Answer the following questions.

1. What is the purpose of the QR Algorithm? Write down its iteration instruction.
2. Let $R = (r_{ij})_{ij} \in \mathbb{R}^{n\times n}$ be a (lower or upper) triangular matrix with $r_{nn} = 0$. Is R invertible? Explain your answer.
3. Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix. Can we generally say that the singular values of A are equal to its eigenvalues? Explain your answer.
4. What is the definition of an orthogonal matrix? What does it mean for the columns of the matrix?
5. Consider the iteration $x_{k+1} = Mx_k + b$ for some matrix $M \in \mathbb{R}^{n\times n}$ and vector $b \in \mathbb{R}^n$. Name a sufficient condition for the convergence of this sequence. What is the limit in this case?

Solution:

1. $(1P)$ Purpose: Compute eigenvalues of a matrix $A \in \mathbb{R}^{n\times n}$

$$
\begin{aligned}
&(1P)\quad A_0 := A\\
&\quad \text{for } i = 1,\dots,n\\
&\qquad Q_iR_i := A_i\\
&\qquad A_{i+1} := R_iQ_i
\end{aligned}
$$

2. $(2P)$ R is not invertible, because triangular matrices are invertible if and only if all diagonal entries are nonzero (see backward/forward substitution)
3. $(2P)$ No! We have

$$\sigma_i := \sqrt{\lambda_i} = |\tilde{\lambda}_i|, \text{ where } \tilde{\lambda}_i \in \sigma(A) \text{ and } \lambda_i = \tilde{\lambda}_i^2 \in \sigma(A^TA) = \sigma(A^2).$$

4. $(1P)$ $Q \in \mathbb{R}^{n\times n}$ orthogonal $:\Leftrightarrow \quad Q^TQ = I$
   $(1P)$ Thus the columns of Q are mutually orthonormal.