Adjoint Method for Periodic Orbits

PDE $\quad \partial_t u = F(u) \quad u = u(x,t) \quad x \in [0, L]$

Periodic orbit of period T:

$u(x,T) = u(x,0)$

$\Rightarrow$ Define $\quad v(x,T) = u(x,T) - u(x,0)$

$$= \int_0^T F(u(x,s))\, ds$$

We look for $\quad v(x,T) = 0$

Note: $\quad v(x,T) = 0 \iff \int_0^L |v(x,T)|^2 dx = 0$

Define $\quad e(T) = \int_0^L |v(x,T)|^2 dx$

$$= \int_0^L \left(\int_0^T F(u(x,s))\, ds \right)^2 dx$$

Now we introduce the fictitious time τ:

$$e(T,\tau) = \int_0^L |v(x,T,\tau)|^2 dx$$

Assume that the period T is known. Then we try to minimize $e(T,\tau)$.

$$\frac{\partial e}{\partial \tau}(T,\tau) = 2\int_0^L v(x,T,\tau)\,\frac{\partial v}{\partial \tau}(x,T,\tau)\,dx$$

$$= 2\int_0^L v(x,T,\tau)\left(\int_0^T \mathcal{L}_{u(x,s,\tau)}\left(\frac{\partial u}{\partial \tau}(x,s,\tau)\right) ds\right) dx$$

$$= 2\int_0^T\int_0^L v(x,T,\tau)\,\mathcal{L}_{u(x,s,\tau)}\left(\frac{\partial u}{\partial \tau}(x,s,\tau)\right) dx\,ds$$

$$= 2\int_0^T\int_0^L \frac{\partial u}{\partial \tau}(x,s,\tau)\,\mathcal{L}^{\dagger}_{u(x,s,\tau)}\left(v(x,T,\tau)\right) dx\,ds$$

Choosing $\boxed{\frac{\partial u}{\partial \tau}(x,s,\tau) = -\mathcal{L}_u^{\dagger}\left(\int_0^T F(u)\,ds\right)}$

ensures $\frac{\partial e}{\partial \tau} \leq 0$. Here, $u := u(x,s,\tau)$