Towards SVD: Properties of $A^T A$ and AA^T
Let $A \in \mathbb{R}^{m \times n}$ be any matrix. Please show:

1. $A^T A \in \mathbb{R}^{n \times n}$ and $AA^T \in \mathbb{R}^{m \times m}$ are symmetric.
2. $A^T A$ and AA^T are positive semi-definite. (*Hint:* $\|x\|_2^2 = x^T x \geq 0 \ \ \forall x \in \mathbb{R}^n$.)
   Remark: Thus, eigenvalues are nonnegative.
3. $A^T A$ and AA^T have the same positive eigenvalues.
4. $\ker(A) = \ker(A^T A)$ and $\ker(A^T) = \ker(AA^T)$.
   Remark: Thus, $v \in \ker(A)$ is eigenvector of $A^T A$ ($u \in \ker(A^T)$ is eigenvector of AA^T) to the eigenvalue $\lambda = 0$.
5. Name two sufficient conditions for the invertibility of $A^T A$ and AA^T.

Solution:

1. *Recall: A matrix B is called symmetric, if $B^T = B$ holds.*
   Here, we find
$$(A^T A)^T = A^T (A^T)^T = A^T A.$$
   and
$$(AA^T)^T = (A^T)^T A^T = AA^T.$$
2. *Recall: A matrix B is called positive semi-definite, if $x^T B x \geq 0$ holds $\forall x \in \mathbb{R}^n$.*
   Here we find
$$x^T (A^T A) x = (Ax)^T Ax = \|Ax\|_2^2 \geq 0 \ \ \forall x \in \mathbb{R}^n.$$
   Next define $C := A^T$ and apply the latter result to C; note that $C^T C = AA^T$.
3. We show mutual subset relation:
   "$\subset$": We first show that all positive eigenvalues of $A^T A$ are positive eigenvalues of AA^T:
   Let $\lambda > 0$ be an eigenvalue of $A^T A$ with eigenvector $v \neq 0$, then we find
$$A^T A v = \lambda v \overset{A\cdot|}{\Rightarrow} AA^T (Av) = \lambda (Av).$$
   In addition we find that $Av \neq 0$, since
$$\|Av\|_2^2 = v^T A^T A v = \lambda v^T v = \underbrace{\lambda}_{>0} \underbrace{\|v\|_2^2}_{>0} > 0.$$
   Thus we can conclude that λ is an eigenvalue of AA^T with eigenvector $Av =: u \neq 0$.

   "$\supset$": Next, we show that all positive eigenvalues of AA^T are also positive eigenvalues of $A^T A$:
   Let $\lambda > 0$ be an eigenvalue of AA^T with eigenvector $u \neq 0$, then we find
$$AA^T u = \lambda u \overset{A^T\cdot|}{\Rightarrow} A^T A (A^T u) = \lambda (A^T u).$$
   In addition we find that $A^T u \neq 0$, since
$$\|A^T u\|_2^2 = u^T \underbrace{AA^T u}_{\lambda u} = \underbrace{\lambda}_{>0} \underbrace{\|u\|_2^2}_{>0} > 0.$$
   Thus we can conclude that λ is an eigenvalue of $A^T A$ with eigenvector $A^T u =: v \neq 0$.

4. We show mutual subset relation:
   (a) $\ker(A) = \ker(A^TA)$:

   <u>"ker$(A) \subseteq$ ker(A^TA)":</u>

   Let $x \in \text{ker}(A) \overset{\text{Def. ker}(A)}{\Rightarrow} Ax = 0 \Rightarrow A^TAx = 0 \overset{\text{Def. ker}(A^TA)}{\Rightarrow} x \in \text{ker}(A^TA)$.

   <u>"ker$(A^TA) \subseteq$ ker(A)":</u>

   Let $x \in \text{ker}(A^TA) \overset{\text{Def.}}{\Rightarrow} A^TAx = 0 \Rightarrow \underbrace{x^TA^TAx}_{=\|Ax\|_2^2} = 0 \overset{\text{norm}\|\cdot\|_2^2 \text{ is definite}}{\Rightarrow} Ax = 0 \overset{\text{Def.}}{\Rightarrow} x \in \text{ker}(A)$.

   (b) $\ker(A^T) = \ker(AA^T)$:

   Define $C := A^T$ and apply result (a) to C; note that $C^TC = AA^T$.

5. For example:

   i) Let A have independent columns ("full column rank").
   This is equivalent to $\underbrace{\text{ker}(A)}_{=\text{ker}(A^TA)} = \{0\}$, thus also the columns of A^TA are independent, which implies that A^TA is invertible.

   ii) Let A^TA be positive definite.
   Then its eigenvalues are strictly positive. Since A^TA is symmetric we can use its eigendecomposition to conclude that A^TA is invertible.