Inverse Power Method

Let $A \in \mathbb{R}^{n\times n}$ be a matrix with eigenvalues λ_j for $j \in \{1,\dots,n\}$ with the property

$$|\lambda_n| \geq \dots \geq |\lambda_{i+1}| > |\lambda_i| > |\lambda_{i-1}| \geq \dots \geq |\lambda_1|.$$

Let $\hat{\lambda} \approx \lambda_i$ be a *good guess* for the eigenvalue λ_i, which means

$$0 < |\lambda_i - \hat{\lambda}| < |\lambda_j - \hat{\lambda}| \ \ \forall\ i \neq j. \tag{1}$$

Or in other words, $\hat{\lambda}$ is closer to λ_i (in absolute terms) than any other eigenvalue.
Let us define

$$B := (A - \hat{\lambda} I).$$

1. Show that $(\lambda_i - \hat{\lambda})^{-1}$ is (in magnitude) the largest eigenvalue of the matrix B^{-1}.
2. Let $x^0 \in \mathbb{R}^n$. With respect to A, the iteration

$$x^{k+1} := \frac{B^{-1}x^k}{\|B^{-1}x^k\|} \tag{2}$$

   for $k \geq 0$ is called *inverse power iteration*. Under which condition on x^0 does this iteration converge and what is the limit?

   Hint: Apply the theorem about the power method from the lecture.

Solution:

The message here is: Under the assumption of having a "good" guess for λ_i, we find the "exact" eigenvector (and -value) with the help of the *inverse power iteration*. However note that it is computationally more expensive, because you have to solve a linear system in each iteration.

We have $A \in \mathbb{R}^{n\times n}$ with eigenvalues $|\lambda_n| \geq \dots \geq |\lambda_{i+1}| > |\lambda_i| > |\lambda_{i-1}| \geq \dots \geq |\lambda_1|$ and corresponding eigenvectors $v_j \neq 0$. Also we have $\hat{\lambda} \approx \lambda_i$ so that

$$0 < |\lambda_i - \hat{\lambda}| < |\lambda_j - \hat{\lambda}| \ \ \forall\ i \neq j\ (*).$$

Now define $B := (A - \hat{\lambda} I)$.

1. We show, that $(\lambda_i - \hat{\lambda})^{-1}$ is the largest eigenvalue of B^{-1}.
   (i) For all j we have that $(\lambda_j - \hat{\lambda})$ is an eigenvalue of B to the same eigenvector v_j (previous sheets).
   (ii) Also B is invertible, otherwise there would exist $v \neq 0$, so that $Bv = 0$, which would imply $\hat{\lambda}$ eigenvalue of A, which would contradict the assumption: $|\lambda_j - \hat{\lambda}| > 0\ \forall j \ \ \Rightarrow \ \ |\lambda_j - \hat{\lambda}| \neq 0\ \forall j$. Thus $\frac{1}{\lambda_j - \hat{\lambda}}$ is an eigenvalue of B^{-1} for all j to the same eigenvector v_j (see previous sheets).
   By $(*)$ we then find that

$$\frac{1}{|\lambda_i - \hat{\lambda}|} > \frac{1}{|\lambda_j - \hat{\lambda}|}\ \forall j \ \ (\sharp).$$

2. Let v_1 be an eigenvector to the eigenvalue $(\lambda_i - \hat{\lambda})^{-1}$, and x^0 an orthogonal initial guess, i.e., $(x^0, v_1) \neq 0$, then due to $(\sharp)$ we find by the theorem about the power method that

$$\frac{B^{-1}x^k}{\|B^{-1}x^k\|} \to v_1.$$

   Note: v_1 is eigenvector of
   - ...A to the eigenvalue λ_i.
   - ...$B = A - \hat{\lambda}_i I$ to the eigenvalue $\lambda_i - \hat{\lambda}$.
   - ...$B^{-1} = (A - \hat{\lambda}_i I)^{-1}$ to the eigenvalue $(\lambda_i - \hat{\lambda})^{-1}$.