Answer the following questions.

1. What does the dimension formula say?
2. Assume you are given the singular value decomposition (SVD) $U\Sigma V^\top = A$ of some matrix $A \in \mathbb{R}^{m\times n}$ with $\operatorname{rank}(A) = r$. Find a singular value decomposition of $A^\top$ as well as a basis for $\operatorname{Im}(A)$ and $\operatorname{Im}(A^\top)$, respectively.
3. Let $R = (r_{ij})_{ij} \in \mathbb{R}^{n\times n}$ be a (lower or upper) triangular matrix with $r_{nn} = 0$. Is R invertible? Explain your answer.
4. What is the definition of an orthogonal matrix? What does it mean for the columns of the matrix?
5. What is the purpose of the QR Algorithm? Write down its iteration instruction.
6. Draw the sets $\{x \in \mathbb{R}^2 \colon \|x\|_p = 1\}$ for $p = 1, 2, \infty$.

Solution:

1. $(1P)$ $\operatorname{rank}(A) + \dim(\ker(A)) = n$ (for $A \in \mathbb{R}^{m\times n}$)
2. (0.5P): $A^\top = (U\Sigma V^\top)^\top = V\Sigma^\top U^\top$
   (0.5P): Since $r = \operatorname{rank}(A) = \operatorname{rank}(A^\top)$ we have
   $$\operatorname{Im}(A) = \operatorname{span}\{u_1, \dots, u_r\}$$
   $$\operatorname{Im}(A^\top) = \operatorname{span}\{v_1, \dots, v_r\}$$
3. $(0.5P)$ Purpose: Compute eigenvalues of a matrix $A \in \mathbb{R}^{n\times n}$
   $(0.5P)$ $A_0 := A$
   for $i = 1, \dots, n$
   $Q_i R_i := A_i$
   $A_{i+1} := R_i Q_i$
4. $(1P)$ R is not invertible, because triangular matrices are invertible if and only if all diagonal entries are nonzero (see backward/forward substitution)
5. (1P)
   

6. $(0.5P)$ $Q \in \mathbb{R}^{n\times n}$ orthogonal $:\Leftrightarrow$ $Q^T Q = I$
   $(0.5P)$ Thus the columns of Q are mutually orthonormal.