, \tag{1.20}$$

where $\Box$ is the D-dimensional Laplacian. Exponentiate the denominator following Schwinger,

$$D_0(x,x') = \int_0^\infty dT\, e^{-m^2T} \langle x|e^{-T(-\Box)}|x'\rangle\,, \tag{1.21}$$

Replace the operator in the exponent by a path integral

$$D_0(x,x') = \int_0^\infty dT\, e^{-m^2T} \int_{x(0)=x'}^{x(T)=x} \mathcal{D}x(\tau)\, e^{-\int_0^T d\tau \frac{1}{4}\dot{x}^2}\,, \tag{1.22}$$

where τ is a proper-time parameter. This is the *worldline path integral* representation of the relativistic propagator of a scalar particle in Euclidean spacetime. It is easily evaluated and leads to the usual space and momentum space free propagators. Adding the QED interaction terms leads to the Feynman's worldline path integral representation [61] of the charged scalar propagator of mass m in the presence of a background field $A(x)$,

$$D(x,x') = \int_0^\infty dT\, e^{-m^2T} \int_{x(0)=x'}^{x(T)=x} \mathcal{D}x(\tau)\, e^{-S_0-S_e-S_i}\,, \tag{1.23}$$

where (0) is the free propagation

$$S_0 = \int_0^T d\tau \frac{1}{4}\dot{x}^2\,, \tag{1.24}$$

(e) is the interaction of the charged scalar with the external field

$$S_e = -ie\int_0^T d\tau\, \dot{x}^\mu A_\mu(x(\tau))\,, \tag{1.25}$$

and (i) are the virtual photons exchanged along the charged particle's trajectory

$$S_i = \frac{e^2}{2}\int_0^T d\tau_1 \int_0^T d\tau_2\, \dot{x}_1^\mu\, D_{\mu\nu}(x_1-x_2)\, \dot{x}_2^\nu\,, \tag{1.26}$$

where $D_{\mu\nu}$ is the x-space photon propagator.

Consider first the charged scalar field in external field, neglecting internal photon loops. By taking the constant external field $A(x)$ to be a sum of N plane waves, one obtains the rule for inserting N external photons:

$$\begin{aligned} D_{(N)}(x,x') &= (-\lambda)^N \int_0^\infty dT\, e^{-m^2T} \int_0^T d\tau_1 \cdots \int_0^T d\tau_N \\ &\quad \times \int_{x(0)=y}^{x(T)=x} \mathcal{D}x\, e^{i\sum_{i=1}^N k_i\cdot x(\tau_i)} e^{-\int_0^T d\tau\, \frac{1}{4}\dot{x}^2}\,. \end{aligned} \tag{1.27}$$

For the spinor case, the magnetic moment will be given by the term linear in a constant external field $A(x)$, and in order to define gauge sets, one will have to distinguish the in- and out-electron lines.

The object of great interest to us is the quenched internal virtual photons term (1.26):

$$\int_{x(0)=x'}^{x(T)=x} \mathcal{D}x(\tau)\, e^{-S_i} = \int_{x(0)=x'}^{x(T)=x} \mathcal{D}x(\tau)\, e^{-\frac{e^2}{2}\int_0^T d\tau_1 \int_0^T d\tau_2\, \dot{x}_1^\mu\, D_{\mu\nu}(x_1 - x_2)\, \dot{x}_2^\nu} . \quad (1.28)$$

(Fried and Gabellini [70] refer to this as the "linkage operator"). Expanded perturbatively in α/π, this yields the usual Feynman-parametric vertex diagrams. However, it is Gaussian in $\dot{x}^\mu$, and if by integration by parts, $\dot{x}^\mu$ are eliminated in favor of x^μ, internal photons can be integrated over directly, prior to an expansion in $(\alpha/\pi)^n$, and one gets integrals in terms of N*-photon propagators*, symmetrized sums over N photons, and not the usual Feynman graphs. Each usual Feynman graph corresponds to one particular permutation of internal photon insertions, and from that comes the factorial growth in the number of graphs.

These integrations by parts lead to the first and second proper-time derivatives of the Green's function, worked out in the literature (for example, in refs. [134, 145]), the details would take too much space to recap here. I find Bastianelli, Huet, Schubert, Thakur and Weber 2014 paper [15] quite inspirational. My notes on these papers are below, around page 42. Apologies, my notes are just a jumble, jottings taken as I try to understand this literature. They might be useful anyway, as pointers to the literature.

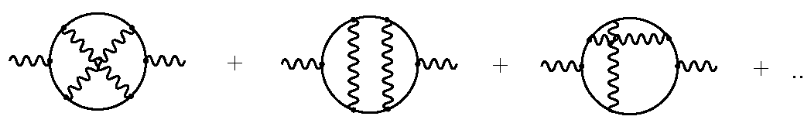


Figure 1.7: Quenched diagrams contributing to the three loop QED photon propagator. From ref. [15].

Thus, for the quenched scalar QED, the worldline integrals are expressed in terms of N-photon propagators, the central ingredient that defines the quenched gauge sets (1.8). Unlike the Feynman parameter integrals for individual vertex graphs, they are independent of the ordering of the momenta $k_1, \ldots, k_N$; the formula (1.28) contains all $\approx N!$ ways of attaching the N photons to the charged particle propagator. The formulation combines combinatorially many Feynman diagrams into a single integral. An example are the quenched contributions to the three-loop photon propagator shown in figure 1.7.

In QED the N-photon propagator formulation combines into one integral all Feynman graphs related by permutations of photon legs along fermion lines, that is, it should yield *one* integral for a gauge set (k, m, m') defined in (1.8).

1.6.1 High-orders QED in worldline formalism

A non-perturbative formula for QED in a constant field, given for scalar QED in 1982 by Affleck, Alvarez, and Manton [1], and for spinor QED by Lebedev and Ritus [109] in 1984, is an example how the worldline formalism can yield high-order information on QED amplitudes. Huet, McKeon, and Schubert [78] continue this in their 2010 study of the 1-electron loop, N-photon amplitudes in the limit of large photon numbers and low photon energies, this time for 1+1 dimensional scalar QED, in order to illustrate the large cancellations inside gauge invariant classes of graphs.

Affleck *et al.* [1] use the Feynman [61] 'worldline path integral' representation of the quenched effective action for scalar QED in the constant electric field, and calculate the amplitude in a stationary path approximation. The stationary trajectory so obtained is a circle with a field dependent radius, called "instanton" in this context. The worldline action on this trajectory yields the correct exponent, and the second variation determinant yields the correct prefactor. Using Borel analysis, they obtain non-perturbative information on the on-shell renormalized N-photon amplitudes at large N and low energies.

For the quenched spinor QED (fermion lines decorated by photon exchanges) closed-form expressions for general N require the worldline super- formalism [133], at the cost of introducing Fradkin 1966 [65] Grassmann path integral, or, alternatively, the second order formalism of Strassler [145].

G. Torgrimsson, Schneider, Oertel and Schützhold [149] 2017 use the N-photon formalism to determine a saddle and the asymptotic form of two types of dynamically assisted Sauter-Schwinger effect.

The 2006 Dunne and Schubert [56] study of scalar and spinor QED N-photon amplitudes, in the quenched approximation (i.e., taking only the diagrams with one electron loop) led to "the following generalization of Cvitanović's conjecture: the perturbation series converges for all on-shell renormalized QED amplitudes at leading order in N_f. It must be emphasized that the on-shell renormalization is essential in all of the above." Unlike Cvitanović [39] purely numerical conjecture, theirs is a sophisticated argument, buttressed by Borel dispersion relations.

1.6.2 Electron magnetic moment in worldline formalism

For quenched QED, the IR and UV structure is very simple. As there are no fermion loops, there is no vacuum polarization contribution to the charge renormalization (1.7), $Z = Z_3 = 1$, so the bare coupling equals the physical coupling, $\alpha_0 = \alpha$. Each order in (1.5) is IR and UV finite, and the UV subdivergences are cancelled by $L^{(2m)}$ counterterms in (1.6). None of this depends on counterterms associated to individual Feynman diagrams, so if both $L^{(2m)}$ and $M^{(2m)}$ can be evaluated in terms of N-photon propagators, the calculation might be UV and IR finite throughout, with no need for constructing counterterms.

1. To proceed, one needs something like a Bern-Kosower [17] type master formula for the electron line dressed with any number of photons, with a single constant external (arbitrarily weak) magnetic field insertion. For the magnetic moment calculation, the external vertex is distinguished by its $\sigma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]$ form (1.2), while all internal, virtual photon vertices are of the usual γ^μ form.

2. Please write down the worldline formula for the anomalous magnetic moment of the electron $a = \tilde{F}_2(0)$, corresponding to Dirac trace expression (1.4) for M.

3. As the external vertex transfers a (vanishing) momentum, the incoming and outgoing electron on-mass shell legs are distinct, and thus there are three kinds of N-photon propagators; k photons crossing the external vertex (cross-photons) and $m[m']$ photons originating and terminating on the incoming [outgoing] electron leg (leg-photons). One needs to prove in the worldline formalism that each (k, m, m') integral (corresponding to a set of quenched set of 1-particle irreducible Feynman vertex diagrams without electron loops) is separately (i) a gauge-invariant set, and (ii) the minimal gauge invariant set.

4. Hopefully the distinction motivates the gauge set sign rule (1.9). Keep in mind, however, that this empirical rule is already violated by the gauge set $(2, 2, 0)$.

5. Please write down the worldline integral for one-loop anomaly $a_0^{(2)}$ in (1.6). The first thing to verify is that the worldline $(1, 0, 0)$ integral reproduces Schwinger's $\frac{1}{2}\left(\frac{\alpha}{\pi}\right)$ result [136], exactly. That is an exercise in converting the integral into Feynman-parametric form, already done several times for other amplitudes.

6. Please write down the worldline integral for 2-loop anomaly $a_0^{(4)} = M^{(4)} - L^{(2)}M^{(2)}$ in (1.6). Can $L^{(2)}M^{(2)}$ be absorbed into the integrand? If cancelations can be made pointwise, that would obviate a need for constructing UV (and IR?) counterterms.

7. For 2-loop anomaly there are only 2 quenched gauge sets (k, m, m'): $(2, 0, 0)$ and $(1, 1, 0)$, which equals $(1, 0, 1)$ by time reversal, see figure 1.2 and figure 1.5. So, reformulate the 2-loop calculation as two worldline integrals, one for each gauge set. Most likely, want to do the gauge set $(2, 0, 0)$ first, as it seems to have simpler subdiagram structure (though not sure about that). Do not attempt (for now) to evaluate these analytically (though Laporta, Kreimer, etc., would be interested to see whether some simplification occurs), main thing is to understand that the UV renormalization works, and that there are no intermediate IR divergences in this reformulation.

1.7 Summary

Everyone makes mistakes—including Feynman
— Toichiro Kinoshita [90]

Currently Sergey A. Volkov is in the best position to check the QED finiteness conjecture numerically, by computing the 5-loops gauge sets.

Worldline formalism could be useful on a qualitative level, as a way of proving the finiteness of QED conjecture,

1. Develop a saddle point expansion for the N-photon propagator integrals, such that the leading term explains the apparent $\approx \pm 1/2$ (or a multiple thereof) size of each quenched gauge set. Affleck *et al.* [1] and G. Torgrimsson *et al.* [149] show the way.

2. Use that to establish bounds on gauge sets for large orders, prove finiteness of quenched QED. If that works, I trust electron loop insertions will be next, and thereafter renormalons [107], etc., will go gently into that good night.

and in a precise way, as a new computational tool:

1. Develop a worldline formulation of spinor QED in which each gauge set is given by a computable integral, in a way to be fleshed out in sect. 1.6.2.
2. Parenthetically, a reformulation of the self-energy diagrams magnetic moment calculation, sect. 1.2.1, would be an even greater computational time saver - all quenched diagrams contributions calculated at one go.
3. In either case, a worldline formulation might make it possible to evaluate orders beyond 5-loops, as the number of gauge sets grows only polynomially. A win-win.
4. A renormalization question: A gauge set is by definition UV and IR finite. Does that mean that its worldline formalism integrand is pointwise finite, no need for counterterms?
5. One electron-loop insertion into $(1, 0, 0)$ might be the easiest worldline integral to evaluate, but I find the quenched sets a higher priority.
6. Things get interesting with reformulating the quenched 3-loop calculation as four worldline integrals / gauge sets, see figure 1.4 and figure 1.5. In particular, the fermion line attachments of different kinds of N-photon propagators now get intertwined.
7. One photon-photon scattering electron-loop insertion into $(2, 0, 0)$ might be the most tempting to evaluate, but I find the quenched sets a higher priority.

My main problem at the moment (well, there are many:) is that nobody seems to have written an explicit formula for the spinor QED anomalous magnetic moment in the worldline formalism.

1.8 Is QED finite? Correspondence

> I am not sure I'll will tell you anything useful any time soon, as I'm a lapsed field theorist, but here is for your amusement my recent rant. To cite my wife: "You may well be right".
> — Keep the Aspidistra Flying!

2007-03-18 Dirk Kreimer I was pretty much looking forward to grab you for a chat on Dyson Schwinger etc whilst in Bonn this starting week. Now I fail to be there myself due to a stupid mechanical problem with a vertebrae. Pondering about other options, let me just point out that the IHES (www.ihes.fr) has an excellent visitor program. If you foresee some time you fancy to pay us a visit, please go ahead.

I am running a little seminar at Boston whilst being there, so besides what works out with IHES, whenever you are available in the above period, you are herewith invited for a talk at BU. Would you be able to come for a seminar to Boston, Thu, Nov 29 this year?

2011-08-23 Predrag to Dirk:

I feel like some stuck-up god knows who for not having answered sooner - but it is not that. I have been focusing on turbulence (with the goal of using what I learn on Yang-Mills, eventually - if it works, Feynman diagrammar will be of secondary importance, the first step is fully and totally non-perturbative) so I feel like fraud talking about QFT when I am suspended between the perturbative past and the elusive non-perturbative future.

If someone wants to hear about turbulence as in ChaosBook.org/tutorials, I can do that honestly - otherwise we wait until I actually do any QFT worth hearing about...

I got birdtracks.eu published, finally - you get 100 birdtracks per 1$, a real deal if you are into that kind of thing.

Guilty as Charged - Predrag

2011-08-26 Dirk June 01-05 2009 I am organizing a workshop at IHES. It has a non-perturbative component via Dyson–Schwinger eqs. Would you be interested to participate and come?

2017-06-16 Dirk I will show your notes to Henry Kissler [94] <kissler@physik.hu-berlin.de> and Michael Borinsky [23, 24] <borinsky@physik.hu-berlin.de> .

There should be a few people at Les Houches June 2018 who have something to say about asymptotics of solutions of Dyson-Schwinger equation for example.

2013-10-23 Warren D. Smith warren.wds@gmail.com
(Predrag: I do not know Dr. Smith, but for his life until 2009, see his resume here.)

I found out interesting info which the textbooks do not know!

Cvitanović 1977 conjectured that gauge-invariant quenched diagram sets always have small sum of diagram values, indeed small enough that he thought quenched QED series would always converge [39] [2]. He apparently originally thought this even for unquenched but Lautrup [107] disproved it and Dyson had long had a highly convincing argument [57] (Reprinted p.255-6 in Selected papers of Freeman Dyson with commentary, AMS 1996) – and I now have even more convincing arguments – that generic QED series diverge for any nonzero α.) This was due to Cvitanović's empirical observation of amazingly huge cancellations within gauge-invariant diagram sets, especially in quenched QED.

However, what Cvitanović did not know, was that Bogomolny and Kubyshin 1981-1982 found estimates of the growth rate of generic QED series, and also for the quenched QED subseries, and indeed for QED for diagrams with k electron loops only (k fixed) [20]. I only found this out the other day, but I had long known about estimates due to Dyson and others predicting divergence for QED series. It is just that this B+K work by permitting arbitrary fixed k, directly addresses Cvitanović's quenched-convergence conjecture and massively conflicts with it – for quenched QED the prediction is that at Nth order we get a quenched diagram sum growing factorially with N:

Evgeny B Bogomolny and Yu A Kubyshin:
1. The choice of the form of the steepest-descent solutions [20]. Asymptotic estimates for diagrams with a fixed number of fermion loops in quantum electrodynamics.
2. The extremal configurations with the symmetry group $O(2) \times O(3)$, Soviet J. Nuclear Phys. 35 ,1 (1982) 114-119. [3]

I believe in the sort of arguments Bo+Ku are making, albeit the details are questionable. (E.g. saddlepoint asymptotic estimates are not rigorous unless you prove stuff about the saddlepoints and about tail estimates, which they never proved, and probably nobody can prove.)

Note that this believed N! growth for fixed-k (including quenched) QED diagrams (Lautrup's diagrams also feature N! growth) is far faster than the believed growth – more like (N/2)! – for the *full* QED series!! This fact that subseries diverge far more rapidly than full series can only be explained by presuming that the values at different k cancel each other amazingly well when we sum over all k. This indicates Cvitanović was extremely wrong asymptotically, and that cancellations quite different than his observations ultimately occur which seem even more dramatic.

However, Cvitanović was correct that amazingly large cancellations (also) occur, empirically, within the quenched diagrams alone, at least for the small N that have been reached by computer. And I presume that Kreiman's knot ideas [102] and my "fractal distribution" empirical observations are a partial explanation of why.

[2] Predrag: 'quenched' = no internal electron loops, i.e., $m_e \to \infty$ approximation

[3] Predrag: there is a hard copy at GT library, 4th Floor East Call Number: QC173.I252X (or microfilm?)

 1.8 *Is QED finite? Correspondence*

So a consistent picture is now developing about how QED perturbative series (and various interesting sub-series, such as quenched) allegedly behave asymptotically (Although I never saw anything saying all that I just said in one place...)

2013-10-23 Warren to Predrag do you have a graph-theoretic characterization of 'gauge invariant diagram set' or know how many such (minimal) sets there are at order N? E.g. does their count grow exponentially, super-exponentially, polynomially or what? You gave a formula for the count of quenched gauge sets which grows only polynomially but I suspect exponential or faster growth for unquenched QED. Even an incomplete graph-theory understanding might be adequate to get good growth bounds.

2013-10-23 Warren Although I do not understand "gauge invariant sets of Feynman diagrams" I have figured out enough now to prove that their count grows ultimately superexponentially in unquenched QED. Specifically I now claim to have a proof that the number of gauge invariant sets of Feynman diagrams in QED at α^n order is

$$\approx 0.01\,(96)^{-n/4}\frac{n!}{(n/2)!(n/4)!}$$

for any integers $n > 0$. And this clearly grows superexponentially. (My bound is very unlikely to be optimal.)

2013-10-23 Predrag to Warren : I'm interested in this discussion, but can you do me a favor and actually read my paper, and edit your initial email accordingly, before we wrangle with further details?

Many of your statements are addressed in my paper, and I can answer them more efficiently if you go through them first. I love Dyson dearly, but his statement is an elementary statement about asymptotic series for factorials. Mine is about mass-shell gauge invariant quantities (perhaps planar?), a topic that is gaining some traction now, unfortunately not applicable to QED.

If you can get gauge sets added up, that would be useful, but Aoyama *et al.* [7, 9] do not have them: they use the second method of computing magnetic moment, eq (6.22) (see sect. 1.2.1 above). That mixes up the gauge sets.

2013-11-23 Warren Smith draft of my notes .

2013-10-24 Warren Predrag: Bogomolny and Kubyshin [20] should find combinatorial growth in all sectors - a perturbation expansion around a saddle point is always asymptotic. For example, a path integral over a non-linear oscillator with quartic potential can be well defined as integral, but saddle point expansion is asymptotic - a simple example is worked out in Sect 3.3 *Saddle-point expansions are asymptotic* (click here).

I agree with Dyson (and Bogomolny & Y. Kubyshin, which is more modern version of the argument).

Essentially, what they did is this. "Instanton" solutions of quantum field theory stationarize the action and hence are "saddlepoints." The full path integral over all quantum field histories, is hopefully dominated by this saddlepoint and nearby configurations. It seems there are more than one kind of instanton, and you need to know which one is the "dominant" saddlepoint, and they may have the wrong one, but if so it hopefully does not matter much (changes some constants, but not the qualitative behavior, of their results, one hopes). So anyhow, under this assumption they actually are able to work out the asymptotics of the Nth term in QFT perturbative series, in the limit where N is large. They find $N!^P$ style growth for positive powers P. Furthermore, for the "quenched" QED sub-series, one might naively think that approach would say nothing about it, but they have further generating functionology tricks which enables them to get conjectured asymptotics for that, and lots of other sub-series, too.

The result is $N!^P$ style growth in all cases. Indicating divergency and with radius of convergence zero.

Suslov [146] and other Russians claim the original idea for this was due to Lev Lipatov, but it has been explored by a fairly large number of papers & authors now, I think mostly French & Russian. My tome has 2 chapters 5 & 6 on Dysonian & other divergence arguments which give pointers into the literature. The Suslov paper you cite also has many such pointers.

> Predrag: What I claim/hope is that gauge invariance + mass-shell condition (neither accounted for in the above asymptotic estimates) induce cancellations that make the theory convergent.

–I think this literature already had invented some way of handling gauge invariance within their saddle pointage. They had thought of it and figured out a way to deal with it. (Mind you, all of this stuff is horribly nonrigorous.) And if by "mass shell" you mean what I am calling "quenched QED" (kind of a battle over which name is worse...) then as I said, Bogol already had a trick for obtaining that from the unquenched analysis.

> Predrag: Mass shell means that all external legs of Feynman diagrams are the physical, asymptotic states satisfying $E = mc^2$. Intermediate virtual states do not do that. Gauge invariance cancellations kick in for the mass shell states, not for the off-mass shell amplitudes. Saddle point estimates do not use mass-shell conditions, to the best of my knowledge.
>
> Suslov [146] *High orders of perturbation theory. Are renormalons significant?*, arXiv:hep-ph/0002051, seems to be about an unphysical but comparatively well behaved ϕ^4 quantum field theory. It seems to argue that renormalons of t'Hooft and Lautrup do not matter. I have not studied Suslov's papers (or the ones that cite them inspire-hep.net/record/510344/citations).

The Lautrup renormalon does not contradict the instanton-based results. And renormalons also lead to yet more kinds of series divergency. By the way my tome also discusses Lautrup renormalon. Suslov in his footnote 4 and nearby points out that the whole saddlepoint approach of Lipatov is nonrigorous (like I didn't know) and in fact may be bogus. Physicists often use saddlepoint analysis but rarely do what it takes to do it rigorously, for example they rarely prove “tail estimates." (And of course when everything is in an infinite dimensional space like it is here, that adds a whole new level of difficulty in trying to get any rigor.) They generally instead just hope for the best. Suslov suggests that the renormalon may be a clue that in this case, the tail may be big enough that the whole Lipatov technique is wrong.

I also think everything Suslov does re Borel transforms, is wrong for QED since QED is not going to be Borel summable. However for ϕ^4 theory Borel might be OK... yes, it is, see table 11 in my tome.

> Predrag: You have to learn all this stuff (including planar field theory) to get people to read you.

I don't know whether anybody will read me, and I've had plenty of trouble learning what i did learn, and I'm not willing to do too much work to overcome that. I am certainly quite ignorant in many respects. I draw some comfort from the fact that many leaders in this area, including many Nobel prize winners, have made plenty of errors, some of which I detected for the first time, and some of which were real howlers. The moral of that for me is, nobody really knows what they are doing about this stuff. You've got to know the right stuff, not all stuff. I hope if my ideas are valid they will eventually gain some attention. If they are wrong I hope this will become clear as soon as possible...

2017-04-30 Predrag to Stefano Laporta <stefano.laporta@bo.infn.it>:

Thanks for the listings of gauge sets, and of course, for your whole amazing project. While in detail my 1977 guesses are wrong, the overall finiteness conjecture still looks good - it's crazy how small all these individual contributions are.

The reason I got into a collaboration with Kinoshita (nominally, Tung-Mow Yan was my adviser) is that he gave me the 3-loop ladder diagram (at 4-loops, it would belong to your gauge set (6) = $(4, 0, 0)$) to evaluate, and I derived a compact formula for the parametric integrand of n-loops ladder diagram. So maybe gauge sets of type (6) = $(4, 0, 0)$ are the easiest to evaluate?

Is evaluating gauge set $(5, 0, 0)$ with 5 loops feasible? If that turns out to be +1/2, that would be sweet.

2017-05-05 Laporta to Predrag

To my knowledge, there are no ideas how to bound the sizes of the gauge sets.

In the set (6) = $(4, 0, 0)$ the ladder is easy to evaluate, but the others are not. The evaluation of the set $(5, 0, 0)$ with five loops would be a real challenge.

Among the sets (1)-(6), the easiest to evaluate is the set (3) = $(1, 2, 1)$.

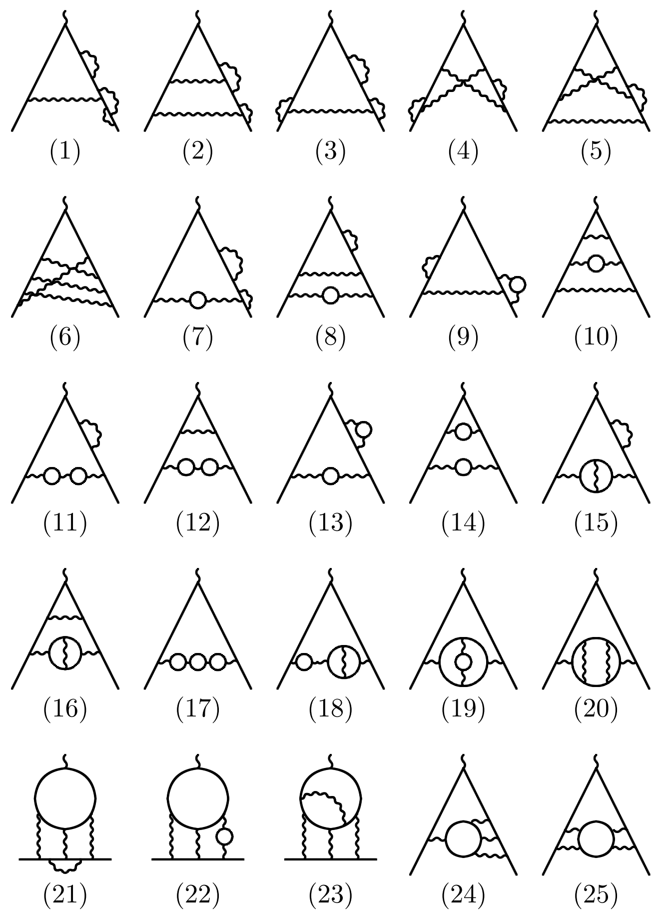


Figure 1.8: Examples of the $A_1^{(8)}$ 4-loop vertex diagrams belonging to the 25 gauge-invariant sets. The number indicates the gauge-invariant set to which the diagram belongs. Gauge sets (1) = $(1, 3, 0)$, (2) = $(2, 2, 0)$, (3) = $(1, 2, 1)$, (4) = $(2, 1, 1)$, (5) = $(3, 1, 0)$, (6) = $(4, 0, 0)$, In the case of the sets 1-16, 24 and 25, remaining diagrams in the set can be obtained by permuting separately the vertices on the left and right side of the main electron line, and considering also the mirror images of the diagrams. From Laporta [104].

 1.8 *Is QED finite? Correspondence*

1	(1,3,0)	- 1.971075616835818943645699655337264406980	- 1/2	
2	(2,2,0)	- 0.142487379799872157235945291684857370994	1/2	(!)
3	(1,2,1)	- 0.621921063535072522104091223479317643540	- 1/2	
4	(2,1,1)	1.086698394475818687601961404690600972373	1/2	
5	(3,1,0)	- 1.040542410012582012539438620994249955094	- 1/2	
6	(4,0,0)	0.512462047967986870479954030909194465565	1/2	
7		0.690448347591261501528101600354802517732		
8		- 0.056336090170533315910959439910250595939		
9		0.409217028479188586590553833614638435425		
10		0.374357934811899949081953855414943578759		
11		- 0.091305840068696773426479566945788826481		
12		0.017853686549808578110691748056565649168		
13		- 0.034179376078562729210191880996726218580		
14		0.006504148381814640990365761897425802288		
15		- 0.572471862194781916152750849945181037311		
16		0.151989599685819639625280516106513042070		
17		0.000876865858889990697913748939713726165		
18		0.015325282902013380844497471345160318673		
19		0.011130913987517388830956500920570148123		
20		0.049513202559526235110472234651204851710		
21		- 1.138822876459974505563154431181111707424		
22		0.598842072031421820464649513201747727836		
23		0.822284485811034346719894048799598422606		
24		- 0.872657392077131517978401982381415610384		
25		- 0.117949868787420797062780493486346339829		

Table 1.2: Contribution to $A_1^{(8)}$ of the 25 gauge-invariant sets of figure 1.8, as reported by Laporta [104].

2017-06-17 Laporta on *transcedentals*: There are no cancellations of transcendentals at 2 loops.

At 3 loops there is only one:
$\zeta(2)\log(2)$ cancels in the sum of the gauge sets (1,2,0)+(2,1,0)+(1,1,1)

At 4 loop there are several:
$\zeta(2)^2\log(2)$ cancels in the sum of the gauge sets 8,9,10 (using my numbering of gauge sets)
$\zeta(2)^2$ cancels in the sum of the gauge sets 5,6,20 (or 10,13,19)
a weight-7 elliptic constant cancels in the sum of the gauge sets 2,3,4; (I have not checked all possible combinations of gauge sets)

2017-06-16 Predrag to Stefano thanks for the info! Apparently there are not so many transcedentals cancellations when one is adding gauge sets...

I was actually thinking of cancellations *within* an individual gauge set. For example, in Laporta and Remiddi [105] *The analytical value of the electron (g-2) at order in QED* you list analytical values for each diagram separately $+\ln\lambda$

terms. They have $\zeta(2)\log(2)$'s and $\zeta(2)\log(2)^2$ and zillion other transcedentals - do many of them cancel within a gauge set (for example, $\ln\lambda$'s certainly do)?

Just trying to figure out if we can use Kreimer kind of ideas to get any bounds.

2017-05-25 Predrag to Stefano:

In the literature there seem to be 2 approaches that might be relevant to establishing bounds on, and perhaps even to actual computation of gauge-invariant sets (ignoring various $N = 2$ and $N = 4$ supersymmetric models), the worldline formalism pursued by Schubert and collaborators, see sect. 1.6, and Hopf algebraic approach of Kreimer and collaborators, see sect. 1.4. The worldline formalism operates with N-photon propagators, the ingredient that defines the quenched gauge-invariant sets for $(g-2)$. The summary of this literature is in sect. 1.2.2.

2017-05-21 Predrag to Sergey A. Volkov <volkoff_sergey@mail.ru>

I have read your *New method of computing the contributions of graphs without lepton loops to the electron anomalous magnetic moment in QED* [151] (and the earlier ref. [150]) with great interest, and I can see that you are set to compute the 5-loop correction to the electron $(g-2)$. This is a very hard calculation, and approaching it strategically is a necessity.

May I suggests that you order your calculation by gauge sets of figure 1.5, as illustrated (for the 4-loop case) in Stefano Laporta's figure 1.6? My hunch is that the gauge set (6) = $(4, 0, 0)$ would be the most interesting, though Stefano thinks it too hard, and suggests starting with a 5-loop relative $(1, 3, 1)$ (or $(1, 2, 2)$?) of the set (3) = $(1, 2, 1)$ instead. While the contributions of individual vertex graphs (and self-energy sets [9]) are all over the place, all gauge-invariant sets are insanely small up to order 8, and it would be very sweet to see that this continues through order 10 (at least for the 5-loop graphs with no electron loops).

By the way, to check my conjecture one needs the gauge sets only to two significant digits or so, no high accuracy is needed.

2017-06-18 Sergey **About gauge sets**: It is possible to calculate the contributions of (k, m, m') by my method. It will require more computer resources than the calculation of the total contribution without splitting into sets, because the sets (k, m, m') cut the families that I use for calculation into many pieces. I will try to calculate 5-loop contributions of (k, m, m') after some refinement of my code (this requires some months). Even a not precise calculation needs a lot of computer time.

2017-06-18 Sergey About **"That QED on-mass shell amplitudes are IR-free must be an old result..."**: Is there a rigorous mathematical proof that they are IR-free? I didn't find such a proof. I saw only some approximate considerations about this. The proof of the Bogoliubov-Parasiuk theorem is quite large and complicated. I think that the total proof of IR and UV cancellation should be larger. However, all considerations that I saw are much simpler. I think that an existence of such a proof should affect computational methods. However, at the present moment, the best calculation of the QED contributions to g-2 (Aoyama, Kinoshita et al.) uses the subtraction procedure that needs a handwork at 5-loop level (see the 2008 Aoyama *et al.* article [10], and the link to it in the end of their 2012 article [8]). On the other hand, all mathematical proofs of IR cancellations that I saw are connected with IR divergences of other nature (not on-shell). Is there a mathematical proof that the QED contributions to $g-2$ are finite at any order of the perturbation series (only for each term, not for the sum)?

2017-06-18 Predrag to Sergey I thought that my thesis argument on IR divergences subtraction in the Feynman-parametric space in [46] establishes IR finitness for (g-2). We show that the UV and IR finite part of amplitude M_G is given by

$$\Delta M_G = \prod_{ij}(1 - I_{G/S_i})(1 - K_{G/S_j})M_G \tag{1.29}$$

where the products are over all self-energy and vertex subdiagrams S_i and S_j. We then use Ward identities to show that UV and IR divergent parts of counterterms $I_{G/S_i}M_G$, $K_{G/S_j}M_G$, etc. cancel on mass shell (and the finite parts contribute to the anomaly). In the paper I check various cases, and I seem to have missed some that do not arise in the 3-loop order. But how could anything be simpler than (1.29)? And how could such elegant formula be wrong?

I think the UV/IR finitness is very clean and clear to see in the dimensional regularization. I have now reread my 1976 *Yang-Mills theory of the mass-shell* [38], and it is a rather amusing, but probably not a persuasive answer to your query. The paper is good and original, so it is never cited. At least it got published. The original draft was to be exactly *one* page of Physical Review Letters, had *one* reference, a homage to Schwinger [136]. Those were happier times - I wrote it in a couple of days while visiting Helsinki Research Institute for Theoretical Physics, mailed it, impressing my hosts infinitely - they have never seen a PR Lett written at that speed, and *bicycled off into Lapland*. But referees made me quadruple the length of the Letter, making it as boring as any other PR Lett. For the proof of absence of on-mass shell $(g-2)$ IR divergences I cite no less than three of my own unpublished papers (!!!), and nobody else. What happened? I had understood that (the finiteness of QED conjecture had much to do with it) that doing more Feynman graphs would be stupid (expanding around the wrong vacuum), and thus started my nonlinear detour. I started thinking about functional equations on May 1, 1976, and wrote down *the period doubling fixed point function equation* on May 3, 1976. In my office I do have a very thick folder marked UNPUBLISHED, and it must contain the drafts of these IR papers. But how would I know that anyone would care 40 years later?

The one that has IAS number was actually published [37], see here, and it says:

> "The anomaly, expressed in terms of the unrenornormalized coupling constant, is free of infrared divergences for QED (i.e., all $a^{(2n)}$ are IR finite). This is particularly easy to demonstrate using the method for separation of IR divergences given in ref. [46]. The analysis is rather technical, but the result is very intuitive: a Feynman integral M_G

> contributing to the magnetic moment is IR divergent whenever the corresponding diagram G can be split into a vertex subdtagram S and a cloud of soft gluons attached to the external quark lines (diagram G/S). In general some gluons in G/S can be hard, provided they are within UV divergent subdiagrams, In all cases, the IR divergent part of M_G factors into M_S times the IR divergent part of $L_{G/S}$, where $L_{G/S}$ is the charge form factor computed from the diagram G/S."

and so on - the paper is focused on QCD, but I believe it establishes IR finiteness of QED on-mass shell amplitudes, though it most likely not as rigorously as you would like it.

Yennie and S. C. Frautschi and H. Suura [153] *The infrared divergence phenomena and high-energy processes* (cited over 800 times!) and Kinoshita [89] *Mass singularities of Feynman amplitudes* say nothing about cancellation of IR divergences by renormalization. Herzog and Ruijl [77] *The R*-operation for Feynman graphs with generic numerators* and Chetyrkin, Tkachov and Smirnov [32, 33, 141] are not of interest, as R*-operation is not about on-shell IR-divergences. (Herzog and Ruijl [77] cite Chetyrkin, Tkachov and Smirnov [32, 33, 141] but no Volkov [150, 151] or Cvitanvić [37, 38, 40, 46, 48] papers.)

In ref. [46] we also cited Johnson and Zumino [85] *Gauge dependence of the wave-function renormalization constant in Quantum Electrodynamics*.

An unrelated IR problem: Kinoshita [88] *Note on the infrared catastrophe*, about which he writes [91]: " I developed a method to handle the infrared problem within the perturbative framework of Feynman–Dyson theory [88]. My insight was not deep enough, however, and I overlooked a subtlety in the infrared problem. This became the source of an error in the calculation of radiative correction to the $\mu - e$ decay. "

2017-06-11 Predrag to
Christian Schubert <schubert@ifm.umich.mx>

Stefano Laporta has recently published analytic values of all 4-loop electron magnetic moment gauge sets, and they look very intriguing. Sergey A. Volkov has started evaluating individual 5-loop vertex diagrams and is in position to estimate numerically the size of 5-loop gauge sets. So it might be a good time to make a new attempt to prove/disprove the QED finiteness conjecture. This set of notes is my current best attempt to motivate this effort.

I do not take my 1977 "gauge-set approximation" very literally - its content is only that if individual gauge sets can be bounded to anything growing slower than combinatorially, quenched QED (and hopefully full QED) is a finite theory, not an asymptotic series. But the form of vertex gauge sets is very suggestive; it is defined in terms of N-photon exchanges. So to me the wordline formalism seem the most promising way forward.

While the quenched QED magnetic moment of electron is the cleanest possible physical calculation one can do, for sociological reasons most effort on high order estimates has gone in other directions. Do you have a formula for spinor QED magnetic moment that one could attempt to analyze more closely?

2017-06-12 Christian I have been fascinated by the finiteness conjecture ever after Gerald Dunne and I were led there, too, from Euler-Heisenberg, Borel analysis, Ritus mass shift and worldline instantons fifteen years ago. The things have picked up a lot just during the last year or so, namely:

1. Idrish Huet, Michel Rausch and I are working on the Affleck, Alvarez and Manton (AAM) exponentiation conjecture [1] in 1+1 QED. We had a nice parameter integral representation for the three-loop Euler-Heisenberg for quite a while, but could not get a sufficient number of weak-field expansion coeffs out of it (so far what we got points to AAM not holding at three loops - rather the coeffs go asymptotically BELOW the AAM prediction - but this is very preliminary). Michel was just visiting, and we now got a really nice algorithm (based on the polynomial invariants of the dihedral group) for the analytical calculation of the coeffs from the more difficult (nonplanar) 3-loop EH diagram.
2. If AAM really fails, then presumably the AAM worldline instanton needs refinement at higher loops. My former student Naser Ahmadiniaz (now postdoc in Korea) has made some progress with this.
3. For many years I am planning to apply the worldline formalism to (g-2), in particular to the important graphs with the light-by-light subdiagram. For this subdiagram I have, since a long time, an off-shell representation that is permutation symmetric, manifestly gauge invariant, without spurious UV poles, and moreover allows one to trivially integrate out the one low-energy photon leg. What held me back was that the various worldline representations that existed hitherto for the electron line were all somewhat cumbersome, and seemed not suitable for high-order calculations. Precisely this problem we have been working on here for the last half year, and things have fallen into place really nicely, we have a Bern-Kosower type master formula for the electron line dressed with any number of photons, in vacuum and in the presence of a constant field, and one of our students is already programming it. I am now definitely trying to assemble a collaboration to attack the QED (g-2), and I anticipate that your notes will be quite useful for motivating my collaborators.

Gies *et al.* [74] have some intriguing numerical results from worldline Monte Carlo (in section 5). Another thing I find quite interesting are Ritus [21] most recent papers [122–124] on the value of the bare fine structure constant - he confirmed to me in an email from last December (at age 90) that he definitely does not think that the bare charge is infinite.

I am appending a summary of our efforts in this line of work [79] which I wrote as a contribution to the 5th Winter Workshop on Non-Perturbative Quantum Field Theory. 22-24 March 2017, Sophia-Antipolis.

2017-06-18 Naser Ahmadiniaz <ahmadiniaz.naser@gmail.com> (currently a postdoctoral fellow at the Center for Relativistic Laser Science (CoReLS), Institute for Basic Science (IBS), Gwangju, South Korea).
My main research interests are in amplitude calculations in QED, QCD and quantum gravity from the worldline formalism. Recently, we have been interested in (off-shell) tree-level amplitudes for QED processes in vacuum as well as in the presence of classical background fields which will also be applied to higher order corrections.

2017-06-20 James P. Edwards <jedwards@ifm.umich.mx>, homepage:
I am currently working with Christian on the worldline approach. In some of our recent work we have been working on more efficient ways to determine g-2 based upon a worldline expression for the dressed spinor propagator (both in vacuum and in a constant background).

2017-06-11 Predrag to Dirk Kreimer <kreimer@physik.hu-berlin.de>

Stefano Laporta has recently published analytic values of all 4-loop electron magnetic moment gauge sets, and they look very intriguing. It might be a good time to make a new attempt to prove/disprove the QED finiteness conjecture. This set of notes is my current best attempt to motivate this effort.

If individual gauge sets can be bounded to anything growing slower than combinatorially, quenched QED (and hopefully full QED) is a finite theory, not an asymptotic series. The form of vertex gauge sets is suggestive of a way forward; it is defined in terms of N-photon exchanges. Hopf algebras might hold the key, but I do not know how to use these ideas.

I would be grateful for any further pointers to the literature. Anybody else I should send these notes to? And I'm finally ready to stand up an be counted - I can fly to Berlin on a short notice if that helps.

2017-06-14 Dirk Kreimer consider coming to Les Houches — June 4-15, 2018 on structures in local quantum field theory.

1.9 QCD gauge sets - a blog

In 1981 Cvitanović *et al.* [48] constructed gauge invariant subsectors in QCD.

2016-12-10 Predrag Penante [117] 2016 On-shell methods for off-shell quantities in N=4 Super Yang-Mills: from scattering amplitudes to form factors and the dilatation operator has an up-to-date review of on-shell methods.

2016-12-26 Predrag Read Cruz-Santiago, Kotko and Staśto [36] 2015 *Scattering amplitudes in the light-front formalism*: "The idea is to divide the process into appropriate gauge invariant components. It turns out that the gauge invariant subsets are invariant under cyclic permutations of the external gluons. This decomposition was proposed in works of [58–61] for the tree level amplitudes. A thorough analysis of the relation between color structures and gauge invariance was done in ref. [48]. The color decomposition principle was extended beyond the tree level to loop amplitudes in [63]."

2016-12-26 Predrag Should also read Dixon [51] 1996 *Calculating scattering amplitudes efficiently*.

2017-05-26 Predrag The decomposition of scattering amplitudes into gauge invariant subsets of diagrams is studied by Boos and Ohl [12, 22]. Boos and Ohl [22] *Minimal gauge invariant classes of tree diagrams in gauge theories*, arXiv:hep-ph/9903357 (see arXiv:hep-ph/9911437 and arXiv:hep-ph/0307057 for more detail) is motivated by applications to Standard Model multi-particle diagrams, mostly at the tree level.

Perturbative calculations require an explicit breaking of gauge invariance for technical reasons and the cancellation of unphysical contributions is not manifest in intermediate stages of calculations. The contribution of a particular Feynman diagram to a scattering amplitude depends in the gauge fixing procedure and has no physical meaning. the identification of partial sums of Feynman diagrams that are gauge invariant by themselves is of great practical importance. Calculation a subset of diagrams that is not gauge invariant has no predictive power, because they depend on unphysical parameters introduced during the gauge fixing.

A *gauge invariance class* is a minimal subset of Feynman diagrams that is independent of the gauge parameter and satisfies the Slavnov-Taylor identities.

The set of diagrams connected by flavor and gauge flips they call *forest*, a set of diagrams connected by gauge flips the call *grove*. They shown that the groves are the minimal gauge invariance classes of tree Feynman diagrams. In unbroken gauge theories, the permutation symmetry of external gauge quantum numbers can be used to subdivide the scattering amplitude corresponding to a grove further into gauge invariant sub-amplitudes.

This (largely uncited) work seems to have no impact on the $(g-2)$ gauge sets discussed here.

2017-05-27 Predrag Reuschle and Weinzierl [121] *Decomposition of one-loop QCD amplitudes into primitive amplitudes based on shuffle relations* cite our ref. [48]. They say:

QCD calculations organise the computation of the one-loop amplitude as a sum over smaller pieces, called *primitive amplitudes*. The most important features of a primitive amplitude are gauge invariance and a fixed cyclic ordering of the external legs. Primitive amplitudes should not be confused with *partial amplitudes* (also referred to as a *dual amplitude* or a *color-ordered amplitude*), which are the kinematic coefficients of the independent colour structures. The first step in a discussion of perturbative Yang-Mills is the decoupling of color from kinematics,

$$A_{tot} = \sum c_J A_J \tag{1.30}$$

where A_{tot} represents the total amplitude for a scattering process, A_J are all the possible color structures, and A_J are partial amplitudes which depend only on the kinematical data (momenta and polarizations). Partial amplitudes are gauge invariant, but not necessarily cyclic ordered. Partial amplitudes are far simpler to calculate than the full amplitude. There exist linear relations among the partial amplitudes, called Kleiss-Kuijf relations, which reduce the number of linearly independent partial amplitudes to $(n-2)!$ The leading contributions in an $1/N$-expansion (with N being the number of colours) are usually cyclic ordered, the sub-leading parts are in general not. The decomposition of the full one-loop amplitude into partial amplitudes is easily derived. However, it is less trivial to find a decomposition of the partial amplitudes into primitive amplitudes.

There are several possible choices for a basis in colour space. A convenient choice is the colour-flow basis [147].

2017-05-27 Predrag Schuster [135] *Color ordering in QCD*: " We derive color decompositions of arbitrary tree and one-loop QCD amplitudes into color-ordered objects called primitive amplitudes. "

2017-05-27 Predrag Zeppenfeld [154] *Diagonalization of color factors* (Georgia Tech has no access to this paper)

2017-05-27 Predrag Edison and Naculich [58] *Symmetric-group decomposition of SU(N) group-theory constraints on four-, five-, and six-point color-ordered amplitudes at all loop orders*: " Color-ordered amplitudes for the scattering of n particles in the adjoint representation of SU(N) gauge theory satisfy constraints that arise from group theory alone. These constraints break into subsets associated with irreducible representations of the symmetric group S_n, which allows them to be presented in a compact and natural way. "

2017-05-27 Predrag Kol and Shir [97, 98] *Color structures and permutations* has a useful overview of the literature in the introduction, and is a very interesting read overall.

We may permute (or re-label) the external legs in the expression for a color structure and thereby obtain another color structure. This means that the space of color structures is a representation of S_n, the group of permutations. A natural question is to characterize this representation including its character and its decomposition into irreducible representations (irreps).

The decomposition of color structures into irreps was suggested by Zeppenfeld [154].

The space of tree-level color structures TCS_n of dimension

$$\dim(TCS_n) = (n-2)! \tag{1.31}$$

is the vector space generated by all diagrams with n external legs and an oriented cubic vertex, which are connected and without loops (trees), where diagrams which differ by the Jacobi identity are to be identified.

The f-based and t-based color structures are related by celebrated Kleiss-Kuijf [96] relations (rederived in this paper).

The original problem, that of capturing symmetries of the partial amplitudes which originate with those of the color structures, is now formulated as the problem of obtaining the S_n character of the space of color structures. It turns out that (at least at tree level) this problem was fully solved in the mathematics literature by Getzler and M. M. Kapranov [72].

The free Lie algebra over some set A, denoted by $L(A)$, is the Lie algebra generated by A with no further relations apart for antisymmetry and the Jacobi identity which are mandated by definition.

Self duality under Young conjugation: for some n values TCS_n is self-dual under Young conjugation, namely under the interchange of rows and columns in the Young diagrams

2017-05-27 Predrag Getzler and Kapranov [72] *Modular operads*: " We develop a 'higher genus' analogue of operads, which we call modular operads, in which graphs replace trees in the definition. We study a functor F on the category of modular operads, the Feynman transform, which generalizes Kontsevich's graph complexes and also the bar construction for operads. We calculate the Euler characteristic of the Feynman transform, using the theory of symmetric functions: our formula is modelled on Wick's theorem. "

2017-05-27 Predrag Maltoni *et al.* [111] *Color-flow decomposition of QCD amplitudes*

The *color-flow* decomposition is based on treating the $SU(N)$ gluon field as an $N \times N$ matrix. (PC: I think that is what I actually do.) It has several nice features. First, a similar decomposition exists for all multiparton amplitudes, like the fundamental-representation decomposition. Second, the color-flow decomposition allows for a very efficient calculation of multiparton amplitudes. Third, it is a very natural way to decompose a QCD amplitude. As the name suggests, it is based on the flow of color, so the decomposition has a simple physical interpretation.

To calculate the amplitude, one orders the gluons clockwise, and draws color-flow lines, with color flowing counterclockwise, connecting adjacent gluons. One then deforms the color-flow lines in all possible ways to form the Feynman diagrams that contribute to this partial amplitude. The Feynman diagrams that contribute to a partial amplitude are planar. This is not due to an expansion in $1/N$; the partial amplitudes are exact. They note (see their Table 1) that the number of Feynman diagrams contributing to an n-gluon partial amplitude grows as $\approx 3 \cdot 8^n$. In contrast, the number of Feynman diagrams contributing to the full amplitude grows factorially, as $\approx (2n)!$.

2017-06-09 Predrag Henn *et al.* [76] *Four-loop photon quark form factor and cusp anomalous dimension in the large-N_c limit of QCD*, arXiv:1612.04389, is a thoroughly modern paper, a listing of different codes used to generate diagrams and evaluate integrals.

2017-06-16 Predrag Chang, Liu and Roberts [31] *Dressed-quark anomalous magnetic moments*

2017-06-16 Predrag Choudhury and Lahiri [34] *Anomalous chromomagnetic moment of quarks*

2017-06-16 Predrag Bermudez *et al.* [16] *Quark-gluon vertex: A perturbation theory primer and beyond*, arXiv:1702.04437: The on-shell limit enables us to compute anomalous chromomagnetic moment of quarks.

[...] we present some "physically" relevant results for the on-shell limit $p^2 = k^2 = m^2$ and $q^2 = 0$. The Dirac and Pauli form factors, $F_1(q^2)$ and $F_2(q^2)$, respectively, define the Gordon decomposition of the quark current as in (1.2). The anomalous chromomagnetic moment (ACM) of quarks can be identified as $F_2(q^2)$ for $q^2 \to 0$. The Abelian version of this decomposition with $C_F = 1$ and $C_A = 0$ is the electron-photon vertex of quantum electrodynamics. The great successes of the Dirac equation is the prediction of the magnetic moment of a charged fermion $\mu = eg/(2m)S$. The radiative corrections lead to [136]

$$\frac{e}{2m} \Rightarrow \left(1 + \frac{\alpha}{2\pi}\right) \frac{e}{2m}. \tag{1.32}$$

Note that the quark-gluon vertex differs from the electron-photon vertex already at one loop, by the contributions of an additional Feynman diagram, involving the triple-gluon vertex. In fact, apart from introducing additional color structure, this non-Abelian diagram introduces, at the one-loop level, a kinematical structure which is absent in the QED.

It is straightforward to see that for the soft gluon limit, $q^2 = 0$, the Abelian contribution for the ACM reduces to the non-Abelian counterpart of Schwinger's result, $F_2^a(0) = -\alpha/12\pi$, already derived in ref. [31]. On the other hand, the corresponding non-Abelian contribution yields a divergence [34], for a non-zero quark mass, $m \neq 0$. We find this divergence to be logarithmic. For deep infrared gluon momenta it behaves as $F_2^b(q^2 \to 0) = C_b \ln\left(-q^2/m^2\right)$. Of course, perturbation theory in QCD is not the way to explore deep infrared region. All perturbative conclusions will be taken over by non-perturbative effects, overshadowing this divergence.

2017-06-16 Predrag Brambilla *et al.* [25] *QCD and strongly coupled gauge theories: challenges and perspectives*

2017-06-16 Predrag Simonov and Tjon [139] *The Feynman–Schwinger representation in QCD*:" The proper time path integral representation is derived for Green's functions in QCD. After an introductory analysis of perturbative properties, the total gluonic field is separated into a nonperturbative background and valence gluon part. For nonperturbative contributions the background perturbation theory is used systematically, yielding two types of expansions. As an application, we discuss the collinear singularities in the Feynman–Schwinger representation formalism. "

1.10 Is QED finite? A blog

1977-03-03 Predrag Drell and Pagels [52] *Anomalous magnetic moment of the electron, muon, and nucleon* attempt got the sign right, but was not successful in predicting the magnitude of the sixth-order magnetic moment; $0.15\left(\frac{\alpha}{\pi}\right)^3$ instead of $1.19\left(\frac{\alpha}{\pi}\right)^3$.

1971-08-01 Lautrup, Peterman, and de Rafael [108] 1972 *Recent developments in the comparison between theory and experiments in quantum electrodynamics* list the 3-loop, no-electron loop "gauge invariant subclasses" (their Fig. 4.3).

1974-01-07 Samuel [130] 1974 *Estimates of the eighth-order corrections to the anomalous magnetic moment of the muon*:

"We speculate that in making radiative corrections to a class of graphs by inserting a single photon in all possible ways, one obtains a contribution which is roughly $-\frac{\alpha}{\pi}$ times the contribution of the class. This seems to be obeyed by the known contributions."

2013-11-24 Predrag As far as I can tell, terminology "of *quenched* type" was first introduced in this context by Marinari, Parisi and Rebbi [112] *Monte Carlo simulation of the massive Schwinger model*, in the context of lattice gauge theory. They write: "A first approximation to the effect of the gauge field on the fermion observables may be achieved by [...] neglecting the contributions from the fermionic vacuum polarization diagrams and, using a terminology developed in the theory of condensed matter, we shall call the expectation values thus obtained 'quenched'",

Parisi is reputable, so in the "quenched approximation" one neglects the fermionic vacuum polarization effects (i.e, the fermion loops) from the fermion determinant in the effective action. If it is good enough for Kinoshita, it is good enough for you.

"In the 'quenched approximation' the quark determinant is set equal to unity, i.e., neglecting the effect of virtual quark loops. In other words, this extreme approximation in terms of heavy quarks with a vanishing number of flavors assumes that gauge fields affect quarks while quarks have no dynamical effect on gauge fields."

A more general usage: "In Quantum Cosmology "quenching," amounts to quantizing a single scale factor thereby selecting a class of cosmological models, for instance, the Friedmann-Robertson-Walker space-time while neglecting the quantum fluctuations of the full metric."

But I still do not like it - it is mostly associated with Kogut, where it means something different (as in "... treating the gauge interaction in the quenched, planar (ladder) approximation"); search for quench here. Or here is what Brezin says:

"concept of quenching is well-known in the statistical mechanics of random media ; consider a system of particles, for instance, an electron gas, interacting with impurities. If these impurities are mobile, they will thermalize with the electron gas and the average physical quantities are obtained by a trace over the electron gas and the impurities degrees of freedom. However if the impurities are frozen, the 'quenched' case, the physical observables are obtained by calculating their value for fixed impurities and then averaging over these impurities. "

That is how I know it - nothing about fermion loops, just dirt physics...

From my point of view, the question is whether the sum of all corrections to (g-2) is a convergent series, or an asymptotic one. If on can prove the convergence for the quenched sector, I would expect each un-quenched sector (diagrams with one, two, ... lops) separately to be convergent, and their sum as well.

Here is something to amuse you: on amplituhedron. More serious: Lance Dixon on calculating amplitudes.

The inventor of the "gauge invariant diagram sets" concept is Benny Lautrup.

2013-12-08 Predrag to Piotr, Wanda and Andrea (Piotr Czerski <piotr.czerski@ifj.edu.pl>, wanda.alberico@to.infn.it, andrea.prunotto@gmail.com):

I'm no fan of Feynman diagrams (my rant is here), and I'm always looking for other ways to look at perturbative expansions. So just a little email - if you have a new angle [120] on subsets of diagrams which are gauge invariant sets, I would be curious to learn how you look at that.

Just something to keep in mind :)

PS to Andrea: I realize you might rather forget this stuff (takes you a decade to write a paper?) but at least I got a ringtone out of you. The only problem is, I do not have a cell phone, so I do not know how to make it ring. At least I'm more technologically savvy than Peter Higgs.

2013-12-10 Andrea Sorry for late reply (well, we're used to longer gaps). Yes! I actually took 10 years to write this paper out of my master thesis, but I have some excuses: I did my PhD in Biochemistry (Zürich) and now I work on genetics (Lausanne). This summer my "old" professor Wanda found my work in a drawer and then contacted me, telling me that it would be a good idea to publish it.

About your request: I'm really interested in seeing if the rooted-map approach to Feynman diagrams can address the problem you've risen. But I have no idea what the "subsets of diagrams which are gauge invariant sets" are. I've checked a bit on the web but I'm sure you can give me better indications (the works I found were too technical: I need to know the basis of the problem). Can you send me some specific link at freshman level, in particular where I can see the geometry of these subclasses of diagrams?

2013-12-11 Predrag Googling is good, but it is faster to click on this link. The article defines the gauge invariant sets.

2014-02-11 M. Borinsky *Feynman graph generation and calculations in the Hopf algebra of Feynman graphs* [23] "Programs for the computation of perturbative expansions of quantum field theory amplitudes are provided. feyngen can be used to generate Feynman graphs for Yang-Mills, QED and phi^k theories. feyncop implements the Hopf algebra of those Feynman graphs which incorporates the renormalization procedure necessary to calculate finite results in perturbation theory of the underlying quantum field theory. "

2016-02-08 Predrag Prunotto [119] *A Homological Approach to Feynman Diagrams in the Quantum Many-Body Theory*, and Prunotto, Alberico and Czerski [120] 2013 *Feynman Diagrams and Rooted Maps* has been submitted to the European Physical Journal A as manuscript ID EPJA-103480, seems not to have been published anywhere by 2017. They write: " The Rooted Maps Theory, a branch of the Theory of Homology, is shown to be a powerful tool for investigating the topological properties of Feynman diagrams, related to the single particle propagator in the quantum many-body systems. The numerical correspondence between the number of this class of Feynman diagrams as a function of perturbative order and the number of rooted maps as a function of the number of edges is studied. A graphical procedure to associate Feynman diagrams and rooted maps is then stated. Finally, starting from rooted maps principles, an original definition of the genus of a Feynman diagram, which totally differs from the usual one, is given. "

2017-03-15 Predrag Dunne and Krasnansky [53] 2006 *"Background field integration-by-parts" and the connection between one-loop and two-loop Heisenberg-Euler effective actions*: " We develop integration-by-parts rules for diagrams involving massive scalar propagators in a constant background electromagnetic field, and use these to show that there is a simple diagrammatic interpretation of mass renormalization in the two-loop scalar QED Heisenberg-Euler effective action for a general constant background field. This explains why the square of a one-loop term appears in the renormalized two-loop Heisenberg-Euler effective action, and dramatically simplifies the computation of the renormalized two-loop effective action for scalar QED, and generalizes a previous result obtained for self-dual background fields. "

2017-05-23 Predrag M. G. Schmidt and C. Schubert [131] 1994 *Multiloop calculations in the string-inspired formalism: the single spinor-loop in QED*, arXiv:hep-th/9410100: They use the worldline path-integral Bern-Kosower formalism for to calculate the sum of all diagrams with one spinor loop and fixed numbers of external and internal photons. Of interest: in this formalism the three 2-loop photon polarization graphs, see figure 1.7, are a single integral, easier to evaluate than any of the three Feynman graphs. They also note an unexplained cancelation not only of poles, but also of "transcedentals." A knot-theoretic explanation for the rationality of the quenched QED beta function is given in ref. [27].

2017-05-23 Predrag Nieuwenhuis and Tjon [116] 1996 *Nonperturbative study of generalized ladder graphs in a $\phi^2\chi$ theory*, arXiv:hep-ph/9606403:

2017-05-23 Predrag Christian Schubert [133] 2001 *Perturbative quantum field theory in the string-inspired formalism*, arXiv:hep-th/0101036:

The Feynman rules for (Euclidean) spinor QED in the second order formalism (see Morgan [115] 1995, Strassler [145] 1992, and references therein) are, up to statistics and degrees of freedom, the ones for scalar QED with the addition of a third vertex. The third vertex involves $\sigma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]$ and corresponds to the $\psi^\mu F_{\mu\nu}\psi^\nu$ – term in the worldline Lagrangian L_{spin}. For the details and for the non-abelian case see Morgan [115]. There also an algorithm is given, based on the Gordon identity, which transforms the sum of Feynman (momentum) integrals resulting from the first order rules into the ones generated by the second order rules.

2017-06-12 Predrag Morgan [115] *Second order fermions in gauge theories*, arXiv:hep-ph/9502230 seems to be the same discussion that I use in my QFT course to define the electron magnetic moment via $\sigma^{\mu\nu}$ (see lectures 25 and 26 here)

2017-06-12 Predrag Gies, Sanchez-Guillen, Vázquez [74] *Quantum effective actions from nonperturbative worldline dynamics*, arXiv:hep-th/0505275: " We demonstrate the feasibility of a nonperturbative analysis of quantum field theory in the worldline formalism with the help of an efficient numerical algorithm. In particular, we compute the effective action for a super-renormalizable field theory with cubic scalar interaction in four dimensions in quenched approximation (small- N f expansion) to all orders in the coupling. We observe that nonperturbative effects exert a strong influence on the infrared behavior, rendering the massless limit well defined in contrast to the perturbative expectation. "

2017-06-12 Predrag Giesand Hämmerling [73] *Geometry of spin-field coupling on the worldline*, arXiv:hep-th/0505072:

2017-03-15 Predrag Huet, McKeon, and Schubert [78] 2010 *Euler-Heisenberg lagrangians and asymptotic analysis in 1+1 QED. Part I: Two-loop* (no GaTech online access, arXiv:1010.5315):" We continue an effort to obtain information on the QED perturbation series at high loop orders, and particularly on the issue of large cancellations inside gauge invariant classes of graphs, using the example of the l-loop N-photon amplitudes in the limit of large photon numbers and low photon energies. The high-order information on these amplitudes can be obtained from a nonperturbative formula, due to Affleck *et al.* [1], for the imaginary part of the QED effective lagrangian in a constant field. The procedure uses Borel analysis and leads, under some plausible assumptions, to a number of nontrivial predictions already at the three-loop level. Their direct verification would require a calculation of this 'Euler-Heisenberg lagrangian' at three-loops, which seems presently out of reach (though see Huet, de Traubenberg, and Schubert [80] below). Motivated by previous work by Dunne and Krasnansky [53] on Euler-Heisenberg Lagrangians in various dimensions, in the present work we initiate a new line of attack on this problem by deriving and proving the analogous predictions in the simpler setting of 1+1 dimensional QED. In the first part of this series, we obtain a generalization of the formula of Affleck *et al.* [1] to this case, and show that, for both scalar and spinor QED, it correctly predicts the leading asymptotic behaviour of the weak field expansion coefficients of the two loop Euler-Heisenberg lagrangians.

> "The present work continues an effort [54–56, 113] to study the multiloop behaviour of the QED N-photon amplitudes using the QED effective lagrangian, and in particular to prove or disprove Cvitanović's conjecture for these amplitudes."

2017-05-24 Predrag Bastianelli, Huet, Schubert, Thakur and Weber [15] 2014 *Integral representations combining ladders and crossed-ladders* write:

> This property is particularly interesting in view of the fact that it is just this type of summation which in QED often leads to extensive cancellations, and to final results which are substantially simpler than intermediate ones (see, e.g., ref. [27, 39]). More recently, similar cancellations have been found also for graviton amplitudes (see, e.g., ref. [11]). Although this property of the worldline formalism is well-known, and has been occasionally exploited [13, 125, 126, 131] (see also ref. [69]) a systematic study of its implications is presently still lacking.
>
> The first classes of Green's functions is the x-space propagator for one scalar interacting with the second one through the exchange of N given momenta.
>
> This object, to be called "N-propagator", is given by a set of $N!$ simple tree-level graphs, is in the worldline formalism combined into a single integral.
>
> The second class are the similarly looking x-space $N+2$ - point functions defined by a line connecting the points x and y and N further points $z_1, \ldots, z_N$ connecting to this line in an arbitrary order.
>
> An advantage of the worldline representation over the usual Feynman parameterization is the automatic inclusion of all possible ways of crossing the "rungs" of the ladders. They obtain such representations in explicit form both in x-space and in momentum space.
>
> The inclusion of the crossed ladder graphs is essential for the consistency of the one-body limit where one of the constituents becomes infinitely heavy, and for maintaining gauge invariance.
>
> They concentrate on the case of infinite N, *i.e.*, the sum over *all* ladder *and* crossed ladder graphs.
>
> As their main application, they consider the case of two massive scalars interacting through the exchange of a massless scalar, obtain an the case of a massless exchanged particle (along the "rungs" of the ladders).
>
> Applying asymptotic estimates and a saddle-point approximation to the N-rung ladder plus crossed ladder diagrams, they derive a semi-analytic approximation formula for the lowest bound state mass in this model.
>
> They use the worldline formalism to derive integral representations for the N-propagators and the N-ladders - in scalar field theory, and give a compact expression combining the $N!$ Feynman diagrams contributing to the amplitude. They give these representations in both x and (off-shell) momentum space. Being off-shell, can be used as building blocks for more complex amplitudes. They derive

a compact expression for the sum of all ladder graphs with N rungs, including all possible crossings of the rungs.

Nieuwenhuis and Tjon [116] 1996 have numerically evaluated the path integrals of the worldline representation for the same scalar model field theory, thus including all ladder *and* crossed ladder graphs.

2017-05-23 Predrag Huet, de Traubenberg, and Schubert [80] 2017 *Multiloop Euler-Heisenberg Lagrangians, Schwinger pair creation, and the photon S-matrix*: "Schwinger pair creation in a constant electric field, may possibly provide a window to high loop orders; simple non-perturbative closed-form expressions have been conjectured for the pair creation rate in the weak field limit, for scalar QED in 1982 by Affleck, Alvarez, and Manton [1], and for spinor QED by Lebedev and Ritus [109] in 1984. Using Borel analysis, these can be used to obtain non-perturbative information on the on-shell renormalized N-photon amplitudes at large N and low energy."

"there is something quite implausible about it: a summation to all loop orders has produced the perfectly analytic factor $e^{\alpha\pi}$! This is certainly contrary to standard QED wisdom"

Preliminary results of a calculation of the three-loop Euler–Heisenberg Lagrangian in two dimensions indicate that the exponentiation conjecture by Affleck *et al.* and Lebedev/Ritus probably fails in $D = 2$.

Dunne and Schubert conjectured in 2005 that the QED $N-$photon amplitudes in the quenched (one electron loop) approximation are convergent in perturbation theory [56]. In this article they say; "Later they learned that Cvitanović in 1977 had already made the analogous conjecture for $(g-2)$ [39]."

2017-06-14 Predrag Academician Ritus [123] writes: " The requirement $e^2/\hbar c = 1$ leads to unique values of the point-like charge and its fine structure constant, $e_0 = \pm\sqrt{\hbar c}$, $\alpha_0 = 1/4\pi$. Arguments are adduced in favor of the conclusion that this value of the fine structure constant is the bare, nonrenormalized value. "

In 1951 Ritus was assigned to what was known at the time as the First Main Directorate of the USSR Council of Ministers, later rechristened the Ministry of Medium Machine Building (Sredmash) – a powerful state body placed above any other in the name of implementing the Soviet Government sponsored program of thermonuclear weapons design. [...] The legend of his infallibility when conducting complicated and cumbersome computations, just started to take root; its protagonist did nothing that would sully this reputation, neither then nor later. Science is unaware of any errors ever made by Ritus! (from ref. [21].)

2017-05-23 Predrag Das, Frenkel and Schubert [50] 2013 *Infrared divergences, mass shell singularities and gauge dependence of the dynamical fermion mass*; arXiv:1212.2057:

2017-05-23 Predrag Ahmadiniaz, Bashir and Schubert [3] 2016 *Multiphoton amplitudes and generalized Landau-Khalatnikov-Fradkin transformation in scalar QED*, arXiv:1511.05087:

1.10 *Is QED finite? A blog*

$D_{\mu\nu}$ is the x-space photon propagator. In D dimensions and arbitrary covariant gauge

$$D_{\mu\nu}(x) = \frac{1}{4\pi^{\frac{D}{2}}}\left\{\frac{1+\xi}{2}\Gamma\left(\frac{D}{2}-1\right)\frac{\delta_{\mu\nu}}{(x^2)^{\frac{D}{2}-1}}+(1-\xi)\Gamma\left(\frac{D}{2}\right)\frac{x_\mu x_\nu}{(x^2)^{\frac{D}{2}}}\right\}. \quad (1.33)$$

Calculation methods:

1. The analytic or "string-inspired" approach, based on the use of worldline Green's functions: all path integrals are brought into Gaussian form; this requires some expansion and truncation. They are then calculated by Gaussian integration.
2. The semi-classical approximation, based on a stationary trajectory ("worldline instanton").

We will focus on the closed-loop case in the following, since it turns out to be simpler than the propagator one. Nevertheless, it should be emphasized that everything that we will do in the following for the effective action can also be done for the propagator.

Some reasonable gymnastics leads to the "Bern-Kosower master formula" [17, 18, 145]

2017-05-23 Predrag Strassler [145] *Field theory without Feynman diagrams: One-loop effective actions*, arXiv:hep-ph/9205205;

2017-05-23 Predrag Ahmad *et al.* [2] 2017 *Master formulas for the dressed scalar propagator in a constant field*, arXiv:1612.02944

2007-01-31 Kurusch Ebrahimi-Fard Here are the links I mentioned:

Anatomy of a gauge theory by Dirk Kreimer, arXiv:hep-th/0509135

Renormalization of gauge fields: A Hopf algebra approach by Walter D. van Suijlekom, arXiv:hep-th/0610137

The Hopf algebra of Feynman graphs in QED by Walter D. van Suijlekom, arXiv:hep-th/0602126

This is Jean-Yves Thibon's web-page, a very good combinatorialist!

2016-11-15 Kevin Hartnett Strange Numbers Found in Particle Collisions

2017-05-23 Predrag Should I talk to Spencer Bloch?

2017-05-23 Predrag Broadhurst, Delbourgo and Kreimer [27] 1996 *Unknotting the polarized vacuum of quenched QED* has lots of magic leading to cancelations of "transcedentals." They say: "Complete cancellation of transcendentals from the beta function, at every order, is to be expected only in quenched QED and quenched SED, where subdivergences cancel between bare diagrams."

Online collection of papers on this topic.

2013-10-23 Warren D. Smith D. J. Broadhurst and D. Kreimer: *Association of multiple zeta values with positive knots via Feynman diagrams up to 9 loops*, Physics Letters B 393 (1997) 403-412, arXiv:hep-th/9609128.

Furthermore, the number of different kinds of knots with N crossings, KnotCount(N), is known asymptotically to be bounded between two simple-exponentials,

$$A^N < \text{KnotCount}(N) < B^N$$

where $B \leq 13.5$ according to
D.J.A. Welsh,*On the number of knots and links, Sets, graphs and numbers* (Budapest, 1991), 713– 718, Colloq. Math. Soc. Janos Bolyai, 60, North-Holland, Amsterdam, 1992,
while $A \geq 2.68$ according to
C.Ernst & D.W.Sumners *The growth of the number of prime knots*, Proc Cambridge Philo Soc 102 (1987) 303-315.

So, that's the funny thing. My attempt to further-destroy Cvitanović, just led to an estimate involving simple exponential growth and NOT superexponential (e.g. factorial style). This is right on the boundary for convergence questions, i.e.

$$\sum A_N x^N$$

has a finite, nonzero radius of convergence if $|A_N|$ grows exponentially. So maybe there remains some hope for some form of Cvitanović conjecture.

The Welsh upper bound also works for links. Which means: if you believe this could rescue Cvitanović's quenched-QED convergence conjecture, that would also presumably mean you believe full unquenched QED series have finite nonzero radius of convergence.

2013-11-25 David Broadhurst <David.Broadhurst@open.ac.uk>

Dirk and kind of gave up when it turned out that a pair of counterterms at 7 loops, unidentified in 1996, have weight 11, whereas our intuition about the knots 10_139 and 10_154 has suggested weight 10.

Maybe there is some sort of connection, but I know not what.

2017-05-23 Predrag Dirk Kreimer and Karen Yeats [103] 2008 *Recursion and growth estimates in renormalizable quantum field theory*

Our method is very different in spirit from the constructive approach or the functional integral approach. It relies on a Hopf algebraic decomposition of terms in the perturbative expansion into primitive constituents, not unlike the decomposition of a ζ function into Euler factors.

Our construction of a basis of primitives with a given Mellin transform resolves overlapping divergences, thanks to the Hochschild cohomology of the relevant Hopf algebras [101].

We next assume there to be $p(k)$ primitives at k loops where p is a polynomial.

Yeats [152] 2017 *A Combinatorial Perspective on Quantum Field Theory*. I have put a copy here.

2016-08-20 Predrag Kißler [94] *Hopf-algebraic renormalization of QED in the linear covariant gauge*: "The possibility of a finite electron self-energy by fixing a generalized linear covariant gauge is discussed. An analysis of subdivergences leads to the conclusion that such a gauge only exists in quenched QED."
2017-06-16 Henry Kißler The term "finite electron self-energy" does not refer to the convergence of the perturbation series, but to an order-by-order cancellation of divergences. The idea was to gauge away all divergences in the self-energy order-by-order as used by Broadhurst [26] in "Four-loop Dyson-Schwinger-Johnson anatomy" for quenched QED.

2017-06-16 Predrag Broadhurst [26] *Four-loop Dyson-Schwinger-Johnson anatomy*: "Dyson–Schwinger equations are used to evaluate the 4-loop anomalous dimensions of quenched QED in terms of finite, scheme-independent, 3-loop integrals. The 4-loop beta function has 24 unambiguous terms. The rational, $\zeta(3)$ and $\zeta(5)$ parts of the other 22 miraculously sum to zero. Vertex anomalous dimensions have 40 terms, with no dramatic cancellations. Our methods come from work by the late Kenneth Johnson, done more than 30 years ago. They are entirely free of the subtractions and infrared rearrangements of later methods."

2016-08-20 Predrag Kißler and Kreimer [95] 2016 *Diagrammatic cancellations and the gauge dependence of QED*: " The perturbative expansion given in terms of Feynman graphs might be rearranged in terms of meta graphs or subsectors with a maximum number of cancellations implemented. "

They discuss how the QED tree-level cancellation identity implies cancellation between Feynman graphs of different topologies and determines the gauge dependence. They parameterize the momentum part of Landau gauge, then start by keeping only a liner term in graphs (i.e., insert only one Landau propagator, rest Feynman). Not sure it is useful to us...
2017-06-16 Henry Your definition of gauge sets differs from the one we use in *Diagr. cancellations and the gauge dependence* [95]. The anomalous mag. moment is gauge independent due to the on-shell electrons, so studying the gauge parameter terms is not important, but I find it interesting to compare both definitions of gauge invariant sets, maybe one can improve the other.

Read also Kreimer [102] 2000 *Knots and Feynman diagrams*.

2017-06-16 Henry Kißler Here an alternative idea to approach the finiteness conjecture: There is a bound for the value of a scalar Feynman diagram due to Eric Panzer, which he called the *Hepp bound*. This bound is derived using the 1974 Cvitanović and Kinoshita [45] Feynman-parametric representation. A natural question is: does this bound generalize to a sum of Feynman graphs when the sum goes over something as your gauge sets. Of cause, the gamma matrices in the numerator make things more complicated, but imposing on-shell conditions and choosing an appropriate gauge might simplify this task. Unfortunately, there is little in the literature about the Hepp bound [132]; the only thing I am aware of is a short section in the thesis of Iain Crump [35].
2017-06-16 Predrag I do not see how this would work: The Hepp invariant

$H(G)$ is defined for a given graph G. Suppose we use instead of a Feynman diagram the (k, m, m') multi-photons gauge set diagram $\tilde{G}$ (for examples, see figure 1.4 and figure 1.5). In worldline formalism one has a proper time parametrization intermingled with Feynman-parametric bits. Any clue how the weights in the Hepp invariant $H(\tilde{G})$ would be defined? Just for a scalar theory, let us forget QED for the time being, along the lines of **2017-05-24 Predrag** entry above, on Bastianelli *et al.* [15]?

2017-06-19 Predrag Iain Crump [35] thesis *Graph Invariants with Connections to the Feynman Period in ϕ^4 Theory* (he follows Yeats [152]):

The *Feynman period* is a simplified version of the Feynman integral. The period is of special interest, as it maintains much of the important number theoretic information from the Feynman integral. It is also of structural interest, as it is known to be preserved by a number of graph theoretic operations.

2017-05-23 Predrag Badger, Bjerrum-Bohr and Vanhove [11] 2009 *Simplicity in the structure of QED and gravity amplitudes*.

2017-05-23 Predrag Rosenfelder and Schreiber [125, 126] 1996, arXiv:nucl-th/9504002, arXiv:nucl-th/9504005:

2017-06-02 Predrag Rosenfelder and Schreiber [127] 2004 *An Abraham-Lorentz-like equation for the electron from the worldline variational approach to QED*:

They discuss the of a spin-1/2 electron dressed by an arbitrary number of photons in the quenched approximation to QED. The approach is patterned after Feynman's celebrated variational treatment of the polaron problem [63], which was first applied by Mano, Progr. Theor. Phys. 14, 435 (1955) [8] to a relativistic scalar field theory and rediscovered and expanded by them in a series of papers [125, 126]. Its main features are the description of relativistic particles by worldlines [133] parametrized by the proper time, an exact functional integration over the photons and a variational approximation of the resulting effective action by a retarded quadratic trial action. In recent work we have extended this approach to more realistic theories, in particular to quenched QED [5] (the divergence structure and renormalization, a compact expression for the anomalous mass dimension of the electron). Here they calculate the finite contributions.

The variational formulation of worldline QED leads to an equation which is similar to Abraham, Lorentz and Dirac description of the electron and its self-interaction with the radiation field. The approach contains (almost) all the ingredients of the relativistic field theory of electrons and photons, in particular its divergence structure. This has been demonstrated by deriving an approximate nonperturbative expression for the anomalous mass dimension of the electron.

2017-05-23 Predrag K. Barro-Bergflödt, R. Rosenfelder and M. Stingl [13] 2006 *Variational worldline approximation for the relativistic two-body bound state in a scalar model*, arXiv:hep-ph/0601220.

2017-05-23 Predrag Fried and Gabellini [68] 2009 *Analytic, nonperturbative, almost exact QED: The two-point functions*. The remarkable (but speculative) result of this paper is that in a convenient gauge, the (unphysical) electron propagator renormalization is a multiplicative, non-perturbative *finite* factor bounded between 0 and 1:

$$Z_2 = \exp\left[-2\gamma\left(\left(\frac{\pi}{2}\right)^2 + \ln^2\left(\frac{\Lambda^2}{\mu^2}\right)\right)\right] , \tag{1.34}$$

where $\gamma = e^2/4\pi^2$ is the fine structure constant (referred to by the vulgar multitudes as α), $\mu \to 0$ is the infrared cutoff, and $\Lambda \to \infty$ is the UV cutoff. One would still need to compute the vertex renormalization Z_1 to get a gauge and renomalization method invariant result. Fried seem to only cite Schwinger, Fradkin and himself, so the similarity of this to 1982 Affleck, Alvarez, and Manton [1] nonperturbative result $e^{\alpha\pi}$ is not remarked upon.

Fried and Gabellini [69] 2012 *On the Summation of Feynman Graphs*, arXiv:1004.2202.

Fried and Gabellini [70] 2013 *QED vacuum loops and vacuum energy*

2017-03-15 Predrag I have tried reading Fried [67] 2014 *Modern Functional Quantum Field Theory: Summing Feynman Graphs*: "a simple, analytic, functional approach to non-perturbative QFT, using a functional representation of Fradkin to explicitly calculate relevant portions of the Schwinger Generating Functional (GF). In QED, this corresponds to

summing all Feynman graphs representing virtual photon exchange

between charged particles. It is then possible to see, analytically, the cancellation of an infinite number of perturbative, UV logarithmic divergences, leading to an approximate but most reasonable statement of finite charge renormalization. A similar treatment of QCD, with the addition of a long-overlooked but simple rearrangement of the Schwinger GF which displays Manifest Gauge Invariance, is then able to produce a simple, analytic derivation of quark-binding potentials without any approximation of infinite quark masses. A crucial improvement of previous QCD theory takes into account the experimental fact that asymptotic quarks are always found in bound state."

This book can be read online via GaTech library link here or here.

Even though I am a grandchild of Schwinger (via Tung Mow Yan), and have written/drawn a book where Schwinger's functional formalism is explained to everywoman, I still find the functional formalism of Schwinger and Fradkin [65] hard to follow. I believe the results are essentially the same as wordline formalism developed by Schubert *et al.*.

2017-06-16 Predrag Jia and Pennington [82] *How gauge covariance of the fermion and boson propagators in QED constrain the effective fermion-boson vertex*

Jia and Pennington [83] *Gauge covariance of the fermion Schwinger–Dyson equation in QED*

Jia and Pennington [84] *Landau-Khalatnikov-Fradkin transformation for the fermion propagator in QED in arbitrary dimensions*

2017-06-16 Christian Melnikov, Vainshtein and Voloshin [114] *Remarks on the effect of bound states and threshold in g-2*: " The appearance of positronium poles in a photon propagator in QED formally requires a summation of an infinite series of terms in perturbation theory. [...] we show how these nonperturbative contributions disappear, using the case of the electron anomalous magnetic moment as an example. [...] it never happens that a summation of infinite classes of Feynman diagrams enhanced at any threshold generates additional effects beyond perturbation theory. The misunderstanding of this fact appears to be quite common. "

The same conclusion is reached by Eides [59] *Recent ideas on the calculation of lepton anomalous magnetic moments* and Fael and Passera [60] *Positronium contribution to the electron $g-2$.*

2017-06-16 Predrag Herzog and Ruijl [77] *The R*-operation for Feynman graphs with generic numerators*: "The R*-operation by Chetyrkin, Tkachov, and Smirnov is a generalisation of the BPHZ R-operation, which subtracts both ultraviolet and infrared divergences of euclidean Feynman graphs with non-exceptional external momenta. It can be used to compute the divergent parts of such Feynman graphs from products of simpler Feynman graphs of lower loops. In this paper we extend the R*-operation to Feynman graphs with arbitrary numerators, including tensors. We also provide a novel way of defining infrared counterterms which closely resembles the definition of its ultraviolet counterpart. We further express both infrared and ultraviolet counterterms in terms of scaleless vacuum graphs with a logarithmic degree of divergence. By exploiting symmetries, integrand and integral relations, which the counterterms of scaleless vacuum graphs satisfy, we can vastly reduce their number and complexity."

Ruijl, Ueda, Vermaseren and Vogt [129] Four-loop QCD propagators and vertices with one vanishing external momentum: " We have computed the self-energies and a set of three-particle vertex functions for massless QCD at the four-loop level. The vertex functions are evaluated at points where one of the momenta vanishes. Analytical results are obtained for a generic gauge group and with the full gauge dependence, which was made possible by extensive use of the Forcer program for massless four-loop propagator integrals. The bare results in dimensional regularization are provided in terms of master integrals and rational coefficients; the latter are exact in any space-time dimension."

Chetyrkin and Tkachov [33] *Infrared R-operation and ultraviolet counterterms in the MS-scheme*, together withWe Chetyrkin and Smirnov [32] *R*-Operation corrected*

Smirnov and Chetyrkin [141] *R* operation in the minimal subtraction scheme*

Johnson and Zumino [85] *Gauge dependence of the wave-function renormalization constant in Quantum Electrodynamics*: " [...] point out the existence of an exact and simple relation between the electron Green's function renormalization constants in the general class of manifestly covariant gauges. "

Korthals Altes and De Rafael [99] 1976 *Infrared structure of non-abelian gauge theories: An instructive calculation*

Korthals Altes and De Rafael [100] 1977 *Infrared structure of non-abelian gauge theories: Comments on perturbation theory calculations*

Frenkel *et al.* [66] 1976 *Infra-red behaviour in non-abelian gauge theories*

2017-06-27 Predrag Søndergaard, Palla, Vattay, and Voros [143] *Asymptotics of high order noise corrections*: "We consider an evolution operator for a discrete Langevin equation with a strongly hyperbolic classical dynamics and noise with finite moments. Using a perturbative expansion of the evolution operator we calculate high order corrections to its trace in the case of a quartic map and Gaussian noise. The asymptotic behaviour is investigated and is found to be independent up to a multiplicative constant of the distribution of noise."

2017-07-03 Predrag First Morelia discussion (all errors are mine):

Christian is inclined to compute (g-2) starting with their two-field spinor QED Bern-Kosower formula. In that formulation all photons are born equal; one of them is kept as the external field (the seagull vertex, or the k^μ coefficient) is the magnetic moment $\sigma_{\mu\nu}$), and the rest are contracted pairwise in all possible ways. In the quenched case, this yields one gauge invariant set, the self-energy set of sect. 1.2.1.

James would like to start with their electron propagator in constant external field, and keep the term linear in the external field. I vastly prefer that, because it should be possible to distinguisn in- and out-legs, and the three kinds of N-photon propagators that yield the minimal gauge sets.

I probably need to got through the proof of gauge invariance with them.

2017-07-04 Predrag Morelia Schubertiad day 1: the lecture written up in Schubert [134] 2012 *Lectures on the worldline formalism*, sects. 1.4 *Gaussian integrals* and 1.5 *The N-photon amplitude*.

Christian was right. One has to start with scalar QED one-loop effective action to understand the Bern-Kosowar type master formulas. That yields a loop with any number of photons attached, each photon vertex carrying a 1D proper time Green's function. This could be computed by usual math methods techniques for computing Green's functions, but they find it useful for reasons that will be understood later to compute it as a sum of Fourier modes. The marginal modes (4 spacetime translations) are fixed in the Gauss way, by shifting the origin to loops center of mass (i.e., different symmetry reduction for each loop).

We then separate the integration over x_0, thus reducing the path integral to an integral over the relative coordinate q:

$$x^\mu(\tau) = x_0^\mu(\tau) + q^\mu(\tau), \tag{1.35}$$

with the relative coordinate q periodic and satisfying constraint

$$\int_0^T d\tau\, q^\mu(\tau) = 0\,. \tag{1.36}$$

In the symmetry-reduce$q$-space the zero-mode integral then yields the energy-momentum conservation δ function. The 1D Laplacian $M = -d^2/d\tau^2$ has positive eigenvalues (the usual k^2 Fourier modes), (do the exercise!)

$$\det M = (4T)^D\,, \tag{1.37}$$

and the bosonic Green's function of $-\frac{1}{2}\frac{d^2}{d\tau^2}$ in the symmetry-reduced space is (this $^{-2}$ should presumably be $^{-1}$?)

$$G_B^c(\tau,\tau') = 2\langle\tau|\left(\frac{d^2}{d\tau^2}\right)^{-2}|\tau'\rangle = |\tau-\tau'| - \frac{(\tau,\tau')^2}{T} - \frac{T}{6}\,. \tag{1.38}$$

The first derivative $\dot{G}$ has a sign function, and $\ddot{G}$ has a $\delta(\tau-\tau')$ (because of the translation invariance, $d/d\tau$ can always e taken to act on the left variable τ).

This results in a Bern-Kosower [17] type master formula

$$\begin{aligned} &\Gamma_{\rm scal}[k_1,\varepsilon_1;\ldots;k_N,\varepsilon_N] = \\ &\quad (-ie)^N(2\pi)^D\delta(\sum k_i)\int_0^\infty \frac{dT}{T}(4\pi T)^{-\frac{D}{2}}e^{-m^2T}\prod_{i=1}^N\int_0^T d\tau_i \\ &\quad \times\exp\Big\{\sum_{i,j=1}^N\Big[\tfrac{1}{2}G_{Bij}k_i\cdot k_j - i\dot{G}_{Bij}\varepsilon_i\cdot k_j + \tfrac{1}{2}\ddot{G}_{Bij}\varepsilon_i\cdot\varepsilon_j\Big]\Big\}\,|_{\rm lin(\varepsilon_i)} \end{aligned} \tag{1.39}$$

for the one-loop N-photon amplitude in scalar QED, with photon momenta k_i and polarization vectors ε_i. m denotes the mass, e the charge and T the total proper time of the scalar loop particle.

This method is not applicable to open fermion lines.

A scary realisation. This is a special case of our general, nonlinear, arbitrary order interaction vertex smooth conjugacy calculation of sect. 1.5. In other words, our calculation is not just for scalar theory in vacuum, it is for any nonconstant background, and any order of interaction, i.e., inter alia general relativity. As we start from a saddlepoint (classical periodic solution) that is not translation invariant, we do not have to worry about fixing the marginal modes - there are none.

I fear having to explain the smooth conjugacy method to civilians.

2017-07-05 Predrag Morelia Schubertiad day 2: the $N = 2$ photon legs case is worked out in Schubert [134] 2012 lectures sect. 1.6 *The vacuum polarization*.

Even though the result is strictly zero (by Furry theorem, or by time reversal of odd number of $\dot{G}$ functions), $N = 3$ is useful to start understanding how integrations by parts work.

For N photon legs, see sect. 2.5 *Integration-by-parts and the replacement rule* and Ahmadiniaz, Schubert and Villanueva [4] *String-inspired representations of photon/gluon amplitudes*, arXiv:1211.1821: "The Bern-Kosower rules provide an efficient way for obtaining parameter integral representations of the one-loop N-photon/gluon amplitudes involving a scalar, spinor or gluon loop, starting from a master formula and using a certain integration-by-parts ("IBP") procedure. Strassler observed that this algorithm also relates to gauge invariance, since it leads to the absorption of polarization vectors into field strength tensors. Here we present a systematic IBP algorithm that works for arbitrary N and leads to an integrand that is not only suitable for the application of the Bern-Kosower rules but also optimized with respect to gauge invariance. In the photon case this means manifest transversality at the integrand level, in the gluon case that a form factor decomposition of the amplitude into transversal and longitudinal parts is generated naturally by the IBP, without the necessity to consider the nonabelian Ward identities. Our algorithm is valid off-shell, and provides an extremely efficient way of calculating the one-loop one-particle-irreducible off-shell Green's functions ("vertices") in QCD. In the abelian case, we study the systematics of the IBP also for the practically important case of the one-loop N-photon amplitudes in a constant field."

2017-07-06 Predrag Morelia Schubertiad day 3:

Idrish Huet explained to me how the numerical Monte-Carlos of worldline path integrals work.

2017-07-05 Christian says there is a relevant new arXiv from some Vietnamese authors, but I couldn't find it.

2017-07-07 Predrag Morelia Schubertiad day 4:

2017-07-09 5:22 am Predrag had a panic attack that we'll never get started on (g-2). Forgot all about going to Patzquaro, spent entire Sunday writing up the new sect. 1.1 *Electron magnetic moment* and sect. 1.6.2 *Electron magnetic moment in worldline formalism*.

2017-07-10 Predrag Morelia Schubertiad day 5:

2017-07-11 Predrag Morelia Schubertiad day 6:

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