Let

$$Q := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

1. What can you say about the columns of Q? Justify your answer rigorously.
2. Compute the determinant of Q and the eigenvalues of Q.
3. Interpret the function $\mathbb{R}^3 \to \mathbb{R}^3, x \mapsto Qx$ geometrically.

Solution:

1.
   - orthonormal columns $\Rightarrow$ Q is orthogonal, i.e., $Q^T Q = QQ^T = I$
   - Q is invertible, $Q^{-1} = Q^T$

2.
   - Determinant (by Sarrus Rule): $\det(Q) \overset{[Sarrus]}{=} 0+0+0-0-0-(-1) = 1$
     (Recall: For any orthogonal matrix Q we have $|\det(Q)| = 1$)
   - Similarly we find the eigenvalues:

$$0 \overset{!}{=} \det(q - \lambda I) \overset{(Sarrus)}{=} (-\lambda)\cdot(-\lambda)\cdot(1-\lambda) + (1-\lambda) = (1-\lambda)(\lambda^2+1)$$
$$\Leftrightarrow (1-\lambda) = 0 \text{ or } (\lambda^2+1) = 0 \Leftrightarrow \lambda = 1 \text{ or } \lambda = \pm i$$

   Thus: The spectrum of Q is given by $\sigma(Q) = \{1, i, -i\}$

3. Q is a rotation of 90^o around the x_3-axis:

$$Q = Q_{\alpha=\frac{\pi}{2}} = \begin{pmatrix} cos(\alpha) & -sin(\alpha) & 0 \\ sin(\alpha) & cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$Q \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 \\ x_3 \end{pmatrix}$$

   We can see, that Q switches x_1 and x_2 with a sign change for x_2 and x_3 remains the same.