1 Minimum Norm Least Squares Solution: Example

Let $b \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m \times n}$. Assume you are given the least squares problem

$$\min_{x \in \mathbb{R}^n} \|Ax - b\|_2^2.$$

1. Which equation does a solution $\hat{x}$ of the above least squares problem solve? Give formula and name of the equation.
2. Assume you are given the following data

z	-1	1
y	2	-1

which you want to explain by a model $f : \mathbb{R} \to \mathbb{R}$ of the form

$$f_c(z) = c_0 + c_1 z + c_2 z^2.$$

Solve the least squares problem

$$\min_{c \in \mathbb{R}^3} \sum_{i=1}^{2} (f_c(z_i) - y_i)^2$$

to determine appropriate coefficients (c_0, c_1, c_2). If there are infinitely many solutions, pick the one with minimal norm.

Hint: Characterize the solution set $\hat{S}$ and derive a formula for the norm of a solution. Then use the fact, that the minimum (or maximum) of a quadratic function $p \colon \mathbb{R} \to \mathbb{R}$, $p(s) = a_0 + a_1 s + a_2 s^2$ can be found by setting the first derivative to zero, i.e., $0 = p'(s) = a_1 + 2a_2 s$.

Solution:

1. $\hat{x}$ solves the normal equation $A^T A \hat{x} = A^T b$.
2. We define

$$A := \begin{pmatrix} z_1^0 & z_1^1 & z_1^2 \\ z_2^0 & z_2^1 & z_2^2 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad b := \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix},$$
$$x = (c_0, c_1, c_2).$$

Then:

$$\min_{c \in \mathbb{R}^3} \sum_{i=1}^{2} (f_c(z_i) - y_i)^2 \quad = \quad \min_{x \in \mathbb{R}^3} \|Ax - b\|_2^2.$$

Thus we can equivalently solve the normal equation:

$$A^TAx = A^Tb \Leftrightarrow \underbrace{\begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}}_{=\begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix}} x = \underbrace{\begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix}}_{=\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}}$$

$$\Leftrightarrow \begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} \begin{matrix} (\mathrm{I}) \\ (\mathrm{II}) \\ (\mathrm{III}) \end{matrix}$$

$$\begin{matrix} (\mathrm{I})' = \frac{1}{2}(\mathrm{I}) \\ (\mathrm{II})' = \frac{1}{2}(\mathrm{II}) \\ (\mathrm{III})' = (\mathrm{III}) - (\mathrm{I}) \end{matrix} \quad \Leftrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \\ 0 \end{pmatrix}$$

$$(\mathrm{I})' \Rightarrow x_1 + x_3 = \frac{1}{2} \Leftrightarrow x_3 = \frac{1}{2} - x_1$$

$$(\mathrm{II})' \Rightarrow x_2 = -\frac{3}{2}.$$

Thus the solution set is given by

$$\hat{S} := \{x \in \mathbb{R}^3 : A^TAx = A^Tb\} = \left\{ \begin{pmatrix} s \\ -\frac{3}{2} \\ \frac{1}{2} - s \end{pmatrix} : s \in \mathbb{R} \right\}.$$

We now determine the solution with minimal norm:

a) For an arbitrary solution $x_s := \begin{pmatrix} s \\ -\frac{3}{2} \\ \frac{1}{2} - s \end{pmatrix}$ for some $s \in \mathbb{R}$ we find:

$$p(s) := \|x_s\|_2^2 = s^2 + \frac{9}{4} + \left(\frac{1}{2} - s\right)^2 = s^2 + \frac{9}{4} + \frac{1}{4} - s + s^2 = \frac{5}{2} - s + 2s^2.$$

b) We determine the parameter $s \in \mathbb{R}$ for which $\|x_s\|_2^2$ is minimal:

$$0 \overset{!}{=} p'(s) = -1 + 4s \quad \Leftrightarrow \quad s = \frac{1}{4}.$$

c) The minimum norm solution is therefore given by

$$x_{s=\frac{1}{4}} = \begin{pmatrix} \frac{1}{4} \\ -\frac{3}{2} \\ \frac{1}{2} - \frac{1}{4} \end{pmatrix} = \begin{pmatrix} 0.25 \\ -1.5 \\ 0.25 \end{pmatrix}.$$

The model function corresponding to the minimum norm solution is then given by

$$f_c(z) = 0.25 - 1.5z + 0.25z^2.$$