Towards continuous symmetry reduction for high dimensional flows

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Continuous symmetry in PDE's

- Continuous symmetry arizes in many PDE's and their truncations, e.g.
  - Kuramoto-Sivashinsky equation in periodic domain ($O(2)$)
  - Plane Couette flow ($O(2) \times O(2)$)
  - Maxwell-Bloch equations for a detuned ring laser ($SO(2)$)
- For any solution a continuum of "rotated" solutions exists
- Phase space structure "blurred" - recurrence up to symmetry
- Poincaré sections hard to construct

Complex Lorenz equations

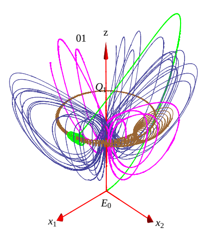
- Drastic truncation of Maxwell-Bloch equations for a unidirectional ring laser with detuning, under slow-varying envelope and rotating wave approximations [4] and also in study of baroclinic instability [3].
- Complex version of Lorenz equations:

$$\begin{aligned} \dot{x} &= -\sigma x + \sigma y\,, \\ \dot{y} &= (r - z)x - ay\,, \\ \dot{z} &= \frac{1}{2}\left(x\bar{y} + \bar{x}y\right) - bz\,, \end{aligned} \tag{1}$$

  where $x, y \in \mathbb{C}$, $z \in \mathbb{R}$ and parameters $r = r_1 + ir_2$, $a = 1 - ie$, σ, b, r_1, r_2, $e \in \mathbb{R}$.
- Equations left invariant (flow is equivariant) under the action of $SO(2)$

$$R(\theta)(x, y, z) = (e^{i\theta}x, e^{i\theta}y, z)\,, \quad \theta \in [0, 2\pi)\,. \tag{2}$$

- For detuned laser case $r_2 = 0$, e proportional to detuning [7]
- Parameters used here: $r_1 = 28$, $b = 8/3$, $\sigma = 10$, $a = 1$, $e = 1/10$, $r_2 = 0$



E_0: The origin is an unstable equilibrium

Q_1: relative equilibrium (group-invariant cycle, traveling wave in PDEs)

01: relative periodic orbit satisfying

$$x(0) = R(\theta_p)x(T)$$

for some θ_p and T_p. Here $T_{01} = 1.542$, $\theta_{01} = 2.953$.

Symmetry Reduction

- *Orbit space projection* (symmetry reduction): All points related by a symmetry operation (on the same group orbit) are mapped to the same point.
- Relative equilibria become equilibria and relative periodic orbits become periodic orbits in reduced space.
- Usual approach: Rewrite the dynamics in a symmetry-invariant polynomial basis (Hilbert basis)
  - Computationally prohibitive as the dimension of the system increases (algebraic geometry algorithms)
  - Phase space portraits often not informative, *cf.* Panel (a)
  - For CLe the polynomial basis is [6]

$$\begin{aligned} u_1 &= x_1^2 + x_2^2\,, \\ u_2 &= y_1^2 + y_2^2\,, \\ u_3 &= x_1 y_2 - x_2 y_1\,, \\ u_4 &= x_1 y_1 + x_2 y_2\,, \\ u_5 &= z\,, \end{aligned} \tag{3}$$

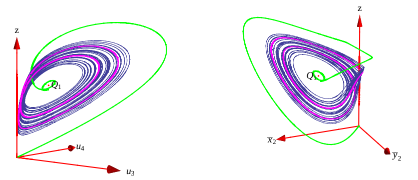
  where we have written x and y in terms of their real and imaginary parts: $x = x_1 + ix_2$, $y = y_1 + iy_2$. The u_i's are functionally related by the *syzygy*

$$u_1 u_2 - u_3^2 - u_4^2 = 0\,.$$

- Fundamental invariants (non-polynomial) can be efficiently generated by the moving frame method of Fels and Olver [5]. For our example:

$$\begin{aligned} \overline{x}_2 &= \sqrt{x_1^2 + x_2^2}\,, \\ \overline{y}_1 &= \frac{x_2 y_1 - x_1 y_2}{\sqrt{x_1^2 + x_2^2}}\,, \\ \overline{y}_2 &= \frac{x_1 y_1 + x_2 y_2}{\sqrt{x_1^2 + x_2^2}}\,, \\ \overline{z} &= z\,. \end{aligned} \tag{4}$$

  Note that no syzygy is present here, the dimension of the system has been reduced by one.
  - Still not efficient for numerical computations in very high dimensional truncations
  - Large jumps in phase portraits due to the denominators, *cf.* Panel (b)
  - Group-theoretic restrictions

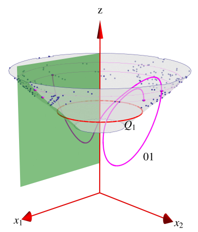


(*a*) (*b*)

An efficient approach

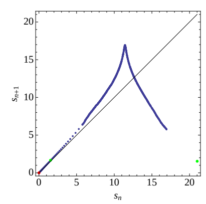
We need a method that can be implemented for *high-dimensional* discretizations of PDE's and it is sufficient for our purposes to be able to construct a Poincaré section.

- Do not attempt to rewrite the dynamics.
- Use geometric interpretation of moving frame method of Fels and Olver.
- Set up a group-invariant (as a set) Poincaré section $\mathcal{P}_1$. Use a few of the first fundamental invariants that can be easily computed by the moving frame method. Here use $\overline{x}_2 = \overline{y}_2$ or in original space $x_1^2 + x_2^2 = x_1 y_1 + x_2 y_2$.
- Think orbits as tori (generated by the action of the group)
- Since $\mathcal{P}_1$ is group-invariant it contains the group orbit of any point of intersection.
- Set up a second section $\mathcal{P}_2$ that intersects each tori once. Here use condition $\theta = 0$ where θ the phase in complex x-plane. In practice instead of tracing the tori we apply the group transformation to set $\theta = 0$ for points on $\mathcal{P}_1$.
- We only need to apply a linear transformation to map a point to $\mathcal{P}_2$ (globally the transformation is still non-linear and equivalent to (4).)



Return map

For CLE reduction through the use of the method proposed here allows the construction of a return map that accurately captures the dynamics



Coordinate s is Euclidean length along unstable manifold of relative equilibrium Q_1. The map is approximatelly 1-dimensional as a result of the very strong contraction along the stable directions. The same map can be obtained through any reduction method.

The map allows the determination of initial guesses for relative periodic orbits of any given length that are then determined through multiple shooting to machine accuracy in the original space.

Future

- Test applicability to Kuramoto-Sivashinsky equation, Plane Couette flow.
- Use relative periodic orbits found in the framework of periodic orbit theory[2] to compute asymptotic statistics of the attractor.

References

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