Let $A \in \mathbb{R}^{n\times n}$ be a *negative* definite matrix, i.e., $x^\top Ax < 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$. Show that the eigenvalues of A are strictly negative, i.e., $\lambda < 0$ for all $\lambda \in \sigma(A)$.

Solution:

Consider the unit vectors $e_i = (0, \ldots, 1, \ldots, 0)^\top \in \mathbb{R}^n$. Then

$$0 < e_i^\top A e_i = a_i i.$$