Properties of $A^T A + \delta I$

Let $A \in \mathbb{R}^{m \times n}$ be any matrix. Please show that $A^T A + \delta I$ is symmetric and positive definite for all $\delta > 0$. Why is $A^T A + \delta I$ invertible for all $\delta > 0$?

Solution:

Symmetry (Recall: A matrix B is called symmetric, if $B^T = B$ holds). Here we find

$$(A^T A + \delta I)^T \quad = \quad (A^T A)^T + \delta I^T \quad = \quad A^T A + \delta I.$$

Positivity (Recall: A matrix B is called positive definite, if $x^T B x > 0$ holds $\forall x \in \mathbb{R}^n \setminus \{0\}$). Here we find

$$x^T(A^T A + \delta I)x \quad = \quad \underbrace{x^T(A^T A)x}_{\geq 0 \text{(Ex. 2.2)}} + \underbrace{\delta}_{>0} \underbrace{x^T x}_{>0} \quad > \quad 0 \ \text{ for all } x \in \mathbb{R}^n \setminus \{0\}.$$

Thus, since $A^T A + \delta I$ is symmetric and positive definite for $\delta > 0$, it is invertible (see previous exercises).