Compute the eigenvalues of the following matrices.

1.
$$A = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$$

2.
$$B = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -9 \\ 0 & 1 & 6 \end{pmatrix}$$

3.
$$C = \begin{pmatrix} \pi & 3 & -1 & 6 \\ 0 & 4 & 1 & 7 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

Solution:

1. A upper triangular $(1P)$ $(\Rightarrow \sigma(A) = \text{diag}(A))$

$$\Rightarrow \quad \det(A - \lambda I) = (\pi - \lambda)(4 - \lambda)(1 - \lambda)\left(\frac{1}{2} - \lambda\right)$$

$$\Rightarrow \quad \sigma(A) = \{\pi, 4, 1, \frac{1}{2}\} \quad (2P)$$

2.

$$0 \overset{!}{=} \det(B - \lambda I) = \det \begin{pmatrix} -\lambda & 0 & 0 \\ 1 & -\lambda & -9 \\ 0 & 1 & 6-\lambda \end{pmatrix} \begin{matrix} -\lambda & 0 \\ 1 & -\lambda \\ 0 & 1 \end{matrix} \quad (1P) \quad \overset{\text{[Sarrus]}}{=} \lambda^2(6-\lambda) - 9\lambda$$

$$\Leftrightarrow \quad 0 = \lambda(\lambda(6-\lambda) - 9) = \lambda(-\lambda^2 + 6\lambda - 9) = -\lambda(\lambda - 3)^2 \quad (1P)$$

$$\Leftrightarrow \quad \lambda = 0 \text{ or } \lambda = 3 \ (\sigma(B) = \{0, 3\}) \quad (1P)$$

3. $p(x) = x^3 + \underbrace{(-6)}_{=c_2} x^2 + \underbrace{9}_{=c_1} x + \underbrace{0}_{=c_0}$

$$\text{companion matrix} = C_p = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -9 \\ 0 & 1 & 6 \end{pmatrix} = B, \text{ see 2.} \quad (3P)$$

4.

$$D - \lambda I = \begin{pmatrix} \sqrt{2} - \lambda & \pi \\ 0 & 1 - \lambda \end{pmatrix} \quad (1P) \quad \Rightarrow \quad 0 = \det(D - \lambda I) = (\sqrt{2} - \lambda)(1 - \lambda) \quad (1P)$$

$$\Rightarrow \quad \sigma(D) = \{\sqrt{2}, 1\} \quad (1P)$$

5.

$$E - \lambda I = \begin{pmatrix} -\lambda & 0 & 0 \\ 1 & (1-\lambda) & 3 \\ 2 & 0 & (1-\lambda) \end{pmatrix} \begin{matrix} -\lambda & 0 \\ 1 & (1-\lambda) \\ 2 & 0 \end{matrix} \quad (1P) \quad \Rightarrow \quad 0 = \det(E - \lambda I) = \lambda(1-\lambda)^2 \quad (1P)$$

$$\Rightarrow \quad \sigma(E) = \{0, 1\} \quad (1P)$$