is space time?
a spatio-temporal theory of transitional turbulence

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Session C58: Large Deviations and the Butterfly Effect
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1. what this talk is about
2. turbulence in small domains
3. turbulence in large domains
4. coupled cat maps lattice
5. space is time
6. bye bye, dynamics

do clouds solve PDEs?

do clouds integrate Navier-Stokes equations?

NO!

$\Longrightarrow$ other swirls $\Longrightarrow$

do clouds satisfy Navier-Stokes equations?

yes!

they satisfy them locally, everywhere and at all times

part 1

1. turbulence in small domains
2. turbulence in large domains
3. coupled cat maps lattice
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5. bye bye, dynamics

goal : enumerate the building blocks of turbulence

Navier-Stokes equations

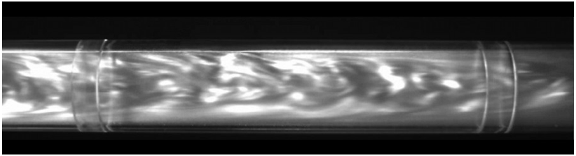
$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{1}{R}\nabla^2 \mathbf{v} - \nabla p + \mathbf{f}\,, \qquad \nabla \cdot \mathbf{v} = 0,$$

velocity field $\mathbf{v} \in \mathbb{R}^3$; pressure field p ; driving force $\mathbf{f}$

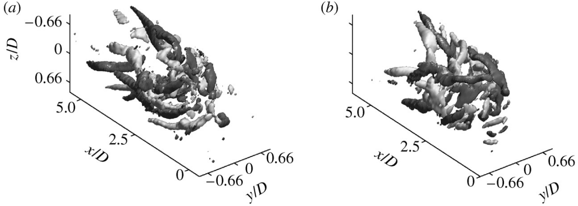
describe turbulence

starting from the equations (no statistical assumptions)

pipe experiments



T. Mullin lab



B. Hof lab

pipe theory and numerics

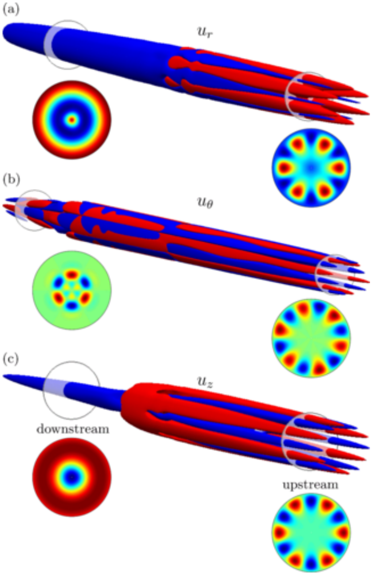
amazing experiments!
amazing numerics!
beautiful visualizations !

“Exact Coherent Structures” :
numerical Navier-Stokes

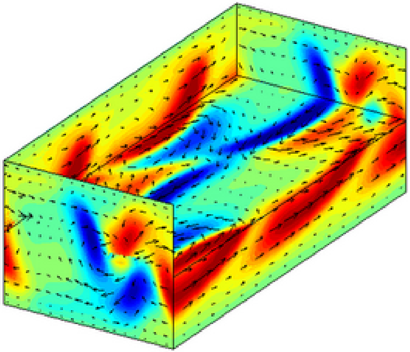
isosurfaces and cross sections
of the streamwise velocity

red (blue) streaks
are faster (slower)
than the base flow

Ritter et al., Phys. Rev. Fluids (2018)



so far, successful only for **small** computational cells

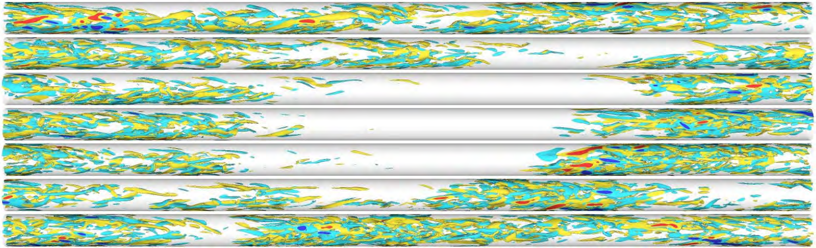


plane Couette minimal cell, etc : 100's of exact solutions

part 2

1. turbulence in small domains
2. **turbulence in large domains**
3. coupled cat maps lattice
4. space is time
5. bye bye, dynamics

can simulate **large** computational domains



pipe flow close to onset of turbulence [1]

but we have hit a wall :

exact coherent structures are too unstable to compute

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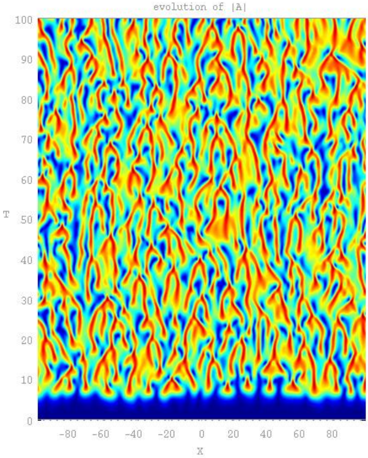
[1] M. Avila and B. Hof, Phys. Rev. **E 87** (2013)

another large space-time domain simulation

complex Ginzburg-Landau

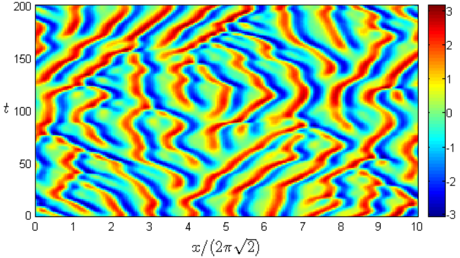


(will return to this)

[horizontal] space $x \in [-L/2, L/2]$ [up] time evolution

codeinthehole.com/static/tutorial/coherent.html

a test bed : Kuramoto-Sivashinsky on a large domain



[horizontal] space $x \in [0, L]$ [up] time evolution

- turbulent behavior
- simpler physical, mathematical and computational setting than Navier-Stokes

(1+1) space-time dimensional "Navier-Stokes"

conceptually not ready yet to explore
$(1+3)$-dimensional turbulence - start instead with

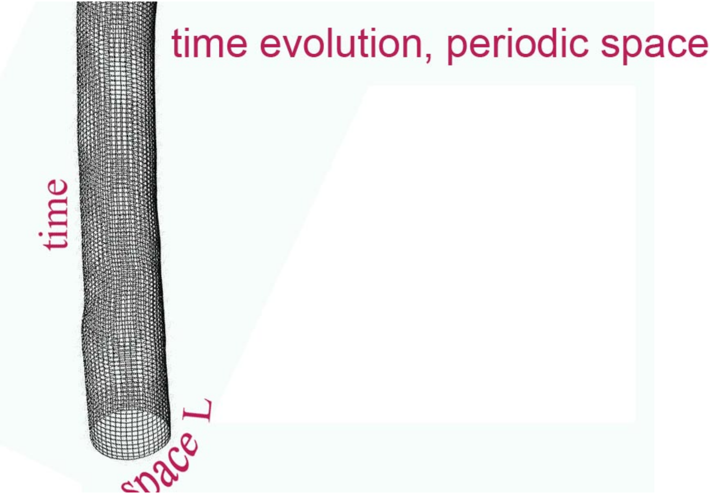
Kuramoto-Sivashinsky $(1+1)$-dimensional PDE

$$u_t + u\nabla u \;=\; -\nabla^2 u - \nabla^4 u\,, \qquad x \in [-L/2, L/2]\,,$$

describes spatially extended systems such as

- flame fronts in combustion
- reaction-diffusion systems
- …

compact space, infinite time cylinder



so far : Navier-Stokes on compact spatial domains, all times

compact space, infinite time

Kuramoto-Sivashinsky equation

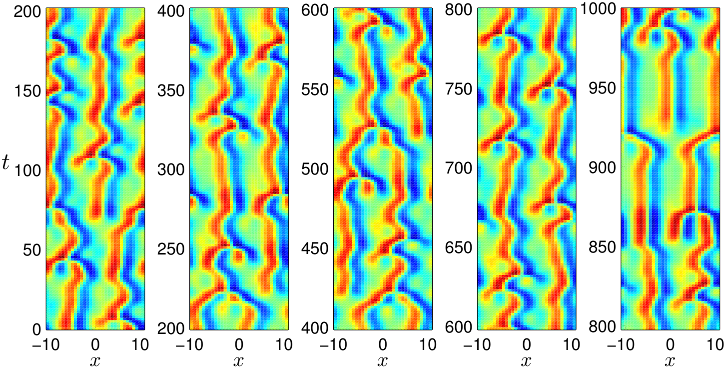
$$u_t = -(+\nabla^2 + \nabla^4)u - u\nabla u \,, \qquad x \in [-L/2, L/2] \,,$$

in terms of discrete spatial Fourier modes

N ordinary differential equations (ODEs) in time

$$\dot{\tilde{u}}_k(t) = (q_k^2 - q_k^4)\,\tilde{u}_k(t) - i\frac{q_k}{2}\sum_{k'=0}^{N-1}\tilde{u}_{k'}(t)\tilde{u}_{k-k'}(t) \,.$$

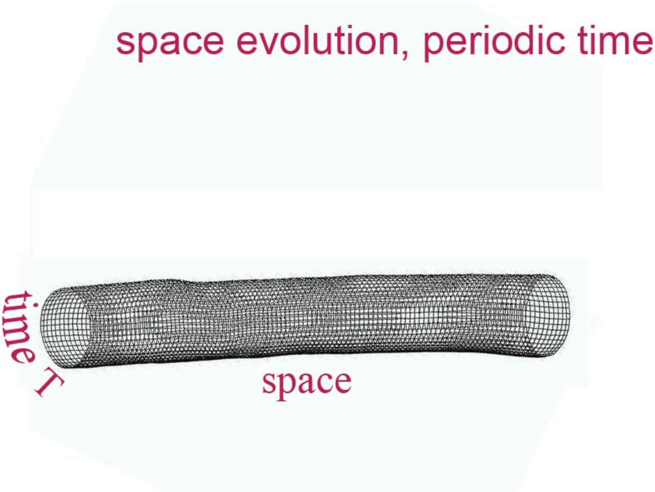
evolution of Kuramoto-Sivashinsky on small $L = 22$ cell



horizontal: $x \in [-11, 11]$
vertical: time
color: magnitude of $u(x, t)$

is space time?

compact time, infinite space cylinder



compact time, infinite space

Kuramoto-Sivashinsky as four fields 1st order in spatial derivatives

$$
\begin{aligned}
u_t &= -uu_x - u_{xx} - u_{xxxx}\,, \\
u^{(0)} &\equiv u\,, \quad u^{(1)} \equiv u_x\,, \quad u^{(2)} \equiv u_{xx}\,, \quad u^{(3)} \equiv u_{xxx}
\end{aligned}
$$

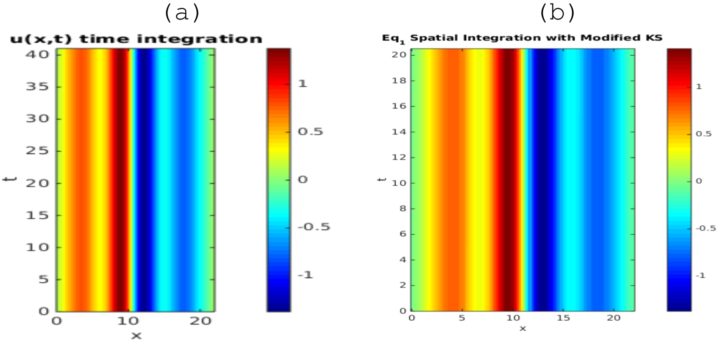
evolve four 1st order PDEs $u^{(j)}(t,x)$ in x,

periodic boundary condition in time $u(x,t) = u(x,t+T)$

$$
\begin{aligned}
u_x^{(0)} &= u^{(1)}\,, \quad u_x^{(1)} = u^{(2)}\,, \quad u_x^{(2)} = u^{(3)} \\
u_x^{(3)} &= -u_t^{(0)} - u^{(2)} - u^{(0)}u^{(1)}
\end{aligned}
$$

initial $u^{(0)}(x_0,t)$, $u^{(1)}(x_0,t)$, $u^{(2)}(x_0,t)$, $u^{(3)}(x_0,t)$ specified for $t \in [0,T)$, at a fixed space point x_0

a time-invariant equilibrium, spatial periodic orbit

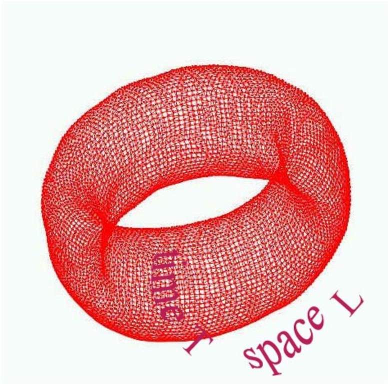


evolution of EQ_1 : (a) in time, (b) in space
initial condition for the spatial integration is the time strip $u(x_0, t)$, $t = [0, T)$, where time period $T = 0$, spatial x period is $L = 22$.

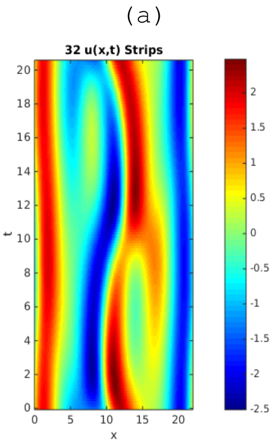
Michelson 1986, Gudorf 2016

spatiotemporally compact solutions

periodic spacetime : 2-torus



a spacetime invariant 2-torus integrated in either time or space



(a) old : time evolution. (b) new : space evolution
$x = [0, L]$ initial condition : time periodic line $t = [0, T]$

Gudorf 2016

but integrations are uncontrollably unstable

neither time nor space integration works
for large domains

rethink the calculation

every compact solution is a fixed point on a discrete lattice

discretize $u_{nm} = u(x_n, t_m)$ over NM points of spatiotemporal periodic lattice $x_n = nL/N$, $t_m = mT/M$, Fourier transform :

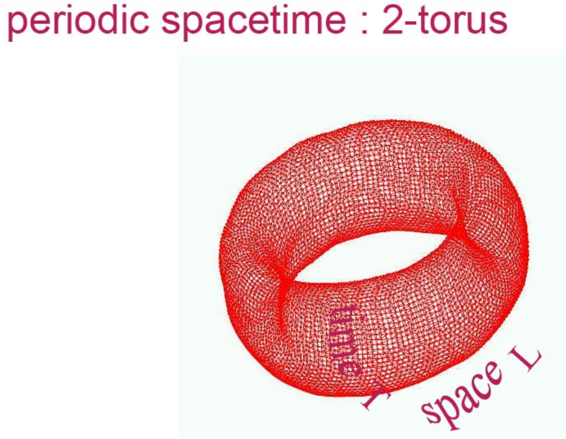
$$\tilde{u}_{k\ell} = \frac{1}{NM} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} u_{nm}\, e^{-i(q_k x_n + \omega_\ell t_m)}\,, \quad q_k = \frac{2\pi k}{L}\,,\ \omega_\ell = \frac{2\pi \ell}{T}$$

Kuramoto-Sivashinsky is no more a PDE / ODE,
but an algebraic $[N \times M]$-dimensional fixed point problem
of determining a solution to

$$\left[-i\omega_\ell - (q_k^2 - q_k^4)\right] \tilde{u}_{k\ell} + i\frac{q_k}{2} \sum_{k'=0}^{N-1} \sum_{m'=0}^{M-1} \tilde{u}_{k'm'} \tilde{u}_{k-k',m-m'} = 0$$

every calculation is a spatiotemporal lattice calculation

field is discretized as $\tilde{u}_{k\ell}$ values
over NM points of a periodic lattice



there is no more space or time evolution

solution to Kuramoto-Sivashinsky is now given as a condition
that at each lattice point $k\ell$
the tangent field at $\tilde{u}_{k\ell}$
satisfies the equation of motion

$$\left[-i\omega_\ell - (q_k^2 - q_k^4)\right] \tilde{u}_{k\ell} + i\frac{q_k}{2}\sum_{k'=0}^{N-1}\sum_{m'=0}^{M-1}\tilde{u}_{k'm'}\tilde{u}_{k-k',m-m'} = 0$$

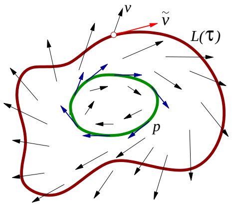
this is a local tangent field constraint on a global solution

robust : no exponential instabilities as there are no finite time / space integrations

how to find solutions ? a 1-dimensional example

guess loop tangent $\tilde{v}(\tilde{x}) \neq v(\tilde{x})$

periodic orbit $\tilde{v}(\tilde{x})$, $v(\tilde{x})$ aligned



cost function

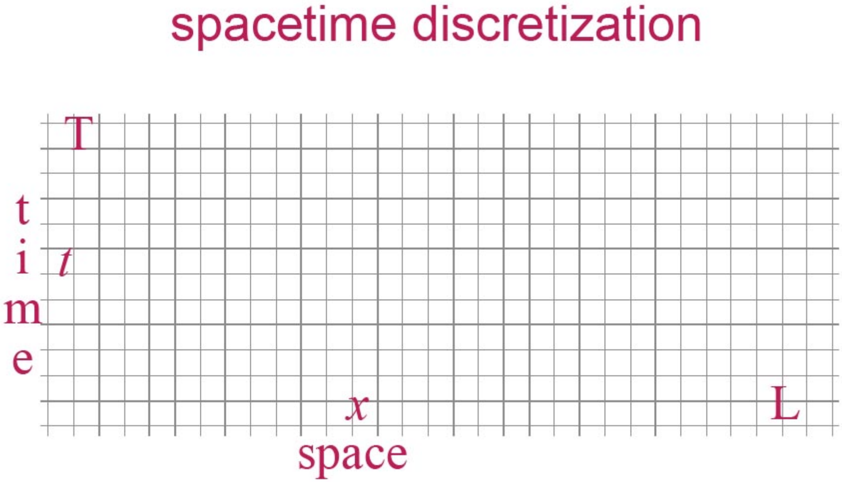
$$F^2[\tilde{x}] = \oint_L ds\,(\tilde{v} - v)^2\,; \quad \tilde{v} = \tilde{v}(\tilde{x}(s,\tau))\,,\ v = v(\tilde{x}(s,\tau))\,,$$

penalize[2] misorientation of the loop tangent $\tilde{v}(\tilde{x})$ relative to the true dynamical flow tangent field $v(\tilde{x})$

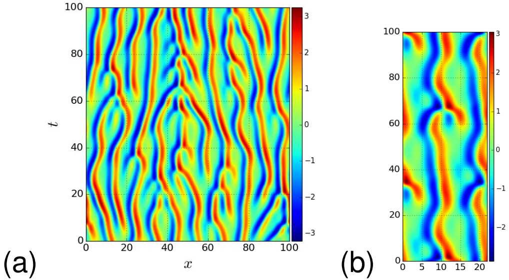
[2] Lan and Cvitanović, Phys. Rev. (2004)

spacetime lattice sites $(x, t) \in (-\infty, \infty) \times (-\infty, \infty)$

continuous symmetries : space, time translations



a crude lattization of turbulence

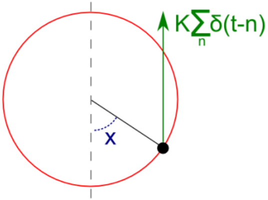


model
(a) global turbulent field
by a lattice of nearest-neighbor coupled
(b) “minimal” turbulent cells

next : pick the simplest such model

example of a “small domain dynamics” : kicked rotor

an electron circling an atom, subject to
a discrete time sequence of angle-dependent kicks $F(x_t)$



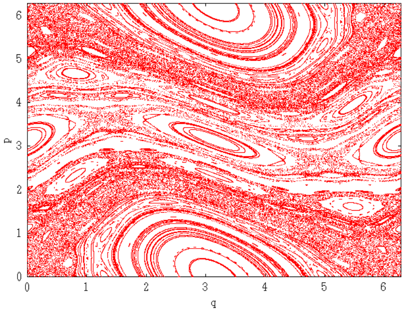
Taylor, Chirikov and Greene standard map

$$
\begin{aligned}
x_{t+1} &= x_t + p_{t+1} \qquad \text{mod } 1, \\
p_{t+1} &= p_t + F(x_t)
\end{aligned}
$$

$\rightarrow$ chaos in Hamiltonian systems

standard map

example of chaos in a Hamiltonian system



the simplest example : a cat map evolving in time

force $F(x) = Kx$ linear in the displacement x , $K \in \mathbb{Z}$

$$\begin{aligned} x_{t+1} &= x_t + p_{t+1} \qquad \bmod 1 \\ p_{t+1} &= p_t + Kx_t \qquad \bmod 1 \end{aligned}$$

Continuous Automorphism of the Torus, or

Hamiltonian cat map

a linear, area preserving map of a 2-torus onto itself

$$\begin{pmatrix} x_{t+1} \\ p_{t+1} \end{pmatrix} = A \begin{pmatrix} x_t \\ p_t \end{pmatrix} \quad \bmod 1 \,, \qquad A = \begin{pmatrix} s-1 & 1 \\ s-2 & 1 \end{pmatrix}$$

for integer $s = \operatorname{tr} A > 2$ the map is
hyperbolic $\to$ a fully chaotic Hamiltonian dynamical system

cat map in Lagrangian form

replace momentum by velocity

$$p_{t+1} = (x_{t+1} - x_t)/\Delta t$$

dynamics in (x_t, x_{t-1}) state space is particularly pretty[3]

2-step difference equation

$$x_{t+1} - s\,x_t + x_{t-1} = -m_t$$

unique integer m_t ensures that

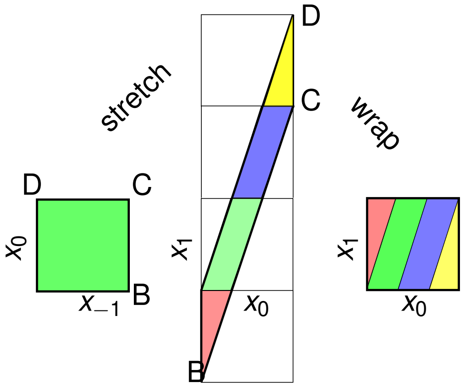
x_t lands in the unit interval at time step t

nonlinearity : mod 1 operation, encoded in

$$m_t \in \mathcal{A}\,, \quad \mathcal{A} \;=\; \text{finite alphabet of possible values for } m_t$$

[3] Percival and Vivaldi, Physica D (1987)

example : $s = 3$ cat map symbolic dynamics



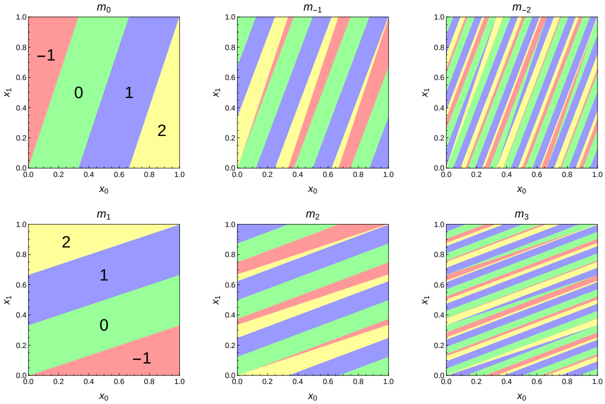
cat map stretches the unit square
translations by

$$m_t \in \mathcal{A} = \{\underline{1}, 0, 1, 2\} = \{\text{red}, \text{green}, \text{blue}, \text{yellow}\}$$

return stray kittens back to the torus

ask for bonus slides to find out why this is stupid :)

cat map (x_0, x_1) state space partition



(a) 4 regions labeled by $m_0.$, obtained from (x_{-1}, x_0) state space by one iteration
(b) 14 regions, 2-steps past $m_{-1}m_0.$ (c) 44 regions, 3-steps past $m_{-2}m_{-1}m_0.$

(d) 4 regions labeled by future $.m_1$

(e) 14 regions, 2-steps future $.m_1 m_2$ (f) 44 regions, 3-steps future block $m_3 m_2 m_1.$

spatiotemporally infinite spatiotemporal cat map

spatiotemporal cat map

consider a 1 spatial dimension lattice, with field x_{nt}
(the angle of a kicked rotor "particle" at instant t, at site n)

require

- each site couples to its nearest neighbors $x_{n\pm1,t}$
- invariance under spatial translations
- invariance under spatial reflections
- invariance under the space-time exchange

obtain[4]

2-dimensional coupled cat map lattice

$$x_{n,t+1} + x_{n,t-1} - s\, x_{nt} + x_{n+1,t} + x_{n-1,t} = -m_{nt}$$

[4] Gutkin and Osipov, Nonlinearity (2016)

write the spatial-temporal differences as discrete derivatives

Laplacian : in $d = 1$ and $d = 2$ dimensions

$$
\begin{aligned}
\Box x_t &= x_{t+1} - 2x_t + x_{t-1} \\
\Box x_{n,t} &= x_{n,t+1} + x_{n,t-1} - 4\, x_{n,t} + x_{n+1,t} + x_{n-1,t}
\end{aligned}
$$

$\rightarrow$ the cat map is thus generalized to

d-dimensional spatiotemporal cat map

$$(\Box - s + 2d)x_z = m_z$$

where $x_z \in \mathbb{T}^1\,, \quad m_z \in \mathcal{A}$ and $z \in \mathbb{Z}^d$ = lattice site label

discretized linear PDE

d-dimensional spatiotemporal cat map

$$(\Box - s + 2d)x_z = m_z$$

is linear and known as

- Helmholtz equation if stretching is weak, $s < 2d$ (sines, cosines)
- damped Poisson equation if stretching is strong, $s > 2d$ (sinches, coshes)

the nonlinearity is hidden in the “source”

$$m_z \in \mathcal{A} \text{ at lattice site } z \in \mathbb{Z}^d$$

solving cat map using Green's functions

the Green's function for a period T solution of

$$(\Box - s + 2)x_z = m_z$$

is a Topelitz matrix g that satisfies

$$(\mathcal{D}\mathrm{g})_{nn'} \;=\; \delta_{nn'}\,, \qquad n, n' \in 0, 1, 2, \cdots, T-1$$

$$\mathcal{D} \;=\; \begin{pmatrix} s & -1 & 0 & 0 & \dots & 0 & -1 \\ -1 & s & -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & s & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \dots & \dots & s & -1 \\ -1 & 0 & \dots & \dots & \dots & -1 & s \end{pmatrix}$$

1) symbolic dynamics for turbulent flows

an insight that applies to
all coupled-map lattices, and
all PDEs with translational symmetries

a d-dimensional spatiotemporal pattern
$\{x_z\} = \{x_z, z \in \mathbb{Z}^d\}$

is labelled by a *d-dimensional* spatiotemporal block of symbols
$\{m_z\} = \{m_z, z \in \mathbb{Z}^d\}$,

rather than a single temporal symbol sequence

(as is done when describing a small coupled few-"particle" system, or a small computational domain).

2) periodic orbits generalize to d-tori

1 time, 0 space dimensions

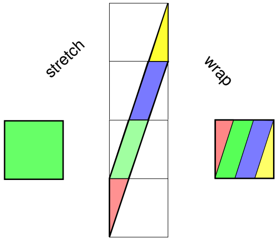
a state space point is *periodic* if its orbit returns to it after a finite time T; such orbit tiles the time axis by infinitely many repeats

1 time, d-1 space dimensions

a state space point is *spatiotemporally periodic* if it belongs to an invariant d-torus $\mathcal{R}$,
i.e., a block $\mathsf{M}_{\mathcal{R}}$ that tiles the lattice state M,
with period ℓ_j in jth lattice direction

remember ? spatiotemporal cat map symbolic dynamics

dynamics at each site



is coded by the dynamical state space partition
with (color) alphabet

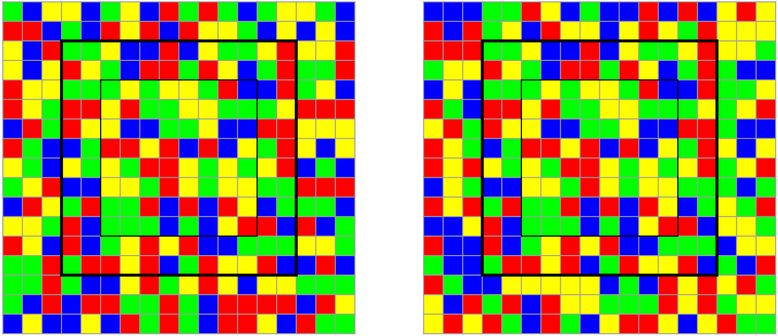
$$m_{t\ell} \in \mathcal{A} = \{\underline{1}, 0, 1, 2\} = \{\textit{red}, \textit{green}, \textit{blue}, \textit{yellow}\}$$

indicating the coarse-grained state of $x_{t\ell}$ at the lattice site $t\ell$

is spatiotemporal cat map ergodic?

yes! any orbit can be shadowed

shadowing, symbolic dynamics space



2d symbolic representation of two invariant 2-tori shadowing each other within the shared block $M_{\mathcal{R}} = M_{\mathcal{R}_0} \cup M_{\mathcal{R}_1}$ (blue)

- border $\mathcal{R}_1$ (thick black), interior $\mathcal{R}_0$ (thin black)
- symbols outside $\mathcal{R}$ differ

$s = 7$ Saremi 2017

solve using the linear equation

$$(\Box - s + 4)x_z = m_z$$

the Green's function g for a $L \times T$ lattice

yields the state x_z at every lattice point $z = \{\ell n\}$

$$x_z = \frac{1}{\Box - s + 4} m_z$$

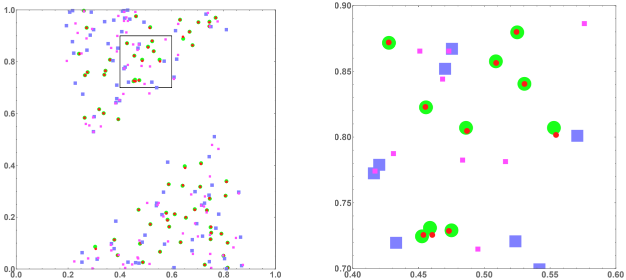
where

$$\mathsf{g} = \frac{1}{\Box - s + 4}$$

is a Topelitz (tensor) matrix, and the integer s is the local stretching rate (how chaotic is each site)

visualize all x_z by plotting them in the $(x_{\ell n}, x_{\ell, n+1})$ unit square

shadowing, state space



(left) state space points $(x_{\ell t}, x_{\ell,t+1})$ of the two invariant 2-tori
(right) a zoom

interior points $\in \mathcal{R}_0$ (large green), (small red) circles
border points $\in \mathcal{R}_1$ (large violet), (small magenta) squares

$\Rightarrow$ within the interior of the shared block
the shadowing is exponentially close

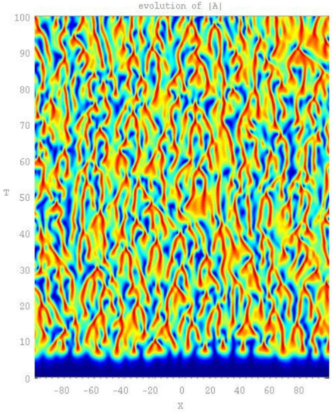
1. turbulence in small domains
2. turbulence in large domains
3. coupled cat maps lattice
4. **space is time**
5. bye bye, dynamics

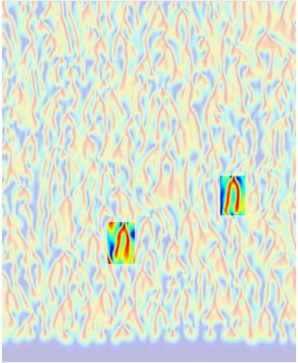
yes, lattice schmatiz, but

does it work for PDEs?

space-time complex Ginzburg-Landau on a large domain

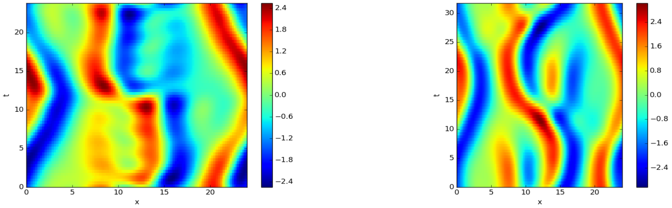
goal : enumerate nearly recurrent chronotopes





[left-right] space $x \in [-L/2, L/2]$ [up] time $t \in [0, T]$

KS invariant 2-tori can be found variationally



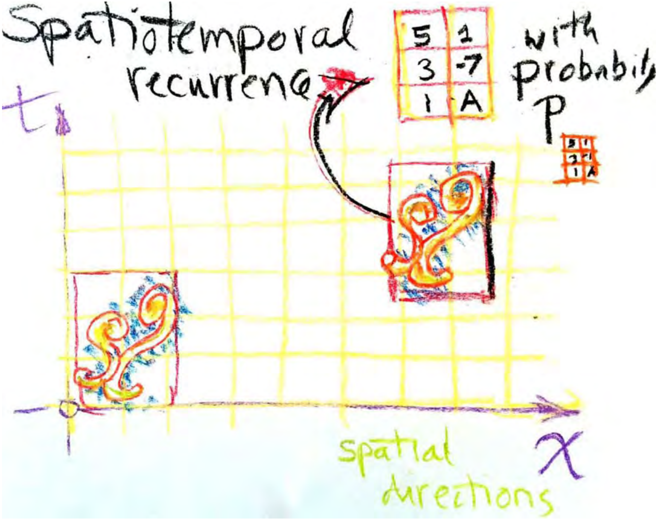
(a) (b)

(a) a rough initial guess from a numerical close recurrence
(b) the spatiotemporal fixed point found variationally
spatial and time periods
$L = 24.0714310445$, $T = 31.8597201649$

- spatial and time periods intrinsic to each solution (no discrete lattice)
- no robust code as yet
- no symbolic dynamics as yet

Gudorf 2018

to be done : 2D symbolic dynamics $\in (-\infty, \infty) \times (-\infty, \infty)$



zeta function for a field theory ? much like Ising model

"periodic orbits" are now spacetime tilings p

$$Z(s) \approx \sum_p \frac{e^{-A_p s}}{\left|\det\left(1 - J_p\right)\right|}$$

tori / spacetime tilings : each of area $A_p = L_p T_p$

symbolic dynamics : d-dimensional

essential to encode shadowing

at this time :

- $d = 1$ cat map zeta function works like charm
- $d = 2$ Spatiotemporal cat map should be within reach
- $d \geq 2$ Navier-Stokes zeta is still but a dream

part 5

1. turbulence in small domains
2. turbulence in large domains
3. coupled cat maps lattice
4. space is time
5. **bye bye, dynamics**

goodbye

to long time and/or space integrators

they never worked and could never work

the trouble:
forward time-integration codes too unstable

multishooting inspiration: replace a guess that a point is on the periodic orbit by a guess of the entire orbit.

$\rightarrow$

a variational method for finding spatio-temporally periodic solutions of classical field theories

compute locally, adjust globally

computing literature

parallelizing spatiotemporal computation is FLOPs intensive, but more robust than integration forward in time

it's rocket science[5]

[5] Q. Wang *et al.*, *Towards scalable parallel-in-time turbulent flow simulations*, Physics of Fluids (2013)

clouds do not solve PDEs

do clouds integrate Navier-Stokes equations?

NO!

$\Longrightarrow$ other swirls $\Longrightarrow$

at any spacetime point Navier-Stokes equations describe the local tangent space

they satisfy them locally, everywhere and at all times

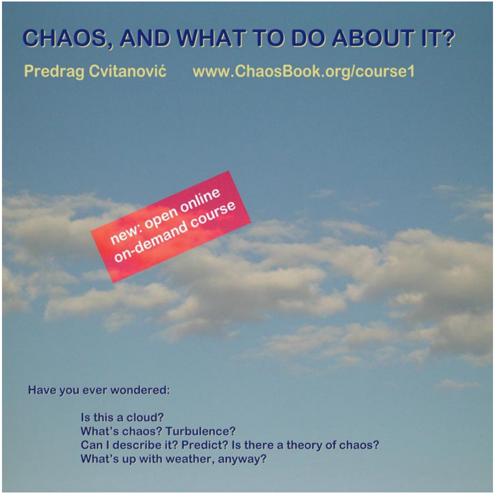
summary

1. small computational domains reduce “turbulence” to “single particle” chaos
2. consider instead turbulence in infinite spatiatemporal domains
3. theory : classify all spatiotemporal tilings
4. numerics : future is spatiotemporal

there is no more time

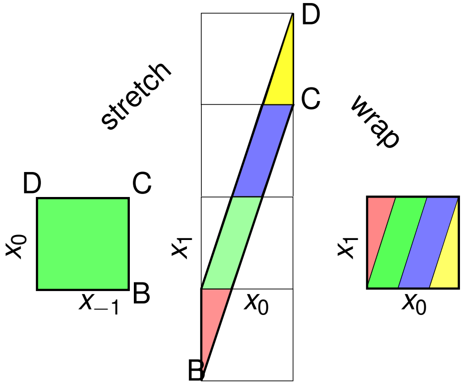
there is only enumeration of spacetime solutions

what is next ? take the course!



student raves :
...10^6 times harder than any other online course...

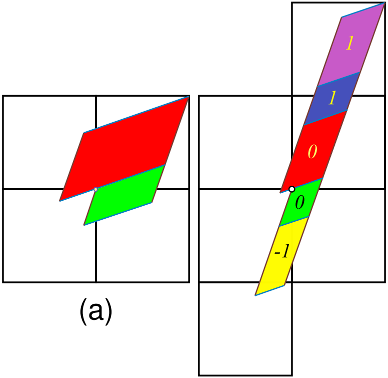
bonus slides : Percival-Vivaldi unit square is not a partition



forward iteration of the unit square yields a grammar with infinity of longer and longer inadmissible sequences (pruning rules)

that's stupid

bonus slides : $s = 3$ cat map generating partition

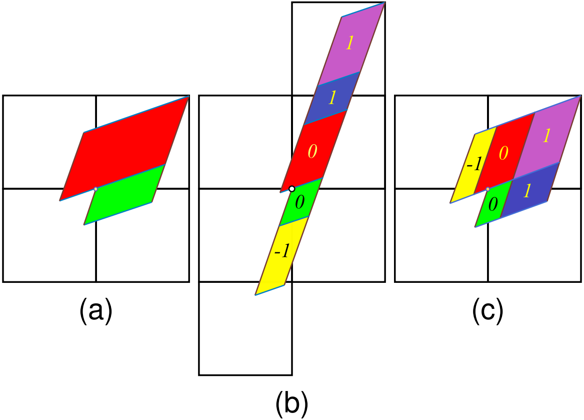


(a)

(b)

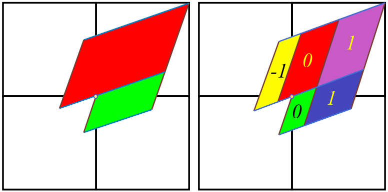
(a) An Adler-Weiss generating partition of the unit torus with rectangles $\mathcal{M}_A$ (red) and $\mathcal{M}_B$ (green) with borders given by the cat map stable (blue) and unstable (dark red) manifolds.

bonus slides : cat map generating partition



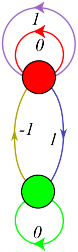
(b) Mapped one step forward in time, the rectangles are stretched along the unstable direction and shrunk along the stable direction. Sub-rectangles $\mathcal{M}_j$ that have to be translated back into the partition are indicated by color and labeled by their lattice translations $m_j \in \mathcal{A} = \{\underline{1}, 0, 1\}$

bonus slides : cat map generating partition



(a)

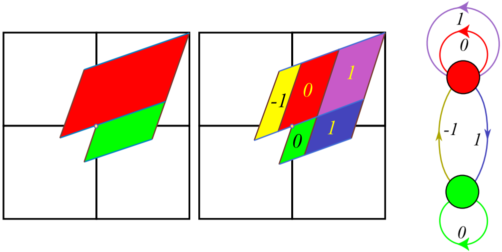
(c)



(d)

(c) The sub-rectangles $\mathcal{M}_j$ yield a generating partition, with
(d) the finite grammar given by the finite transition graph.
The nodes refer to the rectangles A and B,
the five links correspond to the five sub-rectangles
induced by one step forward-time dynamics.

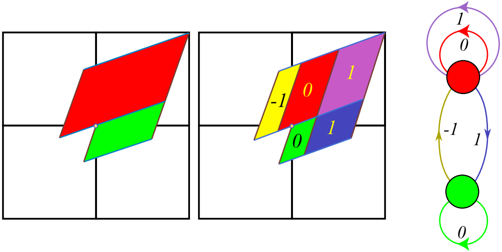
bonus slides : cat map Perron-Frobenious operator



the two-rectangle partition has [2×2] Markov matrix, where one sums over all admissible transitions:

$$
\begin{aligned}
\begin{bmatrix} \phi'_A \\ \phi'_B \end{bmatrix} &= L\phi = \begin{bmatrix} L_{A^0A} + L_{A^1A} & L_{A^{\underline{1}}B} \\ L_{B^1A} & L_{B^0B} \end{bmatrix} \begin{bmatrix} \phi_A \\ \phi_B \end{bmatrix} \\
L &= \frac{1}{\Lambda} \begin{bmatrix} 2 & \Lambda - 2 \\ \Lambda - 1 & 1 \end{bmatrix}
\end{aligned}
$$

bonus slides : cat map Perron-Frobenious eigenvalues



the two-rectangle partition has [2×2] Markov matrix, with eigenvalues given by the zeros of the Fredholm determinant (Zeta function; Markov graph determinant)

$$\det(1 - zL) = 1 - 3\frac{z}{\Lambda} - \frac{z^2}{\Lambda}(\Lambda - 3)$$

that's it! lattice damped Poisson equation

$$(\Box - s + 2d)x_z = m_z$$

solved completely and analytically!