1 Calculate Eigenvalues Exact

choose better numbers next time

Let

$$A_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -0.1 & -1 \\ -1 & 100 \end{pmatrix}.$$

1. Set up the characteristic polynomials p_i of the matrices A_i for $i = 1,2$ and compute their eigenvalues.
2. Determine an eigenvector to the (in magnitude) largest eigenvalue of the matrix A_2 by solving the system

   $$(A_2 - \lambda I)x = 0$$

   using the LU-decomposition.
3. Determine $\text{rank}_\vartheta(A_2)$ for $\vartheta = 10^{-1}, 0$.

Hint: Try to calculate with exact values (i.e., with fractions), since rounding errors can strongly affect the solution in (ii).

Solution:

1. - $p_1(\lambda) = -1 + \lambda^2 \quad \Rightarrow \quad \lambda_{1/2} = \pm i$
   - $p_2(\lambda) = \underbrace{-11}_{q} + \underbrace{99,9}_{p}\lambda + \lambda^2$

     $\Rightarrow \quad \lambda_{1/2} = -\frac{999}{20} \pm \sqrt{\frac{999^2}{20^2} + \frac{20^2}{20^2}11} = \frac{-999 \pm \sqrt{1002401}}{20}$

     $\approx \lambda_1 = -100,01,\ \lambda_2 = -0,10999$

2. $$\begin{aligned} (A_2 - \lambda_1 I) &= \begin{pmatrix} -0,1 & -1 \\ -1 & 100 \end{pmatrix} - \begin{pmatrix} 100,01 & 0 \\ 0 & 100,01 \end{pmatrix} \\ &= \begin{pmatrix} -100,11 & -1 \\ -1 & -0,01 \end{pmatrix} \quad \rightsquigarrow \quad \begin{pmatrix} -100,11 & -1 \\ 0 & 0 \end{pmatrix} \\ &\Rightarrow \quad -100,11x_1 - x_2 = 0 \quad \Rightarrow \quad x_2 = -100,11x_1 \\ &\Rightarrow \quad v \in \left\{ s\begin{pmatrix} 1 \\ -100,11 \end{pmatrix} : \ s \in \mathbb{R} \right\} \end{aligned}$$

3. $\text{rank}_{0,1}(A_2) = |\{\lambda \in \sigma(A_2) : |\lambda| > 0,1\}| = 2$

   $\text{rank}_0(A_2) = 2$