Properties of Eigenvalues
Prove the following statements (see also the corresponding Lemma from the lecture):

1. *The eigenvalues of the powers of a matrix:*
   Let $A \in \mathbb{F}^{n\times n}$, $\lambda \in \sigma(A)$ then $\lambda^k \in \sigma(A^k)$ for any $k \in \mathbb{N}$.
2. *Eigenvalues of invertible Matrices:*
   Let $A \in \mathbb{F}^{n\times n}$ be invertible and $\lambda \in \sigma(A)$, then $\lambda \neq 0$ and $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.
3. *Eigenvalues of a scaled matrix:*
   Let $A \in \mathbb{F}^{n\times n}$ and $\lambda \in \sigma(A)$, then $\alpha\lambda \in \sigma(\alpha A)$ for any $\alpha \in \mathbb{F}$.
4. *Real symmetric matrices have real eigenvalues:* (Not obvious without using properties of complex numbers! Solution not relevant for exam and thus not discussed here.)
   $A \in \mathbb{R}^{n\times n},\ A = A^T \quad \Rightarrow \quad \sigma(A) \subset \mathbb{R}$.
5. *The eigenvalues of real orthogonal matrices:* (Not obvious without using properties of complex numbers! Solution not relevant for exam and thus not discussed here.)
   $Q \in \mathbb{R}^{n\times n}$ be orthogonal, $\lambda = a + ib \in \sigma(Q) \quad \Rightarrow \quad |\lambda| := \sqrt{a^2+b^2} = 1$
6. *The eigenvalues of an upper (or lower) triangular matrix are sitting on its diagonal:*
   Let $U \in \mathbb{F}^{n\times n}$ with $u_{ij} = 0$ for $i > j$. Then $\sigma(U) = \{u_{11}, \ldots, u_{nn}\}$.
7. *Similar matrices have the same eigenvalues:*
   Let $A \in \mathbb{F}^{n\times n}$ and $T \in GL_n(\mathbb{F})$, i.e., T is invertible. Then $\sigma(A) = \sigma(T^{-1}AT)$.
8. *Eigenvalues of a shifted matrix:*
   Let $A \in \mathbb{F}^{n\times n}$ and $\lambda \in \sigma(A)$, then $(\lambda - \alpha)$ is an eigenvalue of $(A - \alpha I)$ for any $\alpha \in \mathbb{F}$.
9. *Symmetric matrices have orthogonal eigenvectors:* (Solution not relevant for exam and thus not discussed here.)
   Let $\lambda_1 \neq \lambda_2$ be two *distinct* eigenvalues of a real *symmetric* matrix $A \in \mathbb{R}^{n\times n}$ (i.e., $A = A^T$), and let $v_1, v_2 \in \mathbb{R}^n$ be corresponding eigenvectors. Proof that v_1 and v_2 are orthogonal, i.e., $v_1^\top v_2 = 0$.

Solution:

1. ***The eigenvalues of the powers of a matrix:***
   For an eigenpair (λ, v) of A we find
   $$A^k v = A^{k-1}Av = A^{k-1}\lambda v.$$
   Iterating k times gives the desired result.
2. ***Eigenvalues of invertible Matrices:***
   Let $A \in GL_n(\mathbb{F})$ and $\lambda \in \sigma(A)$ with eigenvector $v \neq 0$.
   First, we show that $\lambda \neq 0$ holds. For this purpose let us assume $\lambda = 0$. Then $Av = \lambda v = 0$ implies $v = A^{-1} \cdot 0 = 0$, which contradicts that v is an eigenvector (and therefore nonzero).
   Second, we proof $\frac{1}{\lambda} \in \sigma(A^{-1})$. We find
   $$Av = \lambda v \quad \overset{A\in GL_n(\mathbb{F})}{\Leftrightarrow} \quad v = \lambda A^{-1} v \quad \overset{\lambda\neq 0}{\Leftrightarrow} \quad \frac{1}{\lambda} v = A^{-1} v \quad \Leftrightarrow \quad \frac{1}{\lambda} \in \sigma(A^{-1}) \quad \text{(with the same eigenvector } v\text{)}.$$
3. ***Eigenvalues of a scaled matrix:***
   Let $\alpha \in \mathbb{F}$, then for an eigenpair (λ, v) of A we find
   $$Av = \lambda v \Leftrightarrow (\alpha A)v = (\alpha\lambda)v$$
   implying that $(\alpha\lambda, v)$ is an eigenpair of αA.

4. ***Real symmetric matrices have real eigenvalues:***
   We first collect some observations:
   - In general, for $x, y \in \mathbb{F}^n$ and a matrix $A \in \mathbb{F}^{n\times n}$ we easily find by the definition of the matrix product
     $$x^\top Ay = \sum_{i,j} a_{ij}x_i y_j.$$
     If the matrix is symmetric, i.e., $a_{ij} = a_{ji}$, we further find
     $$x^\top Ay = \sum_i a_{ii}x_i y_i + \sum_{i\neq j} a_{ij}(x_i y_j + x_j y_i). \tag{1}$$
   - For $z = x + iy \in \mathbb{C}$ let us define $\overline{z} := x - iy$ (the so-called *complex conjugate* of z). Then we easily find
     i) $z\overline{z} = a^2 + b^2 \in \mathbb{R}$ (real number!),
     ii) for $w = c + id$ we find $\overline{z}w + z\overline{w} = 2(ac + bd) \in \mathbb{R}$ (real number!) and also $\overline{z\cdot w} = \overline{z}\cdot\overline{w}$.

   Now to the task: Let (λ, v) be an eigenpair of $A = A^\top \in \mathbb{R}^{n\times n}$, i.e.,
   $$Av = \lambda v.$$
   Multiplying $\overline{v} := (\overline{v}_1, \dots, \overline{v}_n)$ from the left yields
   $$\overline{v}^\top Av = \lambda\overline{v}^\top v.$$
   Now, if we can show that $\overline{v}^\top Av$ and $\overline{v}^\top v$ are real, then we know that λ is real. First, we observe that
   $$\overline{v}^\top v = \sum_{i=1}^n \overline{v}_i v_i,$$
   which is real, since all summands $\overline{v}_i v_i$ are real by i) above. Secondly, since A is **symmetric** we can apply (**??**) to obtain
   $$\overline{v}^\top Av = \sum_i a_{ii}\overline{v}_i v_i + \sum_{i\neq j} a_{ij}(\overline{v}_i v_j + \overline{v}_j v_i),$$
   which is real, since all summands are real by i) and ii) above and the assumption that the matrix only has **real** coefficients.

   Remarks:
   - **We cannot relax symmetry assumption:** Matrices with just real coefficients can have complex eigenvalues. Take for example the (orthogonal) matrix
     $$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
     for which $\det(A - \lambda I) = \lambda^2 + 1$, so that $\sigma(A) = \{i, -i\}$ with eigenvectors $\{\begin{pmatrix} 1 \\ -i \end{pmatrix}, \begin{pmatrix} 1 \\ i \end{pmatrix}\}$. However, the additional property of symmetry is a sufficient condition for a real matrix A to have solely real eigenvalues!
   - **We cannot relax assumption of real coefficients:** A symmetric matrix with complex coefficients can have complex eigenvalues. Consider for example
     $$A = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$
     for which $\det(A - \lambda I) = \lambda^2 + 1$, so that $\sigma(A) = \{i, -i\}$.
   - **The general result for complex matrices:** Like *symmetry* for real matrices we introduce for complex matrices: A matrix $A \in \mathbb{C}^{n\times n}$ is called *Hermitian* or *self-adjoint* if $A^\top = \overline{A}$. With other words, Hermitian matrices are invariant under transposition *and* (additionally) conjugation. With the same proof as above one can show that such matrices also have real eigenvalues. For example, consider the Hermitian matrix
     $$A = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$
     for which $\det(A - \lambda I) = \lambda^2 - 1$, so that $\sigma(A) = \{1, -1\}$.

5. ***The eigenvalues of real orthogonal matrices:***
We know $Q^\top Q = I$. Now let (λ, v) be an eigenpair of Q, i.e., $Qv = \lambda v$. Using the notation from the previous subtask, i.e., letting $\overline{v} = (\overline{v}_1, \ldots, \overline{v}_n)$ denote the complex conjugate of v, we find on the one hand that
$$(Q\overline{v})^\top (Qv) = \overline{v}^\top Q^\top Qv = \overline{v}^\top v \in \mathbb{R}.$$
On the other hand, since Q is assumed to be real we find
$$(Q\overline{v})^\top (Qv) = (\overline{Qv})^\top (Qv) = (\overline{\lambda v})^\top (\lambda v) = \overline{\lambda}\lambda(\overline{v}^\top v) = |\lambda|^2(\overline{v}^\top v) \in \mathbb{R}.$$
Thus combining these two equations gives
$$|\lambda| = 1.$$

6. ***The eigenvalues of an upper (or lower) triangular matrix are sitting on its diagonal:***
<u>Recall:</u>
1) The determinant of an (upper) triangular matrix is given by the product of its diagonal entries
$$\det(U) = u_{11} \cdot u_{22} \cdot \ldots \cdot u_{nn}.$$
2) The eigenvalues λ of a matrix $A \in \mathbb{R}^{n\times n}$ are the roots of the characteristic polynomial
$$\det(A - \lambda I) = 0.$$
Since $U - \lambda I = \begin{pmatrix} u_{11} - \lambda & & * \\ & \ddots & \\ 0 & & u_{nn} - \lambda \end{pmatrix}$ is also upper triangular we find
$$\det(U - \lambda I) = (u_{11} - \lambda) \cdot (u_{22} - \lambda) \cdot \ldots \cdot (u_{nn} - \lambda) \overset{!}{=} 0 \Leftrightarrow \lambda \in \{u_{11}, \ldots, u_{nn}\}.$$
Analogue proof for lower triangular matrices.

7. ***Similar matrices have the same eigenvalues:***
Let (λ, v) be an eigenpair of A, then since T is invertible we find
$$\underbrace{Av = \lambda v}_{\text{by definition of}(\lambda,v)} \overset{T^{-1}\cdot|}{\Leftrightarrow} \underbrace{T^{-1}Av}_{=T^{-1}AIv} = T^{-1}(\lambda v) \overset{I=TT^{-1}}{\Leftrightarrow} T^{-1}AT(T^{-1}v) = \lambda(T^{-1}v).$$
Thus, $(\lambda, T^{-1}v)$ is an eigenpair for the matrix $T^{-1}AT$ (note that $T^{-1}v \neq 0$, since $v \neq 0$).

Remark: We call two matrices A and B *similar* if there exists an invertible matrix T such that $B = T^{-1}AT$.

8. ***Eigenvalues of a shifted matrix:***
Let $\alpha \in \mathbb{F}$ and (λ, v) be an eigenpair of A. Then $((A - \alpha I) - (\lambda - \alpha)I)v = (A - \lambda I)v \overset{\lambda \in \sigma(A)}{=} 0$
Thus $(\lambda - \alpha)$ is an eigenvalue of $(A - \alpha I)$ with the same eigenvector v.

9. ***Symmetric matrices have orthogonal eigenvectors:***
Remark upfront: Since we are considering a real symmetric matrix, we know that the eigenvalues are real. However, one may still find complex eigenvectors. For example consider the identity matrix I with spectrum $\sigma(I) = \{1\}$. Then obviously any nonzero vector v (complex or not) is an eigenvector to the eigenvalue 1. We now show that real eigenvalues enable us to choose real eigenvectors (as implicitly assumed in the task). To clarify this, let (λ, v) be an eigenpair, where $\lambda \in \mathbb{R}$ but v may potentially have complex coefficients. Let us split v according to the real and imaginary parts of its coefficient, more precisely
$$v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} x_1 + iy_1 \\ \vdots \\ x_n + iy_n \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + i \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} =: x + iy.$$
Then we find
$$Av = \lambda v \Leftrightarrow A(x + iy) = \lambda(x + iy) \Leftrightarrow Ax + iAy = \lambda x + i\lambda y.$$

Note here that λ is real and thus we have the splitting into real λx and imaginary part λy. By comparing real parts in the last equation we obtain

$$Ax = \lambda x,$$

so that the real part of v is also an eigenvector to the eigenvalue λ.

Now to the task: Let A be symmetric, i.e., $A = A^T$ and let (λ_1, v_1) and (λ_2, v_2) be two (real) eigenpairs of A with $\lambda_1 \neq \lambda_2$, then

$$v_1^T \underbrace{Av_2}_{=\lambda_2 v_2} = \lambda_2 v_1^T v_2,$$

and also

$$v_1^T \underbrace{A}_{=A^T} v_2 = \underbrace{v_1^T A^T v_2}_{=(v_1^T A^T v_2)^T} = v_2^T \underbrace{Av_1}_{=\lambda_1 v_1} = \lambda_1 v_2^T v_1 = \lambda_1 v_1^T v_2.$$

Now we subtract these terms and find

$$0 = v_1^T A v_2 - v_1^T A v_2 = \lambda_2 v_1^T v_2 - \lambda_1 v_1^T v_2 = \underbrace{\lambda_2 - \lambda_1}_{\neq 0, \text{ since } \lambda_2 \neq \lambda_1} v_1^T v_2$$

$$\overset{\lambda_1 \neq \lambda_2}{\Rightarrow} v_1^T v_2 = 0.$$