Let $A \in \mathbb{R}^{m\times n}$. Show that $\ker(A)$ and $\operatorname{Im}(A)$ are subspaces of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively.

Solution:

To show:

1. $\ker(A) \subset \mathbb{F}^n$ subspace
2. $\operatorname{Im}(A) \subset \mathbb{F}^m$ subspace

Proof:

1. a) $A \cdot 0 = 0 \in \ker(A)$, thus nonempty

   b) For $i = 1,2$ let $\lambda_i \in \mathbb{F},\ v_i \in \ker(A)$, then by linearity $A(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 \underbrace{Av_1}_{=0} + \lambda_2 \underbrace{Av_2}_{=0} = 0$

   $\Rightarrow \quad \lambda_1 v_1 + \lambda_2 v_2 \in \ker(A)$

2. a) $A \cdot 0 = 0 \in \operatorname{Im}(A)$, thus nonempty

   b) For $i = 1,2$ let $\lambda_i \in \mathbb{F},\ w_i \in \operatorname{Im}(A)$, then

$$\begin{aligned}
& \exists v_1, v_2 \in \mathbb{F}^n :\ w_1 = Av_1,\ w_2 = Av_2 \\
\Rightarrow\quad & \lambda_1 w_1 + \lambda_2 w_2 = \lambda_1 A v_1 + \lambda_2 A v_2 = A(\lambda_1 v_1 + \lambda_2 v_2) \\
\Rightarrow\quad & \lambda_1 w_1 + \lambda_2 w_2 \in \operatorname{Im}(A)
\end{aligned}$$