turbulence
how **fat** is it?

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recurrence, self-organization, and the dynamics of turbulence
KITP, Santa Barbara CA

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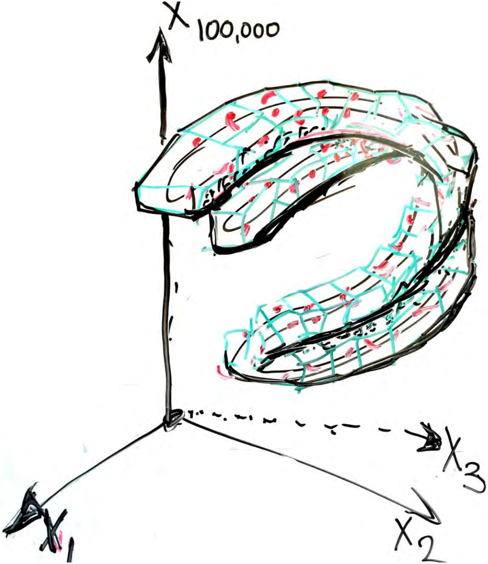
overview

1. what this talk is about
2. why are we here
3. state space
4. dimension of the inertial manifold



inertial manifold

strange attractor stuffed into a finite-dimensional body bag

1. why are we here
2. state space
3. dimension of the inertial manifold

Navier-Stokes equations (1822)

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{1}{R}\nabla^2 \mathbf{v} - \nabla p + \mathbf{f}\,, \qquad \nabla \cdot \mathbf{v} = 0,$$

velocity field $\mathbf{v} \in \mathbb{R}^3$; pressure field p ; driving force $\mathbf{f}$

describe turbulence

starting from the equations (no statistical assumptions)

1. why is Cvitanović talking?
2. state space
3. dimension of the inertial manifold

algorithmic advances

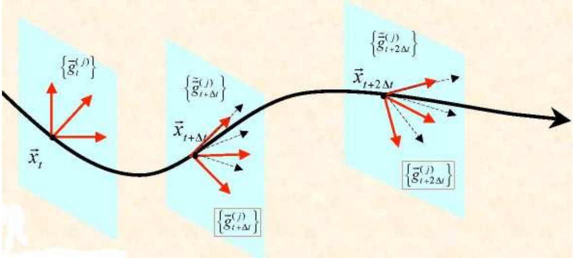
F. Ginelli, H. Chaté, G. Radons, A. Politi, P. Poggi, A. Turchi, R. M. Samelson, C. L. Wolfe:

computation of covariant "Lyapunov" vectors

Phys. Rev. Lett. 99, 130601 (2007); Tellus A 59, 355 (2007);

J. Phys. A 46, 254005 (2013)

covariant vectors are non-normal



(references are hyperlinked)

beautiful insights of

F. Ginelli, H. Chaté, G. Radons, A. Politi, P. Poggi, A. Turchi, H.-l. Yang, K. A. Takeuchi

physical dynamics is hyperbolically separated from the infinity of transient modes :

physical dimension of an inertial manifold

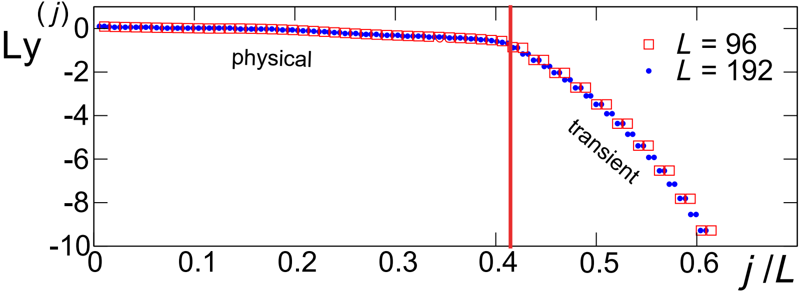
Phys. Rev. Lett. 102, 074102 (2009); Phys. Rev. E 84, 046214 (2011); Phys. Rev. Lett. 117, 024101 (2016)

- Kuramoto-Sivashinsky? OK!
- complex Ginzburg-Landau? OK!
- Navier-Stokes? dunno...

(references are hyperlinked)

the killer slide

Kuramoto-Sivashinsky Lyapunov spectrum
cells $L = 22, 96, 192$: it scales!



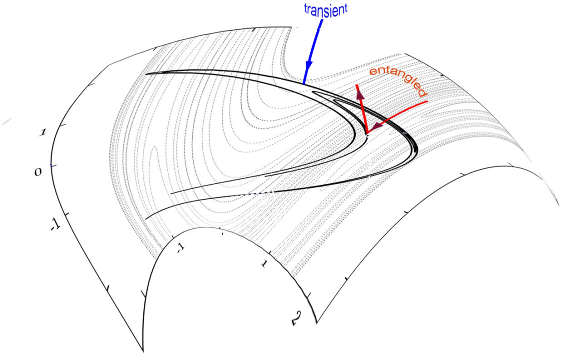
Now double # computational elements, fixed L :
all new ones go to the transient spectrum [1]

[1] Yang et al (Phys. Rev. Lett. 2009)

what this talk is about:

the attracting set of a dissipative flow

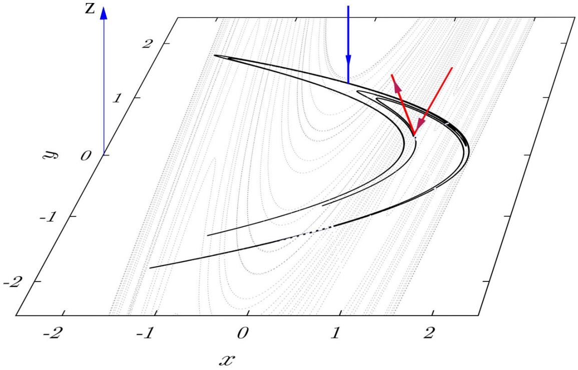
is embedded with the (curvilinear) inertial manifold embedded into ∞-dimensional state space



but try to draw THAT :)

what this talk is about:

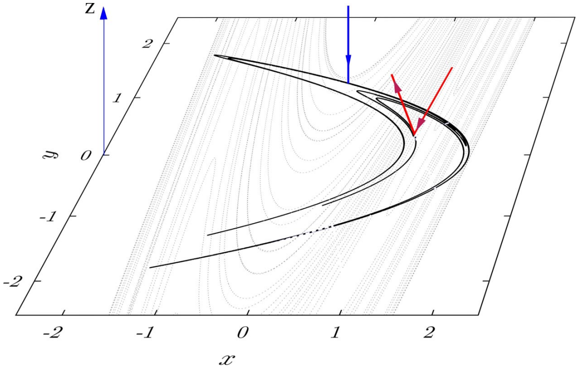
it is believed that the attracting set of a dissipative flow



- is confined to :
  a finite-dimensional smooth *inertial manifold*
- "z" directions :
  the remaining ∞ of *transient dimensions*

what this talk is about:

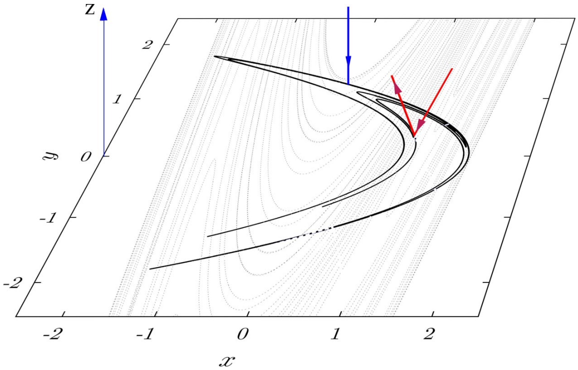
state space of dissipative flow is split into



- inertial manifold : spanned locally by entangled covariant vectors, tangent to unstable / stable manifolds
- the rest : spanned by the remaining ∞ of the contracting, decoupled, transient covariant vectors

what this talk is about:

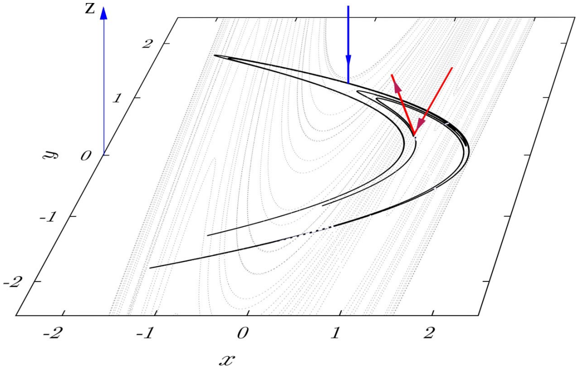
inertial manifold



- dynamics of the vectors that span the inertial manifold is entangled, with small angles and frequent tangencies
- a transient covariant vector : isolated, nearly orthogonal to all other covariant vectors

what this talk is about:

goal : construct inertial manifold for a turbulent flow



- tile it with a finite collection of bricks centered on recurrent states, each brick $\approx 10 - 100$ dimensions
- span of ∞ of transient covariant vectors : no intersection with the entangled modes

if all this works out, it is kinda amazing

computation of turbulent solutions

requires at least
$\rightarrow$ integration of 10^4-10^6 coupled ordinary differential equations

inertial manifold, tiled

50 linear tiles cover the (nonlinear, curved) inertial manifold

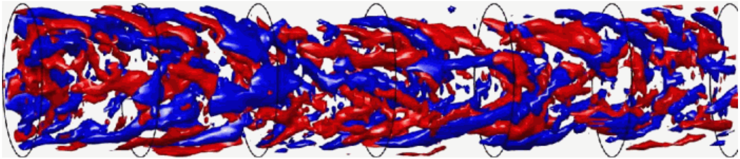
each tile 100 dimensional (fingers crossed :)

part 1

1. why are we here
2. state space
3. dimension of the inertial manifold

a state of turbulent pipe flow at instant in time

Stereoscopic Particle Image Velocimetry $\rightarrow$ 3-*d* velocity field over the entire pipe[2]



[2]Casimir W.H. van Doorne (PhD thesis, Delft 2004)

part 2

1. why are we here
2. **state space**
3. dimension of the inertial manifold

dynamical description of turbulence

state space

a manifold $\mathcal{M} \in \mathbb{R}^d$: d numbers determine the state of the system

representative point

$x(t) \in \mathcal{M}$
a state of physical system at instant in time

integrate the equations

trajectory $x(t) = f^t(x_0)$ = representative point time t later

1 spatial dimension "Navier-Stokes"

computationally not ready yet to explore
the inertial manifold of $3D$ turbulence - we start with

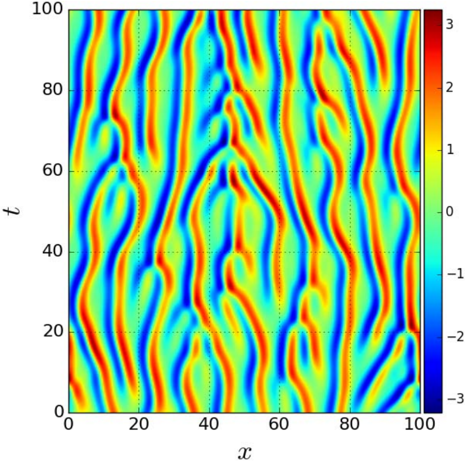
Kuramoto-Sivashinsky equation

$$u_t + u\nabla u = -\nabla^2 u - \nabla^4 u\,, \qquad x \in [-L/2, L/2]\,,$$

describes spatially extended systems such as

- flame fronts in combustion
- reaction-diffusion systems
- ...

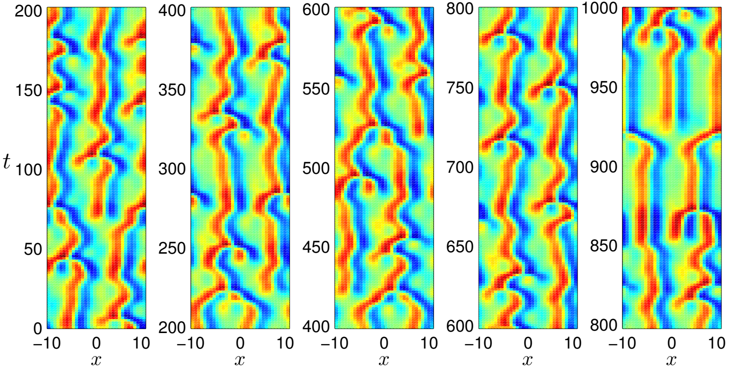
Kuramoto-Sivashinsky on a large spacetime domain



[horizontal] space $x \in [0, 100]$ [up] time evolution

- turbulent behavior
- simpler physical, mathematical and computational setting than Navier-Stokes

evolution of Kuramoto-Sivashinsky on small periodic domain

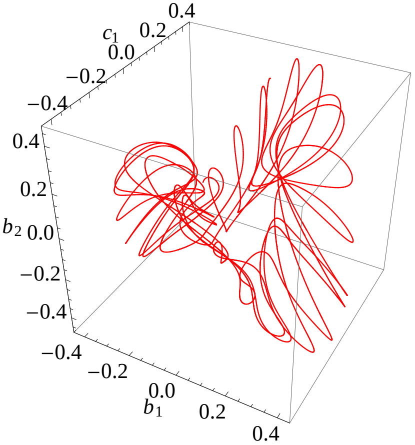


[horizontal] space $x \in [-11, 11]$ [up] time evolution

color: magnitude of $u(x, t)$

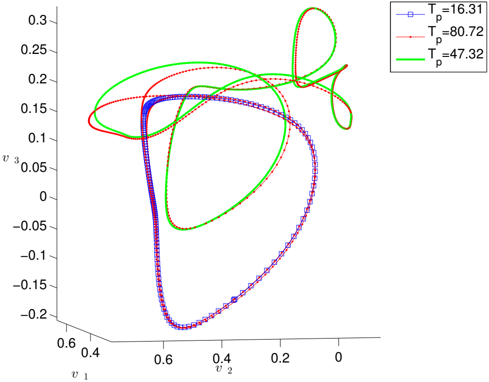
a relative periodic orbit

full state space : many periods



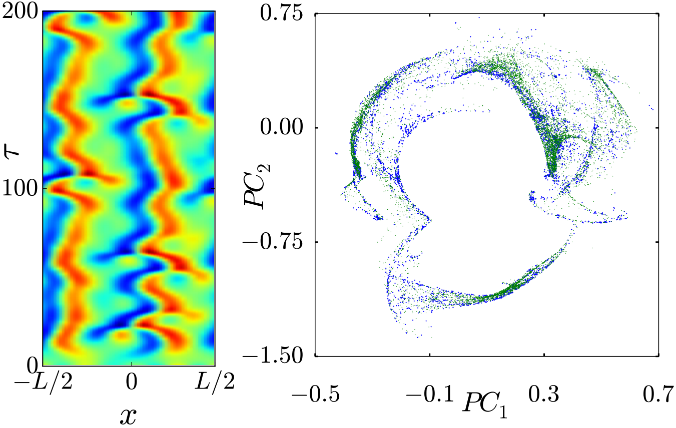
have computed: 60 000 periodic orbits

can explore shadowing



(impossible without symmetry reduction)

periodic orbits are dense in the attractor



- [left] turbulent trajectory segment in [space$\times$time]
- Poincaré section, turbulent trajectory (natural measure)
- periodic points, from 479 periodic orbits[3]

[3] Budanur (PhD thesis 2015)

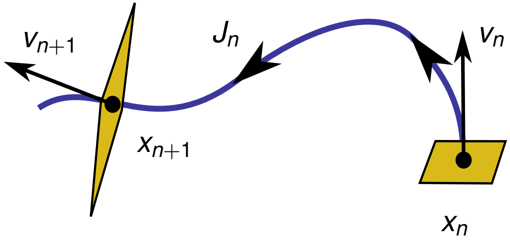
1. why are we here
2. state space
3. dimension of the inertial manifold

what is the dimension of the inertial manifold?

we determine it in 6 independent ways

- Lyapunov exponents (diagnostic only, previous work)
- Lyapunov vectors (sharp, previous work)
- four periodic orbits determinations (presented here)

linearized deterministic flow



$$x_{n+1} + z_{n+1} = f(x_n) + J_n\, z_n\,, \quad J_{ij} = \partial f_i / \partial x_j$$

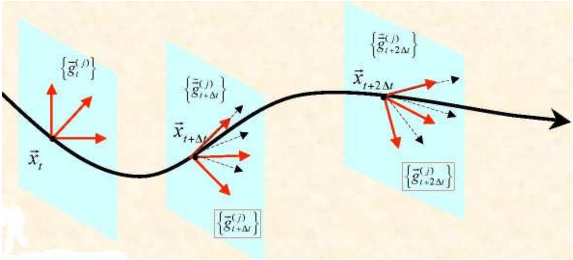
in one time step a linearized neighborhood of x_n is

(1) advected by the flow

(2) transported by the Jacobian matrix J_n into a neighborhood given by the J eigenvalues and eigenvectors

method (0) : global ergodic trajectory, $t \in [-\infty, \infty]$

Ginelli *et al.*, Phys. Rev. Lett. (2007)



Jacobian matrix : orthogonal frame $\rightarrow$ non-orthogonal frame

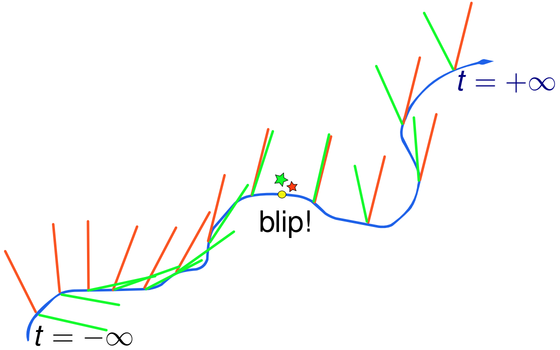
$\rightarrow$

QR decomposed into an *R*-matrix + Gram-Schmidt frame

$\rightarrow$

next Jacobian matrix, and so on

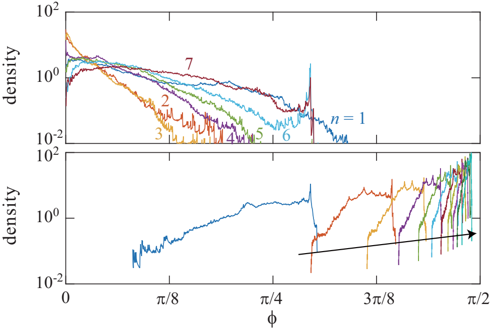
eigenvectors spanning "physical" manifold



a pair of "entangled" eigenvectors
distinct Lyapunov exponents
dance along t from $-\infty$ to $-\infty$ orbit

at the instant "blip!" they are (almost?) collinear

(0) distribution of angles between eigenvectors



histogram of angles between nth leading covariant vector and the next, accumulated over many long orbits :

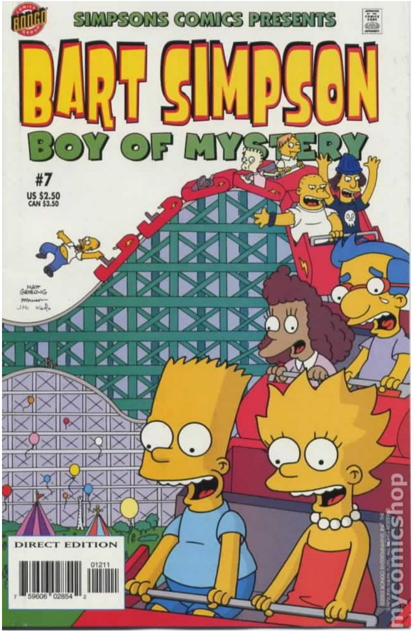
- (top) For $n = 1 \cdots 7$ (eigenvector within the entangled manifold) the angles can be arbitrarily small
- (bottom) For the remaining, transient eigenvectors, $n = 8, 11, 12, \cdots$: angles are bounded away from zero

.

OK, so the the frame is locally flat

but where the (blip) are we in the state space?

we are here



next : cartography of a roller coaster ride

part 4

1. why are we here
2. state space
3. dimension of the inertial manifold
4. new : cartography of the inertial manifold

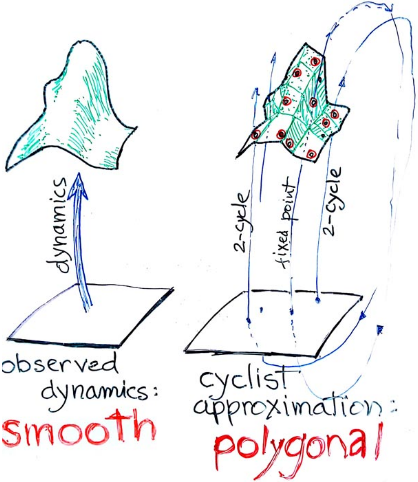
cartography for fluid dynamicists

cover the inertial manifold with a set of flat charts

we can do this with

finite-dimensional bricks embedded in $10^{100\,000}$ dimensions!

tile the inertial manifold by **recurrent flows**

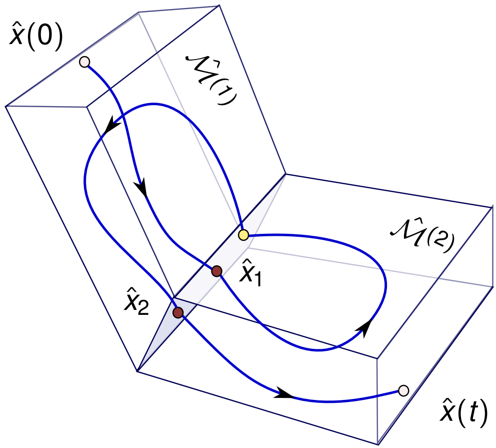


a fixed point

a 2-cycle, etc.

smooth dynamics (left frame)
tesselated by the skeleton of recurrent flows,
together with (right frame) their linearized neighborhoods

charting the inertial manifold

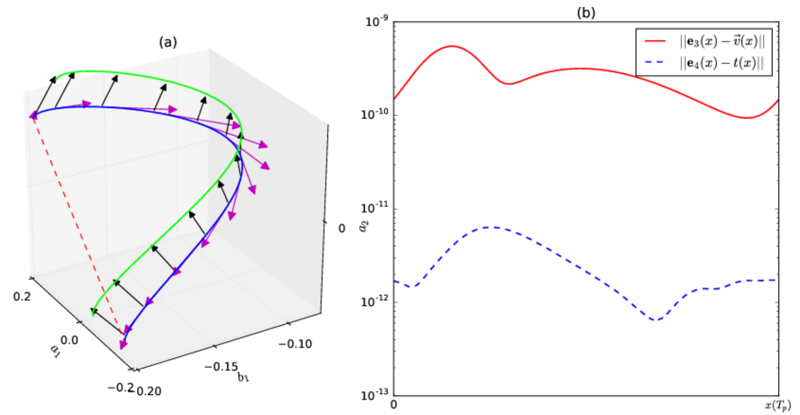


two tangent “entangled” tiles = finite-dimensional bricks

shaded plane :
when integrating your equations, switch the local chart

method (1) : local relative periodic orbit, one period

Ding & Cvitanović, SIAM J. Appl. Dyn. Syst. (2016)



[right panel]
all eigenvectors computed close to the machine precision

(1) algorithmic breakthrough : all Floquet exponents to machine precision

	$\mu^{(i)}$	$e^{iT_p\omega^{(i)}}$
1=2	0.0331970261043278	-0.42330856002164 + i 0.905985575499084
3=4	(2 marginal)	
5	-0.216338085869672	1
6=7	-0.265233812289065	-0.867175421594352 + i 0.49800279937231
...	...	...
29	-316.19797864063	1
30	-320.666664811713	-1

Floquet exponents for the shortest pre-periodic orbit :

$\mu^{(i)}$ = real part of the exponent.
either the multiplier sign for a real exponent, or
$\omega^{(i)} \rightarrow$ the multiplier phase for a complex Floquet exponent

(1) algorithmic breakthrough : all Floquet exponents to machine precision

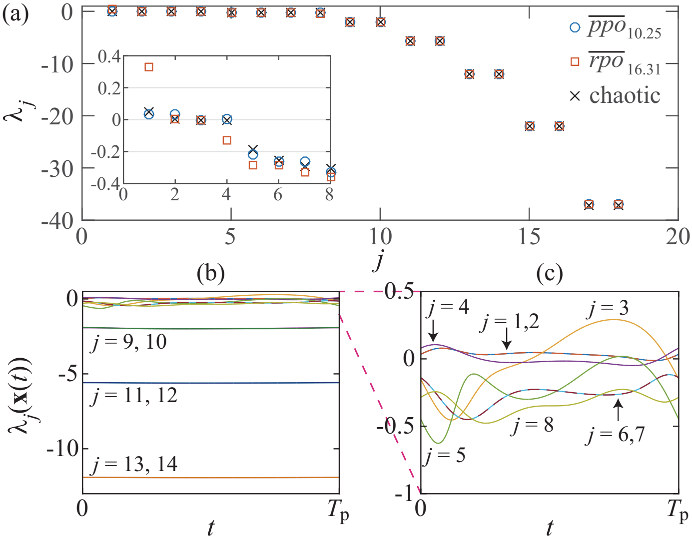
why is this a big deal?

periodic Schur decomposition : resolves Floquet multipliers differing by thousands of orders of magnitude

here the smallest Floquet multiplier for the shortest periodic orbit is

$$|\Lambda_{62}| \simeq e^{-6080.4\times 10.25} \approx 10^{-27069}$$

(1) Floquet and Lyapunov exponents, $L = 22$ small cell



8 entangled modes, rest transient :

inertial manifold is 8 dimensional!

(1) dimension of the inertial manifold from an individual orbit (??)

Floquet exponents separate into entangled vs. transient

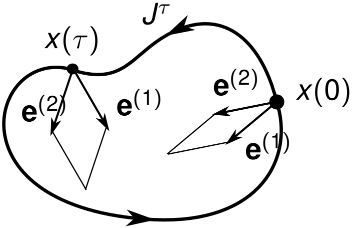
for every single periodic orbit! (checked 500 orbits)

if true for Navier-Stokes, that would make life easy!

(2) dimension of the inertial manifold from ensemble of orbits

- principal angles between hyperplanes spanned by Floquet vectors

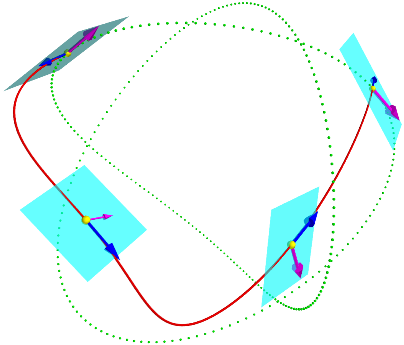
(2) Floquet vectors



a parallelepiped spanned by a pair of Floquet eigenvectors (‘covariant vectors’) transported along the orbit

- Jacobian matrix not self-adjoint : the eigenvectors are not orthogonal, the eigenframe is ‘non-normal’
- Measure the angle between eigenvectors $\mathbf{e}^{(i)}(x(t))$ and $\mathbf{e}^{(j)}(x(t))$

(2) example : Kuramoto-Sivashinsky relative periodic orbit

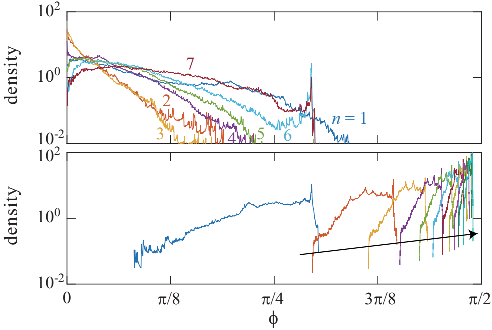


dotted green : a group orbit

solid red : a relative periodic orbit

planes : a tangent space spanned and transported by 2 Floquet vectors

(2) distribution of principal angles between Floquet subspaces



histogram of angles between S_n (n leading Floquet vectors) and $\bar{S}_n$ (the rest), accumulated over the 400 orbits :

- (top) For $n = 1 \cdots 7$ (S_n within the entangled manifold) the angles can be arbitrarily small
- (bottom) For the $\bar{S}_n$ spanned by transient modes, $n = 8, 10, 12, \cdots, 28$: angles bounded away from unity

.

(3), (4) dimension of the inertial manifold from a chaotic trajectory shadowing a given orbit

two independent measurements

(3) shadowing separation vector lies within the orbit's Floquet entangled manifold

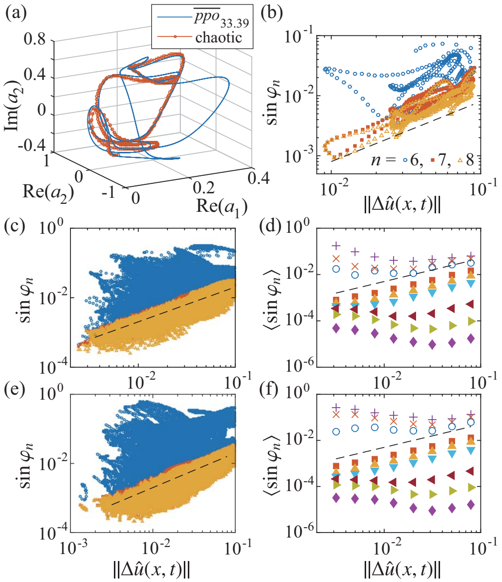
(4) shadowing separation vector lies within the chaotic trajectories covariant vectors' entangled manifold

'separation vector' = difference vector between the chaotic orbit point and periodic orbit point at their (locally) closest passage

accumulate 1000's of near recurrences

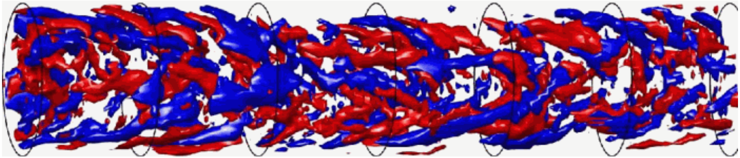
(3)

chaotic trajectory shadows periodic orbits within the entangled subspace

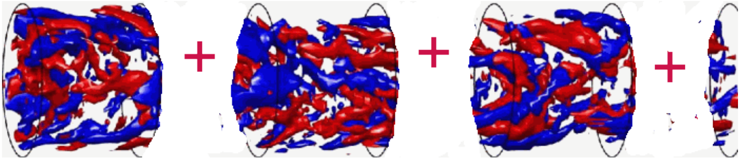


what about large or ∞ domains ?

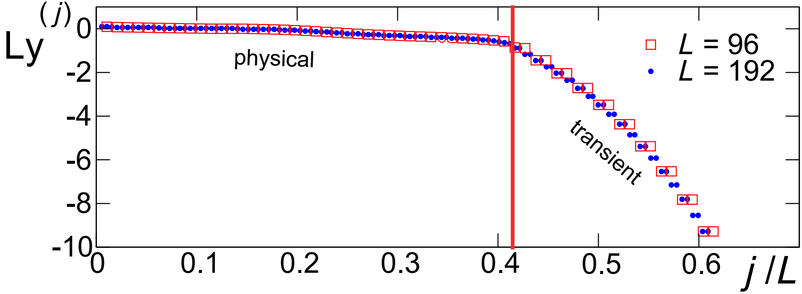
spatiotemporal chaos



spatiotemporal chaos is extensive



Kuramoto-Sivashinsky physical dimension grows linearly with the domain size!



Now double # Fourier modes : all new ones go to the transient spectrum [4]

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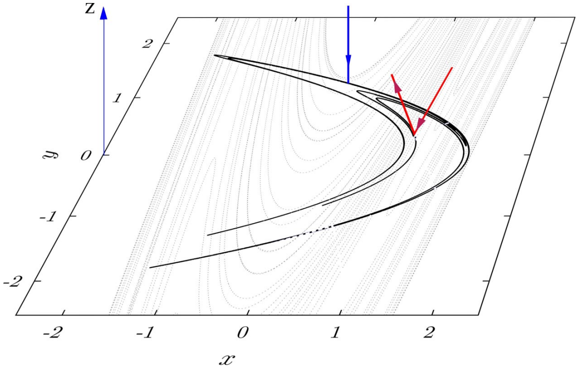
[4] Yang et al (Phys. Rev. Lett. 2009)

summary for the impatient

state space of dissipative flow is split into



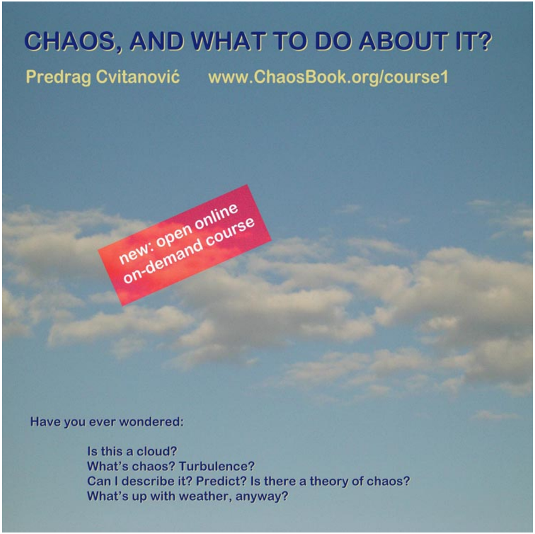
- inertial manifold : spanned locally by entangled covariant vectors, tangent to unstable / stable manifolds
- the rest : spanned by the remaining ∞ of the contracting, decoupled, transient covariant vectors

detailed summary
6 ways to determine the dimension of the inertial manifold

Tangent spaces separate into entangled vs. transient

1. Lyapunov exponents (plausible, previous work)
2. Lyapunov vectors (sharp, previous work)
3. for each individual orbit Floquet exponents separate into entangled vs. transient (new)
4. for an ensemble of orbits principal angles between hyperplanes spanned by Floquet vectors separate into entangled vs. transient (new)
5. for a chaotic trajectory shadowing a given orbit the separation vector lies within the orbit's Floquet entangled manifold (new)
6. for a chaotic trajectory shadowing a given orbit the separation vector lies within the chaotic trajectories covariant vectors' entangled manifold (new)

what next? take the course!



student raves :

...10^6 times harder than any other online course...