Variational Autoencoders

Learning generative models with latent representations

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Deep Generative Models - SoSe 2019

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Inference through optimization

What if we replaced an inference question with optimization?

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- idea: rephrase inference as optimization

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$$P(X|z) = \mathcal{N}(X|f(z;\phi), \sigma^2 \cdot I) \tag{2}$$

- f is deterministic, $\mathcal{N}$ enables optimization

Working through the math - 2

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- $Q(Z)$ should be close to $P(Z|X)$

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$$\mathcal{KL}[Q(Z)||P(Z|X)] = \mathbb{E}_{Z\sim Q}[\log Q(Z) - \log \frac{P(X|Z)P(Z)}{P(X)}] \tag{4}$$

$$= \mathbb{E}_{Z\sim Q}[\log Q(Z) - (\log P(X|Z) + \log P(Z) - \log P(X))] \tag{5}$$

$$= \mathbb{E}_{Z\sim Q}[\log Q(Z) - \log P(X|Z) - \log P(Z)] + \log P(X) \tag{6}$$

Working through the math - 4

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- We can choose Q arbitrarily...
- ... so we can choose an entry which depends on x

Working through the math - 5

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- right hand side: ELBO (Evidence Lower BOund): maximization target

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- right hand side: ELBO (Evidence Lower BOund): maximization target
- $P(X|z)$ and $Q(z|X)$: decoder and encoder learned from the data
- $P(z)$: prior on latent variables
- $P(z|X)$ will hopefully be approximated well by $Q(z|X)$. (discussion later)

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- Σ represents noise

Final math slide!

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  - possible for other PDFs too
- KL depends on learnable functions f and μ
- We can use neural networks to approximate those!
- Are we finished now?

Variational Autoencoders

What is an autoencoder?

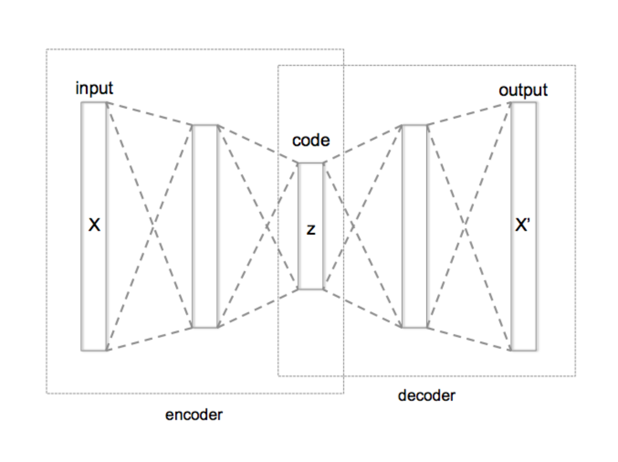


Figure 1: Taken from `https://commons.wikimedia.org/wiki/File:Autoencoder_structure.png`, (CC BY-SA 4.0)

What is a variational autoencoder?

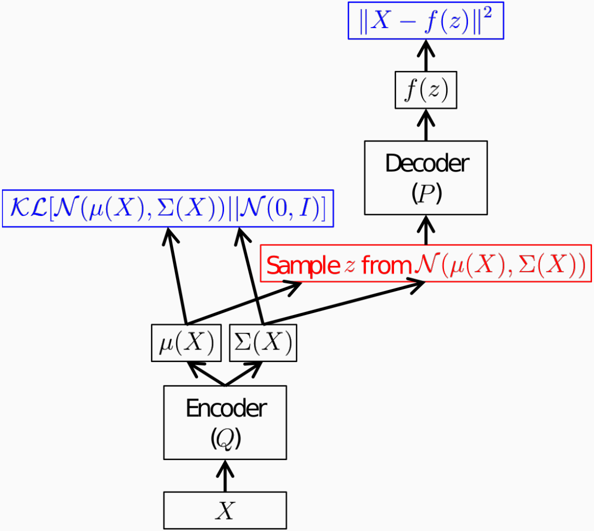
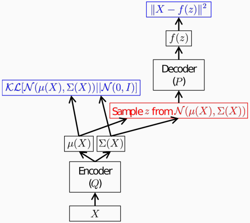


Figure 2: Taken from "Auto-Encoding Variational Bayes", Kingma & Welling, 2014

Where is the trick?



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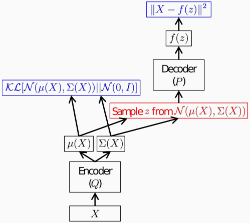

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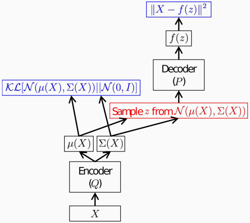

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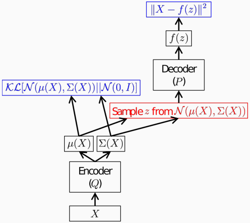

- ELBO is captured in the optimization by loss
- the problem is in backpropagation
- you can't propagate through a sampling layer

All the parts in detail

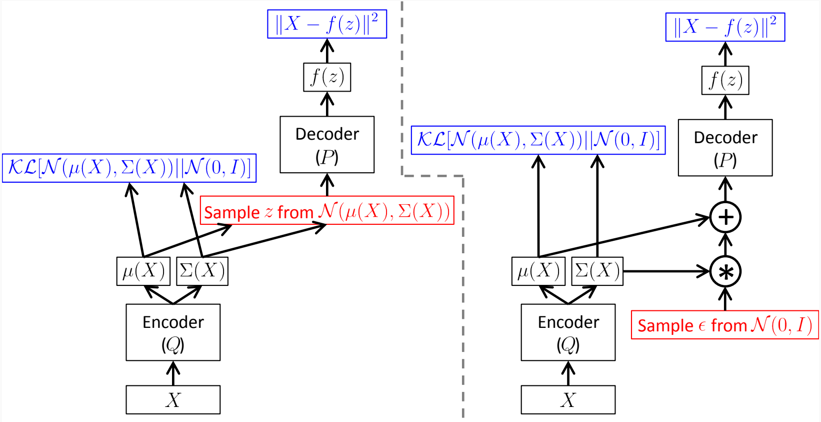

Figure 3: Taken from "Tutorial on Variational Autoencoders", Doersch, 2016

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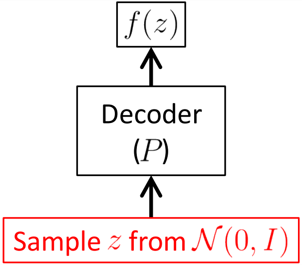

Figure 4: Taken from "Tutorial on Variational Autoencoders", Doersch, 2016

Code!

Code presentation

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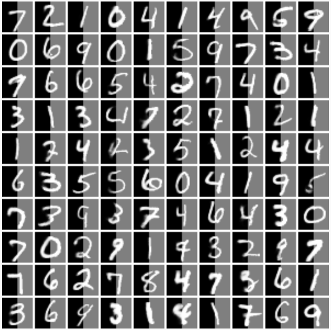
- VAEs are density models, GAN not so much
- but the density is still intractable
- there is no mathematical guarantee for choosing a ”good” latent

Applications of VAE

VAE for generating images

(a) Taken from "Tutorial on Variational Autoencoders", Doersch, 2016



(b) Taken from GitHub `https://github.com/yzwxx/vae-celebA`

VAE for translations

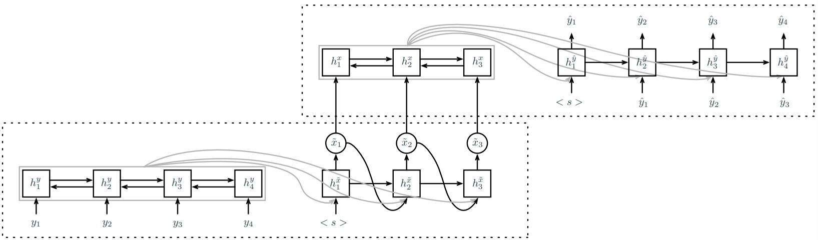


Figure 6: Taken from "Semantic Parsing with Semi-Supervised Sequential Autoencoders", Kocisky et al, 2016

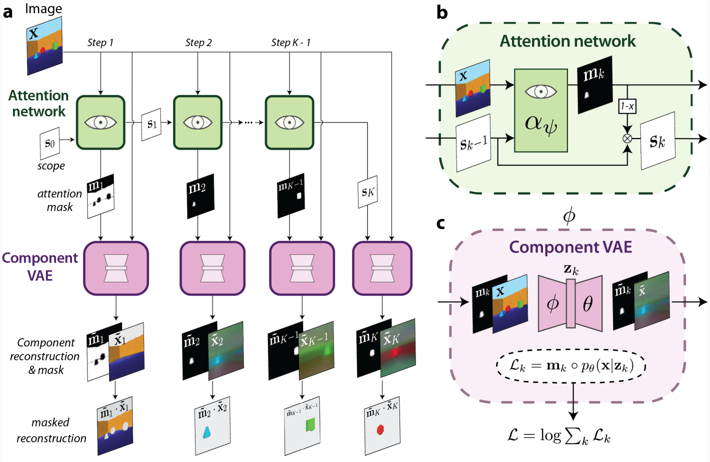


Figure 7: Taken from "MONet: Unsupervised Scene Decomposition and Representation", Burgess et al., 2019

More complex VAE usage

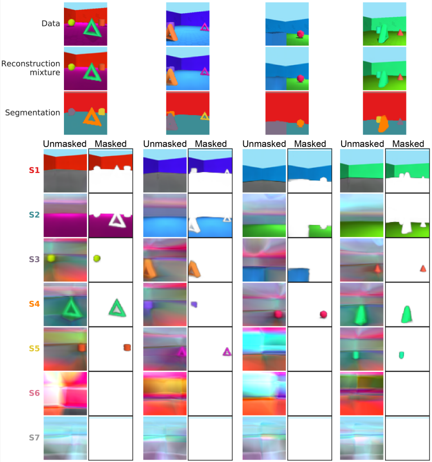


Figure 8: Taken from "MONet: Unsupervised Scene Decomposition and Representation", Burgess et al., 2019

Further reading

- "Stochastic Backpropagation and Approximate Inferencein Deep Generative Models" by Danilo Rezende et al., 2014
- "Auto-Encoding Variational Bayes" by Durk Kingma and Max Welling, 2014
- "Tutorial on Variational Autoencoders" by Carl Doersch, 2016
- "Variational Inference: A Review for Statisticians" by David Blei et al., 2018
- "Improving Variational Inference with Inverse Autoregressive Flow" by Durk Kingma, 2016
- "Attend, Infer, Repeat: Fast Scene Understanding with Generative Models" by Ali Eslami et al., 2016
- "The Dreaming Variational Autoencoder for Reinforcement Learning Environments" by Per-Arne Andersen, 2018

Questions?

Any remaining questions?

Kullback Leibler Divergence

- characterizes the "distance" between distributions
- positive, 0 only if two distributions are equal (almost everywhere)[1]
- not a metric, since it is asymetric, but still useful

$$KL(P||Q) = \int p(x) \log \frac{p(x)}{q(x)} dx \tag{11}$$

$$= \mathbb{E}_{x \sim p}[\log \frac{p(x)}{q(x)}] = \mathbb{E}_{x \sim p}[\log(p(x)) - \log(q(x))] \tag{12}$$

[1] In a mathematical, strict sense, for practical purposes

$$KL(P||Q) = 0 \text{ , iff } P = Q$$

Kullback Leibler Divergence

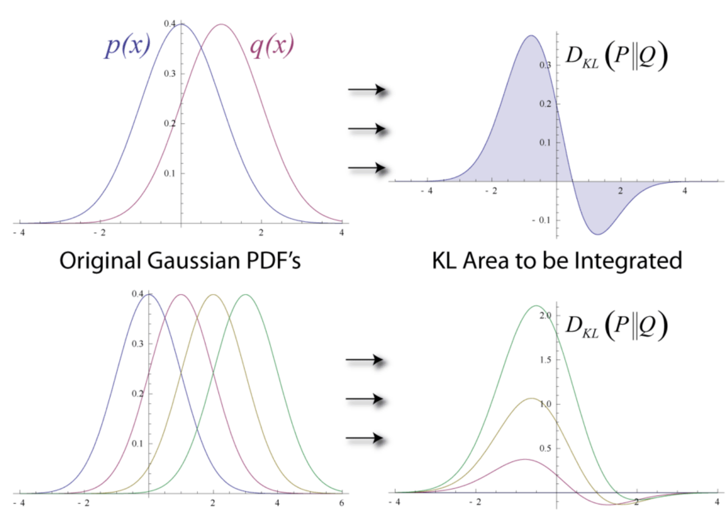


Figure 9: Taken from "Kullback–Leibler divergence", Wikipedia, CC BY-SA 3.0