1 Eigenvalue Characteristics

We denote the identity matrix of $\mathbb{F}^{n\times n}$ by I. Let $A \in \mathbb{F}^{n\times n}$ be a matrix. Some value $\lambda \in \mathbb{C}$ is called eigenvalue of A, if there is a vector $v \neq 0$ in $\mathbb{F}^n$ such that

$$(A - \lambda I)v = 0,$$

where $0 \in \mathbb{F}^n$ denotes the zero vector. Use the above definition to prove the following assertions.

1. If A is invertible, then for all eigenvalues λ of A we have $\lambda \neq 0$ and $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.
2. If λ is an eigenvalue of A, then $(\lambda - \alpha)$ is an eigenvalue of $(A - \alpha I)$ for any $\alpha \in \mathbb{C}$.
3. If λ is an eigenvalue of A, then λ is also an eigenvalue of $Q^\top AQ$ for any orthogonal matrix $Q \in \mathbb{F}^{n\times n}$.

Solution:

Let $A \in \mathbb{F}^{n\times n}$.

1. Let $A \in GL_n(\mathbb{F})$ and $\lambda \in \sigma(A)$ (with eigenvector $v \neq 0$).
   To show, that $\lambda \neq 0$ holds, we assume $\lambda = 0$. $\Rightarrow \quad Av = \lambda v = 0 \quad \Rightarrow \quad v = A^{-1} \cdot 0 = 0$. Here we see the contradiction.
   Now we proof $\frac{1}{\lambda} \in \sigma(A^{-1})$:
   $Av \overset{A\in GL_n(\mathbb{F})}{\Leftrightarrow} v = \lambda A^{-1}v \overset{\lambda\neq 0}{\Leftrightarrow} \frac{1}{\lambda}v = A^{-1}v \Leftrightarrow \frac{1}{\lambda} \in \sigma(A^{-1})$ (with the same eigenvector v).
2. Let $\alpha \in \mathbb{C}$.
   $((A - \alpha I) - (\lambda - \alpha)I)v = (A - \lambda I)v \overset{\lambda\in\sigma(A)}{=} 0 \ \checkmark$
   $\Rightarrow \quad (\lambda - \alpha)$ eigenvalue of $(A - \alpha I)$ with the same eigenvector v
3. $$\begin{aligned} Av = \lambda v \quad &\Leftrightarrow \quad Q^T Av = Q^T \lambda v \\ &\Leftrightarrow \quad Q^T A \underbrace{QQ^T}_{=I} v = \lambda Q^T \underbrace{QQ^T}_{=I} v \\ &\Leftrightarrow \quad Q^T AQ(Q^T v) = \lambda(Q^T v) \\ &\Leftrightarrow \quad \lambda \text{ is eigenvalue of } Q^T AQ \text{ with eigenvector } Q^T v. \end{aligned}$$