a spatiotemporal theory of
turbulence
in terms of exact coherent structures

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SIAM - DS19 Dynamical Systems

May 21, 2019

overview

1. what this talk is about
2. turbulence in large domains
3. space is time
4. bye bye, dynamics

do clouds solve PDEs?

do clouds integrate Navier-Stokes equations?

NO!

 $\Longrightarrow$ other swirls $\Longrightarrow$ 

do clouds satisfy Navier-Stokes equations?

yes!

they satisfy them locally, everywhere and at all times

part 1

1. turbulence in large domains
2. space is time
3. spacetime
4. bye bye, dynamics

goal : enumerate the building blocks of turbulence

Navier-Stokes equations

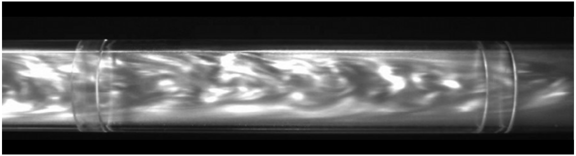
$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{1}{R}\nabla^2 \mathbf{v} - \nabla p + \mathbf{f}, \qquad \nabla \cdot \mathbf{v} = 0,$$

velocity field $\mathbf{v} \in \mathbb{R}^3$; pressure field p ; driving force $\mathbf{f}$

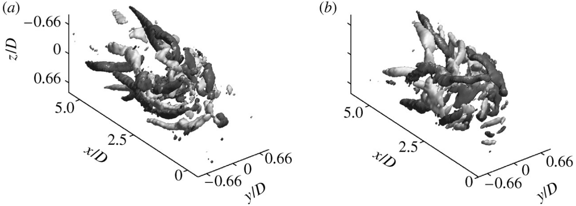
describe turbulence

starting from the equations (no statistical assumptions)

challenge : experiments are amazing

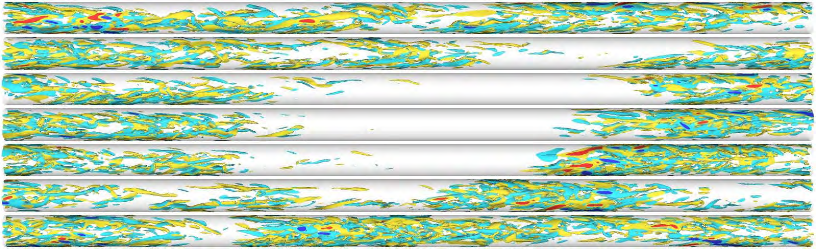


T. Mullin lab



B. Hof lab

can simulate **large** computational domains



pipe flow close to onset of turbulence [1]

but we have **hit a wall**:

exact coherent structures are too unstable to compute

[1] M. Avila and B. Hof, Phys. Rev. **E 87** (2013)

goal : we can do 3D turbulence, but for this talk

Navier-Stokes equations

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{1}{R}\nabla^2 \mathbf{v} + (\cdots)$$

velocity field $\mathbf{v}(\mathbf{x}; t) \in \mathbb{R}^3$

not helpful for developing intuition

we cannot visualize 3D velocity field at every 3D spatial point

look instead at 1D 'flame fronts'

(1+1) spacetime dimensional "Navier-Stokes"

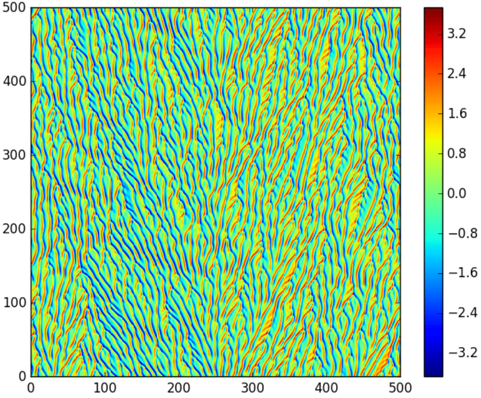
Kuramoto-Sivashinsky $(1+1)$-dimensional PDE

$$u_t + u\nabla u = -\nabla^2 u - \nabla^4 u \,, \qquad x \in \mathbb{R} \,,$$

describes spatially extended systems such as

- flame fronts in combustion
- reaction-diffusion systems
- ...

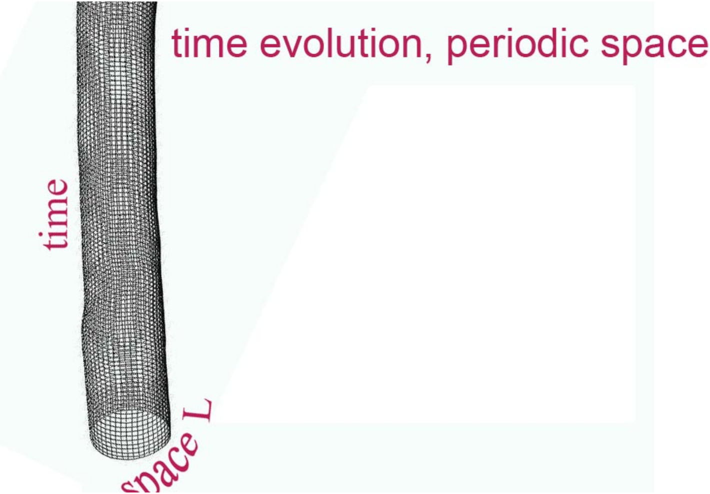
an example : Kuramoto-Sivashinsky on a large domain



[horizontal] space $x \in [0, L]$ [up] time evolution

- turbulent behavior
- simpler physical, mathematical and computational setting than Navier-Stokes

compact space, infinite time cylinder



so far : Navier-Stokes on compact spatial domains, all times

compact space, infinite time

Kuramoto-Sivashinsky equation

$$u_t = -(+\nabla^2 + \nabla^4)u - u\nabla u\,, \qquad x \in [-L/2, L/2]\,,$$

in terms of discrete spatial Fourier modes

N ordinary differential equations (ODEs) in time

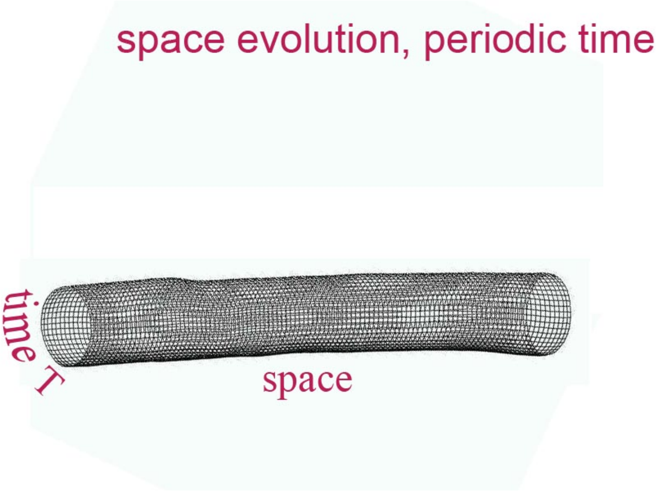
$$\dot{\tilde{u}}_k(t) = (q_k^2 - q_k^4)\,\tilde{u}_k(t) - i\frac{q_k}{2}\sum_{k'=0}^{N-1}\tilde{u}_{k'}(t)\tilde{u}_{k-k'}(t)\,.$$

part 2

1. turbulence in large domains
2. space is time
3. spacetime
4. bye bye, dynamics

is space time?

can do : compact time, infinite space cylinder



compact time, infinite space

Kuramoto-Sivashinsky as four fields 1st order in spatial derivatives

$$
\begin{aligned}
u_t &= -uu_x - u_{xx} - u_{xxxx} \,, \\
u^{(0)} &\equiv u \,, \quad u^{(1)} \equiv u_x \,, \quad u^{(2)} \equiv u_{xx} \,, \quad u^{(3)} \equiv u_{xxx}
\end{aligned}
$$

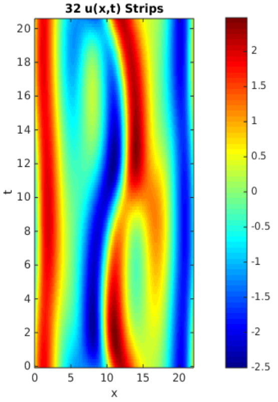
evolve four 1st order PDEs $u^{(j)}(t, x)$ in x,

periodic boundary condition in time $u(x, t) = u(x, t + T)$

$$
\begin{aligned}
u_x^{(0)} &= u^{(1)} \,, \quad u_x^{(1)} = u^{(2)} \,, \quad u_x^{(2)} = u^{(3)} \\
u_x^{(3)} &= -u_t^{(0)} - u^{(2)} - u^{(0)} u^{(1)}
\end{aligned}
$$

initial $u^{(0)}(x_0, t)$, $u^{(1)}(x_0, t)$, $u^{(2)}(x_0, t)$, $u^{(3)}(x_0, t)$
specified for $t \in [0, T)$, at a fixed space point x_0

a solution integrated in either time or space



old : time evolution
$x = [0, L]$ initial condition

new : space evolution
time periodic line $t = [0, T]$

Gudorf 2016

but integrations are uncontrollably unstable

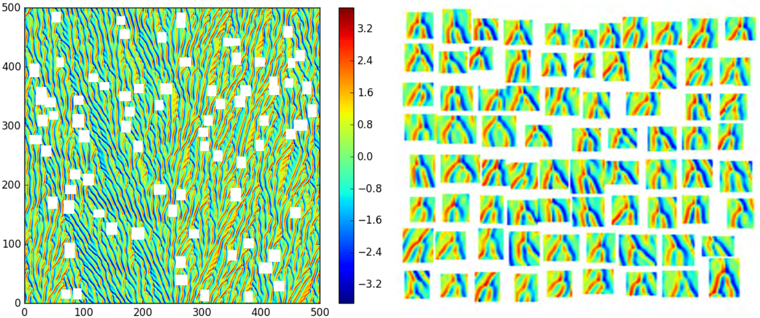
neither time nor space integration works
for large domains

rethink the formulation!

1. turbulence in large domains
2. space is time
3. **spacetime**
4. bye bye, dynamics

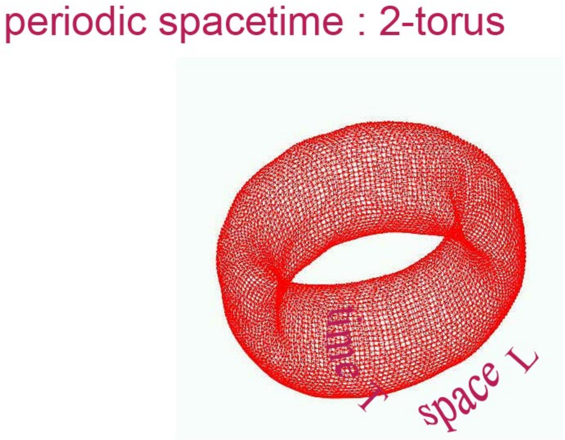
Kuramoto-Sivashinsky on a large spacetime domain

the same small tile recurs often in a turbulent pattern



goal : define, enumerate nearly recurrent tiles

use spatiotemporally compact solutions as spacetime ‘tiles’



shadows a small patch of spacetime

periodic orbits generalize to d-tori

1 time, 0 space dimensions

a state space point is *periodic* if its orbit returns to it after a finite time T ;
such orbit tiles the time axis by infinitely many repeats

1 time, d-1 space dimensions

a state space point is *spatiotemporally periodic* if it belongs to an invariant d-torus $\mathcal{R}$;
such torus tiles the spacetime by infinitely many repeats

every compact solution is a fixed point on a discrete lattice

discretize $u_{nm} = u(x_n, t_m)$ over NM points of spatiotemporal periodic lattice $x_n = nL/N$, $t_m = mT/M$, Fourier transform :

$$\tilde{u}_{k\ell} = \frac{1}{NM} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} u_{nm}\, e^{-i(q_k x_n + \omega_\ell t_m)}\,, \quad q_k = \frac{2\pi k}{L}\,,\ \omega_\ell = \frac{2\pi \ell}{T}$$

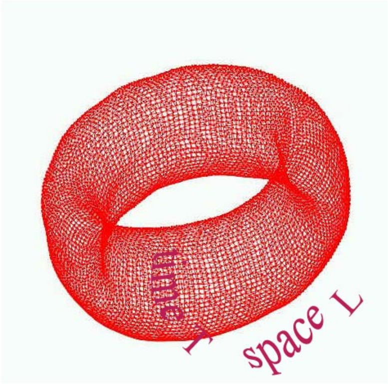
Kuramoto-Sivashinsky is no more a PDE / ODE,
but an algebraic $[N \times M]$-dimensional fixed point problem
of determining a solution to

$$\left[-i\omega_\ell - (q_k^2 - q_k^4)\right] \tilde{u}_{k\ell} + i\frac{q_k}{2} \sum_{k'=0}^{N-1} \sum_{m'=0}^{M-1} \tilde{u}_{k'm'} \tilde{u}_{k-k',m-m'} = 0$$

every calculation is a spatiotemporal lattice calculation

field is discretized as $\tilde{u}_{k\ell}$ values
over NM points of a periodic lattice

periodic spacetime : 2-torus



there is no more time evolution

solution to Kuramoto-Sivashinsky is now given as

condition that

at each lattice point $k\ell$
the tangent field at $\tilde{u}_{k\ell}$

satisfies the equations of motion

$$\left[-i\omega_\ell - (q_k^2 - q_k^4)\right] \tilde{u}_{k\ell} + i\frac{q_k}{2}\sum_{k'=0}^{N-1}\sum_{m'=0}^{M-1}\tilde{u}_{k'm'}\tilde{u}_{k-k',m-m'} = 0$$

this is a local tangent field constraint on a global solution

robust : no exponential instabilities as there are no finite time / space integrations

part 4

1. turbulence in large domains
2. space is time
3. spacetime
4. **spacetime computations**
5. bye bye, dynamics

how do clouds solve PDEs?

clouds do not NOT integrate Navier-Stokes equations

 $\Longrightarrow$ other swirls $\Longrightarrow$ 

do clouds satisfy Navier-Stokes equations?

yes!

they satisfy them locally, everywhere and at all times

the equations imposed as local constraints

Kuramoto-Sivashinsky equation

$$F(u) = u_t + u_{xx} + u_{xxxx} + uu_x = 0$$

for example, minimize over the entire 2-torus

cost function

$$G \equiv \frac{1}{2}|F(u)|^2_{L\times T}$$

need your help !

Adjoint descent

cost function

$$G = \frac{1}{2}\mathbf{F}^\top \mathbf{F}\,.$$

introduce fictitious time (τ) flow by differentiation of cost function.

$$\partial_\tau G = (J^\top \mathbf{F})^\top (\partial_\tau \mathbf{x})$$

“adjoint descent” method defined by chosing[2]

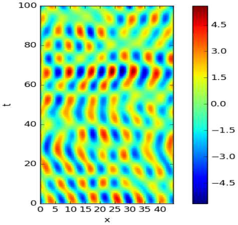
$$\partial_\tau \mathbf{x} = -(J^\top \mathbf{F})$$

[2] M. Farazmand, J. Fluid M. **795**, 278–312 (2016).

initial guess generation ?

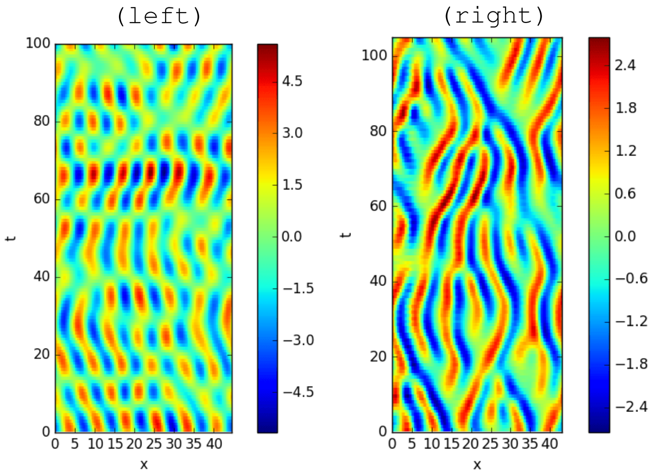
the time scale : the shortest ‘turnover’ scale (?)

the spatial scale : $\bar{L} = 2\pi\sqrt{2}$, the most unstable spatial wavelength of the Kuramoto-Sivashinsky



initial : spatial $\bar{L}$-modulated random guess

KS invariant 2-torus found variationally



(left) initial : $\bar{L} = 2\pi\sqrt{2}$ spatially modulated "noisy" guess
(right) adjoint descent : converged invariant 2-torus

Gudorf 2018

initial guesses, embedded in ergodic sea?

Historically,

guesses are extracted from close recurrences observed in long turbulent simulations

1. inefficient, find only the shortest, least unstable orbits[3,4]
2. cannot integrate that far in time

need spatiotemporal guesses

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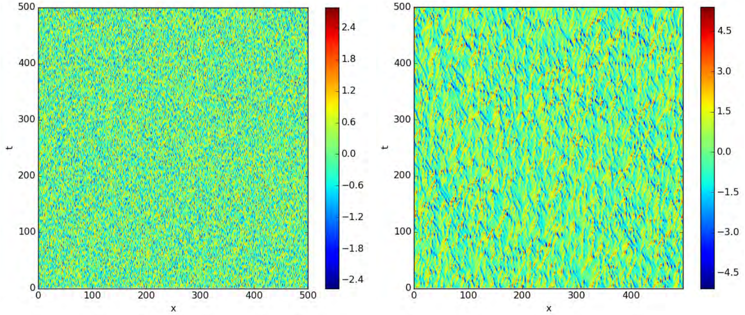
[3] D. Auerbach et al., Phys. Rev. Lett. **58**, 2387–2389 (1987).

[4] J. F. Gibson et al., J. Fluid Mech. **611**, 107–130 (2008).

our guesses are extracted from large spacetime domains



(left) random initial state on $(L_0, T_0) = (500.0, 500.0)$
(right) adjoint descent $\rightarrow$ typical Kuramoto-Sivashinsky state

finite windows are starting guesses for invariant 2-tori

what do do?

Matt Gudorf

has 1 000's of such invariant 2-tori

see Matt's poster, Session PP2

Wednesday, May 22

building blocks of turbulence

how do we recognize a cloud?

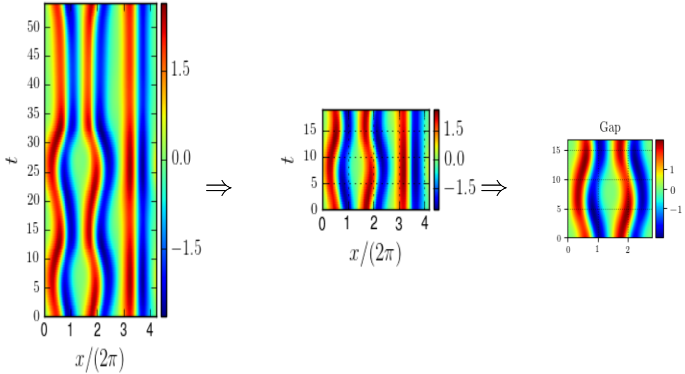
WATCH

$\Longrightarrow$ other swirls $\Longrightarrow$

by recurrent shapes!

so, construct an alphabet of possible shapes

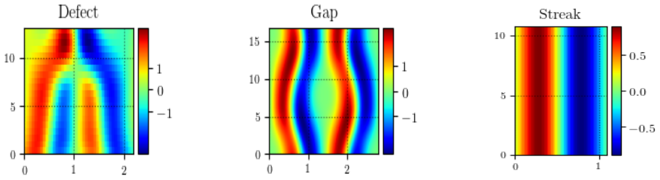
extracting a fundamental tile



1) invariant 2-torus
2) invariant 2-torus computed from initial guess cut out from 1)
3) “gap" invariant 2-torus, cut out from 2)

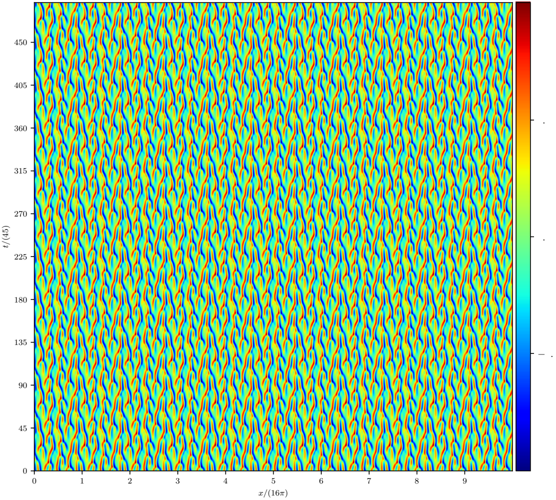
a (too) simple set of prime (rubber) tiles

an alphabet of Kuramoto-Sivashinsky fundamental tiles



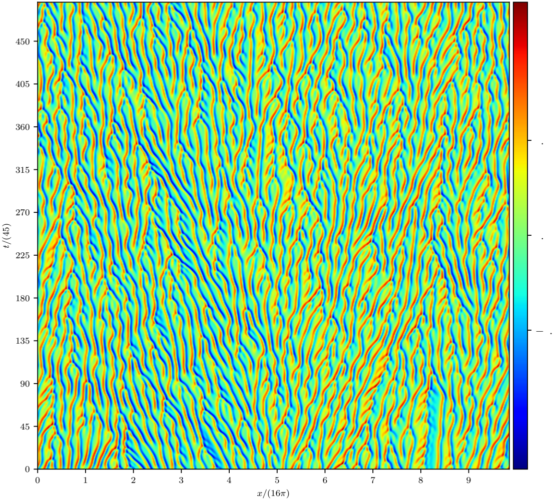
utilize also discrete symmetries :
spatial reflection, spatiotemporal shift-reflect, $\cdots$

spacetime tiled by a small tile



tiling by invariant 2-torus
$(L, T) = (55.83, 24)$

any particular tiling looks nothing like turbulent Kuramoto-Sivashinsky!



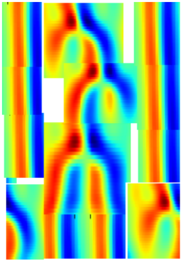
[horizontal] space $x \in [-L/2, L/2]$ [up] time evolution

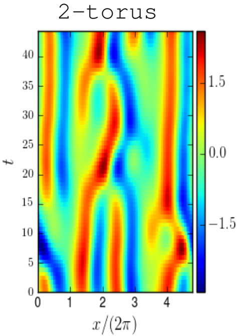
part 5

1. turbulence in large domains
2. space is time
3. spacetime
4. fundamental tiles
5. **gluing tiles**
6. bye bye, dynamics

a qualitative tiling guess

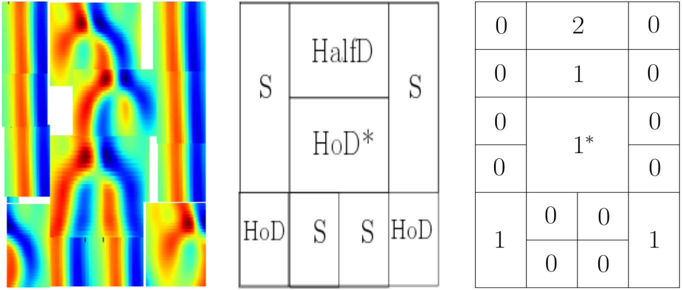
a guess tiling and the resulting solution





turbulence.zip : each solution has a unique symbolic name

symbolic dynamics is 2-dimensional!



- each symbol indicates a corresponding spatiotemporal tile
- these are “rubber” tiles

part 5

1. turbulence in large domains
2. space is time
3. bye bye, dynamics

goodbye

to long time and/or space integrators

they never worked and could never work

life outside of time

the trouble:
forward time-integration codes too unstable

multishooting inspiration: replace a guess that a point is on the periodic orbit by a guess of the entire orbit.

$\rightarrow$

spatio-temporally periodic solutions of classical field theories can be found by variational methods do this ?

the answer is

scalability

compute locally, adjust globally

Navier-Stokes codes

- T. Schneider : developing a matrix-free variational Navier-Stokes code, machine learning initial guesses
- D. Lasagna and A. Sharma : developing variational adjoint solvers to find periodic orbits with long periods
- Q. Wang : parallelizing spatiotemporal computation is FLOPs intensive, but more robust than integration forward in time

it's rocket science[5,6,7]

[5] Q. Wang et al., Phys. Fluids **25**, 110818 (2013).

[6] T. M. Schneider, *Variational adjoint methods coupled with machine learning*, private communication, 2019.

[7] D. Lasagna et al., *Periodic shadowing sensitivity analysis of chaotic systems*, 2018.

towards scalable parallel-in-time turbulent flow simulations

future :

processor speed $\rightarrow$ limit

number of cores $\rightarrow 10^6 \rightarrow \cdots$

Q. Wang[8]*:*

next-generation simulation paradigm : spacetime parallel simulations, on discretized $4D$ spacetime computational domains, with each computing core handling a spacetime lattice cell

compared to time-evolution solvers: significantly higher level of concurrency, reduction the ratio of inter-core communication to floating point operations

$\Rightarrow$ a path towards exascale DNS of turbulent flows

[8] Q. Wang et al., Phys. Fluids **25**, 110818 (2013).

take home : clouds do not solve PDEs

do clouds integrate Navier-Stokes equations?

NO!

$\Longrightarrow$ other swirls $\Longrightarrow$

at any spacetime point Navier-Stokes equations describe the local tangent space

they satisfy them locally, everywhere and at all times

summary

1. study turbulence in infinite spatiatemporal domains
2. theory : classify all spatiotemporal tilings
3. numerics : future is spatiotemporal

there is no more time

there is only enumeration of spacetime solutions

spatiotemporally infinite spatiotemporal cat