is QED finite?

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overview

1. what this is about
2. QED finiteness conjecture
3. bye bye, Feynman diagrams
4. worldline approach

4- and 5-loop contributions to the anomaly

in 2017 Laporta completed
the twenty-year project of computing analytically
the 891 4-loop electron magnetic moment diagrams[1]

here we shall discuss only the quenched set, i.e., the contributions of diagrams with no lepton loops

the 4- and 5-loop contributions to the anomaly $a = \frac{1}{2}(g-2)$:

$$\begin{aligned} a^{(8)} &= -2.17686602773954007744325935589589393867 0 \\ a^{(10)} &= \quad 7.606(192)\ldots \quad \text{Aoyama } \textit{et al.} \text{ 2018}^{2} \\ &= \quad 6.782(113)\ldots \quad \text{Volkov 2019}^{3} \end{aligned}$$

awesome, heroic achievements

[1] **Laporta17**.

[2] **AoKiNi18**.

[3] **Volkov19**.

request #1

please always do look at the quenched set separately

renormalons schnormalons, they will go gently into that good night :)

notation : electron-photon vertex Γ_μ

out-, in- electron momenta : $p_\pm = p \mp q/2$
evaluated on the mass shell $p_\pm^2 = m^2 = 1$

Dirac, Pauli form factors $F_1(q^2)$ and $F_2(q^2)$:

$$\overline{u}(p_+)\Gamma_\mu(p,q)u(p_-) = \overline{u}(p_+)\left\{F_1(q^2)\gamma_\mu - \frac{F_2(q^2)}{2m}\,\sigma_{\mu\nu}q^\nu\right\}u(p_-)\,,$$

spinors $\overline{u}(p_+)$ and $u(p_-)$ satisfy the Dirac equation

$$\overline{u}(p_+)\,\not{p}_+ = \overline{u}(p_+)\,m\,, \qquad \not{p}_-\,u(p_-) = m\,u(p_-)\,.$$

notation : renormalized vertex

$Z_1 = 1 + L$: vertex renormalization constant
Z_2 : electron wave function renormalization constant
Ward identity : $Z_1 = Z_2$.

renormalized vertex

by definition, the renormalized charge form factor $\tilde{F}_1(0) = 1$

The vertex renormalization constant L is given by the on-shell value of the unrenormalized charge form factor[4]

$$1 + L = F_1(0) = \frac{1}{4}\mathrm{tr}\left[(\not{p} + 1)p^\nu \Gamma^\nu\right]_{q=0}$$

(the electron mass set to $m = 1$ throughout)

[4] **BroSul67**.

the anomalous magnetic moment of an electron

$$a = (g-2)/2$$

is given by the static limit of the magnetic form factor $a = \tilde{F}_2(0) = M/(1+L)$, where[5]

$$M = \lim_{q\to 0} \frac{1}{4q^2} \mathrm{tr} \left\{ \left[\gamma^\nu p^2 - (1+q^2/2)p^\nu \right] (\not{p}_+ + 1)\Gamma_\nu(\not{p}_- + 1) \right\}$$

[5] **BroSul67**.

perturbative expansion for the magnetic moment anomaly

$$a = \frac{M(\alpha)}{1 + L(\alpha)} = \sum_{n=1}^{\infty} a^{(2n)} \left(\frac{\alpha}{\pi}\right)^n ,$$

where $1 + L = F_1(0)$, $M = F_2(0)$ are computed from the unrenormalized proper vertex, given by the sum of all one-particle irreducible electron-electron-photon vertex diagrams with internal photons and electron mass counterterms. Expanding M and L we have

$$\begin{aligned}
a^{(2)} &= M^{(2)} \\
a^{(4)} &= M^{(4)} - L^{(2)} M^{(2)} \\
a^{(6)} &= M^{(6)} - L^{(2)} M^{(4)} - (L^{(4)} - (L^{(2)})^2) M^{(2)}
\end{aligned}$$

request #2

look at physical, mass-shell observables

4- and 5-loop contributions to the anomaly

4-loop : 518 diagrams

5-loop : 6 354 diagrams

each of size $\approx \pm 10$, add them up:

$$\begin{aligned} a^{(2)} &= +0.5 \\ a^{(4)} &= -0.33 \\ a^{(6)} &= +0.92 \\ a^{(8)} &= -2.18 \\ a^{(10)} &= +7.60 \end{aligned}$$

(not random graphs sum $\approx \pm 800$!!!)

Q : why are these numbers SO insanely small?

Q : what is the sign of nth contribution?

gauge cancellations?

as a prelude, you might enjoy the Dunne and Schubert[6] historical review of ideas about the QED perturbation series

they note:

a point which remains poorly understood

"is the influence of gauge cancellations on the divergence structure of a gauge theory."

[6] **DunSch06**.

gauge invariance induced cancellations

If gauge invariance of QED guarantees that all UV and on-mass shell IR divergences cancel, could it be that it also enforces cancellations among the finite parts of contributions of different Feynman graphs?

gauge invariance

A gauge change generates a k^μ term in a photon propagator, and that affects a photon-electron vertex in a very simple way.

from $\not{k} = (\not{p} + \not{k} + m) - (\not{p} + m)$ it follows that

$$\frac{1}{\not{p} + \not{k} - m} \not{k} \frac{1}{\not{p} - m} = \frac{1}{\not{p} - m} - \frac{1}{\not{p} + \not{k} - m},$$

neighbouring photon insertions cancel, leading to

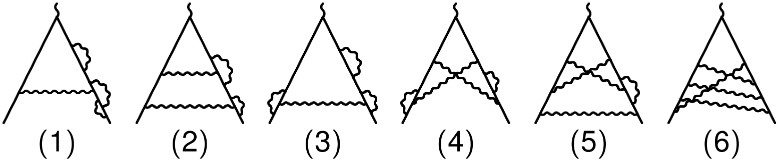
gauge invariant sets

gauge sets

A gauge set kmm' consists of all 1-particle irreducible vertex diagrams, with k photons crossing the external vertex (cross-photons) and m [m'] photons originating and terminating on the incoming [outgoing] electron leg (leg-photons)

$$a = \frac{1}{2}(g-2) = \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} \sum_{m'=0}^{\infty} a_{kmm'} \left(\frac{\alpha}{\pi}\right)^{k+m+m'} \quad .$$

representative 4-loop gauge set graphs



remaining diagrams : permute vertices, mirror diagrams

gauge set	kmm'	Laporta	approx
(1)	130	- 1.9710	- 2
(2)	220	- 0.1424	0
(3)	121	- 0.6219	- 1/2
(4)	211	1.0867	1
(5)	310	- 1.0405	- 1
(6)	400	0.5125	1/2

Laporta[7] gauge-set contributions $a^{(8)}_{kmm'}$; my approximations

Signs are right, and the sets are close to multiples of 1/2

[7] **Laporta17**.

there are very few gauge sets

Order $2n$	Vertex graphs Γ_{2n}	Gauge sets G_{2n}	Anomaly $a^{(2n)}$
2	1	1	1/2
4	6	2	0
6	50	4	1
8	518	6	0
10	6354	9	3/2
12	89 782	12	0
14	1 429 480	16	2

Comparison of the number of vertex diagrams without fermion loops, gauge sets, and the "gauge-set approximation"[8] for the magnetic moment in $2n$th order.

[8] **Cvit77b**.

Feynman's challenge, 12th Solvay Conference

Is there any method of computing the anomalous moment of the electron which, on first approximation, gives a fair approximation to the α term and a crude one to α^2; and when improved, increases the accuracy of the α^2 term, yielding a rough estimate to α^3 and beyond?[9]

[9] **Feynman62**.

the unreasonable smallness of gauge sets

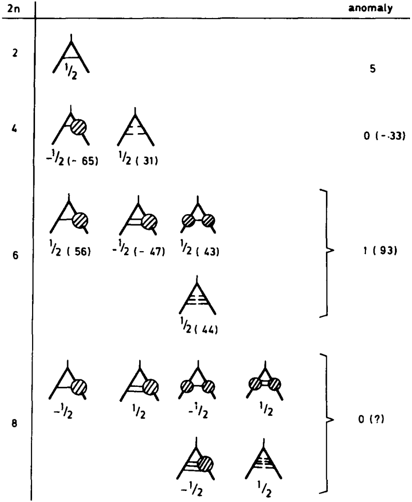
When the diagrams are grouped into gauge sets, a surprising thing happens; while the finite part of each Feynman diagram is of order of 10 to 100, and each one is UV and IR divergent, for $n = 2, 3$ every gauge set adds up to approximately

$$\pm \frac{1}{2} \left(\frac{\alpha}{\pi}\right)^n ,$$

with the sign given by a simple empirical rule

$$a_{kmm'} = (-1)^{m+m'} \frac{1}{2}$$

1977 (slightly wrong) four-loop prediction



new “prediction” : $a^{(8)} = -2$, rather than 0.

2019 five-loop status

$2n$	(k,m,m')					anomaly
2	$(\mathbf{1,0,0})$ $\frac{1}{2}$					$\frac{1}{2}$
4	$(\mathbf{1,1,0})$ $-\frac{1}{2}$ (-.65)	$(\mathbf{2,0,0})$ $\frac{1}{2}$ (.31)				0 (-.33)
6	$(\mathbf{1,2,0})$ $\frac{1}{2}$ (.56)	$(\mathbf{2,1,0})$ $-\frac{1}{2}$ (-.47)	$(\mathbf{3,0,0})$ $\frac{1}{2}$ (.44)			1 (.93)
	$(\mathbf{1,1,1})$ $\frac{1}{2}$ (.43)					
8	$(\mathbf{1,3,0})$ $-\frac{1}{2}\cdot 4$ (-1.97)	$(\mathbf{2,2,0})$ $\frac{1}{2}\cdot 0$ (-0.14)	$(\mathbf{3,1,0})$ $-\frac{1}{2}\cdot 2$ (-1.04)	$(\mathbf{4,0,0})$ $\frac{1}{2}$ (.51)		0 (-2.17)
	$(\mathbf{1,2,1})$ $-\frac{1}{2}$ (-.62)	$(\mathbf{2,1,1})$ $\frac{1}{2}\cdot 2$ (1.08)				
10	$(\mathbf{1,4,0})$ $\frac{1}{2}\cdot 12$ (6.2)	$(\mathbf{2,3,0})$ $-\frac{1}{2}$ (-0.72)	$(\mathbf{3,2,0})$ $\frac{1}{2}\cdot 0$ (-0.40)	$(\mathbf{4,1,0})$ $-\frac{1}{2}\cdot 2$ (-1.02)	$(\mathbf{5,0,0})$ $\frac{1}{2}\cdot 2$ (1.09)	$\frac{3}{2}\cdot 4$ (6.78)
	$(\mathbf{1,3,1})$ $\frac{1}{2}$ (0.90)	$(\mathbf{2,2,1})$ $-\frac{1}{2}\cdot 4$ (-2.16)	$(\mathbf{3,1,1})$ $\frac{1}{2}\cdot 5$ (2.62)			
	$(\mathbf{1,2,2})$ $\frac{1}{2}$ (0.30)					

gauge-set (k, m, m')

[naive ansatz $\pm\frac{1}{2}$] $\cdot$ [integer] $\approx$ [$(\cdots)$ Volkov 2019 numerical value]

an example of (slightly wrong) gauge-set approximation

With prediction $a_{kmm'} = (-1)^{m+m'}/2$, the “zeroth” order estimate of the electron magnetic moment anomaly is given by the “gauge-set approximation,” convergent and summable to all orders

$$a = \frac{1}{2}(g-2) = \frac{1}{2}\frac{\alpha}{\pi}\frac{1}{\left(1-\left(\frac{\alpha}{\pi}\right)^2\right)^2} + \text{“corrections"} .$$

gauge invariance matters

forget Dyson

most colleagues believe that in 1952 Dyson[10] had shown that the QED perturbation expansion is an asymptotic series (for a discussion, see Dunne and Schubert[11]), in the sense that the n-th order contribution should be exploding combinatorially

$$\frac{1}{2}(g-2) \approx \cdots + n^n \left(\frac{\alpha}{\pi}\right)^n + \cdots \,,$$

contrast with my estimate

$$\frac{1}{2}(g-2) \approx \cdots + \frac{n}{2} \left(\frac{\alpha}{\pi}\right)^{2n} + \cdots \,.$$

hence "QED is finite" claim

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[10] **Dyson52**.

[11] **DunSch06**, **HuTrSc17a**.

request #4 : prove that quenched QED is finite

any bound on a gauge set, exponential or slower, will do the trick!

so far facts ; next, speculation with Edwards and Schubert

1. QED finiteness conjecture
2. bye bye, Feynman diagrams
3. worldline approach

bye bye, Feynman diagrams

it’s been a good ride, but there are way too many of you

a fun fact

the idea of how to avoid Feynman diagrams can be traced to 1950 Feynman paper[12], though it took a long time for it to gain traction

by the time I explained the gauge set conjecture to him in 1975, Feynman had forgotten all about it

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[12] **Feynman50**.

worldline path integral for the free scalar propagator

propagator for the Euclidean Klein-Gordon equation[13] is

$$D_0(x,x') = \langle x | \frac{1}{-\Box + m^2} | x' \rangle \,,$$

exponentiate the denominator

$$D_0(x,x') = \int_0^\infty dT \, e^{-m^2 T} \langle x | e^{-T(-\Box)} | x' \rangle \,,$$

replace the the D-dimensional Laplacian by a path integral to obtain

$$D_0(x,x') = \int_0^\infty dT \, e^{-m^2 T} \int_{x(0)=x'}^{x(T)=x} \mathcal{D}x(\tau) \, e^{-\int_0^T d\tau \frac{1}{4} \dot{x}^2} \,,$$

[13] **Schubert12**.

worldline formula for charged propagator

that emits and reabsorbs N photons as it propagates

Adding the QED interaction terms leads to the Feynman's worldline path integral representation[14] of the charged scalar propagator in the presence of a background field $A(x)$,

$$D(x,x') = \int_0^\infty dT\, e^{-m^2 T} \int_{x(0)=x'}^{x(T)=x} \mathcal{D}x(\tau)\, e^{-S_0 - S_e - S_i}\,,$$

where the suffix (0) indicates the free propagation

$$S_0 = \int_0^T d\tau \frac{1}{4}\dot{x}^2\,,$$

(e) is the interaction of the charged scalar with the external field

$$S_e = -ie \int_0^T d\tau\, \dot{x}^\mu A_\mu(x(\tau))\,,$$

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[14] **Feynman50**.

worldline formula for N-photon propagator

and (i) are the virtual photons exchanged along the charged particle's trajectory

$$S_i = \frac{e^2}{2}\int_0^T d\tau_1 \int_0^T d\tau_2\, \dot{x}_1^\mu\, D_{\mu\nu}(x_1 - x_2)\, \dot{x}_2^\nu\,,$$

where $D_{\mu\nu}$ is the x-space photon propagator.

The object of great interest to us is the internal virtual photons term

$$\int_{x(0)=x'}^{x(T)=x} \mathcal{D}x(\tau)\, e^{-S_i} = \int_{x(0)=x'}^{x(T)=x} \mathcal{D}x(\tau)\, e^{-\frac{e^2}{2}\int_0^T d\tau_1 \int_0^T d\tau_2\, \dot{x}_1^\mu\, D_{\mu\nu}(x_1 - x_2)\, \dot{x}_2^\nu}$$

expanded perturbatively in α/π, this yields
the usual n! Feynman-parametric vertex diagrams

however, the path integral is Gaussian in $\dot{x}^\mu$, and if by integration by parts, $\dot{x}^\mu$ are eliminated in favor of x^μ, internal photons can be integrated over directly, prior to an expansion in $(\alpha/\pi)^n$, and one gets integrals in terms of

N-photon propagators

symmetrized sums over N photons

and not the usual Feynman graphs

each usual Feynman graph corresponds to one particular permutation of internal photon insertions, and from that comes the factorial growth in the number of graphs

note : the worldline integrals are expressed in terms of N-photon propagators,
the central ingredient that defines the gauge sets

unlike the Feynman parameter integrals for individual vertex graphs, they are independent of the ordering of the momenta $k_1, \ldots, k_N$; the worline formula contains all $\approx N!$ ways of attaching the N internal photons to the charged particle propagator

worldline representation combines
combinatorially many Feynman diagrams into

a single integral

In QED the N-photon propagator formulation combines into one integral all Feynman graphs related by permutations of photon legs along fermion lines, that is, it yield a *single* integral for a gauge set kmm'

worldline formalism

wordline propagator

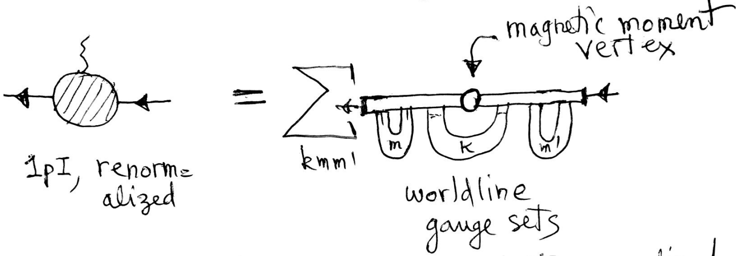
N! Feynman graphs

master formula = sum of all symmetric insertions

"details" for spinor QED : see $\boxed{\text{Edwards}}$ & Schubert

worldline formalism

$\Gamma^{\mu}(x_+, x_-)$ electron - external field - electron vertex



cute fact: not 1pI, but renormalized

worldline formalism

$$a^{(2)} = a_{100} \qquad [\text{note } a_{010}, a_{001} = 0] \qquad \# \; 1$$

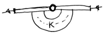
$$a^{(4)} = a_{200} + a_{110} + a_{101} \qquad 3$$

$$a^{(6)} = a_{300} + a_{210} + a_{201} + a_{120} + a_{102} + a_{111} \qquad 6$$

$$a^{(8)} = a_{400} + a_{310} + a_{301} + a_{220} + a_{202} + a_{211} + a_{130} + a_{121} + a_{112} + a_{103} \qquad 10$$

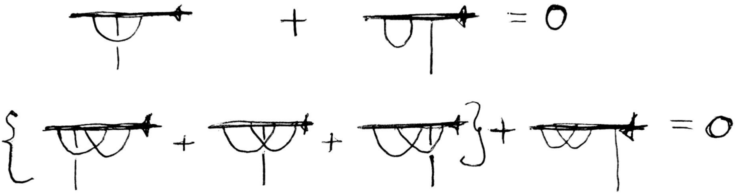
$$a^{(10)} = a_{500} + a_{410} + a_{401} + a_{320} + a_{302} + a_{311} + a_{230} + a_{221} + a_{140} + a_{131} + a_{122} + a_{203} + a_{212} + a_{104} + a_{113} \qquad 15$$

$\downarrow$

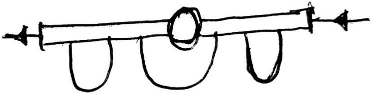


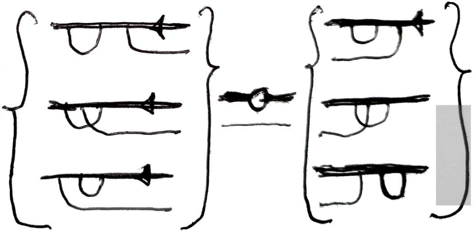
rainbows

worldline formalism

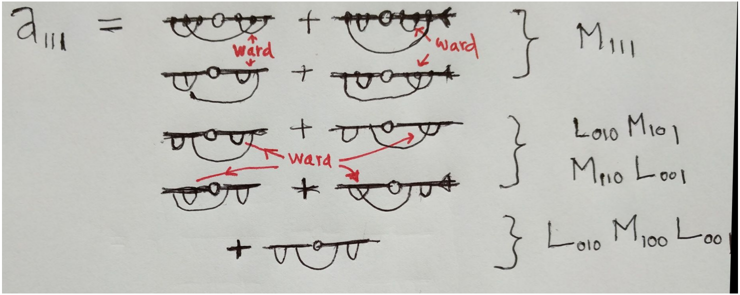


an example gauge set

a_{111} = 

= 

worldline formalism



summary

1. a proof of the QED finiteness conjecture might be within reach
2. so might be methods for computing gauge invariant QFT sets without recourse to Feynman diagrams

you can download the current version of full notes here:
ChaosBook.org/∼predrag/papers/finiteQED.pdf

The source code: GitHub.com/cvitanov/reducesymm/QFT