Projections and Least Squares

Let $a, b \in \mathbb{R}^n \setminus \{0\}$ be two nonzero vectors. Consider the 1-dimensional optimization task

$$\min_{c \in \mathbb{R}} \frac{1}{2} \|ca - b\|_2^2 =: f(c),$$

where $\|x\|_2 := \left(\sum_{i=1}^n x_i^2\right)^{\frac{1}{2}}$ denotes the Euclidean norm of a vector $x \in \mathbb{R}^n$. Determine the parameter $c \in \mathbb{R}$ which minimizes f. Compare your results to the projection of b onto a, i.e., $\text{proj}_a(b) := \frac{a^\top b}{\|a\|_2} \frac{a}{\|a\|_2}$.

Hint: As in high-school, compute the derivative f' of f with respect to c and solve the equation $f'(c) = 0$.

Solution:

First we note that

$$f(c) = \frac{1}{2} \|ca - b\|_2^2 = \frac{1}{2} \left(c^2 a^\top a - 2ca^\top b - b^\top b \right)$$

Thus, for the derivative with respect to the scalar c, we find

$$f'(c) = ca^T a - a^T b.$$

Since $a \neq 0$ and therefore $a^\top a \neq 0$, we find

$$f'(\hat{c}) = 0 \quad \Leftrightarrow \quad \hat{c} = \frac{a^T b}{a^T a}.$$

By convexity of f we can conclude that $\hat{c}$ is a minimizer (you will learn this in the course "Numerical Optimization").

Remark: We will later identity the equation $ca^T a - a^T b = 0$ as the **normal equation**. The vector on the line span(a) closest to b in terms of the Euclidean norm is given by

$$\hat{c}a = \frac{a^T b}{a^T a} a = \frac{a^T b}{\|a\|_2} \frac{a}{\|a\|_2} = \text{proj}_a(b).$$