State space geometry of a spatio-temporally chaotic Kuramoto-Sivashinsky flow

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Introduction

In the dynamical systems approach, the theory of turbulence for a given system, with given boundary conditions, is given by (a) the geometry of the state space and (b) the associated natural measure, that is, the likelihood that asymptotic dynamics visits a given state space region. We pursue this program in the context of Kuramoto-Sivashinsky (KS) equation, one of the simplest physically interesting spatially extended nonlinear systems.

Dynamical state space representation of a PDE is ∞-dimensional, but the KS flow is strongly contracting and its non-wondering set, and, within it, the set of invariant solutions investigated here, is embedded into a finite-dimensional inertial manifold [2] in a non-trivial, nonlinear way. 'Geometry' in the title of this paper refers to our attempt to systematically triangulate this set in terms of dynamically invariant solutions (equilibria, periodic orbits, ...) and their unstable manifolds, in a PDE representation independent way. The goal is to describe a given 'turbulent' flow quantitatively, not model it qualitatively by a low-dimensional model.

In previous work, the state space geometry and the natural measure for this system have been studied [8, 3, 4] in terms of unstable periodic solutions restricted to the antisymmetric subspace of the KS dynamics. Here the dynamics are not restricted to an invariant subspace and as a result of the continuous symmetries the notion of exact periodicity in time is replaced by the notion of relative spatiotemporal periodicity. We present a study of the Kuramoto-Sivashinsky dynamics for a specific system size $L = 22$, sufficiently large to exhibit many of the features typical of 'turbulent' dynamics observed in large KS systems, but small enough to lend itself to a detailed exploration of the geometry of the flow.

Kuramoto-Sivashinsky equation

The Kuramoto-Sivashinsky [henceforth KS] system [6, 7], which arises in the description of stability of flame fronts, reaction-diffusion systems and many other physical settings, is one of the simplest nonlinear PDEs that exhibit spatiotemporally chaotic behavior. In the formulation adopted here, the time evolution of the 'flame front slope' $u = u(x,t)$ on a periodic domain $u(x,t) = u(x+L,t)$ is given by

$$u_t = F(u) = -\frac{1}{2}(u^2)_x - u_{xx} - u_{xxxx}\,, \qquad x \in [-L/2, L/2]\,. \qquad (1)$$

In what follows we shall state results of all calculations either in units of the 'dimensionless system size' $\tilde{L}$, or the system size $L = 2\pi\tilde{L}$. All numerical results presented in this paper are for the system size $\tilde{L} = 22/2\pi = 3.5014\ldots$.

Symmetries of Kuramoto-Sivashinsky equation

G, the group of actions $g \in G$ on a state space (reflections, translations, *etc.*) is a symmetry of the KS flow (1) if $g\,u_t = F(g\,u)$. The KS equation is time translationally invariant, and space translationally invariant on a periodic domain under the 1-parameter group of $O(2) : \{\tau_{\ell/L}, R\}$. If $u(x,t)$ is a solution, then

$$\tau_{\ell/L}\, u(x,t) = u(x+\ell,t) \qquad (2)$$

is an equivalent solution for any shift $-L/2 < \ell \le L/2$, as is the reflection ('parity' or 'inversion')

$$R\,u(x) = -u(-x)\,. \qquad (3)$$

Equilibria and relative equilibria

Equilibria (or the steady solutions) are the fixed profile time-invariant solutions,

$$u(x,t) = u_q(x)\,. \qquad (4)$$

Due to the translational symmetry, the KS system also allows for relative equilibria (traveling waves, rotating waves), characterized by a fixed profile $u_q(x)$ moving with constant speed c, that is

$$u(x,t) = u_q(x-ct)\,. \qquad (5)$$

Because of the reflection symmetry (3), the relative equilibria come in counter-traveling pairs $u_q(x-ct)$, $-u_q(-x+ct)$.

Relative periodic orbits, symmetries and periodic orbits

A relative periodic orbit satisfies

$$g\,u(x,T_p) = u(x,0)\,, \qquad (6)$$

where $g \in G$, with G a symmetry of the flow. Thus, the KS equation can have relative periodic orbits corresponding to

1. Invariance under $\tau_{\ell/L}$

$$\tau_{\ell_p/L}\,u(x,T_p) = u(x+\ell_p,T_p) = u(x,0) = u_p(x)\,. \qquad (7)$$

2. Invariance under reflections

$$Ru(x,T_p) = -u(-x,T_p) = u(x,0) = u_p(x)\,, \qquad (8)$$

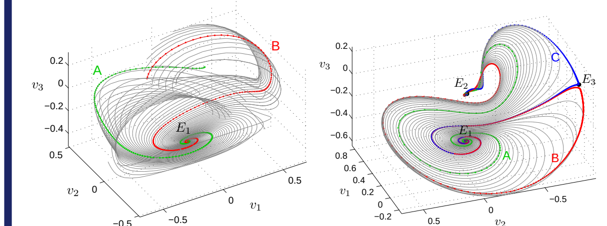
Such an orbit is pre-periodic to a periodic orbit with period $2T_p$.

Equilibria and relative equilibria for $L = 22$

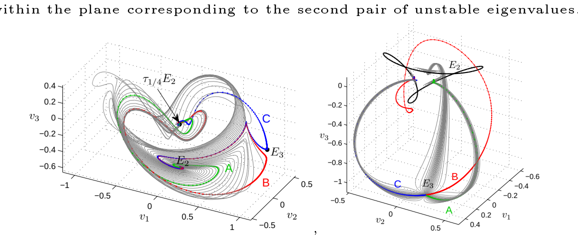
In addition to the trivial equilibrium $u = 0$ (denoted E_0), we find three equilibria with dominant wavenumber k (denoted E_k) for $k = 1, 2, 3$. All equilibria are symmetric with respect to the reflection symmetry (3). In addition, E_2 and E_3 are symmetric with respect to translation (2), by $L/2$ and $L/3$, respectively. E_2 and E_3 essentially lie, respectively, in the 2^{nd} and 3^{rd} Fourier component complex plane, with small $k = 2j$, $k = 3j$ harmonics deformations.

We find two pairs of relative equilibria (5) with velocities $c = \pm 0.73699$ and ± 0.34954 which we label $TW_{\pm 1}$ and $TW_{\pm 2}$, for 'traveling waves.'

All equilibria and relative equilibria found here are unstable. Even though E_1, E_2 and E_3 lie in the antisymmetric subspace, their unstable manifolds are not restricted to this subspace.



The left panel shows the unstable manifold of equilibrium E_1 starting within the plane corresponding to the first pair of unstable eigenvalues. The right panel shows the unstable manifold of equilibrium E_1 starting within the plane corresponding to the second pair of unstable eigenvalues.

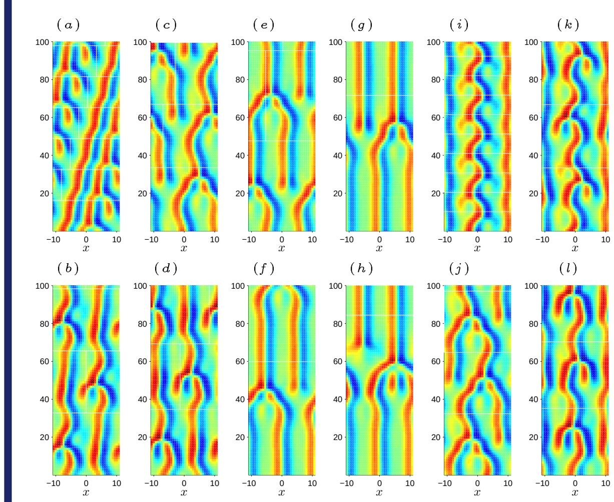


The left panel shows the two-dimensional unstable manifold of equilibrium E_2. The right panel shows the two-dimensional unstable manifold of equilibrium E_3.

The coordinate axes v_1, v_2, and v_3 in all of the above figures are constructed by Gram-Schmidt orthogonalization from the stability eigenvectors corresponding to the leading eigenvalues of the relevant equilibrium.

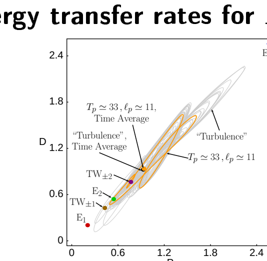
The heteroclinic and homoclinic connections between families of equilibria that can be seen in the figures are connected to the symmetries (2–3) of Kuramoto-Sivashinsky equation and are found to be structurally stable, *cf.* ref. [5].

Relative periodic orbits for $L = 22$



Selected relative periodic and pre-periodic orbits of Kuramoto-Sivashinsky equation with $L = 22$: (a) $T_p = 16.3$, $\ell_p = 2.86$; (b) $T_p = 32.8$, $\ell_p = 10.96$; (c) $T_p = 33.5$, $\ell_p = 4.04$; (d) $T_p = 34.6$, $\ell_p = 9.60$; (e) $T_p = 47.6$, $\ell_p = 5.68$; (f) $T_p = 59.9$, $\ell_p = 5.44$; (g) $T_p = 71.7$, $\ell_p = 5.503$; (h) $T_p = 84.4$, $\ell_p = 5.513$; (i) $T_p = 10.3$; (j) $T_p = 32.4$; (k) $T_p = 33.4$; (l) $T_p = 35.2$. Horizontal and vertical white lines indicate periodicity and phase shift of the orbits, respectively. We have limited our search to orbits with $T_p < 200$ and found over 300 relative periodic orbits with $\ell_p > 0$. The search has not been exhaustive, and there are likely to be more orbits with $T_p < 200$. However, the orbits we have found provide a representative sample of typical periodic and relative periodic orbits and approximate well the chaotic attractor since they were located using seeds obtained from close returns within the chaotic dynamics.

Energy transfer rates for $L = 22$



Power input $P = \langle u_x{}^2 \rangle$ *vs.* dissipation rate $D = \langle u_{xx}{}^2 \rangle$ for several equilibria and relative equilibria, a relative periodic orbit, and a typical 'turbulent' long-time trajectory. The relative periodic orbit $(T_p, \ell_p) = (32.8, 10.96)$ appears well embedded within the turbulent flow. The mean power $\overline{P_p}$ is numerically quite close to the long-time turbulent time average $\overline{P}$. Yet, figures like this can be misleading. As always, here too one needs a hierarchy of periodic orbits of increasing length to obtain accurate predictions [9].

References

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