Consider the system

$$\begin{pmatrix} \frac{1}{2} & -2 & 0 \\ 2 & 8 & -2 \\ 1 & 0 & 2 \end{pmatrix} x = \begin{pmatrix} -1 \\ 10 \\ 4 \end{pmatrix}.$$

1. Define a matrix A and a vector b, so that this system reads as $Ax = b$. Then compute an LU-decomposition of A by applying Gaussian elimination with **row pivoting**. Denote the respective matrices L, U and P, such that $PA = LU$. (*Hint:* Verify the desired properties of the factor matrices and test whether $PA = LU$ holds.)
2. Use the result from the LU-decomposition to determine an x which solves $Ax = b$. (*Hint:* Test whether $Ax = b$ holds.)

Solution:

1.

$$A = \begin{pmatrix} \frac{1}{2} & -2 & 0 \\ 2 & 8 & -2 \\ 1 & 0 & 2 \end{pmatrix},\ b = \begin{pmatrix} -1 \\ 10 \\ 4 \end{pmatrix} \qquad (1+1P)$$

2.

$$\left(\begin{array}{ccc|c} \frac{1}{2} & -2 & 0 & -1 \\ 2 & 8 & -2 & 10 \\ 1 & 0 & 2 & 4 \end{array}\right) \begin{array}{c} \text{(I)} \\ \text{(II)} \\ \text{(III)} \end{array},\ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$\overset{\text{(I)}<-->\text{(II)}}{\rightsquigarrow} \left(\begin{array}{ccc|c} 2 & 8 & -2 & 10 \\ \frac{1}{2} & -2 & 0 & -1 \\ 1 & 0 & 2 & 4 \end{array}\right) \begin{array}{c} \text{(I)} \\ \text{(II)} \\ \text{(III)} \end{array},\ \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

$$\overset{\text{(II')}=\text{(II)}-\frac{1}{4}\text{(I)},\ \text{(III')}=\text{(II)}-\frac{1}{2}\text{(I)}}{\rightsquigarrow} \left(\begin{array}{ccc|c} 2 & 8 & -2 & 10 \\ \frac{1}{4} & -4 & \frac{1}{2} & -3.5 \\ \frac{1}{2} & -4 & 3 & -1 \end{array}\right) \begin{array}{c} \text{(I)} \\ \text{(II)} \\ \text{(III)} \end{array},\ \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

$$\overset{\text{(III)}'=\text{(III)}-\text{(II)}}{\rightsquigarrow} \left(\begin{array}{ccc|c} 2 & 8 & -2 & 10 \\ \frac{1}{4} & -4 & \frac{1}{2} & -3.5 \\ \frac{1}{2} & 1 & 2.5 & 2.5 \end{array}\right) \begin{array}{c} \text{(I)} \\ \text{(II)} \\ \text{(III)} \end{array},\ \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

We obtain

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} (1P),\quad L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & 0 \\ \frac{1}{2} & 1 & 1 \end{pmatrix} (1P),\quad U = \begin{pmatrix} 2 & 8 & -2 \\ 0 & -4 & \frac{1}{2} \\ 0 & 0 & 2.5 \end{pmatrix} (1P)$$

3. Test: (1+1P)
4.

$$\begin{aligned} x_3 = 1 \quad &\Rightarrow \quad -4x_2 = -3.5 - 0.5 = -4 \cdot 1 \quad \Rightarrow \quad x_2 = 1 \\ &\Rightarrow \quad 2x_1 = 10 - 8 + 2 = 4 \quad \Rightarrow \quad x_1 = 2 \\ &\Rightarrow \quad x^* = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \quad (1+1+1P) \end{aligned}$$