1 Weighted Jacobi, Gauß–Seidel and Sucessive Over–Relaxation

9 Bonus points

Let $A \in \mathbb{R}^{n\times n}$ be a matrix with nonzero diagonal entries $a_{ii} \neq 0$ and consider the splitting $A = L + D + U$ into lower triangular, diagonal and upper triangular part of A. Also recall that splitting methods are of the form

$$x^{k+1} = (I - NA)x^k + Nb,$$

where the significant matrix $M := I - NA$ is called iteration matrix.

Show the following:

1. **Weighted Jacobi:** $N = \theta D^{-1}$
   - For the iteration matrix we find
   $$M_{Jac} := I - \theta D^{-1}A = (1-\theta)I - \theta D^{-1}(L+U)$$
   - The i-the component of $x^{k+1} = (I - NA)x^k + Nb$ is given by
   $$x_i^{k+1} = (1-\theta)x_i^k + \frac{\theta}{a_{ii}}\left(b_i - \sum_{j\neq i} a_{ij}\, x_j^{k+1}\right).$$
2. **Gauß–Seidel:** $N = (L+D)^{-1}$
   - For the iteration matrix we find
   $$M_{GS} := I - (L+D)^{-1}A = -(L+D)^{-1}U$$
   - The i-the component of $x^{k+1} = (I - NA)x^k + Nb$ is given by
   $$x_i^{k+1} = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}\, x_j^{k+1} - \sum_{j=i+1}^{n} a_{ij}\, x_j^{k}\right).$$
3. **Sucessive Over–Relaxation (variant of Gauß-Seidel):** $N = \theta \cdot (\theta L + D)^{-1}$
   - For the iteration matrix we find
   $$M_{SOR} := I - \theta(\theta L + D)^{-1}A = (\theta L + D)^{-1}((1-\theta)D - \theta U).$$
   - The i-the component of $x^{k+1} = (I - NA)x^k + Nb$ is given by
   $$x_i^{k+1} = (1-\theta)x_i^k + \frac{\theta}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}\, x_j^{k+1} - \sum_{j=i+1}^{n} a_{ij}\, x_j^{k}\right).$$

   Remark: We observe that SOR for $\theta = 1$ is Gauß–Seidel and otherwise is a combination of the previous step x^k and the Gauß-Seidel update. For spd matrices it allows for $\theta > 1$ which is why it is called over–relaxation.

Hint: For 2. and 3. cast the formulas into the form $x^{k+1} = T^{-1}w$ for some lower triangular matrix T and some vector w and then use forward substitution.

Solution:

We first recall the forward substitution formula for inverting lower triangular matrices: Let $T = (\ell_{ij}) \in \mathbb{R}^{n\times n}$ be triangular and $w \in \mathbb{R}^n$, then the i-th component of $z = T^{-1}w$ is given by

$$z_i = \frac{1}{\ell_{ii}} \left(w_i - \sum_{j=1}^{i-1} \ell_{ij} z_j \right), \quad i \in \{1, \dots, n\}.$$

1. **Jacobi:**
   - Using the splitting $A = L + D + U$ we find
   $$M_{Jac} := I - \theta D^{-1} A = I - \theta D^{-1}(L+D+U) = I - \theta(I + D^{-1}(L+U)) = (1-\theta)I - \theta D^{-1}(L+U)$$
   - The inverse of D is $D^{-1} = \mathrm{diag}(\frac{1}{a_{11}}, \dots, \frac{1}{a_{nn}})$. Thus the i-th component of $\theta D^{-1} b$ is given by $\theta \frac{b_i}{a_{ii}}$. Now applying the definition of the matrix vector product we find for the i-th component of $\theta D^{-1} A x^k$ that $\theta \frac{1}{a_{ii}} \sum_{j=1}^n a_{ij} x_j^k$. Combining this we obtain for the i-th of the Jacobi iterate the searched formula
   $$\begin{aligned} x_i^{k+1} = x_i^k - \theta \frac{1}{a_{ii}} \sum_{j=1}^n a_{ij} x_j^k + \theta \frac{b_i}{a_{ii}} &= x_i^k - \theta x_i^k - \theta \frac{1}{a_{ii}} \sum_{j\neq i} a_{ij} x_j^k + \theta \frac{b_i}{a_{ii}} \\ &= (1-\theta) x_i^k + \frac{\theta}{a_{ii}} \left(b_i - \sum_{j\neq i} a_{ij}\, x_j^{k+1} \right). \end{aligned}$$

2. **Gauß–Seidel**
   - Using the splitting $A = L + D + U$ we find
   $$M_{GS} := I - (L+D)^{-1} A = I - (L+D)^{-1}(L+D+U) = I - (I + (L+D)^{-1}U) = -(L+D)^{-1}U$$
   - We cast the formula into a lower triangular system:
   $$\begin{aligned} x^{k+1} &= (I - NA)x^k + Nb = M_{GS} x^k + Nb \\ &= -(L+D)^{-1} U x^k + (L+D)^{-1} b \\ &= (L+D)^{-1}(b - Ux^k) \end{aligned}$$
   Now we consider $z = x^{k+1}$, $T = (L+D)$ and $w = b - Ux^k$ and apply forward substitution to obtain
   $$\begin{aligned} x_i^{k+1} = z_i &= \frac{1}{\ell_{ii}} \left(w_i - \sum_{j=1}^{i-1} \ell_{ij} z_j \right), \quad i \in \{1, \dots, n\} \\ &= \frac{1}{a_{ii}} \left(b_i - \sum_{j=i+1}^{n} a_{ij} x_j^k - \sum_{j=1}^{i-1} a_{ij} x_j^{k+1} \right), \quad i \in \{1, \dots, n\}. \end{aligned}$$

3. **SOR**
   - We use
   $$N = \theta \cdot (\theta L + D)^{-1} = \left(\tfrac{1}{\theta}\right)^{-1} (\theta L + D)^{-1} = \left(L + \tfrac{1}{\theta} D\right)^{-1}$$
   and the splitting
   $$A = L + D + U = L + D + U \pm \tfrac{1}{\theta} D = (L + \tfrac{1}{\theta} D) + U + (1 - \tfrac{1}{\theta}) D.$$
   Then we find
   $$\begin{aligned} M_{SOR} := I - NA &= I - \left(L + \tfrac{1}{\theta} D\right)^{-1} \left((L + \tfrac{1}{\theta} D) + U + (1 - \tfrac{1}{\theta}) D \right) \\ &= -\left(L + \tfrac{1}{\theta} D\right)^{-1} \left(U + (1 - \tfrac{1}{\theta}) D \right) \\ &= \left(L + \tfrac{1}{\theta} D\right)^{-1} \left(\tfrac{1-\theta}{\theta} D - U \right) \\ &= \theta \cdot (\theta L + D)^{-1} \left(\tfrac{1-\theta}{\theta} D - U \right) \\ &= (\theta L + D)^{-1} \left((1-\theta) D - \theta U \right). \end{aligned}$$

- We cast the formula into a lower triangular system:

$$\begin{aligned}x^{k+1} &= (I - NA)x^k + Nb = M_{SOR}x^k + Nb \\ &= (\theta L + D)^{-1}\left((1-\theta)D - \theta U\right)x^k + \theta \cdot (\theta L + D)^{-1} b \\ &= (\theta L + D)^{-1}(\theta b - ((1-\theta)D - \theta U)x^k)\end{aligned}$$

Now we consider $z = x^{k+1}$, $T = (\theta L + D)$ and $w = \theta b - (1-\theta)Dx^k - \theta U x^k$ and apply forward substitution to obtain

$$\begin{aligned}x_i^{k+1} = z_i &= \frac{1}{\ell_{ii}}\left(w_i - \sum_{j=1}^{i-1} \ell_{ij} z_j\right), \quad i \in \{1, \dots, n\} \\ &= \frac{1}{a_{ii}}\left(\theta b_i - (1-\theta)a_{ii}x_i^k - \theta \sum_{j=i+1}^{n} a_{ij}x_j^k - \sum_{j=1}^{i-1} \theta a_{ij} x_j^{k+1}\right), \quad i \in \{1, \dots, n\} \\ &= (1-\theta)x_i^k + \frac{\theta}{a_{ii}}\left(b_i - \sum_{j=i+1}^{n} a_{ij}x_j^k - \sum_{j=1}^{i-1} a_{ij}\, x_j^{k+1}\right), \quad i \in \{1, \dots, n\}.\end{aligned}$$