1 Vector Space of Polynomials

Let $\mathbb{F}$ be a field. A set V together with a mapping $+$ (sum) and a mapping $\cdot$ (scalar multiplication) with

$$\begin{aligned} + : V \times V &\to V \qquad & \cdot : \mathbb{F} \times V &\to V \\ (v, w) &\mapsto v + w & (\lambda, v) &\mapsto \lambda \cdot v \end{aligned}$$

is called ***vector space (or linear space)*** *over* $\mathbb{F}$, if the following axioms **VR1** and **VR2** hold:

VR1 $(V, +)$ is a commutative (or abelian) group with neutral element 0, i.e.,

- **G1** Associativity: $\forall v_1, v_2, v_3 \in V: \; v_1 + (v_2 + v_3) = (v_1 + v_2) + v_3$
- **G2** Neutral element: $\forall v \in V: \; v + 0 = v$
- **G3** Inverse element: $\forall v \in V \; \exists_1 (-v) \in V: \; v + (-v) = 0$
- **G4** Commutativity: $\forall v_1, v_2 \in V: \; v_1 + v_2 = v_2 + v_1$

VR2 The scalar multiplication is consistent/compatible with $(V, +)$ in the following way:

for $\lambda, \mu \in \mathbb{F}$, $v, w \in V$ it holds that

- (i) $(\lambda + \mu) \cdot v = \lambda \cdot v + \mu \cdot v$
- (ii) $\lambda \cdot (v + w) = \lambda \cdot v + \lambda \cdot w$
- (iii) $\lambda \cdot (\mu \cdot v) = (\lambda \cdot \mu) \cdot v$
- (iv) $1 \cdot v = v$

Furthermore, let $v_1, \ldots, v_n \in V$, then with the summation and scalar multiplication we can more generally define the **span** as

$$\text{span}(v_1, \ldots, v_n) = \{\sum_{i=1}^{n} \lambda_i v_i : \lambda_i \in \mathbb{F}\}.$$

Further we say that $v_1, \ldots, v_n \in V$ are **linearly independent** if

$$\sum_{i=1}^{n} \lambda_i v_i = 0 \quad \Rightarrow \quad \lambda_i = 0 \;\; \forall \, i.$$

If $v_1, \ldots, v_n \in V$ are linearly independent and $\text{span}(v_1, \ldots, v_n) = V$, then we call $v_1, \ldots, v_n$ a **basis of** V.
A mapping $f\colon V_1 \to V_2$ between two vector spaces is called **linear**, if

$$f(\lambda \cdot_1 v +_1 w) = \lambda \cdot_2 f(v) +_2 f(w)$$

for all $v, w \in V$. Here $+_1, \cdot_1$ and $+_2, \cdot_2$ denote the summation and scalar multiplication defined on V_1 and V_2, respectively.
Examples are $\mathbb{R}^n$ and $\mathbb{R}^{m \times n}$ with the usual vector/matrix sum "$+$" and scalar multiplication "$\cdot$".
Now we consider another example of a vector space: Let $n \in \mathbb{N}$ and $P_n(\mathbb{R})$ be the set of all polynomials of degree $\leq n$ on $\mathbb{R}$, i.e., the set $P_n(\mathbb{R})$ contains all functions $p : \mathbb{R} \to \mathbb{R}$ of the form

$$p(x) = \sum_{k=0}^{n} \alpha_k x^k$$

for some $\alpha_0, \ldots, \alpha_n \in \mathbb{R}$. We define a summation and scalar multiplication:

$$\begin{aligned} +\colon P_n(\mathbb{R}) \times P_n(\mathbb{R}) \to P_n(\mathbb{R}), &\quad (p + q)(x) := p(x) + q(x), \\ \cdot\colon \mathbb{R} \times P_n(\mathbb{R}) \to P_n(\mathbb{R}), &\quad (r \cdot p)(x) := r \cdot p(x). \end{aligned}$$

1. **VR axioms:** Please show that $P_n(\mathbb{R})$ together with the above defined summation and scalar multiplication forms a vector space.

   Hint: Check **VR1** and **VR2** with $V = P_n(\mathbb{R})$.

2. Let $k < m \in \mathbb{N}$. Compute

$$\lim_{x\to\infty} \frac{x^k}{x^m}, \quad \text{and} \quad \lim_{x\to\infty} \frac{\sum_{k=0}^{m-1} \alpha_k x^k}{x^m}$$

for arbitrary $\alpha_0, \ldots, \alpha_m \in \mathbb{R}$.

3. **Monomials form a basis:** Please show that the set $B := \{q_0, \ldots, q_n\}$ with

$$q_k : \mathbb{R} \to \mathbb{R}, \;\; x \mapsto x^k,$$

is a basis of $P_n(\mathbb{R})$. What is the dimension of the vector space $P_n(\mathbb{R})$?

Hint: Part (ii) basically provides the proof of linear independence and the other assertion is obvious from the definition of p.

4. **Derivative as linear operator:** Show that the operator $\mathcal{D} : P_n(\mathbb{R}) \to P_n(\mathbb{R}), \;\; p \mapsto p'$, which maps a polynomial to its first derivative, is a $\mathbb{R}$-linear function.

Hint: For $p(x) = \sum_{k=0}^{n} \alpha_k x^k$ we have $p'(x) = \sum_{k=0}^{n} \alpha_k k x^{k-1}$.

5. **Matrix representation of the derivative:** Let Φ be the linear, invertible function which maps a polynomial to its coefficients (coordinates in the above basis), i.e.,

$$\Phi : P_n(\mathbb{R}) \to \mathbb{R}^{n+1}, \;\; \sum_{k=0}^{n} \alpha_k x^k \mapsto (\alpha_0, \ldots, \alpha_n).$$

Please remark shortly why Φ is bijective. What is the matrix representation of the linear function $F : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ defined by

$$F := \Phi \circ \mathcal{D} \circ \Phi^{-1}$$

with respect to the standard basis $\{e_1, \ldots, e_{n+1}\}$? More precisely, derive the matrix $A \in \mathbb{R}^{(n+1)\times(n+1)}$ defined by

$$A := \begin{pmatrix} | & & | \\ F(e_1) & \ldots & F(e_{n+1}) \\ | & & | \end{pmatrix}.$$

Solution:

1. **VR1:** Show, that $(P_n(\mathbb{R}), +)$ is an abelian group.
Let $p(x) = \sum_{k=0}^{n} \alpha_k x^k$, $q(x) = \sum_{k=0}^{n} \beta_k x^k$ and $w(x) = \sum_{k=0}^{n} \gamma_k x^k$ be in $P_n(\mathbb{R})$.

   (i) Associativity:

$$((p+q)+w)(x) = \sum[(\alpha_k + \beta_k) + \gamma_k]x^k = \sum[\alpha_k + (\beta_k + \gamma_k)]x^k = (p + (q+w))(x)$$

   (ii) Neutral Element:
   $0 := \sum_{k=0}^{n} 0 \cdot x^k$, then $\forall \; p \in P_n(\mathbb{R})$:

$$(0+p)(x) = \sum(0 + \alpha_k)x^k = p(x)$$

   (iii) Inverse element:
   For $p(x) = \sum \alpha_k x^k$ define $-p(x) := \sum(-\alpha_k)x^k$,

$$\Rightarrow \quad (p + (-p))(x) = 0.$$

   (iv) Commutativity:

$$(p+q)(x) = \sum \underbrace{(\alpha_k + \beta_k)}_{=\beta_k+\alpha_k} x^k = (q+p)(x)$$

VR2: Consistency properties: Let $r, s \in \mathbb{R}$.

(i) $((r+s)p)(x) = \sum \underbrace{(r+s)\alpha_k}_{=r\alpha_k + s\alpha_k} x^k = (rp)(x) + (sp)(x)$

(ii)-(iv) ✓

2. Let $k < m \in \mathbb{N}$. Then

$$\frac{x^k}{x^m} = x^{k-m} = \frac{1}{x^{m-k}} \overset{x \to +\infty}{\longrightarrow} 0$$

$$\Rightarrow \quad \forall \alpha_0, \dots, \alpha_m : \quad \frac{\sum_{k=0}^{m-1} \alpha_k x^k}{x^m} = \sum_{k=0}^{m-1} \alpha^k \underbrace{\left(\frac{x^k}{x^m}\right)}_{\to 0} \overset{x \to +\infty}{\longrightarrow} 0.$$

In particular we can conclude that

$$\forall \alpha_0, \dots, \alpha_m : \quad \sum_{k=0}^{m-1} \alpha^k x^k \neq x^m$$

because otherwise limit $\equiv 1$.

3. a) Linear independence by 2.
   Asssume $\exists \alpha_0, \dots, \alpha_n$ ($m :=$ max$\{k : \alpha_k \neq 0\}$) not all zero with $\sum_{k=0}^m \alpha_k q_k = 0$

$$\Rightarrow \quad \sum_{k=0}^{m} \alpha_k q_k = \sum_{k=0}^{m-1} \alpha_k q_k + \alpha_m q_m = 0$$

$$\Rightarrow \quad \sum_{k=0}^{m} \alpha_k x^k = (-\alpha_m) x^m$$

contradiction to 2.

   b)

$$\text{span}\{q_0, \dots, q_n\} = P_n(\mathbb{R}) \quad \text{by definition}$$
$$\Rightarrow \quad \dim(P_n(\mathbb{R})) = n+1$$

4. Let $p = \sum_{k=0}^n \alpha_k q_k$ and $w = \sum_{k=0}^n \beta_k q_k$ and $\lambda \in \mathbb{R}$, then:

$$\mathrm{D}(\lambda p + w)(x) = \left(\sum_{k=0}^{n} (\lambda \alpha_k + \beta_k) x^k \right)' = \sum_{k=0}^{n} (\lambda \alpha_k + \beta_k) k x^{k-1}$$
$$= \lambda \sum_{k=0}^{n} \alpha_k k x^{k-1} + \sum_{k=0}^{n} \beta_k k x^{k-1} = \lambda \mathrm{D}(p)(x) + \mathrm{D}(q)(x)$$

5. We have:

$$D: \; P_n(\mathbb{R}) \to P_n(\mathbb{R}), \; p = \sum_{k=0}^{n} \alpha_k q_k \mapsto p' = \sum_{k=0}^{n-1} \alpha_{k+1}(k+1) q_k$$

$$\Phi: \; P_n(\mathbb{R}) \to \mathbb{R}^{n+1}, \; p = \sum_{k=0}^{n} \alpha_k q_k \mapsto (\alpha_0, \dots, \alpha_n)^T$$
$$= (\pi_0(p), \dots, \pi_n(p))$$

$\to$[Φ linear since π_j are linear (see lecture) and bijective since $\{q_0, \dots, q_n\}$ basis, $\Phi^{-1} : \mathbb{R}^{n+1} \to P_n(\mathbb{R})$, $(\alpha_0, \dots, , \alpha_n)^T \mapsto p = \sum \alpha_k q_k$]

Now consider: $F : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1},\ F(\alpha) := \Phi \circ D \circ \Phi^{-1}$

$$\begin{array}{ccccc} (\alpha_o, \dots, \alpha_n)^T & \begin{array}{c}\mathbb{R}^{n+1} \\ \{e_1, \dots, e_{n+1}\}\end{array} & \xrightarrow{A} & \begin{array}{c}\mathbb{R}^{n+1} \\ \{e_1, \dots, e_{n+1}\}\end{array} & (\alpha_1, 2\alpha_2, 3\alpha_3, \dots, n\alpha_n, 0) =: \beta \\ & \downarrow \Phi^{-1} & & \Phi \uparrow & \\ p = \sum_{k=0}^{n} \alpha_k q_k & \begin{array}{c}P_n(\mathbb{R}) \\ \{q_1, \dots, q_{n+1}\}\end{array} & \overrightarrow{D} & \begin{array}{c}P_n(\mathbb{R}) \\ \{q_1, \dots, q_{n+1}\}\end{array} & p' = \sum_{k=0}^{n} \beta_k q_k \end{array}$$

$$\begin{aligned} F(\alpha) &= (\Phi \circ D \circ \Phi^{-1})(\alpha) \\ &= (\Phi \circ D)\left(\sum_{k=0}^{n} \alpha_k q_k\right) \\ &= \Phi\left(\sum_{k=0}^{n-1} \alpha_{k+1}(k+1) q_k\right) \\ &= (\alpha_1, 2\alpha_2, 3\alpha_3, \dots, n\alpha_n, 0)^T \end{aligned}$$

To obtain the matrix representation we have to evaluate F on the standard basis $\{e_1, \dots, e_{n+1}\}$:

$$A = \begin{pmatrix} | & & | \\ F(e_1) & \dots & F(e_{n+1}) \\ | & & | \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 \\ \vdots & \vdots & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \ddots & n-1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & n \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$