Matrix Product as Sum of rank-1 Matrices

1. Let $A \in \mathbb{R}^{m\times k}$ and $B \in \mathbb{R}^{k\times n}$. Show that

$$A \cdot B = \sum_{i=1}^{k} a_i b_i^\top = \sum_{i=1}^{k} \begin{pmatrix} a_{1i} \\ \vdots \\ a_{mi} \end{pmatrix} \cdot \begin{pmatrix} b_{i1} & \dots & b_{in} \end{pmatrix},$$

   where $a_i \in \mathbb{R}^{m\times 1}$ denotes the i-th *column* of A and $b_i^\top \in \mathbb{R}^{1\times n}$ denotes the i-th *row* of B.

2. Construct a few examples with actual numbers.

Remark: Also see p. 11 in Gilbert Strang's "Linear Algebra and Learning from Data".

Solution:

Note that by definition of the matrix product we have that the entry at (μ, ν) of AB is given by

$$(AB)_{\mu\nu} = \sum_{i=1}^{k} a_{\mu i} b_{i\nu}.$$

Again, by definition of the matrix product, for the i-th column $a_i = (a_{1i}, \dots, a_{mi})^\top \in \mathbb{R}^{m\times 1}$ and i-th row $b_i^\top = (b_{i1}, \dots, b_{in}) \in \mathbb{R}^{1\times n}$, we find

$$(a_i b_i^\top)_{\mu\nu} = \sum_{j=1}^{1} (a_i)_{\mu j} (b_i^\top)_{j\nu} = (a_i)_{\mu 1} (b_i^\top)_{1\nu} = a_{\mu i} b_{i\nu}.$$

Thus

$$\left(\sum_{i=1}^{k} a_i b_i^\top \right)_{\mu\nu} = \sum_{i=1}^{k} \left(a_i b_i^\top \right)_{\mu\nu} = \sum_{i=1}^{k} a_{\mu i} b_{i\nu} = (AB)_{\mu\nu}.$$