Inverse Matrix

Please prove the following statements.

1. An invertible matrix $A \in \mathbb{F}^{n\times n}$ has exactly one inverse matrix.
2. The inverse A^{-1} of an invertible matrix $A \in \mathbb{F}^{n\times n}$ is also invertible, with inverse $\left(A^{-1}\right)^{-1} = A$.
3. The product of two invertible matrices, say A and B, is invertible with inverse
$$(AB)^{-1} = B^{-1}A^{-1}.$$
4. A diagonal matrix
$$D = \mathrm{diag}(d_1, \dots, d_n) = \begin{pmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{pmatrix} \in \mathbb{F}^{n\times n}$$
is invertible if and only if $d_i \neq 0$ for all $i = 1, \dots, n$. What is its inverse?
Hint: It may be useful to split up the equivalence $\Leftrightarrow$ into $\Rightarrow$ and $\Leftarrow$ and to prove each of them separately.
5. Construct an example matrix and derive its inverse.

Solution:

1. Suppose $BA = I$ and $AC = I$, then
$$B = BI = B(AC) = (BA)C = IC = C.$$
In the next subtasks we verify that the suggested inverse, say $\widetilde{A}$, satisfies the determining requirement $A\widetilde{A} = A\widetilde{A} = I$.
2. Let $B := A^{-1}$ and $\widetilde{B} := A$, then by definition of the inverse for A we find
$$B\widetilde{B} = A^{-1}A = I$$
and
$$\widetilde{B}B = AA^{-1} = I.$$
Thus $B^{-1} = (A^{-1})^{-1} = \widetilde{B} = A$.
3. Let $C := AB$ and $\widetilde{C} := B^{-1}A^{-1}$, then by exploiting the rules for matrix computations we obtain
$$C\widetilde{C} = (AB)B^{-1}A^{-1} = A(BB^{-1})A^{-1} = A \cdot I \cdot A^{-1} = AA^{-1} = I$$
and similarly
$$\widetilde{C}C = B^{-1}A^{-1}(AB) = B^{-1}(A^{-1}A)B = B^{-1} \cdot I \cdot B = B^{-1}B = I.$$
Thus $C^{-1} = (AB)^{-1} = \widetilde{C} = B^{-1}A^{-1}$.
4. We again split the proof for the equivalence ("$\Leftrightarrow$", "if and only if") into two statements ("$\Rightarrow$", "$\Leftarrow$").
"$\Leftarrow$": First, let $d_i \neq 0$ for all i (thus we can divide by d_i) and set
$$\widetilde{D} := \mathrm{diag}(d_1^{-1}, \dots, d_n^{-1}) = \begin{pmatrix} d_1^{-1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n^{-1} \end{pmatrix} \in \mathbb{F}^{n\times n}.$$
Then by definition of the matrix product we can quickly verify that
$$D\widetilde{D} = I \quad \text{and} \quad \widetilde{D}D = I,$$

implying $D^{-1} = \widetilde{D}$ in this case.
"$\Rightarrow$": Proof by contradiction: Let $d_i = 0$ for at least one i. Then the i-th row (and column) of D solely contains 0 entries. Thus for any $\widetilde{D} \in \mathbb{F}^{n \times n}$ we have that the i-th row of $D\widetilde{D}$ is necessarily a zero row. Thus there cannot be a matrix $\widetilde{D}$ so that $D\widetilde{D} = I$. In particular, there cannot be a matrix $\widetilde{D}$ satisfying the requirements of the inverse matrix for D.

Alternatively:
The invertibility statement also follows from:

$$D \in \mathsf{GL}_n(\mathbb{F}) \quad \Leftrightarrow \quad \det(D) = \prod_{i=1}^{n} d_i \neq 0 \quad \Leftrightarrow \quad \forall 1 \leq i \leq n\colon d_i \neq 0$$

Then with the first part above we can derive the explicit expression for the inverse D^{-1}.

5. Take for example $D = \mathrm{diag}(1, 2, \ldots, n) \in \mathbb{R}^{n \times n}$ for $n \in \mathbb{N}$, then $D^{-1} = \mathrm{diag}(1, \frac{1}{2}, \ldots, \frac{1}{n})$.