Continuous symmetry reduction for high-dimensional flows

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translational symmetry $\Rightarrow$

- traveling wave solutions
- unstable relative periodic orbits

question

what are the invariant objects that organize phase space in a spatially extended system with translational symmetry and how do they fit together to form a skeleton of the dynamics?

state space

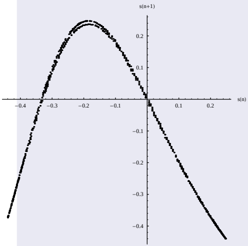
- the space in which all possible states u's live
- ∞-dimensional:
  point $u(x)$ is a function of x on interval $x \in L$.
- in practice:
  a high but finite dimensional space (e.g. through a spectral discretization)

Take the hint from low dimensional systems

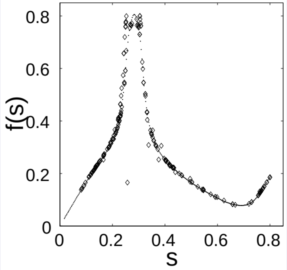
- low dimensional systems:
  equilibria, periodic orbits organize the long time dynamics.
- is this true in extended systems?

Kuramoto-Sivashinsky flow reduced to discrete maps

within the discrete $u(x) = -u(-x)$ invariant subspace



Christiansen et. al. (1996)



Lan and Cvitanović (2004)

- $\infty - d$ PDE state space dynamics can be reduced to low dimensional return maps!

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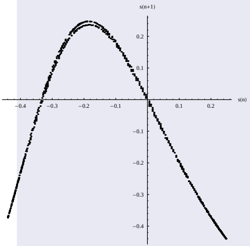

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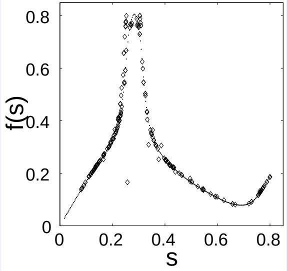


Lan and Cvitanović (2004)

- $\infty - d$ PDE state space dynamics can be reduced to low dimensional return maps!
- BUT! must reduce continuous symmetries first

from complex Lorenz flow $5D$ attractor $\rightarrow$ unimodal map

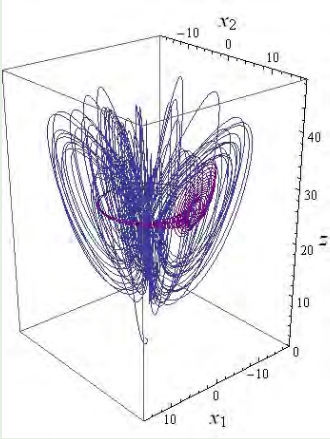
complex Lorenz equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{y}_1 \\ \dot{y}_2 \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -\sigma x_1 + \sigma y_1 \\ -\sigma x_2 + \sigma y_2 \\ (\rho_1 - z)x_1 - \rho_2 x_2 - y_1 - e y_2 \\ \rho_2 x_1 + (\rho_1 - z)x_2 + e y_1 - y_2 \\ -bz + x_1 y_1 + x_2 y_2 \end{bmatrix}$$

$\rho_1 = 28, \rho_2 = 0, b = 8/3, \sigma = 10, e = 1/10$

A typical $\{x_1, x_2, z\}$ trajectory of the complex Lorenz flow
+ a short trajectory of whose initial point is close to the relative equilibrium Q_1 superimposed.

attractor



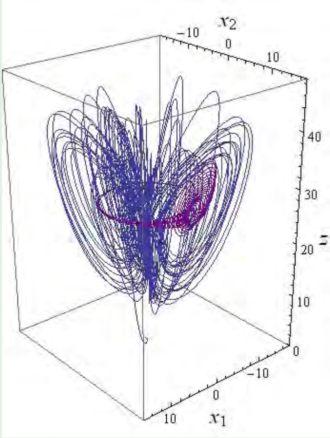
from complex Lorenz flow $5D$ attractor $\to$ unimodal map

what to do?

the goal

reduce this messy strange attractor to a 1-dimensional return map

attractor



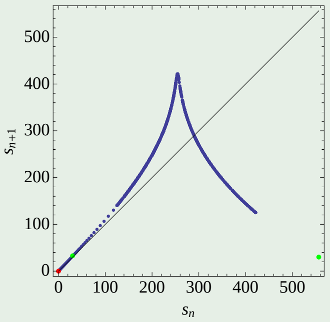
from complex Lorenz flow $5D$ attractor $\rightarrow$ unimodal map

the goal attained

but it will cost you

after symmetry reduction; must learn how to quotient the $SO(2)$ symmetry

1D return map!



symmetries of dynamics

A flow $\dot{x} = v(x)$ is G-equivariant if

$$v(x) = g^{-1}\, v(g\, x)\,, \qquad \text{for all } g \in G\,.$$

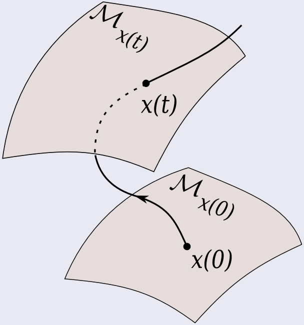
example: $SO(2)$ rotations for complex Lorenz equations

$SO(2)$ rotation by finite angle θ:

$$g(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta & 0 \\ 0 & 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

foliation by group orbits

group orbits

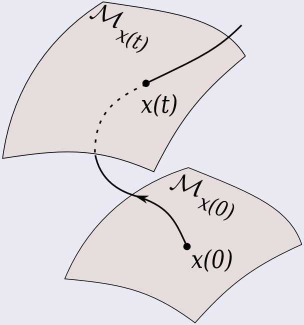


group orbit $\mathcal{M}_x$ of x is the set of all group actions

$$\mathcal{M}_x = \{g\,x \mid g \in G\}$$

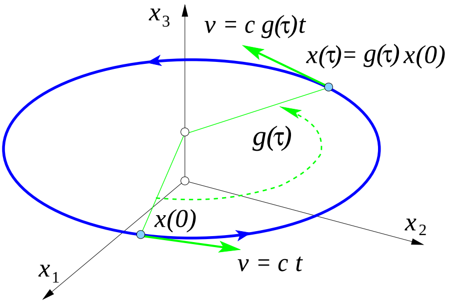
foliation by group orbits

group orbits



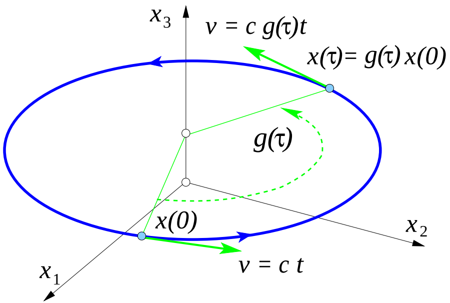
action of a symmetry group endows the state space with the structure of a union of group orbits, each group orbit an equivalence class.

a traveling wave



relative equilibrium
(traveling wave, rotating wave)
$x_{\mathrm{TW}}(\tau) \in \mathcal{M}_{\mathrm{TW}}$: the dynamical flow field points along the group tangent field, with constant 'angular' velocity c, and the trajectory stays on the group orbit

a traveling wave

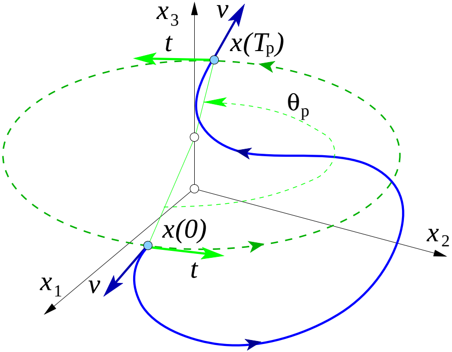


relative equilibrium

$$v(x) = c \cdot t(x), \qquad x \in \mathcal{M}_{\mathrm{TW}}$$

$$x(\tau) = g(-\tau c)\,x(0) = e^{-\tau c \cdot \mathbf{T}} x($$

a relative periodic orbit

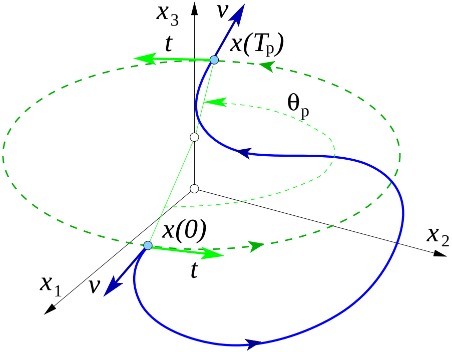


relative periodic orbit

$$x_p(0) = g_p x_p(T_p)$$

exactly recurs at a fixed *relative period* T_p, but shifted by a fixed group action g_p

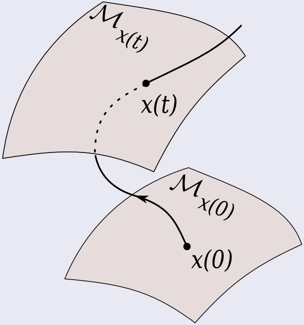
a relative periodic orbit



relative periodic orbit starts out at $x(0)$, returns to the group orbit of $x(0)$ after time T_p, a rotation of the initial point by g_p

foliation by group orbits

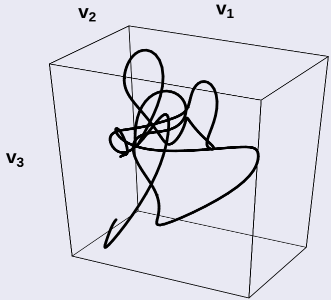
group orbits



the goal:
replace each group orbit by a unique point a lower-dimensional *reduced state space* (or orbit space)

relativity for pedestrians

try a co-moving coordinate frame?

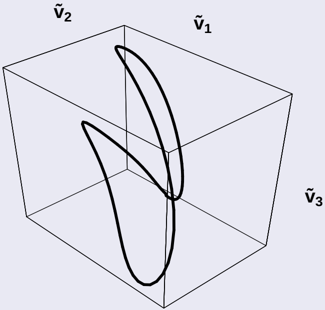


(*a*)

A relative periodic orbit of the Kuramoto-Sivashinsky flow, traced for four periods T_p, projected on
(a) a stationary state space coordinate frame $\{v_1, v_2, v_3\}$;

relativity for pedestrians

try a co-moving coordinate frame?

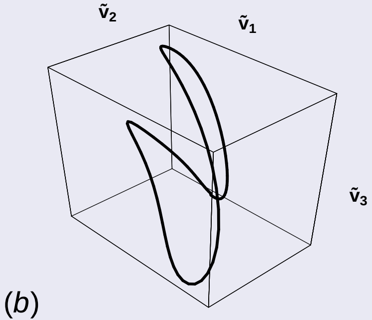


(b)

A relative periodic orbit of the Kuramoto-Sivashinsky flow, traced for four periods T_p, projected on
(b) a co-moving $\{\tilde{v}_1, \tilde{v}_2, \tilde{v}_3\}$ frame

relativity for pedestrians

no good global co-moving frame!



this is no symmetry reduction at all; all other relative periodic orbits require their own frames,moving at different velocities.

symmetry reduction

- all points related by a symmetry operation are mapped to the same point.
- relative equilibria become equilibria and relative periodic orbits become periodic orbits in reduced space.
- families of solutions are mapped to a single solution

reduction methods

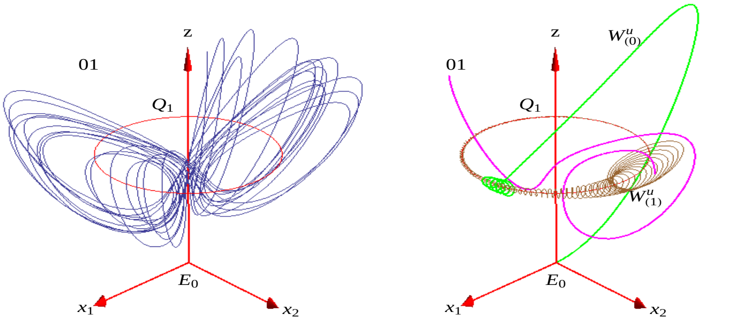
1. **Hilbert polynomial basis**: rewrite equivariant dynamics in invariant coordinates
2. **moving frames, or slices**: cut group orbits by a hypersurface (kind of Poincareé section), each group orbit of symmetry-equivalent points represented by the single point

reduction methods

1. **Hilbert polynomial basis**: rewrite equivariant dynamics in invariant coordinates: global
2. **moving frames, or slices**: cut group orbits by a hypersurface (kind of Poincareé section), each group orbit of symmetry-equivalent points represented by the single point: local

state space portrait of complex Lorenz flow

drift induced by continuous symmetry



A generic chaotic trajectory (blue), the E_0 equilibrium, a representative of its unstable manifold (green), the Q_1 relative equilibrium (red), its unstable manifold (brown), and one repeat of the $\overline{01}$ relative periodic orbit (purple).

Lie groups elements, Lie algebra generators

An element of a compact Lie group:

$$g(\theta) = e^{\theta \cdot \mathbf{T}}, \qquad \theta \cdot \mathbf{T} = \sum \theta_a \mathbf{T}_a, \; a = 1, 2, \cdots, N$$

$\theta \cdot \mathbf{T}$ is a *Lie algebra* element, and θ_a are the parameters of the transformation.

example: $SO(2)$ rotations for complex Lorenz equations

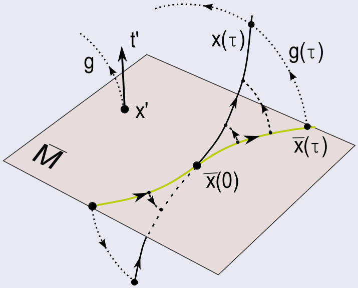
$$\mathbf{T} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

flow field at the state space point x induced by the action of the group is given by the set of N *tangent fields*

$$t_a(x)_i = (\mathbf{T}_a)_{ij} x_j$$

slice & dice

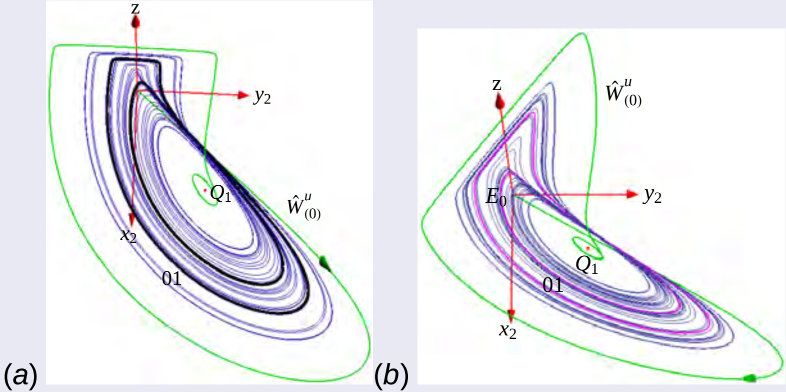
flow reduced to a slice



Slice $\bar{\mathcal{M}}$ through the slice-fixing point x', normal to the group tangent t' at x', intersects group orbits (dotted lines). The full state space trajectory $x(\tau)$ and the reduced state space trajectory $\overline{x}(\tau)$ are equivalent up to a group rotation $g(\tau)$.

slice trouble 1

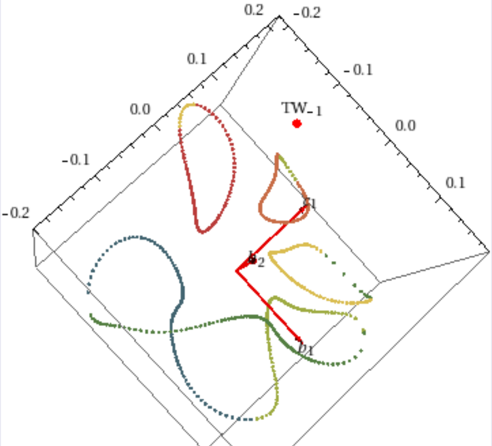
portrait of complex Lorenz flow in reduced state space



all choices of the slice fixing point x' exhibit flow discontinuities / jumps

slice trouble 2

slice cuts an relative periodic orbit multiple times



Relative periodic orbit intersects a hyperplane slice in 3 closed-loop images of the relative periodic orbit and 3 images that appear to connect to a closed loop.

summary

conclusion

- Symmetry reduction: efficient implementation allows exploration of high-dimensional flows with continuous symmetry.
- stretching and folding of unstable manifolds in reduced state space organizes the flow

to be done

- construct Poincaré sections and return maps
- find all (relative) periodic orbits up to a given period.
- use the information quantitatively (periodic orbit theory).