Equivalent Definitions of the Matrix Rank
Let $A \in \mathbb{R}^{m\times n}$, then show that the following statements are equivalent:

i) The maximum number of linearly independent columns of A is r.

ii) The dimension of the image of A is r, i.e., $\dim(\mathrm{Im}(A)) = r$.

iii) The number of positive singular values of A is r.

Hint: You can use the SVD $A = U\Sigma V^\top$ and Lemma 2.29.

As a consequence (since $A^\top = V\Sigma^\top U^\top$), we find

$$\mathrm{rank}(A) = r = \mathrm{rank}(A^\top),$$

so that the dimension formulas for A and $A^\top$ read as

$$\begin{aligned} n &= \mathrm{rank}(A) + \mathrm{nullity}(A) = \mathrm{rank}(A^\top) + \mathrm{nullity}(A), \\ m &= \mathrm{rank}(A^\top) + \mathrm{nullity}(A^\top) = \mathrm{rank}(A) + \mathrm{nullity}(A^\top). \end{aligned}$$

Solution:

i) $\Rightarrow$ ii): Pick r independent columns of $A = [a_1, \dots, a_n]$, say $a_{i_1}, \dots, a_{i_r}$. Then by i) any other column of A can be written as linear combination of $a_{i_1}, \dots, a_{i_r}$, so that

$$\mathrm{Im}(A) = \mathrm{span}(a_1, \dots, a_n) = \mathrm{span}(a_{i_1}, \dots, a_{i_r}).$$

Thus the $a_{i_1}, \dots, a_{i_r}$ are a basis of length r for $\mathrm{Im}(A)$, implying ii) by the definition of "dimension".

ii) $\Rightarrow$ i): Let $\dim(\mathrm{Im}(A)) = r$. Since any basis of a subspace has the same length, any basis of $\mathrm{Im}(A)$ has length r.
Now assume A has more than r independent columns. Then by the reasoning from above, these columns would yield another basis of $\mathrm{Im}(A)$ but with length $> r$, which would contradict the fact that any basis has length r. Therefore implying i): the maximum number of independent columns in A is r.

ii) $\Leftrightarrow$ iii) We show that

$$\dim(\mathrm{Im}(A)) = \text{number of positive singular values of } A.$$

Let us consider the reduced SVD $A = U_r\Sigma_r V_r^\top$ where r denotes the number of (positive) singular values, Σ_r is an invertible diagonal matrix and U_r and $V_r^\top$ have independent (even orthonormal) columns and rows, respectively. Thus

$$\mathrm{Im}(A) = \mathrm{Im}(U_r\Sigma_r V_r^\top).$$

In order to put Lemma 2.29 into position we show that $\Sigma_r V_r^\top$ is surjective, i.e., $\mathrm{rank}(\Sigma_r V_r^\top) = r$. This easily follows from the fact that $(\Sigma_r V_r^\top)^\top = V_r\Sigma_r$ has independent columns and thus

$$\mathrm{nullity}(V_r\Sigma_r) = 0 \Leftrightarrow \mathrm{rank}(\Sigma_r V_r^\top) = r.$$

Therefore by the mentioned Lemma we obtain

$$\mathrm{Im}(A) = \mathrm{Im}(U_r\Sigma_r V_r^\top) = \mathrm{Im}(U_r).$$

Since the columns of U_r are independent they are a basis of length r for $\mathrm{Im}(A)$, which implies $\dim(\mathrm{Im}(A)) = r$=number of (positive) singular values.