spatiotemporal cat

a chaotic field theory

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ChaosBook.org/overheads/spatiotemporal notes

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overview

1. what this is about
2. chaos - a short course
3. temporal cat
4. spatiotemporal cat
5. space is time
6. bye bye, dynamics

what is this? some background

this talk is an introduction to the

spatiotemporal cat[1]

if there is time, will discuss the larger picture

spatiotemporal turbulence[2]

that motivates our study of discrete spatiotemporal lattices

[1] P. Cvitanović and H. Liang, *Spatiotemporal cat: An exact classical chaotic field theory*, in preparation, 2020.

[2] M. Gudorf and P. Cvitanović, *Spatiotemporal tiling of the Kuramoto-Sivashinsky flow*, in preparation, 2020.

the goal of this presentation

build

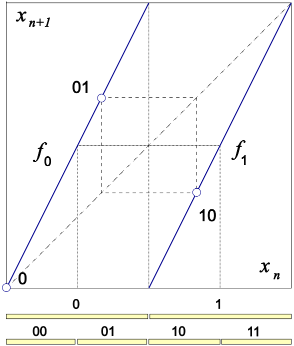
a chaotic field theory

from

the simplest chaotic blocks

the essence of deterministic chaos

fair coin toss (AKA Bernoulli map)



$$x_{t+1} = \begin{cases} f_0(x_t) = 2x_t \\ f_1(x_t) = 2x_t \pmod 1 \end{cases}$$

$\Rightarrow$ fixed point $\overline{0}$, 2-cycle $\overline{01}$, $\cdots$

a coin toss

the simplest example of deterministic chaos

what is (mod 1) ?

map with integer-valued 'stretching' parameter $s \geq 2$:

$$x_{t+1} = s\,x_t$$

(mod 1) : subtract the integer part $m_{t+1} = \lfloor s x_t \rfloor$
to keep fractional part ϕ_{t+1} in the unit interval $[0, 1)$

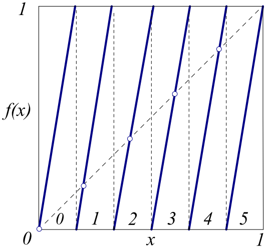
$$\phi_{t+1} = s\phi_t - m_{t+1}\,, \qquad \phi_t \in \mathcal{M}_{m_t}$$

m_t takes values in the s-letter alphabet

$$m \in \mathcal{A} = \{0, 1, 2, \cdots, s-1\}$$

a fair dice throw

slope 6 Bernoulli map



$$\phi_{t+1} = 6\phi_t - m_{t+1}\,, \;\; \phi_t \in \mathcal{M}_{m_t}$$

6-letter alphabet
$m \in \mathcal{A} = \{0, 1, 2, \cdots, 5\}$

6 subintervals $\{\mathcal{M}_{m_1}\}$, 6^2 subintervals $\{\mathcal{M}_{m_1 m_2}\}, \cdots$

$N_n = 6^n$ unstable orbits, each labeled by $\mathsf{M} = m_1 m_2 \cdots m_n$

this is chaos!

positive Lyapunov ($\ln s$) + positive entropy ($\frac{1}{n} \ln N_n$)

the precise sense in which

deterministic chaos is a dice throw

lattice Bernoulli

now recast the time-evolution Bernoulli map

$$\phi_{t+1} = s\phi_t - m_{t+1}$$

as a 1-step difference equation on the temporal lattice

$$\phi_t - s\phi_{t-1} = -m_t \,, \qquad \phi_t \in [0,1)$$

with a field ϕ_t, source m_t
on each site t of a 1-dimensional lattice $t \in \mathbb{Z}$

write an n-sites lattice segment as
the lattice state and the symbol block

$$\mathsf{X} = (\phi_{t+1}, \cdots, \phi_{t+n}) \,, \quad \mathsf{M} = (m_{t+1}, \cdots, m_{t+n})$$

think globally, act locally

Bernoulli equation at every instant t, local in time

$$\phi_t - s\phi_{t-1} = -m_t$$

is enforced by the global equation

$$\left(1 - s\sigma^{-1}\right) \mathrm{X} = -\mathrm{M}\,,$$

where the $[n \times n]$ matrix

$$\sigma_{jk} = \delta_{j+1,k}\,, \qquad \sigma = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & & \ddots & \\ & & & 0 & 1 \\ 1 & & & & 0 \end{pmatrix},$$

implements the 1-time step operation

think globally, act locally

solving the lattice Bernoulli equation

$$\mathcal{J}\mathsf{X} = -\mathsf{M}\,,$$

with the $[n \times n]$ matrix $\qquad \mathcal{J} = 1 - s\sigma^{-1}\,,$

can be viewed as a search for zeros of the function

$$F[\mathsf{X}] = \mathcal{J}\mathsf{X} + \mathsf{M} = 0$$

the entire global lattice state X_M is now

a single fixed point $\mathsf{X}_\mathsf{M} = (\phi_1, \phi_2, \cdots, \phi_n)$
in the n-dimensional unit hyper-cube $\mathsf{X} \in [0, 1)^n$

orbit Jacobian matrix

solving a nonlinear $F[\mathsf{X}] = 0$ fixed point condition with Newton method requires evaluation of the $[n \times n]$ orbit Jacobian matrix

$$\mathcal{J}_{ij} = \frac{\delta F[\mathsf{X}]_i}{\delta \phi_j}$$

what does this global orbit Jacobian matrix do?

1. fundamental fact !
2. global stability of lattice state X, perturbed everywhere

(1) fundamental fact

to satisfy the fixed point condition

$$\mathcal{J}\mathsf{X} + \mathsf{M} = 0$$

the orbit Jacobian matrix $\mathcal{J}$

1. stretches the unit hyper-cube $\mathsf{X} \in [0,1)^n$ into the n-dimensional fundamental parallelepiped
2. maps each periodic point X_M into an integer lattice $\mathbb{Z}^n$ point
3. then translate by integers M into the origin

hence N_n, the total number of solutions = the number of integer lattice points within the fundamental parallelepiped

the fundamental fact[3]

$$N_n = |\mathrm{Det}\, \mathcal{J}|$$

integer points in fundamental parallelepiped = its volume

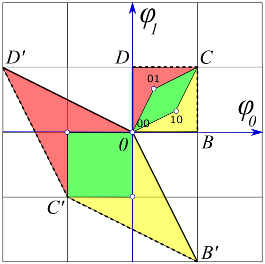
[3] M. Baake et al., J. Phys. A **30**, 3029–3056 (1997).

example : fundamental parallelepiped for $n = 2$

orbit Jacobian matrix, unit square basis vectors, their images :

$$\mathcal{J} = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}; \quad \mathsf{X}_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \mathsf{X}_{B'} = \mathcal{J}\,\mathsf{X}_B = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \cdots,$$

Bernoulli periodic points of period 2



$N_2 = 3$

fixed point X_{00}

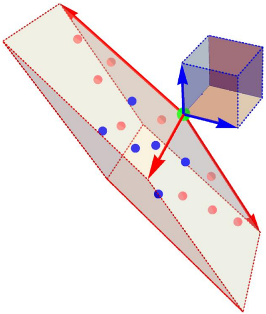
2-cycle $\mathsf{X}_{01}, \mathsf{X}_{10}$

square $[0BCD] \Rightarrow \mathcal{J} \Rightarrow$ fundamental parallelepiped $[0B'C'D']$

fundamental fact for any n

temporal cat $n = 3$ example

$\mathcal{J}$ [unit hyper-cube] = [fundamental parallelepiped]



unit hyper-cube $X \in [0, 1)^n$

$n > 3$ cannot visualize

(2) orbit stability vs. temporal stability

orbit Jacobian matrix

$\mathcal{J}_{ij} = \frac{\delta F[X]_i}{\delta \phi_j}$ stability under global perturbation of the whole orbit

for n large, huge $[d n \times d n]$ matrix

Jacobian matrix

J^n propagates initial perturbation n time steps

small $[d \times d]$ matrix

are related by[4]

Hill's (1886) remarkable formula

$$|\mathrm{Det}\, \mathcal{J}| = |\det(\mathbf{1} - J^n)|$$

[4] G. W. Hill, Acta Math. **8**, 1–36 (1886).

periodic orbit theory

how come that $\mathrm{Det}\,\mathcal{J}$ counts periodic orbits ?

in 1984 Ozorio de Almeida and Hannay[5] related the number of periodic points to a Jacobian matrix by their

principle of uniformity

"periodic points of an ergodic system, counted with their natural weighting, are uniformly dense in phase space"

where

natural weight of periodic orbit M

$$\frac{1}{|\det(1 - J_M)|}$$

[5] A. M. Ozorio de Almeida and J. H. Hannay, J. Phys. A **17**, 3429 (1984).

periodic orbit theory

how come that a $\mathrm{Det}\,\mathcal{J}$ counts periodic orbits ?

this principle is in[6]

periodic orbit theory

known as the flow conservation sum rule :

$$\sum_{\phi_i \in \mathrm{Fix} f^n} \frac{1}{|\det(1 - J_i)|} = \sum_{\phi_i \in \mathrm{Fix} f^n} \frac{1}{|\mathrm{Det}\,\mathcal{J}_i|} = 1$$

state space is divided into
neighborhoods of periodic points of period n

[6] P. Cvitanović, "Why cycle?", in *Chaos: Classical and Quantum*, edited by P. Cvitanović et al. (Niels Bohr Inst., Copenhagen, 2020).

periodic orbit theory

how come that a $\mathrm{Det}\,\mathcal{J}$ counts periodic orbits ?

flow conservation sum rule :

$$\sum_{\phi_i \in \mathrm{Fix} f^n} \frac{1}{|\mathrm{Det}\,\mathcal{J}_i|} = 1$$

Bernoulli system 'natural weighting' is simple :

the determinant $\mathrm{Det}\,\mathcal{J}_i = \mathrm{Det}\,\mathcal{J}$ the same for all periodic points, whose number thus verifies the fundamental fact

$$N_n = |\mathrm{Det}\,\mathcal{J}|$$

the number of Bernoulli periodic lattice states

$N_n = |\mathrm{Det}\,\mathcal{J}| = s^n - 1$ for any n

topological zeta function

the generating function that sums up number of periodic points N_n to all orders is called 'topological zeta function':

$$1/\zeta_{\mathrm{top}}(z) \,=\, \exp\left(-\sum_{n=1}^{\infty}\frac{z^n}{n}N_n\right) \,=\, \frac{1-sz}{1-z}$$

numerator $(1-sz)$ says that Bernoulli orbits are built from s fundamental primitive lattice states,

the fixed points $\{\phi_0, \phi_1, \cdots, \phi_{s-1}\}$

every other lattice state is built from their concatenations and repeats.

solved!

This is 'periodic orbit theory'
And if you don't know, now you know

think globally, act locally - summary

the problem of enumerating and determining all global solutions stripped to its bare essentials :

1. each solution a zero of the global fixed point condition

$$F[\mathrm{X}] = 0$$

2. global stability : the orbit Jacobian matrix

$$\mathcal{J}_{ij} = \frac{\delta F[\mathrm{X}]_i}{\delta \phi_j}$$

3. fundamental fact : the number of period-n orbits

$$N_n = |\mathrm{Det}\,\mathcal{J}|$$

4. zeta function $1/\zeta_{\mathrm{top}}(z)$: all predictions of the theory

coin toss ? that's not physics

a field theory should be Hamiltonian and energy conserving,
so that it can serve as an underpinning for
a Quantum Field Theory

need a system as simple as the Bernoulli, but mechanical

so, we move on from running in circles,
to a mechanical rotor to kick.

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field theory in one spacetime dimension

we start with

cat map in 1 spacetime dimension

then we generalize to

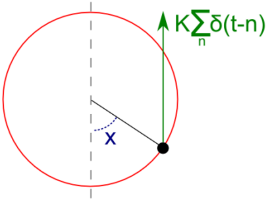
d-dimensional spatiotemporal cat

- cat map in Hamiltonian formulation
- cat map in Lagrangian formulation
  (so much more elegant!)

Hamiltonian formulation

example of a “small domain” dynamics : a single kicked rotor

an electron circling an atom, subject to
a discrete time sequence of angle-dependent kicks $F(x_t)$



Taylor, Chirikov and Greene standard map

$$
\begin{aligned}
x_{t+1} &= x_t + p_{t+1} \qquad \mod 1, \\
p_{t+1} &= p_t + F(x_t)
\end{aligned}
$$

$\rightarrow$ chaos in Hamiltonian systems

the simplest example : a cat map evolving in time

force $F(x) = Kx$ linear in the displacement x , $K \in \mathbb{Z}$

$$
\begin{aligned}
x_{t+1} &= x_t + p_{t+1} \qquad \mod 1 \\
p_{t+1} &= p_t + Kx_t \qquad \mod 1
\end{aligned}
$$

Continuous Automorphism of the Torus, or

Hamiltonian cat map

a linear, area preserving map of a 2-torus onto itself

$$
\begin{pmatrix} \phi_t \\ \phi_{t+1} \end{pmatrix} = J \begin{pmatrix} \phi_{t-1} \\ \phi_t \end{pmatrix} - \begin{pmatrix} 0 \\ m_t \end{pmatrix} , \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & s \end{pmatrix}
$$

for integer "stretching" $s = \mathrm{tr}\, J > 2$ the map is
hyperbolic $\rightarrow$ a fully chaotic Hamiltonian dynamical system

Lagrangian formulation

cat map in Lagrangian form

replace momentum by velocity

$$p_{t+1} = (\phi_{t+1} - \phi_t)/\Delta t$$

formulation on (ϕ_t, ϕ_{t-1}) temporal lattice is particularly pretty[7]

2-step difference equation

$$\phi_{t+1} - s\,\phi_t + \phi_{t-1} = -m_t$$

integer m_t ensures that

ϕ_t lands in the unit interval

$$m_t \in \mathcal{A}\,, \quad \mathcal{A} = \{\text{finite alphabet}\}$$

[7] I. Percival and F. Vivaldi, Physica D **27**, 373–386 (1987).

think globally, act locally

temporal cat at every instant t, local in time

$$\phi_{t+1} - s\,\phi_t + \phi_{t-1} = -m_t$$

is enforced by the global equation

$$(\sigma - s\mathbb{1} + \sigma^{-1})\,\mathsf{X} = -\mathsf{M}\,,$$

where

$$\mathsf{X} = (\phi_{t+1}, \cdots, \phi_{t+n})\,, \quad \mathsf{M} = (m_{t+1}, \cdots, m_{t+n})$$

are lattice state and symbol block

think globally, act locally

solving the temporal cat equation

$$\mathcal{J}\mathsf{X} = -\mathsf{M}\,,$$

with the $[n\times n]$ matrix $\qquad \mathcal{J} = \sigma - s\mathbb{1} + \sigma^{-1}\,,$

can be viewed as a search for zeros of the function

$$F[\mathsf{X}] = \mathcal{J}\mathsf{X} + \mathsf{M} = 0$$

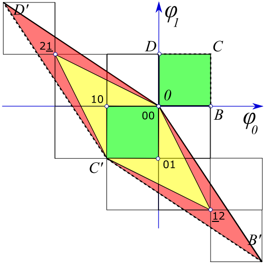
where the entire global lattice state X_M is

a single fixed point $\mathsf{X}_\mathsf{M} = (\phi_1, \phi_2, \cdots, \phi_n)$
in the n-dimensional unit hyper-cube $\mathsf{X} \in [0,1)^n$

fundamental fact in action

temporal cat fundamental parallelepiped for period 2

square $[0BCD] \Rightarrow \mathcal{J} \Rightarrow$ fundamental parallelepiped $[0B'C'D']$



$N_2 = |\mathrm{Det}\,\mathcal{J}| = 5$

fundamental parallelepiped
$=$ 5 unit area quadrilaterals

again, one periodic point per each unit volume

temporal cat topological zeta function

again, can evaluate

$$N_n = |\mathrm{Det}\,\mathcal{J}|$$

substitute into the generating function for numbers of solutions:

substitute the number of periodic points N_n into the topological zeta function

$$\begin{aligned} 1/\zeta_{\mathrm{top}}(z) &= \exp\left(-\sum_{n=1} \frac{z^n}{n} N_n\right) \\ &= \frac{1 - sz + z^2}{(1-z)^2} \end{aligned}$$

solved!

what continuum theory is temporal cat discretization of?

have

2-step difference equation

$$\phi_{t+1} - s\,\phi_t + \phi_{t-1} = -m_t$$

use discrete lattice derivatives

Laplacian in 1 dimension

$$\phi_{t+1} - 2\phi_t + \phi_{t-1} = \Box\,\phi_t$$

to rewrite cat map as an (anti)oscillator chain

$d = 1$ damped Poisson equation (!)

$$(\Box - s + 2)\,\phi_t = -m_t$$

did you know that a cat map can be so cool?

that's it! for spacetime of 1 dimension

lattice damped Poisson equation

$$(\Box - s + 2)\phi_z = -m_z$$

solved completely and analytically!

think globally, act locally - summary

the problem of enumerating and determining all global solutions stripped to its bare essentials :

1. each solution a zero of the global fixed point condition

$$F[\mathsf{X}] = 0$$

2. compute the orbit Jacobian matrix

$$\mathcal{J}_{ij} = \frac{\delta F[\mathsf{X}]_i}{\delta \phi_j}$$

3. fundamental fact $\qquad N_n = |\mathrm{Det}\, \mathcal{J}|$
   counts the number N_n of period n orbits
4. construct zeta function $1/\zeta_{\mathrm{top}}(z)$

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spatiotemporally infinite ‘spatiotemporal cat’

herding cats in d spacetime dimensions

start with

a cat map at each lattice site

talk to neighbors

spacetime d-dimensional   spatiotemporal cat

- Hamiltonian formulation is awkward, forget about it
- Lagrangian formulation is elegant

spatiotemporal cat

consider a 1 spatial dimension lattice, with field ϕ_{nt}
(the angle of a kicked rotor “particle” at instant t, at site n)

require

- each site couples to its nearest neighbors $\phi_{n\pm1,t}$
- invariance under spatial translations
- invariance under spatial reflections
- invariance under the space-time exchange

obtain[8]

2-dimensional coupled cat map lattice

$$\phi_{n,t+1} + \phi_{n,t-1} - 2s\,\phi_{nt} + \phi_{n+1,t} + \phi_{n-1,t} = -m_{nt}$$

[8] B. Gutkin and V. Osipov, Nonlinearity **29**, 325–356 (2016).

spatiotemporal cat : a strong coupling field theory

symmetries : translational and time-reversal, spatial reflection

the key assumption

- invariance under the space-time exchange

- spatiotemporal cat is a Euclidean field theory
- in Lagrangian formulation

herding cats : a discrete Euclidean space-time field theory

write the spatial-temporal differences as discrete derivatives

Laplacian : in $d = 1$ and $d = 2$ dimensions

$$
\begin{aligned}
\square\,\phi_t &= \phi_{t+1} - 2\phi_t + \phi_{t-1} \\
\square\,\phi_{nt} &= \phi_{n,t+1} + \phi_{n,t-1} - 4\,\phi_{nt} + \phi_{n+1,t} + \phi_{n-1,t}
\end{aligned}
$$

$$-m_{nt} = \phi_{n,t+1} + \phi_{n,t-1} - 2s\,\phi_{nt} + \phi_{n+1,t} + \phi_{n-1,t}$$

the cat map is thus generalized to

***d*-dimensional spatiotemporal cat**

$$(\square - d(s-2))\phi_z = -m_z$$

where $\phi_z \in \mathbb{T}^1\,, \quad m_z \in \mathcal{A}$ and $z \in \mathbb{Z}^d$ = lattice sites

discretized linear PDE

d-dimensional spatiotemporal cat

$$(\Box - d(s-2))\,\phi_z = -m_z$$

is linear and known as

- Helmholtz equation if stretching is weak, $s < 2$ (oscillatory sine, cosine solutions)
- damped Poisson equation if stretching is strong, $s > 2$ (hyperbolic sinches, coshes)

the nonlinearity is hidden in the “source”

$$m_z \in \mathcal{A} \text{ at lattice site } z \in \mathbb{Z}^d$$

the simplest of all ‘turbulent’ field theories !

spatiotemporal cat

$$(\Box - d(s-2))\phi_z = -m_z$$

can be solved completely (?) and analytically (!)

assign to each site z a letter m_z from the alphabet $\mathcal{A}$.

a particular fixed set of letters m_z (a lattice state)

$$\mathsf{M} = \{m_z\} = \{m_{n_1 n_2 \cdots n_d}\} \,,$$

is a complete specification of the corresponding lattice state X

from now on work in $d = 2$ dimensions, ‘stretching parameter’ $s = 5/2$

think globally, act locally

solving the spatiotemporal cat equation

$$\mathcal{J}\mathsf{X} = -\mathsf{M}\,,$$

with the $[n\times n]$ matrix $\qquad \mathcal{J} = \sum_{j=1}^{2}\left(\sigma_j - s\mathbb{1} + \sigma_j^{-1}\right)$

can be viewed as a search for zeros of the function

$$F[\mathsf{X}] = \mathcal{J}\mathsf{X} + \mathsf{M} = 0$$

where the entire global lattice state X_M is

a single fixed point $\mathsf{X}_\mathsf{M} = \{\phi_z\}$
in the LT-dimensional unit hyper-cube $\mathsf{X} \in [0,1)^{LT}$

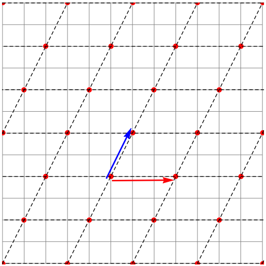
L is the 'spatial', T the 'temporal' lattice period

Bravais lattices

2-dimensional *Bravais lattice* is an infinite array of points

$$\Lambda = \{n_1\mathbf{a}_1 + n_2\mathbf{a}_2 \,|\, n_i \in \mathbb{Z}\}$$

example : $[3\times2]_1$ Bravais tile



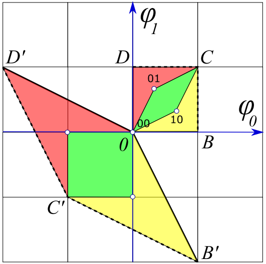
basis vectors
$\mathbf{a}_1 = (3,0)$, $\mathbf{a}_2 = (1,2)$

6 field values, on 6 lattice sites $z = (n,t)$, $[3\times2]$ rectangle:

$$\begin{bmatrix} \phi_{01} & \phi_{11} & \phi_{21} \\ \phi_{00} & \phi_{10} & \phi_{20} \end{bmatrix}$$

fundamental fact works in spacetime (!)

recall Bernoulli example ?



$[0BCD]$:
unit hyper-cube $\mathsf{X} \in [0,1)^n$

$[0B'C'D']$:
fundamental parallelepiped

$\mathcal{J}\,[0BCD] =$ fundamental parallelepiped $[0B'C'D']$

any spacetime, fundamental parallelepiped basis vectors $\mathsf{X}^{(j)}$
$=$ columns of the orbit Jacobian matrix

$$\mathcal{J} = (\mathsf{X}^{(1)}|\mathsf{X}^{(2)}|\cdots|\mathsf{X}^{(n)})$$

example : spacetime periodic $[3\times2]$ Bravais block

$$F[\mathsf{X}] = \mathcal{J}\mathsf{X} + \mathsf{M} = 0$$

6 field values, on 6 lattice sites $z = (n, t)$, $[3\times2]$ rectangle:

$$\left[\begin{array}{ccc} \phi_{01} & \phi_{11} & \phi_{21} \\ \phi_{00} & \phi_{10} & \phi_{20} \end{array}\right]$$

$$z = (\ell t), z' = (\ell' t') \in T^2_{[3\times2]}$$

vectors and matrices are written in block-matrix form, vectors as 1-dimensional arrays,

$$\mathsf{X}_{[3\times2]} = \begin{array}{c} \left[\begin{array}{c} \phi_{01} \\ \phi_{00} \end{array}\right] \\ \left[\begin{array}{c} \phi_{11} \\ \phi_{10} \end{array}\right] \\ \left[\begin{array}{c} \phi_{21} \\ \phi_{20} \end{array}\right] \end{array}, \qquad \mathsf{M}_{[3\times2]} = \begin{array}{c} \left[\begin{array}{c} m_{01} \\ m_{00} \end{array}\right] \\ \left[\begin{array}{c} m_{11} \\ m_{10} \end{array}\right] \\ \left[\begin{array}{c} m_{21} \\ m_{20} \end{array}\right] \end{array}$$

and the orbit Jacobian matrix as

$$
\mathcal{J}_{[3\times2]} = \begin{pmatrix}
-2s & 2 & 1 & 0 & 0 & 0 & 1 & 0 \\
2 & -2s & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & -2s & 2 & 1 & 0 & 0 & 0 \\
0 & 1 & 2 & -2s & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & -2s & 2 & 1 & 0 \\
0 & 0 & 0 & 1 & 2 & -2s & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & -2s & 2 \\
0 & 1 & 0 & 0 & 0 & 1 & 2 & -2s
\end{pmatrix}
$$

The ‘fundamental fact’ now yields the number of solutions

$$
N_{[3\times2]} = |\mathrm{Det}\,\mathcal{J}_{[3\times2]}| = 4(s-2)s(2s-1)^2(2s+3)^2
$$

counting spatiotemporal cat solutions

1. can construct Bravais spacetime tilings, from small tiles to as large as you wish
2. for each Bravais spacetime tile $[L \times T]_S$, can evaluate

$$N_{[L \times T]_S}$$

the number of doubly-periodic lattice states for a Bravais tile

short tiles are exponentilly good approximations to longer ones (shadowing), so can attain any desired accuracy

spatiotemporal cat topological zeta function

again, can evaluate

$$N_{[L\times T]_S}$$

number of doubly-periodic lattice states for any $[L \times T]_S$ Bravais tile

the generating function for solution counting?

substitute the number of periodic points N_n into the topological zeta function

$$\begin{aligned} 1/\zeta_{\mathrm{top}}(z) &= \exp\left(-\sum_{n=1} \frac{z^n}{n} N_n\right) \\ &= ?? \end{aligned}$$

not solved :(

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insight 1 : how is turbulence described?

not by the evolution of an initial state

exponentially unstable system have finite (Lyapunov) time and space prediction horizons

but

by enumeration of admissible field configurations

and their natural weights

insight 2 : symbolic dynamics for turbulent flows

applies to all coupled-map lattices, and
all PDEs with translational symmetries

a *d*-dimensional spatiotemporal field configuration

$$\{\phi_z\} = \{\phi_z, z \in \mathbb{Z}^d\}$$

is labelled by a *d-dimensional* spatiotemporal block of symbols

$$\{m_z\} = \{m_z, z \in \mathbb{Z}^d\}\,,$$

rather than a single temporal symbol sequence

(as is done when describing a small coupled few-“particle” system, or a small computational domain).

insight 3 : description of turbulence by invariant 2-tori

1 time, 0 space dimensions

a phase space point is *periodic* if its orbit returns to it after a finite time T; such orbit tiles the time axis by infinitely many repeats

1 time, d-1 space dimensions

a phase space point is *spatiotemporally periodic* if it belongs to an invariant d-torus $\mathcal{R}$,
i.e., a block $\mathsf{M}_{\mathcal{R}}$ that tiles the lattice state M,
with period ℓ_j in jth lattice direction

but, is this

chaos?

is spatiotemporal cat ergodic?

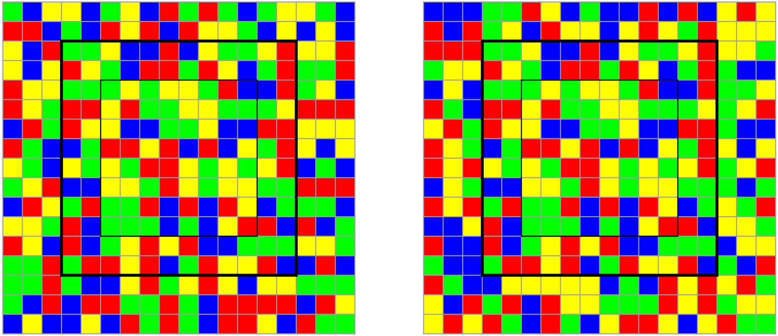
the state at each site is coded with (color) alphabet

$$m_{t\ell} \in \mathcal{A} = \{\underline{1}, 0, 1, 2, \cdots\} = \{\textit{red}, \textit{green}, \textit{blue}, \textit{yellow}, \cdots\}$$

indicating the state $\phi_{t\ell}$ at the lattice site $t\ell$

in deterministic chaos any non-wandering set orbit can be shadowed

shadowing, symbolic dynamics space

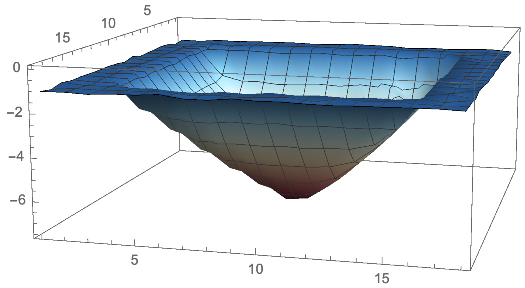


2d symbolic representation of two invariant 2-tori shadowing each other within the shared block $M_{\mathcal{R}}$

- border $\mathcal{R}$ (thick black)
- symbols outside $\mathcal{R}$ differ

$s = 7$ Saremi 2017

shadowing



the logarithm of the average of the absolute value of site-wise distance

$$\ln |X_{2,z} - X_{1,z}|$$

averaged over 250 solution pairs
emphasizes the exponential falloff of the distance around the center of the shared block $\mathcal{R}$

shadowing, phase space

$\Rightarrow$ within the interior of the shared block
the shadowing is exponentially close

zeta function for a field theory ???

"periodic orbits" are now spacetime tilings p

$$Z(s) \approx \sum_p \frac{e^{-A_p s}}{\left|\det\left(1 - J_p\right)\right|}$$

tori / spacetime tilings : each of area $A_p = L_p T_p$

symbolic dynamics : d-dimensional

essential to encode shadowing

at this time :

- $d = 1$ cat map zeta function works like charm
- $d = 2$ spatiotemporal cat works
- $d \geq 2$ Navier-Stokes zeta is still but a dream

summary

spatiotemporal cat

summary

1. goal : describe states of turbulence in infinite spatiatemporal domains
2. theory : classify, enuremate all spatiotemporal tilings
3. the simplest model of “turbulence” : spatiotemporal cat

there is no more time

there is only enumeration of admissible spacetime field configurations

goodbye

to long time and/or space integrators

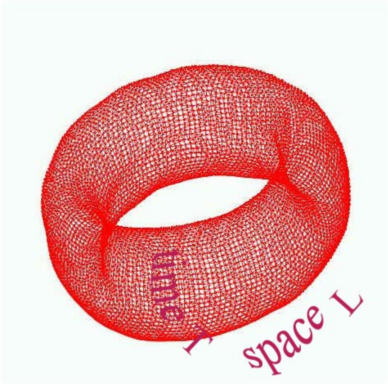
they never worked and could never work

XXX

XXX

take chronotopes to be spatiotemporally compact solutions

periodic spacetime : 2-torus



after the space and time Fourier transforms, obtain

the simplest of chaotic field theories ?

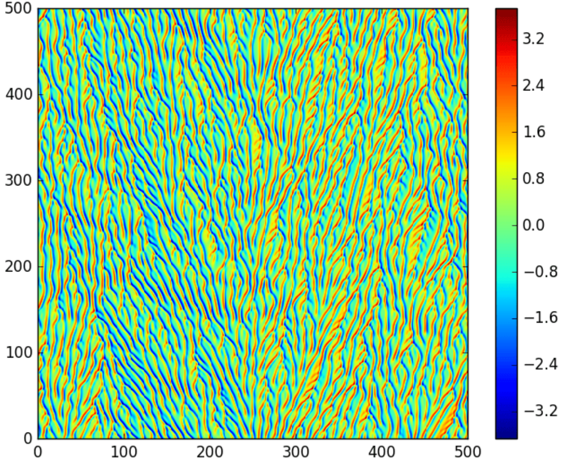
a description of
the admissible Kuramoto-Sivashinsky,
complex Ginzburg-Landau or Navier-Stokes
field configurations is still out of our reach

we need a simple exact model to hone our intuition

spatiotemporal cat does that

an example of large spacetime domain field configuration

Kuramoto-Sivashinsky



[horizontal] space $\phi \in [0, L]$ [up] time evolution

describe this!

now and forever you will be able to distinguish
a Kuramoto-Sivashinsky field configuration vs.
a complex Ginzburg-Landau field configuration

we need the corresponding

alphabets of spatiotemporal patterns (chronotopes)

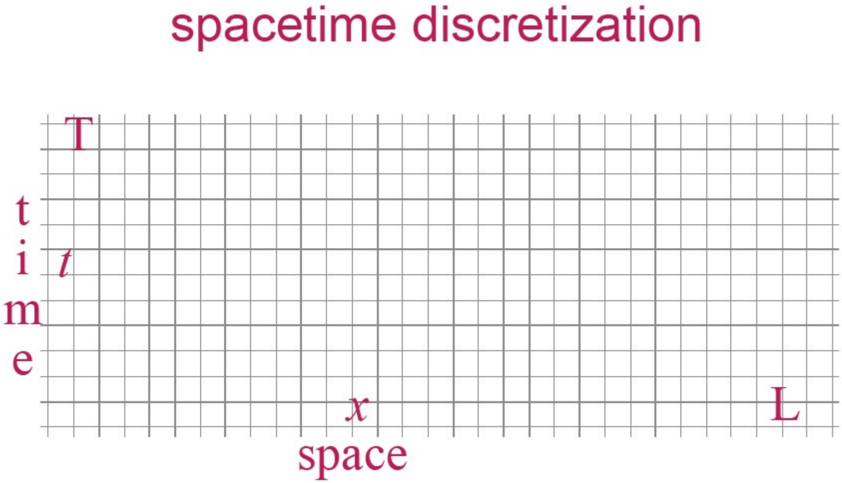
and grammars of admissible ways of joining them

spatiotemporal cat

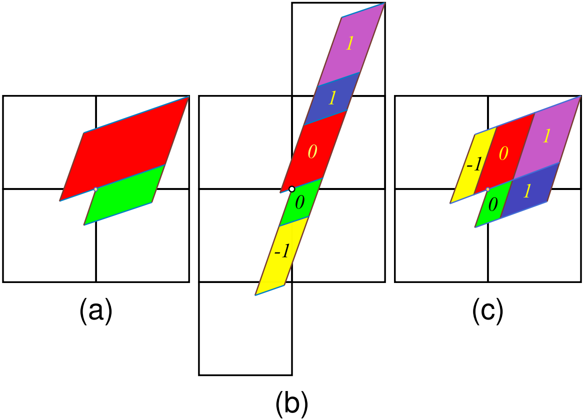
teaches us that

spacetime lattice sites $z = (x, t) \in (-\infty, \infty) \times (-\infty, \infty)$

continuous symmetries : space, time translations



cat map generating partition of the unwrapped torus



(b) mapped step forward in time, the rectangles are stretched along the unstable direction and shrunk along the stable direction

(c) sub-rectangles $\mathcal{M}_j$ that have to be translated back into the partition are indicated by color and labeled by their lattice translations $m_j \in \mathcal{A} = \{\underline{1}, 0, 1\}$