1 Linear Least Squares

z	0	1	2	2.5	3	4
y	0.15	1	0.84	0	-0.2	-0.96

Table 1: This table contains a dataset of pairs (z_i, y_i) for $i = 1, \ldots, 6$.

Let $c_0, c_1 \in \mathbb{R}$ and let the data points (z_i, y_i) for $i = 1, \ldots, 6$ be given as in Table 1. Please solve the problem

$$\min_{c_0, c_1} \sum_{i=1}^{6} (c_0 + c_1 z_i - y_i)^2$$

in the following steps.

1. Set up the design matrix A and compute $A^\top A$ as well as $A^\top y$.
2. Solve the normal equation $A^\top A \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = A^\top y$.

Solution:

Model: $f(x) = c_0 + c_1 x$

1.
$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 2,5 \\ 1 & 3 \\ 1 & 4 \end{pmatrix} \quad \Rightarrow \quad \underbrace{A^T A}_{\text{[always symmetric]}} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2,5 & 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 2,5 \\ 1 & 3 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 6 & 12,5 \\ 12,5 & 36,25 \end{pmatrix}$$

2. $A^T y = \begin{pmatrix} 0,83 \\ -1,76 \end{pmatrix}$, $x := \begin{pmatrix} c_0 \\ c_1 \end{pmatrix}$

   Thus $A^T A x = A^T y$ is equivalent to:

$$\left(\begin{array}{cc|c} 6 & 12,5 & 0,83 \\ 12,5 & 36,25 & -1,76 \end{array} \right) \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\underset{\leadsto}{\text{(I)}\leftrightarrow\text{(II)}} \left(\begin{array}{cc|c} 12,5 & 36,25 & -1,76 \\ 6 & 12,5 & 0,83 \end{array} \right) \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\underset{\leadsto}{\text{(II)'}=\text{(II)}-\frac{6}{12,5}\text{(I)}} \left(\begin{array}{cc|c} 12,5 & 36,25 & -1,76 \\ 0,48 & -4,9 & 1,67 \end{array} \right) \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\Rightarrow \quad c_0 = 0,85, \; c_1 = -0,34$$