**Frobenius Matrices**

Let $\ell_j := (0, \ldots, 0, \ell_{j+1,j}, \ldots, \ell_{m,j})^\top \in \mathbb{R}^m$, $e_j \in \mathbb{R}^m$ be the $j$-th unit vector and $I \in \mathbb{R}^{m\times m}$ be the identity matrix. Then show that the matrix

$$L_j := I + \ell_j e_j^\top \in \mathbb{R}^{m\times m}$$

satisfies:

1. The matrix $L_j$ is an invertible lower triangular matrix.
2. The inverse of $L_j$ is given by $L_j^{-1} := I - \ell_j e_j^\top \in \mathbb{R}^{m\times m}$.
3. For $i \le j$ it holds that $L_i L_j = I + \ell_j e_j^\top + \ell_i e_i^\top$ and $L_i^{-1} L_j^{-1} = I - \ell_j e_j^\top - \ell_i e_i^\top$.

**Solution:**

1. First note that $\ell_j e_j^\top$ is a lower triangular matrix with zeroes on its diagonal because $\ell_{i,j} = 0$ for $i \le j$. Therefore $L_j$ is a lower triangular matrix with ones on its diagonal and thus invertible (note, e.g., that $\det(L_j) = 1 \neq 0$).
2. Since the inverse matrix is unique it is sufficient to show that $L_j(I - \ell_j e_j^\top) = I$. By inserting the definition we find that

$$\begin{aligned} L_j(I - \ell_j e_j^\top) &= (I + \ell_j e_j^\top)(I - \ell_j e_j^\top) \\ &= I + \ell_j e_j^\top - \ell_j e_j^\top - \ell_j e_j^\top \ell_j e_j^\top \\ &= I - \ell_j (e_j^\top \ell_j) e_j^\top \\ &= I, \end{aligned}$$

   where we have exploited $e_j^\top \ell_j = 0$ which follows from $\ell_{j,j} = 0$.
3. We insert definitions and compute the products. First,

$$\begin{aligned} L_i L_j &= (I + \ell_i e_i^\top)(I + \ell_j e_j^\top) \\ &= I + \ell_i e_i^\top + \ell_j e_j^\top + \ell_i e_i^\top \ell_j e_j^\top \\ &= I + \ell_i e_i^\top + \ell_j e_j^\top + \ell_i (e_i^\top \ell_j) e_j^\top \\ &= I + \ell_i e_i^\top + \ell_j e_j^\top, \end{aligned}$$

   where we have exploited $e_i^\top \ell_j = 0$, which follows from $\ell_{i,j} = 0$ for all $i \le j$. The second statement follows along the same lines.