The Subspaces Kernel and Image

1. Let $A \in \mathbb{F}^{m \times n}$. Show that $\ker(A)$ and $\operatorname{Im}(A)$ are subspaces of $\mathbb{F}^n$ and $\mathbb{F}^m$, respectively.
2. Construct two example matrices and consider their kernel and image.

Solution:

1. a) To show: $\ker(A) \subset \mathbb{F}^n$ subspace

   Proof:

   i. $A \cdot 0 = 0$, so that $0 \in \ker(A)$, thus $\ker(A) \neq \emptyset$.

   ii. For $i = 1,2$ let $\lambda_i \in \mathbb{F},\ v_i \in \ker(A),$ then by linearity $A(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 \underbrace{Av_1}_{=0} + \lambda_2 \underbrace{Av_2}_{=0} = 0$

   $\Rightarrow \quad \lambda_1 v_1 + \lambda_2 v_2 \in \ker(A)$

   b) To show: $\operatorname{Im}(A) \subset \mathbb{F}^m$ subspace

   Proof:

   i. $A \cdot 0 = 0 \in \operatorname{Im}(A)$, thus nonempty.

   ii. For $i = 1,2$ let $\lambda_i \in \mathbb{F},\ w_i \in \operatorname{Im}(A)$, then

$$\begin{aligned} & \exists v_1, v_2 \in \mathbb{F}^n: \ w_1 = Av_1,\ w_2 = Av_2 \\ \Rightarrow \quad & \lambda_1 w_1 + \lambda_2 w_2 = \lambda_1 A v_1 + \lambda_2 A v_2 = A(\lambda_1 v_1 + \lambda_2 v_2) \\ \Rightarrow \quad & \lambda_1 w_1 + \lambda_2 w_2 \in \operatorname{Im}(A) \end{aligned}$$

2. Examples:

   a) Consider the matrix composed of ones from the previous exercise.

   b) Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$. Then $Ax = 0$ implies $x = 0$, so that $\ker(A) = \{0\}$. In particular, the columns are linearly independent, so that they form a basis of $\mathbb{R}^2$, with other words: $\mathbb{R}^2 = \operatorname{span}(a_1, a_2) = \operatorname{Im}(A)$.