Compute the SVD

Consider the matrix

$$A = \begin{pmatrix} 3 & 0 \\ 4 & 5 \end{pmatrix}.$$

i) Compute its SVD $A = U\Sigma V^T$.

ii) Write A as a sum of rank-1 matrices.

iii) Is A invertible?

Hint: For i) follow this recipe:

1. Compute the eigenvalues λ_i with eigenvectors v_i of A^TA. Index the eigenvalues so that $\lambda_1 \geq \cdots \geq \lambda_r > 0$, where $r :=$"number of positive eigenvalues". Normalize the eigenvectors v_i.
2. For $i = 1, \dots, r$: Set $\sigma_i := \sqrt{\lambda_i}$ and $u_i := \frac{1}{\sigma_i} A v_i$.
   [Until here we will already have the reduced SVD]
3. Extend the bases:
   - If $r < n$: Find orthonormal $v_{r+1}, \dots, v_n \in \ker(A)$ by solving $Av_i = 0$ and orthogonalizing.
   - If $r < m$: Find orthonormal $u_{r+1}, \dots, u_m \in \ker(A^T)$ by solving $A^T u_i = 0$ and orthogonalizing.

Solution:

$$A = \begin{pmatrix} 3 & 0 \\ 4 & 5 \end{pmatrix}, \quad A^TA = \begin{pmatrix} 25 & 20 \\ 20 & 25 \end{pmatrix}$$

i) <u>COMPUTE SVD:</u>

1. Compute $\sigma(A^TA)$ and corresponding eigenvectors.
   - Eigenvalues:

$$\begin{aligned} 0 &\stackrel{!}{=} \det(A^TA - \lambda I) = \det \begin{pmatrix} 25-\lambda & 20 \\ 20 & 25-\lambda \end{pmatrix} = (25-\lambda)^2 - 400 \\ \Leftrightarrow \quad & 25 - \lambda = \pm\sqrt{400} = \pm 20 \\ \Leftrightarrow \quad & \lambda_1 = 45, \quad \lambda_2 = 5. \end{aligned}$$

   - Eigenvectors v_1 and v_2 are solutions of $(A^TA - \lambda_i I)v_i = 0$.

     a)

$$(A^TA - \lambda_1 I) = \begin{pmatrix} -20 & 20 \\ 20 & -20 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} -20 & 20 & 0 \\ 20 & -20 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} -20 & 20 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \quad -x_1 + x_2 = 0 \quad \Rightarrow \quad x_1 = x_2$$

$$\Rightarrow \quad v_1 \in \{s \begin{pmatrix} 1 \\ 1 \end{pmatrix} : s \in \mathbb{R}\}$$

Choose $s = \frac{1}{\sqrt{2}}$ and $v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

b)

$$(A^TA - \lambda_2 I) = \begin{pmatrix} 20 & 20 \\ 20 & 20 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 20 & 20 & 0 \\ 20 & 20 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 20 & 20 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \quad x_1 + x_2 = 0 \quad \Rightarrow \quad x_1 = -x_2.$$

$\Rightarrow \quad v_2 \in \{s \begin{pmatrix} -1 \\ 1 \end{pmatrix} : s \in \mathbb{R}\}$

Choose $s = \frac{1}{\sqrt{2}}$ and $v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

2. Set $\sigma_1 := \sqrt{45} = 3\sqrt{5}$, $\sigma_2 := \sqrt{5}$.
   Compute u_i:

$$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{3\sqrt{5}} \frac{1}{\sqrt{2}} \begin{pmatrix} 3 & 0 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{3\sqrt{1}} \begin{pmatrix} 3 \\ 9 \end{pmatrix} = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 3 \end{pmatrix},$$

$$u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{\sqrt{5}} \frac{1}{\sqrt{2}} \begin{pmatrix} 3 & 0 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{10}} \begin{pmatrix} -3 \\ 1 \end{pmatrix}.$$

3. Since $2 = r = m = n$, we are done.

ii) <u>A as sum of rank-1 matrices:</u>

$$A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T = \underbrace{\frac{3\sqrt{5}}{\sqrt{10}\sqrt{2}}}_{\frac{3}{2}} \begin{pmatrix} 1 \\ 3 \end{pmatrix} (1\ 1) + \underbrace{\frac{\sqrt{5}}{\sqrt{10}\sqrt{2}}}_{\frac{1}{2}} \begin{pmatrix} -3 \\ 1 \end{pmatrix} (-1\ 1)$$

$$= \frac{3}{2} \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 3 & -3 \\ -1 & 1 \end{pmatrix}.$$

iii) <u>A invertible?</u>
Yes, since $\sigma_1 \neq 0 \neq \sigma_2$ and thus Σ is invertible implying that the product $U\Sigma V^T = A$ is invertible with inverse $A^{-1} = V\Sigma^{-1}U^T$.