Turbulence, shmurbulence how fat is it?

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part 1

1. dynamical theory of turbulence
2. state space
3. symmetry reduction
4. dimension of the inertial manifold

since 1822 have Navier-Stokes equations

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{1}{R}\nabla^2 \mathbf{v} - \nabla p + \mathbf{f}\,, \qquad \nabla \cdot \mathbf{v} = 0,$$

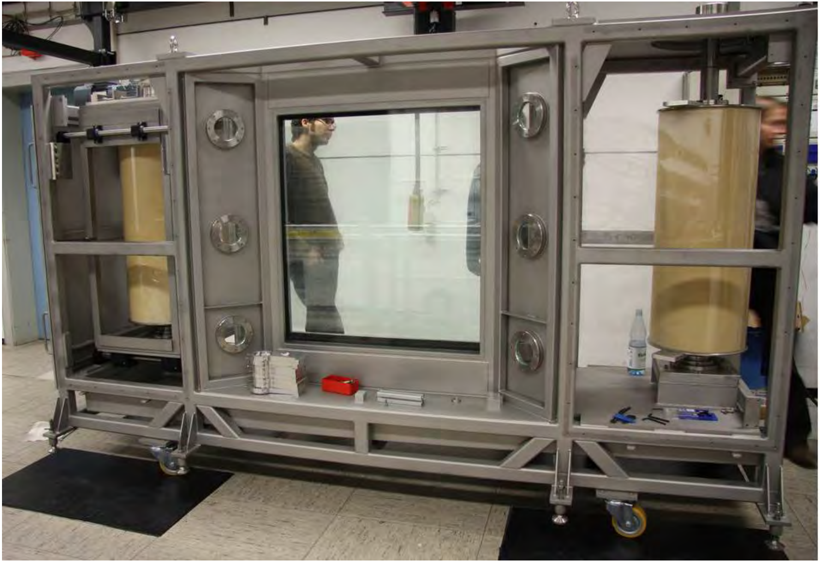
velocity field $\mathbf{v} \in \mathbb{R}^3$; pressure field p ; driving force $\mathbf{f}$

since 1883 Osborne Reynolds experiments

an outstanding problem of classical physics :

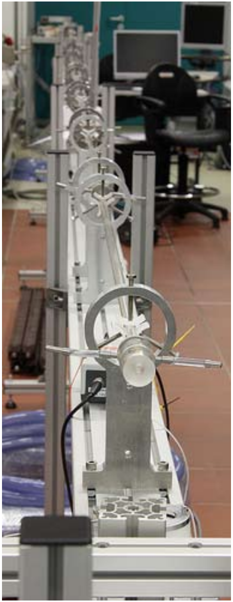
describe turbulence

plane Couette experiment



B. Hof lab

pipe experiment

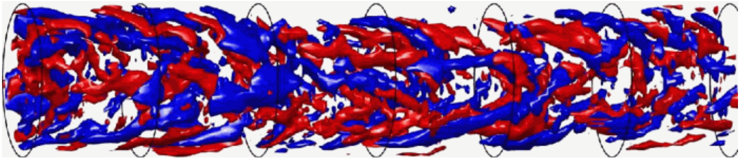


B. Hof lab

pipe experiment data point

a state of turbulent pipe flow at instant in time

Stereoscopic Particle Image Velocimetry → 3-*d* velocity field over the entire pipe[1]



[1] Casimir W.H. van Doorne (PhD thesis, Delft 2004)

numerical challenges

computation of turbulent solutions

requires 3-dimensional volume discretization
$\rightarrow$ integration of 10^4-10^6 coupled ordinary differential equations

challenging, but today possible

J.F. Gibson ChannelFlow.org
A. P. Willis OpenPipeFlow.org

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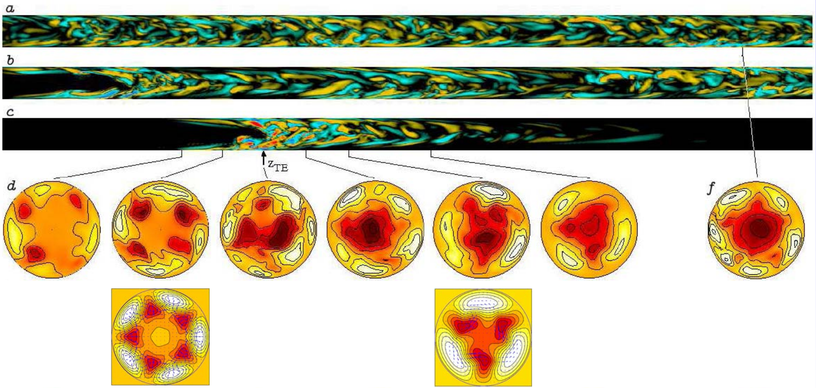
typical simulation

each instant of the flow $>$ Megabytes
a video of the flow $>$ Gigabytes

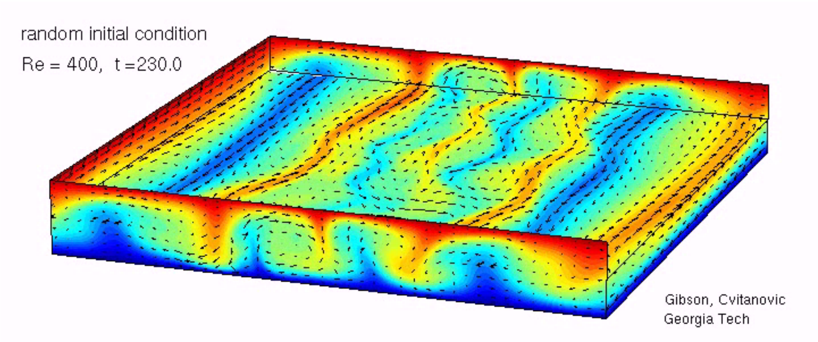
example : pipe flow

amazing data! amazing numerics!

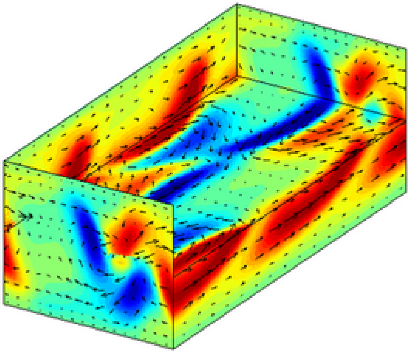


- here each instant of the flow $\approx$ 2.5 MB
- videos of the flow $\approx$ GBs

plane Couette velocity visualization



computational cell, velocity visualization



next - the same solution, different visualization

plane Couette isovorticity visualization

PRL **102,** 114501 (2009) PHYSICAL REVIEW LETTERS week ending 20 MARCH 2009

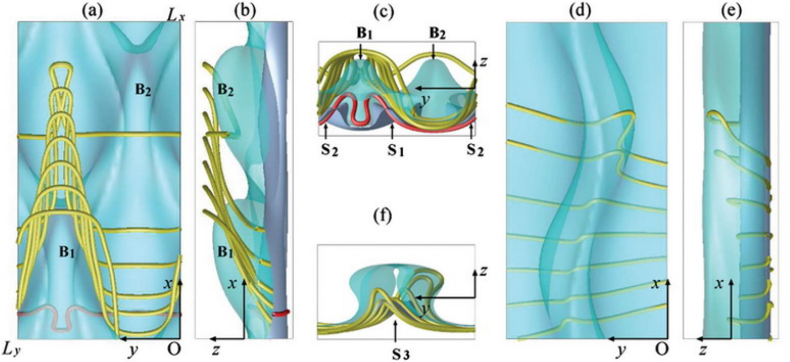


FIG. 3 (color). (a), (b), (c) The x-y, x-z, and y-z projections of the new state (upper branch) at Re = 200 in PCF. Yellow curves are vortex lines across the channel midplane (visualized over B_1, but not over B_2), underneath which there are low-speed structures visualized as isosurfaces of $u_x = -0.1$ and -0.4, colored by cyan [$z \approx \frac{1}{2}$ for the peaks of (B_1, B_2)] and blue [$z \approx -\frac{1}{2}$ for (S_1, S_2)], respectively. (d), (e), (f) Correspond to the same projections but for the upper branch of the NBW state. The vortex lines are integrated from the equivalent points located at $|z| = 0.8$ for both HVS and NBW.

part 2

1. dynamical theory of turbulence
2. **state space**
3. symmetry reduction
4. dimension of the inertial manifold

new : look at it in

state space!

E. Hopf 1948

dynamical description of turbulence

state space

a manifold $\mathcal{M} \in \mathbb{R}^d$: d numbers determine the state of the system

representative point

$x(t) \in \mathcal{M}$
a state of physical system at instant in time

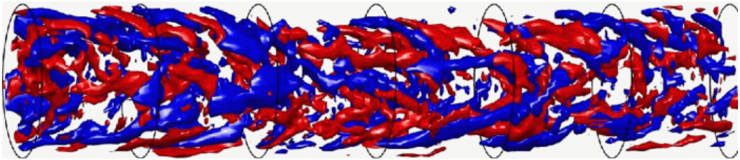
deterministic dynamics

trajectory $x(t) = f^t(x_0)$ = representative point time t later

example of a representative point

$x(t) \in \mathcal{M}$, $d = \infty$
a state of turbulent pipe flow at instant in time

Stereoscopic Particle Image Velocimetry $\rightarrow$ 3-d velocity field over the entire pipe[2]

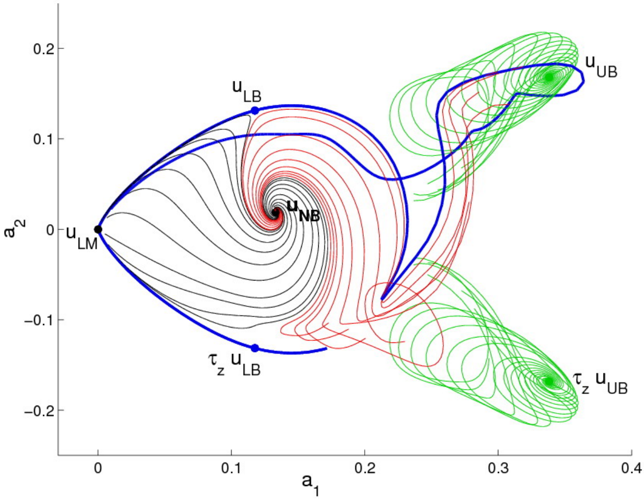


[2] Casimir W.H. van Doorne (PhD thesis, Delft 2004)

charting the state space of a turbulent flow

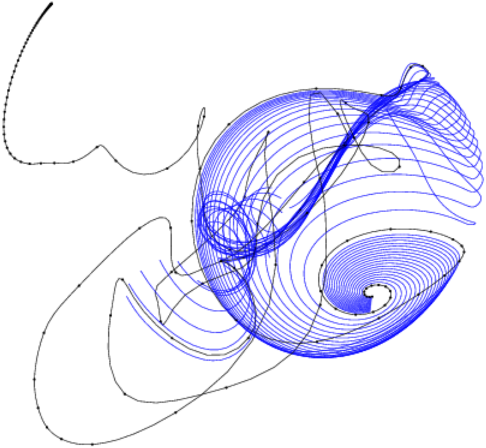
John F Gibson (U New Hampshire)
Jonathan Halcrow (Google)

can visualize 61,506 dimensional state space of turbulent flow

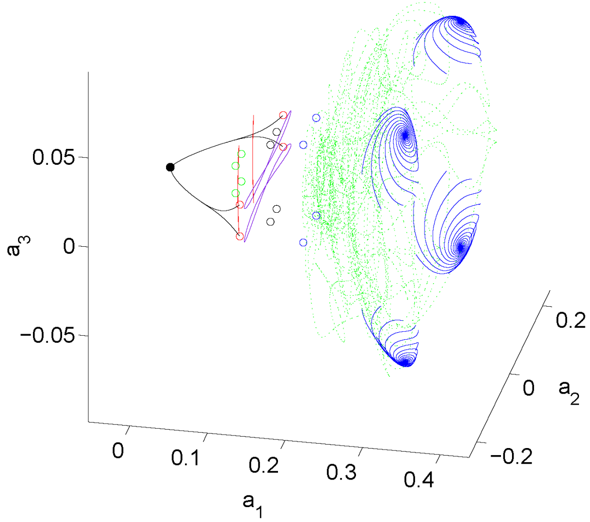


equilibria of turbulent plane Couette flow,
their unstable manifolds, and
myriad of turbulent videos mapped out as one happy family

plane Couette state space, an equilibrium unstable manifold



plane Couette state space $10^5 \to 3D$



nature loves symmetry

or does she?

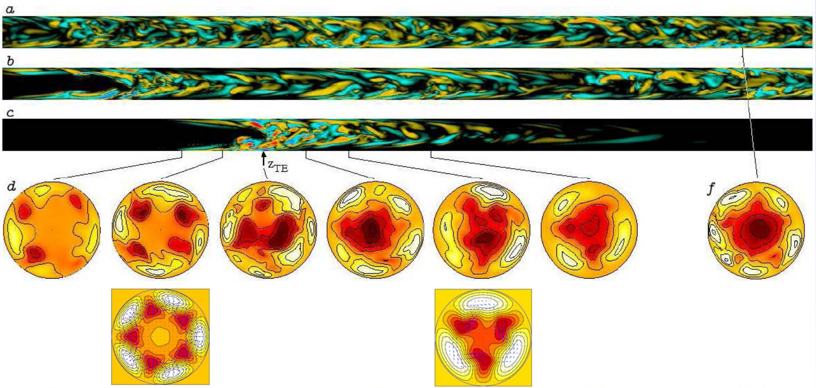
problem

physicists like symmetry more than Nature

Rich Kerswell

nature : turbulence in pipe flows

top : experimental / numerical data
bottom : theorist's solutions



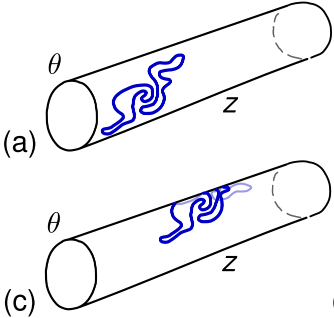
Nature, she don't care : turbulence breaks all symmetries

in turbulence,

use of symmetries is subtle

Elie Cartan 1926 : slice it!

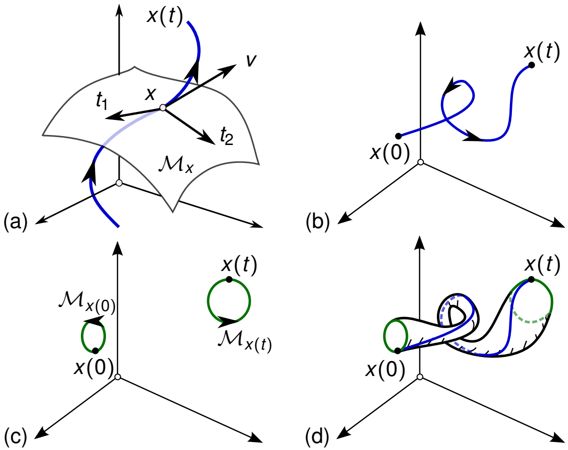
example : $\mathrm{SO}(2)_z \times \mathrm{O}(2)_\theta$ symmetry of pipe flow



a fluid state, shifted by a stream-wise translation, azimuthal rotation g_p is a physically equivalent state

- **b)** stream-wise
- **c)** stream-wise, azimuthal
- **d)** azimuthal flip

state space trajectories, group orbits



(a) x tangent vectors:

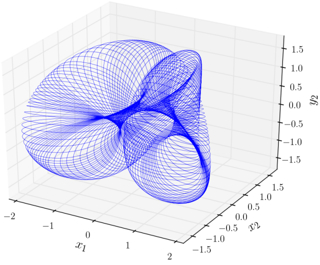
$v(x)$ along time flow $x(t)$
$t_1(x), \cdots, t_N(x)$ group tangents

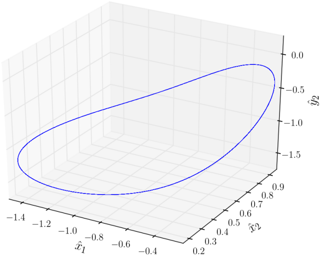
(b) trajectory $x(t)$

(c) group orbits $g\,x(t)$

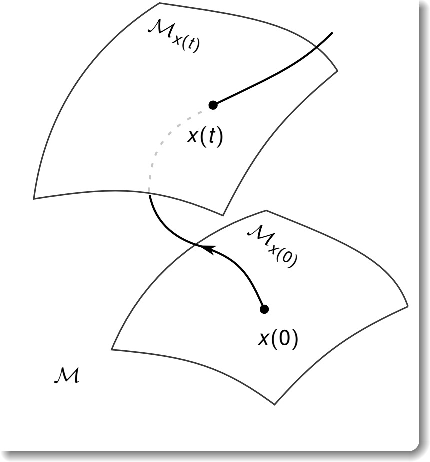
(d) wurst $g\,x(t)$

relative periodic orbit, in full state space and in slice





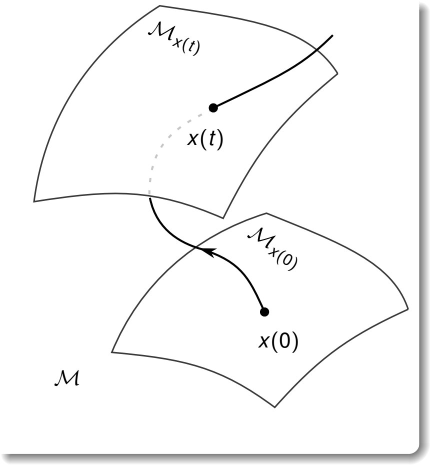
foliation by group orbits



group orbit $\mathcal{M}_x$ of x is the set of all group actions

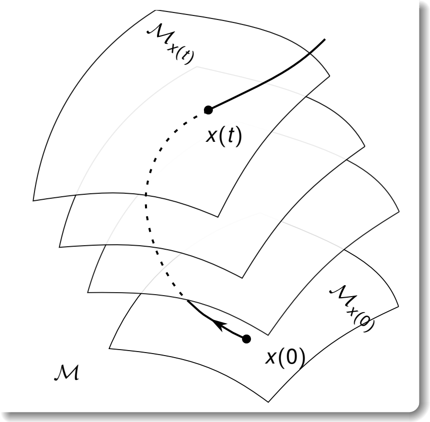
$$\mathcal{M}_x = \{g\, x \mid g \in G\}$$

foliation by group orbits



any point on the manifold $\mathcal{M}_{x(t)}$ is equivalent to any other

foliation by group orbits



actions of a symmetry group foliates the state space $\mathcal{M}$ into a union of group orbits $\mathcal{M}_x$

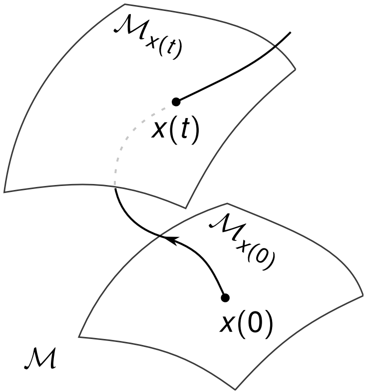
each group orbit $\mathcal{M}_x$ is an equivalence class

the goal

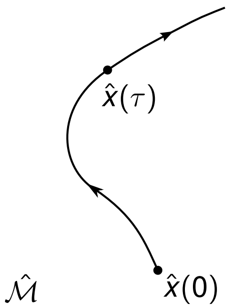
replace each group orbit by a unique point in a lower-dimensional

symmetry reduced state space $\mathcal{M}/G$

full state space



reduced state space



replace each group orbit by a unique point in a lower-dimensional

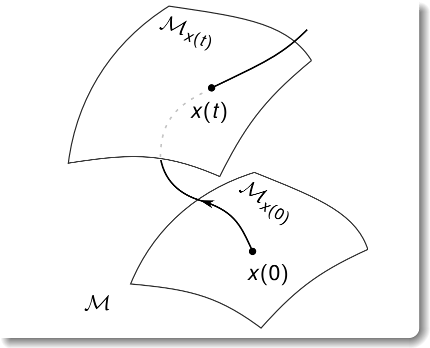
symmetry reduced state space $\mathcal{M}/G$

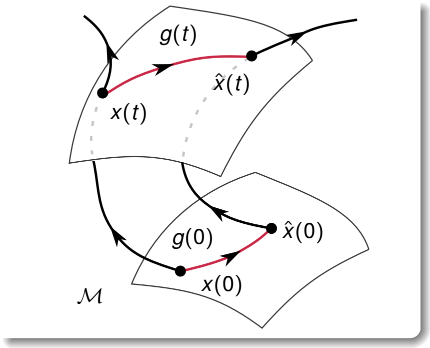
continuous symmetry reduction in high-dimensional flows

with

the method of slices

Cartan's idea : moving frame





free to redefine the flow any time instant
by transformation to
a frame moving along symmetry directions

method of slices

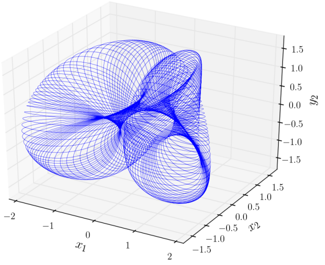
cut group orbits by a hypersurface (not a Poincaré section), each group orbit of symmetry-equivalent points represented by the single point

cut how?

geometers'choice

chose the frames so that distances are minimized

relative periodic orbit, in full state space and in slice



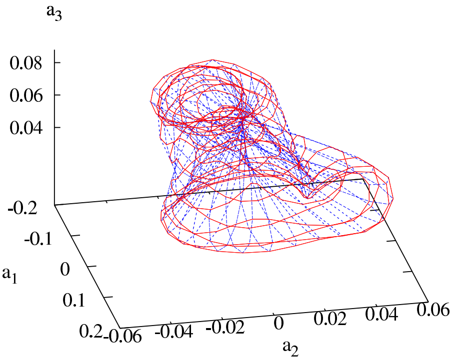
group orbits are NOT circles

nonlinearities couple many Fourier modes

group orbit manifolds of highly nonlinear states are smooth, but not nice

example : group orbit of a pipe flow turbulent state

$\mathrm{SO}(2) \times \mathrm{SO}(2)$ **symmetry**
$\Rightarrow$ **group orbit is topologically 2-torus, but**
a mess in any projection

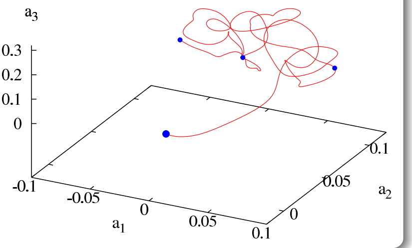


group orbits of highly nonlinear states are topologically tori, but highly contorted tori

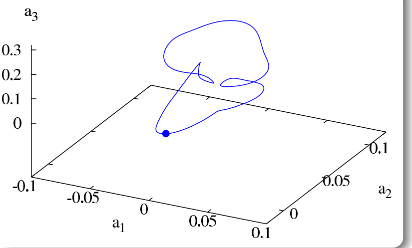
Ashley Willis

example : pipe flow relative periodic orbit

3 repeats, full space



reduced space



Ashley Willis

take home :

if you have a symmetry, reduce it!

your quandry

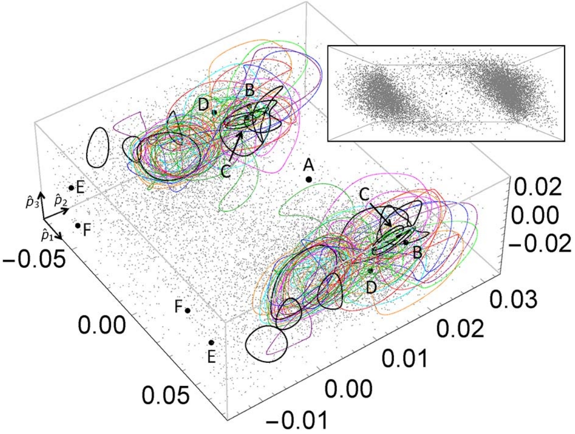
mhm - seems this would require extra thinking
what's the payoff?

it works !

Kimberly Short
Ashley Willis

it works : all pipe flow solutions in one happy family

symmetry-reduced infinite-dimensional slice : a 3D projection



grey cloud : the natural measure

32 relative periodic orbits, 6 relative equilibria

periodic orbits capture the natural measure density well

could not find without symmetry reduction :

1. dynamical theory of turbulence
2. state space
3. symmetry reduction
4. **dimension of the inertial manifold**

turbulence.zip

or ‘equation assisted’ data compression:

replace the ∞ of turbulent videos by the best possible

small finite set

of videos encoding all physically distinct motions of the turbulent fluid

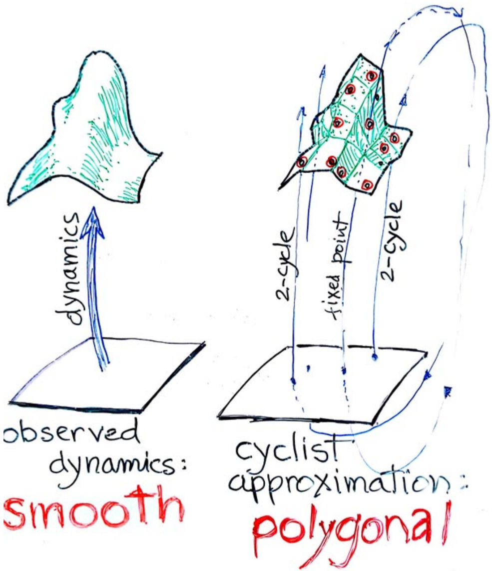
dynamical description of turbulence

dynamical system

the pair $(\mathcal{M}, f)$

the problem

enumerate, classify all solutions of $(\mathcal{M}, f)$



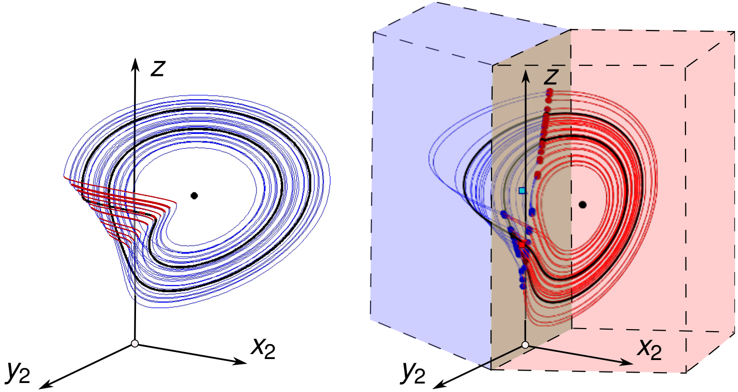
tessellate the state space by recurrent flows

cartography for geometers

cover the reduced manifold with a set of flat charts

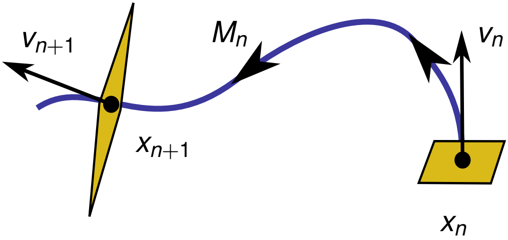
yes, we can do this with 8-dimensional brick embedded in 10^6 dimensions

tiling the inertial manifold



The N-chart atlas of the same strange attractor stays in the physical manifold.

linearized deterministic flow



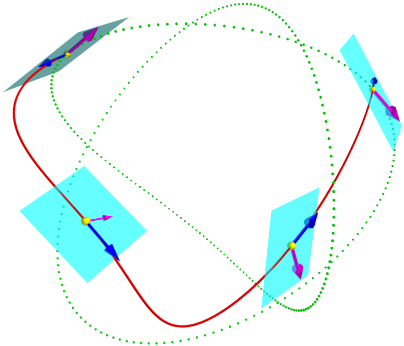
$$x_{n+1} + z_{n+1} = f(x_n) + M_n \, z_n \,, \quad M_{ij} = \partial f_i / \partial x_j$$

in one time step a linearized neighborhood of x_n is

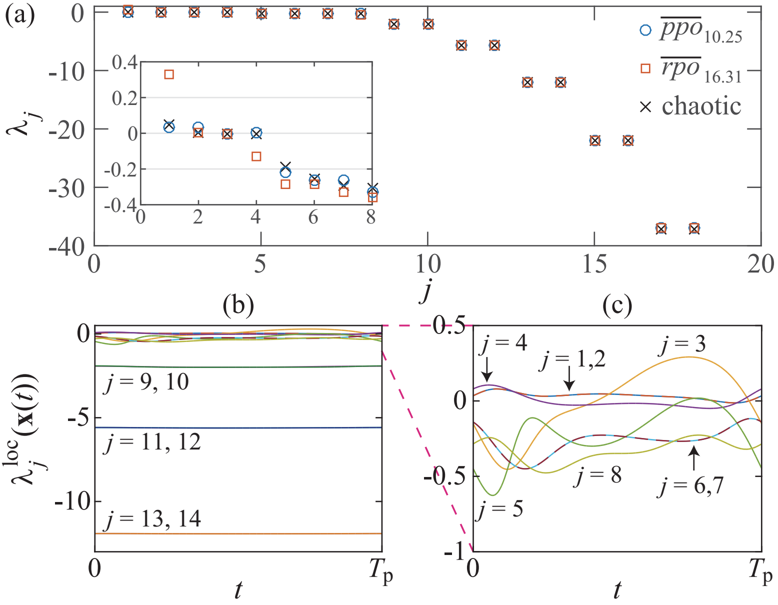
(1) advected by the flow

(2) transported by the Jacobian matrix M_n into a neighborhood given by the M eigenvalues and eigenvectors

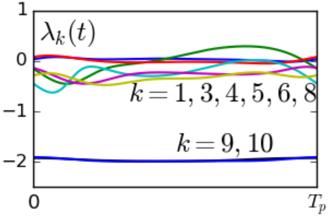
relative periodic orbit with a pair of Floquet vectors



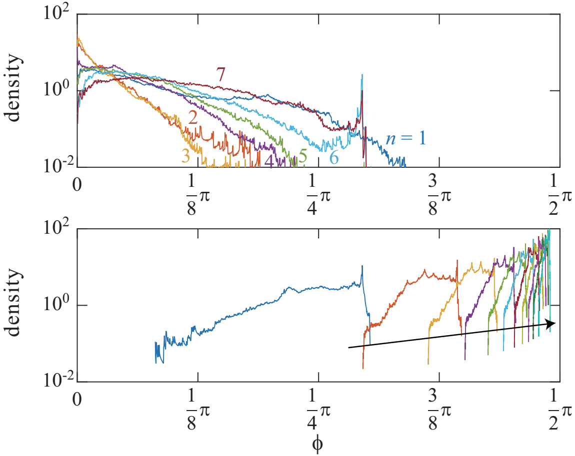
Floquet and Lyapunov exponents



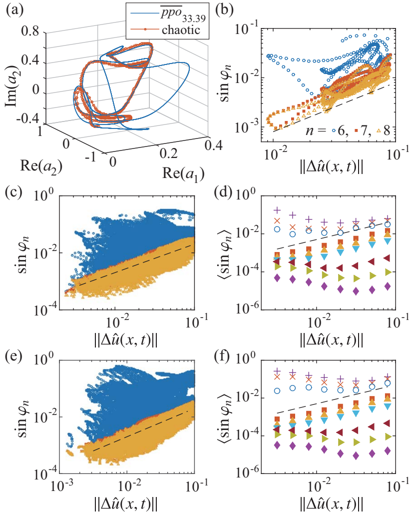
entangled Floquet modes



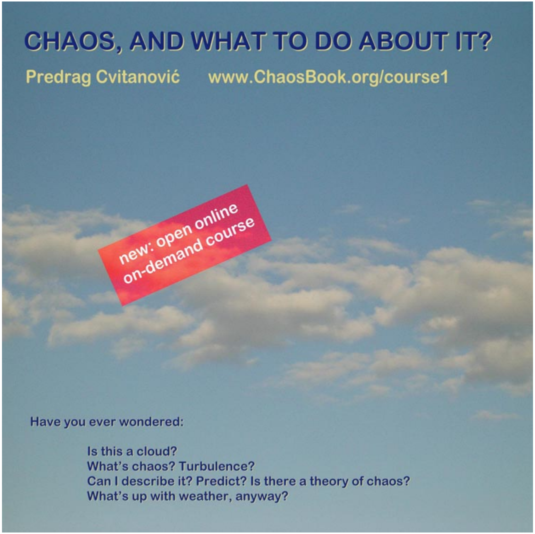
distribution of principal angles between Floquet subspaces



ergodic trajectory shadows periodic orbits within the entangled subspace



what next? take the course!



student raves :

...10^6 times harder than any other online course...