1 Gram-Schmidt-Algorithm

The Gram-Schmidt algorithm is an algorithm to compute a QR-decomposition of some matrix A. The basic idea is to successively built up an orthogonal system from a given set of linearly independent vectors. Those are, in this case, given by the columns of an invertible matrix $A = [a_1, \dots, a_n] \in \mathbb{R}^{n \times n}$. We choose the first column as starting point for the algorithm and set $\widetilde{q}_1 := a_1$. Of course, in order to generate an orthogonal matrix Q we have to rescale the vector and set $q_1 := \frac{\widetilde{q}_1}{\|\widetilde{q}_1\|}$. The successive vectors $\widetilde{q}_k$ are generated by subtracting all the shares $a_k^\top q_\ell$ of the previous vectors q_ℓ from the column a_k, i.e.

$$\widetilde{q}_k := a_k - \sum_{\ell=1}^{k-1} a_k^\top q_\ell \, q_\ell.$$

The following algorithm computes a QR-decomposition of some matrix $A \in \mathbb{R}^{n \times n}$.

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r_11 ← ||a_1||;
q_1 ← a_1 / r_11;
for k = 2, ..., n do
    for ℓ = 1, ..., k−1 do
        r_ℓk ← a_k^⊤ q_ℓ
    end
    q̃_k ← a_k − Σ_{ℓ=1}^{k−1} r_ℓk q_ℓ;
    r_kk ← ||q̃_k||;
    q_k ← q̃_k / r_kk;
end
```

Algorithm 1: Gram-Schmidt algorithm

1. Please check that the matrix $Q := [q_1, \dots, q_n] \in \mathbb{R}^{n \times n}$ is orthogonal, i.e. that $Q^T Q = I_n$.

   Hint: Show $\|q_i\| = 1$ first, and then perform an induction proof using the induction assumption that q_k is orthogonal to all previous $q_1, \dots, q_{k-1}$.

2. Let $R := (r_{\ell k})_{\ell \leq k}$ be the upper triangular matrix which results from the Gram-Schmidt algorithm. Please show that the algorithm provides a QR decomposition, i.e., that $QR = A$.

   Hint: It suffices to show $Qr_k = a_k$, where r_k (a_k) is the k-th column of R (A).

Solution:

1. $Q = \begin{pmatrix} | & & | \\ q_1 & \dots & q_n \\ | & & | \end{pmatrix}$ orthogonal $\Leftrightarrow$ $Q^T Q = I$ $\Leftrightarrow$ $q_i^T q_j = \delta_{ij}$

   a) $\|q_i\| = \|\frac{\tilde{q}_i}{\|\tilde{q}_i\|}\| = \frac{\|\tilde{q}_i\|}{\|\tilde{q}_i\|} = 1 \;\; \forall i$
   $\Rightarrow$ diagonal of $Q^T Q$ contains only 1's.

   b) **Induction Basis** $(k = 1)$ $\{q_1\}$ trivially orthogonal system
   Induction Step $(k \mapsto k+1)$
   Assume $\{q_1, \dots, q_k\}$ are pairwise orthogonal and let

$$\tilde{q}_{k+1} := a_{k+1} - \sum_{l=1}^{k} r_{lk+1} q_l \text{ and } q_{k+1} := \frac{1}{\tilde{q}_{k+1}} \|\tilde{q}_{k+1}\|.$$

Now let $j \in \{1, \ldots, k\}$, then

$$q_j{}^T q_{k+1} = \left[q_j{}^T a_{k+1} - \sum_{l=1}^{k} r_{lk+1} \underbrace{q_j{}^T q_l}_{=\delta_{jl} \text{ by induction assumption}} \right] \frac{1}{\|\tilde{q}_{k+1}\|}$$

$$= [q_j^T a_{k+1} - \underbrace{r_{jk+1}}_{=q_j^T a_{k+1}}] \frac{1}{\|\tilde{q}_{k+1}\|} = 0$$

2. Show hint: For all k we have:

$$Qr_k = \begin{pmatrix} | & & | \\ q_1 & \ldots & q_n \\ | & & | \end{pmatrix} \begin{pmatrix} r_{1k} \\ \vdots \\ r_{kk} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \sum_{l=1}^{k} r_{lk} q_l = \underbrace{r_{kk} q_k}_{=\tilde{q}_k = a_k - \sum_{l=1}^{k-1} r_{lk} q_l} + \sum_{l=1}^{k-1} r_{lk} q_l$$

$$= a_k - \sum_{l=1}^{k-1} r_{lk} q_l + \sum_{l=1}^{k-1} r_{lk} q_l = a_k$$