Please show that the eigenvalues of an upper triangular matrix $A \in \mathbb{R}^{n\times n}$ are given on its diagonal.

Solution:

Show: $A \in \mathbb{R}^{n\times n}$ upper triangular $\Rightarrow \quad \sigma(A) = \{a_{11}, \ldots, a_{nn}\}$
Proof:

$$\begin{aligned} \lambda \in \sigma(A) \quad &\Leftrightarrow \quad \det(A - \lambda I) = 0 \quad (1P) \\ &\Leftrightarrow \quad \prod_{i=1}^{n}(a_{ii} - \lambda) = 0 \quad (1P) \\ &\left[\uparrow \text{ since } A - \lambda I =: U \text{ is upper triangular and } \det(U) = \prod_{i=1}^{n} u_{ii}\right] \quad (1P) \end{aligned}$$

Now we show $\det(U) = \prod_{i=1}^{n} u_{ii}$:

$$\begin{aligned} \det(U) &= u_{11} \cdot \underbrace{\det(U_1)}_{=u_{22}\cdot\det(U_2)} \\ (3P) \quad &\vdots \\ &= \prod_{i=1}^{n} u_{ii} \end{aligned}$$

$\square$