Answer the following questions.

1. When is a diagonal matrix invertible? Write down the inverse in this case.
2. What is the purpose of the QR Algorithm? Write down its iteration instruction.
3. Consider the iteration $x_{k+1} = Mx_k + b$ for some matrix $M \in \mathbb{R}^{n\times n}$ and vector $b \in \mathbb{R}^n$. Name a sufficient condition for the convergence of this sequence. What is the limit in this case?
4. What is the definition of an orthogonal matrix? What does it mean for the columns of the matrix?

Solution:

1. $(1P)$ $D = \operatorname{diag}(d_{ii})$ invertible $\Leftrightarrow$ $d_{ii} \neq 0 \ \forall i$
   $(1P)$ Then $D^{-1} = \operatorname{diag}(\frac{1}{d_{ii}})$
2. $(1P)$ Purpose: Compute eigenvalues of a matrix $A \in \mathbb{R}^{n\times n}$

   $(1P)$ $A_0 := A$
   for $i = 1, \dots, n$
   $Q_i R_i := A_i$
   $A_{i+1} := R_i Q_i$
3. $\rho(M) < 1$ $(1P)$ $\Rightarrow$ $(x_k)_k$ converges to fixed point $x^* = Mx^* + b$ $(1P)$
4. $(1P)$ $Q \in \mathbb{R}^{n\times n}$ orthogonal $:\Leftrightarrow$ $Q^T Q = I$
   $(1P)$ Thus the columns of Q are mutually orthonormal.