$2D$ Ising Model through a Graph Zeta Function

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Outline

Part I

i. Ihara's graph zeta function

ii. Onsager's solution for planar Ising models

iii. The Feynman - Sherman theorem
(Ihara + planarity + a simplifying algebraic observation)

iv. Interpreting the Kac - Ward resolvents
(it yields an "order - disorder" correlation function)

Part II

v. Pfaffian structure of related correlation functions

vi. A step beyond planarity: emergent planarity
at the critical points of non-planar 2D models

vii. Question: how far do emergent structures populate our world?

Based on joint works with:
Hugo Duminil- Copin, Vincent Tassion, Simone Warzel.

Graph zeta function

Let $\mathcal{M}$ be a matrix indexed by the vertex set $V_{\mathbb{G}}$ of a graph $\mathbb{G}$.
The traces $\operatorname{tr}\mathcal{M}^n$ are given by weighted sums over closed paths on $V_{\mathbb{G}}$.
A closed path is called primitive if if it is not a k^{th} fold repetition of a shorter closed path.

Theorem (Ihara '66, ...)

For any finite graph $\mathbb{G}$, and a $\mathbb{G}\times\mathbb{G}$ matrix satisfying

$$\|\mathcal{M}\|_{\infty,\infty} \;\equiv\; \max_{x\in\mathcal{G}} \sum_{y\in\mathbb{G}} |\mathcal{M}_{x,y}| \;<\; 1\,, \tag{1}$$

$$\boxed{\det(1-u\mathcal{M}) \;=\; \prod_{p^*}\left[1-u^{|p^*|}\chi_{\mathcal{M}}(p^*)\right]} := [\zeta_M(u)]^{-1} \tag{2}$$

with p^* ranging over equivalence classes of primitive closed paths, w.r.t. cyclic reparametrization, and

$$\chi_{\mathcal{M}}(p^*) := \prod_{j=1}^{n} \mathcal{M}_{x_{j+1},x_j}\,. \tag{3}$$

Some notable features:

- While the product on the right expands into an infinite collection of terms, with arbitrarily high powers of u, on the left is a polynomial of a finite degree!
- Such functions satisfy functional relations.
- Notion of Ramanujan graphs (Bass, Hashimoto,.., Stark - Terras, ...)

Onsager's solution of the n.n. Ising model on $\mathbb{Z}^2$

The model: a system of ± 1 valued spins (σ_x) on a graph $\mathcal{G} = V(\mathcal{G}), \mathcal{E}(\mathcal{G})$

$$H(\sigma) \;=\; -\sum_{\{x,y\}\in\mathcal{E}(\mathcal{G})} J_{x,y}\sigma_x\sigma_y - h\sum_{x\in V(\mathcal{G})}$$

Partition function: $Z_{\mathcal{G}}(\beta,h) \;=\; \sum_\sigma e^{-\beta H(\sigma)}$

Pressure: $\boxed{\psi(\beta,h) \;=\; \frac{1}{|\mathcal{G}|}\log Z_{\mathcal{G}}(\beta,h)}$ (or limit thereof)

Theorem (Onsager '44)

On $\mathbb{Z}^2$, the infinite-volume pressure of the 2D ferromagnetic ($J > 0$) Ising model is

$$\boxed{\psi(\beta,0) = \frac{1}{2}\int \ln\left[\left(Y + Y^{-1} - 2\right) + E(k_1,k_2)\right]\frac{dk_1\,dk_2}{(2\pi)^2} \;+\; \frac{1}{2}\ln\cosh(2\beta J)}$$

with

$$Y = Y(\beta) \;:=\; \sinh(2\beta J)\,, \qquad E(k_1,k_2) \;:=\; 2 - [\cos(k_1) + \cos(k_2)] \approx |k|^2/2\,.$$

The free energy for general planar graphs

Theorem

For any planar graph $\mathbb{G}$

$$\boxed{Z_{\mathbb{G}}(\beta) \;/ \prod_{\{x,y\}\subset E_{\mathbb{G}}} \cosh \beta J_{x,y} \;=\; \det(\mathbb{1} - \mathcal{W}K)^{1/2}} \tag{4}$$

with the Kac-Ward matrices defined over the set of oriented edges $\mathcal{E}_0^*$ *as:*

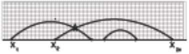
$$\mathcal{W}_{e',e} \;:=\; \exp\left(\tfrac{i}{2}\angle(e',e)\right) \mathbb{1}[o(e') = t(e)\,;\, e' \neq \overline{e}]$$

$$K_{e',e} \;:=\; \delta_{e',e} K_e \qquad \text{at} \quad K_e \;=\; \tanh(\beta J_{x,y}).$$

Kac-Ward '52, Feynman, Sherman '60, Hurst-Green '60, Burgoyne, .., Cimasoni '12, ..

The answer can also be expressed as a Pfaffian.
Recall that for an anti-symmetric $2N \times 2N$ matrix $\mathcal{A}$:

$$\det(\mathcal{A})^{1/2} \;=\; \mathrm{Pf}[\mathcal{A}] \;:=\; \sum_{\text{pairings } \pi} \varepsilon(\pi) \prod_{j=1}^{N} \mathcal{A}_{\pi(2j-1),\pi(2j)}$$



Two implications of planarity

1) **Lemma 1** *For any planar graph, and a symmetric edge function ($K_e = K_{\overline{e}}$)*

$$\mathcal{F}(\{K_e\}) \; = \; \det(\mathbb{1} - K\mathcal{W})$$

is the square of a multilinear function of the parameters $\{K_e\}_{e \in \mathcal{E}_0}$.

(Proven through a reduction to an antisymmetric matrix, and: $\det(\mathcal{A}) = \mathrm{Pf}[\mathcal{A}]^2$.)

2) For any planar loop of oriented non-backtracking edges $\{e_1, e_2, ...\}$

$$\prod_{j=1}^{n} \mathcal{W}_{e_{j+1}, e_j} = (-1)^{w(p^*)} \; = \; (-1)^{n(p^*)} \, .$$

with $w(p^*) =$ winding number, and $n(p^*) = \#\{$ self crossings $\}$ [Whitney's Thm].

This would be combined with the Ihara relation, for matrices indexed by oriented edges:

$$\det(\mathbb{1} - KW)_{\mathcal{E}_0 \times \mathcal{E}_0} \; = \; \prod_{\ell} \left[1 + (-1)^{n(\ell)} \chi_K(\ell) \right]^2$$

the product being over unoriented loops on $\mathbb{G}$ (hence the power 2).

Ihara meets planarity

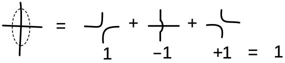
Expanding :

$$\det(\mathbb{1} - KW)^{1/2}_{\mathcal{E}_0 \times \mathcal{E}_0} = \sum_{\mathcal{P}=\{\ell_1,\ell_2,\ldots\}} (-1)^{\sum_j n(\ell_j)} \chi_K(\mathcal{P})$$

where the sum is over finite collections of loops, and $\chi_K(\mathcal{P}) = \prod_j \chi_K(\ell_j)$.

From Lemma 1 we may deduce that there are plenty of cancellations. Denoting $M(\mathcal{P}) =$ the maximal edge multiplicity: $\boxed{\text{only terms with } M(\mathcal{P}) = 1 \text{ survive !}}$

Each multigraph Γ with $M(\Gamma) = 1$ can be realized in multiple ways with differing factors. But at each vertex:



One may conclude that the net factor for each subgraph Γ with no odd end-points the sum of the contributing weights is 1.

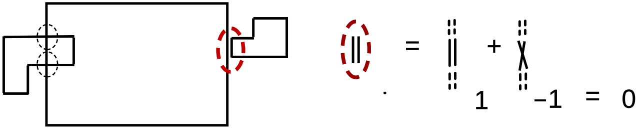
Hence the Kac-Ward conjectured identity: *for any planar graph, and set of symmetric edge weights K*

$$\boxed{\det(\mathbb{1} - KW)^{1/2}_{\mathcal{E}_0 \times \mathcal{E}_0} = \sum_{\mathcal{P}} (-1)^{n(\mathcal{P})} \chi(\mathcal{P}) = \sum_{\substack{\Gamma\,:\,\partial\Gamma=\emptyset \\ M(\Gamma)\le 1}} \chi(\Gamma) = Z_{\mathcal{G}}(\beta, 0)} \tag{5}$$

(The last equality by the 'high temperature' expansion for Z.) Q.E.D

Remark on the previous argument by Feynman

Going beyond the previous calculation of non-repeating diagrams, Feynman started to show cancellations of higher order terms, order by order, using arguments such as:



His conjecture, that at each order the total sum is 0, was established by Sherman, using a well organized combinatorial argument.

The proof outlined above takes a short-cut through the observation that the zeta function relation implies an infinite collection of cancellations.

(Hmm ... could a similar tactic be applied to other problems with zeta function relations (¿ quantum chaos?) which have been studied order by order through partial sums ?)

The Kac-Ward matrix resolvent kernel:

An object of related interest is the resolvent kernel:

$$
\begin{aligned}
G(e_2, e_1) &= \langle e_2 | \frac{1}{\mathbb{1} - WK} | e_1 \rangle \\
&= \sum_{\gamma : e_1 \to e_2} e^{i \Delta \arg(\gamma)/2} \prod_{n=1}^{|\gamma|-1} K_{e_n} \qquad (6)
\end{aligned}
$$

This quantity has played an essential role in many of the recent developments in the subject.

Its interpretation in terms which are intrinsic to the model is closely linked to the next discussion. But let us skip it here.