what is ‘chaos’?
a field theorist stroll through Bernoullistan

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ChaosBook.org/overheads/spatiotemporal/kittens/ notes
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Mephistopheles knocks at Faust’s door and says, “Du mußt es dreimal sagen!"

“You have to say it three times"

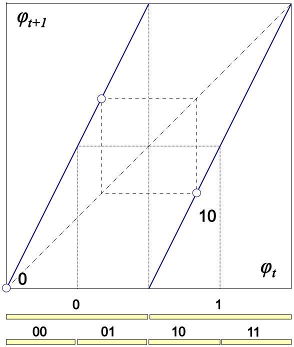
— Johann Wolfgang von Goethe
Faust I - Studierzimmer 2. Teil

1. coin toss
2. temporal cat
3. spatiotemporal cat
4. bye bye, dynamics

time-evolution formulation

fair coin toss

Bernoulli map



$$\phi_{t+1} = \begin{cases} 2\phi_t \\ 2\phi_t \pmod 1 \end{cases}$$

$\Rightarrow$ fixed point $\overline{0}$, 2-cycle $\overline{01}$, $\cdots$

a coin toss

the essence of deterministic chaos

what is (mod 1) ?

map with integer-valued 'stretching' parameter $s \geq 2$:

$$x_{t+1} = s\, x_t$$

(mod 1) : subtract the integer part $m_{t+1} = \lfloor s x_t \rfloor$
so fractional part ϕ_{t+1} stays in the unit interval $[0, 1)$

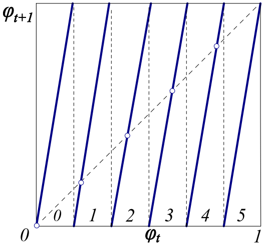
$$\phi_{t+1} = s\phi_t - m_{t+1}\,, \qquad \phi_t \in \mathcal{M}_{m_t}$$

m_t takes values in the s-letter alphabet

$$m \in \mathcal{A} = \{0, 1, 2, \cdots, s-1\}$$

a fair dice throw

slope 6 Bernoulli map



$$\phi_{t+1} = 6\phi_t - m_{t+1}\,,\ \phi_t \in \mathcal{M}_{m_t}$$

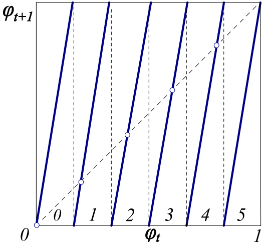
6-letter alphabet
$m_t \in \mathcal{A} = \{0, 1, 2, \cdots, 5\}$

6 subintervals $\{\mathcal{M}_0, \mathcal{M}_1, \cdots, \mathcal{M}_5\}$

what is chaos ?

a fair dice throw

6 subintervals $\{\mathcal{M}_{m_t}\}$, 6^2 subintervals $\{\mathcal{M}_{m_1 m_2}\}, \cdots$



each subinterval contains a periodic point, labeled by $\mathrm{M} = m_1 m_2 \cdots m_n$

$N_n = 6^n$ unstable orbits

definition : chaos is

positive Lyapunov ($\ln s$) - positive entropy ($\frac{1}{n} \ln N_n$)

definition : chaos is

positive Lyapunov ($\ln s$) - positive entropy ($\frac{1}{n} \ln N_n$)

the precise sense in which dice throw
is an example of deterministic chaos

lattice formulation

lattice Bernoulli

recast the time-evolution Bernoulli map

$$\phi_{t+1} = s\phi_t - m_{t+1}$$

as 1-step difference equation on the temporal lattice

$$\phi_t - s\phi_{t-1} = -m_t \,, \qquad \phi_t \in [0, 1)$$

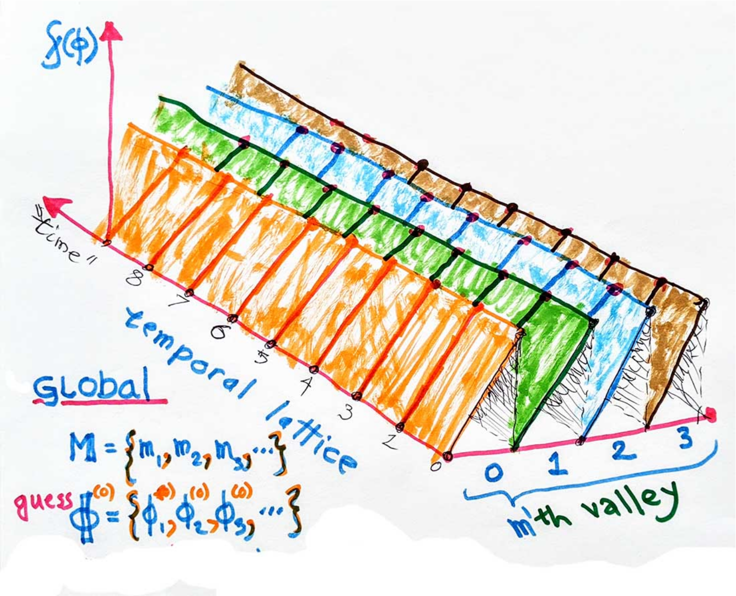
field ϕ_t, source m_t
on each site t of a 1-dimensional lattice $t \in \mathbb{Z}$

write an n-sites lattice segment as
the lattice state and the symbol block

$$\Phi = (\phi_{t+1}, \cdots, \phi_{t+n}) \,, \quad \mathsf{M} = (m_{t+1}, \cdots, m_{t+n})$$

'M' for 'marching orders' : come here, then go there, $\cdots$

exponentially many distinct walks through Bernoullistan



think globally, act locally

Bernoulli condition at every lattice site t, local in time

$$\phi_t - s\phi_{t-1} = -m_t$$

is enforced by the global equation

$$\left(1 - s\,\sigma^{-1}\right)\,\Phi = -\mathrm{M}\,,$$

$[n \times n]$ shift matrix

$$\sigma_{jk} = \delta_{j+1,k}\,, \qquad \sigma = \begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & & \ddots & & \\ & & & & 0 & 1 \\ 1 & & & & & 0 \end{pmatrix}$$

compares the neighbors

think globally, act locally

solving the lattice Bernoulli system

$$\mathcal{J}\Phi = -\mathsf{M}\,,$$

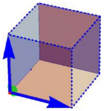
$[n \times n]$ Hill matrix $\qquad \mathcal{J} = 1 - s\,\sigma^{-1}\,,$

is a search for zeros of the function

$$F[\Phi] = \mathcal{J}\Phi + \mathsf{M} = 0$$

the entire global lattice state Φ_{M} is now

a single fixed point $(\phi_1, \phi_2, \cdots, \phi_n)$



in the n-dimensional unit hyper-cube $\qquad \Phi \in [0,1)^n$

Hill matrix

solving a nonlinear $F[\Phi] = 0$ fixed point condition with Newton method requires evaluation of the $[n \times n]$ Hill matrix

$$\mathcal{J}_{ij} = \frac{\delta F[\Phi]_i}{\delta \phi_j}$$

what does this global Hill matrix do?

1. fundamental fact !
2. global stability of lattice state Φ, perturbed everywhere

(1) fundamental fact

to satisfy the fixed point condition

$$\mathcal{J}\Phi + \mathrm{M} = 0$$

the Hill matrix $\mathcal{J}$

1. stretches the unit hyper-cube $\Phi \in [0,1)^n$ into the n-dimensional fundamental parallelepiped
2. maps each periodic point $\Phi_{\mathrm{M}} \;\Rightarrow\;$ integer lattice $\mathbb{Z}^n$ point
3. then translate by integers $\mathrm{M} \;\Rightarrow\;$ into the origin

hence $N_n =$ total $\sharp$ solutions = the $\sharp$ integer lattice points within the fundamental parallelepiped

the fundamental fact[1] : Hill determinant counts solutions

$$N_n = |\mathrm{Det}\,\mathcal{J}|$$

$\sharp$ integer points in fundamental parallelepiped $=$ its volume

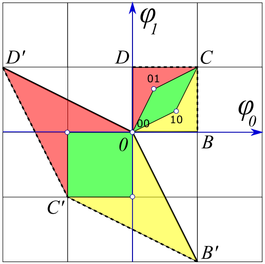
[1] M. Baake et al., J. Phys. A **30**, 3029–3056 (1997).

example : fundamental parallelepiped for $n = 2$

Hill matrix, $s = 2$ unit square basis vectors, their images :

$$\mathcal{J} = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix} ; \quad \Phi_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \Phi_{B'} = \mathcal{J}\,\Phi_B = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \cdots ,$$

Bernoulli periodic points of period 2



$N_2 = 3$

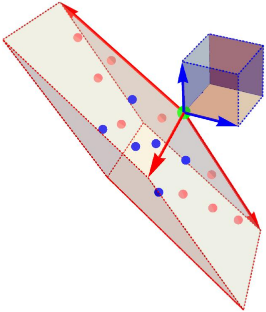
fixed point Φ_{00}
2-cycle Φ_{01}, Φ_{10}

square $[0BCD] \Rightarrow \mathcal{J} \Rightarrow$ fundamental parallelepiped $[0B'C'D']$

fundamental fact for any n

an $n = 3$ example

$\mathcal{J}$ [unit hyper-cube] = [fundamental parallelepiped]



unit hyper-cube $\Phi \in [0, 1)^n$

$n > 3$ cannot visualize

a periodic point $\rightarrow$ integer lattice point, ● on a face, ● in the interior

(2) orbit stability vs. temporal stability

Hill matrix

$\mathcal{J}_{ij} = \frac{\delta F[\Phi]_i}{\delta \phi_j}$ stability under global perturbation of the whole orbit

for n large, a huge $[dn \times dn]$ matrix

temporal Jacobian matrix

J propagates initial perturbation n time steps

small $[d \times d]$ matrix

J and $\mathcal{J}$ are related by[2]

Hill's (1886) remarkable formula

$$|\mathrm{Det}\,\mathcal{J}_\mathrm{M}| = |\det(\mathbf{1} - J_\mathrm{M})|$$

$\mathcal{J}$ is huge, even ∞-dimensional matrix

J is tiny, few degrees of freedom matrix

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[2] G. W. Hill, Acta Math. **8**, 1–36 (1886).

field theorist's chaos

definition

expanding	Hill determinants	$\mathrm{Det}\,\mathcal{J}$
exponential	$\sharp$ of field configurations	N_n

the precise sense in which
a (discretized) field theory is deterministically chaotic

how come Hill determinant $\mathrm{Det}\,\mathcal{J}$ counts periodic points ?

in 1984 Ozorio de Almeida and Hannay[3] related the number of periodic points to a Jacobian matrix by their

principle of uniformity

"periodic points of an ergodic system, counted with their natural weighting, are uniformly dense in phase space"

where

'natural weight' of periodic orbit M

$$\frac{1}{|\det(1 - J_{\mathrm{M}})|}$$

[3] A. M. Ozorio de Almeida and J. H. Hannay, J. Phys. A **17**, 3429 (1984).

how come a $\mathrm{Det}\,\mathcal{J}$ counts periodic points ?

“principle of uniformity” is in[4]

periodic orbit theory

known as the flow conservation sum rule :

$$\sum_{M} \frac{1}{|\det(1 - J_M)|} = \sum_{M} \frac{1}{|\mathrm{Det}\,\mathcal{J}_M|} = 1$$

sum over periodic points Φ_M of period n

state space is divided into

neighborhoods of periodic points of period n

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[4] P. Cvitanović, “Why cycle?”, in *Chaos: Classical and Quantum*, edited by P. Cvitanović et al. (Niels Bohr Inst., Copenhagen, 2020).

periodic orbit theory

how come a $\mathrm{Det}\,\mathcal{J}$ counts periodic points ?

flow conservation sum rule :

$$\sum_{\phi_i \in \mathrm{Fix} f^n} \frac{1}{|\mathrm{Det}\,\mathcal{J}_i|} = 1$$

Bernoulli system 'natural weighting' is simple :

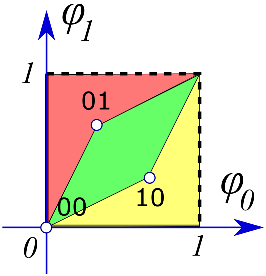
the determinant $\mathrm{Det}\,\mathcal{J}_i = \mathrm{Det}\,\mathcal{J}$ the same for all periodic points, whose number thus verifies the fundamental fact

$$N_n = |\mathrm{Det}\,\mathcal{J}|$$

the number of Bernoulli periodic lattice states

$N_n = |\mathrm{Det}\,\mathcal{J}| = s^n - 1$ for any n

example : lattice states of period 2



$N_2 = 3$

fixed point Φ_{00}
2-cycle Φ_{01}, Φ_{10}

flow conservation sum rule

$$\frac{1}{|\mathrm{Det}\,\mathcal{J}_{00}|} + \frac{1}{|\mathrm{Det}\,\mathcal{J}_{01}|} + \frac{1}{|\mathrm{Det}\,\mathcal{J}_{10}|} = 1$$

sum over periodic points Φ_M of period $n = 2$

state space is divided into

neighborhoods of periodic points of period n

periodic orbit theory

how does 1-time step transition matrix T count periodic lattice states ? For any matrix $\ln\det = \mathrm{tr}\,\ln$, so

$\ln\det(1 - zT) = \mathrm{tr}\,\ln(1 - zT) =$ **sum over loops**

$$\det(1 - zT) = \exp\left(-\sum_{n=1} \frac{z^n}{n} \mathrm{tr}\,T^n\right)$$

AKA

'topological zeta function'

$$1/\zeta_{\mathrm{top}}(z) = \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} N_n\right)$$

1. weight $1/n$ as by (cyclic) translation invariance, n lattice states are equivalent
2. zeta function counts prime orbits, one per each set of equivalent lattice states

topological zeta function

counts prime orbits, one per each set of Bernoulli periodic states $N_n = s^n - 1$

$$1/\zeta_{\text{top}}(z) \,=\, \exp\left(-\sum_{n=1}^{\infty}\frac{z^n}{n}N_n\right) \,=\, \frac{1-sz}{1-z}$$

numerator $(1-sz)$ says that Bernoulli orbits are built from s fundamental primitive lattice states,

the fixed points $\{\phi_0, \phi_1, \cdots, \phi_{s-1}\}$

every other lattice state is built from their concatenations and repeats.

solved!

this is ‘periodic orbit theory’

And if you don’t know, now you know

think globally, act locally - summary

the problem of enumerating and determining all global solutions stripped to its essentials :

1. each solution is a zero of the global fixed point condition

$$F[\Phi] = 0$$

2. global stability : the Hill matrix

$$\mathcal{J}_{ij} = \frac{\delta F[\Phi]_i}{\delta \phi_j}$$

3. fundamental fact : the number of period-n orbits

$$N_n = |\mathrm{Det}\,\mathcal{J}|$$

4. zeta function $1/\zeta_{\mathrm{top}}(z)$: all predictions of the theory

a kicked rotor - templatt.tex next

Du mußt es dreimal sagen!

— Mephistopheles

1. coin toss
2. kicked rotor
3. spatiotemporal cat
4. bye bye, dynamics