Let $T \in \mathbb{R}^{n\times n}$ be an invertible matrix and $A \in \mathbb{R}^{n\times n}$ be any matrix. Show that A and TAT^{-1} have the same eigenvalues.

Solution:

$Q \in \mathbb{R}^{n\times n}$ orthogonal, $A \in \mathbb{R}^{n\times n}$

1. Show: $\sigma(A) = \sigma(QAQ^T)$
   Proof:

$$\begin{aligned}
\lambda \in \sigma(A) \quad &\Leftrightarrow \quad \exists v \in \mathbb{R}^n \setminus \{0\} : \quad Av = \lambda v \\
&\Leftrightarrow \quad \exists v \in \mathbb{R}^n \setminus \{0\} : \quad QA \underbrace{Q^T Q}_{=\mathrm{I}} v = \lambda \underbrace{Qv}_{=\tilde{v}} \\
&\Leftrightarrow \quad \exists \tilde{v} \in \mathbb{R}^n \setminus \{0\} : \quad QAQ^T \tilde{v} = \lambda \tilde{v} \\
&\Leftrightarrow \quad \lambda \in \sigma(QAQ^T) \quad (3P)
\end{aligned}$$

$\square$

2. Show: $|\det(Q)| = 1$
   Proof:

$$\begin{aligned}
1 = \ \det(\mathrm{I}) \ &= \ \det(Q^T Q) \ = \ \det(Q^T)\det(Q) \ = \ (\det(Q))^2 \\
\Rightarrow \quad |\det(Q)| \ &= \ \sqrt{1} \ = \ 1 \quad (3P)
\end{aligned}$$

$\square$