TITLE:	*Linear encoding of the spatiotemporal cat*
AUTHOR:	*B Gutkin, P Cvitanovic, R Jafari, A Saremi, L Han*
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This work is concerned with the computation of probabilities of local spatio-temporal patterns in a family of coupled map lattices, consisting of a one-dimensional array of hyperbolic toral automorphisms. The latter are chosen from a much studied one-parameter family —the 'cat maps'.

The authors construct a two-dimensional spatio-temporal shift space, where the symbolic language, generated from a finite integer alphabet, has the property that orbits and symbolic orbits are related by a linear transformation. (This is a generalisation of an established procedure.) The shift space is decomposed into the union of a full shift and a domain where pruning occurs, with a non-trivial grammar (not of finite type).

The main object of study are finite two-dymensional convex blocks of symbols, representing spatio-temporal patterns that specify the orbit with exponential accuracy. The main result is a numerically stable procedure for computing the probability of the occurrence of such blocks, combining Green's function technique with a geometrical approach.

To the best of my knowledge, these results are new and correct. Even though —for my taste— some physical motivations are overstated (how physically significant are the cat maps?), the questions addressed are compelling and the advances made are sound.

For these reasons, I believe that this work deserves publication in *Nonlinearity*. But before that, besides the customary small corrections, two substantial problems must be properly addressed:

1. The meaning of some key constructs is obscured by poor notation and terminology.

2. Some key results should be formalised as theorems; some numerical results should be distilled into conjectures.

It is hoped that the second item will help the authors address the first. Details follow (line numbers refer to those on left margins).

MAIN POINTS

1. The definition and usage of the main objects of study is confused by inappropriate or ambiguous notation and unsettled terminology, which make reading tiresome. I provide a sample of the problems.

On p.3, the quantity $\{x_z \in \mathbb{T} : z \in \mathbb{Z}^d\}$ is first a 'lattice state' (shorthanded $\{x_z\}$), and few lines below a 'spatiotemporal solution', called X. As written, X is rather a countable collection of circles, indexed by $\mathbb{Z}^2$. If its elements x_z are to be connected by (3), then surely this must be part of the symbolic definition. For the latter, the use a Zermelo-type set notation is plainly inappropriate, since X is not a set (see below).

The quantity M suffers from the same problem, compounded by an excessive overloading of the symbol. Initially (bottom of p.3) M is referred to as a 'block', then (p. 17) as an 'integer lattice', a 'code', and then again a 'block'. On page 4 and beyond, the term 'block' means a finite symbolic words. These blocks appear as $\mathrm{M}_{\mathcal{R}}$, to become just b in one dimension (bottom of p.11).

The nature of the subscripts of M fluctuates throughout the paper. In one dimension b is a block, but in two dimensions $\mathcal{R}$ is not a block but rather a domain. On p.28, two lines after being reminded that the substcript $\mathcal{R}_i$ denotes a restriction, the stranded reader will encounter integer subscripts.

Let me indicate how to restore some order. X and M are *functions* $\mathrm{X} : \mathbb{Z}^2 \to [0, 1)$, and $\mathrm{M} : \mathbb{Z}^2 \to \mathcal{A}$. The former is defined recursively from an initial set of data (which, incidentally, needs to be described), the latter is defined from X via a linear relation. A block B (b in one dimension) is also a function $\mathrm{B}_{\mathcal{R}} : \mathcal{R} \to \mathcal{A}$, whose finite and convex domain $\mathcal{R}$ —the *base* of the block— is inscribed in a rectangle $\Pi_{\mathcal{R}}$ of minimal size. The latter is then embedded canonically in $\mathbb{Z}^2$, to allow comparisons with the restriction $\mathrm{M}|_{\mathcal{R}}$, using, say, Hamming's distance.

2. Since the overall level of rigour is considerable, the authors should attempt to convert the main results into theorems. If gaps of rigour obstruct this process, these should be spelled out clearly.

Moreover, given the advertised effectiveness of the computation of exact probabilities, I find it disappointing that the numerical experiments did not result in the formulation of conjectures. In fact, I don't even know if and where the authors have pushed their computations to the limit, and in this respect some estimate of the complexity of the algorithms would also be welcome.

DETAILS

At the start, this paper gives the impression to cover the d-dimensional case. On page p.3, the setting is still d-dimensional, but the formula for the d-dimensional Laplacian is not given. Two pages later (p. 5, line 55), we learn that to 'streamline the exposition', only the $d = 2$ case will be treated. The actual scope of the higher-dimensional analysis should be spelled out from the beginning.

[p. 2, l.9] The meaning of the symbols in the displayed formula is clarified only in the following section. Symbols must be explained before, or immediately after, their first appearance.

[p. 3, l.16–17] Avoid using juxtaposed subscripts, e.g., x_{nt} in place of $x_{n,t}$. They create notational schizophrenia, as in equations (3) and (6).

[p. 3, formula (4)] Explain here the rationale for the choice of the alphabet. At the moment one has to wait until page 8.

[p. 3, last paragraph] The second sentence is badly phrased. The fact that given X, the block M is unique and admissible is a property of every symbolic dynamics. You could be referring to the injectivity brought about by linearity. I found the converse statement misleading, since not all Ms are admissible. Please clarify.

[p. 4, l.9,12,33] Explain the meaning of the term 'generic' in the present context. Is the measure μ appearing at line 48 connected to this?

[p. 5, l.26] '$d_{\mathcal{R}}$ depends only on the shape of $\mathcal{R}$' The term 'shape' is vague, and it shouldn't be difficult to make it precise using, say, an embedding rectangle. Is $|\mathcal{R}|$ fixed here?

[p. 5, l.32] 'small $\mathcal{R}$' should be 'small $|\mathcal{R}|$'.

[p. 6, l.25] 'length of the block'. A block does not have a length, unless you are in one dimension, in which case this must be stated.

[p. 9, table 1] The usefulness of displaying approximate numerical values is unclear to me, and even to the authors themselves (p. 12).

[p. 10, caption of fig.2] The labels (a) to (i) do not appear in the figure. Also, the information displayed on page 10 should not require an entire page.

[p. 12, l.14] The 'unwieldy and unilluminating' computations may nonetheless be telling an interesting story. Is there a more informative description of the authors' findings?

[p. 16, table 4] Pity that you don't attempt any arithmetical justification. Is this a pure exponential? Can you formulate a conjecture?

[p. 18, l.18] Explain precisely what 'simply connected' means for a subset of $\mathbb{Z}^2$.

[p. 37, l.17] The verb 'verify' suggests that the vanishing of the geometrical part of the entropy is heuristic. If this is not the case, then replace it with 'illustrate'. In any case, fig C1 is rather uninformative; one would like some information of the rate of convergence.

[p. 37, last formula] How fast is this limit approached?

GRAMMAR/TYPOS

[p. 11, l.41-42] Replace ‘(x_0, p_0) phase space’ with ‘phase space coordinates (x_0, p_0)’

[p. 17, l.32] [p. 8, eq (17)] If you write $\underline{1}$ to mean -1, then $\underline{|m_z|}$ means $-|m_z|$, not $-m_z$ as stated. [For consistency, you should also write $\underline{\square} + s\underline{4}$, and 1$\underline{\text{d}}$imensional.]

[p. 18, l.50] If ℓ_1 or ℓ_2 is even, the barycentre z is not a lattice point; either round it to a lattice point, or use a different symbol.

[p. 31, l.22] Shouldn’t g be $\mathbf{g}$? Also, ‘Green’s function g’ should be ‘the Green’s function g’, but on p.34,l.9, ‘the Green’s theorem’ should be ‘Green’s theorem’.