**Thin and Fat Full Rank Matrices**

Answer the following questions without using the SVD. Instead, exploit the orthogonality relation between the four fundamental subspaces and the dimension formula.

1. What is the orthogonal complement of $\{0\}$ and $\mathbb{R}^n$ in $\mathbb{R}^n$, respectively?
2. Give an example for a matrix $C \in \mathbb{R}^{m\times n}$ with nullity$(C) = 0$ (injective).
   a) What do we know about the order relation between $m$ and $n$? (which one is greater or equal than the other?)
   b) What do we know about the columns of $C$?
   c) Let $b \in \mathrm{Im}(C)$. Can we find two distinct $x_1 \neq x_2 \in \mathbb{R}^n$ such that $Cx_1 = b = Cx_2$? Explain your answer.
   d) What do we know about the matrix $C^\top C \in \mathbb{R}^{n\times n}$?
3. Give an example for a matrix $A \in \mathbb{R}^{m\times n}$ with $n > m$ and rank$(A) < m$.
4. Give an example for a matrix $R \in \mathbb{R}^{m\times n}$ with rank$(R) = m$ (surjective).
   a) What do we know about the order relation between $m$ and $n$?
   b) What do we know about the rows of $R$?
   c) What do we know about the matrix $RR^\top \in \mathbb{R}^{m\times m}$?
   d) Let $b \in \mathbb{R}^m$. Can we find an $x \in \mathbb{R}^n$ such that $Rx = b$? Explain your answer and give an example for your matrix.
5. Give an an example for a matrix $A \in \mathbb{R}^{n\times n}$ with rank$(A) = n$ (invertible).
   a) What do we know about the dimensions of $\ker(A)$, $\ker(A^\top)$ and $\mathrm{Im}(A^\top)$?
   b) What do we know about the columns and rows of $A$?
   c) Let $b \in \mathbb{R}^n$. Can we find an *unique* $x \in \mathbb{R}^n$ such that $Ax = b$? Explain your answer and give an example for your matrix.
6. **Bonus*:** Give an an example for a matrix $A \in \mathbb{R}^{m\times n}$ with rank$(A) =$ rank$(A^\top)$ and nullity$(A) \neq$ nullity$(A^\top)$.

**Solution:**

Let us recall the dimension formula here

$$\begin{aligned} n &= \text{rank}(A) + \text{nullity}(A), \\ m &= \text{rank}(A^\top) + \text{nullity}(A^\top). \end{aligned}$$

1. We have
$$\{0\}^\perp = \{x \in \mathbb{R}^n : x^\top v = 0 \ \ \forall v \in \{0\}\} = \{x \in \mathbb{R}^n : x^\top 0 = 0\} = \mathbb{R}^n$$
and
$$(\mathbb{R}^n)^\perp = \{x \in \mathbb{R}^n : x^\top y = 0 \ \ \forall y \in \mathbb{R}^n\} = \{x \in \mathbb{R}^n : \ker(x^\top) = \{0\}\} = \{0\}.$$

   [Not part of the exercise:]
   Together with $(U^\perp)^\perp = U$ this gives in general
$$\begin{aligned} \mathbb{R}^m = \mathrm{Im}(A) = \ker(A^\top)^\perp &\iff \ker(A^\top) = \{0\}, \quad \text{or} \\ \text{rank}(A) = m &\iff \text{nullity}(A^\top) = 0, \quad \text{or} \\ A \ \text{ surjective} &\iff A^\top \ \text{ injective.} \end{aligned}$$

Analogously for the transpose

$$\mathbb{R}^n = \mathrm{Im}(A^\top) = \ker(A)^\perp \iff \ker(A) = \{0\}, \quad \text{or}$$
$$\mathrm{rank}(A^\top) = n \iff \mathrm{nullity}(A) = 0, \quad \text{or}$$
$$A^\top \text{ surjective} \iff A \text{ injective.}$$

2. Example:

$$C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

   a) By dimension formula we know

$$m \geq \dim(\mathrm{Im}(C)) = \mathrm{rank}(C) = n - \mathrm{nullity}(C) = n.$$

**Only square or thin matrices can be injective!**

   b) Since $\mathrm{nullity}(C) = 0$ we know $\ker(C) = \{0\}$ and thus the $n$ columns of $C$ are independent (in fact, only the zero combination gives the zero vector).

   c) No (because $f_C$ injective). Recall the proof: Assume yes, then $Cx_1 = b = Cx_2 \iff C(x_1 - x_2) = 0$, where $x_1 - x_2 \neq 0$ due to $x_1 \neq x_2$. This contradicts the fact that $\ker(C) = \{0\}$.

**Independent columns assure that a solution to $Cx = b$ is unique (if it exists).**

   d) Since $\ker(C^\top C) = \ker(C) = \{0\}$, we have that the $(n \times n)$-matrix $C^\top C$ is invertible.

3. Example: Take for two nonzero vectors $u \in \mathbb{R}^m, v \in \mathbb{R}^n$ with $n > m > 1$ the outer product

$$A := uv^\top,$$

so that each column is a scaling of $u$ and thus $\mathrm{rank}(A) = 1 < m$.

**Simply many columns are not enough for surjectivity!**
**We need independence to get a "larger" subspace.**

4. Example: $R := C^\top$.

   a) From above recall: $\mathrm{rank}(R) = m \iff \mathrm{nullity}(R^\top) = 0$. Now by dimension formula we get

$$n \geq \dim(\mathrm{Im}(R^\top)) = \mathrm{rank}(R^\top) = m - \mathrm{nullity}(R^\top) = m.$$

**Only square or fat matrices can be surjective!**

   b) Again, by $\mathrm{nullity}(R^\top) = 0$, they are independent.

   c) Since $\ker(RR^\top) = \ker(R^\top) = \{0\}$, it's invertible.

   d) Yes, since $\mathrm{Im}(R) = \mathbb{R}^m$, any $b \in \mathbb{R}^m$ is of the form $Rx = b$ for some $x \in \mathbb{R}^n$.

**Independent rows assure that a solution to $Rx = b$ exists.**

5. a) By dimension formula

$$\mathrm{nullity}(A) = n - \mathrm{rank}(A) = 0,$$

   Then using from above $\mathrm{rank}(A^\top) = n \iff \mathrm{nullity}(A) = 0$, we find $\mathrm{rank}(A^\top) = n$, and therefore also

$$\mathrm{nullity}(A^\top) = n - \mathrm{rank}(A^\top) = 0.$$

   b) Since $\ker(A) = \{0\} = \ker(A^\top)$ the $n$ rows and the $n$ columns are independent.

   c) Yes, because from above we know: Independent rows give existence and independent columns uniqueness.

6. Take $A = C$ with $C$ from above. Then

$$\mathrm{rank}(A) = 2 = \mathrm{rank}(A^\top).$$

However

$$\mathrm{nullity}(C) = 0 \neq 1 = \mathrm{nullity}(C^\top).$$

We will learn below:
**Always rank$(A)$ = rank$(A^\top)$,**
**but a similar result is not true in general for nullity$(A)$.**