is space time?

a spatio-temporal theory of turbulence

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overview

1. what this talk is about
2. “turbulence” in small domains
3. “turbulence” in infinite spatial domains
4. space is time
5. bye bye, dynamics

this talk is about [1]

how to solve

strongly nonlinear field theories

[1] references in this presentation are hyperlinked

do clouds solve PDEs?

do clouds integrate Navier-Stokes equations?

NO!

 $\Longrightarrow$ other swirls $\Longrightarrow$ 

do clouds satisfy Navier-Stokes equations?

yes!

they satisfy them locally, everywhere and at all times

part 1

1. “turbulence” in small domains
2. “turbulence” in infinite spatial domains
3. space is time
4. bye bye, dynamics

goal : go from equations to turbulence

Navier-Stokes equations

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{1}{R}\nabla^2 \mathbf{v} - \nabla p + \mathbf{f}\,, \qquad \nabla \cdot \mathbf{v} = 0,$$

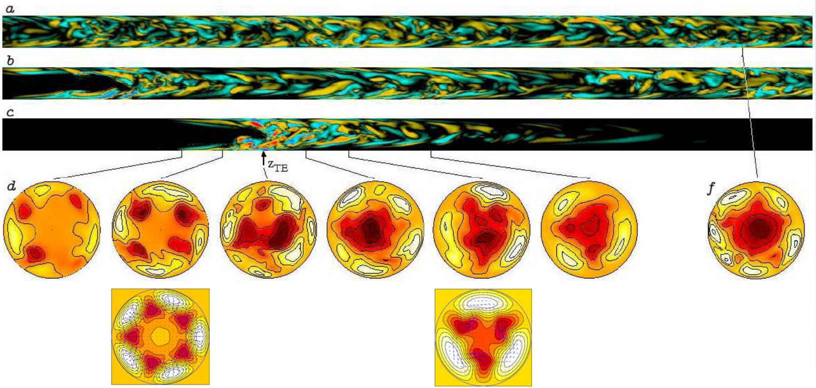
velocity field $\mathbf{v} \in \mathbb{R}^3$; pressure field p ; driving force $\mathbf{f}$

describe turbulence

starting from the equations (no statistical assumptions)

example : pipe flow[2]

amazing experimental data! amazing numerics!



[2] **science04**.

dynamical description of turbulence

state space

a manifold $\mathcal{M} \in \mathbb{R}^d$: d numbers determine the state of the system

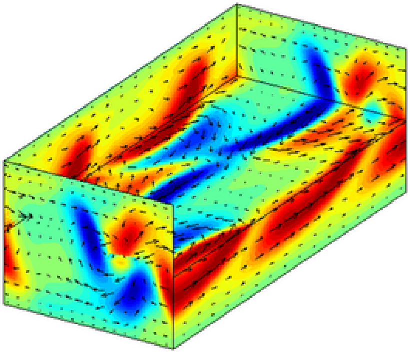
representative point

$x(t) \in \mathcal{M}$
a state of physical system at instant in time

integrate forward in time

trajectory $x(t) = f^t(x_0)$ = representative point time t later

plane Couette : so far, small computational cells[3]



velocity field visualization

John F Gibson (U New Hampshire)
Jonathan Halcrow (Google)

P. C. (Georgia Tech)

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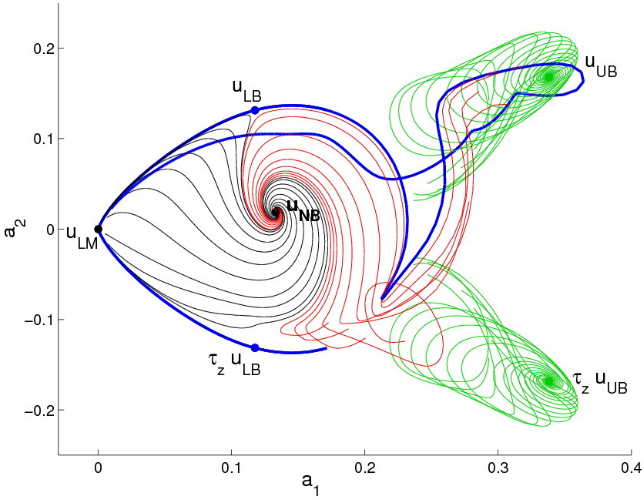
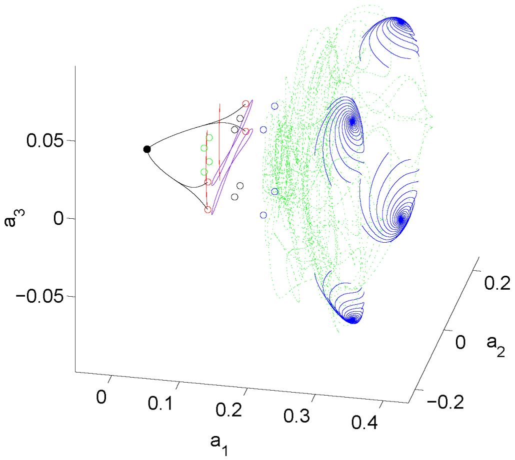
[3] **GHCW07**.

can visualize 61,506 dimensional state space of turbulent flow



equilibria of turbulent plane Couette flow,
their unstable manifolds, and
myriad of turbulent videos mapped out as one happy family

plane Couette state space $10^5 \to 3D$



equilibria, periodic orbits, their (un)stable manifolds

shape the turbulence

unable to compute invariant solutions for large spatial domains[4]

solutions on large domains are too unstable

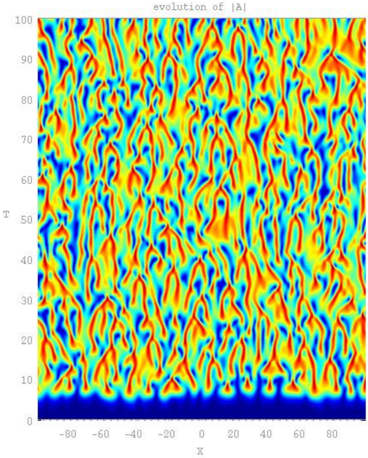
[4] **WFSBC15**.

part 2

1. “turbulence” in small domains
2. **“turbulence” in infinite spatial domains**
3. space is time
4. bye bye, dynamics

next: large space-time domains

example : complex Ginzburg-Landau on a large domain

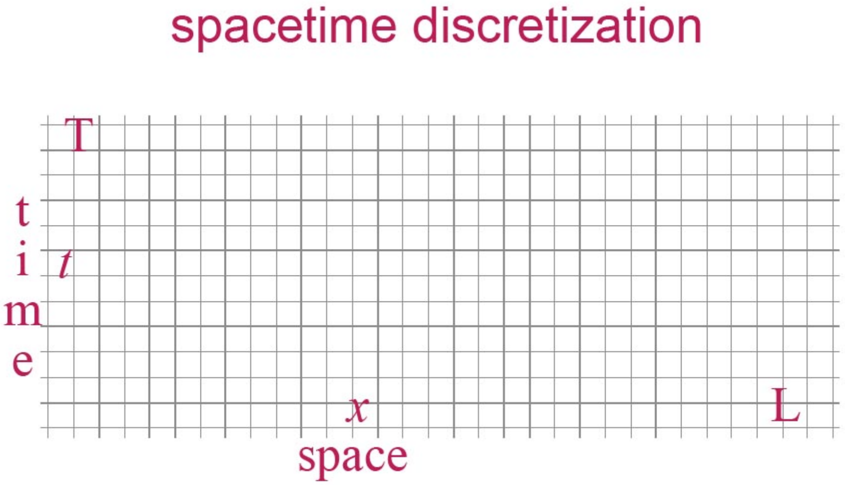


[horizontal] space x [up] time evolution

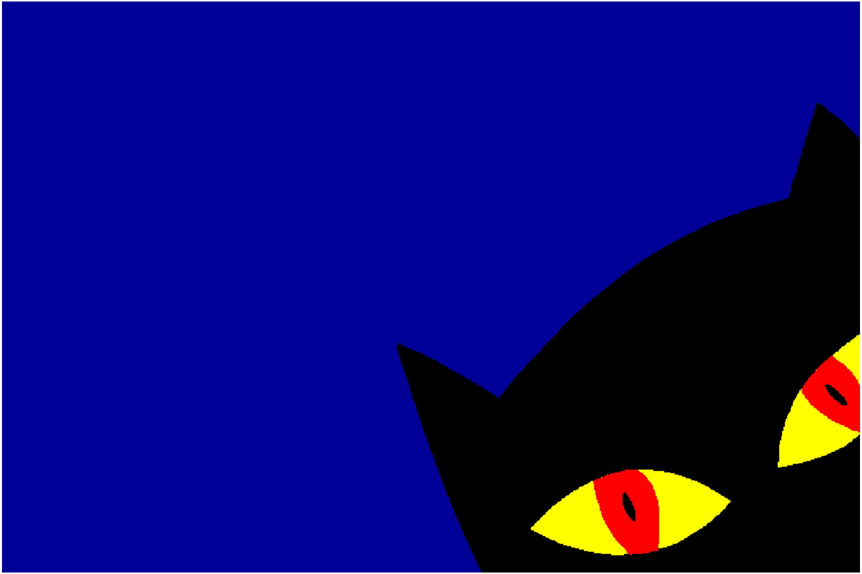
codeinthehole.com/static/tutorial/coherent.html

challenge : describe $(x, t) \in (-\infty, \infty) \times (-\infty, \infty)$

continuous symmetries : space, time translations strategy : first study dicretized spacetime

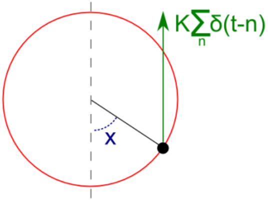


1) chaos and a single kitten



example of a “small domain dynamics” : kicked rotor

an electron circling an atom, subject to
a discrete time sequence of angle-dependent kicks $F(x_t)$



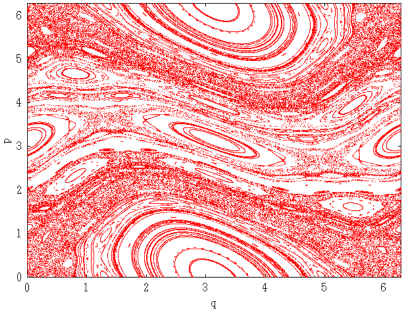
Taylor, Chirikov and Greene standard map

$$
\begin{aligned}
x_{t+1} &= x_t + p_{t+1} \qquad \text{mod } 1, \\
p_{t+1} &= p_t + F(x_t)
\end{aligned}
$$

$\rightarrow$ chaos in Hamiltonian systems

standard map

example of chaos in a Hamiltonian system



the simplest example : a single kitten in time

force $F(x) = Kx$ linear in the displacement x , $K \in \mathbb{Z}$

$$
\begin{aligned}
x_{t+1} &= x_t + p_{t+1} \qquad \text{mod } 1 \\
p_{t+1} &= p_t + Kx_t \qquad \text{mod } 1
\end{aligned}
$$

Continuous Automorphism of the Torus, or
(after same algebra, replacing $K \to s$, etc)

Hamiltonian cat map

a linear, area preserving map of a 2-torus onto itself

$$
\begin{pmatrix} x_{t+1} \\ p_{t+1} \end{pmatrix} = A \begin{pmatrix} x_t \\ p_t \end{pmatrix} \qquad \text{mod } 1 \,, \qquad A = \begin{pmatrix} s-1 & 1 \\ s-2 & 1 \end{pmatrix}
$$

for integer $s = \mathrm{tr}\, A > 2$ the map is hyperbolic $\to$ a fully chaotic Hamiltonian dynamical system

cat map in Lagrangian form[5]

replace momentum by velocity

$$p_{t+1} = (x_{t+1} - x_t)/\Delta t$$

dynamics in (x_t, x_{t-1}) state space is particularly simple

2-step difference equation

$$x_{t+1} - s\, x_t + x_{t-1} = -m_t$$

unique integer m_t ensures that

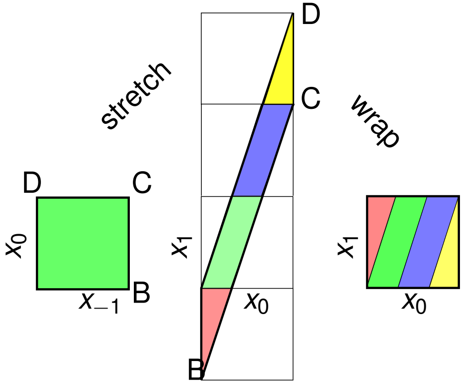
x_t lands in the unit interval at every time step t

nonlinearity : mod 1 operation, encoded in

$$m_t \in \mathcal{A}\,, \quad \mathcal{A} \;=\; \text{finite alphabet of possible values for } m_t$$

[5] **PerViv**.

example : $s = 3$ cat map symbolic dynamics

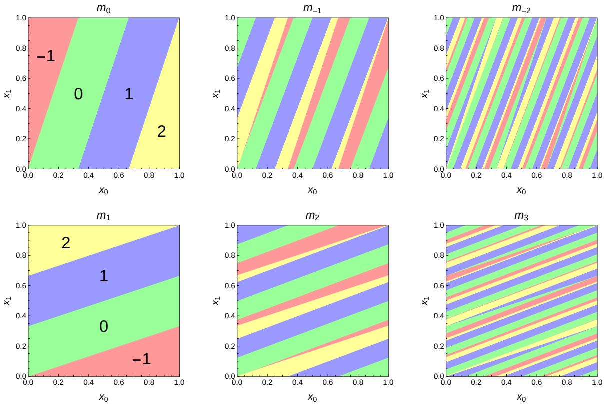


cat map stretches the unit square translations by

$$m_0 \in \mathcal{A} = \{\underline{1}, 0, 1, 2\} = \{\text{red}, \text{green}, \text{blue}, \text{yellow}\}$$

return stray kittens back to the torus

cat map (x_0, x_1) state space partition



(a) 4 regions labeled by m_0. , obtained from (x_{-1}, x_0) state space by one iteration

(b) 14 regions, 2-steps past $m_{-1}m_0$.   (c) 44 regions, 3-steps past $m_{-2}m_{-1}m_0$.

(d) 4 regions labeled by future $.m_1$

(e) 14 regions, 2-steps future $.m_1m_2$ (f) 44 regions, 3-steps future block $m_3m_2m_1$.

conclusion : a single kitten and free will

each possible kitten life x_t

is recorded in the Book of Life by a unique admissible symbol sequence m_t

2) chaos and the spatiotemporally infinite cat

interacting kittens all over the space

spatiotemporal cat map[6]

Consider a 1-dimensional spatial lattice, with field $x_{n,t}$ (the angle of a kicked rotor “particle” at instant t) at site n.

require

(0) each site couples to its nearest neighbors $x_{n\pm1,t}$
(1) invariance under spatial translations
(2) invariance under spatial reflections
(3) invariance under the space-time exchange

obtain

2-dimensional coupled cat map lattice

$$x_{n,t+1} + x_{n,t-1} - s\, x_{n,t} + x_{n+1,t} + x_{n-1,t} = -m_{n,t}$$

[6] **GutOsi15**.

herding cats : a Euclidean field theory[7]

convert the spatial-temporal differences to discrete derivatives

discrete d-dimensional Euclidean space-time Laplacian in $d = 1$ and $d = 2$ dimensions

$$
\begin{aligned}
\Box x_t &= x_{t+1} - 2x_t + x_{t-1} \\
\Box x_{n,t} &= x_{n,t+1} + x_{n,t-1} - 4\, x_{n,t} + x_{n+1,t} + x_{n-1,t}
\end{aligned}
$$

$\rightarrow$ the cat map equations generalized to

d-dimensional spatiotemporal cat map

$$(\Box - s + 2d)\, x_z = m_z$$

where $x_z \in \mathbb{T}^1$, $\quad m_z \in \mathcal{A}$ and $z \in \mathbb{Z}^d$ = lattice site label

[7] **GHJSC16**.

deep insight, derived from observing kittens

an insight that applies to all coupled-map lattices, and all PDEs with translational symmetries

a *d*-dimensional spatiotemporal pattern
$\{x_z\} = \{x_z, z \in \mathbb{Z}^d\}$

is labelled by a *d-dimensional* spatiotemporal block of symbols
$\{m_z\} = \{m_z, z \in \mathbb{Z}^d\}$,

rather than a single temporal symbol sequence

(as is done when describing a small coupled few-“particle” system, or a small computational domain).

“periodic orbits” are now invariant *d*-tori

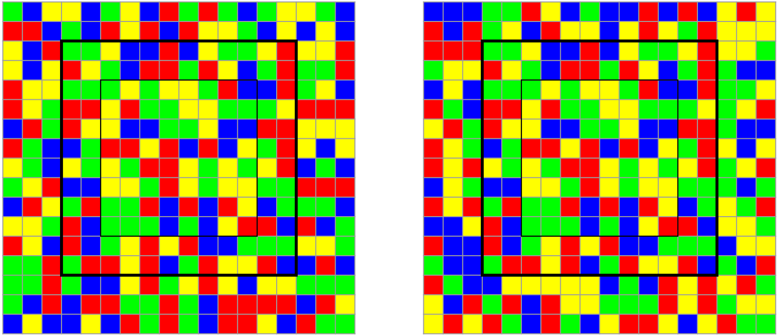
1 time, 0 space dimensions

a state space point is *periodic* if its orbit returns to it after a finite time T; in time direction such orbit tiles the time axis by infinitely many repeats

1 time, *d*-1 space dimensions

a state space point is *spatiotemporally periodic* if it belongs to an invariant d-torus $\mathcal{R}$, i.e., a block $\mathsf{M}_{\mathcal{R}}$ that tiles the lattice state M periodically, with period ℓ_j in jth lattice direction

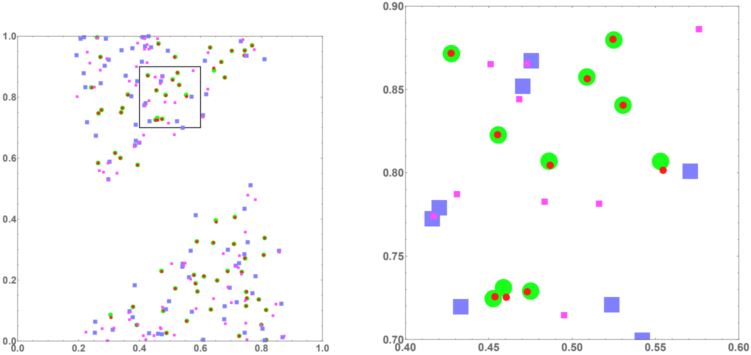
an example of invariant 2-tori : shadowing, symbolic dynamics space



2d symbolic representation of two invariant 2-tori shadowing each other within the shared block $\mathsf{M}_{\mathcal{R}} = \mathsf{M}_{\mathcal{R}_0} \cup \mathsf{M}_{\mathcal{R}_1}$ (blue)

- border $\mathcal{R}_1$ (thick black), interior $\mathcal{R}_0$ (thin black)
- symbols outside $\mathcal{R}$ differ

shadowing, state space



(left) state space points $(x_{0,t}, x_{0,t-1})$ of the two invariant 2-tori
(right) zoom into the small rectangular area
interior points $\in \mathcal{R}_0$ (large green), (small red) circles
border points $\in \mathcal{R}_1$ (large violet), (small magenta) squares
within the interior of the shared block,

the shadowing is exponentially good

conclusion

space, time merely parametrize a given invariant solution

what matters is

the enumeration of distinct invariant solutions

part 3

1. “turbulence” in small domains
2. “turbulence” in infinite spatial domains
3. **space is time**
4. bye bye, dynamics

yes, lattice schmatiz, but

does it work for PDEs?

chronotope[8]

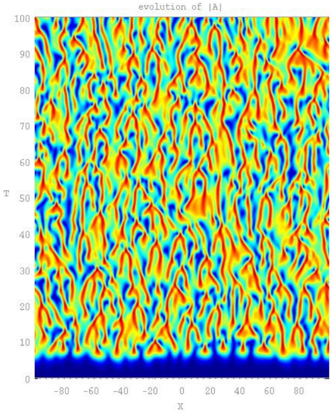
> In literary theory and philosophy of language, the chronotope is how configurations of time and space are represented in language and discourse.
>
> — Wikipedia : Chronotope

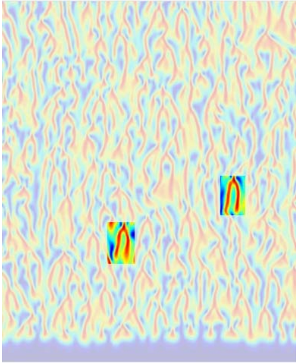
- Mikhail Mikhailovich Bakhtin (1937)

[8] **LePoTo96**.

space-time complex Ginzburg-Landau on a large domain

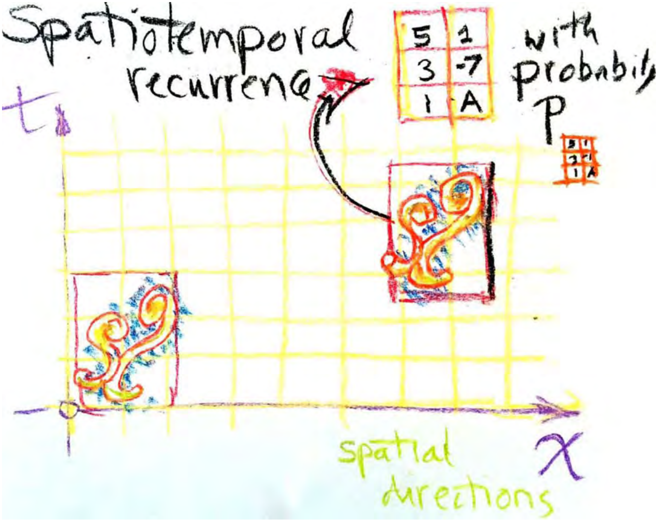
a nearly recurrent chronotope





[horizontal] space $x \in [-L/2, L/2]$ [up] time evolution

goal[9]: determine symbolic dynamics, construct Book of Life



[9] GutOsi15.

(1+1) space-time dimensional “Navier-Stokes”

computationally not ready yet to explore the inertial manifold of $(1+3)$-dimensional turbulence - start instead with $(1+1)$-dimensional

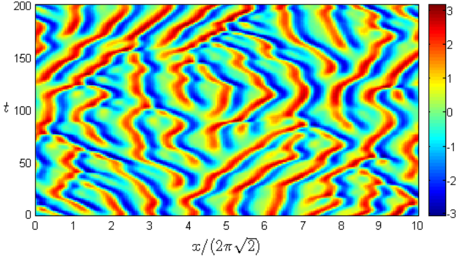
Kuramoto-Sivashinsky time evolution equation

$$u_t + u\nabla u = -\nabla^2 u - \nabla^4 u \,, \qquad x \in [-L/2, L/2]\,,$$

describes spatially extended systems such as

- flame fronts in combustion
- reaction-diffusion systems
- ...

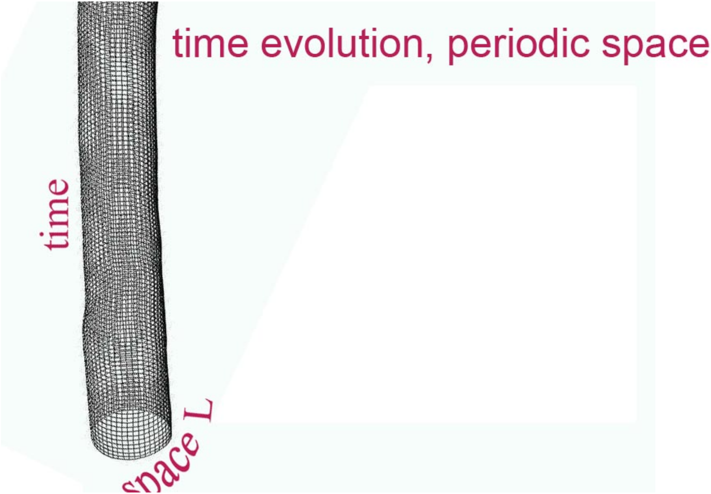
a test bed : Kuramoto-Sivashinsky on a large domain



[horizontal] space $x \in [0, L]$ [up] time evolution

- turbulent behavior
- simpler physical, mathematical and computational setting than Navier-Stokes

compact space, infinite time cylinder



so far : Navier-Stokes on compact spatial domains, all times

compact space, infinite time

Kuramoto-Sivashinsky on spatial domain L

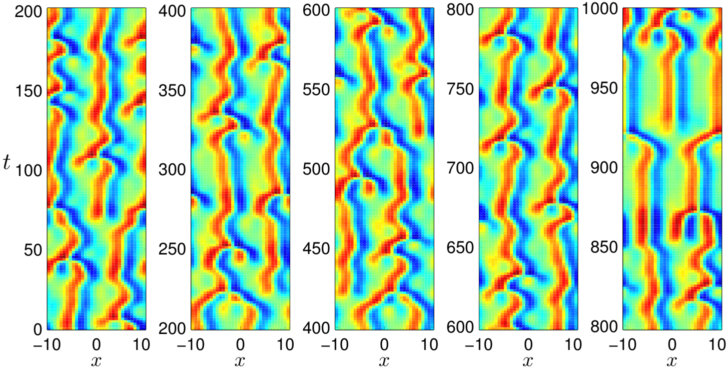
$$u_t + u\nabla u = -\nabla^2 u - \nabla^4 u\,, \qquad x \in [-L/2, L/2]\,,$$

in terms of discrete spatial Fourier modes

N ordinary differential equations (ODEs) in time

$$\dot{\tilde{u}}_k(t) = (q_k^2 - q_k^4)\,\tilde{u}_k(t) - i\frac{q_k}{2}\sum_{k'=0}^{N-1}\tilde{u}_{k'}(t)\tilde{u}_{k-k'}(t)\,.$$

evolution of Kuramoto-Sivashinsky on small $L = 22$ cell

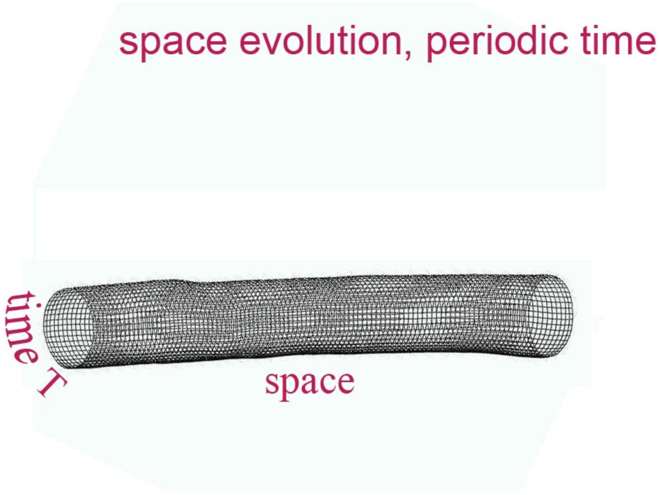


horizontal: $x \in [-11, 11]$
vertical: time
color: magnitude of $u(x, t)$

yes, but

is space time?

compact time, infinite space cylinder



compact time, infinite space Kuramoto-Sivashinsky[10]

$$
\begin{aligned}
u_t &= -uu_x - u_{xx} - u_{xxxx} \,, \\
u^{(0)} &\equiv u \,, \quad u^{(1)} \equiv u_x \,, \quad u^{(2)} \equiv u_{xx} \,, \quad u^{(3)} \equiv u_{xxx}
\end{aligned}
$$

periodic boundary condition in time $u(x,t) = u(x, t+T)$

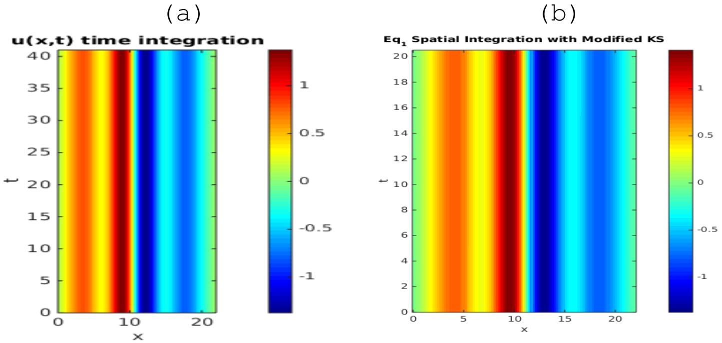
evolve $u(t,x)$ in x, 4 equations, 1st order in spatial derivatives

$$
\begin{aligned}
u_x^{(0)} &= u^{(1)} \,, \quad u_x^{(1)} = u^{(2)} \,, \quad u_x^{(2)} = u^{(3)} \\
u_x^{(3)} &= -u_t^{(0)} - u^{(2)} - u^{(0)} u^{(1)}
\end{aligned}
$$

initial values $u(x_0,t)$, $u_x(x_0,t)$, $u_{xx}(x_0,t)$, $u_{xxx}(x_0,t)$, for all $t \in [0,T)$ at a space point x_0

[10] **GuBuCv17**.

a time-invariant equilibrium, spatial periodic orbit

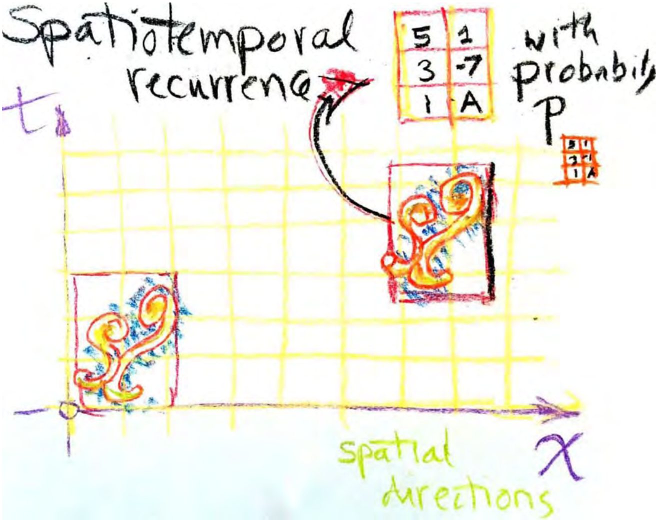


evolution of EQ_1 : (a) in time, (b) in space
initial condition for the spatial integration is the time strip $u(x_0, t)$, $t = [0, T)$, where time period $T = 0$, spatial x period is $L = 22$.

Michelson 1986

chronotope :

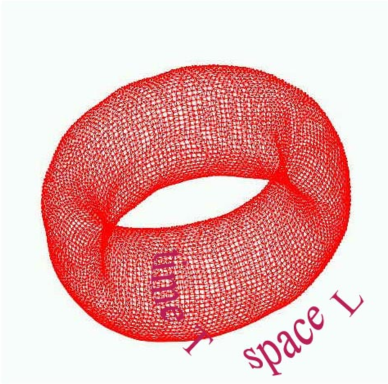
a finite $(1 + D)$-dimensional symbolic dynamics rectangle



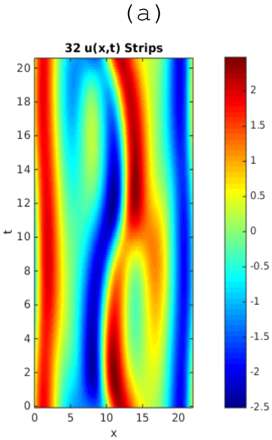
make it doubly periodic

compact space and time chronotope

periodic spacetime : 2-torus



a spacetime invariant 2-torus[11]



(a) old : time evolution.

(b) new : space evolution

$x = [0, L]$ initial condition : time periodic line $t = [0, T]$

Gudorf 2016

[11] **GuBuCv17**.

conclusion

space, time coordinates merely parametrize
a given invariant solution

what matters is

the enumeration of distinct invariant solutions

problem

unable to integrate the equations for times beyond Lyapunov time

unable to integrate the equations for large spatial domains

spatial integration is ill-posed, wildly unstable[12]

[12] **WGBGQ13**.

part 4

1. “turbulence” in small domains
2. “turbulence” in infinite spatial domains
3. space is time
4. **bye bye, dynamics**

computing spacetime solutions

kiss your DNS codes [13]

goodbye

for long time and/or space integrations

they never worked and could never work

[13] DNS : direct numerical simulation

life outside of time

the trouble:
forward time-integration codes too unstable

multishooting inspiration: replace an initial guess point on the periodic orbit by a guess of the entire orbit

$1d$ example : variational principle for any periodic orbit[14,15]

N guess points $\to \infty$ points along a smooth loop

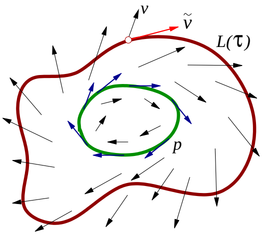
[14] **IanVar1**.

[15] **CvitLanCrete02**.

extremal principle for a general flow

initial guess $\tilde{v}(\tilde{x}) \neq v(\tilde{x})$

periodic orbit $\tilde{v}(\tilde{x})$, $v(\tilde{x})$ aligned



cost function

$$F^2[\tilde{x}] = \oint_L ds\, (\tilde{v} - v)^2 \,; \quad \tilde{v} = \tilde{v}(\tilde{x}(s,\tau))\,,\ v = v(\tilde{x}(s,\tau))\,,$$

penalizes misorientation of the loop tangent $\tilde{v}(\tilde{x})$ relative to the true dynamical flow $v(\tilde{x})$

variational methods

cost minimization

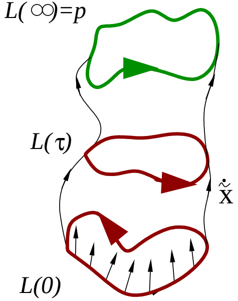
drives

initial guess $L(0)$

$\rightarrow$

exact solution $p = L(\infty)$

as fictitious time $\tau \rightarrow \infty$



life outside of time

the trouble:
Navier-Stokes forward time-integration codes too unstable

replace a guess that an initial fluid state by a guess of the entire spatiotemporal state (snapshots of the pattern at successive time instants)

a variational method then drives the initial guess

$\rightarrow$

an exact spatio-temporally periodic solution of a field theory[16]

---

[16] **LCC06**.

do this ?

the answer is

scalability

compute locally, adjust globally

Computing literature : parallelizing spatiotemporal computation is FLOPs intensive, but more robust than integration forward in time[17]

[17] **WGBGQ13**.

how do clouds solve field equations?

do clouds integrate Navier-Stokes equations?

NO!

$\Longrightarrow$ other swirls $\Longrightarrow$

at any spacetime point Navier-Stokes equations describe the local tangent space

they satisfy them **locally**, everywhere and at all times

summary

1. small computational domains reduce “turbulence” to “single particle” chaos
2. consider instead turbulence in infinite spatiatemporal domains
3. theory : classify all spatiotemporal tilings
4. numerics : parallelize spatiotemporal computations

there is no more time

there is only enumeration of spacetime solutions

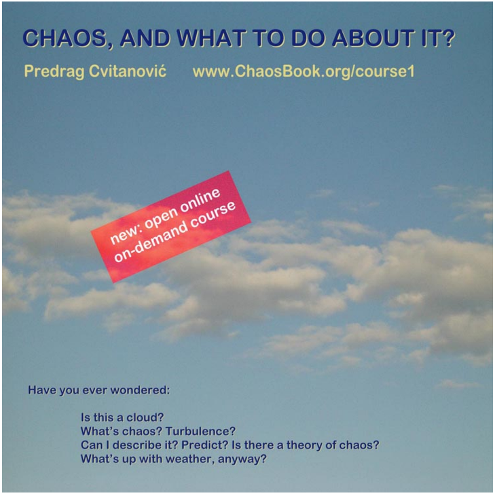
conclusion : turbulence and free will

any possible cloud shape $u(x, t)$

is recorded in the Book of Life
by a unique admissible symbol sequence $m_{t,x}$

what is next for the students of Landau's Theoretical Minimum?

take the course!



student raves :

...10^6 times harder than any other online course...

bonus slide : each chronotope is a fixed point

discretize $u_{n,m} = u(x_n, t_m)$ over NM points of spatiotemporal periodic lattice $x_n = nT/N$, $t_m = mT/M$, Fourier transform :

$$\tilde{u}_{k,\ell} = \frac{1}{NM} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} u_{n,m}\, e^{-i(q_k x_n + \omega_\ell t_m)}\,, \quad q_k = \frac{2\pi k}{L}\,,\ \omega_\ell = \frac{2\pi\ell}{T}$$

Kuramoto-Sivashinsky is no more a PDE / ODE, but a fixed point problem of determining all invariant unstable 2-tori

$$\left[-i\omega_\ell - (q_k^2 - q_k^4)\right] \tilde{u}_{k,\ell} + i\frac{q_k}{2} \sum_{k'=0}^{N-1} \sum_{m'=0}^{M-1} \tilde{u}_{k',m'} \tilde{u}_{k-k',m-m'} = 0$$

Newton method for a NM-dimensional fixed point :
invert $1 - J$,
where J is the 2-torus Jacobian matrix, yet to be elucidated

zeta function for a field theory ? much like Ising model[18,19]

"periodic orbits" are now spacetime tilings

$$Z(s) \approx \sum_p \frac{e^{-A_p s}}{\left|\det\left(1 - J_p\right)\right|}$$

count all tori / spacetime tilings : each of area $A_p = L_p T_p$

symbolic dynamics : $(1 + D)$-dimensional

essential to encoded shadowing

at this time : this zeta is still but a dream

[18] **KacWar52**.

[19] **Ihara66**.

bonus slide : dynamical zeta function for a field theory

∞ of spacetime tilings

$$Z(s) \approx \sum_p \frac{e^{-A_p s}}{|\det(1 - J_p)|}$$

tori / plane tilings
each of area $A_p = L_p T_p$

trace formula for a field theory

References I