symmetry
equivariant
Lorenz
flow!symmet

CHAPTER II

Say your groups act linearly on $\mathbb{R}^n$, globally, and

~~DE~~SYMMETRIZATION AND ITS DISCONTENTS

In this chapter we blah and blah (from Civilization and its discontents) ‡PRIVATE

The subject of symmetries of dynamical systems is vast and covered in many monographs and review articles. In this chapter we summarize the results from this literature that will be needed in applications to the problem at hand, ... making no attempt at completeness or derivation of well established results.

2.1 *Symmetries of dynamical systems*

~~2.1.1 Definition of symmetry~~

check

We consider a system of ODE's of the form

$$\dot{x} = v(x, \lambda) \tag{4}$$

where $v : \mathbb{R}^n \times \mathbb{R}^r \dashrightarrow \mathbb{R}^n$ is a C^∞ mapping. W~~hen not important we~~ will often suppress the r-dimensional vector of parameters λ in the notation.

Any compact Lie group acting on $\mathbb{R}^n$ can be identified with a subgroup of $O(n)$, *cf.* for example ref. [34] for a sketch of the proof. Therefore, without loss of generality we will concentrate on subgroups $\Gamma \subseteq O(n)$ in the following.

Das Buch

Definition 2.1 *We call a group element $\gamma \in O(n)$ a symmetry of (4) if for every solution $x(t)$, $\gamma x(t)$ is also a solution.*

[1]

The question now arises on how to check for symmetries of (4) since we generally do not have knowledge of the set of solutions[2]. Let $x(t)$ be a solution of 4. Then by Definition 2.1 $y(t) = \gamma x(t)$ is another solution and therefore satisfies (4):

$$\dot{y}(t) = v(y(t)) = v(\gamma x(t)) \, .$$

On the other hand

$$\dot{y}(t) = \gamma \dot{x} = \gamma v(x(t)) \, ,$$

for any solution $x(t)$. Since solutions exist for any $x \in \mathbb{R}^n$ we are led to the following condition for γ to be a symmetry of (4):

$$v(\gamma x) = \gamma v(x) \tag{5}$$

for all $x \in \mathbb{R}^n$. We say that v *commutes* with γ or that v is γ-*equivariant.* When v commutes with all $\gamma \in \Gamma$ we say that v is Γ-equivariant. Clearly the finite time flow $f^t(\gamma x_o)$ through γx_o satisfies the equivariance condition $f^t(\gamma x_o) = \gamma f^t(x_o)$ from definition of symmetry and uniqueness of solutions.

For example, the vector field in Lorenz equations (2) is equivariant under the group ? $D_1 \cong D_1$ acting on $\mathbb{R}^3$ by

$$R(\pi)(x, y, z) = (-x, -y, z) \, .$$

[1] Vaggelis: Refer to point vs contact symmetries?

[2] Vaggelis: Rather naively put.

Complex Lorenz flow!symmet
Armbruster-Guckenheim Holmes flow!symmet
group orbit
isotropy subgroup
stabilizer
isotropy subgroup!glo

? Note that this transformation can be considered either as rotation by π around the z axis (hence the group D_1) or as reflection about the origin in a plane perpendicular to the z-axis (hence the group D_1.)

As another example, the vector field in Complex Lorenz equations (28) is equivariant under the group $SO(2)$ acting on $\mathbb{R}^5 \cong \mathbb{C}^2 \times \mathbb{R}$ by

$$R(\theta)(x, y, z) = (e^{i\theta}x, e^{i\theta}y, z), \quad \theta \in [0, 2\pi). \tag{6}$$

Finally, the symmetry group of the Armbruster-Guckenheimer-Holmes flow (3) is $O(2)$ acting by

$$\begin{aligned} R(\theta)(z_1, z_2) &= (e^{i\theta}z_1, e^{i2\theta}z_2), \quad \theta \in [0, 2\pi), \\ \kappa(z_1, z_2) &= (\bar{z}_1, \bar{z}_2). \end{aligned}$$ (6a)

2.1.2 Phase space stratification

In order to understand the implications of equivariance for the solutions of (4) we first have to examine the way a compact Lie group acts on $\mathbb{R}^n$.

The *group orbit* of $x \in \mathbb{R}^n$ is the set

$$\Gamma x = \{\gamma x : \gamma \in \Gamma\}. \tag{7}$$

Definition 2.2 *The isotropy subgroup or stabilizer of a subset $S \subset \mathcal{M}$ is the set*

$$\Sigma_S = \{\gamma \in \Gamma : \gamma S = S\}. \tag{8}$$

Thus the isotropy subgroup describes the symmetries of a set S. Note that by Definition 2.2 the isotropy subgroup is the largest subgroup (in the sense of set inclusion, *cf.* (10)) that leaves S fixed.

?) **Definition 2.3** *The global isotropy subgroup of a subset $S \subset \mathcal{M}$ is the set*

$$\Sigma^*_S = \bigcap_{x \in S} \Sigma_x = \{\gamma \in \Gamma : \gamma x = x, \ \forall x \in S\}.$$

why not Lem 2.1? **Lemma 2.4** *Points on the same group orbit of Γ have conjugate isotropy subgroups:*

$$\Sigma_{\gamma x} = \gamma \Sigma_x \gamma^{-1}. \tag{9}$$

(See ref. [34] for the proof.) Thus we can characterize a group orbit by its *type*, defined as the conjugacy class of its isotropy subgroups.

Why not 2.1? using the same words as for **Proposition 2.5** *Let Γ be a compact Lie group acting on $\mathbb{R}^n$. Then*

1. *If Γ is finite then* $|\Gamma| = |\Sigma_x||\Gamma x|$.
2. *If Γ is continuous then* $\dim \Gamma = \dim \Sigma_x + \dim \Gamma x$.

The proof can be found in ref. [34]. We note that $\dim \Gamma x = \dim(\Gamma/\Sigma_x)$, where the *coset space* of a subgroup Σ of Γ is defined as $\Gamma/\Sigma = \{\gamma\Sigma | \gamma \in \Gamma\}$. Also recall that the (left) *cosets* of Σ in Γ are the sets $\gamma\Sigma = \{\gamma\sigma | \sigma \in \Sigma\}$.

stratum
fixed-point subspace

Therefore, when Γ is continuous each group orbit is a smooth compact manifold of dimension $\dim \Gamma x = \dim \Gamma - \dim \Sigma_x$. The union of orbits of the same type is called a *stratum* and is itself a smooth manifold. Thus $\mathbb{R}^n$ is stratified by the action of Γ into a disjoint union of strata $\mathcal{S}_i$ which are in an $1-1$ correspondence to the group orbit types (*cf.* ref. [14] for proof.) Note that in general the strata do not have the same dimension. There exists a unique stratum $\mathcal{S}_0$ of maximal dimension that is called *principal stratum*[28]. The principal stratum is open, dense and if Γ is connected then $\mathcal{S}_0$ is also connected[14].

Definition 2.6 *Let Σ be a subgroup of Γ acting on $\mathbb{R}^n$. The fixed-point subspace of Σ, denoted by $Fix(\Sigma)$, is the subspace of $\mathbb{R}^n$ containing all fixed points of Σ:*

$$Fix(\Sigma) = \{x \in \mathbb{R}^n \mid \sigma x = x \,, \ \forall \sigma \in \Sigma\} \,.$$

Fixed-point subspaces are invariant under equivariant dynamics. The following theorem applies:

Theorem 2.7 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be Γ-equivariant and let Σ be a subgroup of Γ. Then*

$$f\,(Fix(\Sigma)) \subseteq Fix(\Sigma) \,.$$

3

This leads to:

Proposition 2.8 *Let $x(t)$ be a solution trajectory of an equivariant ODE. Then $\Sigma_{x(t)} = \Sigma_{x(0)}$ for all t.*

4

We can define a partial ordering $\preceq$ on conjugacy classes of subgroups of Γ. Let $H = \{H_i\}$ and $K = \{K_j\}$ be two such conjugacy classes. Then

$$H \preceq K \Leftrightarrow H_i \subseteq K_j \qquad (10)$$

for some representatives H_i, K_j. We refer to the partially ordered set that results from this ordering as the *subgroup lattice.*

Definition 2.9 *The isotropy lattice of Γ is the set of all conjugacy classes of isotropy subgroups partially ordered by $\preceq$.*

5

use the same center as others

Example 2.1 ~~As an example we will~~ C*onsider D_3, the symmetry group of the equilateral triangle figure 1, acting on $\mathbb{R}^2 \cong \mathbb{C}$ by*

$$\begin{aligned} \zeta z &= e^{i\frac{2\pi}{3}} z \,, \\ \kappa z &= \bar{z} \,. \end{aligned} \qquad (10a)$$

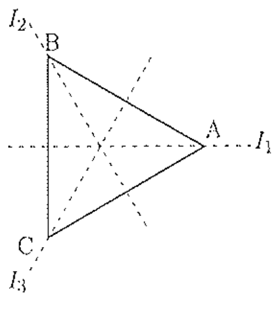

Figure 1: D_3 leaves the equilateral triangle setwise fixed. The reflection symmetry axes have been denoted I_i.

There are, up to conjugacy, three subgroups: $1 = \{e\}$, $D_1(\kappa)$, which is isomorphic[6] *to the subgroups generated by $\kappa\zeta$ and by $\kappa\zeta^2$, and $C_3 = \{e, \zeta, \zeta^2\}$. The subgroup lattice is shown in figure 2(a). There are three classes, each corresponding to a distinct geometrical operation: $\{e\}, \{\kappa, \kappa\zeta, \kappa\zeta^2\}$ and $\{\zeta, \zeta^2\}$.*

Example 2.2 *We now examine how the vertex A of the triangle transforms under the axion of D_3. Under ζ and ζ^2 it is mapped to the vertices B and C, respectively. Thus all vertices belong to the same group orbit and have conjugate isotropy subgroups, from Lemma 2.4. Under κ vertex A remains fixed. Thus the isotropy subgroup of point A is $\Sigma_A = D_1(\kappa)$. By Lemma 2.4 we have $\Sigma_B = \zeta\, D_1(\kappa)\,\zeta^{-1} = D_1(\kappa\zeta)$ and $\Sigma_C = \zeta^{-1}\, D_1(\kappa)\,\zeta = D_1(\kappa\zeta^2)$. Next, note that $D_1(\kappa)$ fixes any point on the symmetry axis I_1, while ζ and ζ^2 map it to I_2 and I_3, respectively. The origin is the only point fixed by any group operation, i.e. has isotropy subgroup D_3. Finally, any point that is not on one of the symmetry axes I_1, I_2, I_3 has trivial isotropy subgroup. Thus we arrive to the following conclusions:*

The isotropy subgroups are: $\Sigma_{\{0\}} = D_3$, $\Sigma_{I_1^*} = D_1(\kappa) \simeq \Sigma_{I_2^*} \simeq \Sigma_{I_3^*}$, $\Sigma_{\mathbb{R}^2\backslash\{\cup I_i\}} = 1$, where I_i^* ~~is used as a shortcut for~~ $I_i\backslash\{0\}$. The fixed point subspaces of D_3, $D_1(\kappa)$, $D_1(\kappa\zeta)$ and $D_1(\kappa\zeta^2)$ are the origin, I_1, I_2 and I_3, respectively. The fixed point subspace of C_3 is the origin but, since C_3 is a proper subgroup of D_3, it does not qualify as isotropy subgroup of the origin (cf. *Definition 2.2.) Thus C_3 is not in the isotropy lattice of D_3 acting on $\mathbb{R}^2$,* cf. *figure 2(b).* [7]

There are three strata in correspondence with the orbit types (and with isotropy subgroups): the origin (type D_3), $\{\cup I_i^*\}$ (type $D_1(\kappa)$), and the principal stratum $\mathbb{R}^2\backslash\{\cup I_i\}$ (type 1).

If we now consider a two-dimensional system of ODEs equivariant under the action
— (10a) *(11) of D_3 we can conclude immediately that the fixed-point subspaces are flow invariant by Proposition 2.8. Thus the origin has to be a fixed point of the flow. Moreover the principal*

[3] Vaggelis: write the proof

[4] Vaggelis: write the proof

[5] Vaggelis: I copy Golubitsky and Stewart's definition of isotropy lattice and partial ordering (but not of subgroup lattice) here because I am somewhat confused. Are the conjugacy classes of subgroups distinct from the classes defined by conjugate elements of the group? For instance in the example of D_3 below, $D_1(\kappa)$, $D_1(\kappa\zeta)$ and $D_1(\kappa\zeta^2)$ are conjugate subgroups. My interpretation is that we call this a conjugacy class of isotropy subgroups. On the other hand, D_3 is partitioned in three classes: $\{e\}, \{\kappa, \kappa\zeta, \kappa\zeta^2\}$ and $\{\zeta, \zeta^2\}$ which of course are not subgroups, except for $\{e\}$.

[6] *Vaggelis: or say conjugate?*

[7] *Predrag: something is screwy with the plane Couette flow analysis - there we keep C_3, I think...*

$$
\begin{tikzcd}[column sep=small]
 & D_3 \arrow[dl, dash] \arrow[dr, dash] & \\
D_1(\kappa) \arrow[dr, dash] & & C_3 \arrow[dl, dash] \\
(a) & \mathbf{1} &
\end{tikzcd}
\qquad
\begin{tikzcd}
D_3 \arrow[d, dash] \\
D_1(\kappa) \arrow[d, dash] \\
(b)\ \mathbf{1}
\end{tikzcd}
$$

Figure 2: (a) D_3 subgroup lattice, (b) D_3 isotropy lattice

stratum $\mathbb{R}^2\backslash\{\cup I_i\}$ is partitioned by the symmetry axes I_i that are flow invariant into six disjoined pieces on the same group orbit of D_3.

Example 2.3 *Consider $\Sigma = SO(2)$ acting on $\mathbb{R}^5$ by*

$$x \longmapsto R(\theta)x\,, \tag{11}$$

where

$$R(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 & 0 & 0 \\ \sin(\theta) & \cos(\theta) & 0 & 0 & 0 \\ 0 & 0 & \cos(\theta) & -\sin(\theta) & 0 \\ 0 & 0 & \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \theta \in [0, 2\pi)\,. \tag{12}$$

Note that this is the same action as in (6) but now we do not make the identification $\mathbb{R}^5 \cong \mathbb{C}^2 \times \mathbb{R}$. As we will see in chapter 3 the following have direct application in symmetry reduction of Complex Lorenz equations. Choose coordinates x_1, x_2, y_1, y_2, z on $\mathbb{R}^5$, related to the complex coordinates of (6) by $x = x_1 + ix_2$, $y = y_1 + iy_2$. ~~Note that~~ The fixed-point subspace of the action of $SO(2)$ ~~defined above~~ is the z-axis. The isotropy subgroup of the z-axis is thus $SO(2)$, while the isotropy subgroup of $\mathcal{M}^* \equiv \mathbb{R}^5\backslash\{x_1 = x_2 = y_1 = y_2 = 0\}$ is the identity element.

Example 2.4 *Consider the action of $O(2)$ on $\mathbb{C}^n$ by*

$$
\begin{aligned}
R(\theta)z_k &= e^{ik\theta}z_k\,,\ \theta \in [0, 2\pi)\,, & (13)\\
\kappa z &= \overline{z}\,,\ z = (z_1, \ldots z_n)\,, & (14)
\end{aligned}
$$
— one

where $z_k \in \mathbb{C}$. The subgroup lattice is drawn in figure 3. The subgroup D_m, $m > 0$, of $O(2)$ is generated by the reflection κ and a rotation $R_m \equiv R(2\pi/m)$. The discrete cyclic *subgroup C_q, $q > 0$ of $SO(2)$ is generated by* discrete *a rotation R_q. If q divides m then we have the following containment relations $C_q \prec C_m \prec D_m$ and $D_q \prec D_m$[8]. Note that $C_1 \cong \mathbf{1}$.* (?)

The fixed-point subspace of $O(2)$ and $SO(2)$ is the origin and thus $SO(2)$ is not in the isotropy lattice. The fixed-point subspace of C_q is given by the condition

$$z_k = 0 \text{ unless } k = qj\,,\ j = 1, \ldots \lfloor n/q \rfloor \tag{15}$$

and is thus a $2\lfloor n/q\rfloor$-dimensional subspace (here we count real dimensions.) For the fixed-point subspace of D_q we have condition (15) and additionaly that

$$Im(z_k) = 0 \text{ for } k = 1, \ldots, n\,. \tag{16}$$

[8] *Vaggelis: Do we also have $C_2 \prec D_m$ for $n \leq 2$?*

free

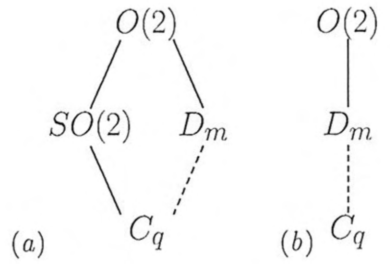

Figure 3: (a) $O(2)$ subgroup lattice, (b) $O(2)$ isotropy lattice for the action defined by (14). In both figures dashed lines indicate containement only when q divides m, while the relations $C_q \prec D_q$ and $D_q \prec D_m$ when q divides m are not drawn. For the isotropy lattice $m, q \leq n$.

Therefore $\mathrm{Fix}\,(D_q)$ *is an* $\lfloor n/q \rfloor$*-dimensional subspace. Note that for* $q > n$ *condition (15) cannot be satisfied and all* z_k*'s have to be equal to zero. This implies that the fixed-point subspace of* D_q *and* C_q *for* $q > n$ *is the origin and as a result those subgroups are not in the isotropy lattice.* $\mathrm{Fix}\,(\mathbf{1}) = \mathbb{R}^n$ *is the principal stratum.* (not $\mathbb{R}^n/\{0\}$?)

Points on the group orbit of a point $x \in \mathrm{Fix}\,(D_m\,(\kappa, R_m))$ *have, by Lemma 2.4 conjugate isotropy subgroups:* $\Sigma_{R(\theta)x} = R(\theta) D_m\,(\kappa, R_m)\, R(-\theta) \simeq D_n\,(\kappa R(\theta), R_m)$*. The fixed-point subspace of* $D_n\,(\kappa R(\theta), R_m)$ *is obtained by rotation* $R(\theta)$ *of* $\mathrm{Fix}\,(D_m\,(\kappa, R_m))$*.*

On the other hand $\mathrm{Fix}\,(C_q)$ *is invariant as a set under the action of* $O(2)$*, i.e.* $\Sigma_{\mathrm{Fix}(C_q)} = O(2)$*.* [9]

(how can this be: 1 point, expect a discrete orbit!)

A general procedure exists [28] to determine which subgroups in the subgroup lattice of a group Γ are isotropy subgroups when Γ acts faithfully on $\mathbb{R}^n$. For each subgroup K (or more precisely for each subgroup class represented by K) determine the dimension of the fixed-point subspace. Then we trace the subgroup lattice: For each subgroup K we compare $dim(Fix(K))$ to $dim(Fix(H))$ for every $H \subseteq \Gamma$ for which $K \subset H$. If $dim(Fix(K)) = dim(Fix(H))$ then K is not an isotropy subgroup. The determination of the dimension of the fixed-point subspace of a subgroup K can be done by means of the following trace formula if an explicit representation $\rho(K)$ of K is known[10]:

$$dim(Fix(K)) = \frac{1}{|K|} \sum_{\kappa \in K} \mathrm{trace}\,(\rho(\kappa))\,. \tag{17}$$

2.1.3 Group Action types

In this section we ~~recall~~ state definitions ~~on the~~ for different types of group actions.

Definition 2.10 *A group* Γ *acts freely on* $\mathcal{M}$ *if all isotropy subgroups are trivial:* $\Sigma_x = \{e\}$ *for all* $x \in \mathcal{M}$*.* Γ *acts locally freely if all isotropy subgroups are discrete subgroups of* Γ*.*

Example 2.5 *The action (11) of* $SO(2)$ *on* $\mathbb{R}^5$ *is not free (or even locally free), while the same action on* $\mathcal{M}^*$ *is free. If we don't restrict* θ *in* $[0, 2\pi)$ *then the group* $\mathbb{R}$ *acts locally freely on* $\mathcal{M}^*$ *since the isotropy subgroup is the discrete subgroup* $2\pi\mathbb{Z}$ *of integer multiples of* 2π*.*

[9] *Vaggelis: Try to understand why* $SO(2)$ *leaves* $\mathrm{Fix}\,(C_m)$ *but not* $\mathrm{Fix}\,(D_m)$ *invariant without reference to its representation.*

[10] Vaggelis: Or if one is able to determine the character of the representation by other means.

Example 2.6 *The action (14) of $O(2)$ is locally free on $\mathbb{C}^n \setminus \{0\}$ but not on $\mathbb{C}^n$.*

faithful
regular
semi-regular

Definition 2.11 *A group Γ acts faithfully (or effectively) on $\mathcal{M}$ if and only if it has trivial global isotropy subgroup.*

Example 2.7 *The actions (11) of $SO(2)$ and (14) of $O(2)$ are ~~both~~ faithfull.*

Definition 2.12 *A group Γ acts semi-regularly on $\mathcal{M}$ if all its orbits have the same dimension. If in addition for each point $x \in \mathcal{M}$ there exists an arbitrarily small neighborhood U such that each orbit of Γ intersects U in a pathwise connected subset, then the group acts regularly.*

[11] [12]

Example 2.8 *Action (11) of $SO(2)$ is regular on $\mathcal{M}^*$ but not on $\mathbb{R}^5$.*

Example 2.9 *Since the action (14) of $O(2)$ on $\mathbb{C}^n \setminus \{0\}$ is free it is also semi-regular. Indeed, from Proposition 2.5 all group orbits of points $x \in \mathbb{C}^n \setminus \{0\}$ are 1-dimensional.* [13].

The group orbits of an effective and regural or semi-regular action of a Lie group Γ on a manifold $\mathcal{M}$ form a foliation of $\mathcal{M}$[14].

2.1.4 Principal Fibre Bundles

Assume a Lie group Γ acts[15] on a topological space E. We call the bundle (E, π, X) a Γ-*bundle* if it is isomorphic to $(E, \rho, E/\Gamma)$. The fibres over the points of the base space E/Γ are the group orbits of points in E. Thus in general [16] the bundle E is not a fibre bundle since the fibres are not homeomorphic to each other. If Γ acts freely on E then the Γ-bundle is a fibre bundle with fibre Γ and is called a *principal* Γ-*bundle*, while Γ is called the structure group of the bundle.

2.1.5 Symmetries of solutions

In the preceding section we concentrated on symmetries of the space on which the group acts. We now proceed to examine the symmetries of solutions of equivariant dynamical systems. [to be write?] This is a section or a subsection?]

2.1.6 Moving frames

In this section we present the method of moving coframes [17] of Fels and Olver [25; 26], also *cf.* ref. [58] for a pedagogical exposition. The method can be used to generate functionally

[11] Vaggelis: Think of regular versus semi-regular action as the group orbit being circle versus a spiral. I'll try to find a specific example of semiregular action.

[12] Vaggelis: We can write in summary: free $\Rightarrow$ locally free $\Rightarrow$ semi-regular.

[13] *Vaggelis: This will turn out to be important when applying moving frame method.*

[14] Vaggelis: Is this related to us being able to construct a local cross-section?

[15] Vaggelis: Any restriction on how it acts? In definition in Isham the term G-space is used, which I don't understand since he doesn't define it.

[16] Vaggelis: Can we substitute "in general" with "if Γ does not act freely on E then"

[17] Predrag: do they really say "moving coframes" rather than "comoving frames"? **ES**: Yes, they really do. They refer to Cartan's method as method of "moving frames" and they claim it to be a special (and less rigorous) case of the moving coframe method. I don't know Cartan's method and the two papers of Fels and Olver [25; 26] are lengthy and technical. Olver's book is readable but I it doesn't describe Cartan's method. I think they say "moving" rather than "comoving" frames because one only comoves in the direction of group action.

into remarks

independent fundamental invariants for the action of a group Γ on a manifold $\mathcal{M}$ under certain conditions. It is a generalization of the method of moving frames of Cartan. The name "moving coframes" ~~is due to the fact that the justification of the method relies on~~ arises through use of Maurer-Cartan form which ~~are~~ is a coframe on the Lie group Γ, in the sense that they form a poinwise basis for the cotangent space. [18] Here we follow ref. [58] but first motivate the general method through an example. *cross-section*

For groups acting regularly we can define a cross-section for the group orbits.

Definition 2.13 *Let Γ act regularly on a n-dimensional manifold $\mathcal{M}$ with r-dimensional orbits. Define a (local)* cross-section *to be an $(n-r)$-dimensional submanifold K of $\mathcal{M}$ such that K intersects each orbit transversally and at most once.*

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Proposition 2.14 ([58]) *If a Lie group Γ acts regularly on a manifold $\mathcal{M}$, then one can construct a local cross-section passing through any point $x \in \mathcal{M}$.*

A cross-section K can be defined by means of level sets of functions $K_i(x) = c_i$, where $x \in V$ and $i = 1, \ldots, r$. If the $K_i(x)$ coincide with the local coordinates x_i on the manifold V, i.e. $K_i(x) = x_i$, then we call K a *coordinate cross section.* In our example we can define a coordinate cross-section for the action of $\Sigma = SO(2)$ on $\mathbb{R}^5 \backslash \{x_1 = x_2 = y_1 = y_2 = 0\}$ by, for instance, $x_1 = 0$.

Example CL Eqs

We can now construct a moving frame for the action (11) of $SO(2)$ as follows. We write out explicitly the group transformations:

$$\begin{aligned}
\overline{x}_1 &= x_1 \cos\theta - x_2 \sin\theta\,, \\
\overline{x}_2 &= x_1 \sin\theta + x_2 \cos\theta\,, \\
\overline{y}_1 &= y_1 \cos\theta - y_2 \sin\theta\,, \\
\overline{y}_2 &= y_1 \sin\theta + y_2 \cos\theta\,, \\
\overline{z} &= z\,.
\end{aligned} \tag{18}$$

Then make $\overline{x}_1$ equal to the constant in the choice of cross-section, i.e. set $\overline{x}_1 = 0$. Thus, we can solve the first of (18) for the group parameter θ and substitute in the remaining equations. We get

$$\begin{aligned}
\theta &= 2\tan^{-1}\frac{-x_2 + \sqrt{x_1^2 + x_2^2}}{x_1}\,, \\
\overline{x}_2 &= \sqrt{x_1^2 + x_2^2}\,, \\
\overline{y}_1 &= \frac{x_2 y_1 - x_1 y_2}{\sqrt{x_1^2 + x_2^2}}\,, \\
\overline{y}_2 &= \frac{x_1 y_1 + x_2 y_2}{\sqrt{x_1^2 + x_2^2}}\,.
\end{aligned} \tag{19}$$

gm $r = \sqrt{x_1^2 + x_2^2}$

[20] ~~We note that~~ By construction $\overline{x}_2, \overline{y}_1, \overline{y}_2, \overline{z}$ are $SO(2)$ invariant. In fact they are the *fundamental invariants* for our problem: any other invariant can be expressed as a function of $\overline{x}_2, \overline{y}_1, \overline{y}_2, \overline{z}$ and they

[18] Vaggelis: cite Cartan

[19] Vaggelis: Is this cross-section related to a cross-section in a Γ-bundle? In other words, can we interprete the latter as a submanifold in total space E of a Γ-bundle $(E, \pi, E/\Gamma)$?

[20] Vaggelis: The solution $\theta = 2\tan^{-1}\frac{-x_2+\sqrt{x_1^2+x_2^2}}{x_1}$ was returned by Mathematica. If we use $\theta = \tan^{-1}\frac{x_2}{x_1}$ our results are multiplied by $sgn(x_2)$.

are functionally independent. Thus they serve to distinguish orbits in the neighborhood of the cross-section, i.e. two points lie on the same group orbit if and only if all the fundamental invariants agree.

The *normalization* procedure for the computation of invariants applied in the example of $SO(2)$ can be applied in much more general situations as follows. Assume Γ acts (locally) freely[21] on $\mathcal{M}$ and thus Γ-orbits have the same dimension, say r, as Γ. Choose a coordinate cross-section $K = \{x_1 = c_1, \ldots, x_r = c_r\}$ defined by the first r coordinates (relabel coordinates as necessary). Introduce local coordinates $g = (g_1, \ldots, g_r)$ on Γ in the neighborhood of the identity. Write out explicitly the group transformations:

$$\bar{x} = g \cdot x = w(g, x)\,. \tag{20}$$

Equating the first r components of the function w to the constants in the definition of the cross-section $K_i(x) = c_i$ yields the *normalization equations* for K:

$$\bar{x}_1 = w_1(g, x) = c_1, \ldots, \bar{x}_r = w_r(g, x) = c_r\,. \tag{21}$$

From the definition of cross-section and the Implicit Function Theorem the normalization equations (21) can always be solved for the group parameters in terms of x, yielding the *moving frame* associated with K: $g = \gamma(x)$. Substitution of the moving frame equation back to (20) will yield the $n - r$ fundamental invariants. For proof *cf.* refs. [25, 26].

The power of the method shows in the calculation of invariants for groups acting on a high-dimensional space. Consider for example the task of computing invariants for the standard action of $SO(2)$ on $\mathbb{C}^n \cong \mathbb{R}^{2n}$ defined by

$$\begin{pmatrix} \bar{b}_k \\ \bar{c}_k \end{pmatrix} = \begin{pmatrix} \cos(k\theta) & -\sin(k\theta) \\ \sin(k\theta) & \cos(k\theta) \end{pmatrix} \begin{pmatrix} b_k \\ c_k \end{pmatrix}, \quad k = 1, \ldots n\,. \tag{22}$$

with $b_k, c_k \in \mathbb{R}$. The Gröbner basis methods ~~usually~~ perform poorly as n becomes larger than six[22]. On a 1GHz Pentium III processor the fundamental invariants for $n = 16$, for the cross section $b_1 = 0$, were computed in approximately 130s. Most importantly the time was mostly spend in simplification of expressions. If one only wants to project equivariant dynamics on the orbit space then the moving frame method, through its geometric interpretation, can be used to perform the projection without explicit knowledge of the fundamental invariants. This idea will become clear in the example of orbit space projection for Complex Lorenz equations.[23] The fundamental invariants for $SO(2)$ acting as in (22) with $n = 6$ calculated with the method of moving frames are listed in table 2.1.6. Note another advantage of the method: no syzygies are present.

refer to it

2.2 *Symmetries imply possible existence of relative periodic orbits*

In a dynamical system $\dot{a} = v(a)$ with a strange invariant set, there exist infinitely many periodic orbits

$$f^T(a) = a$$

characterized by period T, which are dense within the invariant set. Here f^t is the flow map of the flow v, i.e., $a(t) = f^t(a)$ is the solution of the flow v with initial condition $a(0) = a$.

[21] Vaggelis: The condition of free action can be relaxed [58].

[22] Vaggelis: I need to quote the literature or try it to fully justify this.

[23] Vaggelis: refer to appropriate section when written.

$r_1 = \sqrt{b_1^2+c_1^2}$	
$\frac{-b_1^2 b_2+b_2 c_1^2-2b_1 c_1 c_2}{\cancel{b_1^2+c_1^2}\ r^2}$	$\frac{2b_1 b_2 c_1-b_1^2 c_2+c_1^2 c_2}{\cancel{b_1^2+c_1^2}\ r^2}$
$\frac{-3b_1^2 b_3 c_1+b_3 c_1^3+b_1^3 c_3-3b_1 c_1^2 c_3}{\cancel{(b_1^2+c_1^2)^{3/2}}\ r^3}$	$\frac{-b_1^3 b_3+3b_1 b_3 c_1^2-3b_1^2 c_1 c_3+c_1^3 c_3}{\cancel{(b_1^2+c_1^2)^{3/2}}\ r^3}$
$\frac{b_4(b_1^4-6b_1^2c_1^2+c_1^4)+4b_1c_1(b_1^2-c_1^2)c_4}{\cancel{(b_1^2+c_1^2)^2}\ r^4}$	$\frac{4b_1b_4c_1(-b_1^2+c_1^2)+(b_1^4-6b_1^2c_1^2+c_1^4)c_4}{\cancel{(b_1^2+c_1^2)^2}\ r^4}$
$\frac{b_5c_1(5b_1^4-10b_1^2c_1^2+c_1^4)-b_1(b_1^4-10b_1^2c_1^2+5c_1^4)c_5}{\cancel{(b_1^2+c_1^2)^{5/2}}\ r^5}$	$\frac{b_1b_5(b_1^4-10b_1^2c_1^2+5c_1^4)+c_1(5b_1^4-10b_1^2c_1^2+c_1^4)c_5}{\cancel{(b_1^2+c_1^2)^{5/2}}\ r^5}$
$\frac{b_6(-b_1^6+15b_1^4c_1^2-15b_1^2c_1^4+c_1^6)-2b_1c_1(3b_1^4-10b_1^2c_1^2+3c_1^4)c_6}{\cancel{(b_1^2+c_1^2)^3}\ r^6}$	$\frac{2b_1b_6c_1(3b_1^4-10b_1^2c_1^2+3c_1^4)+(-b_1^6+15b_1^4c_1^2-15b_1^2c_1^4+c_1^6)c_6}{\cancel{(b_1^2+c_1^2)^3}\ r^6}$

Table 1: Fundamental invariants for the standard action of $SO(2)$ on $\mathbb{R}^6$

Let the dynamical system have symmetries represented by the operators $\Sigma_{k,s}$, where $k \in \mathcal{K} \subset \mathbb{Z}^p$ are parameters of discrete symmetries and $s \in \mathcal{S} \subset \mathbb{R}^q$ are parameters of continuous symmetries. In other words,

$$f^t(\Sigma_{k,s}a) = \Sigma_{k,s}f^t(a)\,.$$

In this case it is likely that, in addition to periodic orbits, the dynamical system also has relative periodic orbits, characterized by the condition

$$\Sigma_{k,s}f^T(a) = a\,,$$

where, in addition to the period T, the relative periodic orbit is also characterized by the symmetry parameters k and/or s.

In fact, if the symmetry is continuous, then ~~it is~~ much more likely ~~to find~~ relative periodic orbits are than ~~it is to find exact~~ periodic orbits, since $s = 0$, corresponding to a ~~the~~ periodic orbit, is only one specific value in the continuum of possible values of parameter s.

In the case of KS equation, which has continuous symmetry $\tau_{d/L}$ and discrete symmetry κ, it is possible to find relative periodic orbits that satisfy one of the following conditions

$$\tau_{d/L}f^T(a) = a\,, \quad \text{or} \quad \kappa\tau_{d/L}f^T(a) = a\,.$$

The first condition is satisfied by relative periodic orbits with shift d, while relative periodic orbits that satisfy the second condition are exactly periodic ($d = 0$) with period $2T$.

move to a different chapter

2.2.1 Multipoint shooting for relative periodic orbits

Here we will present the one-parameter Lie group case. The generalization to ~~modifications for~~ multi-parameter groups is straightforward ~~are rather obvious~~. Let the symmetry group Γ parametrized by the real parameter θ[24]. Assume that we have an initial guess for a relative periodic orbit of period T_p and phase shift ϕ_p. Let the guess be given as N initial conditions x_i, $i = 1 \ldots N$ for each segment of the RPO, along with the flight times T_i, such that $\sum_i^N T_i = T_p$, and the phase shift ϕ_p. For the true relative periodic orbit we have

defined?

$$\begin{aligned} f^{\tilde{T}_i}(\tilde{x}_i) &= \tilde{x}_{i+1}\,,\ i = 1, \ldots N-1\,, \\ R(\tilde{\phi}_p)f^{\tilde{T}_N}(\tilde{x}_N) &= \tilde{x}_1\,. \end{aligned} \qquad (23)$$

[24] Vaggelis: Should compactness be emphasized?

Assuming that our guess is in the linear neighborhoud of the relative periodic orbit we can Taylor expand (24) around ~~our~~ the guess to linear order in the small quantities $\Delta x_i = \tilde{x}_i - x_i$, $\Delta T_i = \tilde{T}_i - T_i$, $\Delta\phi_p = \tilde{\phi}_p - \phi_p$ to get

$$
\begin{aligned}
J^{T_i}(x_i)\Delta x_i + v_{T_i}\Delta T_i - \Delta x_{i+1} &= x_{i+1} - f^{T_i}(x_i) \\
R(\phi_p)J^{T_N}(x_N)\Delta x_N + R(\phi_p)v_{T_N}\Delta T_N + \mathfrak{g}R(\phi_p)f^{T_N}(x_N)\Delta\phi - \Delta x_1 &= x_1 - R(\phi_p)f^{T_N}(x_N) \qquad (24)
\end{aligned}
$$

where v_{T_i} denotes v evaluated at $f^{T_i}(x_i)$ and $\mathfrak{g}$ denotes the Lie algebra generator of the group. Interpreting Δx_i, ΔT_i, $\Delta\phi_p$ as corrections to our guess solution we iteratively improve our approximation of $\tilde{x}_p$. To ~~overcome the difficulties associated with the two unit~~ exclude variations along the eigenvalues eigendirections of $R(\tilde{\phi}_p)J^{\tilde{T}_p}(\tilde{x})$ we impose the conditions

(what's x_N?)

$$
\begin{aligned}
v(x_i)\cdot\Delta x_i &= 0\,, && (25)\\
(\mathfrak{g}x_N)\cdot\Delta x_N &= 0\,. && (26)
\end{aligned}
$$

Conditions (25) ~~assures~~ ensures that the correction will be transverse to the eigendirection associated with time translational invariance, while condition (25) prohibits correction along the direction of infinitesimal group action. In matrix form we have the system

reform to this fit on a page

$$
\left(\begin{array}{cccc|ccc|c}
\mathbf{J}^{T_1}(x_1) & -\mathbf{1} & & & v_{T_1} & & & \\
 & \ddots & -\mathbf{1} & & & \ddots & & \\
 & & \mathbf{J}^{T_i}(x_i) & -\mathbf{1} & & v_{T_i} & & \\
 & & \ddots & -\mathbf{1} & & & \ddots & \\
-\mathbf{1} & & & R(\phi_p)\mathbf{J}^{T_N}(x_N) & & & R(\phi_p)v_{T_N} & \mathfrak{g}R(\phi_p)f^{T_N}(x_N) \\
\hline
v(x_1) & & & & & & & \\
 & \ddots & & & & & & \\
 & v(x_i) & & & & & & \\
 & & \ddots & & & & & \\
 & & & v(x_N) & & & & \\
\hline
 & & & \mathfrak{g}x_N & & & &
\end{array}\right)
\begin{pmatrix}
\Delta x_1 \\ \vdots \\ \Delta x_i \\ \vdots \\ \Delta x_N \\ \Delta T_1 \\ \vdots \\ \Delta T_i \\ \vdots \\ \Delta T_N \\ \Delta\phi
\end{pmatrix}
=
\begin{pmatrix}
x_2 - f^{T_1}(x_1) \\ \vdots \\ x_{i+1} - f^{T_i}(x_i) \\ \vdots \\ x_1 - R(\phi_p)f^{T_N}(x_N) \\ 0 \\ \\ \vdots \\ \\ \\ 0
\end{pmatrix}
\qquad (27)
$$

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2.3 Commentary

Remark 2.1 Symmetries of the Lorenz equation: After having studied example ?? you will appreciate why `ChaosBook.org` starts out with the symmetry-less Rössler flow (??), instead of the better known Lorenz flow (??) (indeed, getting rid of symmetry was one of Rössler's motivations). He threw the baby out with the water; for Lorenz flow dimensionalities of stable/unstable manifolds make possible a robust heteroclinic connection absent from Rössler flow, with unstable manifolds of an equilibrium flowing into the stable manifold of another equilibria. How such connections are forced upon us is best grasped by perusing the chapter 13 "Heteroclinic tangles" of the inimitable Abraham and Shaw illustrated classic [?]. Their beautiful hand-drawn sketches elucidate the origin of heteroclinic connections in the Lorenz flow (and its high-dimensional Navier-Stokes relatives) better than any computer simulation. Miranda and Stone [?] were first to quotient the D_1 symmetry and explicitly construct the desymmetrized, "proto-Lorenz system," by a nonlinear coordinate transformation into the Hilbert-Weyl polynomial basis invariant under the action of the symmetry group [?]. For in-depth discussion of symmetry-reduced ("images") and symmetry-extended ("covers") topology, symbolic dynamics, periodic orbits, invariant polynomial bases etc., of Lorenz, Rössler and many other low-dimensional systems there is no better reference than the Gilmore and Letellier monograph [31?]. They interpret the proto-Lorenz and its "double cover" Lorenz as "intensities" being the squares of "amplitudes," and call quotiented flows such as (Lorenz)/D_1 "images." Our "doubled-polar angle" visualization figure ?? is a proto-Lorenz in disguise, with the