An ill-conditioned Diagonal Matrix

For $n \in \mathbb{N}$ consider the diagonal matrix

$$D_n = \text{diag}\left(1, \frac{1}{2}, \dots, \frac{1}{n}\right) = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{n} \end{pmatrix} \in \mathbb{R}^{n \times n}.$$

Tasks:

1. Is D_n invertible? Explain your answer.
2. For a given $b \in \mathbb{R}^n$, determine the corresponding $x_b \in \mathbb{R}^n$, such that $D_n x_b = b$. Is x_b uniquely defined? Explain your answer.
3. Determine the spectrum $\sigma(D_n)$ of D_n.
4. Find a singular value decomposition of D_n.
5. What is the condition number $\text{cond}_2(D_n)$ of D_n? Determine $\lim_{n\to\infty} \text{cond}_2(D_n)$.
6. Let us assume $b \in \mathbb{R}^n$ is the true right-hand side and $\tilde{b}$ would be our measured right-hand side, which is prone to some error. For simplicity let us assume $\tilde{b} = b + \varepsilon e$ for some fixed small $\varepsilon > 0$ and $e = (1, \dots, 1)^T \in \mathbb{R}^n$ (i.e., each component of b is equally perturbed by ε). Consider the difference $\Delta x := x_b - x_{\tilde{b}}$ and estimate the relative error $\frac{\|\Delta x\|}{\|x\|}$. What happens for large n?

Solution:

1. Yes, because diagonal entries are nonzero (then, e.g., $\det(D_n) \neq 0$).
2. We find
$$x_b = (b_1, 2b_2, \dots, nb_n),$$
which is uniquely determined because D_n is invertible.
3. We find
$$0 = \det(D_n - \lambda I) = \prod_{i=1}^{n}(d_{ii} - \lambda) \quad \Leftrightarrow \quad \lambda \in \{1, \frac{1}{2}, \dots, \frac{1}{n}\}.$$
4. Set $V := U := I_n$ (orthogonal) and $\Sigma := D_n$ (diagonal with positive entries), then obviously
$$D_n = U \Sigma V^T.$$
5. We find
$$\text{cond}_2(D_n) = \frac{\sigma_{max}}{\sigma_{min}} = \frac{1}{\frac{1}{n}} = n \quad \to \infty \text{ (as } n \to \infty).$$
6. From the lecture
$$\frac{\|\Delta x\|}{\|x\|} \leq \text{cond}_2(D_n) \frac{\|\Delta b\|}{\|b\|} = \frac{\varepsilon n \sqrt{n}}{\|b\|}.$$
Thus for fixed b and ε, the relative error can get arbitrarily large as n increases.