1 Gershgorin Disks

Let $A \in \mathbb{C}^{n\times n}$ be a matrix with entries a_{ij} for $i, j = 1, \ldots, n$. Let $R_i := \sum_{j\neq i} |a_{ij}|$ be the sum of the absolute values of the non-diagonal entries in the i-th row. Moreover, let

$$D(a_{ii}, R_i) := \{z \in \mathbb{C} \mid \|z - a_{ii}\| \leq R_i\}$$

be the disk of radius R_i and centre a_{ii} in $\mathbb{C}$. Prove the following theorem.

Theorem: Every eigenvalue of A lies within at least one of the Gershgorin disks $D(a_{ii}, R_i)$, i.e., $\forall\ \lambda \in \sigma(A)\ \exists\ i \in \{1, \ldots, n\} : \quad \lambda \in D(a_{ii}, R_i)$.

Solution:

We show, that $\forall\ \lambda \in \sigma(A)\ \exists\ i \in \{1, \ldots, n\} : \quad \lambda \in D_i$

- Let $\lambda \in \sigma(A)$, then choose an eigenvector $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$, so that
  $\exists\ i \in \{1, \ldots, n\} :\ v_i = 1$ and $|v_j| \leq 1 \quad \forall i \neq j$
  $\Big($choose any eigenvector $\tilde{v}$ and then set $v := \frac{\tilde{v}}{\tilde{v}_i}$, $\|\tilde{v}\|_\infty = |\tilde{v}_i|\Big)$.
- We know: $Av = \lambda v$, in particular (componentwise)

$$\begin{aligned}
&\underbrace{(Av)_i}_{=\sum_{j=1}^n a_{ij}v_j = a_{ii}\cdot 1 + \sum_{j\neq i} a_{ij}v_j} = (\lambda v)_i = \lambda \underbrace{v_i}_{=1} = \lambda \\
\Leftrightarrow \quad & |\lambda - a_{ii}| = \left|\sum_{j\neq i} a_{ij}v_j\right| \overset{\text{[triangle inequality]}}{\leq} \sum_{j\neq i} |a_{ij}| \underbrace{|v_j|}_{\leq 1} \leq R_i \\
\Leftrightarrow \quad & \lambda \in D_i
\end{aligned}$$

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