Consider the system

$$\begin{pmatrix} 0 & -2 & 1 \\ 2 & 2 & 0 \\ -2 & -\frac{3}{2} & 0 \end{pmatrix} x = \begin{pmatrix} -5 \\ 6 \\ -5 \end{pmatrix}.$$

1. Define a matrix A and a vector b, so that this system reads as $Ax = b$. Then compute an LU-decomposition of A by applying Gaussian elimination with **row pivoting**. Denote the respective matrices L, U and P, such that $PA = LU$. (*Hint: Verify the desired properties of the factor matrices and test whether $PA = LU$ holds.*)

2. Use the result from the LU-decomposition to determine an x which solves $Ax = b$. (*Hint: Test whether $Ax = b$ holds.*)

Solution:

$$\left(\begin{array}{ccc|c} 0 & -2 & 1 & -5 \\ 2 & 2 & 0 & 6 \\ -2 & -1{,}5 & 0 & -5 \end{array}\right) \begin{array}{c} \text{(I)} \\ \text{(II)} \\ \text{(III)} \end{array} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$\underset{\rightsquigarrow}{\text{(III)}\leftrightarrow\text{(I)}} \left(\begin{array}{ccc|c} -2 & -1{,}5 & 0 & -5 \\ 2 & 2 & 0 & -6 \\ 0 & -2 & 1 & -5 \end{array}\right) \begin{array}{c} \text{(I)} \\ \text{(II)} \\ \text{(III)} \end{array} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

$$\underset{\rightsquigarrow}{\text{(II)}'=\text{(II)}+\text{(I)}} \left(\begin{array}{ccc|c} -2 & -1{,}5 & 0 & -5 \\ -1 & 0{,}5 & 0 & 1 \\ 0 & -2 & 1 & -5 \end{array}\right) \begin{array}{c} \text{(I)} \\ \text{(II)}' \\ \text{(III)} \end{array} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

$$\underset{\rightsquigarrow}{\text{(III)}'=\text{(III)}+4\text{(II)}} \left(\begin{array}{ccc|c} -2 & -1{,}5 & 0 & -5 \\ -1 & 0{,}5 & 0 & 1 \\ 0 & -4 & 1 & -1 \end{array}\right) \begin{array}{c} \text{(I)} \\ \text{(II)}' \\ \text{(III)}' \end{array} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \ (1P)\ , L = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -4 & 1 \end{pmatrix} \ (3P)\ , U = \begin{pmatrix} -2 & -1{,}5 & 0 \\ 0 & 0{,}5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \ (3P)$$

$$\begin{aligned} x_3 = -1 \quad &\Rightarrow \quad 0{,}5x_2 = 1 \quad \Rightarrow \quad x_2 = 2 \\ &\Rightarrow \quad -2x_1 - 1{,}5 \cdot 2 = -5 \quad \Rightarrow \quad x_1 = 1 \\ &\Rightarrow \quad x^* = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \quad (1+1+1P) \end{aligned}$$