Spatiotemporal Tiling of the Kuramoto-Sivashinsky equation

Matthew Gudorf

Center for Nonlinear Science
School of Physics
Georgia Institute of Technology

Adviser: Predrag Cvitanović

Motivation

- High dimensional fluid flows are hard. Most results in "minimal flow units"[1]
- Conventional computational tools are providing diminishing returns
- Spatiotemporal methods are scalable[2]

[1] J. Jiménez and P. Moin, J. Fluid Mech. **225**, 213–240 (1991).
[2] Q. Wang et al., Phys. Fluids **25**, 110818 (2013).

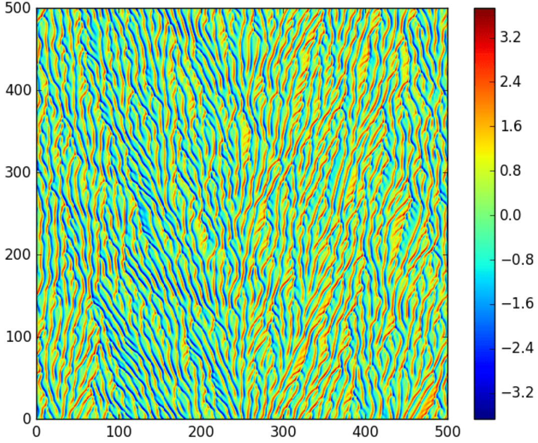
Doubly Periodic Kuramoto-Sivashinsky equation

$$\underbrace{u_t(x,t) = -u_{xx}(x,t)}_{\text{Diffusion (wrong sign)}} - \underbrace{u_{xxxx}(x,t)}_{\text{“Hyper”-diffusion}} - \underbrace{u(x,t)u_x(x,t)}_{\text{Advection}}$$

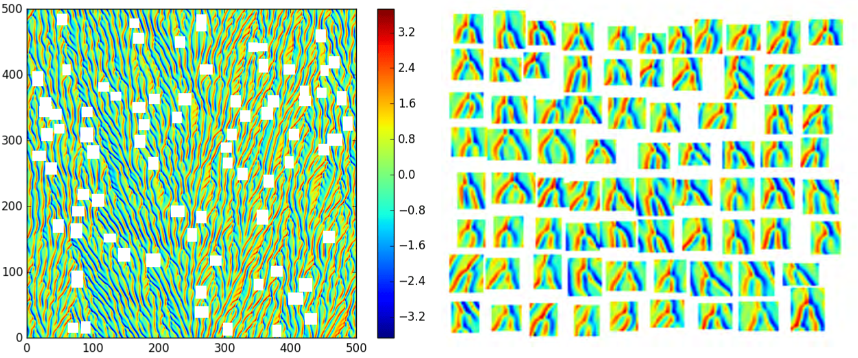
Boundary conditions

$u(x,t) = u(x,t+T_p) = u(x+L,t) = u(x+L,t+T_p)$

Large spatiotemporal simulation



Shadowing


500
400
300
200
100
0
0
100
200
300
400
500
3.2
2.4
1.6
0.8
0.0
−0.8
−1.6
−2.4
−3.2

Fourier-Fourier Basis

Substituting

$$u(x,t) = \sum_{m,n} \tilde{u}_{m,n} e^{\mathrm{i}\omega_n t + \mathrm{i} q_m x}\,, \tag{1}$$

into the Kuramoto-Sivashinsky equation gives,

$$\mathbf{F}(\mathbf{x}) = (\mathrm{i}\frac{2\pi n}{T} + (\frac{2\pi m}{L})^2 - (\frac{2\pi m}{L})^4)\tilde{u}_{m,n} + \frac{1}{2}\frac{2\pi m}{L}\mathcal{F}((\mathcal{F}^{-1}(\tilde{u}_{m,n}))^2) \tag{2}$$

Solve nonlinear algebraic equations for $\mathbf{x} = (\tilde{\mathbf{u}}, T, L)$. Similar to formulation of Lopez[3]. We use real-valued Fourier basis in computations.

[3] V. López, *Numerical continuation of invariant solutions of the complex Ginzburg–Landau equation*, 2015.

Disadvantages of spatiotemporal methods

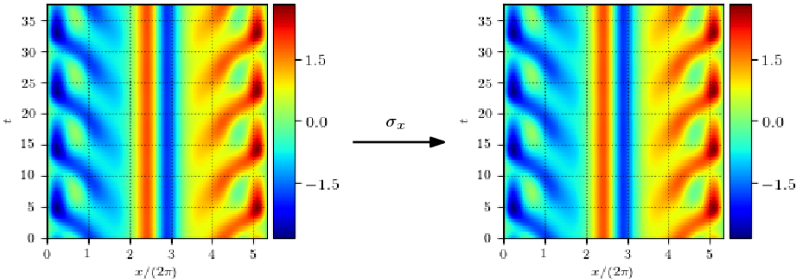
1. High dimensionality
2. Numerically challenging
3. Possible to find isolated solutions that are unimportant
4. Find different members of same continuous families (redundancy)

Advantages of spatiotemporal methods

1. Can use variational methods
2. Fourier-Fourier basis
3. Topological constraint (boundary conditions) provides access to potent methods
4. Stability does not play a role in finding solutions.
5. Can exploit spatiotemporal symmetries

Reflection invariance

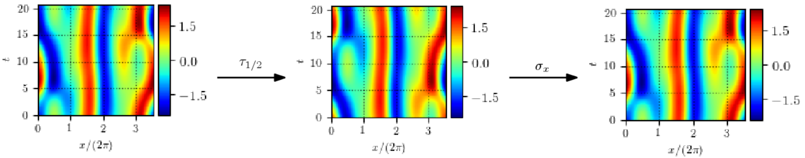
$$u(x,t) = \sigma_x u(x,t)$$



Spatiotemporal Symmetries

Shift-Reflection Invariance

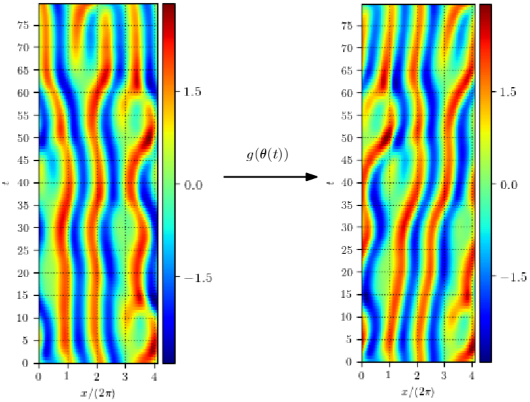
$$u(x,t) = \sigma_x \tau_{1/2} u(x,t)$$



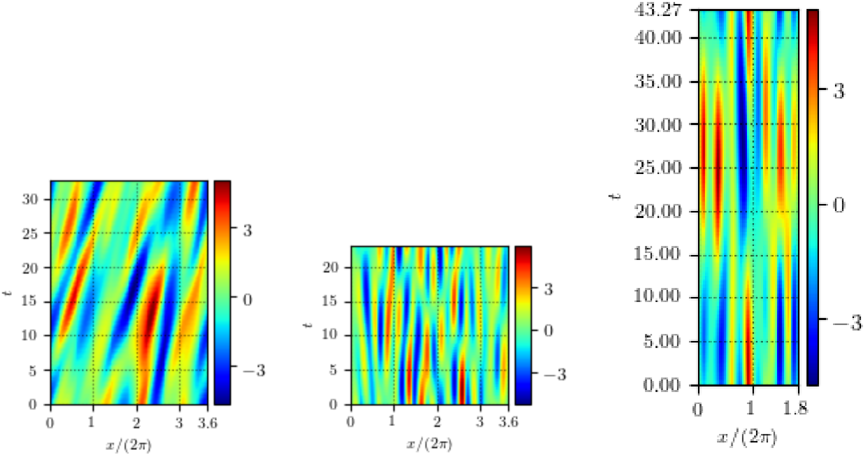
Spatiotemporal Symmetries

Spatial Translation Symmetry

$$u(x,t) = g(\theta(t)) \circ u(x, t+T_p)$$



Initial conditions



Adjoint Descent Method

$$\mathbf{F}(\mathbf{x}) = (\mathrm{i}\frac{2\pi n}{T} + (\frac{2\pi m}{L})^2 - (\frac{2\pi m}{L})^4)\tilde{u}_{m,n} + \frac{1}{2}\frac{2\pi m}{L}\mathcal{F}((\mathcal{F}^{-1}(\tilde{u}_{m,n}))^2) \quad (3)$$

$$I = \frac{1}{2}\mathbf{F}^\top\mathbf{F}\,. \quad (4)$$

Adjoint Descent Method

introduce fictitious time (τ) flow by differentiation of cost function,

$$\partial_\tau I = (J^\top \mathbf{F})^\top (\partial_\tau \mathbf{x}) \tag{5}$$

"adjoint descent" method is defined by choice[4]

$$\partial_\tau \mathbf{x} = -(J^\top \mathbf{F}) \tag{6}$$

[4] M. Farazmand, J. Fluid M. **795**, 278–312 (2016).

Damped Gauss-Newton Method

Solve $\mathbf{F}(\mathbf{x}^*) = 0$ via linearized system

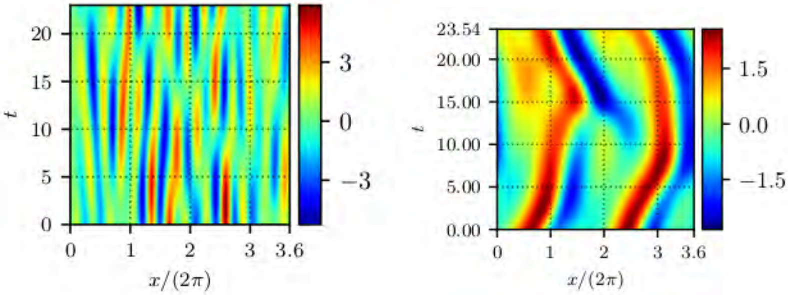
$$\mathbf{F}(\mathbf{x}^*) = \mathbf{F}(\mathbf{x}) + J\delta\mathbf{x} \tag{7}$$

$$J = \begin{bmatrix} \frac{\partial \mathbf{F}}{\partial \mathbf{x}} & \frac{\partial \mathbf{F}}{\partial T} & \frac{\partial \mathbf{F}}{\partial L} \end{bmatrix} \tag{8}$$

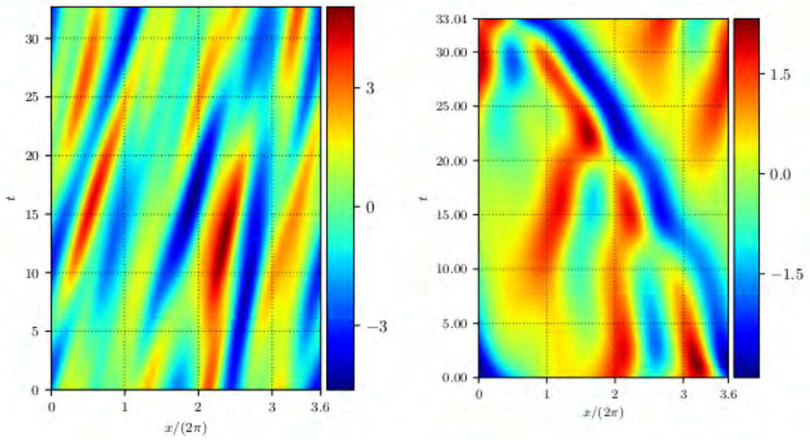
$$J\delta\mathbf{x} = -\mathbf{F}(\mathbf{x}) \tag{9}$$

Shift-reflect symmetry

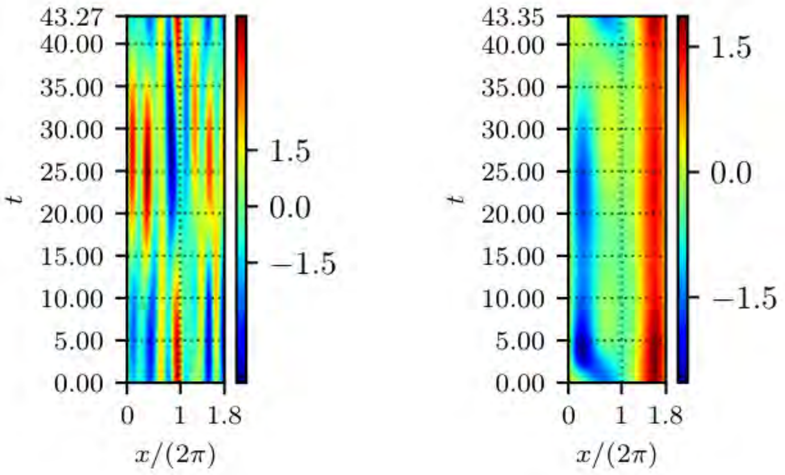
A couple thousand new invariant 2-torus solutions, examples:



Spatial translation symmetry

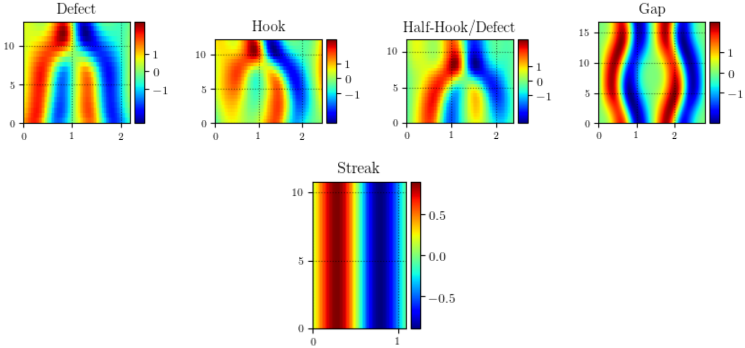


Reflection symmetry

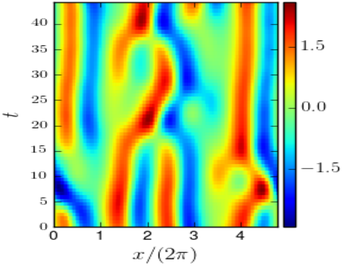


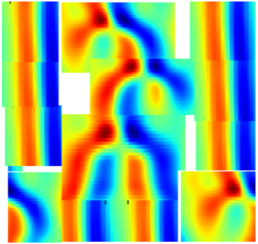
Qualitative guesses for tiles

Sift through collection to find frequently occurring spatiotemporal patterns

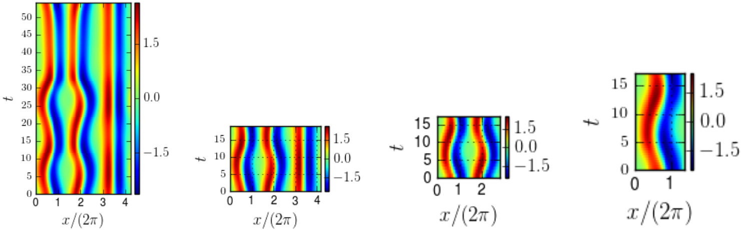


Qualitative tiling of actual solution

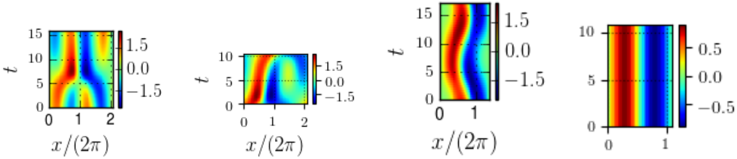




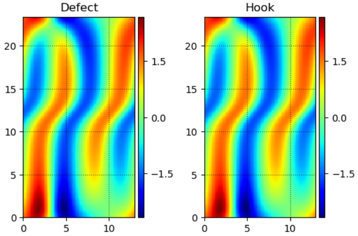
Finding tiles by converging subdomains



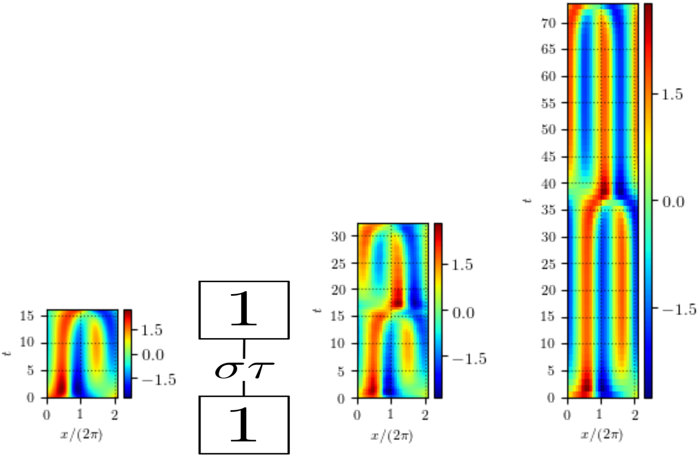
Converged tiles

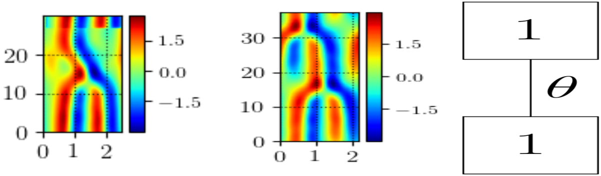


Continuous families and numerical continuation

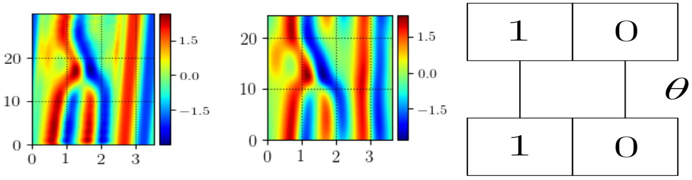


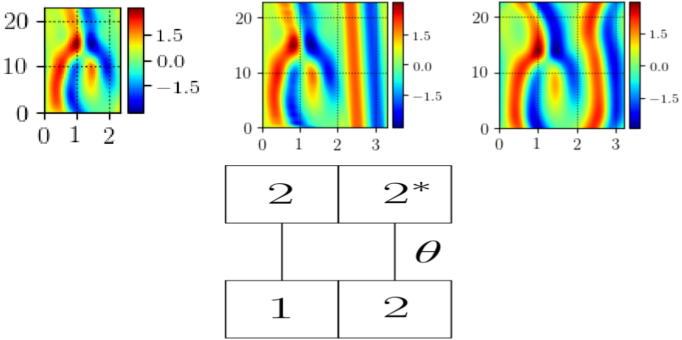
How do we use tiles?



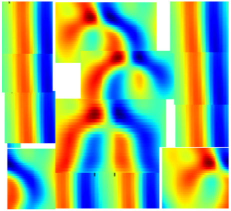

0
10
20
0
1
2
1.5
0.0
−1.5
0
10
20
30
0
1
2
1.5
0.0
−1.5
1
θ
1

Spatiotemporal symbolic dynamics



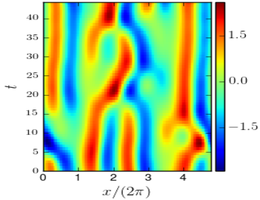

20
10
0
0 1 2
1.5
0.0
−1.5
0 1 2 3
2
2*
θ
1
2

Quantitative tiling of actual solution



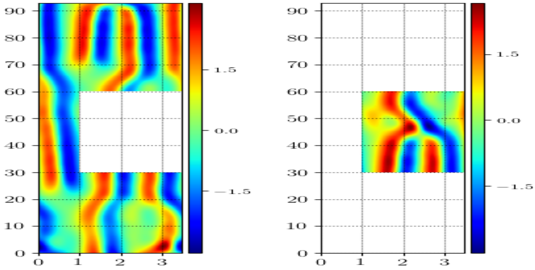
<table>
<tr><td rowspan="2">S</td><td colspan="2">HalfD</td><td rowspan="2">S</td></tr>
<tr><td colspan="2">HoD*</td></tr>
<tr><td>HoD</td><td>S</td><td>S</td><td>HoD</td></tr>
</table>

<table>
<tr><td>0</td><td colspan="2">2</td><td>0</td></tr>
<tr><td>0</td><td colspan="2">1</td><td>0</td></tr>
<tr><td>0</td><td colspan="2" rowspan="2">1*</td><td>0</td></tr>
<tr><td>0</td><td>0</td></tr>
<tr><td rowspan="2">1</td><td>0</td><td>0</td><td rowspan="2">1</td></tr>
<tr><td>0</td><td>0</td></tr>
</table>

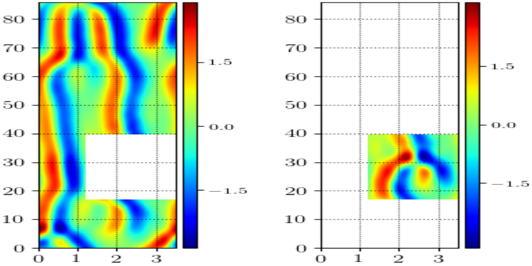


[4] Each symbol is technically paired with spatiotemporal parameters (T, L)

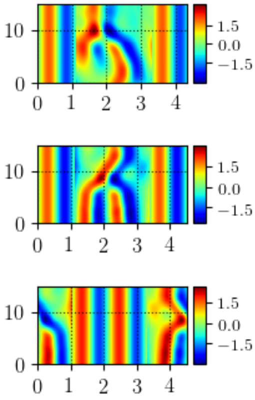
New solutions from gluing together old



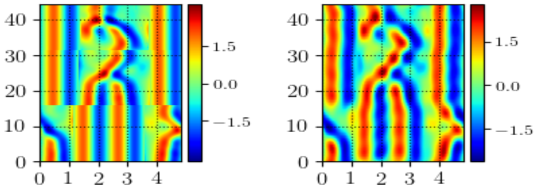
New solutions from gluing together old



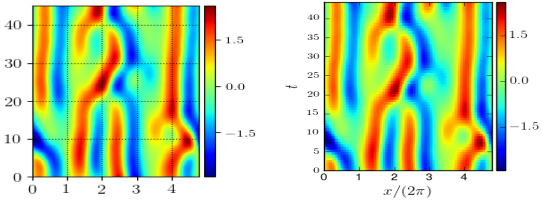
New solutions from gluing together old



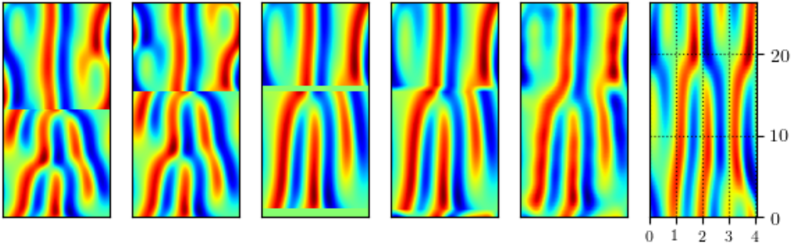
New solutions from gluing together old



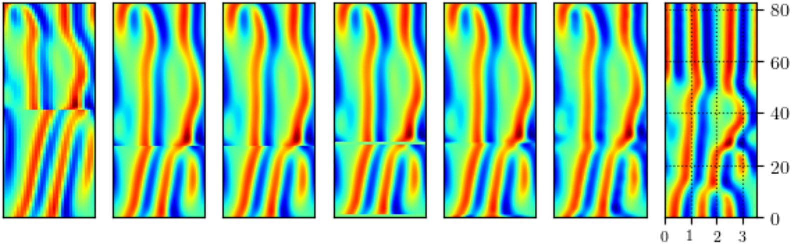
New solutions from gluing together old



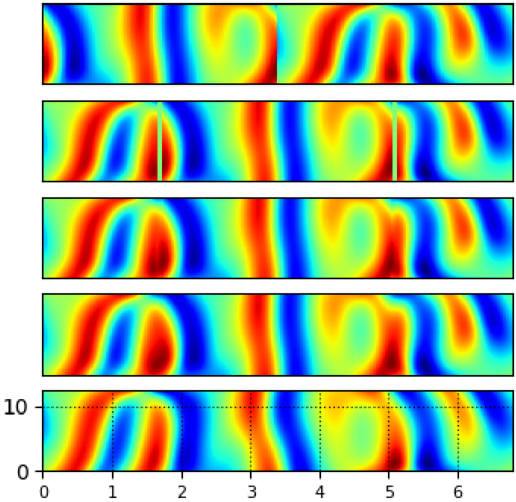
Gluing (larger) solutions, pre-periodic orbit time example



Gluing (larger) solutions, relative periodic orbit time example



Gluing (larger) solutions, pre-periodic orbit space example



Future Work

- Investigate Lagrangian formulation and application of Hill's formula[5,6]
- Complete spatiotemporal symbolic dynamics
- Continuing improvement of numerical methods
- Application of theory to PDEs with more continuous spatial dimensions such as quasi 2-D Kolmogorov flow[7,8].

[5] N. H. Ibragimov and T. Kolsrud, Nonlin. Dyn. **36**, 29–40 (2004).

[6] S. V. Bolotin and D. V. Treschev, Russ. Math. Surv. **65**, 191 (2010).

[7] B. Suri et al., Phys. Rev. E **98**, 023105 (2018).

[8] G. J. Chandler and R. R. Kerswell, J. Fluid Mech. **722**, 554–595 (2013).

Conclusion

- Developed spatiotemporal methods for finding invariant 2-tori of the Kuramoto-Sivashinsky equation
- Beginning to develop symbolic dynamics for the spatiotemporal Kuramoto-Sivashinsky equation
- Developed new method for finding larger (domain) solutions by combining smaller (domain) solutions