Proof for LU decomposition (with row pivoting)

Let $m \in \mathbb{N}$. As above, let $P_{ik_i} \in \mathbb{R}^{m\times m}$ be the permutation matrix which results from interchanging the i-th and k_i-th column ($k_i \geq i$) of the identity matrix in $\mathbb{R}^{m\times m}$. Further for $\ell_j := (0, \dots, 0, \ell_{j+1,j}, \dots, \ell_{m,j})^\top \in \mathbb{R}^m$ and the j-th unit vector $e_j \in \mathbb{R}^m$, let $L_j := I + \ell_j e_j^\top \in \mathbb{R}^{m\times m}$. Then show that for all $1 \leq j < i \leq k_i \leq m$ we have

$$P_{ik_i} L_j = \widehat{L}_j P_{ik_i}$$

where $\widehat{L}_j := I + (P_{ik_i}\ell_j)e_j^\top$.

Solution:

We find

$$\begin{aligned} P_{ik_i} L_j &= P_{ik_i}\left(I + \ell_j e_j^\top\right) \\ &= P_{ik_i} + P_{ik_i}\ell_j e_j^\top \\ &= P_{ik_i} + P_{ik_i}\ell_j e_j^\top P_{ik_i}^\top P_{ik_i} \\ &= (I + P_{ik_i}\ell_j e_j^\top P_{ik_i}^\top) P_{ik_i} \\ &= (I + P_{ik_i}\ell_j (P_{ik_i} e_j)^\top) P_{ik_i}. \\ &= (I + P_{ik_i}\ell_j e_j^\top) P_{ik_i}. \end{aligned}$$

Since $j < i \leq k_i$ we find that $P_{ik_i} e_j = e_j$, since only zeroes are swapped.