1. Let $A \in \mathbb{R}^{n\times n}, b \in \mathbb{R}^n$ be given.

   [i]General linear iterations are of the form $x_{k+1} = (I - NA)x_k + Nb$ for some invertible matrix $N \in \mathbb{R}^{n\times n}$ and $x_0 \in \mathbb{R}^n$. Give a sufficient criteria for the convergence of the sequence $(x_k)_k$. How is the Jacobi iteration without relaxation defined? What is the purpose of the Jacobi iteration?

2) Assume you want to solve the system $Ax = b$, where

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

   [i]Determine the iteration matrix $(I - NA)$ of the Jacobi iteration (without relaxation) in this example. Perform 3 iteration steps of the Jacobi iteration with initial guess $x_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, i.e., compute the iterates x_1, x_2, x_3. Does the Jacobi iteration converge here? Justify your answer.
   (*Hint: Look at task a.i again.*)

Solution:

1) a) (1P) $x_{k+1} = (I - D^{-1}A)x_k + D^{-1}b$, where $D = \text{diag}(A) \in \mathbb{R}^{n\times n}$

   b) (1P) The purpose is to solve a linear system of the form $Ax = b$

   c) (1P) The iteration converges to a fixed point if the spectral radius $\rho(I - D^{-1}A) < 1$

2. a) Here we have

$$D^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D^{-1}A = \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix},$$

$$\text{thus} \quad I - D^{-1}A = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \qquad (2P)$$

   and therefore the Jacobi iteration is given here by

$$x_{k+1} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} x_k = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}^k x_0.$$

   b) We obtain

$$(1P)\; x_1 = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$$

$$(1P)\; x_2 = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{4} \end{pmatrix}$$

$$(1P)\; x_3 = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}\begin{pmatrix} 0 \\ \frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{1}{8} \\ 0 \end{pmatrix}$$

   c) (1P) Yes, the iteration converges.

   (1P) because $\rho(I - D^{-1}A) = \rho\left(\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}\right) = \frac{1}{4} < 1.$