1 Proof via Induction

Please prove the following assertions by induction.

1. $\sum_{k=0}^{n}(2k+1) = (n+1)^2$
2. $2^n = \sum_{k=0}^{n}\binom{n}{k}$
3. Let $q \neq 1$ then

$$\sum_{k=0}^{n} q^k = \frac{q^{n+1}-1}{q-1}.$$

Hint: Use Exercise 12 for (ii).

Solution:

1. **Induction Basis** $(n = 0)$

$$\sum_{k=0}^{0}(2k+1) = 1 = (0+1)^2.$$

   Hence, *induction hypothesis* can be used.
   Induction Step $(n \mapsto n+1)$

$$\sum_{k=0}^{n+1}(2k+1) = (2(n+1)+1) + \sum_{k=0}^{n}(2k+1) = (2(n+1)+1) + (n+1)^2 =$$
$$= 2n+2+1+n^2+2n+1 = n^2+4n+4 = (n+2)^2$$

$\square$

2. **Induction Basis** $(n = 0)$

$$\sum_{k=0}^{0}\binom{n}{k} = \binom{0}{0} = 1 = 2^0.$$

   Induction Step $(n \mapsto n+1)$

$$\sum_{k=0}^{n+1}\binom{n+1}{k} = \sum_{k=0}^{n+1}\left[\binom{n}{k+1} + \binom{n}{k}\right] =$$
$$= \frac{n!}{k!(n-k)!} + \frac{n!}{(k+1)!(n-(k+1))!} = \frac{n!(k+1)+n!(n-k)}{(k+1)!(n-k)!}$$
$$= \frac{n!+n!n}{(k+1)!(n-k)!} = \frac{(n+1)!}{(k+1)!(n+1-(k+1))!} = \binom{n+1}{k+1}.$$

3. **Induction Basis** $(n = 0)$
   $\sum_{k=0}^{0} q^k = 1 = \frac{q^1-1}{q-1}$.

Induction Step $(n \mapsto n+1)$

$$\sum_{k=0}^{n+1} q^k = \frac{q^{n+2}-1}{q-1} + \sum_{k=0}^{n} q^k = q^{n+1} + \frac{q^{n+1}-1}{q-1} = \frac{q^{n+2}-1}{q-1}.$$