LINEAR SYMBOLIC DYNAMICS FOR COUPLED CAT MAPS

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1. Motivation and Goals

Standard symbolic dynamics for cat maps of arbitrary dimension $2N$ uses Markov partition of the phase space. To this end the phase space of the system is partitioned in a number of domains according to stable and unstable directions. Each of these domains is then labeled by a symbol from an alphabet $\mathcal{A}$. Accordingly, points in the phase space can be uniquely labeled by 1D infinite sequences of symbols from $\mathcal{A}$ in accordance to their backward and forward histories. The details of this construction can be found e.g., in Arnold&Avez, ChaosBook and everywhere. This symbolic dynamics has a plenty of nice properties (e.g., finite grammar rules), but one crucial drawback - the size of the alphabet $\mathcal{A}$ grows exponentially with N.

In [1] we introduced symbolic dynamics for N coupled cat model which is free of the last catch. Rather then using 1D linear sequences, the N-particle trajectories are labeled there by $2D$ arrays of symbols from finite alphabet whose size is independent of N. A variant of this construction is described below.

1.1. 2D symbolic dynamics.

The equation of motion for coupled cat map reads in the Newtonian form as

$$x_{n,t-1} + x_{n,t+1} + x_{n+1,t} + x_{n-1,t} = sx_{n,t} + m_{n,t}, \tag{1.1}$$

or equivalently:

$$(\Delta - s + 4)x_{n,t} = m_{n,t}, \qquad (n,t) \in \mathbb{Z}^2 \tag{1.2}$$

with Δ being discrete Laplacian. Here $x_{n,t}$ is a coordinate of the particle and s is an integer larger then 4 which defines the map. The integers $m_{n,t}$ take the values from the set $\mathcal{A} = [3, 2, \cdots - s + 1]$. The construction of symbolic dynamics uses integer winding numbers $m_{n,t}$. Each solution of an infinite system of coupled cat map equations is uniquely (since $\mathcal{D} = \Delta - s + 4$ is invertible) labeled by an infinite array of symbols $\{m_{n,t} | (n,t) \in \mathbb{Z}^2\}$. In particular, N-particle periodic orbits of period T, or more precisely - $N \times T$ tori, correspond to 2D periodic sequences satisfying $m_{n,t} = m_{n+N,t}$, $m_{n,t} = m_{n,t+T}$.

1.2. Periodic orbits.

As explained above, a N-particle periodic orbit of period T (we will refer to it as $\mathrm{PO}(N,T)$) is uniquely encoded by $N \times T$ array of symbols $M = \{m_{n,t} | (n,t) \in \mathbb{Z}^2_{\mathrm{NT}}\}$, where $\mathbb{Z}^2_{\mathrm{NT}} = \{n = 1, \dots N; t = 1, \dots T\}$. Furthermore its coordinates $x_{n,t}$ can be easily restored from M by applying inverse of the linear operator $\mathcal{D}_{\rm NT} := \mathcal{D}|_{\mathbb{Z}^2_{\rm NT}}$ with periodic boundary conditions:

$$X = \mathcal{D}_{\rm NT}^{-1} M, \tag{1.3}$$

where $X = \{x_{n,t} | (n,t) \in \mathbb{Z}^2_{\rm NT}\}$ stands for the array of the particle coordinates between times 1 and T.

Note, that even without any knowledge of grammar rules, it is easy to generate random $\mathrm{PO}(N,T)$ just by taking an arbitrary array of symbols M_0 and then applying modulus one operation:

$$X_{\text{ rand}} = \mod (\mathcal{D}_{\rm NT}^{-1} M_0)\ 1. \tag{1.4}$$

The associated symbolic representation is then recovered by inverse operation $M_{\rm rand} = \mathcal{D}_{\rm NT} X_{\rm rand}$.

Remark 1.1. *To be on the safe side we should check the above statements numerically. The construction in* [1] *was somewhat different. It used winding numbers for both coordinate and momentum of the particle.*

1.3. **Main problems.** The central question which we are going to address here is about probabilities of having prescribed pattern of symbols.

Question 1.2. *Let $M_{r\times p}$ be an array of $r \times p$ integers from the set $\mathcal{A}$. How often this pattern will occur in the symbolic representation of generic trajectory of time T in N-particle coupled cat map with $T \gg r$, $N \gg p$? To put it differently, we are interested in the measures $\mu([M_{r\times p}])$ of the cylinder sets $[M_{r\times p}]$, where only symbols from $r \times p$ rectangular domain are fixed, while the rest can be arbitrary.*

Second, presumably easier question is:

Question 1.3. *Let $X = \{x_{n,t} | (n,t) \in \mathbb{Z}^2\}$ be trajectory (periodic or not) and let $M = \{m_{n,t} | (n,t) \in \mathbb{Z}^2\}$ be its symbolic representation. Let $M_{\mathcal{R}}$ be a rectangular sub-array of $r \times p$ symbols from M in the rectangular (or more general) domain $\mathcal{R} = \{n = 1, \dots r; t = 1, \dots p\}$. To what extent $M_{\mathcal{R}}$ defines position of the points inside $\mathcal{R}$?*

2. Single cat map

Before turning to many-particle case it is instructive to study similar symbolic dynamics just for a single cat map. In this case the equation of motion is:

$$x_{t-1} + x_{t+1} = s x_t + m_t, \tag{2.1}$$

where integers m_t take the values from the set $\mathcal{A} = [1, 0, \cdots - s + 1]$. Note that for T-periodic orbits this equation can be also cast into matrix form:

$$D x_t = m \tag{2.2}$$

with $x = (x_1, x_2, \dots x_T)$, $m = (m_1, m_2, \dots m_T)$ and D_T is tridiagonal matrix of the size $T \times T$:

$$D_T = \begin{pmatrix} -s & 1 & 0 & 0 & \dots & 0 & 1 \\ 1 & -s & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -s & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & \dots & \dots & \dots & 1 & -s \end{pmatrix}.$$

Let $m_1 m_2 \dots m_p$ be a sub-sequence of symbols of length p. We will assume that each $m_i, i = 1, \dots, p$ is fixed to some prescribed value $a_i \in \mathcal{A}$. By inverting eq. 2.2 we have

$$x_i = \sum_{j=1}^{p} (\mathcal{D}_p^{-1})_{i,j} a_j - (\mathcal{D}_p^{-1})_{i,1} x_0 - (\mathcal{D}_p^{-1})_{i,p} x_{p+1}, \qquad i = 1, \dots p, \tag{2.3}$$

where $\mathcal{D}_p$ is $p \times p$ matrix with the same structure as D_T but without lower and upper corners (i.e., $\mathcal{D}$ corresponds to Dirichlet rather then periodic boundary conditions). The matrix elements of $\mathcal{D}^{-1}$ can be expressed through the Chebyshev Polynomials $u_n(s/2) = \frac{\sinh(n+1)\lambda}{\sinh \lambda}$, $s = 2\cosh\lambda$:

$$(\mathcal{D}_p^{-1})_{i,j} = \begin{cases} \frac{u_{i-1}(s/2) u_{p-j}(s/2)}{u_p(s/2)}, & \text{for } i \le j \\ \frac{u_{j-1}(s/2) u_{p-i}(s/2)}{u_p(s/2)}, & \text{for } i > j. \end{cases} \tag{2.4}$$

Defining the "average position" of the ith point as

$$\bar{x}_i(a) = \sum_{j=1}^{p} (\mathcal{D}_p^{-1})_{i,j} a_j, \qquad a = a_1 \dots a_p, \tag{2.5}$$

one immediately gets the error

$$|x_i - \bar{x}_i(a)| = \left| \frac{u_{p-i}(s/2)}{u_p(s/2)} x_0 + \frac{u_{i-1}(s/2)}{u_p(s/2)} x_{p+1} \right| \le \frac{\cosh(\frac{1}{2}(p+1) - i)\lambda}{\cosh \frac{1}{2}(p+1)\lambda}. \tag{2.6}$$

As expected, the minimal error $\sim e^{-\lambda/2}$ is at the center of sequence, where $i = (p+1)/2$ (for odd p).

Eq. (2.6) basically answers question (1.3) in the context of single cat map. We are turning now to a more difficult one (1.2). Let $a = a_1 \dots a_p$ be a fixed sequence of symbols. With what frequency $p(a)$ we will find a as a subsequence of symbolic representation of a generic trajectory? To address this problem we will consider periodic orbits of a large period T. The basic assumption is that $p(a)$ is given by the ratio:

$$p(a) = \lim_{T \to \infty} \frac{\mathcal{N}(a, T)}{\mathcal{N}(T)}, \tag{2.7}$$

where $\mathcal{N}(a, T)$ is the number of periodic orbits with the period T such that $m_i = a_i$, $i = 1, \dots p$ are fixed and $\mathcal{N}(T)$ is the total number of periodic orbits with the same period. Since $\mathcal{N}(T) = \det D_T$, which can be easily calculated, the problem boils down to finding $\mathcal{N}(a, T)$.

Consider all variables x_i, such that i is outside of the set $1, \dots p$. By eq. 2.2 and eq. 2.3 the system of the corresponding equations can be written as:

$$x_{t-1} + x_{t+1} = sx_t + m_t, \qquad t \notin \{0, \dots, p+1\}$$

$$x_{-1} + \frac{1}{u_p(s/2)} \sum_{j=1}^{p} u_{p-j}(s/2) a_j = sx_0 - \frac{u_{p-1}(s/2)}{u_p(s/2)} x_0 - \frac{1}{u_p(s/2)} x_{p+1} + m_0$$

(2.8)

$$x_{p+2} + \frac{1}{u_p(s/2)} \sum_{j=1}^{p} u_{j-1}(s/2) a_j = sx_{p+1} - \frac{u_{p-1}(s/2)}{u_p(s/2)} x_{p+1} - \frac{1}{u_p(s/2)} x_0 + m_{p+1}.$$

which can be also cast into the matrix form:

$$D'_{T-p}(x_0, x_{p+1}, x_{p+2}, \dots, x_{T-1})^\intercal = (m_0 + A, m_{p+1} + A', m_{p+2}, \dots, m_{T-1})^\intercal \tag{2.9}$$

where A, A' are constant numbers depending on sequence a and D'_{T-p} is $T - p \times T - p$ matrix:

$$D'_T = \begin{pmatrix} -s + \frac{u_{p-1}(s/2)}{u_p(s/2)} & \frac{1}{u_p(s/2)} & 0 & 0 & \dots & 0 & 1 \\ \frac{1}{u_p(s/2)} & -s + \frac{u_{p-1}(s/2)}{u_p(s/2)} & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -s & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & \dots & \dots & \dots & 1 & -s \end{pmatrix}.$$

The number of solutions for this system of equations satisfying additional constrain $0 \leq x_i < 1, i = 1, \dots, p$ defines $\mathcal{N}(a, T)$. This number is just proportional to the volume $\mathrm{Vol}(a)$ of the Polytope

$$0 \leq x_i < 1, \qquad i \notin \{1, \dots, p\} \tag{2.10}$$

$$0 \leq -\bar{x}_i(a) + \frac{u_{p-i}(s/2)}{u_p(s/2)} x_0 + \frac{u_{i-1}(s/2)}{u_p(s/2)} x_{p+1} < 1, \qquad i = 1, \dots, p. \tag{2.11}$$

From this follows that $\mathrm{Vol}(a)$ is just defined by the area of the two dimensional domain defined by $0 \leq x_0, x_{p+1} < 1$ and the second condition (2.11). Combining all together we have:

$$p(a) = \lim_{T \to \infty} \frac{\mathrm{Vol}(a) \det D'_{T-p}}{\det D_T}. \tag{2.12}$$

The ratio between two determinants turns out to be independent on T and can be easily evaluated:

$$d_0 = \frac{\det D'_3}{\det D_{3+p}} = \frac{1}{4 \sinh^2 \frac{\lambda(p+3)}{2}} \det \begin{pmatrix} -s + \frac{u_{p-1}(s/2)}{u_p(s/2)} & \frac{1}{u_p(s/2)} & 1 \\ \frac{1}{u_p(s/2)} & -s + \frac{u_{p-1}(s/2)}{u_p(s/2)} & 1 \\ 1 & 1 & -s \end{pmatrix}. \tag{2.13}$$

In the case $s = 3$, one gets $1/d_0 = 8$ for $p = 2$, $1/d_0 = 21$ for $p = 3$, $1/d_0 = 55$ for $p = 4$, $1/d_0 = 144$ for $p = 5$ etc. As a result we have:

$$p(a) = d_0 \mathrm{Vol}(a). \tag{2.14}$$

Note that for sequences of symbols a which contain no "extreme" symbols $-s+1, 1$, $\mathrm{Vol}(a) = 1$ and we have $p(a) = d_0$ in this case.

Remark 2.1. *What has been will be again, what has been done will be done again; there is nothing new under the sun.-Ecclesiastes 1:9. After writing this part I "discovered that this symbolic dynamics was already introduced by Percival and Vivaldi for a single cat in* [2]*, see also* [3]*. It is called linear coding and was used in* [4] *to treat diffusion problem. So I changed the title accordingly. To a large extent the material in this section is related (overlaps?) to one from these papers. On the other hand (almost surely) nothing was done on the 2D front - the main goal of the project.*

2.1. Example: Frequency of single symbol.

Question 2.2. *Consider generic trajectory generated by iteration of cat map on some initial data. Let $M = \dots m_1 m_2 m_3 \dots$ be its symbolic representation. What is the probability that at fixed time $t = k$ symbol m_k takes specific value $a \in A$? Put it differently – What is the frequency of occurrence of a in M?*

We will answer this question by considering the ratio between number of periodic orbits (PO(T)) of period T:

$$p(a) = \frac{\#\text{ of PO(T) with } m_k = a}{\text{Total }\#\text{ of PO(T)}}.$$

Let us first evaluate the total number of PO with period T. To this end we need to find periodic solutions of eq. 2.1 i.e., $x_{t+T} = x_t$. Since this equation is linear the number of such solutions is just given by the number of integer lattice points inside of the Parallelotope obtained from unite Hypercube by action of D in the T -dimension Euclidean space. In other words the number of periodic orbits is given by its volume i.e., $\det D$.

$$\{\text{Total }\#\text{ of PO(T) }\} = \det D.$$

The number of periodic orbits with fixed symbol at k'th place can be evaluated by excluding x_k out of the system of equations. So we have

$$\begin{aligned} x_{t-1} + x_{t+1} &= s x_t + m_t, \quad |t-k| > 1 \\ x_{k-2} + \frac{1}{s} x_{k+1} &= \left(s - \frac{1}{s}\right) x_{k-1} + m_{k-1} + \frac{a}{s} \\ x_{k+2} + \frac{1}{s} x_{k-1} &= \left(s - \frac{1}{s}\right) x_{k+1} + m_{k+1} + \frac{a}{s}. \end{aligned} \tag{2.15}$$

In matrix form this system of equations can be written as:

$$D' x_t = m + \frac{a}{s}(0, \dots 1, 1, 0, \dots)^{\intercal}, \tag{2.16}$$

where D' is now the tridiagonal matrix of the dimension $T-1 \times T-1$. The number of solutions with constrain $m_k = a$ is therefore

$$\{\ \#\text{ of PO with } m_k = a\} = \det D' \quad \mathrm{Vol}(a),$$

where $\mathrm{Vol}(a)$ is the volume of restricted hypercube

$$0 \le x_t < 1, \text{ for all } t \neq k\ , \quad \& \quad 0 \le x_{k-1} + x_{k+1} - a < s. \tag{2.17}$$

Note that the second constraint is fulfilled automatically for $a \neq -s+1, 1$. As a result

$$\mathrm{Vol}(a) = 1/2 \text{ for } a = -s+1, 1 \text{ and } \mathrm{Vol}(a) = 1 \text{ otherwise.}$$

For symbol frequency we obtain:

$$p(a) = 1/2s \text{ for } a = \{-s+1, 1\}, \quad p(a) = 1/s \text{ otherwise.} \tag{2.18}$$

2.2. **Example: Frequency of symbol pairs.** We are now going to extend this result for pairs of symbols.

Question 2.3. *Given generic trajectory with symbolic representation $M = \dots m_1 m_2 m_3 \dots$. With what frequency a pair of symbols $a_1 a_2$, $a_1, a_2 \in A$ occurs in M?*

For the sake of simplicity of presentation we fix here $s = 3$. As before we need to evaluate the number of PO(T) with two fixed consecutive symbols e.g., m_2, m_3. The number of solutions with this constrain is

$$\{\ \#\text{ of PO with } m_2 = a, m_3 = b\} = \det D'' \quad \mathrm{Vol}(ab),$$

where D'' is $T-2 \times T-2$ matrix. $\mathrm{Vol}(ab)$ is the volume of the hypercube

$$0 \le x_t < 1, \text{ for all } t \neq 2, 3 \tag{2.19}$$

restricted by the conditions (coming from $0 \le x_2, x_3 < 1$):

$$\begin{aligned} 0 &\le \frac{1}{8}(3x_1 + x_4 - 3a - b) < 1, \\ 0 &\le \frac{1}{8}(3x_4 + x_1 - 3b - a) < 1. \end{aligned} \tag{2.20}$$

This gives:

$$\mathrm{Vol}(ab) = 1, \text{for all combinations of } 0, -1$$

$$\mathrm{Vol}(ab) = 1/6, \text{for } ab = \{-1-2, -2-1, 01, 10\}$$

$$\mathrm{Vol}(ab) = 1/2, \text{for } ab = \{0-2, -20, -11, 1-1\}$$

$$\mathrm{Vol}(ab) = 2/3, \text{for } ab = \{1-2, -21\}$$

Sequences 11, $-2-2$ are forbidden.

Remark 2.4. *Since $\sum_{a,b} p(ab) = 1$ we do not need even to calculate $\det D''$. But the connection between spectral and volume counting problems is interesting and might be useful for larger sequences of symbols.*

2.3. **Symbol frequency - rigorous derivation.** There is a more straightforward, simple and (most important) rigorous derivation of the results of the previous sections. Note that fixing two coordinates x_0, x_{p+1} defines the whole trjectory. So we need to evaluate the phase space volume $\mathrm{Vol}(a)$ corresponding to the domain:

$$
\begin{aligned}
&0 \le -\bar{x}_i(a) + \frac{u_{p-i}(s/2)}{u_p(s/2)} x_0 + \frac{u_{i-1}(s/2)}{u_p(s/2)} x_{p+1} < 1, \qquad i = 1, \dots, p, \\
&0 \le x_0 < 1, \qquad 0 \le x_{p+1} < 1,
\end{aligned} \tag{2.21}
$$

in the x_0, x_{p+1} plane. To this end we pass from x_0, x_{p+1} to momentum-coordinate x_k, p_k plane at some moment of time k between 0 and $p+1$:

$$
\begin{aligned}
x_k &= -\bar{x}_k(a) + \frac{u_{p-k}(\frac{s}{2})}{u_p(\frac{s}{2})} x_0 + \frac{u_{k-1}(\frac{s}{2})}{u_p(\frac{s}{2})} x_{p+1}, \\
p_k &= -\bar{x}_{k+1}(a) + \bar{x}_k(a) + \frac{u_{p-k-1}(\frac{s}{2}) - u_{p-k}(\frac{s}{2})}{u_p(\frac{s}{2})} x_0 + \frac{u_k(\frac{s}{2}) - u_{k-1}(\frac{s}{2})}{u_p(\frac{s}{2})} x_{p+1}.
\end{aligned} \tag{2.22}
$$

Since this is a linear map, we only need to evaluate the Jakobian:

$$
d_0 = \frac{1}{u_p^2(s/2)} \det \begin{pmatrix} u_{p-k-1}(\frac{s}{2}) - u_{p-k}(\frac{s}{2}) & u_{p-k}(\frac{s}{2}) \\ u_k(\frac{s}{2}) - u_{k-1}(\frac{s}{2}) & u_{k-1}(\frac{s}{2}) \end{pmatrix} = \frac{\sinh \lambda}{\sinh (p+1)\lambda}. \tag{2.23}
$$

Numerical check shows that it is equivalent to (2.13).

2.4. **Cat (map) with the soul of baker (map).** The linear symbolic dynamics of a cat map posseses the following amazing property. Let T be the cat map with parameter s. After setting absorbtion on the parts of the phase space corresponding to the external symbols $-s, 1$ the resulting map T' (with absorbtion) acts on the Cantor set defined by the sequences of remaining symbols. It turns out that any sequence of symbols $[m_1 m_2 \dots m_p]$, where $m_i \notin \{-s, 1\}$ is admissible and the Lebeague measures of the corresponding cylinder sets are eual. *To be continued ...*

3. Coupled cat maps

As a first step let us show that any trajectory i.e., solution of eq. (1.1) can be uniquely restored from its symbolic representation. By inverting eq. (1.1) we obtain

$$
x_{n,t} = \sum_{(n',t') \in \mathbb{Z}^2} g_{n-n',t-t'} m_{n',t'}, \tag{3.1}
$$

where $g_{n-n',t-t'}$ is the Green function solving the equation:

$$
(\Delta - s + 4) g_{n,t} = \delta_{n,0} \delta_{t,0}, \qquad (n,t) \in \mathbb{Z}^2. \tag{3.2}
$$

The solution is given by the double integral:

$$
g_{n,t} = \frac{1}{\pi^2} \int_0^\pi \int_0^\pi \frac{\cos(nx)\cos(ty)}{-s + 2\cos x + 2\cos y} dx dy. \tag{3.3}
$$

In particular, the diagonal elements can be calculated explicitly [5] (should be re-checked):

$$g_{n,n} = \frac{-1}{4\pi i}\left(2iQ_{n-1/2}(z) + \pi P_{n-1/2}(z)\right), \quad z = s^2/8 - 1 > 1.$$

Remark 3.1. *1) For single cat map the corresponding Green's function is given by:*

$$g_n = \frac{1}{\pi}\int_0^{\pi} \frac{\cos(nx)}{-s + 2\cos x}dx = \lambda^{-|n|}/(\lambda^{-1} - \lambda),$$

with $s = \lambda + \lambda^{-1}$, $\lambda > 1$. *Combining this with* $x_n = \sum_{n'\in\mathbb{Z}} g_{n-n'}m_{n'}$ *leads to the grammar rules of* [2]. *2) For periodic orbits of period* T *the connection between symbolic representation and coordinates is given by finite sum* $x_n = \sum_{n'=1}^{T} \tilde{g}_{n-n'}m_{n'}$, *where* $\tilde{g}_n$ *is the Greens function for periodic boundary conditions. It it is easy to find it by using* g_n*:*

$$\tilde{g}_n = \sum_{j=-\infty}^{\infty} g_{n-jT} = \frac{\lambda^{-|n|} + \lambda^{-|n-T|}}{(1-\lambda^{-T})(\lambda^{-1} - \lambda)}.$$

The condition for admissible code of a periodic orbit is:

$$0 \le \sum_{n=1}^{T} \tilde{g}_n m_{n-k} < 1, \qquad \textit{for all } k = 1, \dots T.$$

As an example, for 2-periodic trajectories $s = 3$, *we have:*

$$-5 < 3a_1 + 2a_2 \le 0, \qquad -5 < 3a_2 + 2a_1 \le 0,$$

which admits 5 *solutions:* $(-2,-1), (-1,-2), (-1,0), (0,-1), (0,0)$.

In general any code which does not contain extreme symbols $-s+1, 1$ *is admissible. Indeed if* $-s+2 \le m_n \le 0$*:*

$$0 \le \sum_{n=-\infty}^{\infty} m_n\lambda^{-|n|}/(\lambda^{-1} - \lambda) \le \sum_{n=-\infty}^{\infty} (\lambda^{-1} + \lambda - 2)\lambda^{-|n|}/(\lambda^{-1} - \lambda) = 1.$$

3) As in the single cat case, for large n, t *from 3.3 an exponential decay* $g_{n,t} \sim \lambda^{-\sqrt{n^2+t^2}}$ *is expected to follow (but still should be demonstrated explicitly).*

The grammar rules are fixed by the conditions $0 \le x_{n,t} < 1$. In other words the allowed symbolic representations $\{m_{n,t}\}$ of valid trajectories satisfy the following condition:

$$0 \le \sum_{(n',t')\in\mathbb{Z}^2} g_{n-n',t-t'}m_{n',t'} < 1, \qquad (n,t) \in \mathbb{Z}^2. \tag{3.4}$$

Remark 3.2. *It would be interesting to get a more explicit information on these grammar rules. Are they finite? Put it differently, whether listing of forbidden sequences for finite set of rectangles determines selection rules for infinite sequences? Probably not in the direct sense. But it might be possible that forbidden sequences which are not prohibited by finite grammar rules are very "rare".*

3.1. **Determining particle positions.** Here we provide an answer for the second question. As we show below, an array of symbols $\{m_{n,t} \in \mathcal{R}\}$ from some (connected) domain $\mathcal{R} \in \mathbb{Z}^2$ determines positions x_{n_0,t_0} of the particles at $O = (n_0, t_0) \in \mathcal{R}$ up to an exponentially small corrections which depend on the distance between O and the boundary of $\mathcal{R}$. To this end we will derive a discrete analog of Green's theorem. To simplify the exposition we fix $O = (0, 0)$. As a first step we multiply eq. (2.1) by $g_{n,t}$ and eq. (3.2) by $x_{n,t}$. After summation over $(n, t) \in \mathcal{R}$ and subtracting two sums from each other we obtain:

$$x_{0,0} - \sum_{(n,t)\in\mathcal{R}} g_{n,t} m_{n,t} = \sum_{(i,j)\in\partial\mathcal{R}} \partial_N x_{i,t} g_{i,t} - \partial_N g_{i,t} x_{i,t}, \tag{3.5}$$

where the right hand sum is over the boundary $\partial\mathcal{R}$ and ∂_N stands for normal derivative with respect to it. The normal derivative is defined as $\partial_N b_{n,t} = b_{n\pm1,t} - b_{n,t}$ along the vertical pieces of $\partial\mathcal{R}$ (the sign depends whether $n+1 \in \mathcal{R}$, or $n-1 \in \mathcal{R}$) and $\partial_N b_{n,t} = b_{n,t\pm1} - b_{n,t}$ along the horizontal pieces of $\partial\mathcal{R}$. (Some adjustments of the above definition are needed for the corner points). Taking in account that $g_{n,t}$ is exponentially decaying function one has the following bound:

$$|x_{0,0} - \sum_{(n,t)\in\mathcal{R}} g_{n,t} m_{n,t}| \leq C\, |\partial\mathcal{R}| e^{-L(O,\partial\mathcal{R})}, \tag{3.6}$$

with $|\partial\mathcal{R}|$ being the "length" of the boundary i.e, number of points belonging to $\partial\mathcal{R}$ and $L(O, \partial\mathcal{R})$ being the minimal distance between O and $\partial\mathcal{R}$.

3.2. **Symbol arrays probabilities.** For many particle setting the problemof finding frequency for a given array of symbols can be solved in the same spirit as for single cat map.

3.2.1. *Single symbol frequency.* We need to evaluate the volume of the hypercube

$$0 \leq x_{n,t} < 1, \text{ for all } (n,t) \neq (k,k) \tag{3.7}$$

restricted by the conditions (coming from $0 \leq x_{k,k} < 1$):

$$a \leq x_{k,k+1} + x_{k,k-1} + x_{k-1,k} + x_{k+1,k} < s + a. \tag{3.8}$$

This volume depends on a. For $a \in (-s + 3, \dots 0]$ the volume is just 1. Other "boundary" values $a \in (1, 2, 3) \cup (-s+3, -s+2, -s+1)$ appear with less frequency: 1/4!, 1/2 and 23/4!.

3.2.2. 2×2 *square of symbols.* Here $\{q_1, q_2, q_3, q_4\}$, $\{x_1, \dots x_8\}$ are internal and boundary coordinates respectively, with $\{m_1, m_2, m_3, m_4\}$ being internal symbols. The connection between them is given by:

$$q_1 = \frac{1}{s(s^2-4)}((x_1 + x_8)(s^2 - 2) + (x_3 + x_2 + x_6 + x_7)s + (x_4 + x_5)2 - \\ - m_1(s^2 - 2) - (m_2 + m_3)s - 2m_4) \tag{3.9}$$

$$q_2 = \frac{1}{s(s^2-4)}((x_2+x_3)(s^2-2)+(x_1+x_8+x_4+x_5)s+(x_7+x_6)2- \\ -m_2(s^2-2)-(m_1+m_4)s-2m_3) \tag{3.10}$$

$$q_3 = \frac{1}{s(s^2-4)}((x_7+x_6)(s^2-2)+(x_1+x_8+x_4+x_5)s+(x_2+x_3)2- \\ -m_3(s^2-2)-(m_1+m_4)s-2m_2) \tag{3.11}$$

$$q_4 = \frac{1}{s(s^2-4)}((x_4+x_5)(s^2-2)+(x_3+x_2+x_6+x_7)s+(x_1+x_8)2- \\ -m_4(s^2-2)-(m_2+m_3)s-2m_1) \tag{3.12}$$

To find relative frequencies we need to estimate volume of the eight dimensional polytope in $x_1, \dots x_8$ Euclidean space which is determined by the following inequalities:

$$0 \le x_i < 1, \qquad i = 1, \dots 8, \tag{3.13}$$

$$0 \le q_i < 1, \qquad i = 1, \dots 4. \tag{3.14}$$

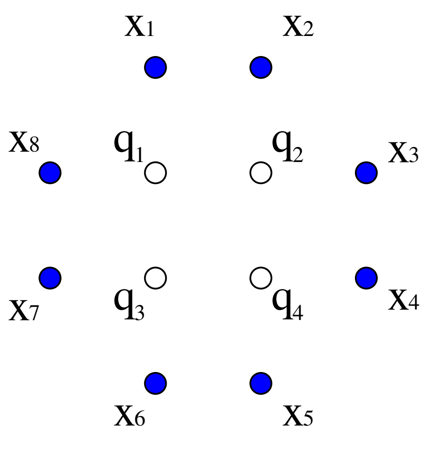

FIGURE 1. 2×2 symbolic array. Set of the internal and boundary coordinates.

It is easy to check that if a symbolic array is composed of internal symbols only, then inequalities (3.14) are satisfied automatically. Indeed, if all symbols satisfy $-s+4 \le m_i \le 0$, $i = 1, \dots 4$ then by eqs. (3.9-3.12) we have

$$0 \le q_i \le \frac{(s-4)(s^2+2s)+2s^2+4s}{s(s^2-4)} = 1, \qquad i = 1, \dots 4$$

This implies that all 2×2 arrays composed of internal symbols appear with one and the same frequency.

In addition, it seems to be straightforward to establish forbidden symbolic arrays. It requires finding combinations of m_i's which provide empty sets. For instance it is easy to see that arrays composed of the most extreme symbols $-s+1$, $-s+2$ or $2,3$ are forbidden. Indeed, if $m_i \in \{-s+1, -s+2\}$, $i = 1, \dots 4$ by eqs. (3.9-3.12) we have

$$q_i \geq \frac{(s-2)(s^2+2s)}{s(s^2-4)} = 1, \qquad i = 1, \dots 4$$

which contradicts (3.14).

3.2.3. *General rectangular symbolic arrays.* For a general array of symbols $\{a_k | k \in \mathcal{R}\}$ we follow the same procedure. Given a domain $\mathcal{R}$ on $\mathbb{Z}^2$ lattice (where array of symbols is defined) one needs to express the internal coordinates $q_i \in \mathcal{R}$ through the boundary ones $x_i \in \partial\mathcal{R}$. Explicitely this connection is given by:

$$q_i = -\sum_{j\in\partial\mathcal{R}} \left(\Delta_D^{-1}\right)_{i,j} x_j + \sum_{k\in\mathcal{R}} \left(\Delta_D^{-1}\right)_{i,k} a_k, \tag{3.15}$$

where Δ_D is discrete Laplacian with Dirichlet (Neumann ?) boundary conditions. *To be continued ...*

Remark 3.3. It is possible to reduce problem for general domain $\mathcal{R}$ to the volume counting of a "cut" hypercube. The dimension of such hypercube is defined by the length of the boundary of $\mathcal{R}$. It is clear from this that the occurrence frequency of any pattern is given by some rational number.

Remark 3.4. Wrong remark (left as historic evidence)! Locality of symbolic dynamics. If $\mathcal{R}_1$ and $\mathcal{R}_2$ are two non-overlapping (well separated) regions of $\mathbb{Z}^2$ then

$$p(M_{\mathcal{R}_1\cup\mathcal{R}_2}) = p(M_{\mathcal{R}_1})p(M_{\mathcal{R}_2}).$$

4. Application to diffusion problem

We are going now to apply the above results to the problem of diffusion in coupled cat model. To make a connection with the diffusion problem we make a standard modification of the original setting. First the coordinates $x_{n,t}$ are shifted by $1/2$: $x_{n,t} \to x_{n,t} + 1/2$ and $m_{n,t} \to m_{n,t} - s/2 + 2$. This preserves the form of eq. 1.1, but changes the range for $x_{n,t}, m_{n,t}$. Now $x_{n,t} \in [-1/2, 1/2)$ and $m_{n,t}$ are half-integers between $-s/2+2$ and $s/2-2$. Second, we allow the momentum of the particle run from $-\infty$ to $+\infty$.

After separating the integer part of the momentum from its fractional part $p_{n,t} \in [-1/2, 1/2)$ the equations of motion take the form:

$$\begin{aligned} q_{n,t+1} &= (p_{n,t} + (s-1)q_{n,t}) - (q_{n+1,t} + q_{n-1,t}) - m^q_{n,t+1} \\ p_{n,t+1} &= (p_{n,t} + (s-2)q_{n,t}) - (q_{n+1,t} + q_{n-1,t}) - m^p_{n,t+1}. \end{aligned} \tag{4.1}$$

In the coordinate (Newtonian) form they can be written as:

$$q_{n,t+1} + q_{n,t-1} + q_{n+1,t} + q_{n-1,t} = sq_{n,t} - m_{n,t}, \qquad (4.2)$$
$$m_{n,t} = -m^q_{n,t} + m^q_{n,t+1} + m^p_{n,t}.$$

The total shift in the momentum of L neighboring particles after time T is given by:

$$\Delta P(L,T) = \sum_{n=1}^{L}\sum_{t=1}^{T} m^p_{n,t} \approx \sum_{n=1}^{L}\sum_{t=1}^{T} m_{n,t}, \qquad (4.3)$$

where we dropped boundary terms (vanishing exactly for periodic orbits).

We are interested in the probability $\mathcal{P}_{L,T}(\sigma)$ that after time T there will be an a shift σ in the total momentum of L neighboring particles. It can be related to the ratio:

$$\mathcal{P}_{L,T}(\sigma) = \lim_{N\to\infty} \frac{\mathcal{N}(\sigma,N,T)}{\mathcal{N}(N,T)}, \qquad (4.4)$$

where $\mathcal{N}(\sigma,N,T)$ is the number of PO(N,T) with $\sum_{n=1}^{L}\sum_{t=1}^{T} m_{n,t} = \sigma$ and $\mathcal{N}(N,T)$ is the total number of periodic orbits of the period T for the N-particle map. This can be represented as the sum:

$$\mathcal{N}(\sigma,N,T) = \sum_{m_{n,t}\in\mathcal{X}} \frac{1}{2\pi}\int_{-\infty}^{+\infty} dz\, e^{iz(\sigma - \sum_{n=1}^{L}\sum_{t=1}^{T} m_{n,t})}, \qquad (4.5)$$

where $\mathcal{X}$ is the set of all admissible sequences. To evaluate this quantity we now go from discrete sum over $m_{n,t}$ to the integral over continues variables $x_{n,t}$ with the help of linear transformation (4.2):

$$\mathcal{N}(\sigma,N,T) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dz\, e^{iz\sigma} \det(\Delta - s + 4) \qquad (4.6)$$
$$\int_{-\frac{1}{2}}^{+\frac{1}{2}} \cdots \int_{-\frac{1}{2}}^{+\frac{1}{2}} \prod_{(n,t)\in\mathbb{Z}^2_{\mathrm{NT}}} dx_{n,t} e^{-iz\sum_{n=1}^{L}\sum_{t=1}^{T}(\Delta - s+4)x_{n,t}}.$$

Since $\mathcal{N}(N,T) = \det(\Delta - s + 4)$, we have for $L \geq 2$:

$$\mathcal{P}_{L,T}(\sigma) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dz\, e^{iz\sigma}[F(z(s-4))]^{(L-2)T}[F(z(s-3))F(z)]^{2T}, \qquad (4.7)$$

where the two factors in the integrand are due to the bulk and boundaries of the domain $L \times T$, respectively. For $L = 1$ we have

$$\mathcal{P}_{1,T}(\sigma) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dz\, e^{iz\sigma}[F(z(s-2))]^{T}[F(z)]^{2T}, \qquad (4.8)$$

with $F(2z)$ being sinc function:

$$F(z) = \int_{-\frac{1}{2}}^{+\frac{1}{2}} e^{izx} dx = \frac{2\sin\frac{z}{2}}{z}.$$

For $T \gg 1$ this gives:

$$\mathcal{P}_{L,T}(\sigma) \simeq \frac{1}{\sqrt{4\pi\mathcal{D}_{\rm dif}T}} e^{-\sigma^2/4\mathcal{D}_{\rm dif}T}, \tag{4.9}$$

with the diffusion constant given by

$$\mathcal{D}_{\rm dif} = \begin{cases} \frac{1}{24}\left((L-1)(s-4)^2 + 4(s-3)\right) & \text{for } L \geq 2 \\ \frac{1}{24}\left((s-2)^2 + 2\right) & \text{for } L = 1. \end{cases}$$

Asymptotically we have:

$$\lim_{T\to\infty} \frac{1}{T} \Delta P(L,T)^2 = \int_{-\infty}^{+\infty} \mathcal{P}_{L,T}(\sigma)\sigma^2 d\sigma = 2\mathcal{D}_{\rm dif}. \tag{4.10}$$

Remark 4.1. 1) For a single cat map one gets similar result, but with the diffusion constant given by $\mathcal{D}_{\rm dif} := (s-2)^2/24$, see [4].

2) Most of the manipulations with matrices in [4] (taking inverses etc.) are completely unnecessary for diffusion. I have no idea what for they are doing this stuff.

3) The result can be straightforwardly extended to the case when cats are coupled along $\mathbb{Z}^n, n > 1$ lattice or (probably) any other network. More exciting is to stick with $\mathbb{Z}$, but consider the case of random couplings, see below for reasons.

4) In quantum mechanics the diffusion in kicked systems is typically suppressed by the Anderson localization effect. It seems to be interesting to study what happens in the case of coupled cat maps with random couplings/kicked parameters i.e., many-body localization.

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