Let $Q \in \mathbb{R}^{n \times n}$ be an orthogonal matrix and $A \in \mathbb{R}^{n \times n}$ be any matrix. Show that A and QAQ^T have the same eigenvalues.

$Q \in \mathbb{R}^{n \times n}$ orthogonal, $A \in \mathbb{R}^{n \times n}$

1. <u>Show:</u> $\sigma(A) = \sigma(QAQ^T)$
   <u>Proof:</u>

$$
\begin{aligned}
\lambda \in \sigma(A) \quad &\Leftrightarrow \quad \exists v \in \mathbb{R}^n \setminus \{0\}: \quad Av = \lambda v \\
&\Leftrightarrow \quad \exists v \in \mathbb{R}^n \setminus \{0\}: \quad QA \underbrace{Q^T Q}_{=\mathrm{I}} v = \lambda \underbrace{Qv}_{=\tilde{v}} \\
&\Leftrightarrow \quad \exists \tilde{v} \in \mathbb{R}^n \setminus \{0\}: \quad QAQ^T \tilde{v} = \lambda \tilde{v} \\
&\Leftrightarrow \quad \lambda \in \sigma(QAQ^T) \quad (3P)
\end{aligned}
$$

$\square$

2. <u>Show:</u> $|\det(Q)| = 1$
   <u>Proof:</u>

$$
\begin{aligned}
1 = \ \det(\mathrm{I}) \ &= \ \det(Q^T Q) \ = \ \det(Q^T)\det(Q) \ = \ (\det(Q))^2 \\
\Rightarrow \ |\det(Q)| \ &= \ \sqrt{1} \ = \ 1 \quad (3P)
\end{aligned}
$$

$\square$