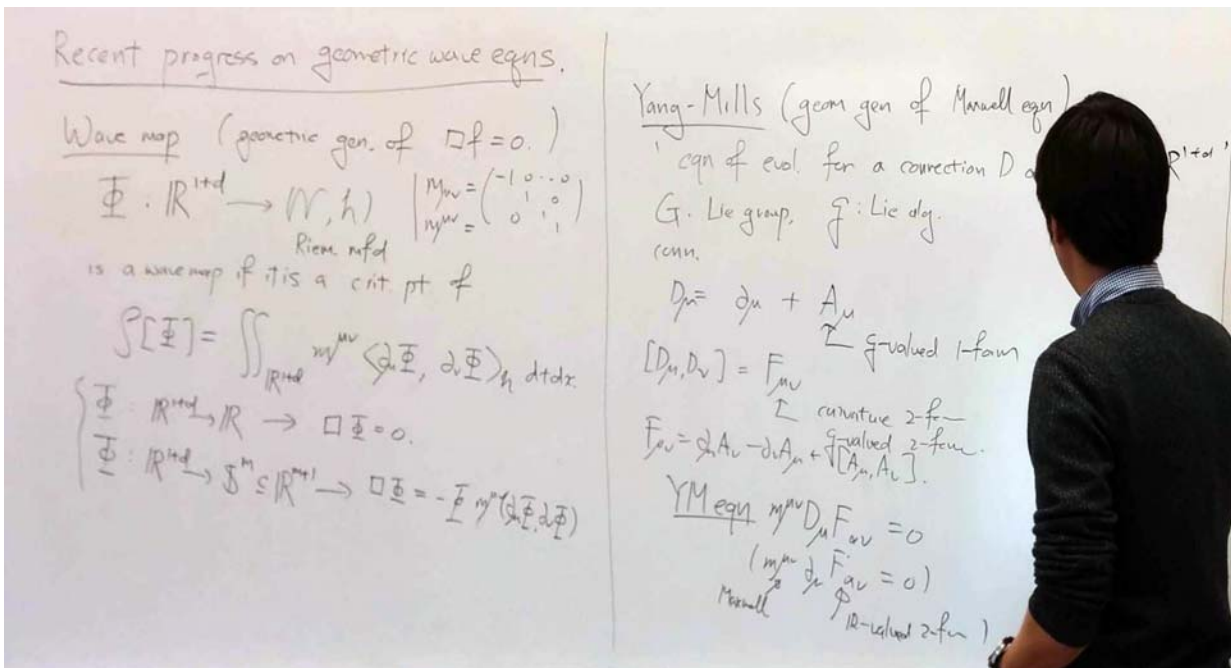


# "Recent progress on geometric wave equations" Sung-Jin Oh (University of California, Berkeley)

**Notebook:** First Notebook

**Created:** 4/6/2016 2:23 PM

**Updated:** 4/7/2016 9:20 AM



<https://math.berkeley.edu/people/faculty/sung-jin-oh>

Wednesday, April 6, 2016

Sung-Jin Oh has PDE focus: are there finite time singularities?

The subject of this talk is wave equations that arise from geometric considerations. Prime examples include the wave map equation and the Yang-Mills equation on the Minkowski space. On one hand, these are fundamental field theories arising in physics; on the other hand, they may be thought of as the hyperbolic analogues of the harmonic map and the elliptic Yang-Mills equations, which are interesting geometric PDEs on their own. I will discuss the recent progress on the problem of global regularity and asymptotic behavior of solutions to these PDEs.

Conserved energy enables one to define a "ground state" (the lowest energy state). "Threshold" theorems: for energies below the ground state, scattering amplitudes are smooth. Above: one has many counter examples.

Recent progress on geometric wave eqns.

Wave map (geometric gen. of  $\square \Phi = 0$ )  
 $\Phi: \mathbb{R}^{1+d} \rightarrow (M, h) \mid \begin{matrix} h_{\mu\nu} = (-1, 0, \dots, 0) \\ h_{\mu\nu} = (0, 1, \dots, 1) \end{matrix}$   
 is a wave map if it is a crit. pt. of  

$$S[\Phi] = \int_{\mathbb{R}^{1+d}} g^{\mu\nu} \langle \partial_\mu \Phi, \partial_\nu \Phi \rangle_h dx$$
  
 $\Phi: \mathbb{R}^{1+d} \rightarrow \mathbb{R} \rightarrow \square \Phi = 0$   
 $\Phi: \mathbb{R}^{1+d} \rightarrow \mathbb{S}^m \subset \mathbb{R}^{m+1} \rightarrow \square \Phi = -\Gamma^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi$   
 $E_{\text{static}}[\Phi] = \int_{\mathbb{R}^d} \frac{1}{2} |\partial_t \Phi|^2 + \dots + \frac{1}{2} |\partial_d \Phi|^2 dx$

Yang-Mills (geom. gen. of Maxwell eqn)  
 'eqn of evol. for a connection  $D$  on a v.b. on  $\mathbb{R}^{1+d}$ '  
 $G$ : Lie group,  $\mathfrak{g}$ : Lie alg.  
 conn.  
 $D_\mu = \partial_\mu + A_\mu$   
 $[D_\mu, D_\nu] = F_{\mu\nu}$   
 $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$   
 $\text{YM eqn } g^{\mu\nu} D_\mu F_{\nu\lambda} = 0$   
 $(\text{Maxwell } \partial_\mu F^{\mu\nu} = 0)$   
 $(\text{10-valued 2-form})$

$$\square_A F_{\mu\nu} = -[F, F]$$

cons. ev.  
 $E_{\text{static}}[A] = \frac{1}{4} \int_{\mathbb{R}^d} |F_{\mu\nu}|^2 dx$   
 $(\approx \int |\partial A|^2 dx)$   
 inu scaling  
 $A \mapsto \hat{A} = \frac{1}{\lambda} A(\lambda \cdot)$

Rank time-indep sol'n to  
 WM  $\rightarrow$  Harmonic map  
 YM  $\rightarrow$  elliptic YM connection

Questions ① global regularity  
 ② asymptotic behavior

non pert. WM  
 conserved ev.

Critical dimension is 4+1 for YM and QED, 2+1 for the wave map equation

A refined threshold theorem for (1+2)-dimensional wave maps into surfaces, with A. Lawrie, to appear in **Comm. Math. Phys.** [arXiv:1502.03435 \[math.AP\]](https://arxiv.org/abs/1502.03435)

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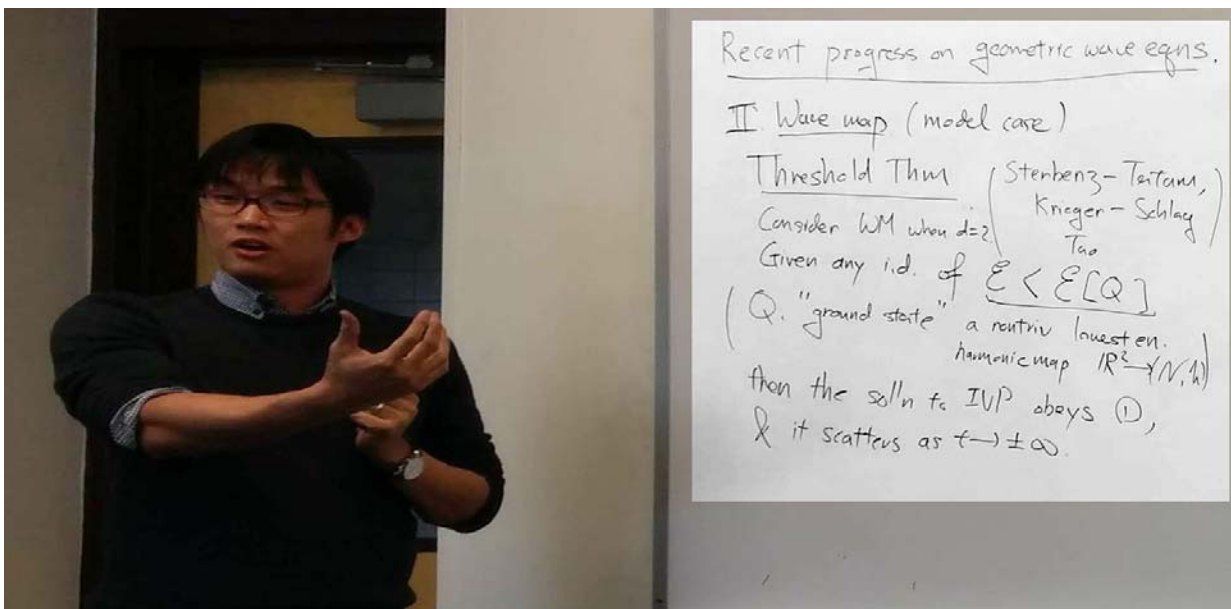
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non pert. en. crit

WM	YM
d=2	d=4

Threshold theorem (Tao, Krieger, ...)



Finite time blowup - bubbling

Rank  $\mathcal{E}[Q]$  is sharp

Bubbling For concreteness, let us study finite time blow up.

By pert analysis

$\Rightarrow$  conc. of en.  $\Phi: [0, T_+) \times \mathbb{R}^2 \rightarrow N$

$\exists$  fin. many pts  $X_i \in \mathbb{R}^2$

$\dots (T_i, X_i)$

$\Delta$   $\mathcal{E}_{S_t}[\Phi] \xrightarrow{t \rightarrow T_+} 0$

Bubbling

$\Rightarrow$   $\Phi = \sum \Phi^i(t, \frac{x - x_0}{\lambda(t)})$

$\gamma^i \bar{\gamma}^i = 0$   $+ \text{err}$

$\gamma^i = d_t$ , time indep sol'n  $\mathcal{E}(\cdot) \rightarrow 0$  as  $t \rightarrow T_+$

$\rightarrow d_t \Phi = 0 \Rightarrow \Phi$  is a HM.

$\Phi$  is a Lorentz boost of a HM.

Maxwell-Klein-Gordon (ie, scalar QED, but with massless scalar particle) in 4+1 OK, but YM not. Perhaps in these references:

- 7 Local well-posedness of the (4+1)-dimensional Maxwell-Klein-Gordon equation, with D. Tataru, to appear in **Annals of PDE**. [arXiv:1503.01560](https://arxiv.org/abs/1503.01560) [math.AP]
- 8 Energy dispersed large energy solutions to the (4+1) dimensional Maxwell-Klein-Gordon equation, with D. Tataru. [arXiv:1503.01561](https://arxiv.org/abs/1503.01561) [math.AP]
- 9 Finite energy global well-posedness and scattering of the (4+1) dimensional Maxwell-Klein-Gordon equation, with D. Tataru, to appear in **Invent. Math.** [arXiv:1503.01562](https://arxiv.org/abs/1503.01562) [math.AP]

Recent progress on geometric wave eqns.

### III YM & Maxwell-Klein-Gordon

$$\begin{cases} \partial^\mu F_{\mu\nu} = \text{Im}(\phi \bar{D}_\nu \phi) \\ D^\mu D_\mu \phi = m^2 \phi \end{cases} \quad | m=0.$$

$$D_\mu = \partial_\mu + iA_\mu$$

$$\mathcal{E} \simeq \| \partial A \|_{L^2}^2 + \| \partial \phi \|_{L^2}^2$$

$$A, \phi \mapsto (\frac{1}{\lambda} A(\cdot/\lambda), \frac{1}{\lambda} \phi(\cdot/\lambda))$$

on crit  $d=4$

(simpler than YM, since it is abelian gauge  $U(1)$ )

Thm (O. Tataru) Consider MKG on  $\mathbb{R}^{1+4}$ .

Given any fin. en. data, the sol'n to IVP, is globally reg. & scatters as  $t \rightarrow \pm\infty$ .

Rank  $\nexists$  ground state for MKG.

Rank Indep pf by Lührmann-Krieger.

YM vs. MKG

gauge transform  
is non/linear

linear

Coulomb gauge (Tao-Wu, 0.)    Coulomb gauge  $\text{div}_x A = 0$

Maxwell-Klein-Dirac (ie, spin 1/2 QED, massless only), with energy 'ground states' critical dimension is  $4+1$ . With conserved charge critical dim is 3, but results are only perturbative.

Gauss=0.

Thm for small crit  
data for Maxwell  
- Dirac.

$|m=0|$

$d=4$

charge crit:  $|d=3|$

From talking to Sung-Jin Oh: there is advance in finding non-singular surfaces within Kerr black holes. The results are on weak solutions (not smooth), not clear to Predrag where this work is going.

Predrag's (undigested) impressions: This might be a path to computing non-perturbative classical solutions of Yang-Mills. Talking to Sung-Jin Oh I kept confusing the quantum and the classical problem - it is for a reason that PDE people do not know what expressions like "on-mass-shell amplitudes" and "Ward identities" mean.