Spatiotemporal Tiling of the Kuramoto-Sivashinsky equation

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Spatiotemporal Chaos

- What do I mean by spatiotemporal chaos?
- Why study spatiotemporal chaos?

Doubly Periodic Kuramoto-Sivashinsky equation

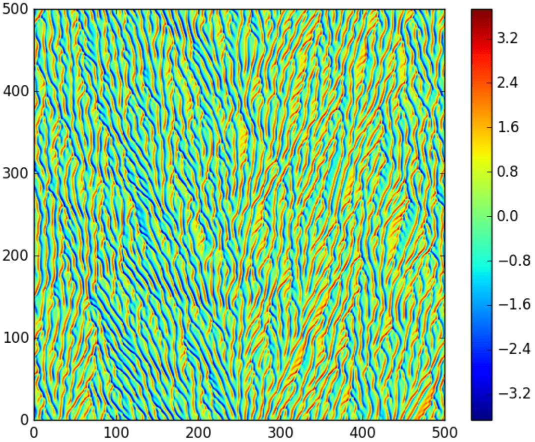
$$u_t(x,t) = \underbrace{-u_{xx}(x,t)}_{\text{Diffusion (wrong sign)}} - \underbrace{u_{xxxx}(x,t)}_{\text{``Hyper''-diffusion}} - \underbrace{u(x,t)u_x(x,t)}_{\text{Advection}}$$

Boundary conditions

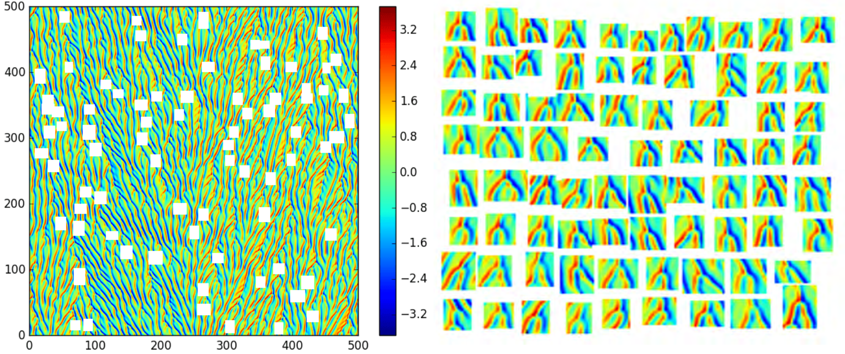
$u(x,t) = u(x,t+T_p) = u(x+L,t) = u(x+L,t+T_p)$

Dynamical Systems Theory: A Cartoon

Large spatiotemporal simulation



Shadowing



500
400
300
200
100
0
0
100
200
300
400
500
3.2
2.4
1.6
0.8
0.0
−0.8
−1.6
−2.4
−3.2

(Computational) Motivating claim

"Time domain parallel methods, such as Parareal, scale poorly in simulation of chaotic,multiscale dynamical systems, e.g., turbulent flows. This poor scalability is because the initial value problem of a chaotic dynamical system is **ill-conditioned**. *A perturbation of magnitude ϵ at the beginning of the simulation can cause a difference of $\epsilon e^{\lambda t}$ at time t later, where $lambda$ is the maximal Lyapunov exponent."*

[Wang et al., 2013]

Fourier-Fourier Basis

Substituting

$$u(x,t) = \sum_{m,n} \tilde{u}_{m,n} e^{\mathrm{i}\omega_n t + \mathrm{i} q_m x} \,, \tag{1}$$

into the Kuramoto-Sivashinsky equation gives,

$$\mathbf{F}(\mathbf{x}) = (\mathrm{i}\frac{2\pi n}{T} + (\frac{2\pi m}{L})^2 - (\frac{2\pi m}{L})^4)\tilde{u}_{m,n} + \frac{1}{2}\frac{2\pi m}{L}\mathcal{F}((\mathcal{F}^{-1}(\tilde{u}_{m,n}))^2) \tag{2}$$

Solve nonlinear algebraic equations for $\mathbf{x} = (\tilde{\mathbf{u}}, T, L)$. Similar to formulation in [López, 2015].

Disadvantages of Spatiotemporal Methods

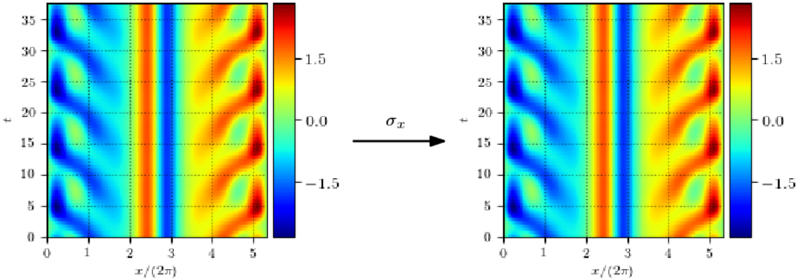
1. Strong coupling
2. Many degrees of freedom
3. Linearized system is ill-conditioned

Advantages of Spatiotemporal Methods

1. Can use variational methods
2. Fourier-Fourier basis makes computation easier and faster
3. Can use (spatiotemporal) symmetries to reduce degrees of freedom

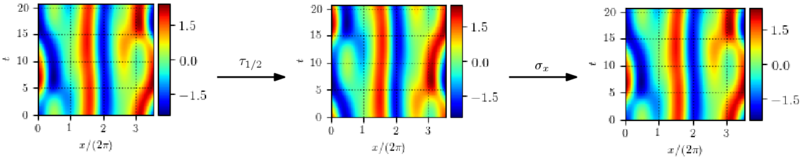
Subgroup $\mathbb{Z}_2 \times \mathbb{I}$

$$u(x,t) = \sigma_x u(x,t)$$



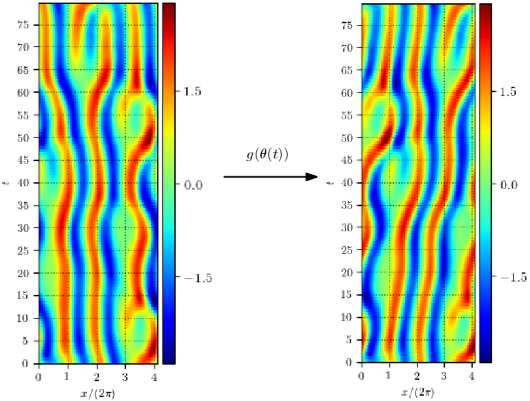
Subgroup $\{e, \sigma_x \tau_{1/2}\}$ (itself a subgroup of Subgroup $Z_2 \times C_2$)

$$u(x,t) = \sigma_x \tau_{1/2} u(x,t)$$

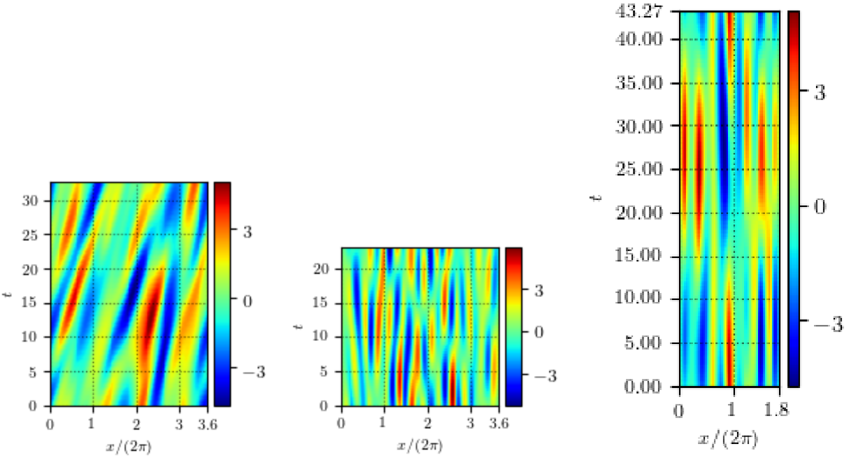


Subgroup $C_M \times \mathbb{I}$

$$u(x,t) = g(\theta(t)) \circ u(x, t + T_p)$$



Initial conditions



Adjoint Descent Method

$$\mathbf{F}(\mathbf{x}) = (\mathrm{i}\frac{2\pi n}{T} + (\frac{2\pi m}{L})^2 - (\frac{2\pi m}{L})^4)\tilde{u}_{m,n} + \frac{1}{2}\frac{2\pi m}{L}\mathcal{F}((\mathcal{F}^{-1}(\tilde{u}_{m,n}))^2) \quad (3)$$

$$I = \frac{1}{2}\mathbf{F}^\top\mathbf{F}\,. \quad (4)$$

Adjoint Descent Method

Develop fictitious time (τ) flow by differentiation of cost function.

$$\partial_\tau I = (J^\top \mathbf{F})^\top (\partial_\tau \mathbf{x}) \tag{5}$$

"adjoint descent" method defined by choice [Farazmand, 2016],

$$\partial_\tau \mathbf{x} = -(J^\top \mathbf{F}) \tag{6}$$

Backtracking Gauss-Newton Method

Solve $\mathbf{F}(\mathbf{x}^*) = 0$ via linearized system

$$\mathbf{F}(\mathbf{x}^*) = \mathbf{F}(\mathbf{x}) + J\Delta\mathbf{x} \tag{7}$$

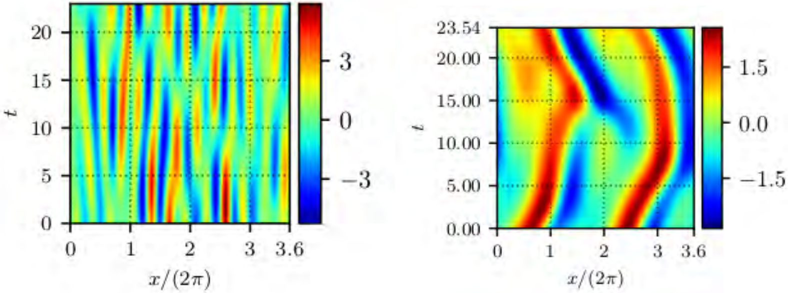
$$J = \begin{bmatrix} \frac{\partial \mathbf{F}}{\partial \mathbf{x}} & \frac{\partial \mathbf{F}}{\partial T} & \frac{\partial \mathbf{F}}{\partial L} \end{bmatrix} \tag{8}$$

$$J\Delta\mathbf{x} = -\mathbf{F}(\mathbf{x}) \tag{9}$$

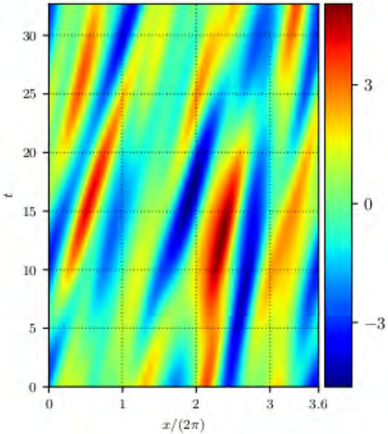
- Least-squares solver, minimizes norm of $\Delta\mathbf{x}$
- Line-search and or backtracking improves convergence.

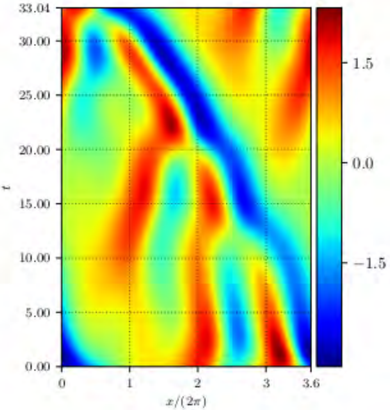
Shift-reflect symmetry

A couple thousand new invariant 2-torus solutions, examples:

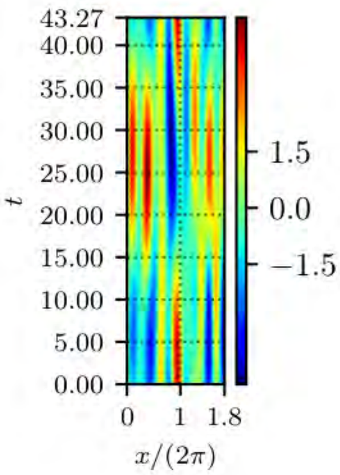


Spatial translation symmetry

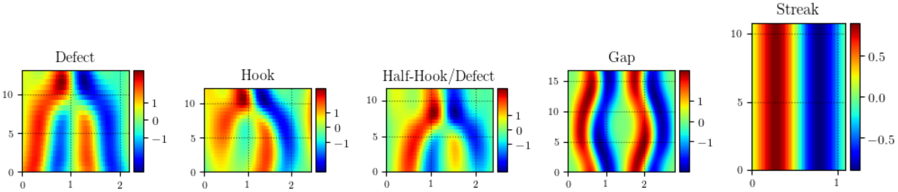




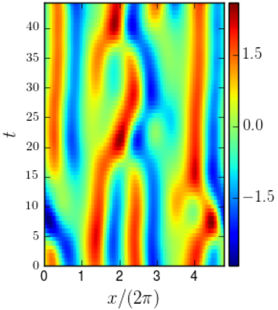
Reflection symmetry

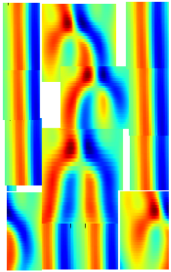


Qualitative guesses for tiles

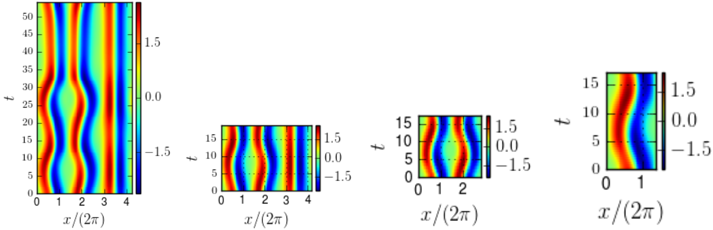


Qualitative tiling of actual solution

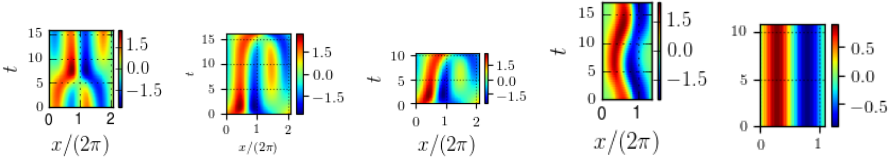




Finding tiles by converging subdomains



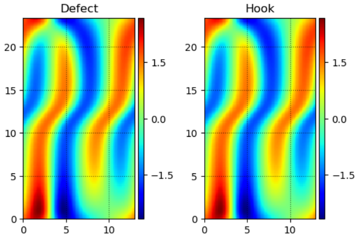
Tiles



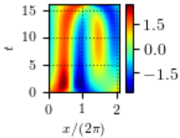
15
10
5
0
t
0 1 2
$x/(2\pi)$
1.5
0.0
−1.5
15
10
5
0
0 1 2
$x/(2\pi)$
1.5
0.0
−1.5
10
5
0
t
0 1 2
$x/(2\pi)$
1.5
0.0
−1.5
15
10
5
0
t
0 1
$x/(2\pi)$
1.5
0.0
−1.5
10
5
0
0 1
0.5
0.0
−0.5

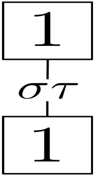
Continuous families and numerical continuation

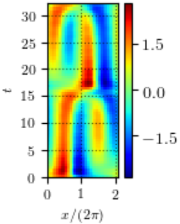
Evidence that "hook" equals "defect". Numerically continue each to the same domain size, quotient continuous symmetries.

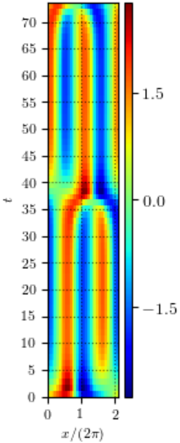


Homoclinic connection from tiles

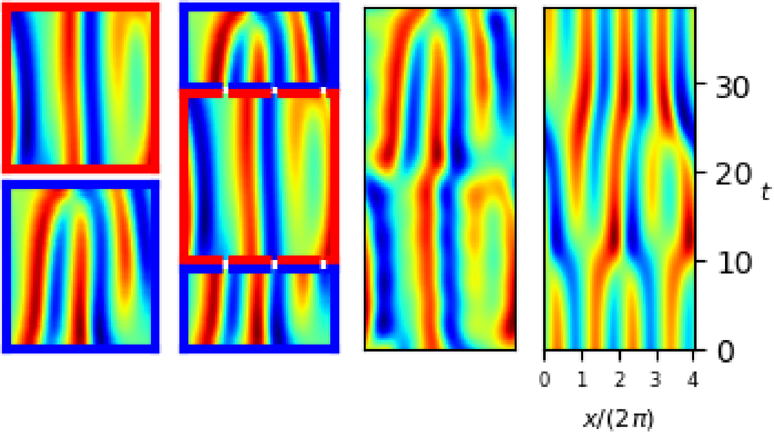




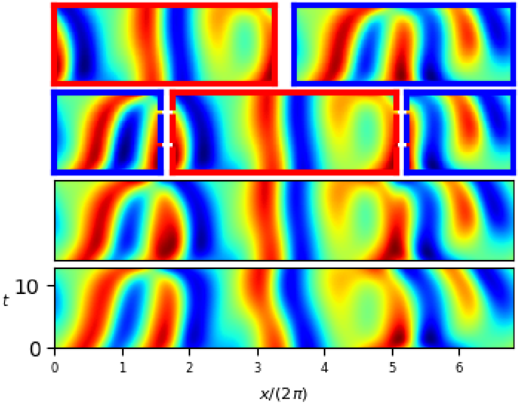




Gluing (larger) solutions, time example



Gluing (larger) solutions, space example



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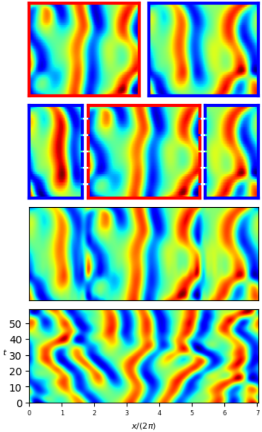
Gluing (larger) solutions, space example



Summary and important contributions

- New nonlinear optimization routines
- Larger (domain size) periodic solutions
- Connection between small domains and large domains (shadowing) via tiles
- Spatiotemporal gluing of solutions

Future Work I

- Need to find all “tiles”, hope for small finite number.
- Need to deal with continuous families.
- Need to figure out spatiotemporal symbolic dynamics

References I

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[Wang et al., 2013] Wang, Q., Gomez, S. A., Blonigan, P. J., Gregory, A. L., and Qian, E. Y. (2013).
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Thank you for listening!

Questions?