1 Proof of Lemma 11.15

Let $f \in \mathrm{Hom}_{\mathbb{F}}(\mathbb{F}^n, \mathbb{F}^m)$ with representing matrix $A \in \mathbb{F}^{m\times n}$. Please show:

1. The sets $\ker(f)$ and $\mathrm{Im}(f)$ are subspaces of $\mathbb{F}^n$ and $\mathbb{F}^m$, respectively.
2. $\mathrm{rank}(A) = \dim(\mathrm{Im}(f))$.
3. $\ker(f) = \{0\} \quad \Leftrightarrow \quad f$ is injective.

Solution:

Let $f : \mathbb{F}^n \to \mathbb{F}^m$ be linear with matrix representation $A \in \mathbb{F}^{m\times n}$.

1. Show:
   a) $\ker(f) \subset \mathbb{F}^n$ subspace
   b) $\mathrm{Im}(f) \subset \mathbb{F}^m$ subspace

   Proof:

   a) i. $0 \in \ker(f)$

      ii. $\lambda \in \mathbb{F},\ u, v \in \ker(f)$, then $f(\lambda u + v) = \lambda \underbrace{f(u)}_{=0} + \underbrace{f(v)}_{=0} = 0$

      $\Rightarrow \quad \lambda u + v \in \ker(f)$

   b) i. $A_0 = 0 \in \mathrm{Im}(f)$

      ii.
$$\begin{aligned} & \lambda \in \mathbb{F},\ w_1, w_2 \in \mathrm{Im}(f) \subset W \\ \Rightarrow\ & \exists v_1, v_2 \in V:\ w_1 = f(v_1),\ w_2 = f(v_2) \\ \Rightarrow\ & \lambda w_1 + w_2 = \lambda f(v_1) + f(v_2) = f(\lambda v_1 + v_2) \\ \Rightarrow\ & \lambda w_1 + w_2 \in \mathrm{Im}(A) \end{aligned}$$

2. Show: $\mathrm{rank}(A) = \dim(\mathrm{Im}(f))$

   Proof:

   Recall:

   - $\mathrm{rank}(A_f) = |\{\sigma \neq 0 :\ \sigma \text{ singular value of } A\}|$

     $A = U\Sigma V^T,\ U \in \mathbb{R}^{m\times m},\ V \in \mathbb{R}^{n\times n}$ orthogonal, $\Sigma = \begin{pmatrix} \ddots & 0 \\ 0 & 0 \end{pmatrix}$
   - $\dim(V) :=$ length of a basis

   Let $\mathrm{rank}(A_f) =: k$, then

$$\begin{aligned} \mathrm{Im}(f) &= \mathrm{Im}(A_f) = \{A_f x :\ x \in \mathbb{F}^n\} = \{U \underbrace{\Sigma V^T x}_{=:\lambda = (\underbrace{\tilde{\lambda}}_{\in \mathbb{R}^k}, \underbrace{0, \dots, 0}_{m-k})} :\ x \in \mathbb{F}^n\} \\ &= \{U_k \tilde{\lambda} :\ \tilde{\lambda} \in \mathbb{F}^k\} \\ \Rightarrow \quad \mathrm{Im}(f) &= \mathrm{span}(u_1, \dots, u_k) \text{ and } u_1, \dots, u_k \text{ linearly independent (even orthogonal)} \\ \Rightarrow \quad \dim(\mathrm{Im}(f)) &= k = \mathrm{rank}(A_f) \end{aligned}$$

3. <u>Show:</u> $\ker(f) = \{0\} \quad \Leftrightarrow \quad f$ injective <u>Proof:</u>

$$\begin{aligned} \ker(f) = \{0\} \quad &\Leftrightarrow \quad \{f(x) = 0 \;\Rightarrow\; x = 0\} \\ &\Leftrightarrow \quad \forall x, y \in \mathbb{F}^n : \; f(x) - f(y) = f(x - y) = 0 \;\Rightarrow\; x - y = 0 \\ &\Leftrightarrow \quad f \text{ injective} \end{aligned}$$

$\square$