Permutation Matrices and Row Swap

A matrix $P \in \mathbb{R}^{m\times m}$ is called *permutation matrix* if it has exactly one entry of 1 in each row and each column and 0s elsewhere.

1. **Sparse representation (CSR Format):** A permutation matrix $P = (p_{ij})_{ij} \in \mathbb{R}^{m\times m}$ can be represented by an m-dimensional vector piv$\in \{1,2,\dots,m\}^m$ in the following way:

   $$\texttt{piv}_i = j \quad :\Leftrightarrow \quad p_{ij} = 1.$$

   Given a permutation matrix P with its sparse representation `piv`. Determine the sparse representation, say pivT$\in \{1,2,\dots,m\}^m$, of the transpose $P^T = (\widetilde{p}_{ij})_{ij}$, so that

   $$\texttt{pivT}_i = j \quad :\Leftrightarrow \quad \widetilde{p}_{ij} = 1.$$

   Hint: Have a look at the routine `numpy.argsort()`.

2. **Inverse:** Show that the inverse of a permutation matrix $P \in \mathbb{R}^{m\times m}$ is given by its transpose, i.e., $P^T = P^{-1}$.

3. **Row Swap**: Let $P_{jk} \in \mathbb{R}^{m\times m}$ be the permutation matrix which results from interchanging the j-th and k-th column ($k \geq j$) of the identity matrix in $\mathbb{R}^{m\times m}$. Thus if its applied to a matrix $A \in \mathbb{R}^{m\times n}$ it interchanges the j-th and k-th row of A. Show that

   $$P_{jk}^\top = P_{jk}.$$

   In particular we find $P_{jk} = P_{jk}^{-1}$, i.e., P_{jk} is self-inverse.

Solution:

1. $\texttt{piv}_i = j \Leftrightarrow 1 = p_{ij} = \widetilde{p}_{ji} \Leftrightarrow: \texttt{pivT}_j = i$

2. By definition, the columns of a permutation matrix are given by the m unit vectors (in potentially permuted order), which are orthonormal.

3. Let $P_{jk} = (q_{i\ell})_{i\ell}$ and $P_{jk}^T = (\widetilde{q}_{i\ell})_{i\ell}$, then by definition we find

   $$q_{i\ell} = \begin{cases} 1: & (i = \ell, k \neq i \neq j) \text{ or } (i = j, \ell = k) \text{ or } (i = k, \ell = j) \\ 0: & \text{else} \end{cases}$$

   and therefore

   $$\widetilde{q}_{i\ell} = q_{\ell i} = \begin{cases} 1: & (\ell = i, k \neq \ell \neq j) \text{ or } (\ell = j, i = k) \text{ or } (\ell = k, i = j) \\ 0: & \text{else} \end{cases}.$$

   Thus, we obviously find $q_{i\ell} = \widetilde{q}_{i\ell}$.