1 Matrix-Norm

Let $A \in \mathbb{R}^{n \times m}$. Please show that $\|A\|_1 := \max\{\|Ax\|_1 : \|x\|_1 = 1\} = \max\limits_j \sum_{i=1}^n |a_{ij}|$.

Hint: As usual it is helpful to split up the equality sign into $\leq$ and $\geq$ and treat the parts separately.

Solution:

<u>**"≥":**</u> Since $\{x : \|x\|_1 = 1\} \supset \{e_1, \dots, e_m\}$, we have

$$\max_{\|x\|_1=1} \|A\|_1 \geq \max_{x \in \{e_1,\dots,e_m\}} \|Ax\|_1.$$

<u>**≤:**</u> Let $x \in \{x : \|x\|_1 = 1\}$, $x \in \mathbb{R}^m$ then

$$\begin{aligned}
\|\underbrace{Ax}_{\in \mathbb{R}^n}\|_1 &= \sum_{i=1}^n \left| \sum_{j=1}^m a_{ij} x_j \right| \leq \sum_{j=1}^m \underbrace{\left[\sum_{i=1}^n |a_{ij}| \right]}_{\leq \max_j \sum_{i=1}^n |a_{ij}| =: m} |x_j| \\
&\leq m \underbrace{\sum_{j=1}^m |x_j|}_{=\|x\|_1 = 1} = m \\
&\overset{x \text{ arbitrary}}{\Rightarrow} \max_{\|x\|_1=1} \|Ax\|_1 \leq m.
\end{aligned}$$

$\square$