Revolution will not be Twitterized!

Spatiotemporal Cat

A Theory of Turbulence

a spatiotemporal theory of
turbulence
computational challenges

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working notes
Georgia Tech

October 9, 2020

overview

1. what this talk is about
2. turbulence in large domains
3. space is time
4. bye bye, dynamics

how do clouds solve PDEs?

do clouds integrate Navier-Stokes equations?

?

$\Longrightarrow$ other swirls $\Longrightarrow$

are clouds Navier-Stokes supercomputers in the sky?

1. turbulence in large domains
2. space is time
3. spacetime
4. bye bye, dynamics

goal : enumerate the building blocks of turbulence

Navier-Stokes equations

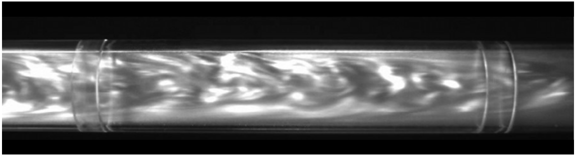
$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{1}{R}\nabla^2 \mathbf{v} - \nabla p + \mathbf{f}\,, \qquad \nabla \cdot \mathbf{v} = 0,$$

velocity field $\mathbf{v} \in \mathbb{R}^3$; pressure field p ; driving force $\mathbf{f}$

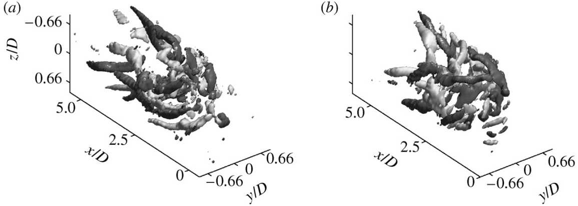
describe turbulence

starting from the equations (no statistical assumptions)

challenge : experiments are amazing

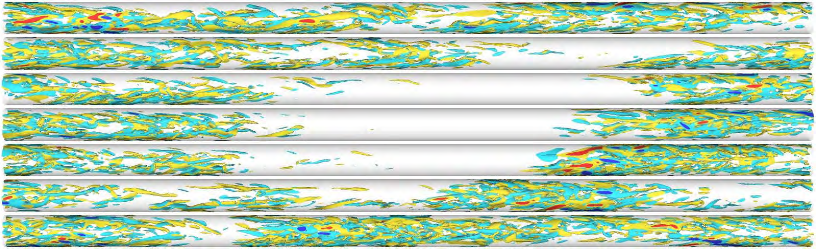


T. Mullin lab



B. Hof lab

can simulate **large** computational domains



pipe flow close to onset of turbulence [1]

but we have **hit a wall**:

exact coherent structures are too unstable to compute

[1] M. Avila and B. Hof, Phys. Rev. **E 87** (2013)

goal : we can do 3D turbulence, but for this presentation

Navier-Stokes equations

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{1}{R}\nabla^2 \mathbf{v} + (\cdots)$$

velocity field $\mathbf{v}(\mathbf{x}; t) \in \mathbb{R}^3$

not helpful for developing intuition

we cannot visualize 3D velocity field at every 3D spatial point

look instead at 1D ‘flame fronts’

(1+1) spacetime dimensional “Navier-Stokes”

Navier-Stokes equations **(1822)**

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{1}{R}\nabla^2 \mathbf{v} + (\cdots)$$

▼
▼
▼

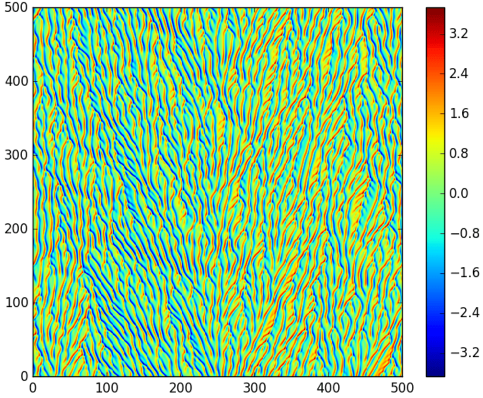
Kuramoto-Sivashinsky (1+1)-dimensional PDE **(1975)**

$$u_t + u\nabla u = -\nabla^2 u - \nabla^4 u \,, \qquad x \in \mathbb{R}\,,$$

describes spatially extended systems such as

- flame fronts in combustion
- reaction-diffusion systems
- …

an example : Kuramoto-Sivashinsky on a large domain

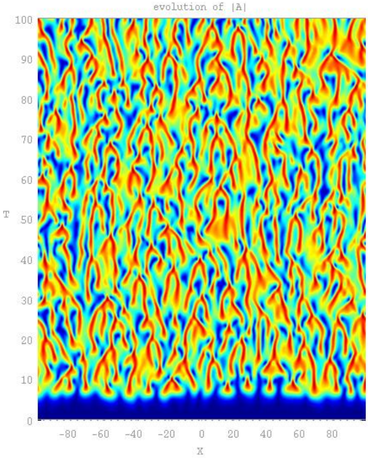


[horizontal] space $x \in [0, L]$ [up] time evolution

- turbulent behavior
- simpler physical, mathematical and computational setting than Navier-Stokes

another example of large spacetime domain simulation

complex Ginzburg-Landau

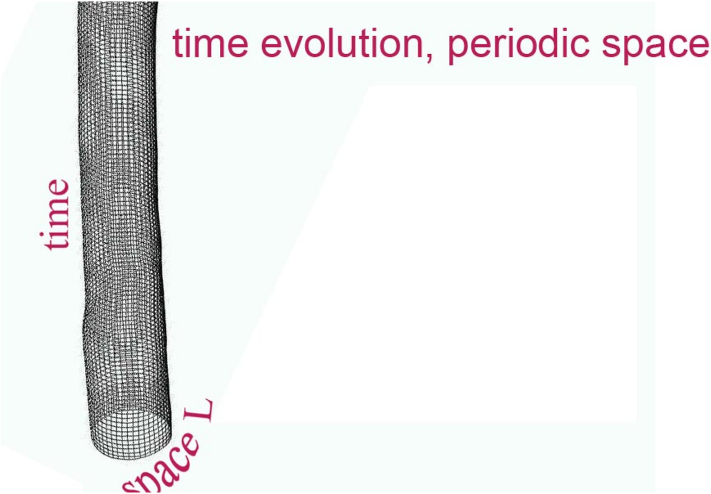


(will return to this)

[horizontal] space $x \in [-L/2, L/2]$ [up] time evolution

codeinthehole.com/static/tutorial/coherent.html

compact space, infinite time cylinder



so far : Navier-Stokes on compact spatial domains, all times

compact space, infinite time

Kuramoto-Sivashinsky equation

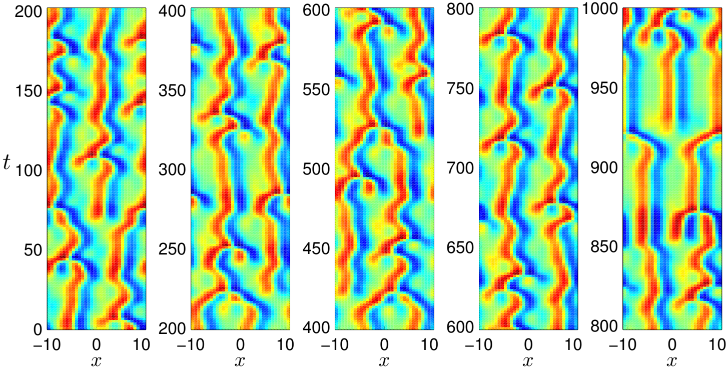
$$u_t = -(+\nabla^2 + \nabla^4)u - u\nabla u\,, \qquad x \in [-L/2, L/2]\,,$$

in terms of discrete spatial Fourier modes

N ordinary differential equations (ODEs) in time

$$\dot{\tilde{u}}_k(t) = (q_k^2 - q_k^4)\,\tilde{u}_k(t) - i\frac{q_k}{2}\sum_{k'=0}^{N-1}\tilde{u}_{k'}(t)\tilde{u}_{k-k'}(t)\,.$$

evolution of Kuramoto-Sivashinsky on small $L = 22$ cell



horizontal: $x \in [-11, 11]$
vertical: time
color: magnitude of $u(x, t)$

part 2

1. turbulence in large domains
2. space is time
3. spacetime
4. bye bye, dynamics

is space time?

compact time, infinite space

rewrite Kuramoto-Sivashinsky

$$u_t \;=\; -uu_x - u_{xx} - u_{xxxx}$$

as 4-fields vector

$$\begin{aligned}\mathbf{u}^\top \;&=\; (u, u', u'', u''') \\ &\quad \text{where } u' \equiv u_x\,,\;\; u'' \equiv u_{xx}\,,\;\; u''' \equiv u_{xxx}\end{aligned}$$

equation $\frac{d}{dx}\mathbf{u}(x) = \mathbf{v}(x)$ now 1st order in spatial derivative

Kuramoto-Sivashinsky = four coupled 1st order PDEs

$$\begin{aligned}\frac{d\,u}{dx} \;&=\; u'\,, \qquad \frac{d\,u'}{dx} = u'' \\ \frac{d\,u''}{dx} \;&=\; u'''\,, \qquad \frac{d\,u'''}{dx} = -u_t - u'' - u\,u'\end{aligned}$$

compact time, infinite space

1st order in spatial derivative

evolve four 1st order PDEs for $\mathbf{u}(x)$ in x,

- $$\frac{d}{dx}\mathbf{u}(x) = \mathbf{v}(x)$$
- compact in time, periodic boundary condition

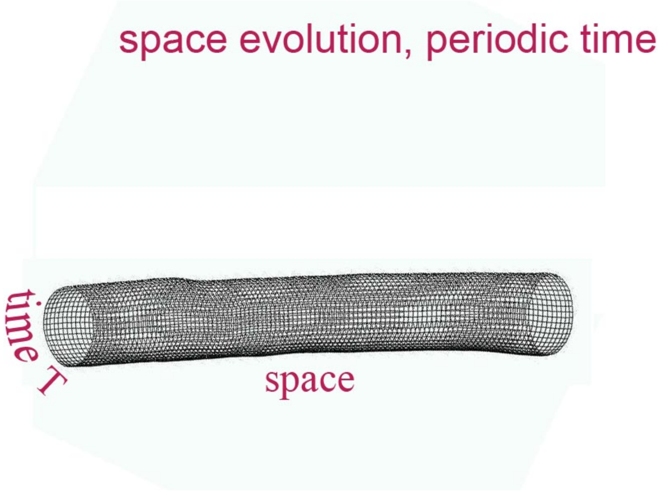
$$u(x,t) = u(x,t+T)$$

- initial data

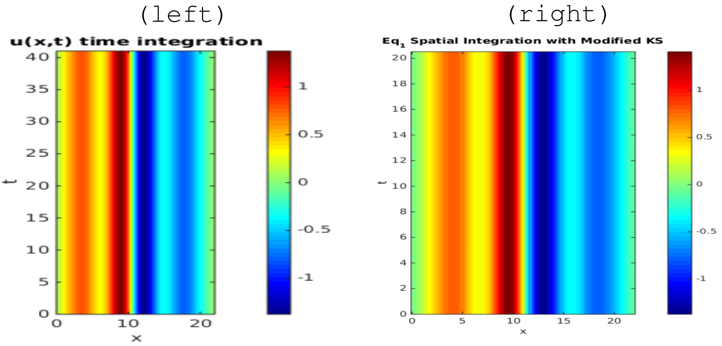
$$\mathbf{u}_0^\top = (u(x_0,t), u'(x_0,t), u''(x_0,t), u''(x_0,t))$$

specified for all $t \in [0, T)$, at a fixed space point x_0

can do : compact time, infinite space cylinder



a time-invariant equilibrium, spatial periodic orbit

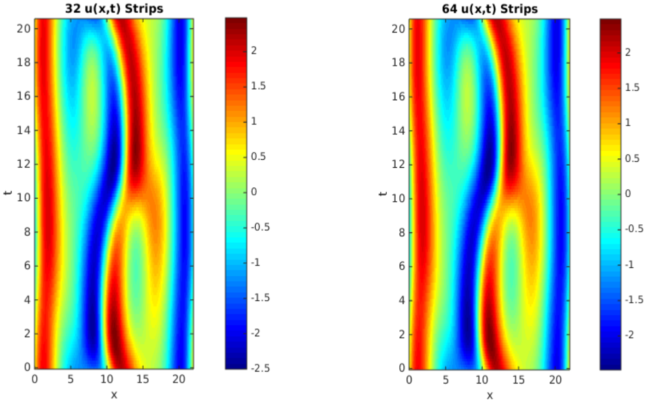


evolution of EQ_1 : (left) in time, (right) in space
initial condition for the spatial integration is the time strip $u(x_0, t)$, $t = [0, T)$, where time period $T = 0$, spatial x period is $L = 22$.

Michelson 1986

a spacetime invariant 2-torus integrated in either time or space

(left) (right)



(left) old : time evolution. (right) new : space evolution
$x = [0, L]$ initial condition : time periodic line $t = [0, T]$

Gudorf 2016

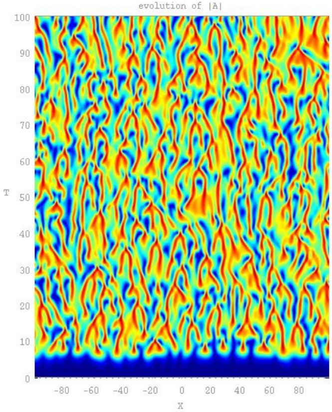
but integrations are uncontrollably unstable

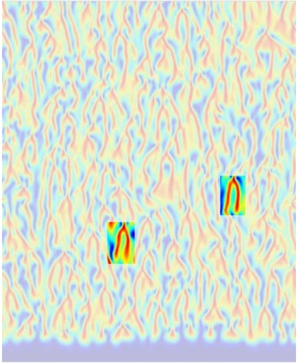
neither time nor space integration works
for large domains

rethink the calculation

complex Ginzburg-Landau on a large spacetime domain

goal : enumerate nearly recurrent chronotopes

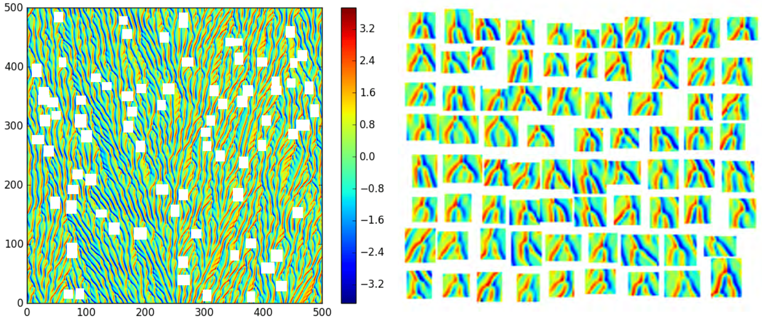




[left-right] space $x \in [-L/2, L/2]$ [up] time $t \in [0, T]$

Kuramoto-Sivashinsky on a large spacetime domain

the same small tile recurs often in a turbulent pattern



goal : define, enumerate nearly recurrent tiles

chronotope[2]

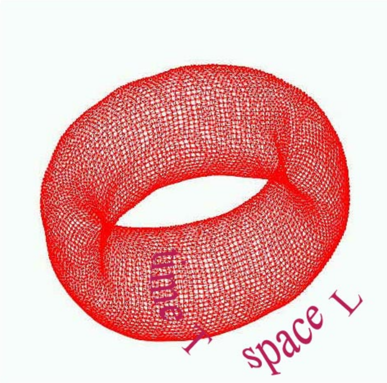
> In literary theory and philosophy of language, the chronotope is how configurations of time and space are represented in language and discourse.
>
> — Wikipedia : Chronotope

- Mikhail Mikhailovich Bakhtin (1937)

[2] S. Lepri et al., J. Stat. Phys. **82**, 1429–1452 (1996).

use spatiotemporally compact solutions as chronotopes

periodic spacetime : 2-torus



this ‘exact coherent structure’
shadows a small patch of spacetime solution $u(x,t)$

periodic orbits generalize to d-tori

1 time, 0 space dimensions

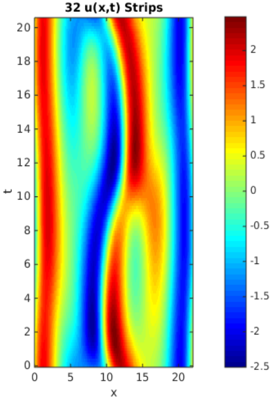
a state space point is *periodic* if its orbit returns to it after a finite time T ;
such orbit tiles the time axis by infinitely many repeats

1 time, d-1 space dimensions

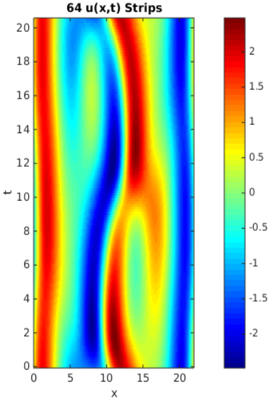
a state space point is *spatiotemporally periodic* if it belongs to an invariant d-torus $\mathcal{R}$;
such torus tiles the spacetime by infinitely many repeats

a spacetime invariant 2-torus integrated in either time or space

(left)



(right)



(left) old : time evolution $t = [0, T]$

initial condition : space periodic line $x = [0, L]$

(right) new : space evolution $x = [0, L]$

initial condition : time periodic line $t = [0, T]$

Gudorf 2016

every compact solution is a fixed point on a discrete lattice

discretize $u_{nm} = u(x_n, t_m)$ over NM points of spatiotemporal periodic lattice $x_n = nL/N$, $t_m = mT/M$, Fourier transform :

$$\tilde{u}_{k\ell} = \frac{1}{NM} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} u_{nm}\, e^{-i(q_k x_n + \omega_\ell t_m)} \,, \quad q_k = \frac{2\pi k}{L} \,,\ \omega_\ell = \frac{2\pi \ell}{T}$$

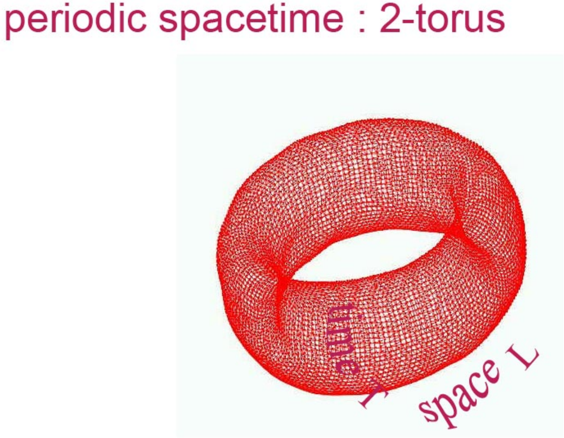
Kuramoto-Sivashinsky is no more a PDE,
but an algebraic $[N \times M]$-dimensional problem
of determining global solution $\mathbf{u}$ to

fixed point condition

$$\left(-i\omega_\ell - (q_k^2 - q_k^4)\right) \tilde{u}_{k\ell} + i\frac{q_k}{2} \sum_{k'=0}^{N-1} \sum_{m'=0}^{M-1} \tilde{u}_{k' m'} \tilde{u}_{k-k', m-m'} = 0$$

every calculation is a spatiotemporal lattice calculation

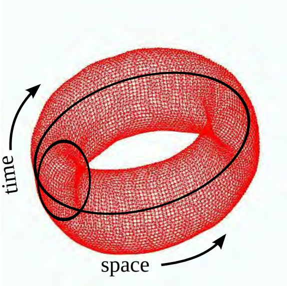
field is discretized as $\tilde{u}_{k\ell}$ values
over NM points of a periodic lattice



professor Zweistein forgets to take his meds

statement : HA!

You are imposing by hand the space & time periods L, T !



answer : NO!

nature chooses L & T, they are free parameters.

there is no more time evolution

solution to Kuramoto-Sivashinsky is now given as

condition that

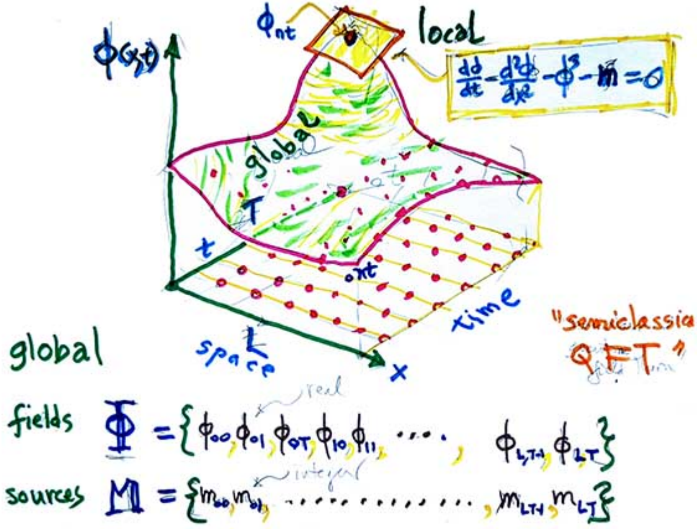
at each lattice point $k\ell$
the tangent field at $\tilde{u}_{k\ell}$

satisfies the equations of motion

$$\left[-i\omega_\ell - (q_k^2 - q_k^4)\right] \tilde{u}_{k\ell} + i\frac{q_k}{2}\sum_{k'=0}^{N-1}\sum_{m'=0}^{M-1}\tilde{u}_{k'm'}\tilde{u}_{k-k',m-m'} = 0$$

this is a local tangent field constraint on a global solution

think globally, act locally



for each symbol array M, a periodic lattice state X_M

unexpected gift from nature

robust : no exponential instabilities

as there are no finite time / space integrations

no need for $\sim 10^{-11}$ accuracies,

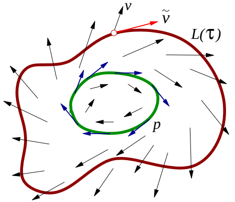
SO

accuracy to a few % suffices,

you only need to get the shape of a solution right

how to find solutions ? an ODE example

the law of motion : $\dot{x} = v(x)$



guess loop tangent $\tilde{v}(\tilde{x}) \neq v(\tilde{x})$

periodic orbit $\tilde{v}(\tilde{x})$, $v(\tilde{x})$ aligned

cost function

$$F^2[\tilde{x}] = \oint_L ds\,(\tilde{v} - v)^2\,; \quad \tilde{v} = \tilde{v}(\tilde{x}(s,\tau))\,,\ v = v(\tilde{x}(s,\tau))\,,$$

penalize[3] misorientation of the loop tangent $\tilde{v}(\tilde{x})$ relative to the true dynamical flow tangent field $v(\tilde{x})$

[3] Y. Lan and P. Cvitanović, Phys. Rev. E **69**, 016217 (2004).

how do clouds solve PDEs?

clouds do not **NOT** integrate Navier-Stokes equations

$\Longrightarrow$ other swirls $\Longrightarrow$

do clouds satisfy Navier-Stokes equations?

yes!

they satisfy them locally, everywhere and at all times

the equations imposed as local constraints

Kuramoto-Sivashinsky equation

$$F(u) = u_t + u_{xx} + u_{xxxx} + uu_x = 0$$

for example, minimize over the entire 2-torus

cost function

$$G \equiv \frac{1}{2}|F(u)|^2_{L\times T}$$

need your help !

adjoint descent

cost function

$$G = \frac{1}{2}\mathbf{F}^\top \mathbf{F}\,.$$

introduce fictitious time (τ) flow by differentiation of cost function.

$$\partial_\tau G = (J^\top \mathbf{F})^\top (\partial_\tau \mathbf{x})$$

"adjoint descent" method defined by chosing[4]

$$\partial_\tau \mathbf{x} = -(J^\top \mathbf{F})$$

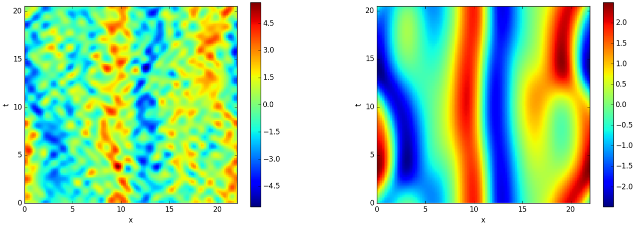
[4] M. Farazmand, J. Fluid M. **795**, 278–312 (2016).

does it work at all ?

add strong noise to a *known* solution,
twice the typical amplitude

only the first test

(not how we actually generate guesses)



(left) initial guess: a known invariant 2-torus

$(L_0, T_0) = (22.0, 20.5057459345)$ + strong random noise

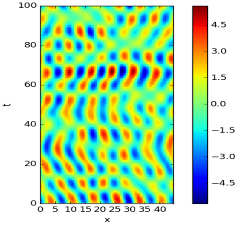
(right) the resulting adjoint descent converged invariant 2-torus

$(L_f, T_f) = (21.95034935834641, 20.47026321555662)$

initial guess generation ?

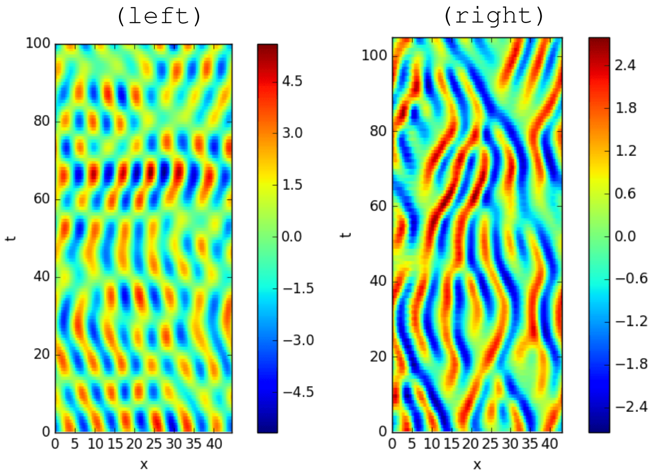
the time scale : the shortest ‘turnover’ scale characterized by the period of the shortest periodic orbit? Or perhaps the Lyapunov time?

the spatial scale : $\bar{L} = 2\pi\sqrt{2}$, the most unstable spatial wavelength of the Kuramoto-Sivashinsky



initial : spatial $\bar{L}$-modulated random guess

KS invariant 2-torus found variationally



(left) initial : $\bar{L} = 2\pi\sqrt{2}$ spatially modulated “noisy” guess
(right) adjoint descent : converged invariant 2-torus

Gudorf 2018

initial guesses, embedded in ergodic sea?

Historically,

guesses extracted from close recurrences
observed in long turbulent simulations

1. inefficient, finds only the shortest, least unstable orbits[5,6]
2. can integrate only not far in time

need spatiotemporal guesses

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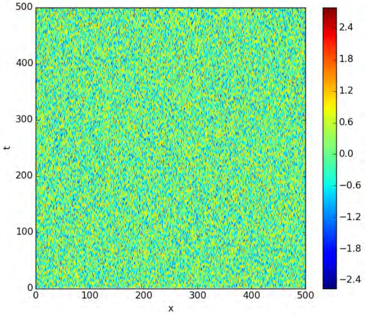
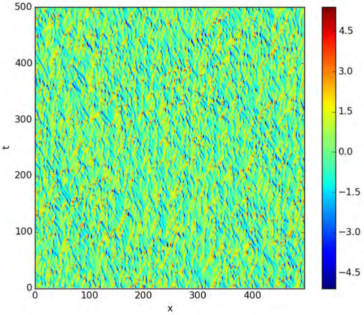
[5] D. Auerbach et al., Phys. Rev. Lett. **58**, 2387–2389 (1987).

[6] J. F. Gibson et al., J. Fluid Mech. **611**, 107–130 (2008).

guesses extracted from large spacetime domains



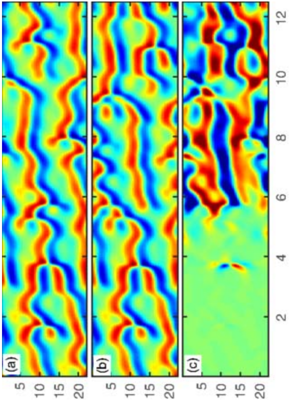


(left) random initial state on $(L, T) = (500, 500)$
(right) adjoint descent $\rightarrow$ typical Kuramoto-Sivashinsky state

finite windows are our starting guesses for invariant 2-tori

another, much twittered : machine learning guesses

"reservoir computing" example[7]



(a) data: Kuramoto-Sivashinsky simulation

(b) reservoir computing prediction

(c) two subtracted agree to ~ 5 Lyapunov times

Q : how would you learn this data?

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[7] J. Pathak et al., Phys. Rev. Lett. **120**, 024102 (2018).

what to do?

Matthew N. Gudorf

has 1 000's of such invariant 2-tori

how do we recognize a cloud?

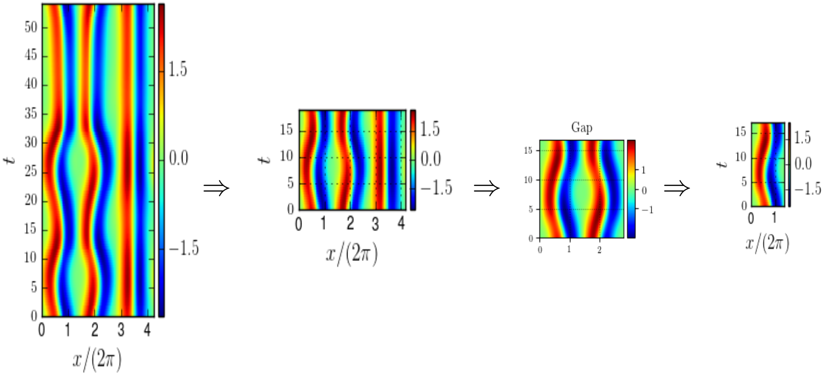
WATCH

$\Longrightarrow$ other swirls $\Longrightarrow$

by recurrent shapes!

so, construct an alphabet of possible shapes

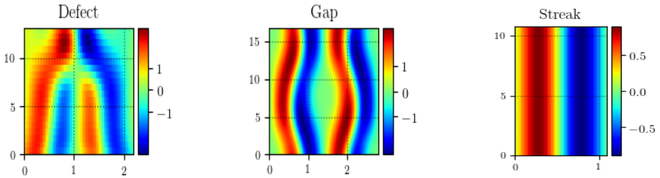
extracting a fundamental tile



1) invariant 2-torus
2) invariant 2-torus computed from initial guess cut out from 1)
3) “gap" invariant 2-torus, initally cut out from 2)
4) the “gap" prime invariant 2-torus fundamental domain

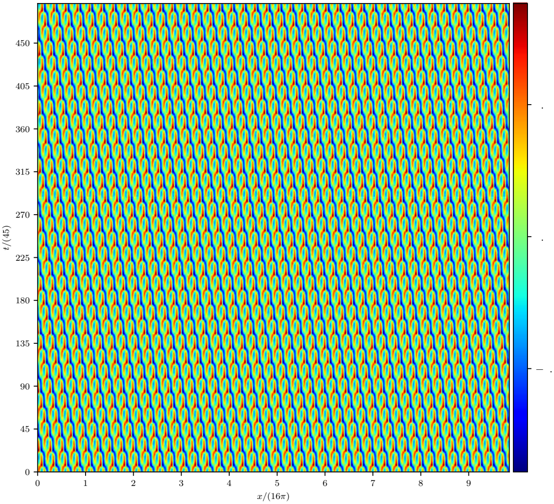
a trial set of prime (rubber) tiles

an alphabet of Kuramoto-Sivashinsky fundamental tiles



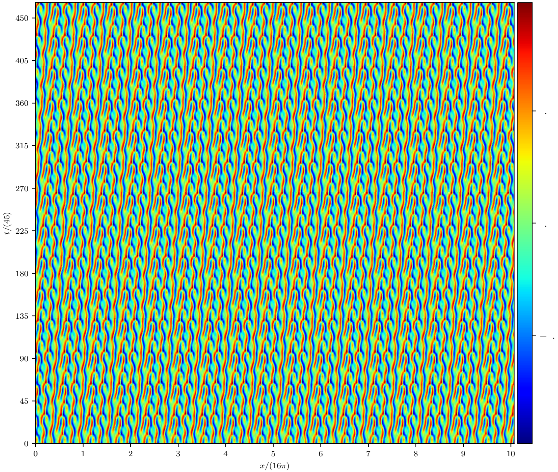
utilize also discrete symmetries :
spatial reflection, spatiotemporal shift-reflect, $\cdots$

Kuramoto-Sivashinsky tiled by a small tile



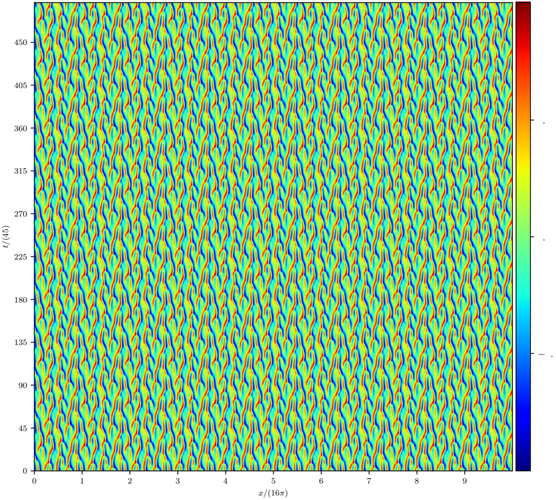
tiling by relative periodic invariant 2-torus
$(L, T) = (13.02, 15)$

spacetime tiled by a larger tile



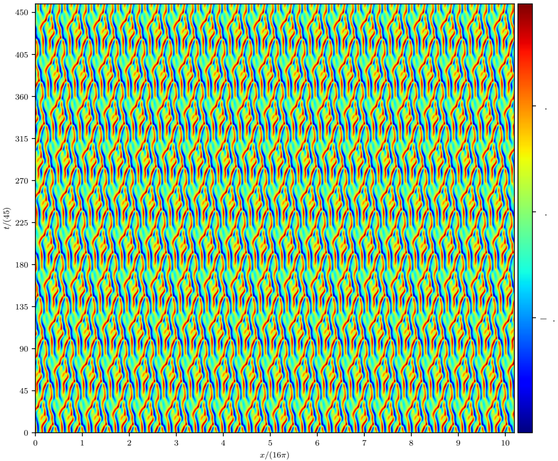
tiling by relative periodic invariant 2-torus
$(L, T) = (33.73, 35)$

spacetime tiled by a tall tile



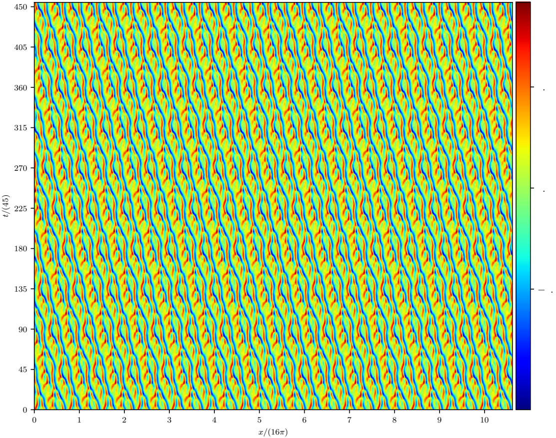
tiling by shift-reflect invariant 2-torus
$(L, T) = (55.83, 24)$

spacetime tiled by a larger tile



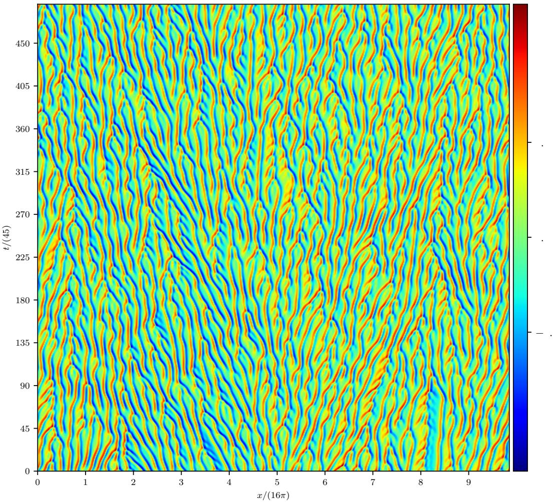
tiling by relative periodic invariant 2-torus
$(L, T) = (32.02, 51)$

spacetime tiled by a larger tile



tiling by relative periodic invariant 2-torus
$(L, T) = (44.48, 50)$

any particular tiling looks nothing like turbulent Kuramoto-Sivashinsky!

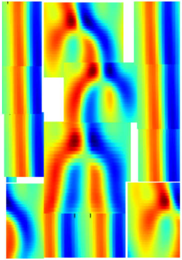


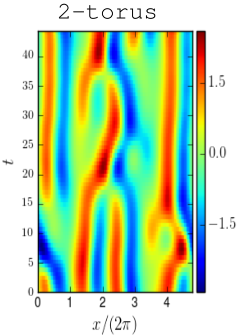
[horizontal] space $x \in [-L/2, L/2]$ [up] time evolution

part 6

a qualitative tiling guess

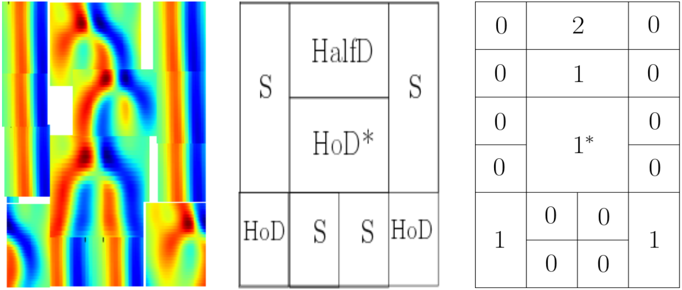
a tiling and the resulting solution





turbulence.zip : each solution has a unique symbolic name

symbolic dynamics is 2-dimensional!



- each symbol indicates a corresponding spatiotemporal tile
- these are “rubber” tiles

part 7

1. turbulence in large domains
2. space is time
3. bye bye, dynamics

in future there will be no future

goodbye

to long time and/or space integrators

they never worked and could never work

life outside of time

the trouble:
forward time-integration codes too unstable

multishooting inspiration: replace a guess that a point is on the periodic orbit by a guess of the entire orbit.

$\rightarrow$

spatio-temporally periodic solutions of classical field theories can be found by variational methods

the equations solved as global optimization problems

impose the equations as local constraints

$$F(u) = u_t + u_{xx} + u_{xxxx} + uu_x = 0$$

minimize globally

perhaps using cost function

$$G \equiv \frac{1}{2} |F(u)|^2_{L \times T}$$

do this ?

the answer is

scalability

compute locally, adjust globally

Navier-Stokes codes

- T. M. Schneider : developing a matrix-free variational Navier-Stokes code, machine learning initial guesses
- D. Lasagna and A. Sharma : developing variational adjoint solvers to find periodic orbits with long periods
- Q. Wang : parallelizing spatiotemporal computation is FLOPs intensive, but more robust than integration forward in time

it's rocket science[8,9,10]

[8] T. M. Schneider, *Variational adjoint methods coupled with machine learning*, private communication, 2019.

[9] D. Lasagna et al., *Periodic shadowing sensitivity analysis of chaotic systems*, 2018.

[10] Q. Wang et al., Phys. Fluids **25**, 110818 (2013).

future :

processor speed $\rightarrow$ limit

number of cores $\rightarrow 10^6 \rightarrow \cdots$

Wang et al (2013)[11] *:*
next-generation : spacetime parallel simulations,
on discretized $4D$ spacetime computational domains,
with each computing core handling a spacetime lattice cell

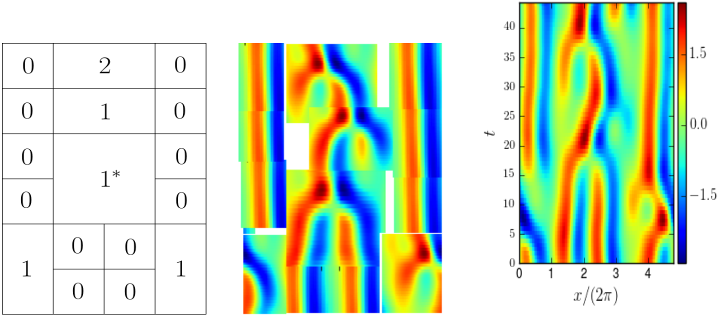
compared to time-evolution solvers: significantly higher level of concurrency, reduction the ratio of inter-core communication to floating point operations

$\Rightarrow$ a path towards exascale DNS of turbulent flows

[11] Q. Wang et al., Phys. Fluids **25**, 110818 (2013).

enumerate hierarchically spatiotemporal patterns

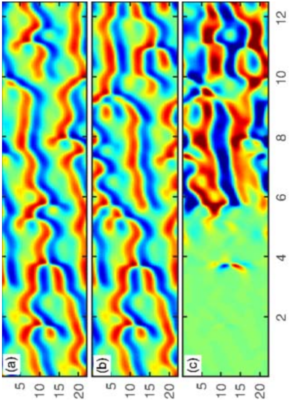
2D symbolic encoding $\Rightarrow$ admissible solutions



- each symbol indicates a minimal spatiotemporal tile
- glue them in all admissible ways

machine learning will be needed

"reservoir computing" example[12]



- **(a)** data: Kuramoto-Sivashinsky simulation
- **(b)** reservoir computing prediction
- **(c)** two subtracted agree to $\sim$ 5 Lyapunov times

Q : how would you learn this data?

---

[12] J. Pathak et al., Phys. Rev. Lett. **120**, 024102 (2018).

take home : clouds do not integrate PDEs

do clouds integrate Navier-Stokes equations?

NO!

$\Longrightarrow$ other swirls $\Longrightarrow$

at any spacetime point Navier-Stokes equations describe the local tangent space

they satisfy them locally, everywhere and at all times

summary

1. study turbulence in infinite spatiatemporal domains
2. theory : classify all spatiotemporal tilings
3. numerics : future is spatiotemporal

there is no more time

there is only enumeration of spacetime solutions

spatiotemporally infinite spatiotemporal cat

part 8

1. turbulence in large domains
2. space is time
3. spacetime
4. fundamental tiles
5. gluing tiles
6. bye bye, dynamics
7. **theory of turbulence ?**

a theory of turbulence ?

1. (semi-)classical field theories
2. state space
3. symbolic dynamics

Dreams of Grand Schemes : solve

Navier-Stokes

$$\rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} = \rho X_i - \frac{\partial p}{\partial x_i} + \mu \nabla^2 u_i .$$

Einstein

$$R_{ik} - \frac{1}{2} g_{ik} R = \frac{8\pi k}{c^4} T_{ik}$$

$$R^i_{\ klm} = \frac{\partial \Gamma^i_{km}}{\partial x^l} - \frac{\partial \Gamma^i_{kl}}{\partial x^m} + \Gamma^i_{nl}\Gamma^n_{km} - \Gamma^i_{nm}\Gamma^n_{kl}$$

Yang-Mills

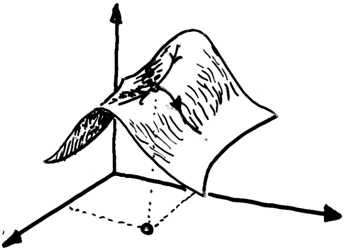
$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F_a^{\mu\nu}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\, C_{abc} A^b_\mu A^c_\nu$$

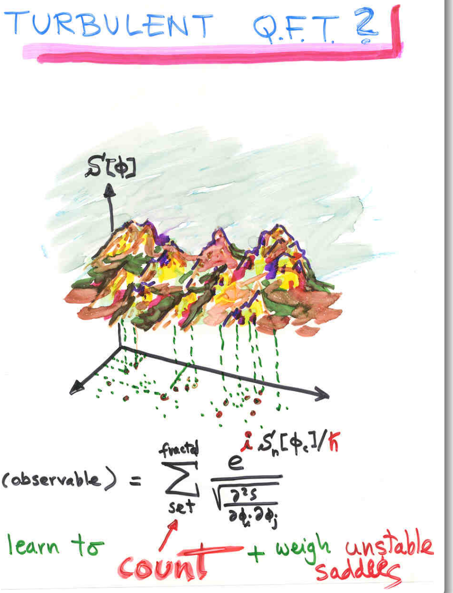
Quantum?

QFT path integrals : semi-classical quantization

a local unstable extremum



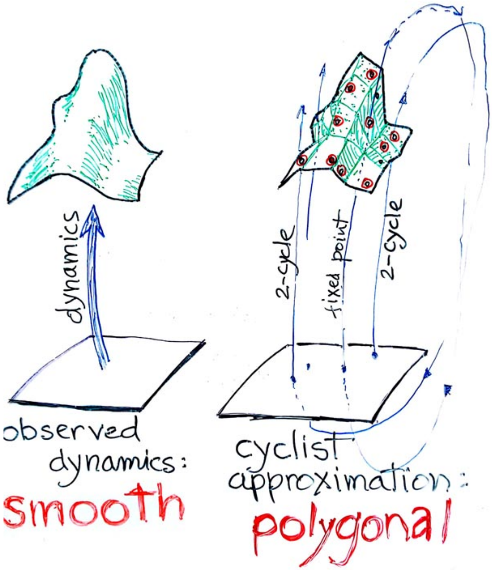
a fractal set of saddles



the very short answer : POT

if you win : I teach you how

(for details, see ChaosBook.org)



tessellate the state space by recurrent flows

classical trace formula for continuous time flows

$$\sum_{\alpha=0}^{\infty} \frac{1}{s - s_\alpha} = \sum_p T_p \sum_{r=1}^{\infty} \frac{e^{r(\beta A_p - s T_p)}}{\left|\det\left(\mathbf{1} - M_p^r\right)\right|}$$

relates the spectrum of the evolution operator

$$\mathcal{L}(x', x) \,=\, \delta\big(x' - f^t(x)\big)\; e^{\beta A(x,t)}$$

to the unstable periodic orbits p of the flow $f^t(x)$.

classical trace formula for averaging over 2-tori

we conjecture

$$\sum_{\alpha=0}^{\infty} \frac{1}{s - s_\alpha} = \sum_p T_p L_p \sum_{r=1}^{\infty} \frac{e^{r(\beta A_p - s T_p L_p)}}{(\det H_p)^r}$$

weighs the unstable relative prime (all symmetries quotiented) d-torus p by its variational Hessian H via Hill's formula

$$\det H_p = \det(1 - J_p)$$

speculation : code discrete Lagrangian methods?

the idea : construct a discrete counterpart to the considered system

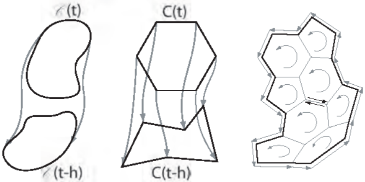
variational integrator : evolution map that corresponds to the discrete Euler–Lagrange equations

Discrete Lagrangian methods

action $S(q) = \int_0^T dt\, L(q, \dot{q})$ + Hamilton's principle $\delta S(q) = 0$

$$\text{discretize} \int_{t_k}^{t_{k+1}} L(q, \dot{q})\, dt \approx \Delta t L(q_k, q_{k+1})\,.$$

symplectic methods preserve phase-space areas[13]



(left) Kelvin's circulation advected by the flow is constant
(middle) the discrete version, on a Voronoi loop
(right) circulation is constant on any discrete loop.

[13] J. E. Marsden and M. West, Acta Numerica **10**, 357–514 (2001).

Discrete Lagrangian codes ?

so far, no codes for
discretized spatiotemporal action / Lagrangian density

$$S = \int dq^d \, \mathcal{L}(q)$$

symplectic Euler incompressible fluid dynamics time-evolution codes exist[14]
claim : can apply to non-conservative system

Navier-Stokes?

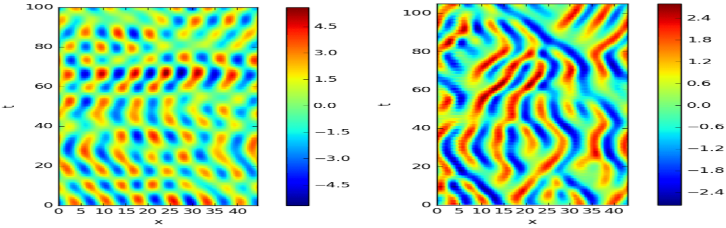
[14] D. Pavlov et al., Physica D **240**, 443–458 (2011).

XXX

XXX

XXX

an intermediate spacetime domain

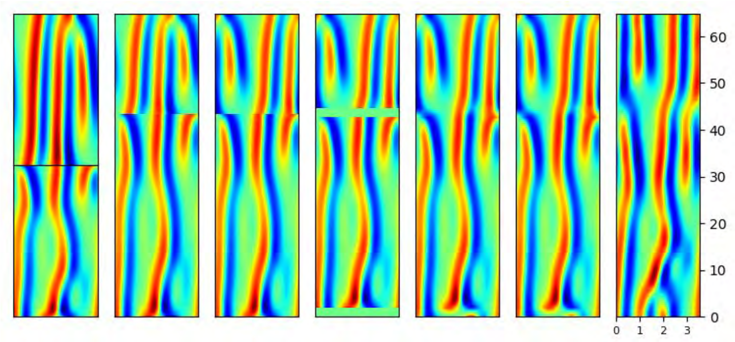


(left) $\bar{L} = 2\pi\sqrt{2}$ modulated initial random guess
$(L_0, T_0) = (5\bar{L}, 100) = (44.4, 100)$

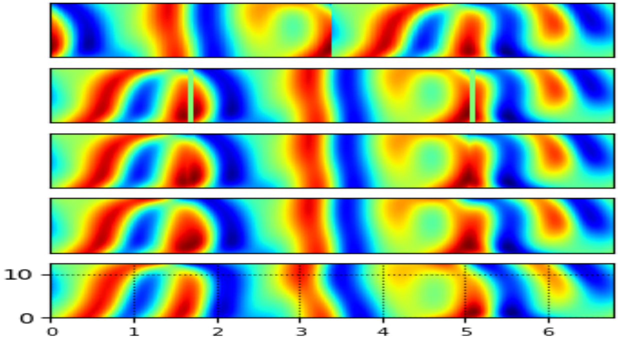
(right) Resulting invariant 2-torus
$(L_f, T_f) = (43.066, 105.08) = (L_0 - 1.363, T_0 + 5.08)$

Adjoint descent took only 7 laptop CPU seconds

temporally glued Frankenstein

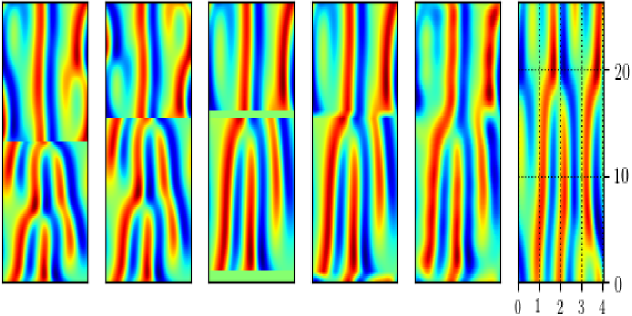


spatial gluing of two invariant 2-tori



1) two invariant 2-tori side by side
2) initial invariant 2-tori split into smaller tiles
3) a guess invariant 2-torus obtained by gluing / smoothing
4) converges to a larger invariant 2-torus

temporal gluing of two invariant 2-tori



1) an invariant 2-torus atop another invariant 2-torus
2) initial invariant 2-tori split into smaller tiles
3) a guess invariant 2-torus obtained by gluing / smoothing
4) converges to a larger invariant 2-torus

KS invariant 2-tori found by rocket science

15

the initial guess

the converged solution $u(x, t)$

15 Q. Wang et al., Phys. Fluids **25**, 110818 (2013).