Continuous symmetry reduction for high-dimensional flows

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30 Jun 2010

Die Vorlesung
Das Problem

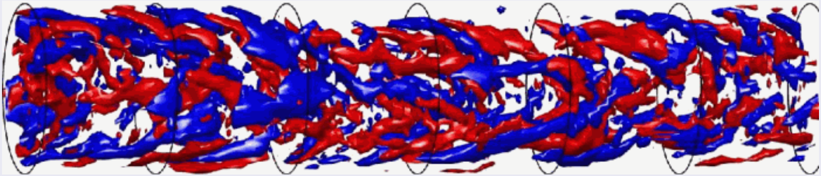
dynamical description of turbulence?

progress 1990-2010:

- flames
- hearts
- pipes
- planes
- cosmos
- gluons

amazing data! amazing numerics!

3D turbulent pipe flow



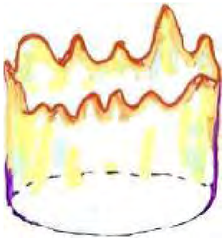
flames: Kuramoto-Sivashinsky equation

1-dimensional "Navier-Stokes"

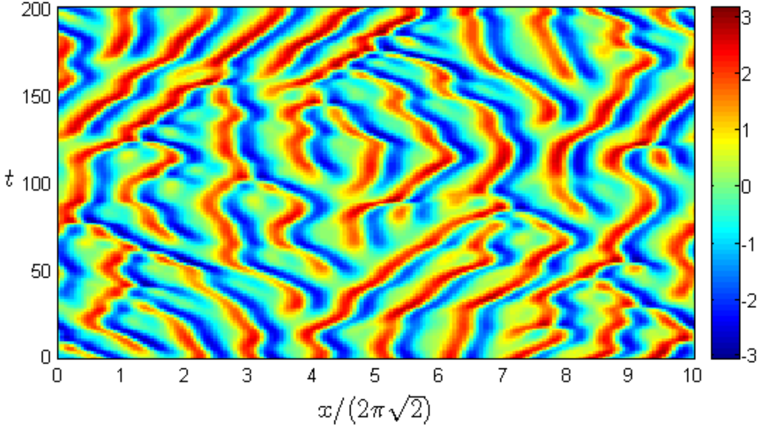
$$u_t + u\nabla u = -\nabla^2 u - \nabla^4 u\,, \qquad x \in [-L/2, L/2]\,,$$

describes extended systems such as

- reaction-diffusion systems
- flame fronts in combustion
- drift waves in plasmas, ...



a turbulent flame



a peak at a pipe flow experiment

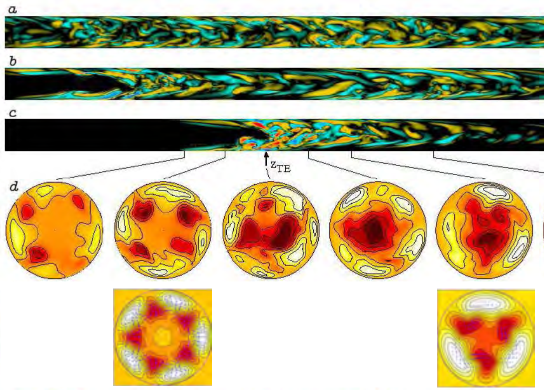
a modern pipe flow experiment

B. Hof, "Complex Dynamics and Turbulence,"

Ludwig Prandtl's office in 2009

one of the groups keeping the Ludwig Prandtl's flame alive in Göttingen.

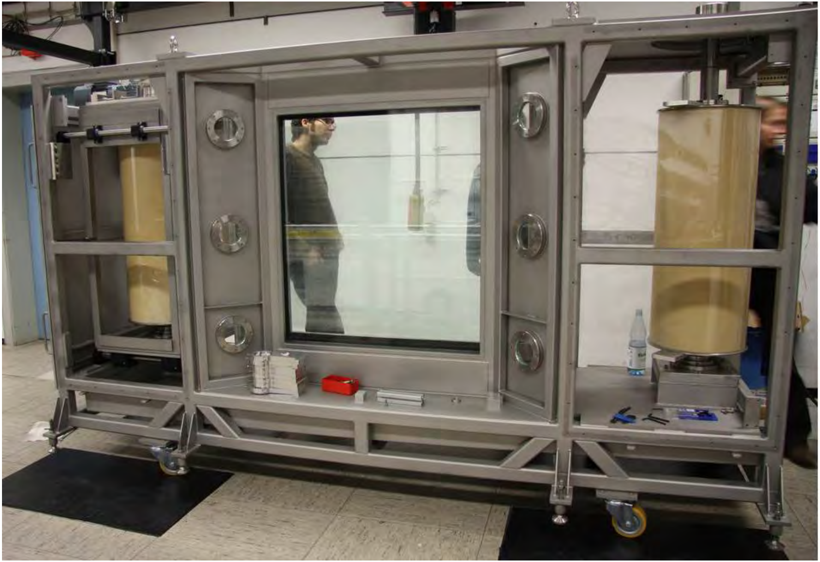
am



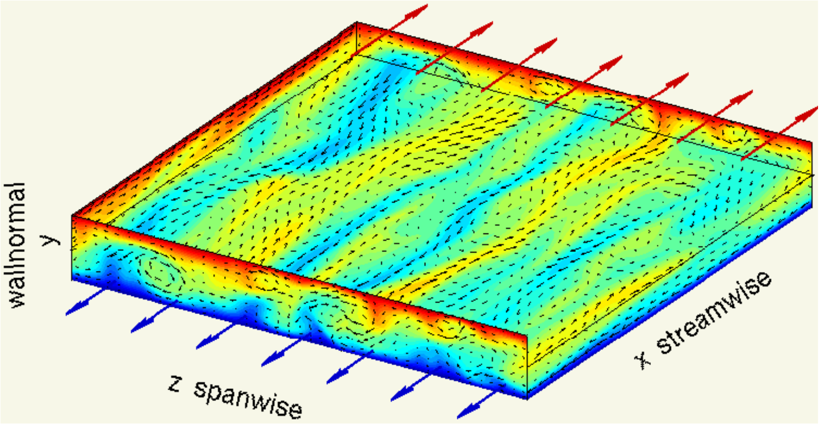
solutions are

- rotationally equivariant
- translationally equivariant

plane Couette flow: Göttingen experiment



plane Couette flow: fully resolved simulations



symmetries of Kuramoto-Sivashinsky equation

with periodic boundary condition

$$u(x,t) = u(x+L,t)$$

the symmetry group is $O(2)$:

- translations: $\tau_{\ell/L}\, u(x,t) = u(x+\ell,t)\,, \qquad \ell \in [-L/2, L/2]\ ,$
- reflections: $\kappa\, u(x) = -u(-x)\,.$

translational symmetry $\to$ traveling wave solutions

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translational symmetry $\rightarrow$ traveling wave solutions

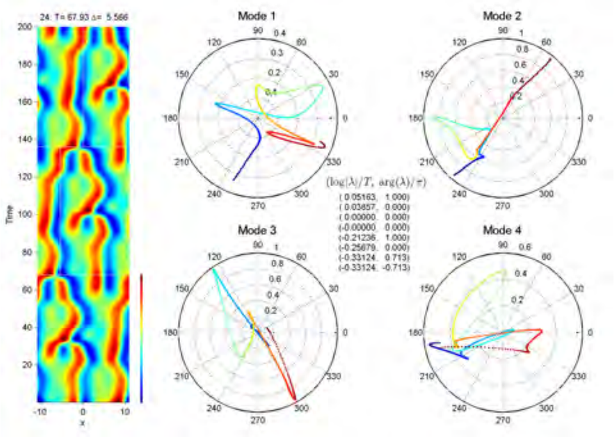
traveling (or relative) unstable coherent solutions are ubiquitous in turbulent hydrodynamic flows

continuous symmetries

translational symmetry $\Rightarrow$

- traveling wave solutions
- unstable relative periodic orbits

unstable relative periodic orbits



- have computed 40,000 unstable periodic and relative periodic orbits.
- how are they organized?

continuous symmetries

question

what are the invariant objects that organize phase space in a spatially extended system with translational symmetry and how do they fit together to form a skeleton of the dynamics?

state space

- the space in which all possible states u's live
- ∞-dimensional:
  point $u(x)$ is a function of x on interval $x \in L$.
- in practice:
  a high but finite dimensional space (e.g. through a spectral discretization)

state space

take the hint from low dimensional systems

- low dimensional systems:
  equilibria, periodic orbits organize the long time dynamics.
- is this true in extended systems?

from complex Lorenz flow $5D$ attractor $\rightarrow$ unimodal map

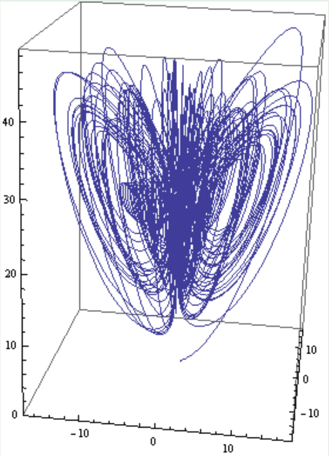
complex Lorenz equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{y}_1 \\ \dot{y}_2 \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -\sigma x_1 + \sigma y_1 \\ -\sigma x_2 + \sigma y_2 \\ (\rho_1 - z)x_1 - \rho_2 x_2 - y_1 - e y_2 \\ \rho_2 x_1 + (\rho_1 - z)x_2 + e y_1 - y_2 \\ -bz + x_1 y_1 + x_2 y_2 \end{bmatrix}$$

$\rho_1 = 28, \rho_2 = 0, b = 8/3, \sigma = 10, e = 1/10$

- A typical $\{x_1, x_2, z\}$ trajectory
- superimposed: a trajectory whose initial point is close to the relative equilibrium Q_1

attractor



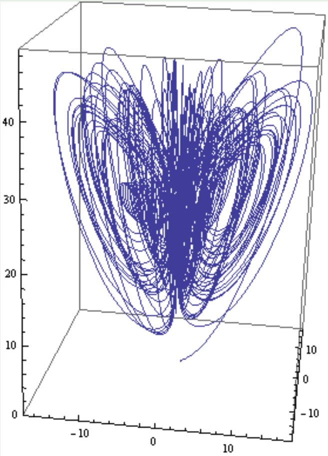
from complex Lorenz flow $5D$ attractor $\to$ unimodal map

what to do?

the goal

reduce this messy strange attractor to a 1-dimensional return map

attractor



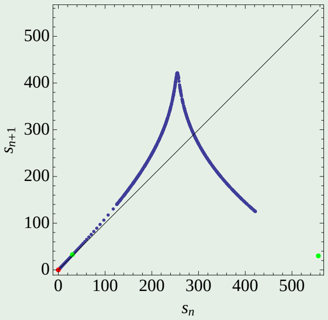
from complex Lorenz flow $5D$ attractor $\rightarrow$ unimodal map

the goal attained

but it will cost you

after symmetry reduction; must learn how to quotient the $SO(2)$ symmetry

1D return map!



Lie groups elements, Lie algebra generators

An element of a compact Lie group:

$$g(\theta) = e^{\theta \cdot \mathbf{T}}\,, \qquad \theta \cdot \mathbf{T} = \sum \theta_a \mathbf{T}_a,\ a = 1, 2, \cdots, N$$

$\theta \cdot \mathbf{T}$ is a *Lie algebra* element, and θ_a are the parameters of the transformation.

example: $SO(2)$ rotations for complex Lorenz equations

$SO(2)$ rotation by finite angle θ:

$$g(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta & 0 \\ 0 & 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

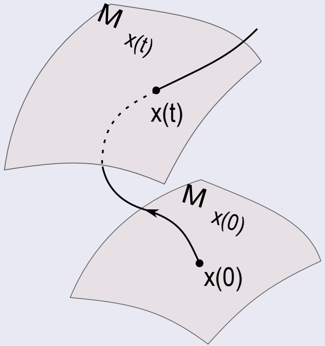
symmetries of dynamics

A flow $\dot{x} = v(x)$ is G-equivariant if

$$v(x) = g^{-1}\, v(g\,x)\,, \qquad \text{for all } g \in G\,.$$

foliation by group orbits

group orbits

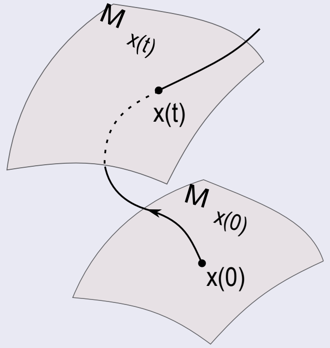


group orbit $\mathcal{M}_x$ of x is the set of all group actions

$$\mathcal{M}_x = \{g\,x \mid g \in G\}$$

foliation by group orbits

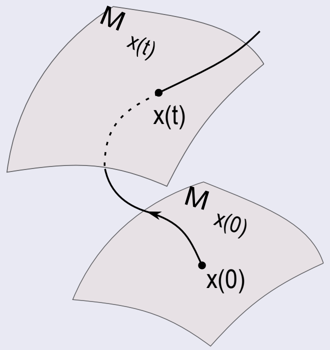
group orbits



group orbit $\mathcal{M}_{x(0)}$ of state space point $x(0)$, and the group orbit $\mathcal{M}_{x(t)}$ reached by the trajectory $x(t)$ time t later.

foliation by group orbits

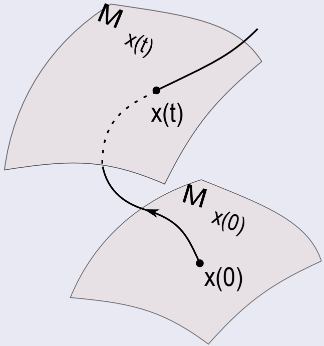
group orbits



any point on the manifold $\mathcal{M}_{x(t)}$ is equivalent to any other.

foliation by group orbits

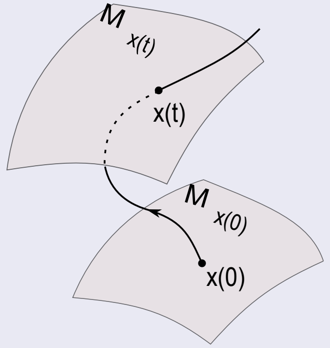
group orbits



action of a symmetry group endows the state space with the structure of a union of group orbits, each group orbit an equivalence class.

foliation by group orbits

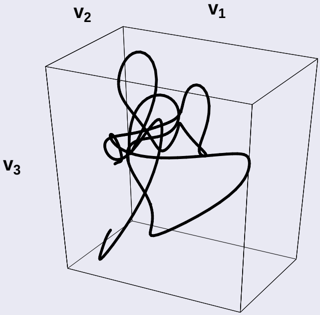
group orbits



the goal:
replace each group orbit by a unique point a lower-dimensional *reduced state space* (or orbit space)

relativity for pedestrians

try a co-moving coordinate frame?

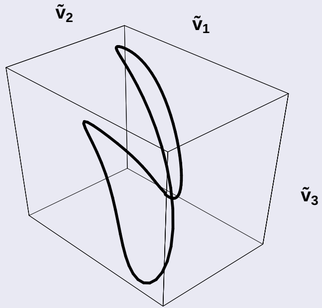


(*a*)

A relative periodic orbit of the Kuramoto-Sivashinsky flow, traced for four periods T_p, projected on
(a) a stationary state space coordinate frame $\{v_1, v_2, v_3\}$;

relativity for pedestrians

try a co-moving coordinate frame?

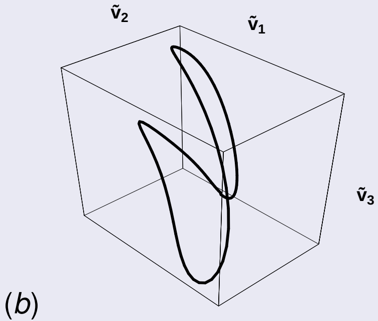


(*b*)

A relative periodic orbit of the Kuramoto-Sivashinsky flow, traced for four periods T_p, projected on
(b) a co-moving $\{\tilde{v}_1, \tilde{v}_2, \tilde{v}_3\}$ frame

relativity for pedestrians

no good global co-moving frame!



this is no symmetry reduction at all;
all other relative periodic orbits require their own frames, moving at different velocities.

symmetry reduction

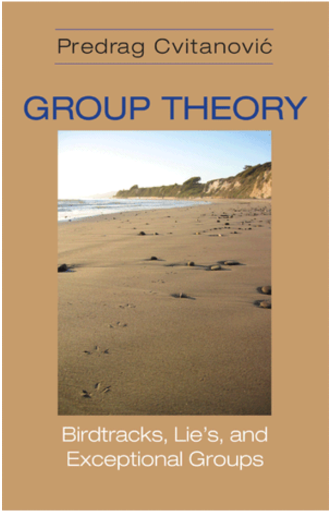
- all points related by a symmetry operation are mapped to the same point.
- relative equilibria become equilibria and relative periodic orbits become periodic orbits in reduced space.
- families of solutions are mapped to a single solution

Happy families are all alike; every unhappy family is unhappy in its own way

everybody, her mother,
and Robert MacKay knows how to do this

except the author of

masters of group theory



reduction methods

1. **Hilbert polynomial basis**: rewrite equivariant dynamics in invariant coordinates
2. **moving frames, or slices**: cut group orbits by a hypersurface (kind of Poincaré section), each group orbit of symmetry-equivalent points represented by the single point

reduction methods

1. **Hilbert polynomial basis**: rewrite equivariant dynamics in invariant coordinates: global
2. **moving frames, or slices**: cut group orbits by a hypersurface (kind of Poincaré section), each group orbit of symmetry-equivalent points represented by the single point: local

invariant polynomials

- rewrite the equations in variables invariant under the symmetry transformation

invariant polynomials

- rewrite the equations in variables invariant under the symmetry transformation
- or compute solutions in original space and map them to invariant variables

invariant polynomials basis

Hilbert basis for complex Lorenz equations

$$
\begin{aligned}
u_1 &= x_1^2 + x_2^2 \,, & u_2 &= y_1^2 + y_2^2 \\
u_3 &= x_1 y_2 - x_2 y_1 \,, & u_4 &= x_1 y_1 + x_2 y_2 \\
u_5 &= z
\end{aligned}
$$

invariant under $SO(2)$ action on a 5-dimensional state space

polynomials related through 1 syzygy:

$$u_1 u_2 - u_3^2 - u_4^2 = 0$$

invariant polynomials basis

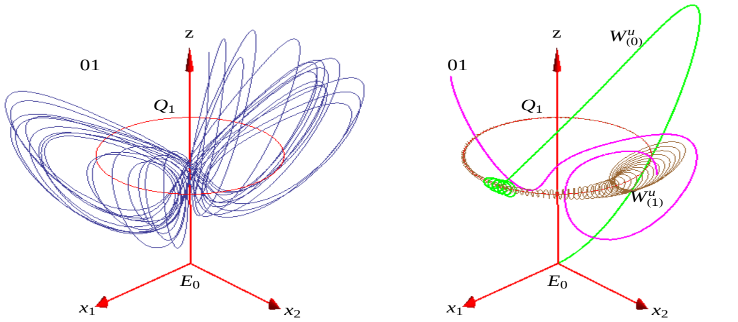
complex Lorenz equations in invariant polynomial basis

$$
\begin{aligned}
\dot{u}_1 &= 2\,\sigma\,(u_3 - u_1) \\
\dot{u}_2 &= -2\,u_2 - 2\,u_3\,(u_5 - \rho_1) \\
\dot{u}_3 &= \sigma\,u_2 - (\sigma - 1)\,u_3 - e\,u_4 + u_1\,(\rho_1 - u_5) \\
\dot{u}_4 &= e\,u_3 - (\sigma + 1)\,u_4 \\
\dot{u}_5 &= u_3 - b\,u_5
\end{aligned}
$$

A 4-dimensional $\mathcal{M}/SO(2)$ reduced state space, a symmetry-invariant representation of the 5-dimensional $SO(2)$ equivariant dynamics

state space portrait of complex Lorenz flow

drift induced by continuous symmetry



A generic chaotic trajectory (blue), the E_0 equilibrium, a representative of its unstable manifold (green), the Q_1 relative equilibrium (red), its unstable manifold (brown), and one repeat of the $\overline{01}$ relative periodic orbit (purple).

invariant polynomials basis

complex Lorenz equations in invariant polynomial basis

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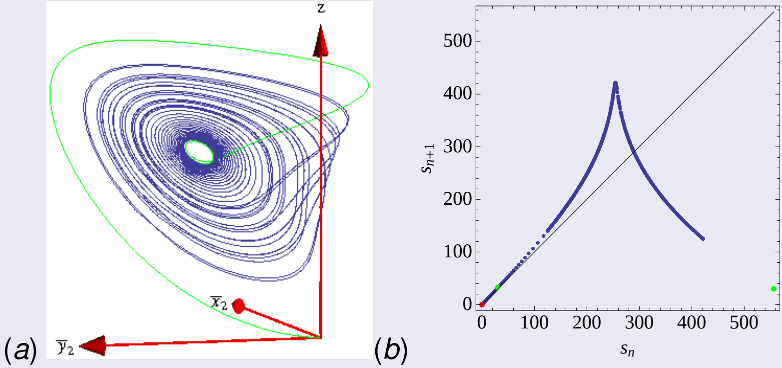
\dot{u}_1 &= 2\,\sigma\,(u_3 - u_1) \\
\dot{u}_2 &= -2\,u_2 - 2\,u_3\,(u_5 - \rho_1) \\
\dot{u}_3 &= \sigma\,u_2 - (\sigma - 1)\,u_3 - e\,u_4 + u_1\,(\rho_1 - u_5) \\
\dot{u}_4 &= e\,u_3 - (\sigma + 1)\,u_4 \\
\dot{u}_5 &= u_3 - b\,u_5
\end{aligned}
$$

the image of the full state space relative equilibrium Q_1 group orbit is an equilibrium point, while the image of a relative periodic orbit, such as $\overline{01}$, is a periodic orbit

Hilbert invariant coordinates

projected onto invariant polynomials basis



(a) The unstable manifold connection from the equilibrium E_0 at the origin to the strange attractor controlled by the rotation around the reduced state space image of relative equilibrium Q_1;

(b) The return map projected on invariant polynomials.

higher-dimensional invariant bases? an example

first 11 invariants for the standard action of $SO(2)$

$$
\begin{aligned}
u_1 &= r_1 = \sqrt{b_1^2 + c_1^2} \\
u_3 &= \frac{b_2\left(b_1^2 - c_1^2\right) + 2 b_1 c_1 c_2}{r_1^2} \\
u_4 &= \frac{-2 b_1 b_2 c_1 + \left(b_1^2 - c_1^2\right) c_2}{r_1^2} \\
u_5 &= \frac{b_1 b_3 \left(b_1^2 - 3 c_1^2\right) - c_1 \left(-3 b_1^2 + c_1^2\right) c_3}{r_1^3} \\
u_6 &= \frac{-3 b_1^2 b_3 c_1 + b_3 c_1^3 + b_1^3 c_3 - 3 b_1 c_1^2 c_3}{r_1^3}
\end{aligned}
$$

higher-dimensional invariant bases? an example

first 11 invariants for the standard action of $SO(2)$

$$
\begin{aligned}
u_7 &= \frac{b_4\left(b_1^4-6b_1^2c_1^2+c_1^4\right)+4b_1c_1\left(b_1^2-c_1^2\right)c_4}{r_1^4}\\
u_8 &= \frac{4b_1b_4c_1\left(-b_1^2+c_1^2\right)+\left(b_1^4-6b_1^2c_1^2+c_1^4\right)c_4}{r_1^4}\\
u_9 &= \frac{b_1b_5\left(b_1^4-10b_1^2c_1^2+5c_1^4\right)+c_1\left(5b_1^4-10b_1^2c_1^2+c_1^4\right)c_5}{r_1^5}\\
u_{10} &= \frac{-b_5c_1\left(5b_1^4-10b_1^2c_1^2+c_1^4\right)+b_1\left(b_1^4-10b_1^2c_1^2+5c_1^4\right)c_5}{r_1^5}\\
u_{11} &= \frac{b_6\left(b_1^6-15b_1^4c_1^2+15b_1^2c_1^4-c_1^6\right)+2b_1c_1\left(3b_1^4-10b_1^2c_1^2+3c_1^4\right)c_6}{r_1^6}\\
u_{12} &= \frac{-2b_1b_6c_1\left(3b_1^4-10b_1^2c_1^2+3c_1^4\right)+\left(b_1^6-15b_1^4c_1^2+15b_1^2c_1^4-c_1^6\right)c_6}{r_1^6}
\end{aligned}
$$

invariant polynomials - how to find them?

- invariant polynomials (Hilbert basis)

invariant polynomials - how to find them?

- invariant polynomials (Hilbert basis): computationally prohibitive for high-dimensional flows

invariant polynomials - how to find them?

- invariant polynomials (Hilbert basis)
- Cartan moving frame method / method of slices

invariant polynomials - how to find them?

- invariant polynomials (Hilbert basis)
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invariant polynomials - how to find them?

- invariant polynomials (Hilbert basis)
- Cartan moving frame method / method of slices: singularities

Lie algebra generators

$\mathbf{T}_a$ generate infinitesimal transformations: a set of N linearly independent $[d \times d]$ anti-hermitian matrices, $(\mathbf{T}_a)^\dagger = -\mathbf{T}_a$, acting linearly on the d-dimensional state space $\mathcal{M}$

example: $SO(2)$ rotations for complex Lorenz equations

$$\mathbf{T} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The action of $SO(2)$ on the complex Lorenz equations state space decomposes into $m = 0$ G-invariant subspace (z-axis) and $m = 1$ subspace with multiplicity 2.

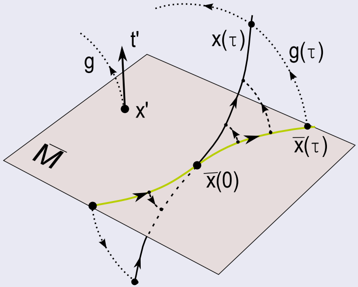
group tangent fields

flow field at the state space point x induced by the action of the group is given by the set of N *tangent fields*

$$t_a(x)_i = (\mathbf{T}_a)_{ij} x_j$$

slice & dice

flow reduced to a slice



Slice $\bar{\mathcal{M}}$ through the slice-fixing point x', normal to the group tangent t' at x', intersects group orbits (dotted lines). The full state space trajectory $x(\tau)$ and the reduced state space trajectory $\overline{x}(\tau)$ are equivalent up to a group rotation $g(\tau)$.

flow within the slice

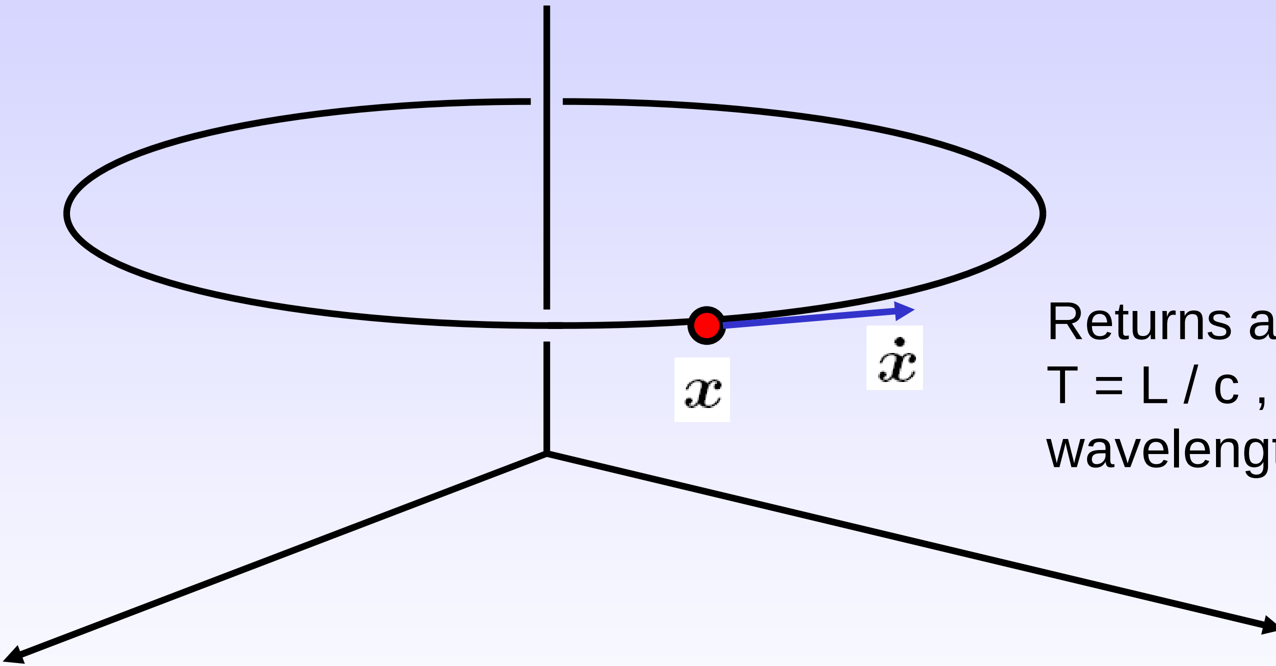
slice fixed by x'

reduced state space $\bar{\mathcal{M}}$ flow $u(\overline{x})$

$$
\begin{aligned}
u(\overline{x}) &= v(\overline{x}) - \dot{\theta}(\overline{x}) \cdot t(\overline{x}) \,, \qquad \overline{x} \in \bar{\mathcal{M}} \\
\dot{\theta}_a(\overline{x}) &= (v(\overline{x})^T t'_a)/(t(\overline{x})^T \cdot t') \,.
\end{aligned}
$$

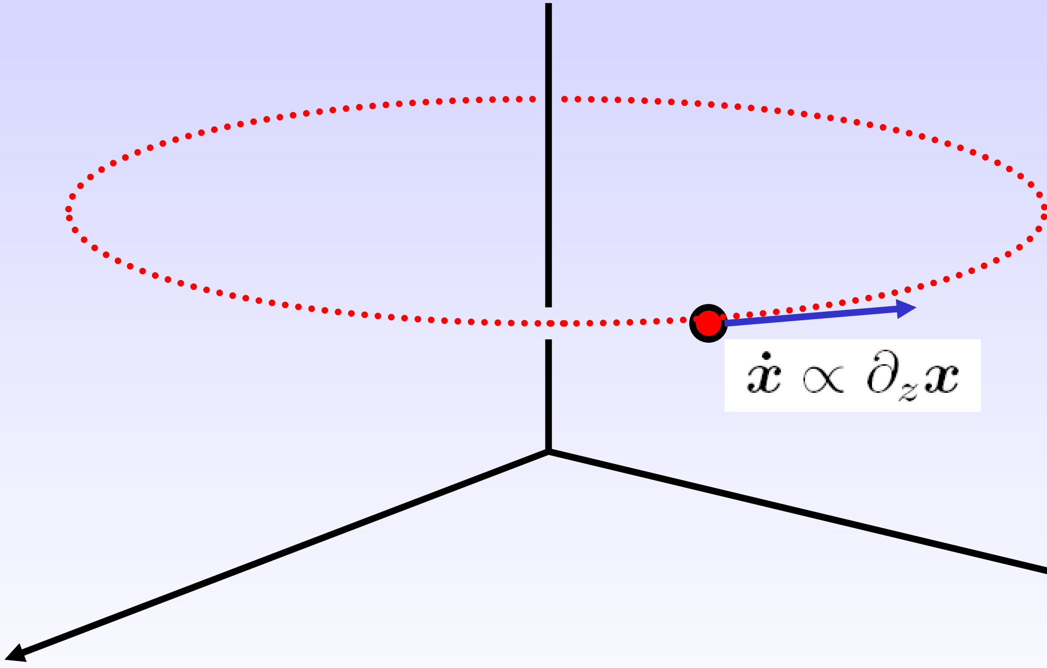
together with the reconstruction equation for the group phases flow $\dot{\theta}$

TW orbit in phase space



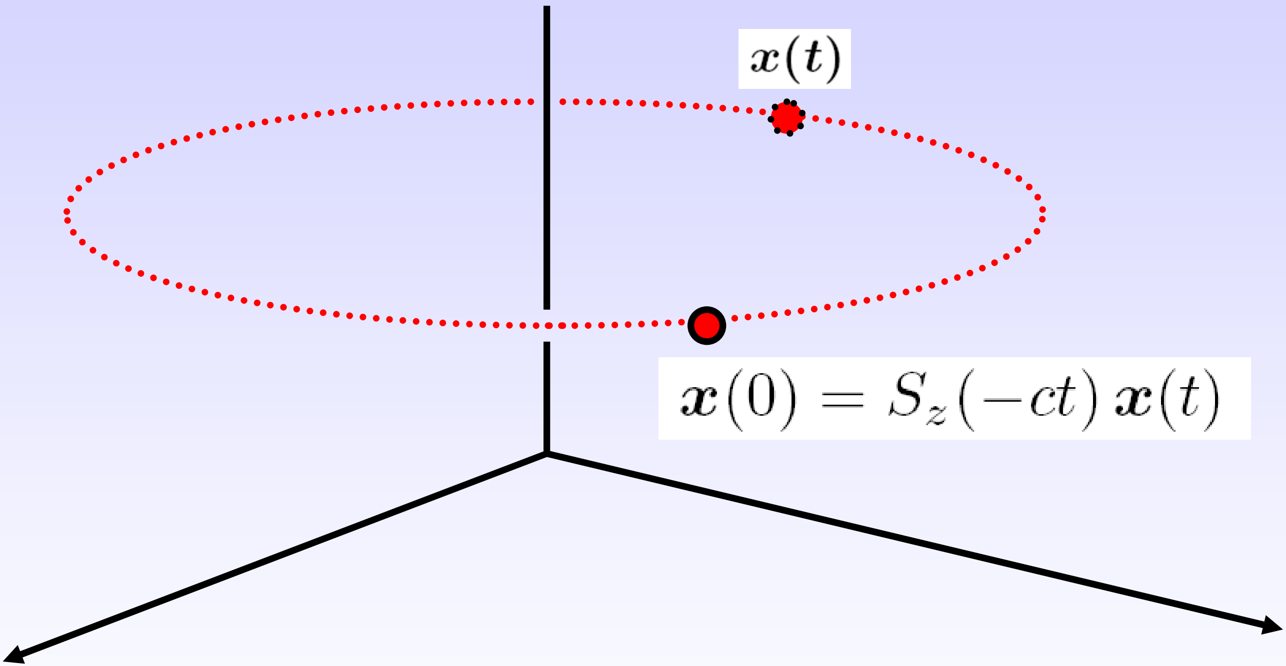
Returns at
T = L / c ,
wavelength L

Axial shifts of TW state



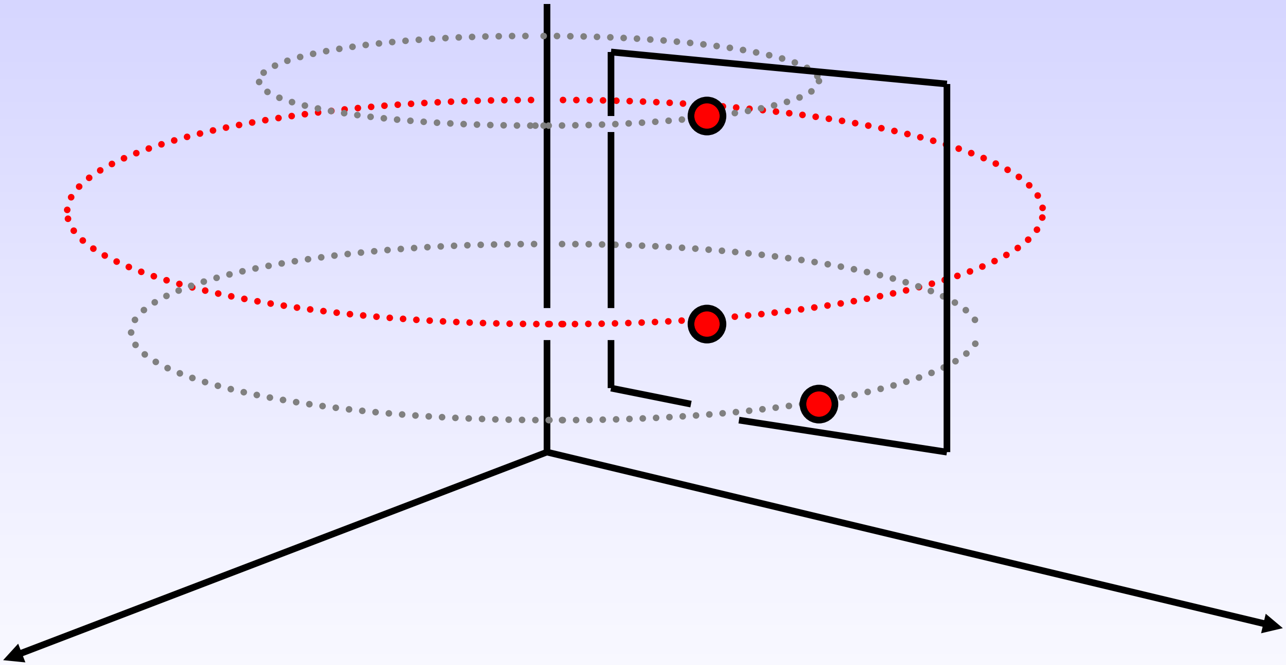
TW orbit coincides with its group orbit

Reduction of TW orbit to point by shifts



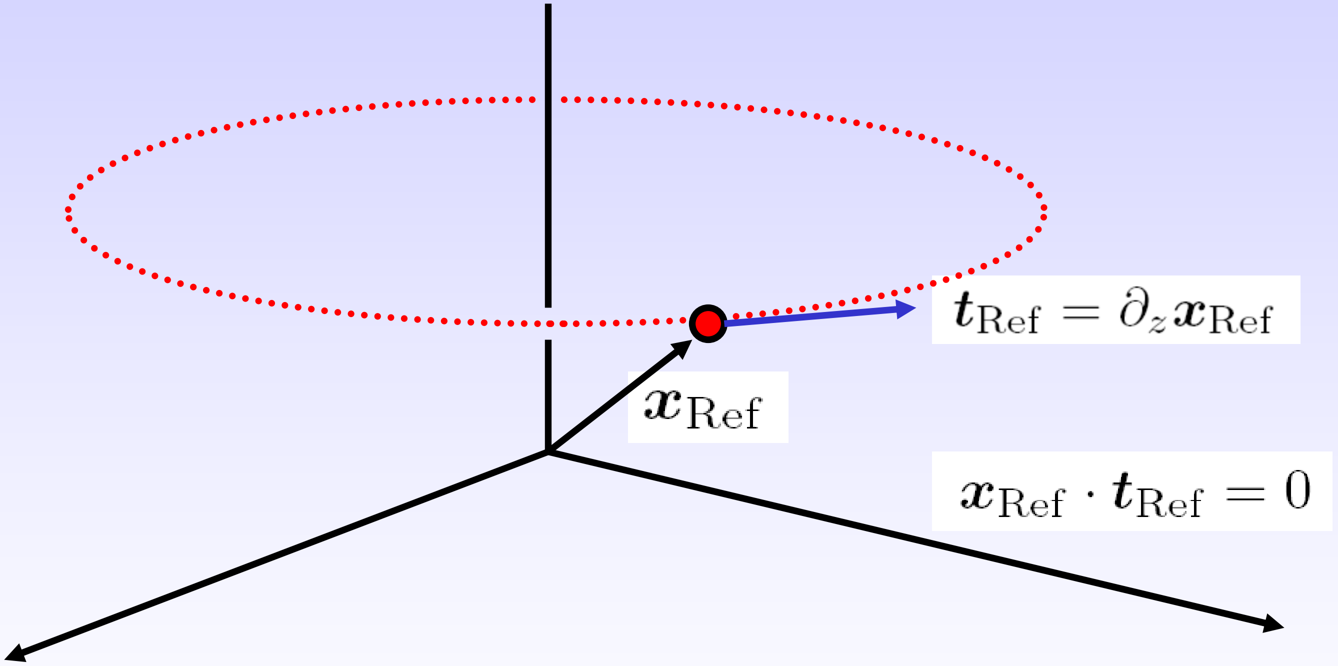
$$\boldsymbol{x}(t) \to \bar{\boldsymbol{x}}$$

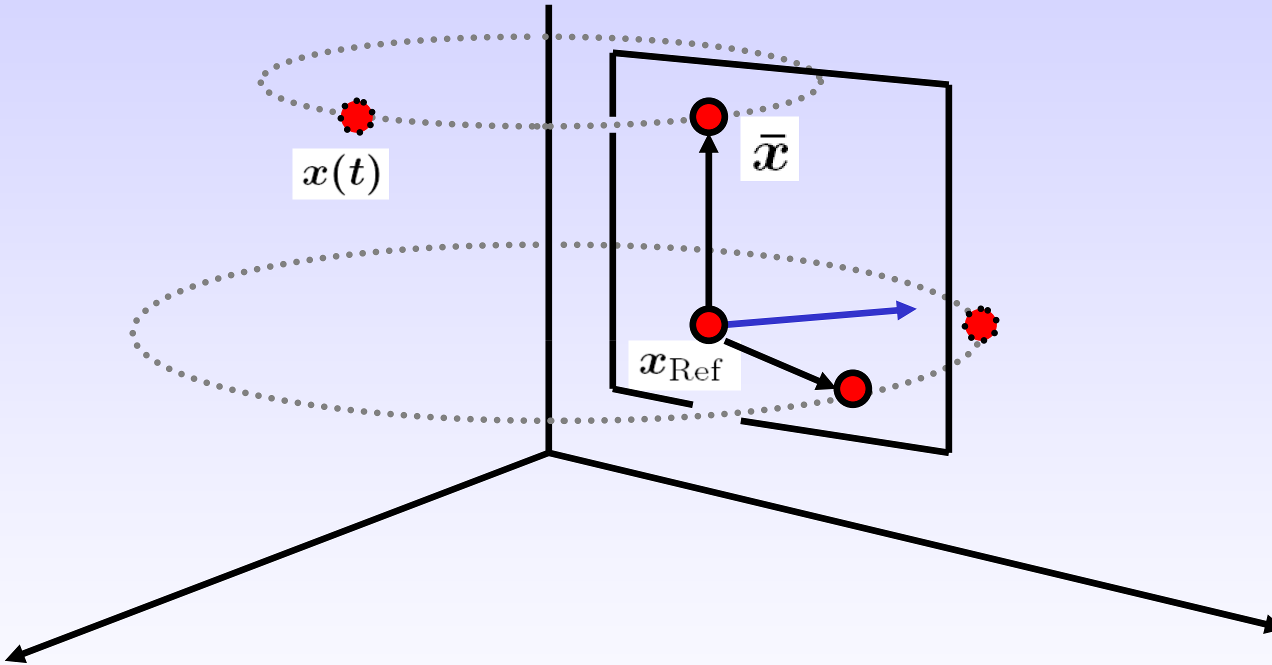
Reduce all TWs into a single slice



$$\boldsymbol{x}_i(0) = S_z(-c_i\,t)\,\boldsymbol{x}_i(t)$$

How? - several speeds *c*, possibly unknown

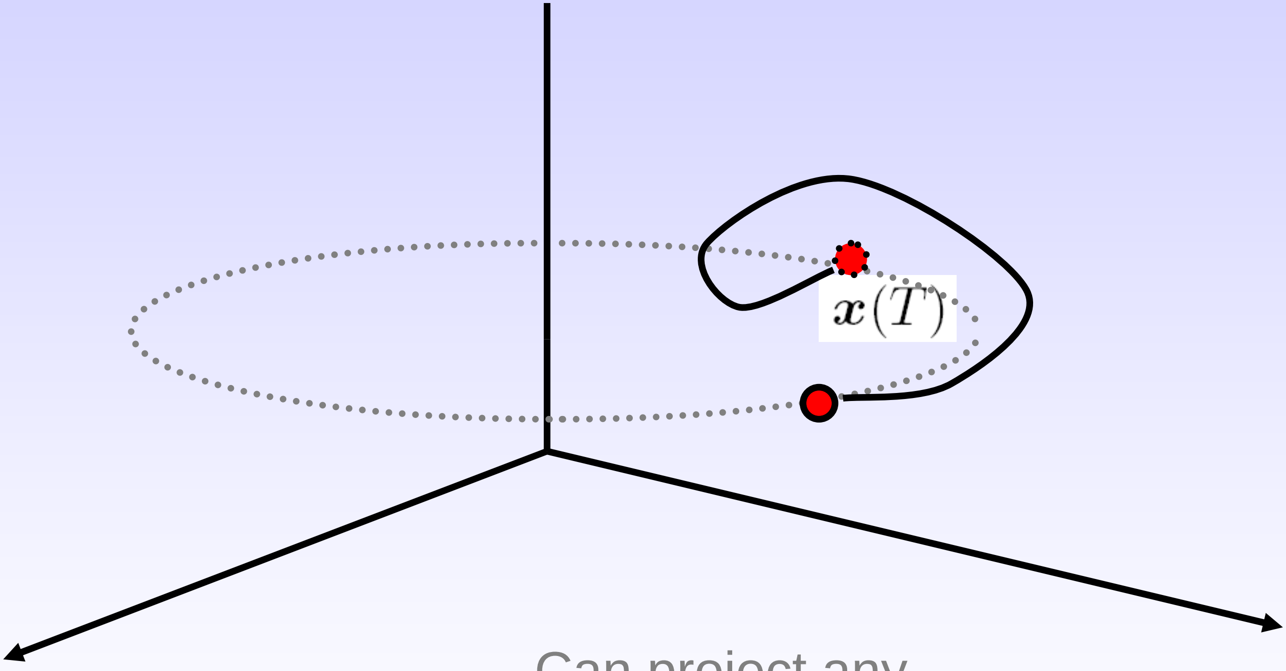

$\boldsymbol{t}_{\mathrm{Ref}} = \partial_z \boldsymbol{x}_{\mathrm{Ref}}$
$\boldsymbol{x}_{\mathrm{Ref}}$
$\boldsymbol{x}_{\mathrm{Ref}} \cdot \boldsymbol{t}_{\mathrm{Ref}} = 0$



$$\bar{x} = g\, x(t) \;:\; (\bar{x} - x_{\mathrm{Ref}}) \cdot t_{\mathrm{Ref}} = 0$$

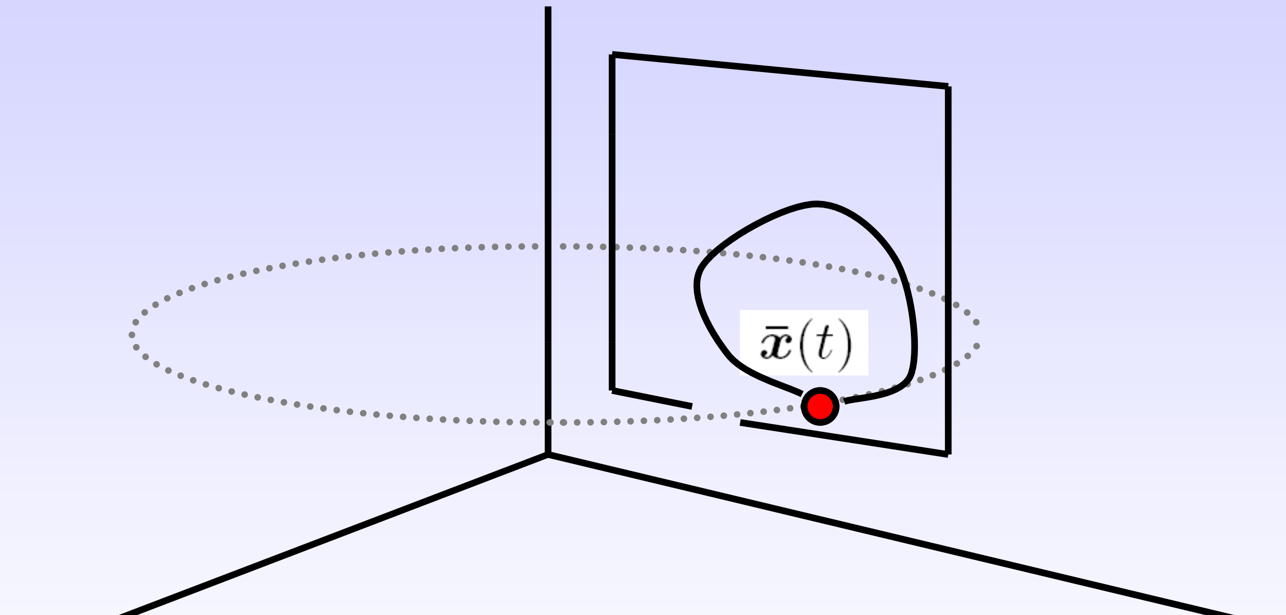
All TWs → points in the slice

Application to a relative periodic orbit (RPO)

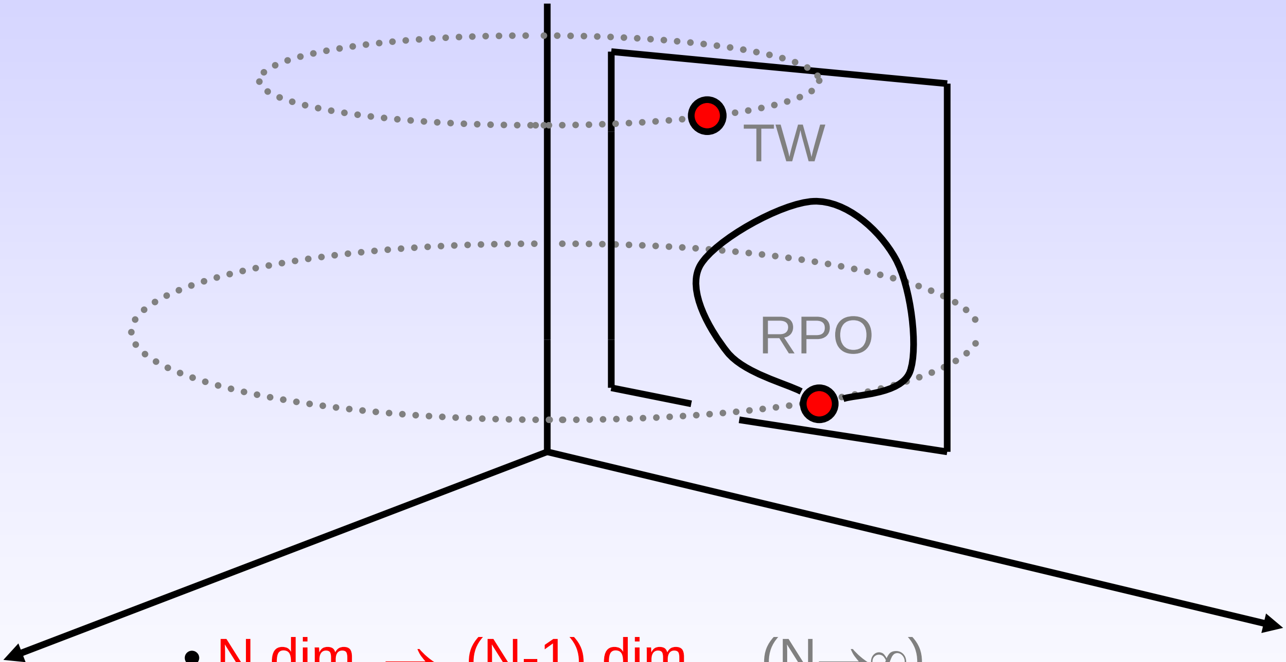


Can project any trajectory into the slice

Application to a relative periodic orbit (RPO)

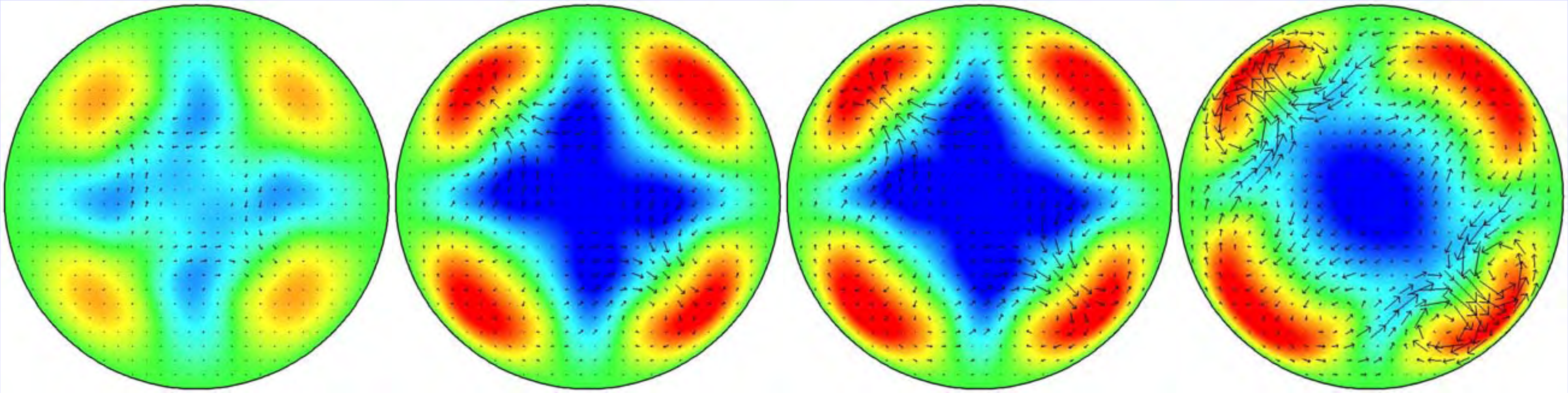


RPO $\rightarrow$ PO within the slice



- N dim $\rightarrow$ (N-1) dim   ($N \rightarrow \infty$)
- Automatic removal of strong shift (gives c for TW)
- TW $\rightarrow$ point
- RPO $\rightarrow$ PO

Application to Pipe Flow, N2 L=2.5 D states



LB | M1 | M2 | UB

1 real | 1 complex | 2 real | 4 complex

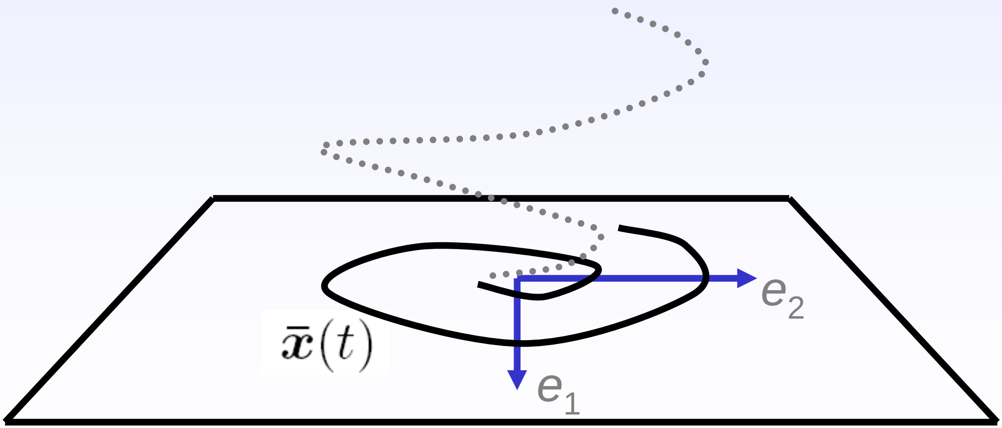
1 real (under M1)

unstable eigenvalues

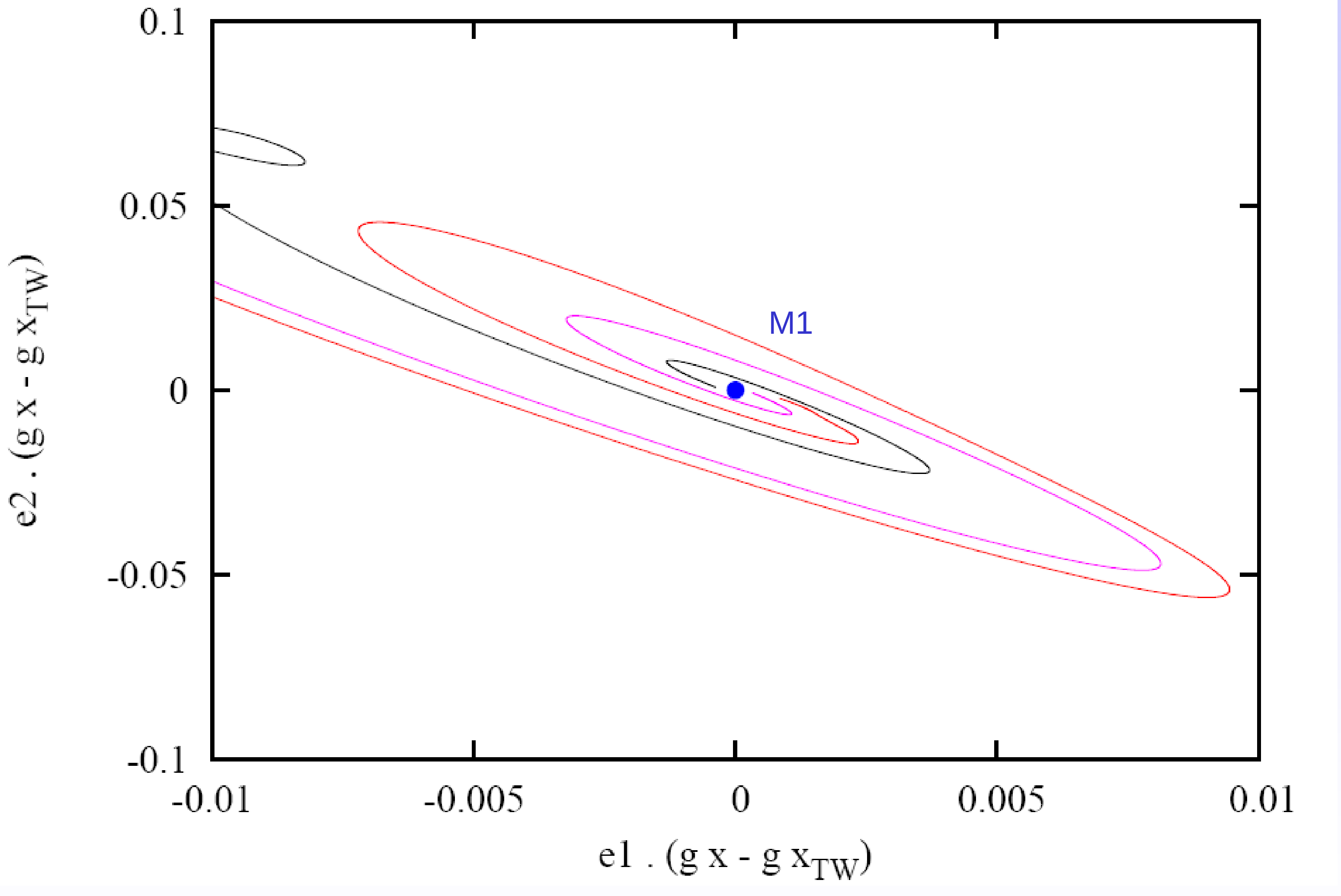
Projection within slice, (N-1) dim $\rightarrow$ 2 dim

$$\boldsymbol{e}_1 \cdot (\bar{\boldsymbol{x}}(t) - \boldsymbol{x}_{\mathrm{Ref}})$$
$$\boldsymbol{e}_2 \cdot (\bar{\boldsymbol{x}}(t) - \boldsymbol{x}_{\mathrm{Ref}})$$

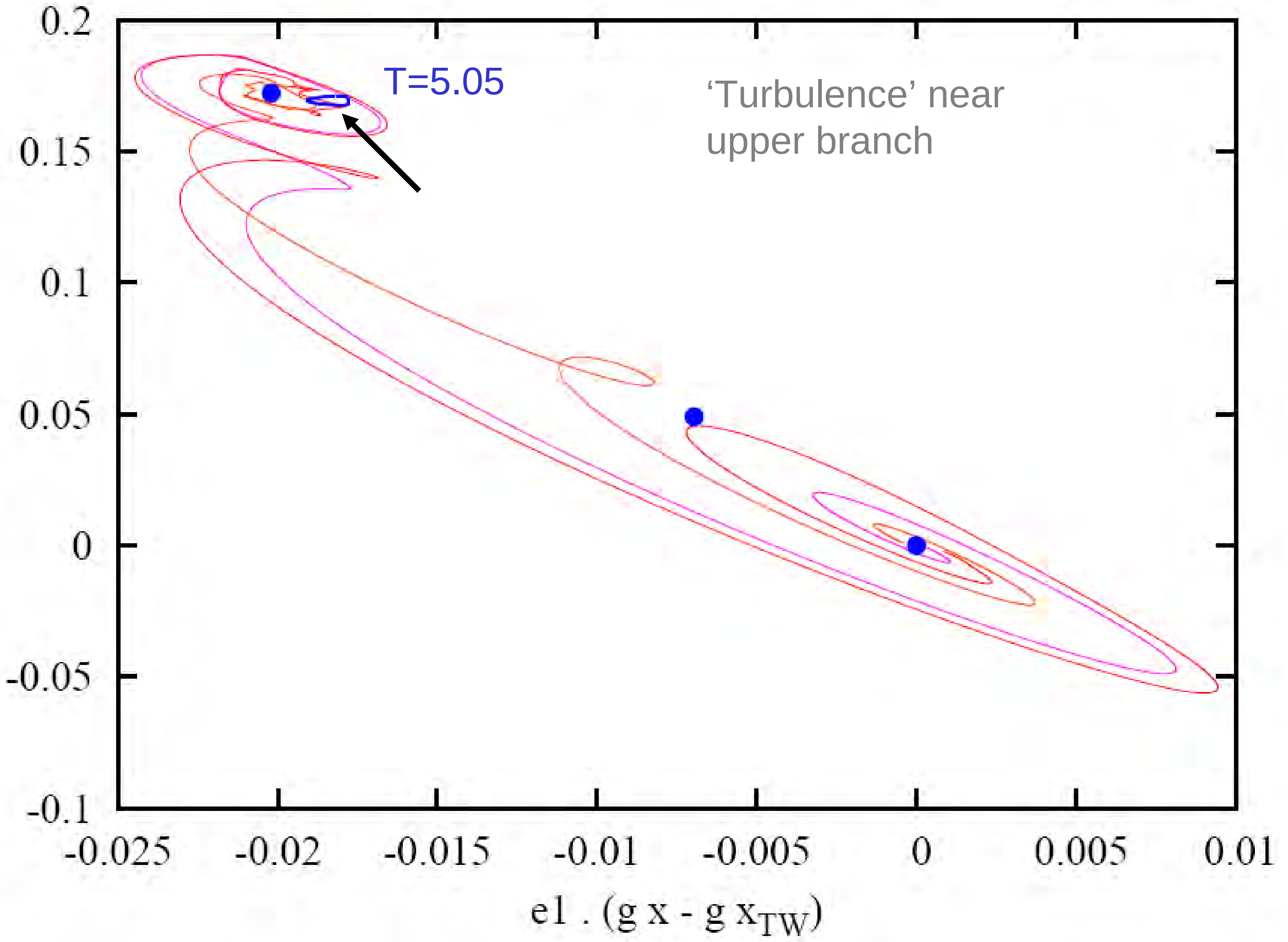
unstable directions of $\boldsymbol{x}_{\mathrm{Ref}}$ (M1 state)



Local dynamics, projection within slice



Embedded RPO



slice trouble 1

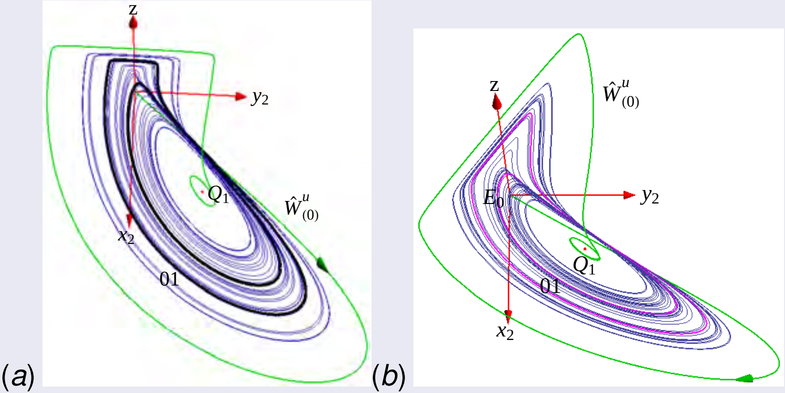
glitches!

group tangent of a generic trajectory orthogonal to the slice tangent at a sequence of instants τ_k

$$t(\tau_k)^T \cdot t' = 0$$

slice trouble 1

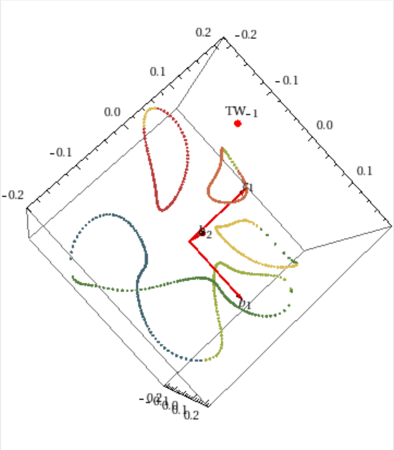
portrait of complex Lorenz flow in reduced state space



all choices of the slice fixing point x' exhibit flow discontinuities / jumps

slice trouble 2

slice cuts a relative periodic orbit multiple times



Relative periodic orbit intersects a hyperplane slice in 3 closed-loop images of the relative periodic orbit and 3 images that appear to connect to a closed loop.

summary

conclusion

- Symmetry reduction by slicing:
  efficiently implemented, allows exploration of high-dimensional flows with continuous symmetry.
- stretching and folding of unstable manifolds in reduced state space organizes the flow

to be done

- construct Poincaré sections and return maps
- find all (relative) periodic orbits up to a given period.
- use the information quantitatively (periodic orbit theory).

amazing data! amazing numerics! frustration...

"Ask your doctor if taking a pill to solve all your problems is right for you."