Let $b \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m \times n}$. Assume you are given the least squares problem

$$\min_{x \in \mathbb{R}^n} \|Ax - b\|_2^2.$$

1. Which equation does a solution $\hat{x}$ of the above least squares problem solve?
2. Assume you are given the following data

z	-2	-1	0	1	2
y	-1	0	0	2	9

Solve the least squares problem

$$\min_{c_0, c_1 \in \mathbb{R}} \sum_{i=1}^{5} (c_0 + c_1 z_i - y_i)^2.$$

Solution:

1. $\hat{x}$ solves the normal equation $(1P)$: $A^T A\hat{x} \overset{(1P)}{=} A^T y$
2. In this case: $A = \begin{pmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}$ $(2P)$,

$$A^T A = \begin{pmatrix} 5 & 0 \\ 0 & 10 \end{pmatrix} \quad (2P),$$

$$A^T y = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 0 \\ 2 \\ 9 \end{pmatrix} = \begin{pmatrix} 10 \\ 22 \end{pmatrix} \quad (2P)$$

Normal equation:

$$\begin{pmatrix} 5 & 0 \\ 0 & 10 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 10 \\ 22 \end{pmatrix} \quad \Leftrightarrow \quad c_0 = 2,\ c_1 = 2.2 \quad (1+1P)$$

$$\Rightarrow \quad \hat{x} = \begin{pmatrix} 2 \\ 2.2 \end{pmatrix}$$