Geometry of inertial manifolds

in nonlinear dissipative dynamical systems

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Main contribution

(Our simulations use 62 degrees of freedom.)

For one-dimensional Kuramoto-Sivashinsky equation defined on a periodic domain of size $L = 22$, the dimension of the inertial manifold is 8.

X. Ding and P. Cvitanović, "Periodic eigendecomposition and its application in Kuramoto-Sivashinsky system," *SIAM J. Appl. Dyn. Syst.* **15**, 1434–1454 (2016)

Ding, X., Chaté, H., Cvitanović, P., Siminos, E., and Takeuchi, K. A., "Estimating the dimension of the inertial manifold from unstable periodic orbits," *Phys. Rev. Lett.* **117**, 024101 (2016)

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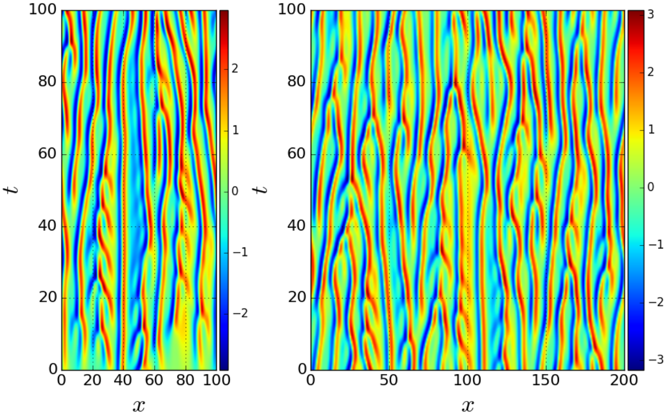
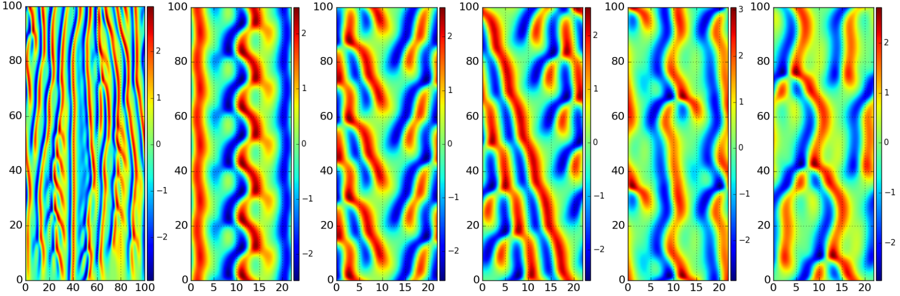


$$u_t + \frac{1}{2}(u^2)_x + u_{xx} + u_{xxxx} = 0\,,\ x \in [0, L].$$

Recurrent patterns not only show up along the temporal axis
but also along the spatial axis

minimal domain captures essential features observed in large domains



horizontally : $x \in [0, L]$

(leftmost) $L = 100$ (the rest) $L = 22$

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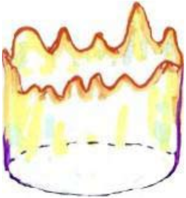
Kuramoto-Sivashinsky equation[1,2]

$$u_t + \frac{1}{2}(u^2)_x + u_{xx} + u_{xxxx} = 0\,,\ x \in [0, L] \qquad (1)$$

$u(x,t)$ represents the flame front velocity. $u(x,t) = u(x+L,t)$.

- Galilean invariance:
  $u(x,t) \to u(x-ct,t) + c$.
- Reflection invariance:
  $R\,u(x,t) = -u(-x,t)$.
- Translation invariance:
  $g(\phi)u(x,t) = u(x+\ell,t)$. Here $\phi = 2\pi\ell/L$.



Reduce Galilean invariance : set $\int dx\,u = 0$.

[1] Y. Kuramoto and T. Tsuzuki. "On the formation of dissipative structures in reaction–diffusion systems". In: *Progr. Theor. Phys.* 54 (1975), pp. 687–699. DOI: 10.1143/PTP.54.687.

[2] D. M. Michelson and G. I. Sivashinsky. "Nonlinear analysis of hydrodynamic instability in laminar flames—II. Numerical experiments". In: *Acta Astronaut.* 4 (1977), pp. 1207–1221. DOI: 10.1016/0094-5765(77)90097-2.

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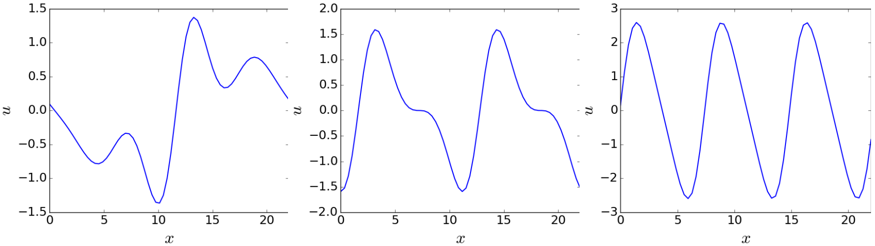
Invariant solutions : equilibria

$$R\,u(x,t) = -u(-x,t)$$
$$g(\phi)u(x,t) = u(x + \tfrac{\phi}{2\pi}L, t)$$

Defintion: $u(t) = u(0)$, i.e., $v(u) = 0$

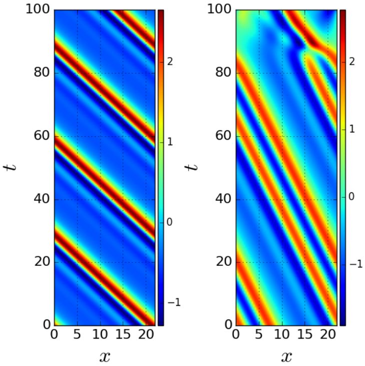


Three equilibria: (a) E_1, (b) E_2, and (c) E_3

Invariant solutions: relative equilibria

$$R\,u(x,t) = -u(-x,t)$$
$$g(\phi)u(x,t) = u(x + \tfrac{\phi}{2\pi}L, t)$$

Defintion: $u(t) = g(t\,c)\,u(0)$ (also called *traveling waves*)

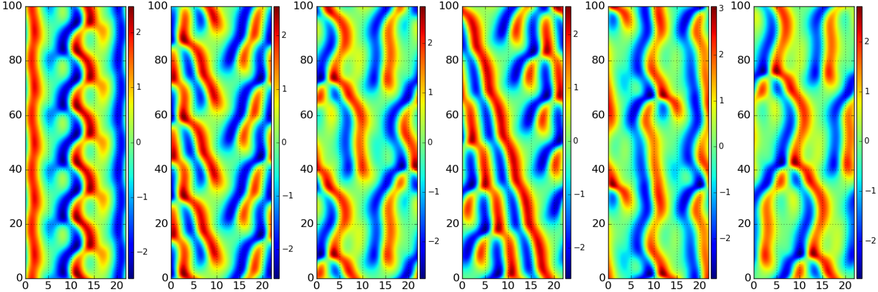


Four relative equilibria: TW_{+1}; TW_{+2}; and their reflections

$$R\,u(x,t) = -u(-x,t)$$
$$g(\phi)u(x,t) = u(x + \tfrac{\phi}{2\pi}L, t)$$

preperiodic orbit: $u(0) = Ru(T_p)$

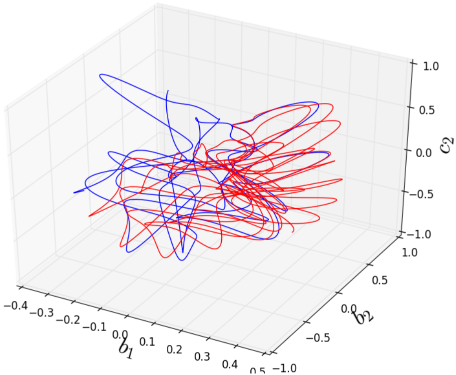
relative periodic orbit: $u(0) = g_p u(T_p)$, with $g_p = g(\phi_p)$



Left three: $\overline{ppo}_{10.25}$, $\overline{ppo}_{14.33}$ and $\overline{ppo}_{32.36}$
Right three: $\overline{rpo}_{16.31}$, $\overline{rpo}_{32.80}$ and $\overline{rpo}_{33.50}$

State space

$$R\,u(x,t) = -u(-x,t)$$
$$g(\phi)u(x,t) = u(x + \tfrac{\phi}{2\pi}L, t)$$



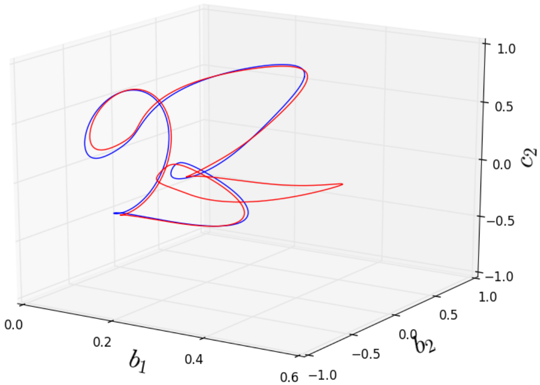
$\overline{rpo}_{35.97}$ and $\overline{rpo}_{57.59}$.

Fourier mode : $a_k = b_k + ic_k$.

state space: $\tilde{u} = (b_1, c_1, b_2, c_2, \cdots, b_{N/2-1}, c_{N/2-1})^\top$.

SO(2)-reduced state space

$$R\,u(x,t) = -u(-x,t)$$
$$g(\phi)u(x,t) = u(x + \frac{\phi}{2\pi}L, t)$$



$\overline{rpo}_{35.97}$ and $\overline{rpo}_{57.59}$.

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Continuous symmetry reduction

A flow $\dot{u} = v(u)\,, u(x,t) \in \mathbb{R}^n$ with $u(t) = f^t(u_0)$ is equivariant under a continuous symmetry group G if

$$gv(u) = v(gu)\,, \quad gf^t(u) = f^t(gu) \quad \text{for any} \quad g \in G \tag{2}$$

$$g(\phi) = e^{\phi \cdot \mathbf{T}}\,, \qquad \phi \cdot \mathbf{T} = \sum_{a=1}^{s} \phi_a \mathbf{T}_a\,, \tag{3}$$

A flow $\dot{u} = v(u)\,, u(x,t) \in \mathbb{R}^n$ with $u(t) = f^t(u_0)$ is equivariant under a continuous symmetry group G if

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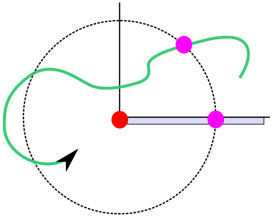
$$g(\phi) = e^{\phi \cdot \mathbf{T}}\,, \qquad \phi \cdot \mathbf{T} = \sum_{a=1}^{s} \phi_a \mathbf{T}_a\,, \tag{3}$$

Slice :

$$\langle \hat{u} - \hat{u}' | t' \rangle = \langle \hat{u} | t' \rangle = 0$$

$$t' = t(\hat{u}') = \mathbf{T}\,\hat{u}'$$

$$\hat{u} = g^{-1}(\phi)\,u\,.$$



1st mode slice[3]

$$a_k = b_k + i c_k$$
$$\tilde{u} = (b_1, c_1, b_2, c_2, \cdots, b_{N/2-1}, c_{N/2-1})^\top$$

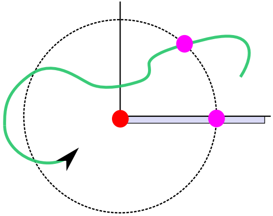
$$R\,u(x,t) = -u(-x,t) \iff (b_k, c_k) \to (-b_k, c_k)$$

$$g(\phi)u(x,t) = u(x + \frac{\phi}{2\pi}L, t) \iff a_k \to e^{iq_k\phi} a_k$$

1st mode slice :

$c_1 = 0,\ b_1 > 0\,,$ that is, $a_k \to e^{-iq_k\phi_1} a_k$

ϕ_1 : the phase of a_1.

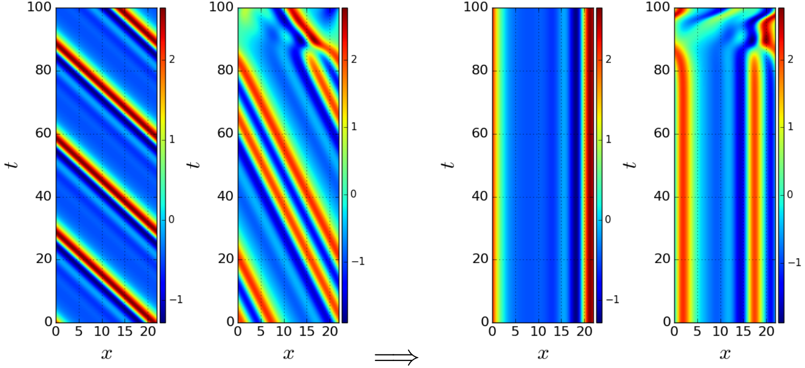


[3] N. B. Budanur et al. "Reduction of the SO(2) symmetry for spatially extended dynamical systems". In: *Phys. Rev. Lett.* 114 (2015), p. 084102. DOI: 10.1103/PhysRevLett.114.084102.

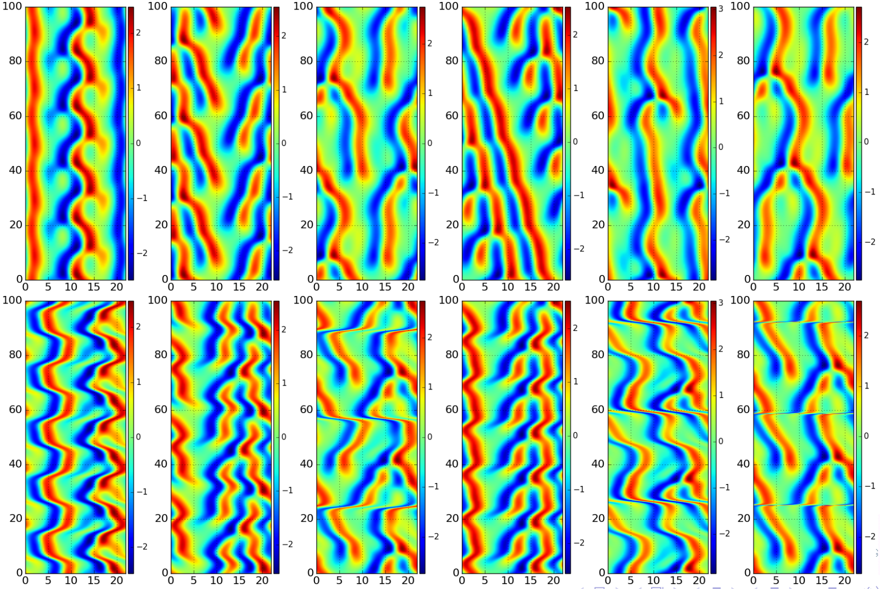
Relative equilibria in the 1st mode slice

$$R\,u(x,t) = -u(-x,t)$$
$$g(\phi)u(x,t) = u(x + \tfrac{\phi}{2\pi}L, t)$$

relative equilibrium: $u(t) = g(t\,c)\,u(0)$



Preperiodic orbits and relative periodic orbits



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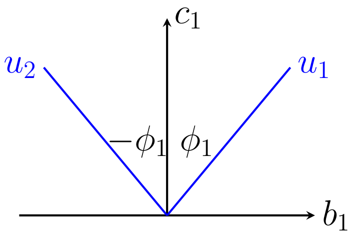
O(2) symmetry reduction

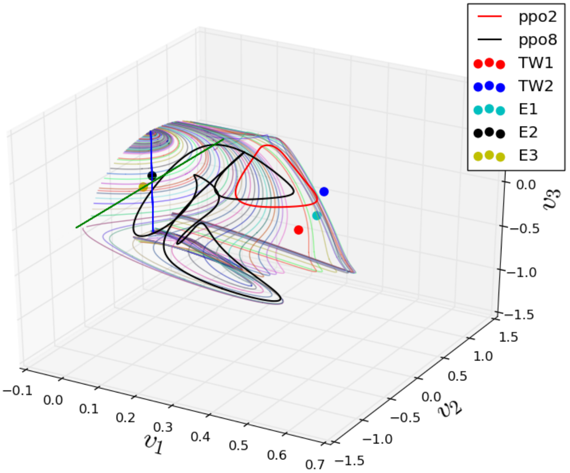
$$(b_k, c_k) \to (-b_k, c_k)$$
$$a_k \to e^{iq_k\phi} a_k$$



Fundamental domain in the slice:

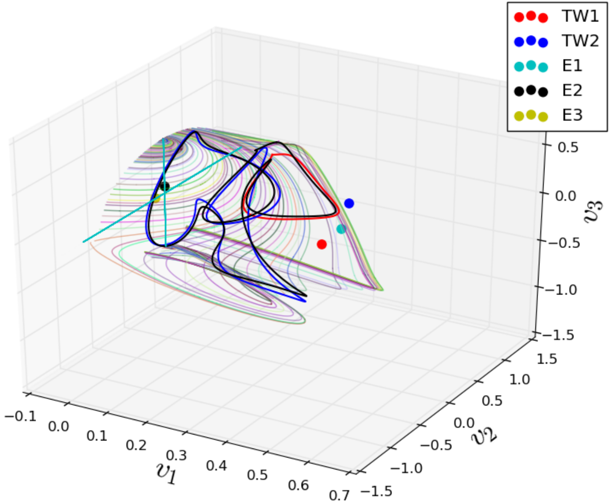
$$\hat{b}_2 > 0\,. \tag{4}$$

Unstable manifold of E_2



The dense set of thin curves is the unstable manifold of E_2. The blue and green straight lines are the group orbits of E_2 and E_3 respectively. ppo2 is $\overline{ppo}_{14.33}$. ppo8 is $\overline{ppo}_{41.08}$. $[v_1, v_2, v_3] = [\hat{c}_1, \hat{c}_3, \hat{c}_2]$.

Shadowing among orbits



(red) $\overline{rpo}_{16.31}$, (blue) $\overline{rpo}_{35.97}$ and (black) $\overline{rpo}_{57.59}$.

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Inertial manifold[4,5]

An inertial manifold is a positively invariant and exponentially attracting manifold in the state space.

Why do we care about inertial manifolds?

[4] C. Foias, G. R. Sell, and R. Temam. "Inertial manifolds for nonlinear evolutionary equations". In: J. Diff. Equ. 73 (1988), pp. 309–353. DOI: 10.1016/0022-0396(88)90110-6.

[5] J. C. Robinson. "Finite-dimensional behavior in dissipative partial differential equations". In: Chaos 5 (1995), pp. 330–345. DOI: 10.1063/1.166081.

Inertial manifold[4,5]

An inertial manifold is a positively invariant and exponentially attracting manifold in the state space.

Why do we care about inertial manifolds?

Because it captures the asymptotic dynamics and it is finite-dimensional.

Its dimension is likely much smaller than the number of degrees of freedom used in simulations (in principle infinite:)

[4] C. Foias, G. R. Sell, and R. Temam. "Inertial manifolds for nonlinear evolutionary equations". In: J. Diff. Equ. 73 (1988), pp. 309–353. DOI: 10.1016/0022-0396(88)90110-6.

[5] J. C. Robinson. "Finite-dimensional behavior in dissipative partial differential equations". In: Chaos 5 (1995), pp. 330–345. DOI: 10.1063/1.166081.

General setup[6]

Given autonomous flow $\dot{u} = v(u)\,, \quad u(x,t) \in \mathbb{R}^n$.

Tangent dynamics:

$$\frac{d}{dt}\delta u = A\delta u\,, \quad A = \frac{\partial v}{\partial u}\,. \tag{5}$$

Jacobian matrix:

$$\delta u(u,t) = J^t(u_0,0)\,\delta u(u_0,0) \tag{6}$$

$$\frac{d}{dt}J = AJ\,, \quad J_0 = I\,. \tag{7}$$

[6] P. Cvitanović et al. *Chaos: Classical and Quantum*. Copenhagen: Niels Bohr Inst., 2016. URL: http://ChaosBook.org.

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Floquet vectors:

$$J_p = J^{T_p}(u,t) \tag{8}$$

$$J_p\,\mathbf{e}_j = \Lambda_j \mathbf{e}_j \tag{9}$$

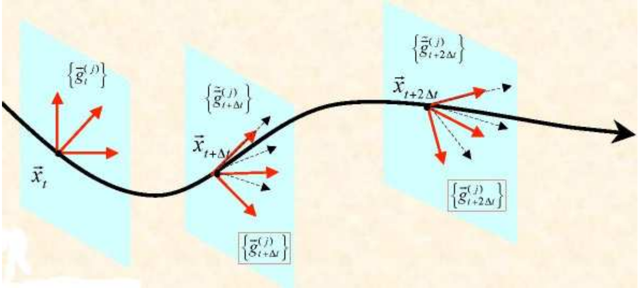
$$\Lambda_j = \exp(T_p\lambda_p^{(j)}) \;= \exp(T_p\mu^{(j)} + i\theta_j)\,.$$

[6] P. Cvitanović et al. *Chaos: Classical and Quantum*. Copenhagen: Niels Bohr Inst., 2016. URL: http://ChaosBook.org.

Covariant (Lyapunov) vectors[7,8,9]

Oseledec matrices [Oseledec 1968; Ruelle 1979]:

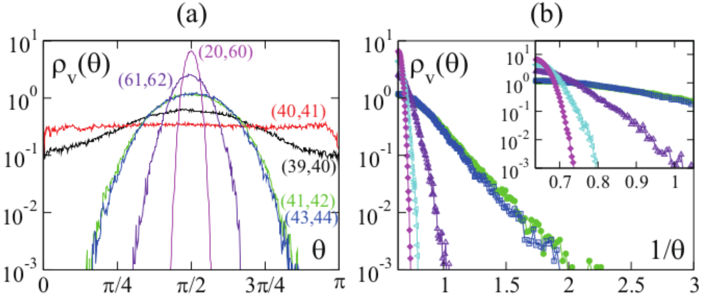
$$\Xi^{\pm}(u) := \lim_{t\to\pm\infty} [J^t(u)^\top J^t(u)]^{1/2t} \tag{10}$$



[7] C. L. Wolfe and R. M. Samelson. "An efficient method for recovering Lyapunov vectors from singular vectors". In: *Tellus A* 59 (2007), pp. 355–366. DOI: 10.1111/j.1600-0870.2007.00234.x.

[8] F. Ginelli, H. Chaté, et al. "Covariant Lyapunov vectors". In: *J. Phys. A* 46 (2013), p. 254005. DOI: 10.1088/1751-8113/46/25/254005. http://arXiv.org/abs/1212.3961.

[9] P. V. Kuptsov and U. Parlitz. "Theory and computation of covariant Lyapunov vectors". In: *J. Nonlinear Sci.* 22 (2012), pp. 727–762. DOI: 10.1007/s00332-012-9126-5. http://arXiv.org/abs/1105.5228.



(from Ref. [Takeuchi et al. 2011])

[10] F. Ginelli, P. Poggi, et al. "Characterizing dynamics with covariant Lyapunov vectors". In: *Phys. Rev. Lett.* 99 (2007), p. 130601. DOI: 10.1103/PhysRevLett.99.130601. http://arXiv.org/abs/0706.0510.

[11] H. L. Yang et al. "Hyperbolicity and the effective dimension of spatially extended dissipative systems". In: *Phys. Rev. Lett.* 102 (2009), p. 074102. DOI: 10.1103/PhysRevLett.102.074102. http://arXiv.org/abs/0807.5073.

[12] K. A. Takeuchi et al. "Hyperbolic decoupling of tangent space and effective dimension of dissipative systems". In: *Phys. Rev. E* 84 (2011), p. 046214. DOI: 10.1103/PhysRevE.84.046214. http://arXiv.org/abs/1107.2567.

Floquet vectors

How about using Floquet vectors of periodic orbits ?

Floquet vectors

How about using Floquet vectors of periodic orbits ?

How to get Floquet vectors?

Floquet vectors

How about using Floquet vectors of periodic orbits ?

How to get Floquet vectors?

Chain rule :

$$J^{t-t_0}(u(t_0), t_0) = J^{t-t_1}(u(t_1), t_1) J^{t_1-t_0}(u(t_0), t_0)$$

$$\Downarrow$$

$$\mathbf{J}^{(0)} = J_m J_{m-1} \cdots J_1 \,, \quad J_i \in \mathbb{R}^{n \times n}, \ i = 1, 2, \cdots, m \,.$$

Periodic eigendecomposition

X. Ding and P. Cvitanović, "Periodic eigendecomposition and its application in Kuramoto-Sivashinsky system," *SIAM J. Appl. Dyn. Syst.* **15**, 1434–1454 (2016)

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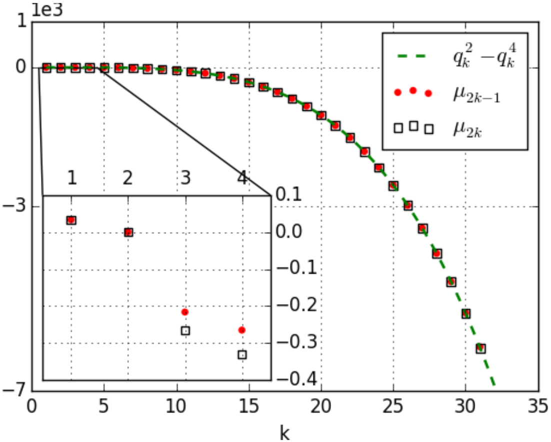
Floquet exponents

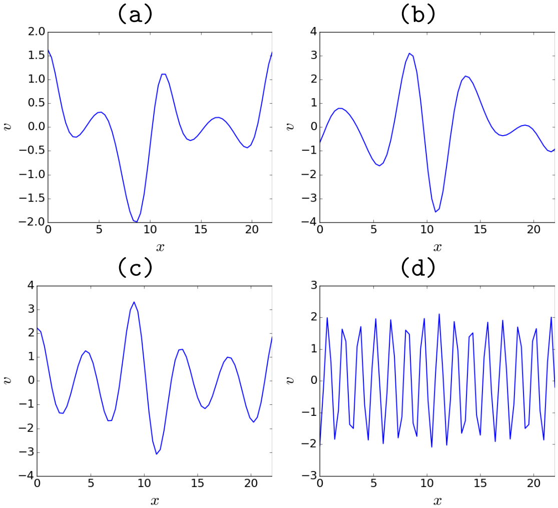


Floquet exponents of $\overline{ppo}_{10.25}$. The dashed line (green) is $q_k^2 - q_k^4$. The inset is a magnification of the region containing the 8 leading exponents.

The smallest multiplier : $10^{-27\,067}$.

$$\Lambda_j = \exp(T_p \lambda_p^{(j)}) \ = \exp(T_p \mu^{(j)} + i\theta_j)$$

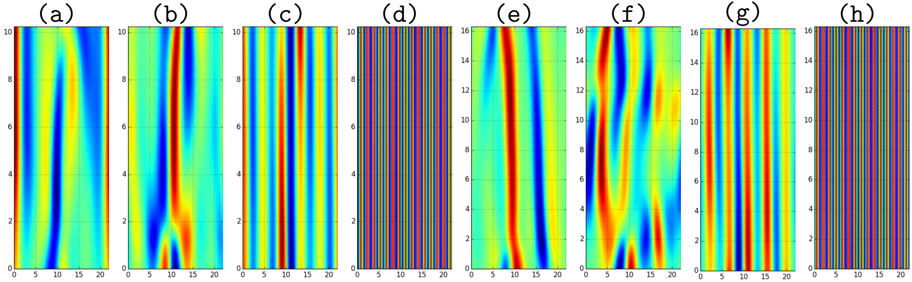
Floquet vectors



Floquet vectors of $\overline{ppo}_{10.25}$ at $t = 0$.

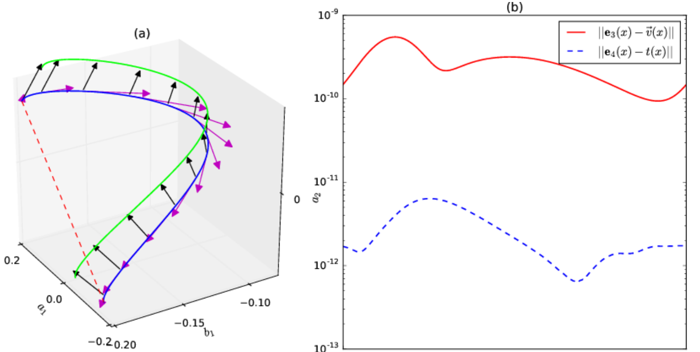
The real part of the (a) 1st, (b)5th, (c) 10th, and (d) 30th Floquet vectors.

Floquet vectors along the orbit



(a) $\sim$ (d) : the 1st (real part), 5th, 10th and 30th Floquet vector along $\overline{ppo}_{10.25}$ for one prime period. (e) $\sim$ (h) : the 1st, 4th (real part), 10th (imaginary part) 30th (imaginary part) Floquet vector along $\overline{rpo}_{16.31}$ for one prime period.

Accuracy of Floquet vectors

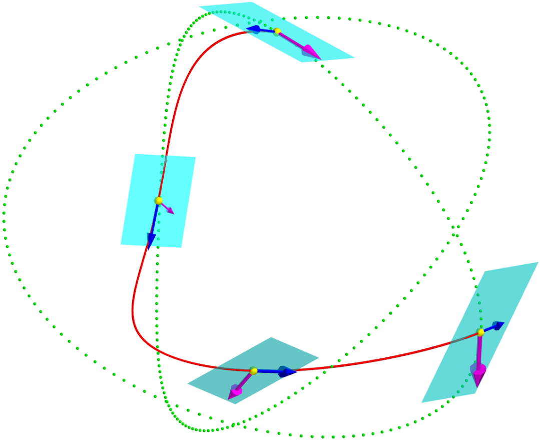


$\overline{ppo}_{10.25}$

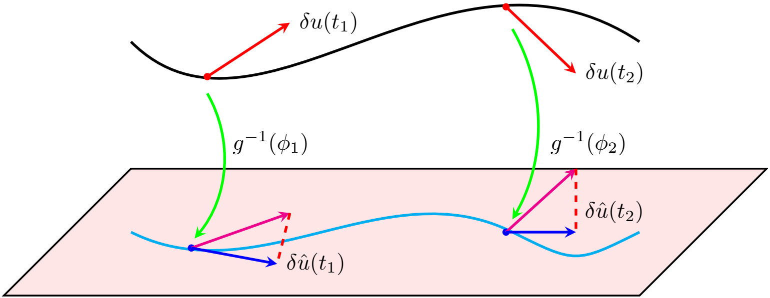
Marginal vectors and the associated errors. (a) $\overline{ppo}_{10.25}$ in one period projected onto $[b_1, c_1, b_2]$

Accuracy of Floquet vectors

$\overline{rpo}_{16.31}$



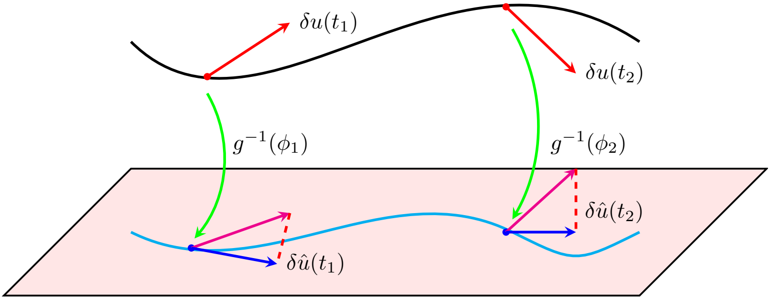
In-slice Jacobian matrix



Deformations in the full state space and in the slice.

$$\hat{J}h^{(-)}(\hat{u}(t_1))g(\phi_1)^{-1} = h^{(-)}(\hat{u}(t_2))g(\phi_2)^{-1}J\,.$$

In-slice Jacobian matrix



Deformations in the full state space and in the slice.

$$\hat{J} h^{(-)}(\hat{u}(t_1)) g(\phi_1)^{-1} = h^{(-)}(\hat{u}(t_2)) g(\phi_2)^{-1} J \,.$$

$$\hat{\Lambda}_j = \Lambda_j \,, \quad \hat{\mathbf{e}}_j = h^{(-)}(\hat{u}_p) g(\phi_p)^{-1} \hat{\mathbf{e}}_j \,.$$

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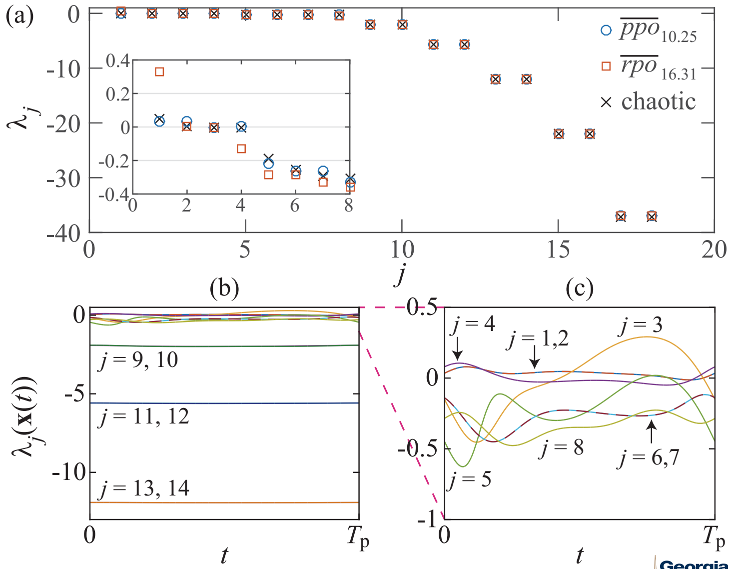
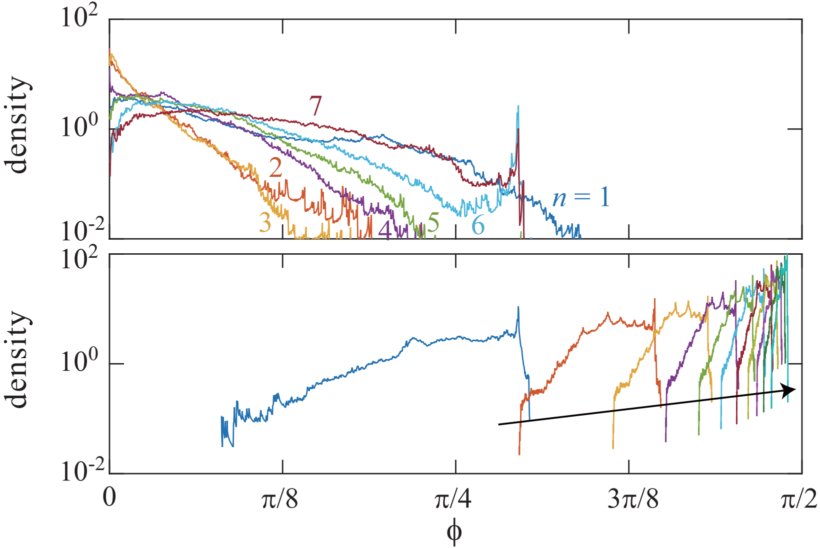
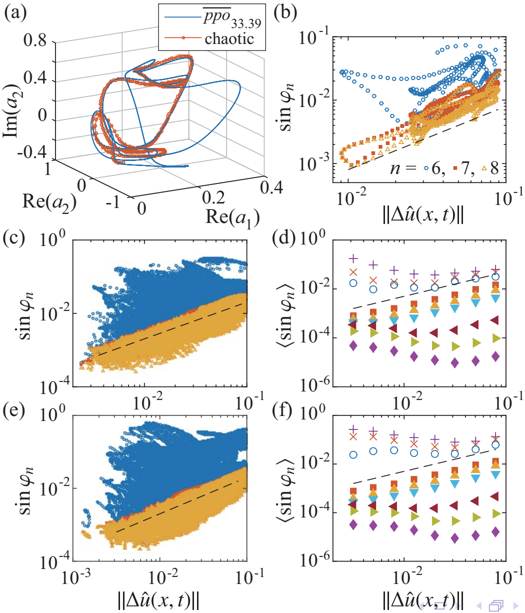
Decoupling of local Floquet exponents



Decoupling of Floquet vectors



Shadowing controlled by Floquet vectors



Main result

(Our simulations use 62 degrees of freedom.)

For one-dimensional Kuramoto-Sivashinsky equation defined on a periodic domain of size $L = 22$, the dimension of the inertial manifold is 8.

Ding, X., Chaté, H., Cvitanović, P., Siminos, E., and Takeuchi, K. A., "Estimating the dimension of the inertial manifold from unstable periodic orbits," *Phys. Rev. Lett.* **117**, 024101 (2016)

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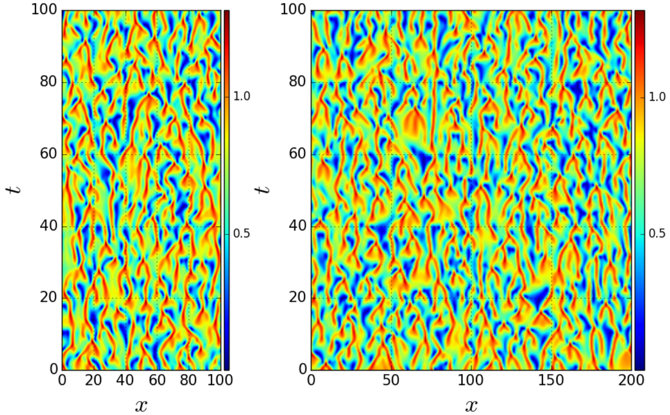
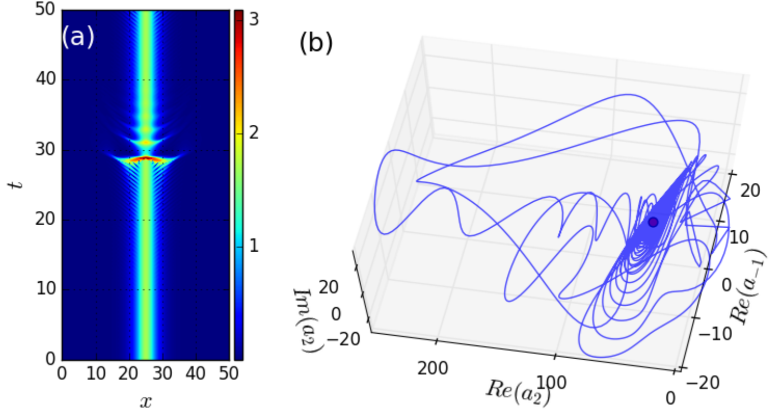
Spatially-extended systems : complex Ginzburg-Landau



$$A_t = A + (1 + i\alpha)A_{xx} - (1 + i\beta)|A|^2 A\,, \quad \alpha = 2 \text{ and } \beta = -2.$$

Recurrent patterns not only show up along the temporal axis
but also along the spatial axis

Following the localized solutions theory of transitional turbulence on large domains[13], here I focus on dissipative *soliton* solutions

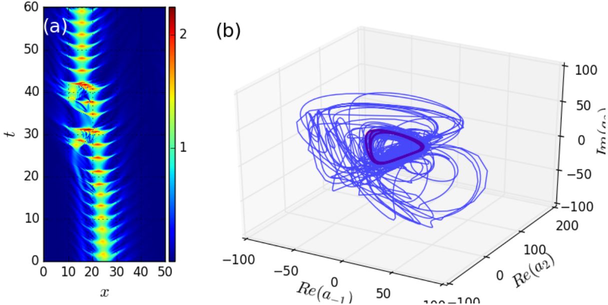


$$A_t = \mu A + (D_r + iD_i)A_{xx} + (\beta_r + i\beta_i)|A|^2 A + (\gamma_r + i\gamma_i)|A|^4 A$$

[13] J. Barnett, D. R. Gurevich, and R. O. Grigoriev. *Streamwise localization of traveling wave solutions in channel flow.* 2016. URL: https://arxiv.org/abs/1609.06608.

Discovery : exploding relative periodic orbit phase

A new class of *exploding solitons* !



Two completed papers :

"Relative periodic orbit explosion in cubic-quintic complex Ginzburg–Landau equation" [Ding and Cvitanović 2017]

"Integration of a cubic-quintic complex Ginzburg–Landau exploding soliton" [Ding, Cvitanović, and Kang 2017]

but no time to discuss this here

Table of Contents

Summary

- Determination of the dimension of the inertial manifold by periodic orbits.
- In-slice dynamics (in-slice Jacobian); $\mathrm{O}(2)$ symmetry reduction;
- Code repository: periodic eigendecomposition.

Future work

- spatiotemporal average by periodic orbits
- dimensions of inertial manifold of other systems


Thanks for your time !
Any question ?
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