chaotic / turbulent field theory?

herding cats audience reviews:[1]

Arnd Bäcker:

"even faster than Tomaž Prosen"

Martina Hentschel:

"Amazing :)"

Arnd Bäcker:

"... maybe I even understood a single word..."

[1] the limiting speed for transfer of quantum-chaotic information is 1 Prozen.

Amazing! I did not understand a single word.

—Fritz Haake

chaotic field theory

Predrag Cvitanović and Han Liang

ChaosBook.org/overheads/spatiotemporal

"Modern Developments in Quantum Chaos"
Physikzentrum Bad Honnef

November 17, 2021

the goal was (and is)

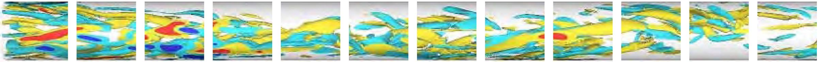
build a chaotic field theory
from simple chaotic blocks

using

- time invariance
- space invariance

a motivation : need a theory of **large** turbulent domains

pipe flow close to onset of turbulence [2]



we have a detailed theory of small turbulent fluid cells

can we can we construct the infinite pipe by coupling small turbulent cells ?

what would that theory look like ?

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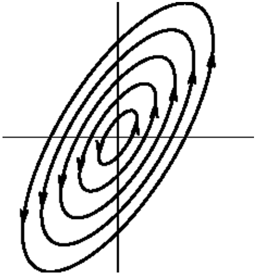
[2] M. Avila and B. Hof, Phys. Rev. **E 87** (2013)

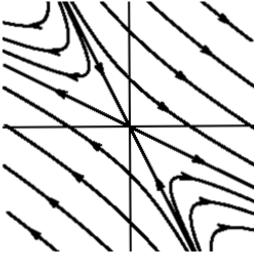
intuition

harmonic field theory



oscillatory eigenmodes

chaotic field theory



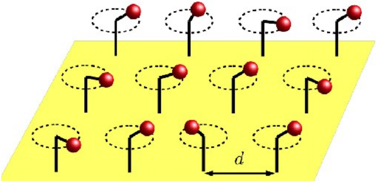
hyperbolic instabilities

take-home from today's talk :

harmonic field theory

tight-binding model
(Helmholtz)

chaotic field theory



Euclidean Klein-Gordon
(damped Poisson)

Mephistopheles knocks at Faust’s door and says, “Du mußt es dreimal sagen!"

. *“You have to say it three times"*

— Johann Wolfgang von Goethe

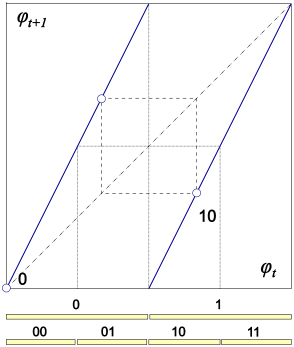
. *Faust I - Studierzimmer 2. Teil*

1. coin toss
2. spatiotemporal Bernoulli
3. chaotic field theory
4. devil is in the details

what we understood that day in 1988 :

time-evolution formulation

Bernoulli map



$$\phi_{t+1} = \begin{cases} 2\phi_t \\ 2\phi_t \pmod 1 \end{cases}$$

$\Rightarrow$ fixed point $\overline{0}$, 2-cycle $\overline{01}$, $\cdots$

a coin toss

the essence of deterministic chaos

symbolic dynamics

map with a 'stretching' parameter $s \geq 2$:

$$x_{t+1} \;=\; s\, x_t$$

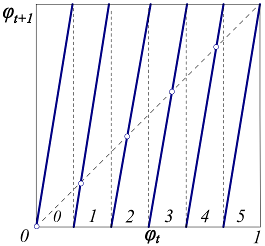
(mod 1) : subtract the integer part $m_{t+1} = \lfloor s x_t \rfloor$
so fractional part ϕ_{t+1} stays in the unit interval $[0, 1)$

$$\phi_{t+1} = s\phi_t - m_{t+1}\,, \qquad \phi_t \in \mathcal{M}_{m_t}$$

m_t takes values in the s-letter alphabet

$$m \in \mathcal{A} = \{0, 1, 2, \cdots, s-1\}$$

slope 6 Bernoulli map



$$\phi_{t+1} = 6\phi_t - m_{t+1}\,,\ \phi_t \in \mathcal{M}_{m_t}$$

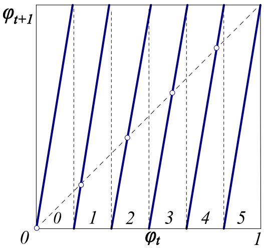
6-letter alphabet
$m_t \in \mathcal{A} = \{0, 1, 2, \cdots, 5\}$

6 subintervals $\{\mathcal{M}_0, \mathcal{M}_1, \cdots, \mathcal{M}_5\}$

what is chaos ?

a fair dice throw

6 subintervals $\{\mathcal{M}_{m_t}\}$, 6^2 subintervals $\{\mathcal{M}_{m_1 m_2}\}, \cdots$



each subinterval contains a periodic point, labeled by $\mathrm{M} = m_1 m_2 \cdots m_n$

$N_n = 6^n - 1$ unstable orbits

definition : chaos is

positive Lyapunov ($\ln s$) - positive entropy ($\frac{1}{n} \ln N_n$)

lattice formulation

lattice Bernoulli

recast the time-evolution Bernoulli map

$$\phi_{t+1} = s\phi_t - m_{t+1}$$

as 1-step difference equation on the
spatiotemporal lattice

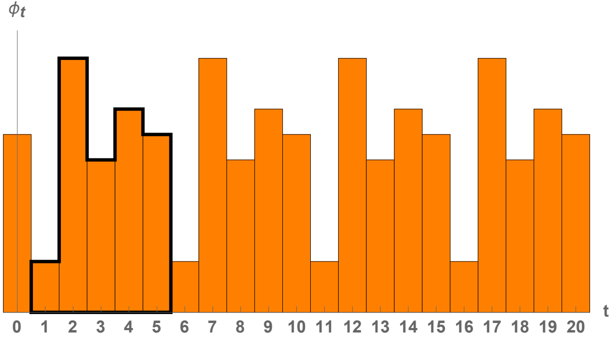
$$\phi_t - s\phi_{t-1} = -m_t\,, \qquad \phi_t \in [0, 1)$$

field ϕ_t, source m_t
on each site t of a 1-dimensional lattice $t \in \mathbb{Z}$

1-dimensional lattice field theory

write a periodic field over n-sites Bravais cell as
the lattice state and the symbol block (sources)

$$\Phi = (\phi_{t+1}, \cdots, \phi_{t+n}), \quad \mathsf{M} = (m_{t+1}, \cdots, m_{t+n})$$



'M' for 'marching orders' : come here, then go there, $\cdots$

think globally, act locally

Bernoulli condition at every lattice site t, local in time

$$\phi_t - s\phi_{t-1} = -m_t$$

is enforced by the global equation

$$\left(1 - s\,r^{-1}\right)\,\Phi = -\mathrm{M}\,,$$

where $[n\times n]$ shift matrix

$$r_{jk} = \delta_{j+1,k}\,, \qquad r = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & & \ddots & \\ & & & 0 & 1 \\ 1 & & & & 0 \end{pmatrix}$$

compares the neighbors

think globally, act locally

solving the lattice Bernoulli system

$$\mathcal{J}\Phi = -\mathsf{M}\,,$$

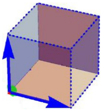
$[n\times n]$ orbit Jacobian matrix $\qquad \mathcal{J} = 1 - s\,r^{-1}\,,$

is a search for zeros of the function

$$F[\Phi] = \mathcal{J}\Phi + \mathsf{M} = 0$$

the entire global lattice state Φ_{M} is now

a single fixed point $(\phi_1, \phi_2, \cdots, \phi_n)$



in the n-dimensional unit hyper-cube $\qquad \Phi \in [0, 1)^n$

orbit stability

orbit Jacobian matrix

solving

$$F[\Phi] = 0 \quad \text{fixed point condition}$$

with Newton method requires evaluation of the $[n \times n]$

orbit Jacobian matrix

$$\mathcal{J}_{ij} = \frac{\delta F[\Phi]_i}{\delta \phi_j}$$

what does this global orbit Jacobian matrix do?

1. global stability of lattice state Φ, perturbed everywhere

orbit stability vs. temporal stability

orbit Jacobian matrix

$\mathcal{J}_{ij} = \frac{\delta F[\Phi]_i}{\delta \phi_j}$ stability under global perturbation of the whole orbit

for n large, a huge $[dn \times dn]$ matrix

temporal Jacobian matrix

J propagates initial perturbation n time steps

small $[d \times d]$ matrix

J and $\mathcal{J}$ are related by[3]

Hill's 1886 remarkable formula

$$|\mathrm{Det}\,\mathcal{J}_\mathrm{M}| = |\det\left(\mathbf{1} - J_\mathrm{M}\right)|$$

$\mathcal{J}$ is huge, even ∞-dimensional matrix
J is tiny, few degrees of freedom matrix

---

[3] G. W. Hill, Acta Math. **8**, 1–36 (1886).

definition : chaos is

expanding	Hill determinants	$\mathrm{Det}\,\mathcal{J}$
exponential $\sharp$	field configurations	N_n

the precise sense in which
a (discretized) field theory is deterministically chaotic

note : there is no 'time' in this definition

what did Fritz not understand that day in 1988?

periodic orbit theory

volume of a periodic orbit neighborhood

Ozorio de Almeida and Hannay[4] 1984 :
$\sharp$ of periodic points is related to their temporal stability by

principle of uniformity

"periodic points of an ergodic system, counted with their natural weighting, are uniformly dense in phase space"

where

'natural weight' of periodic orbit M

$$\frac{1}{|\det(1 - J_{\mathrm{M}})|}$$

[4] A. M. Ozorio de Almeida and J. H. Hannay, J. Phys. A **17**, 3429 (1984).

orbits partition state space into neighborhoods

how come Hill determinant $\mathrm{Det}\,\mathcal{J}$ counts lattice states ?

'principle of uniformity' is in[5]

periodic orbit theory

known as the flow conservation sum rule :

$$\sum_{\mathsf{M}} \frac{1}{|\det(1 - J_{\mathsf{M}})|} = \sum_{\mathsf{M}} \frac{1}{|\mathrm{Det}\,\mathcal{J}_{\mathsf{M}}|} = 1$$

sum over periodic lattice states Φ_{M} of period n

phase space is divided into

neighborhoods of states of period n

[5] P. Cvitanović, "Why cycle?", in *Chaos: Classical and Quantum*, edited by P. Cvitanović et al. (Niels Bohr Inst., Copenhagen, 2020).

Amazing! I did not understand a single word.

—Fritz Haake 1988

zeta function

periodic orbit theory, version (1) : counting lattice states

topological zeta function

$$1/\zeta_{\mathrm{top}}(z) \;=\; \exp\left(-\sum_{n=1}^{\infty}\frac{z^n}{n}N_n\right)$$

1. weight $1/n$ as by (cyclic) translation invariance, n lattice states are equivalent
2. zeta function counts orbits, one per each set of equivalent lattice states

Bernoulli topological zeta function

counts orbits, one per each set of lattice states $N_n = s^n - 1$

$$1/\zeta_{\text{top}}(z) \;=\; \exp\left(-\sum_{n=1}^{\infty}\frac{z^n}{n}N_n\right) \;=\; \frac{1-sz}{1-z}$$

numerator $(1-sz)$ says that Bernoulli orbits are built from s fundamental primitive lattice states,

the fixed points $\{\phi_0, \phi_1, \cdots, \phi_{s-1}\}$

every other lattice state is built from their concatenations and repeats.

solved!

this is 'periodic orbit theory'
And if you don't know, now you know

summary : think globally, act locally

the problem of enumerating and determining all global solutions stripped to its essentials :

1. each solution is a zero of the global fixed point condition

$$F[\Phi] = 0$$

2. global stability : the orbit Jacobian matrix

$$\mathcal{J}_{ij} = \frac{\delta F[\Phi]_i}{\delta \phi_j}$$

3. lattice state neighborhood : the Hill determinant

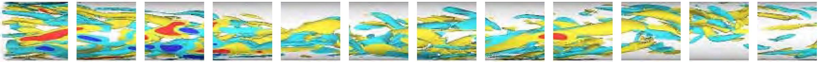
$$1/|\mathrm{Det}\,\mathcal{J}|$$

4. zeta function $1/\zeta_{\mathrm{top}}(z)$: all predictions of the theory

Du mußt es dreimal sagen!
— Mephistopheles

1. coin toss
2. kicked rotor
3. spatiotemporal field theory
4. devil is in the details

building blocks od turbulence



have : a detailed theory of small turbulent cells

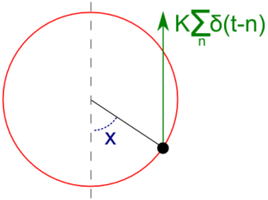
construct : the infinite state by coupling turbulent cells[6]

what would that theory look like ?

[6] M. N. Gudorf et al., *Spatiotemporal tiling of the Kuramoto-Sivashinsky flow*, In preparation, 2021.

example of a “small turbulent cell” : a single kicked rotor

an electron circling an atom, subject to
a discrete time sequence of angle-dependent kicks $F(x_t)$



Taylor, Chirikov and Greene standard map

$$\begin{aligned} x_{t+1} - x_t &= p_{t+1} \qquad \mod 1 \\ p_{t+1} - p_t &= F(x_t) \end{aligned}$$

→ chaos in Hamiltonian systems

(1) what we understood that day in 1988?

time-evolution formulation

the simplest example : a cat map evolving in time

force $F(x) = Kx$ linear in the displacement x , $K \in \mathbb{Z}$

$$\begin{aligned} x_{t+1} &= x_t + p_{t+1} && \mod 1 \\ p_{t+1} &= p_t + Kx_t && \mod 1 \end{aligned}$$

Continuous Automorphism of the Torus, or

time-evolution cat map

a linear, area preserving map of a 2-torus onto itself

$$\begin{bmatrix} \phi_t \\ \phi_{t+1} \end{bmatrix} = J \begin{bmatrix} \phi_{t-1} \\ \phi_t \end{bmatrix} - \begin{bmatrix} 0 \\ m_t \end{bmatrix}, \qquad J = \begin{bmatrix} 0 & 1 \\ -1 & s \end{bmatrix}$$

for integer ‘stretching’ $s = \mathrm{tr}\, J > 2$ the map is
beloved by ergodicists :
hyperbolic $\Rightarrow$ perfect chaotic Hamiltonian dynamical system

a cat is literally Hooke's wild, 'anti-harmonic' sister

for $s < 2$ Hooke rules

local restoring oscillations
around the sleepy z-z-z-zzz resting state

for $s > 2$ cats rule

exponential runaway
wrapped global around a phase space torus

cat is to chaos what harmonic oscillator is to order

there is no more fundamental example of chaos in mechanics

lattice formulation

cat map in lattice formulation

replace momentum by velocity

$$p_{t+1} = (\phi_{t+1} - \phi_t)/\Delta t$$

obtain

$$\left[\begin{array}{c} \phi_t \\ \phi_{t+1} \end{array}\right] = \left[\begin{array}{cc} 0 & 1 \\ -1 & s \end{array}\right] \left[\begin{array}{c} \phi_{t-1} \\ \phi_t \end{array}\right] - \left[\begin{array}{c} 0 \\ m_t \end{array}\right]$$

temporal lattice formulation is pretty[7] :

2-step difference equation

$$\phi_{t+1} - s\,\phi_t + \phi_{t-1} = -m_t$$

integer m_t ensures that

ϕ_t lands in the unit interval

$$m_t \in \mathcal{A}\,, \quad \mathcal{A} = \{\text{finite alphabet}\}$$

[7] I. Percival and F. Vivaldi, Physica D **27**, 373–386 (1987).

think globally, act locally

spatiotemporal cat at every instant t, local in time

$$\phi_{t+1} - s\,\phi_t + \phi_{t-1} = -m_t$$

is enforced by the global equation

$$\mathcal{J}\,\Phi = -\mathsf{M}\,,$$

where

orbit Jacobian matrix

$$\mathcal{J}\,\Phi + \mathsf{M} = 0$$

with

$$\Phi = (\phi_{t+1}, \cdots, \phi_{t+n})\,, \quad \mathsf{M} = (m_{t+1}, \cdots, m_{t+n})$$

a lattice state, and a symbol block

and $[n \times n]$ orbit Jacobian matrix $\mathcal{J}$ is

$$r - s\,\mathbb{1} + r^{-1} = \begin{pmatrix} -s & 1 & & & 1 \\ 1 & -s & 1 & & \\ & 1 & & \ddots & \\ & & & -s & 1 \\ 1 & & & 1 & -s \end{pmatrix}$$

think globally, act locally

solving the spatiotemporal cat equation

$$\mathcal{J}\Phi = -\mathsf{M}\,,$$

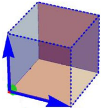
with the $[n\times n]$ matrix $\qquad \mathcal{J} = r - s\,\mathbb{1} + r^{-1}$

can be viewed as a search for zeros of the function

$$F[\Phi] = \mathcal{J}\Phi + \mathsf{M} = 0$$

where the entire global lattice state Φ_M is

a single fixed point $\Phi_\mathsf{M} = (\phi_1, \phi_2, \cdots, \phi_n)$



in the n-dimensional unit hyper-cube $\qquad \Phi \in [0,1)^n$

what continuum theory is spatiotemporal cat discretization of?

have

2-step difference equation

$$\phi_{t+1} - s\,\phi_t + \phi_{t-1} = -m_t$$

discrete lattice

Laplacian in 1 dimension

$$\phi_{t+1} - 2\phi_t + \phi_{t-1} = \Box\,\phi_t$$

so spatiotemporal cat is an (anti)oscillator chain, known as

$d = 1$ Klein-Gordon (or damped Poisson) equation (!)

$$(-\Box + \mu^2)\,\phi_t = m_t\,, \qquad \mu^2 = s - 2$$

did you know that a cat map can be so cool?

what did Fritz not understand that day in 1988?

spatiotemporal cat topological zeta function is the generating function that counts orbits

substituting the Hill determinant count of periodic lattice states

$$N_n = |\mathrm{Det}\,\mathcal{J}|$$

into the topological zeta function

$$1/\zeta_{\mathrm{top}}(z) = \exp\left(-\sum_{n=1} \frac{z^n}{n} N_n\right)$$

leads to the elegant explicit formula[8]

$$1/\zeta_{\mathrm{top}}(z) = \frac{1 - sz + z^2}{1 - 2z + z^2}$$

solved!

[8] S. Isola, Europhys. Lett. **11**, 517–522 (1990).

that's it! for spacetime of 1 dimension

lattice Klein-Gordon equation

$$(-\Box + \mu^2)\,\phi_t = m_t$$

solved completely and analytically!

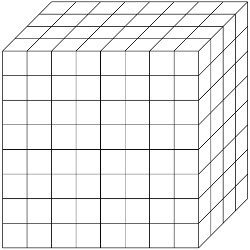
Du mußt es dreimal sagen!
— Mephistopheles

1. coin toss
2. kicked rotor
3. **chaotic field theory**
4. devil is in the details

Euclidean lattice field theory

scalar *field* $\phi(x)$

evaluated on lattice points



$$\phi_z = \phi(x)$$
$$x = az = \text{lattice point}$$
$$z \in \mathbb{Z}^d / L^d$$

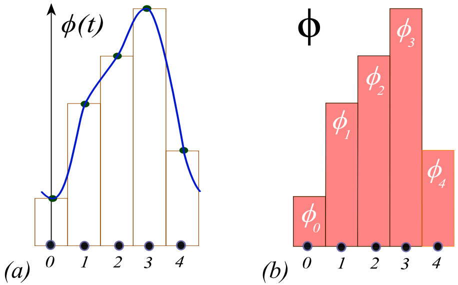
a periodic point per each unit cell

9

[9] G. Münster and M. Walzl, *Lattice gauge theory - A short primer*, 2000.

Discretization of a $1d$ field theory

scalar *field* $\phi(x)$ evaluated on lattice points



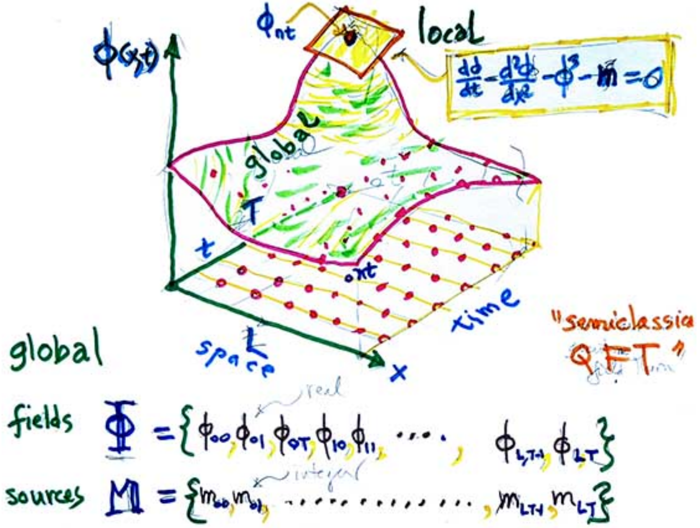
periodic field $\phi(t)$ is a function of continuous coordinate t

corresponding discretized period-5 lattice state $\Phi = \overline{\phi_0\phi_1\phi_2\phi_3\phi_4}$,

Horizontal: t coordinate, lattice sites marked by dots, labelled by $t \in \mathbb{Z}$

the value of the discretized field ϕ_t is plotted as a bar centred at lattice site t

think globally, act locally



for each symbol array M, a periodic lattice state Φ_M

example : Euclidean ϕ^4 theory

lattice action

$$
\begin{aligned}
S &= \sum_x a^d \left\{ \frac{1}{2} \sum_{i=1}^{d} (\partial_\mu \phi(x))^2 + \frac{\mu^2}{2} \phi(x)^2 + \frac{g}{4!} \phi(x)^4 \right\} \\
&= \sum_{z,z'} \frac{1}{2} \left\{ \phi_z \left(-\Box + \mu^2 \right)_{zz'} \phi_{z'} \right\} + \sum_z \frac{g}{4!} \phi_z^4 \, .
\end{aligned}
$$

examples : 1d lattice field theories

spatiotemporal lattice field theory

$$\phi_{t+1} - S'[\phi_t] + \phi_{t-1} \quad = \quad -m_t$$

spatiotemporal Bernoulli

$$\phi_{t+1} - s\,\phi_t \qquad\qquad = \quad -m_t$$

spatiotemporal cat

$$\phi_{t+1} - s\,\phi_t + \phi_{t-1} \quad = \quad -m_t$$

spatiotemporal Hénon

$$\phi_{t+1} - a\,\phi_t^2 + \phi_{t-1} \quad = \quad -m_t$$

spatiotemporal ϕ^4 theory

$$\phi_{t+1} - \frac{g}{3!}\phi_t^3 + \phi_{t-1} \quad = \quad -m_t$$

herding cats in d spacetime dimensions : ‘spatiotemporal cat’

spatiotemporal cat

consider a 1 spatial dimension lattice, with field ϕ_{nt}
(the angle of a kicked rotor "particle" at instant t, at site n)

require

- each site couples to its nearest neighbors $\phi_{n\pm1,t}$
- invariance under spatial translations
- invariance under spatial reflections
- invariance under the space-time exchange

Gutkin & Osipov[10] obtain[11,12]

2-dimensional coupled cat map lattice

$$\phi_{n,t+1} + \phi_{n,t-1} - 2s\,\phi_{nt} + \phi_{n+1,t} + \phi_{n-1,t} = -m_{nt}$$

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[10] B. Gutkin and V. Osipov, Nonlinearity **29**, 325–356 (2016).

[11] B. Gutkin et al., Nonlinearity **34**, 2800–2836 (2021).

[12] P. Cvitanović and H. Liang, *Spatiotemporal cat: a chaotic field theory*, In preparation, 2021.

herding cats : a discrete Euclidean space-time field theory

write the spatial-temporal differences as discrete derivatives

Laplacian in $d = 2$ dimensions

$$\Box\,\phi_{nt} \;=\; \phi_{n,t+1} + \phi_{n,t-1} - 4\,\phi_{nt} + \phi_{n+1,t} + \phi_{n-1,t}$$

subtract 2-dimensional coupled cat map lattice equation

$$-m_{nt} = \phi_{n,t+1} + \phi_{n,t-1} - 2s\,\phi_{nt} + \phi_{n+1,t} + \phi_{n-1,t}$$

cat herd is thus governed by the law of

d-dimensional spatiotemporal cat

$$(-\Box + \mu^2)\,\phi_z = m_z\,, \qquad \mu^2 = d(s-2)$$

where $\phi_z \in [0,1)\,, \quad m_z \in \mathcal{A}$ and $z \in \mathbb{Z}^d$ = integer lattice

discretized linear PDE

d-dimensional spatiotemporal cat

$$(-\Box + \mu^2)\,\phi_z = m_z$$

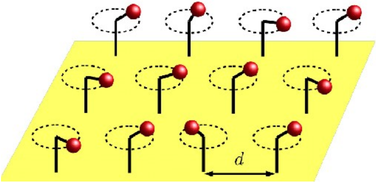
is linear and known as

- tight-binding model or Helmholtz equation
  if stretching is weak, $s < 2$
  [oscillatory sine, cosine solutions]
- Euclidean Klein-Gordon or (damped Poisson)
  if stretching is strong, $s > 2$
  [hyperbolic sinches, coshes, 'mass' $\mu^2 = d(s-2)$]

nonlinearity is hidden in the 'sources'

$m_z \in \mathcal{A}$ at lattice site $z \in \mathbb{Z}^d$

spring mattress vs field of rotors

traditional field theory	chaotic field theory
	
Helmholtz	damped Poisson

our song of chaos has been sang – what next ?

1. coin toss
2. kicked rotor
3. spatiotemporal cat
4. devil is in the details

2.15 Integer lattices literature

There are many reasons why one needs to compute an "orbit Jacobian matrix" Hill determinant $|\mathrm{Det}\,\mathcal{J}|$, in fields ranging from number theory to engineering, and many methods to accomplish that:

discretizations of Helmholtz [58] and screened Poisson [59, 80, 96, 97] (also known as Klein–Gordon or Yukawa) equations

Green's functions on integer lattices [5, 8, 24, 33, 37, 40, 63, 67, 78, 92, 93, 115–117, 135, 140, 143, 149, 150, 159, 180, 196]

Gaussian model [71, 111, 139, 172]

linearized Hartree-Fock equation on finite lattices [121]

quasilattices [29, 69]

circulant tensor systems [33, 37, 146, 164, 166, 200]

Ising model [19, 88, 89, 98, 100, 103–105, 128, 136, 141, 153, 161, 199], transfer matrices [154, 199]

lattice field theory [108, 144, 148, 151, 168, 175, 176, 192]

modular transformations [34, 205]

lattice string theory [77, 157]

random walks, resistor networks [9, 25, 49, 50, 60, 81, 86, 99, 122, 163, 183, 188, 198]

spatiotemporal stability in coupled map lattices [4, 75, 203]

Van Vleck determinant, Laplace operator spectrum, semiclassical Gaussian path integrals [47, 125, 126, 187]

Hill determinant [26, 47, 137]; discrete Hill's formula and the Hill discriminant [186]

Lindstedt-Poincaré technique [189–191]

heat kernel [38, 61, 64, 110, 114, 143, 159, 201]

lattice points enumeration [15, 16, 20, 56]

primitive parallelogram [10, 30, 152, 193]

difference equations [55, 68, 181]

digital signal processing [62, 130, 197]

generating functions, Z-transforms [64, 194]

integer-point transform [20]

graph Laplacians [41, 79, 134, 162]

graph zeta functions [7, 13, 18, 27, 42–44, 57, 61, 83, 87, 94, 101, 123, 124, 162, 165, 169, 171, 179, 184, 185, 204]

zeta functions for multi-dimensional shifts [12, 132, 133, 147]

zeta functions on discrete tori [38, 39, 201] symmetry reduction

spatiotemporal cat : a strong coupling field theory

spatiotemporal cat symmetries :

translations $\circ$ time-reversal $\circ$ spatial reflections

point-group of the square lattice:
rotations by $\pi/2$
reflections across axes and diagonals,

$$\mathrm{D}_4 = \{1, r, r^2, r^3, \sigma, \sigma_1, \sigma_2, \sigma_3\}\,.$$

international crystallographic notation[13],
the square lattice space group $p4mm$.

not a traditional
spatially weakly coupled lattice model[14]

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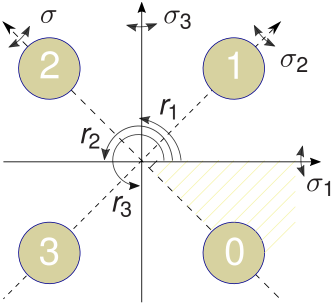
[13] M. S. Dresselhaus et al., *Group Theory: Application to the Physics of Condensed Matter*, (Springer, New York, 2007).

[14] L. A. Bunimovich and Y. G. Sinai, Nonlinearity **1**, 491 (1988).

symmetries of a square lattice unit cell



4 rotations r_j, 4 shift-reflections σ_k of dihedral group

$$\mathrm{D}_4 = \{1, r, r^2, r^3, \sigma, \sigma_1, \sigma_2, \sigma_3\}$$

overly the square onto itself.
They also tile it with the 8 copies $\hat{\mathcal{M}}_\ell$ of the fundamental domain (the shaded wedge)

retreat to : 1d lattice field theories

spatiotemporal cat

$$\phi_{t+1} - s\,\phi_t + \phi_{t-1} \;=\; -m_t$$

spatiotemporal Hénon

$$\phi_{t+1} - a\,\phi_t^2 + \phi_{t-1} \;=\; -m_t$$

spatiotemporal ϕ^4 theory

$$\phi_{t+1} - \frac{g}{3!}\phi_t^3 + \phi_{t-1} \;=\; -m_t$$

orbit Jacobian (Hill, Hessian, ...) matrix

each lattice state has its own

$$\mathcal{J}[\Phi] = \begin{pmatrix} -s_0 & 1 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & -s_1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -s_2 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -s_{n-2} & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 1 & -s_{n-1} \end{pmatrix},$$

stretching factor $s_t = S''[\phi_t]$ is
function of the site field ϕ_t for the given lattice state Φ

1. can compute Hill determinant $\mathrm{Det}\,\mathcal{J}$
2. Hill-Lindstedt-Poincaré :
   all calculations should be done on reciprocal lattice
3. toolbox : discrete Fourier transforms, irreps of D_n

Symmetries of 1-dimensional lattices

There are only two 1-dimensional space groups G:
$p1$ *infinite cyclic group* C_∞ of all lattice translations,

$$\mathrm{C}_\infty = \{\cdots, r_{-2}, r_{-1}, 1, r_1, r_2, r_3, \cdots\}$$

$p1m$, the *infinite dihedral group* D_∞ of all translations and reflections[15],

$$\mathrm{D}_\infty = \{\cdots, r_{-2}, \sigma_{-2}, r_{-1}, \sigma_{-1}, 1, \sigma, r_1, \sigma_1, r_2, \sigma_2, \cdots\}$$

group multiplication $g_i g_j$

	r_j	σ_j
r_i	r_{i+j}	σ_{j-i}
σ_i	σ_{i+j}	r_{j-i}

either adds up translations,
or shifts and then reverses their direction

[15] J. L. Y.-O. Kim and K. K. Park, Pacific J. Math. **209**, 289–301 (2003).

Symmetries of 1-dimensional Bravais sublattices

Bravais cell of *period* n : given by vector $\mathbf{a}$ of length n
Bravais sublattice generated by translations

$$r_j \to r_{j\mathbf{a}}$$

symmetry : translation subgroup of C_∞

$$H_{\mathbf{a}} = \{\cdots, r_{-2\mathbf{a}}, r_{-\mathbf{a}}, 1, r_{\mathbf{a}}, r_{2\mathbf{a}}, \cdots\},$$

and

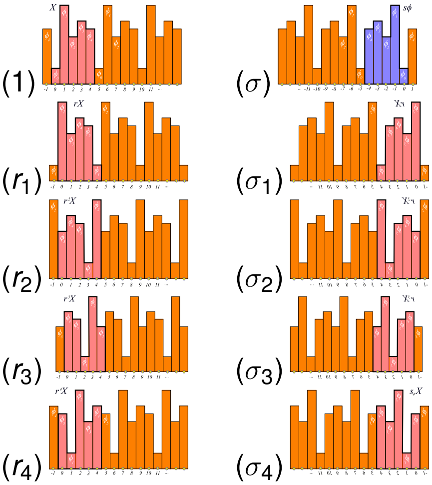
$$r_j \to r_{j\mathbf{a}}, \qquad \sigma \to \sigma_k \quad 0 \leq k < n,$$

symmetry : n infinite dihedral subgroups of D_∞

$$H_{\mathbf{a},k} = \{\cdots, r_{-2\mathbf{a}}, \sigma_k r_{-2\mathbf{a}}, r_{-\mathbf{a}}, \sigma_k r_{-\mathbf{a}}, 1, \sigma_k, r_{\mathbf{a}}, \sigma_k r_{\mathbf{a}}, r_{2\mathbf{a}}, \sigma_k r_{2\mathbf{a}}, \cdots\},$$

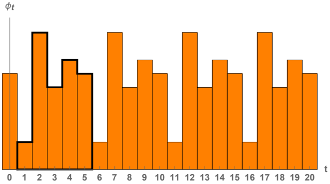
Bravais cell of period n,
with reflection across a symmetry point shifted k 1/2 steps

D_∞ orbit of a generic lattice state

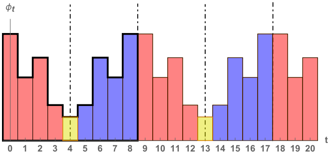


lattice state $\Phi = \overline{\phi_0\phi_1\phi_2\phi_3\phi_4}$, no reflection symmetry
translation group H_5 invariant
D_∞-orbit is isomorphic to D_5 : 10 distinct lattice states

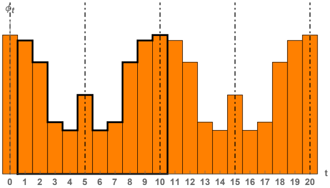
4 kinds of Bravais lattice states



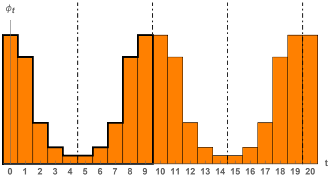
(n)



(o)



(ee)

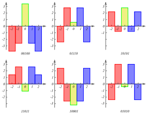


(eo)

(n) *no reflection symmetry:* H_5 invariant period-5 lattice state
(o) *odd period, symmetric:* an $H_{9,8}$ invariant period-9
(ee) *even period, even symmetric:* $H_{10,0}$ invariant period-10
(eo) *even period, odd symmetric:* $H_{10,9}$ invariant period-10

example : 5-period Bravais lattice site, with reflection

scalar *field* $\phi(x)$



temporal Hénon period-5 lattice state $\phi_{-2}\phi_{-1}\phi_0\,\phi_1\phi_2$

$$\phi_i = \phi_{i+5}\,, \quad \phi_{-i} = \phi_i\,.$$

reflection symmetric; fixed lattice field ϕ_0 colored gold

$$\begin{aligned} -S'[\phi_0] + 2\phi_1 &= -m_0 \\ \phi_0 - S'[\phi_1] + \phi_2 &= -m_1 \\ \phi_1 - S'[\phi_2] + \phi_2 &= -m_2 \end{aligned}$$

with a 3-dimensional orbit Jacobian matrix

$$\mathcal{J} = \begin{pmatrix} -s_0 & 2 & 0 \\ 1 & -s_1 & 1 \\ 0 & 1 & -s_2 + 1 \end{pmatrix}$$

what neither Fritz nor Predrag understood that day in 1988

periodic orbit theory, version (1) : counting lattice states[16]

Lind zeta function

$$\zeta_{Lind}(t) = \exp\left(\sum_{H} \frac{N_H}{|G/H|} t^{|G/H|}\right)$$

sum is over all subgroups H of space group G
N_H is the number of fixed points of H
$|G/H|$ is the number of states in H orbit

1. Lind zeta function counts group orbits, one per each set of equivalent lattice states

[16] D. A. Lind, “A zeta function for Z^d-actions”, in *Ergodic Theory of Z^d Actions*, edited by M. Pollicott and K. Schmidt (Cambridge Univ. Press, 1996), pp. 433–450.

what neither Fritz nor Predrag understood that day in 1988

periodic orbit theory, version (1) :
counting lattice states for reflection-symmetric systems[17,18]

Kim-Lee-Park zeta function

$$\zeta_\sigma(t) = \sqrt{\zeta_{top}(t^2)}\, e^{h(t)},$$

where ζ_{top} is the Artin-Mazur zeta function, and the counts of the 3 kinds of symmetric orbits are

$$h(t) = \sum_{m=1}^{\infty} \left\{ N_{2m-1,0}\, t^{2m-1} + \left(N_{2m,0} + N_{2m,1}\right) \frac{t^{2m}}{2} \right\}$$

[17] M. Artin and B. Mazur, Ann. Math. **81**, 82–99 (1965).

[18] J. L. Y.-O. Kim and K. K. Park, Pacific J. Math. **209**, 289–301 (2003).

the goal was (and still is)

build
a chaotic field theory

from
the simplest chaotic blocks

using

- time invariance
- space invariance

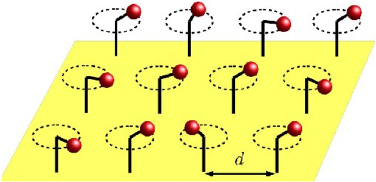
of the defining partial differential equations

take-home :

harmonic field theory

tight-binding model

chaotic field theory



herding cats

what Eugene and Predrag still do not understand today

1. solved so far only 1-dimensional spatiotemporal lattice, point group D_1 (!)
2. all time-reversal symmetric systems should be analyzed this way
3. all dynamical systems should be solved on reciprocal lattice (?!)
4. for 2-dimensional spatiotemporal chaotic field theory, still have to do this for square lattice point group D_4
5. then, solve the problem of turbulence (Navier-Stokes, Yang-Mills, general relativity)

Bye Fritz Haake. You were so brave.