Consider the matrix

$$A := \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}.$$

1. Compute a QR-decomposition of A using the Gram–Schmidt algorithm.
   (*Hint: Verify the desired properties of the factor matrices and test* $QR = A$.))

2. Is A invertible? Use your QR decomposition to explain your answer.

Solution:

$A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$

1. Compute QR-decomposition via Gram-Schmidt:

$$\begin{aligned}
\tilde{q}_1 &:= A_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
r_{11} &:= \|\tilde{q}_1\| = \sqrt{2} \\
q_1 &:= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
r_{12} &:= A_2^T q_1 = (2, 0) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}} = \sqrt{2} \\
\tilde{q}_2 &:= A_2 - r_{12} q_1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix} - \sqrt{2} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
r_{22} &:= \|\tilde{q}_2\| = \sqrt{2} \\
q_2 &:= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
\Rightarrow \quad Q &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad (2P), \quad R = \begin{pmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{2} \end{pmatrix} \quad (2P)
\end{aligned}$$

2. Test:

$$QR = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{2} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} = A \quad \checkmark \quad (2P)$$

3. Yes, because R has nonzero diagonal entries.