Periodic orbits theory of turbulent flows

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Stratified turbulence in the 21st century
Kavli Royal Society Centre, Chicheley Hall, Newport Pagnell

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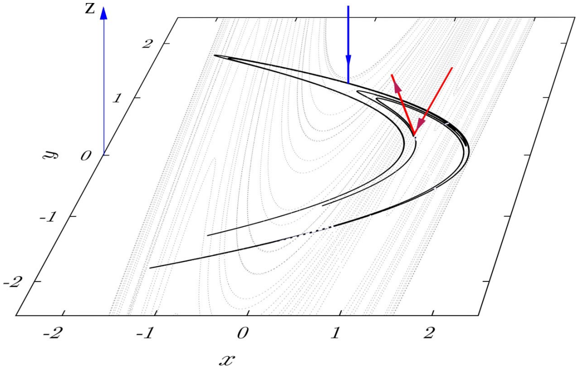
overview

1. what this talk is about
2. dynamical theory of turbulence
3. state space
4. symmetry reduction
5. dimension of the inertial manifold

what this talk is about:

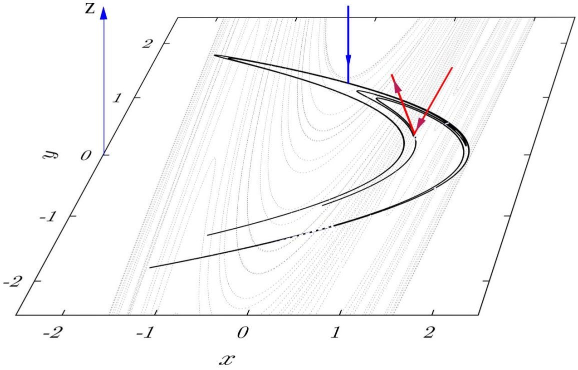
it is believed that the attracting set of a dissipative flow



- is confined to "*x*-*y* plane" :
  a finite-dimensional smooth *inertial manifold*
- "*z*" directions :
  the remaining ∞ of *transient dimensions*

what this talk is about:

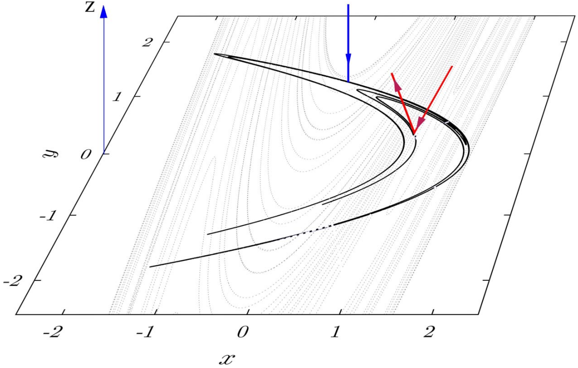
state space of dissipative flow is split into



- inertial manifold : spanned locally by entangled covariant vectors, tangent to unstable / stable manifolds
- the rest : spanned by the remaining ∞ of the contracting, decoupled, transient covariant vectors

what this talk is about:

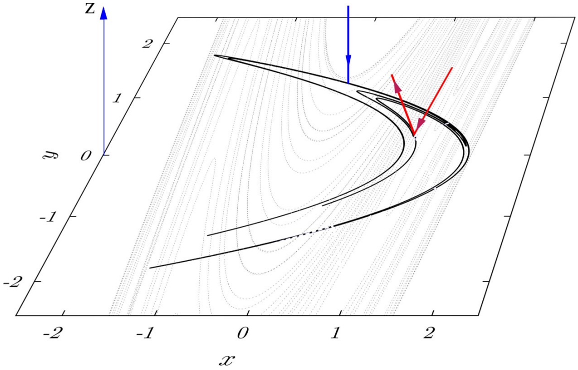
inertial manifold



- dynamics of the vectors that span the inertial manifold is entangled, with small angles and frequent tangencies
- a transient covariant vector : isolated, nearly orthogonal to all other covariant vectors

what this talk is about:

goal : construct inertial manifold for a turbulent flow



- tile it with a finite collection of bricks centered on recurrent states, each brick $\approx 10 - 100$ dimensions
- span of ∞ of transient covariant vectors : no intersection with the entangled modes

part 1

1. dynamical theory of turbulence
2. state space
3. symmetry reduction
4. dimension of the inertial manifold

a life in extreme dimensions

Navier-Stokes equations (1822)

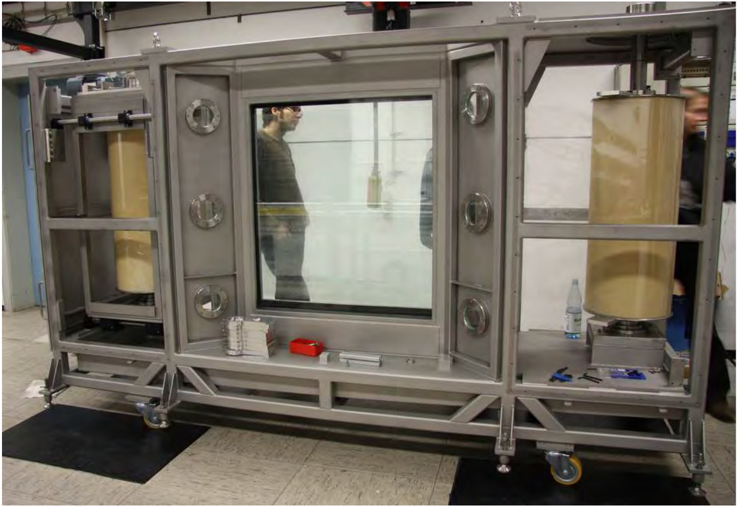
$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{1}{R}\nabla^2 \mathbf{v} - \nabla p + \mathbf{f}\,, \qquad \nabla \cdot \mathbf{v} = 0,$$

velocity field $\mathbf{v} \in \mathbb{R}^3$; pressure field p ; driving force $\mathbf{f}$

describe turbulence

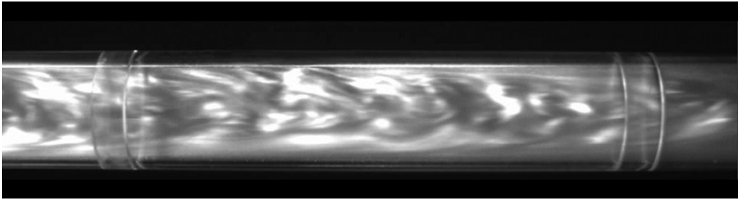
starting from the equations (no statistical assumptions)

plane Couette experiment



B. Hof lab

pipe experiment

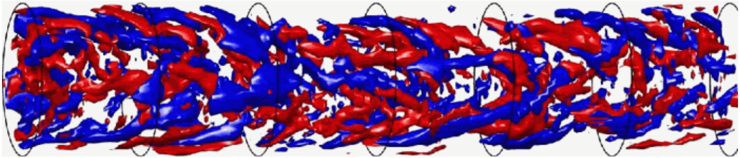


T. Mullin lab

pipe experiment data point

a state of turbulent pipe flow at instant in time

Stereoscopic Particle Image Velocimetry $\rightarrow$ 3-*d* velocity field over the entire pipe[1]



[1] Casimir W.H. van Doorne (PhD thesis, Delft 2004)

numerical challenges

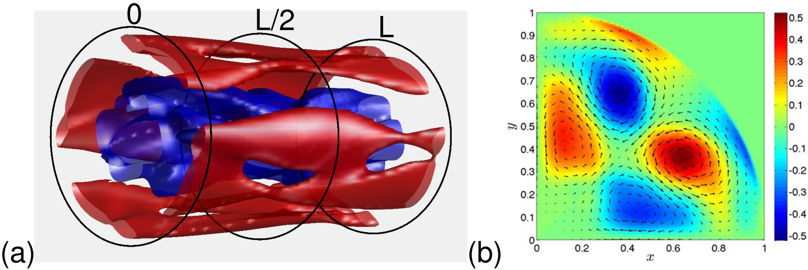
computation of turbulent solutions

requires 3-dimensional volume discretization
$\rightarrow$ integration of 10^4-10^6 coupled ordinary differential equations

challenging, but today possible

J.F. Gibson ChannelFlow.org
A. P. Willis OpenPipeFlow.org

openpipeflow.org simulation

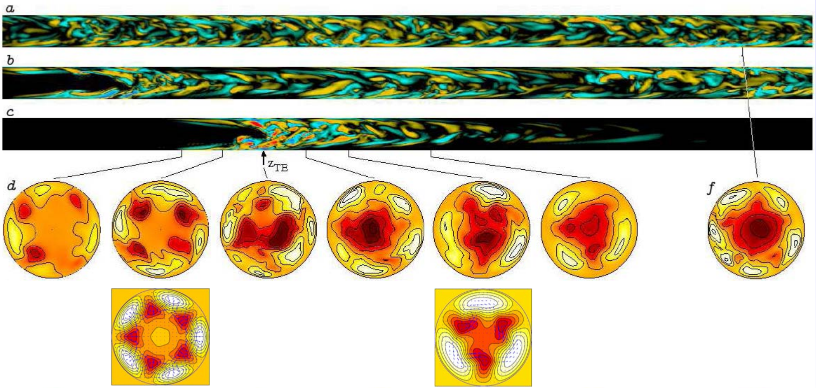


(a) A snapshot of a turbulent state in a pipe flow simulation. Fast ($0.1U$) (red) and slow ($-0.1U$) stream-wise velocity isosurfaces (blue).

b) color : averaged streamwise vorticity $\bar{\omega}_z$
arrows : $\bar{u}_r$ and $\bar{u}_\theta$ are averaged radial and azimuthal velocities

example : pipe flow

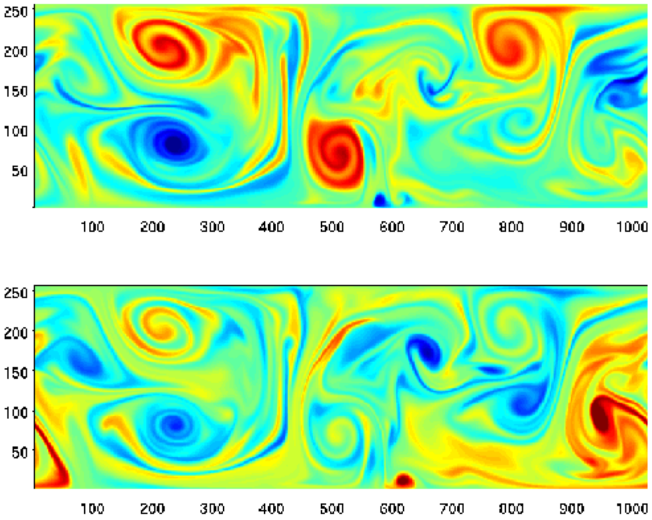
amazing data! amazing numerics!



- here each instant of the flow $\approx$ 2.5 MB
- videos of the flow $\approx$ GBs

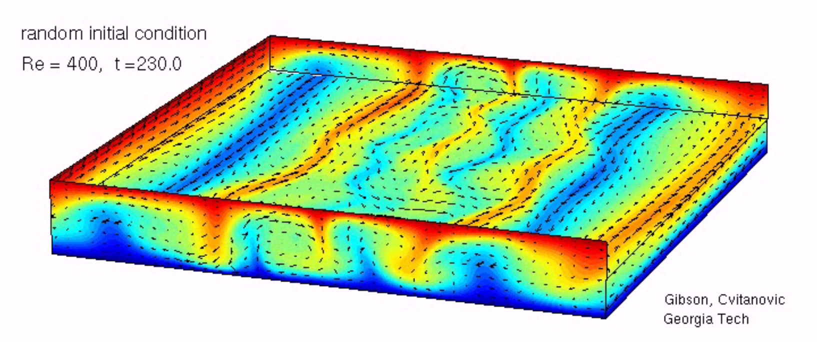
example : stratified flows in 20th century

2-layer baroclinic instability model



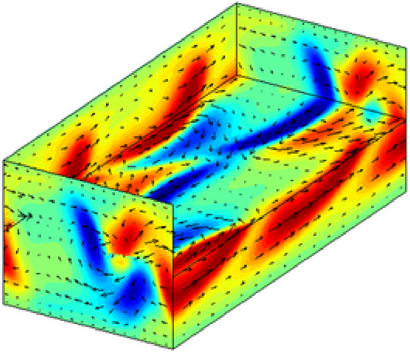
no success so far : please join us

example : plane Couette



velocity visualization

small computational cell, Eulerian velocity visualization



next - the same solution, different visualization

plane Couette isovorticity visualization

PRL **102,** 114501 (2009) PHYSICAL REVIEW LETTERS week ending 20 MARCH 2009

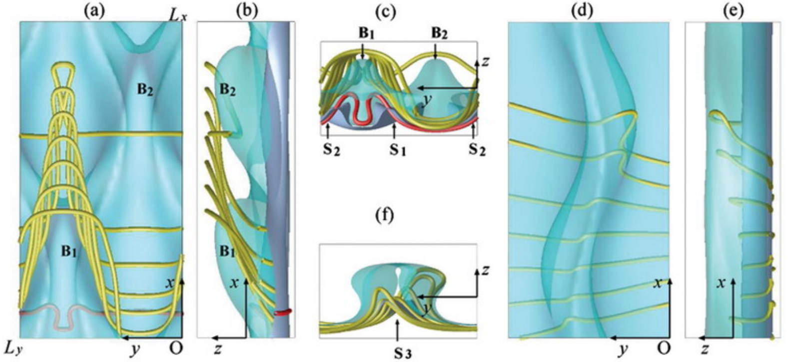


FIG. 3 (color). (a), (b), (c) The x-y, x-z, and y-z projections of the new state (upper branch) at Re $= 200$ in PCF. Yellow curves are vortex lines across the channel midplane (visualized over B_1, but not over B_2), underneath which there are low-speed structures visualized as isosurfaces of $u_x = -0.1$ and -0.4, colored by cyan [$z \approx \frac{1}{2}$ for the peaks of (B_1, B_2)] and blue [$z \approx -\frac{1}{2}$ for (S_1, S_2)], respectively. (d), (e), (f) Correspond to the same projections but for the upper branch of the NBW state. The vortex lines are integrated from the equivalent points located at $|z| = 0.8$ for both HVS and NBW.

part 2

1. dynamical theory of turbulence
2. state space
3. symmetry reduction
4. dimension of the inertial manifold

MUST look at it in

state space

E. Hopf 1948

dynamical description of turbulence

state space

a manifold $\mathcal{M} \in \mathbb{R}^d$: d numbers determine the state of the system

representative point

$x(t) \in \mathcal{M}$
a state of physical system at instant in time

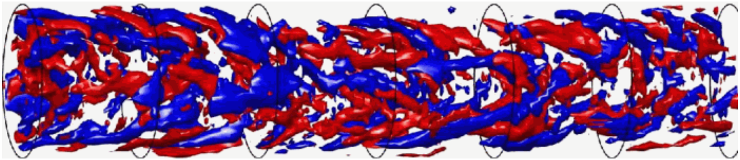
integrate the equations

trajectory $x(t) = f^t(x_0)$ = representative point time t later

example of a representative point (experimment)

$x(t) \in \mathcal{M}$, $d = \infty$
a state of turbulent pipe flow at instant in time

Stereoscopic Particle Image Velocimetry $\rightarrow$ 3-d velocity field over the entire pipe[2]

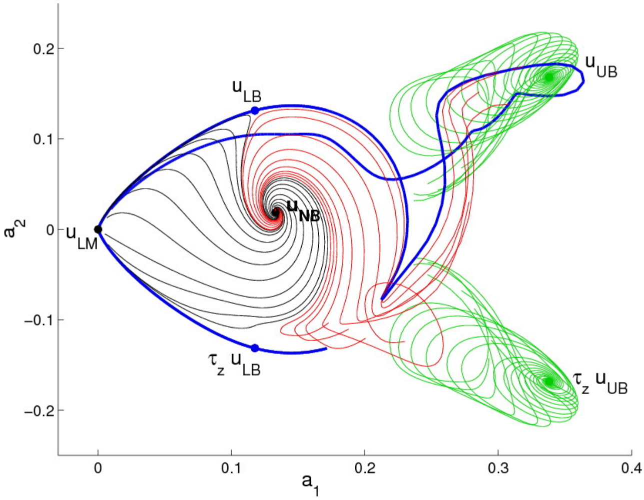


[2] Casimir W.H. van Doorne (PhD thesis, Delft 2004)

charting the state space of a turbulent flow

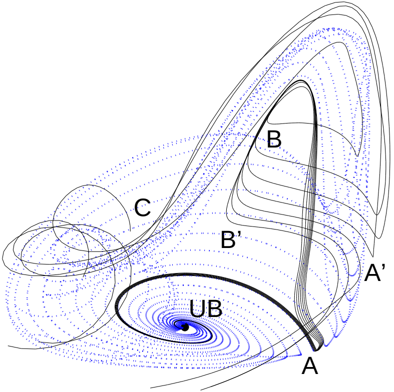
John F Gibson (U New Hampshire)
Jonathan Halcrow (Google)

can visualize 61,506 dimensional state space of turbulent flow



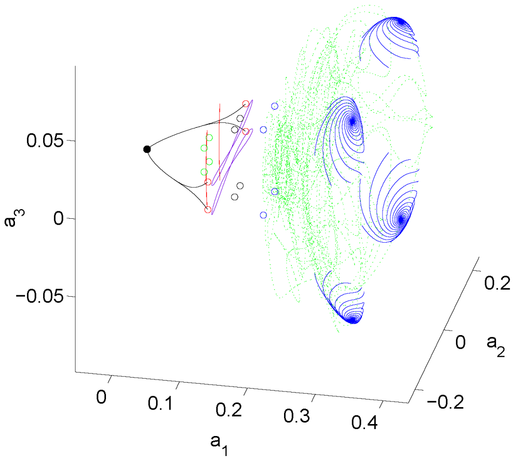
equilibria of turbulent plane Couette flow,
their unstable manifolds, and
myriad of turbulent videos mapped out as one happy family

61,506 dimensional : Nagata's "upper branch"



unstable manifold of an exact plane Couette equilibrium solution

plane Couette state space $10^5 \to 3D$



equilibria, periodic orbits, their (un)stable manifold

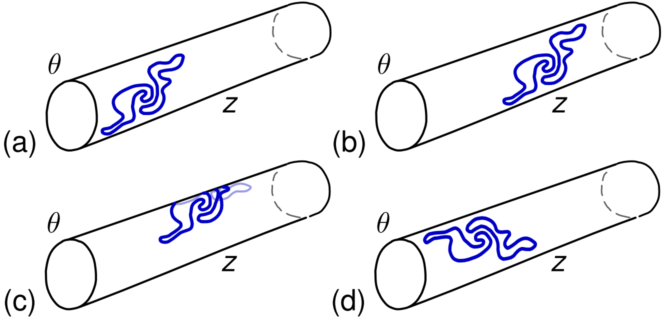
shape the turbulence

1. dynamical theory of turbulence
2. state space
3. **symmetry reduction**
4. dimension of the inertial manifold

symmetries help us

or do they?

example : $\mathrm{SO}(2)_z \times \mathrm{O}(2)_\theta$ symmetry of pipe flow



a fluid state, shifted by a stream-wise translation, azimuthal rotation g_p is a physically equivalent state

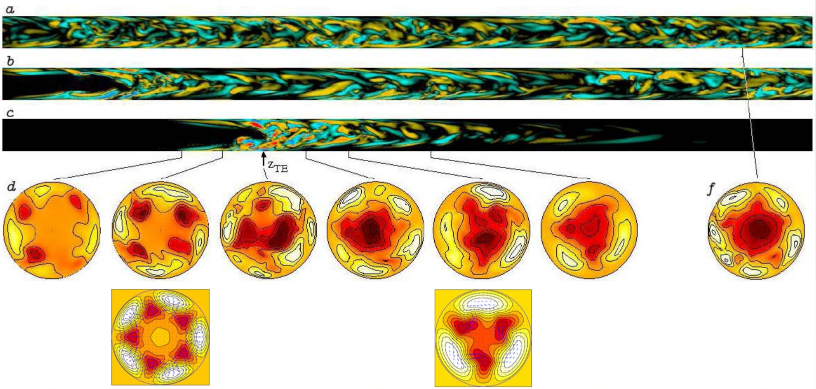
b) stream-wise

c) stream-wise, azimuthal

d) azimuthal flip

nature : turbulence in pipe flows

top : experimental / numerical data
bottom : theorist's solutions

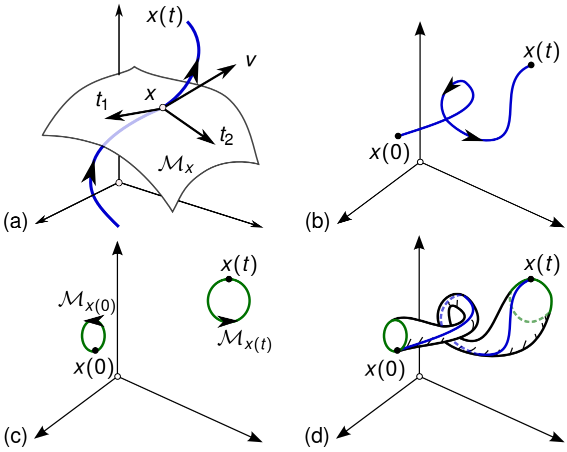


turbulence breaks all symmetries

in turbulence,

use of symmetry is subtle

state space trajectories, group orbits



(a) x tangent vectors:

$v(x)$ along time flow $x(t)$
$t_1(x), \cdots, t_N(x)$ group tangents

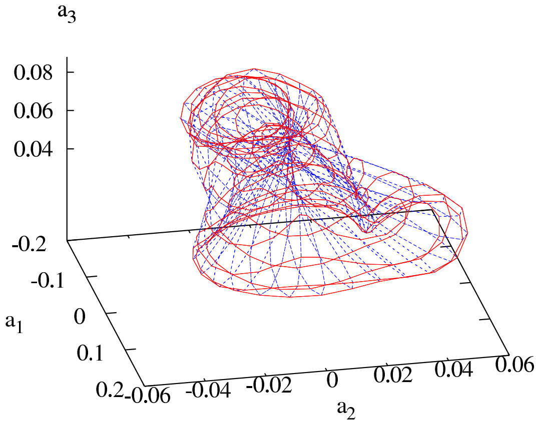
(b) trajectory $x(t)$

(c) group orbits $g\,x(t)$

(d) ‘tube’ $g\,x(t)$

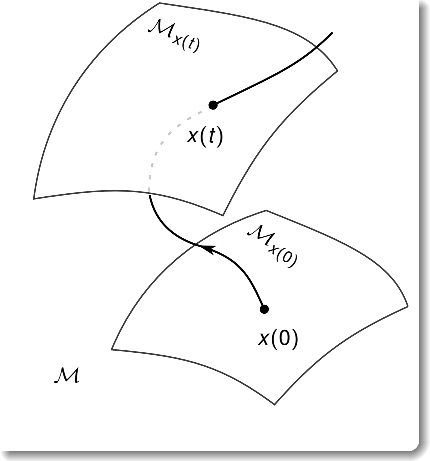
example : group orbit of a pipe flow turbulent state

$SO(2) \times SO(2)$ symmetry $\Rightarrow$ group orbit is topologically 2-torus



group orbits of highly nonlinear states are topologically tori
but highly contorted tori

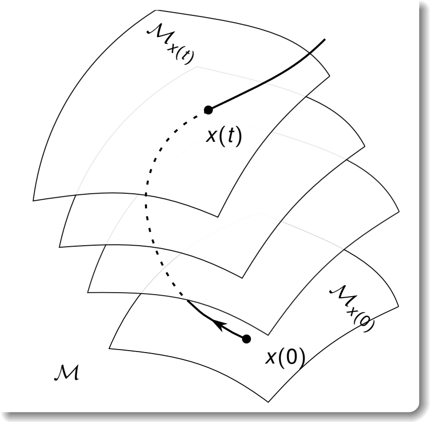
foliation by group orbits



group orbit $\mathcal{M}_x$ of x is the set of all group actions

$$\mathcal{M}_x = \{g\,x \mid g \in G\}$$

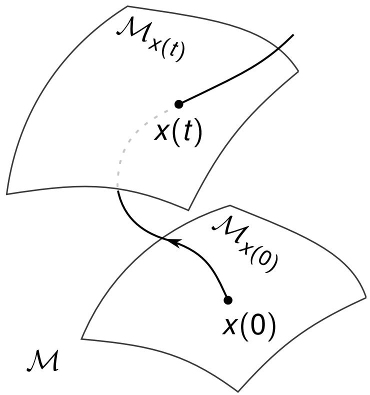
foliation by group orbits



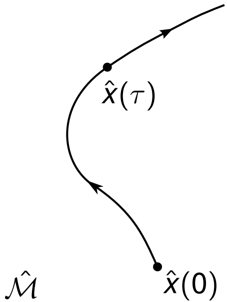
actions of a symmetry group foliates the state space $\mathcal{M}$ into a union of group orbits $\mathcal{M}_x$

each group orbit $\mathcal{M}_x$ is an equivalence class

full state space



reduced state space



the goal : replace each group orbit by a unique point in a lower-dimensional

symmetry reduced state space $\mathcal{M}/G$

symmetry reduction : how?

continuous symmetry reduction in high-dimensional flows

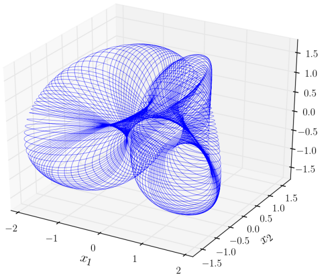
with

the method of slices

- cut group orbits by hypersurfaces
- each group orbit of symmetry-equivalent points represented by the single point

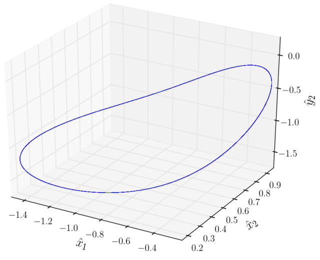
relative periodic orbit, in full state space and in slice

full state space



repeats of a relative periodic orbit trace out a torus

reduced state space



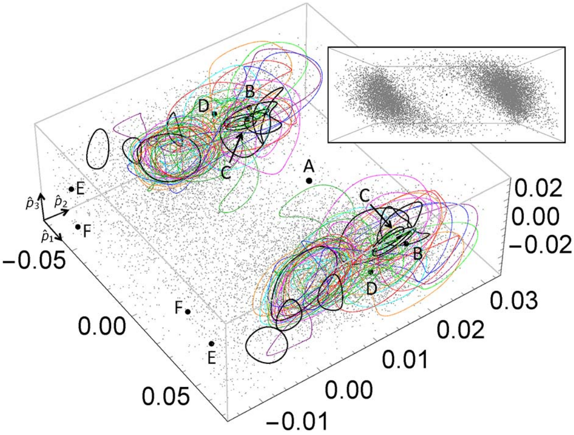
relative periodic orbit
is reduced to periodic orbit

it works !

Kimberly Short
Ashley Willis

it works : all pipe flow solutions in one happy family

symmetry-reduced infinite-dimensional slice : a 3D projection



grey cloud : the natural measure

32 relative periodic orbits, 6 relative equilibria

periodic orbits capture the natural measure density well

could not find without symmetry reduction :

1 spatial dimension “Navier-Stokes”

computationally not ready yet to explore the inertial manifold of $3D$ turbulence - start with

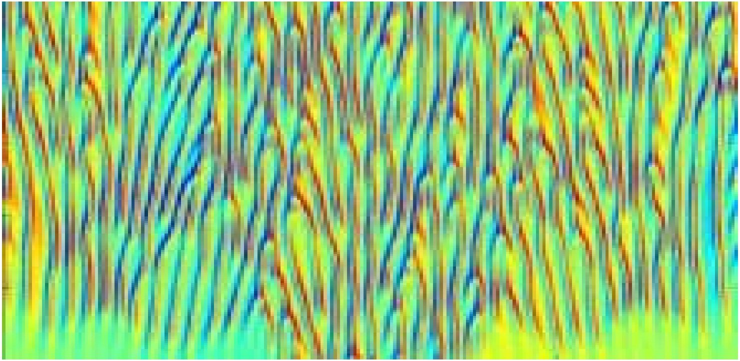
Kuramoto-Sivashinsky equation’

$$u_t + u\nabla u = -\nabla^2 u - \nabla^4 u \,, \qquad x \in [-L/2, L/2] \,,$$

describes spatially extended systems such as

- flame fronts in combustion
- reaction-diffusion systems
- ...

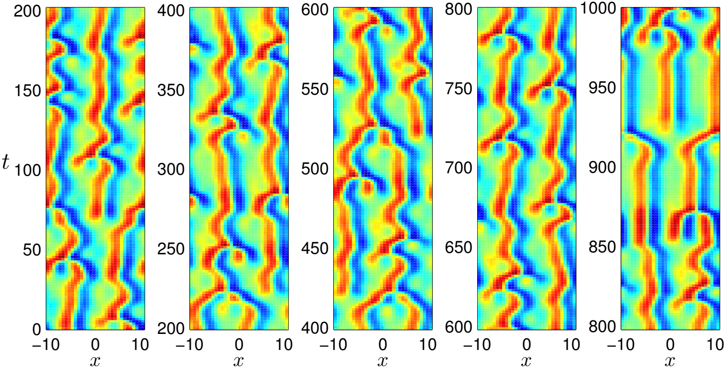
Kuramoto-Sivashinsky on a large domain



[horizontal] space $x \in [0, L]$ [up] time evolution

- turbulent behavior
- simpler physical, mathematical and computational setting than Navier-Stokes

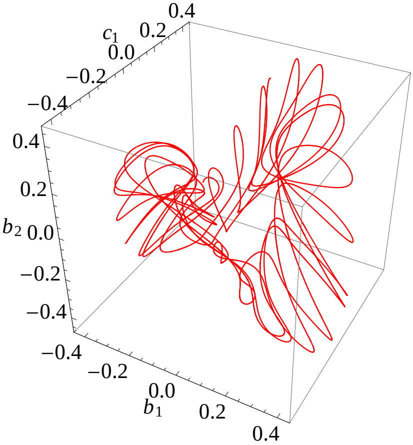
evolution of Kuramoto-Sivashinsky on small $L = 22$ cell



horizontal: $x \in [-11, 11]$
vertical: time
color: magnitude of $u(x, t)$

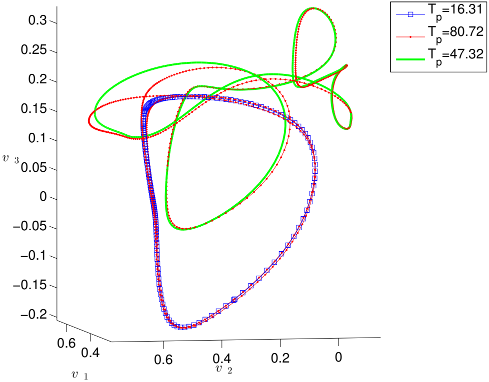
a relative periodic orbit

full state space : many periods



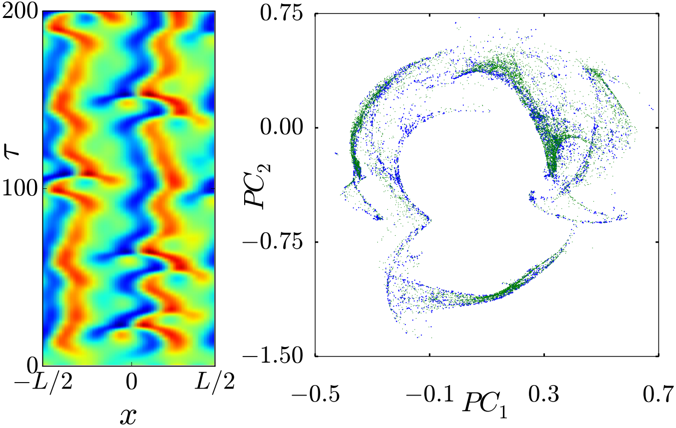
have computed: 60 000 periodic orbits

can explore shadowing



(impossible without symmetry reduction)

periodic orbits are dense in the attractor



- [left] turbulent trajectory segment in [space×time]
- Poincaré section, turbulent trajectory (natural measure)
- periodic points, from 479 periodic orbits[3]

[3] Budanur (PhD thesis 2015)

part 4

1. dynamical theory of turbulence
2. state space
3. symmetry reduction
4. **dimension of the inertial manifold**

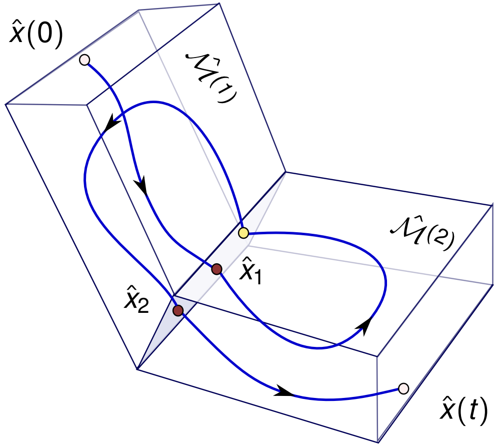
cartography for fluid dynamicists

cover the inertial manifold with a set of flat charts

we can do this with

finite-dimensional bricks embedded in 10^2 dimensions!

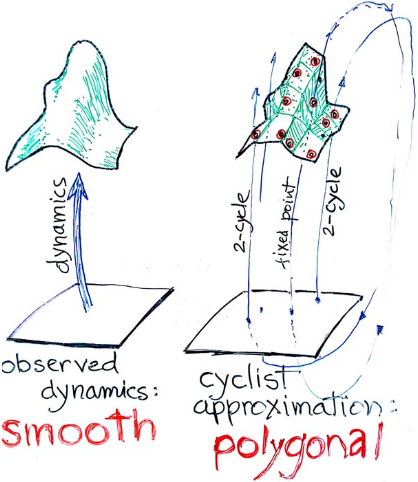
charting the inertial manifold



two tangent “entangled” tiles = finite-dimensional bricks

shaded plane : when integrating your equations, switch the local chart

tessellate the state space by **recurrent flows**



a fixed point

a 2-cycle alternates between neighborhoods

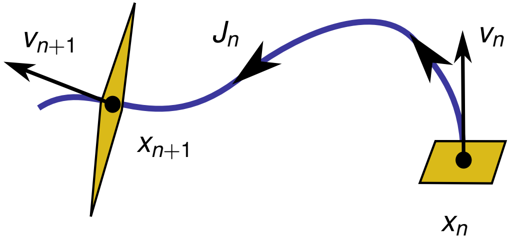
smooth dynamics (left frame)
tesselated by the skeleton of recurrent flows,
together with (right frame) their linearized neighborhoods

what is the dimension of the inertial manifold?

we determine it in 6 independent ways

- Lyapunov exponents (plausible, previous work)
- Lyapunov vectors (sharp, previous work)
- four periodic orbits determinations (presented here for the first time :)

linearized deterministic flow



$$x_{n+1} + z_{n+1} = f(x_n) + J_n\, z_n\,, \quad J_{ij} = \partial f_i / \partial x_j$$

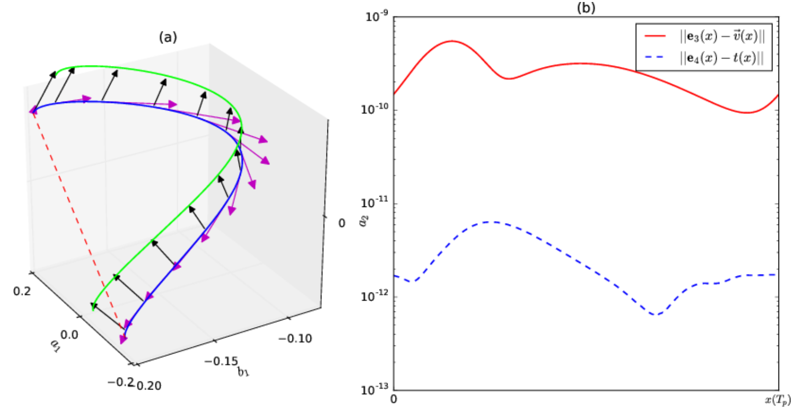
in one time step a linearized neighborhood of x_n is

(1) advected by the flow

(2) transported by the Jacobian matrix J_n into a neighborhood given by the J eigenvalues and eigenvectors

a relative periodic orbit : one period

marginal modes Floquet frame



[right panel]
all eigenvectors computed close to the machine precision

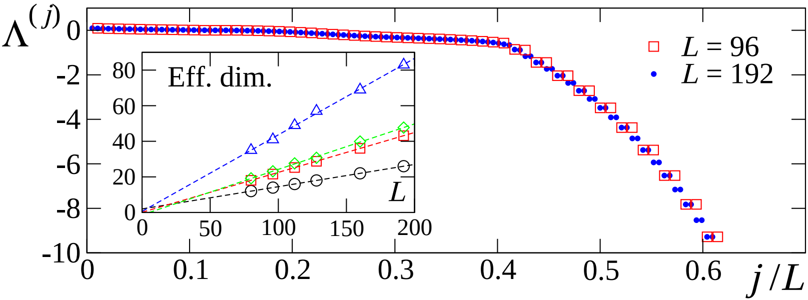
(1) algorithmic breakthrough : all Floquet exponents to machine precision

	$\mu^{(i)}$	$e^{iT_p\omega^{(i)}}$
1=2	0.0331970261043278	-0.42330856002164 + i 0.905985575499084
3=4	(2 marginal)	
5	-0.216338085869672	1
6=7	-0.265233812289065	-0.867175421594352 + i 0.49800279937231
...	...	...
29	-316.19797864063	1
30	-320.666664811713	-1

Floquet exponents for the shortest pre-periodic orbit :

$\mu^{(i)}$ = real part of the exponent.
either the multiplier sign for a real exponent, or
$\omega^{(i)} \rightarrow$ the multiplier phase for a complex Floquet exponent

(1) Kuramoto-Sivashinsky Lyapunov spectrum two large cells : it scales!



Now double # Fourier modes : all new ones go to the transient spectrum [4]

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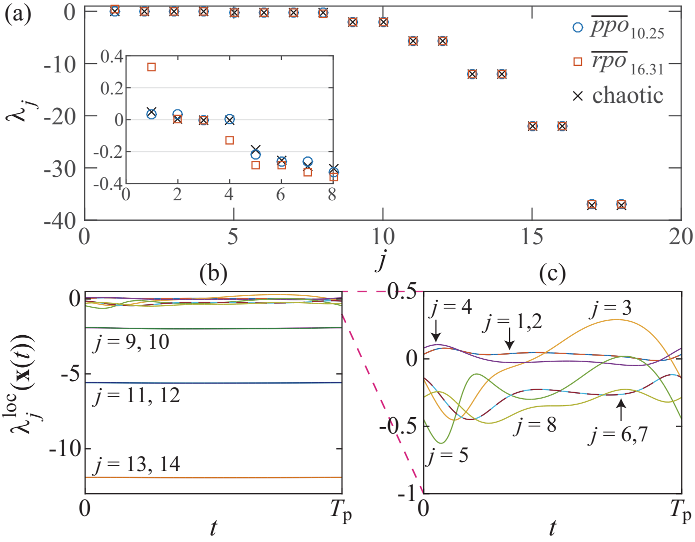
[4] Yang et al (Phys. Rev. Lett. 2009)

(1) dimension of the inertial manifold from an individual orbit

- Floquet exponents separate into entangled vs. transient

(1) Floquet and Lyapunov exponents, $L = 22$ small cell



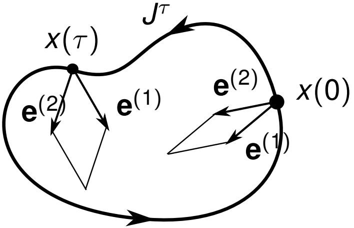
8 entangled modes, rest transient :

inertial manifold is 8 dimensional!

(2) dimension of the inertial manifold from ensemble of orbits

- principal angles between hyperplanes spanned by Floquet vectors

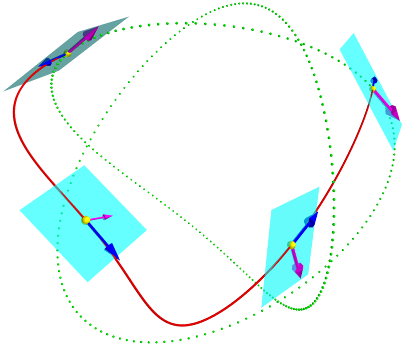
(2) Floquet vectors



a parallelepiped spanned by a pair of Floquet eigenvectors (‘covariant vectors’) transported along the orbit

- Jacobian matrix not self-adjoint : the eigenvectors are not orthogonal, the eigenframe is ‘non-normal’
- Measure the angle between eigenvectors $\mathbf{e}^{(i)}(x(t))$ and $\mathbf{e}^{(j)}(x(t))$

(2) example : Kuramoto-Sivashinsky relative periodic orbit

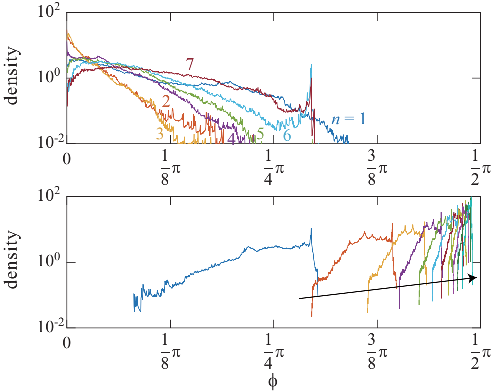


dotted green : a group orbit

solid red : a relative periodic orbit

planes : a tangent space spanned and transported by 2 Floquet vectors

(2) distribution of principal angles between Floquet subspaces



histogram of angles between S_n (n leading Floquet vectors) and $\bar{S}_n$ (the rest), accumulated over the 400 orbits :

- (top) For $n = 1 \cdots 7$ (S_n within the entangled manifold) the angles can be arbitrarily small
- (bottom) For the $\bar{S}_n$ spanned by transient modes, $n = 8, 10, 12, \cdots, 28$: angles bounded away from unity

(3), (4) dimension of the inertial manifold from a chaotic trajectory shadowing a given orbit

two independent measurements

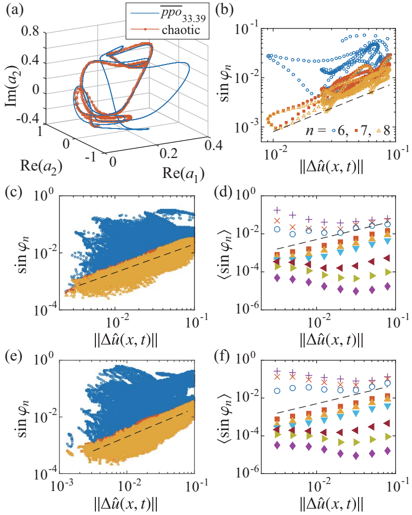
(3) shadowing separation vector lies within the orbit's Floquet entangled manifold

(4) shadowing separation vector lies within the chaotic trajectories covariant vectors' entangled manifold

'separation vector' = difference vector between the chaotic orbit point and periodic orbit point at their (locally) closest passage

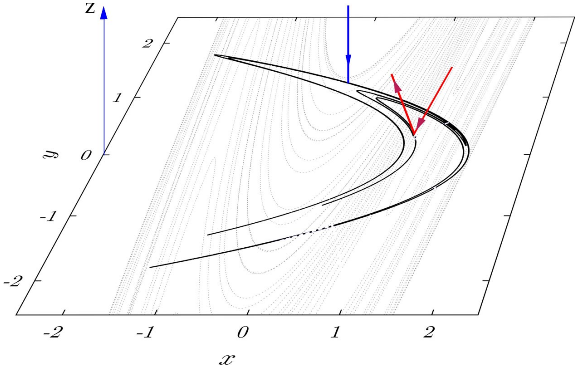
accumulate 1000's of near recurrences

(3) chaotic trajectory shadows periodic orbits within the entangled subspace



summary for the impatient

state space of dissipative flow is split into



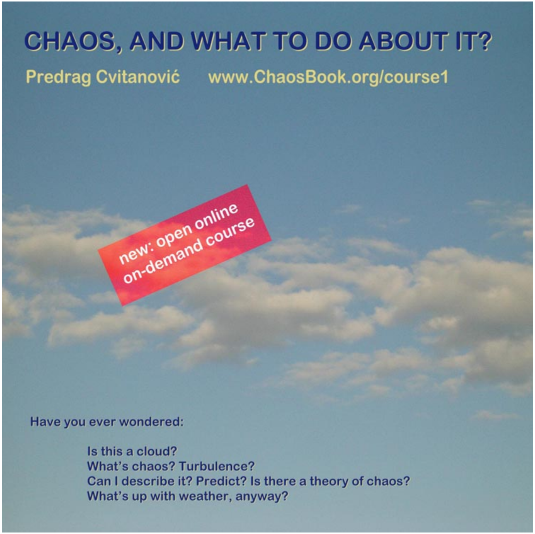
- inertial manifold : spanned locally by entangled covariant vectors, tangent to unstable / stable manifolds
- the rest : spanned by the remaining ∞ of the contracting, decoupled, transient covariant vectors

detailed summary
6 ways to determine the dimension of the inertial manifold

Tangent spaces separate into entangled vs. transient

1. Lyapunov exponents (plausible, previous work)
2. Lyapunov vectors (sharp, previous work)
3. for each individual orbit Floquet exponents separate into entangled vs. transient (new)
4. for an ensemble of orbits principal angles between hyperplanes spanned by Floquet vectors separate into entangled vs. transient (new)
5. for a chaotic trajectory shadowing a given orbit the separation vector lies within the orbit's Floquet entangled manifold (new)
6. for a chaotic trajectory shadowing a given orbit the separation vector lies within the chaotic trajectories covariant vectors' entangled manifold (new)

what next? take the course!



student raves :

...10^6 times harder than any other online course...