Eigendecomposition
Let

$$A := \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} \in \mathbb{R}^{2\times 2}.$$

Why does this matrix possess an eigendecomposition $A = Q\Lambda Q^T$? Compute the matrices Λ and Q, by following this recipe:

1. Determine its eigenvalues λ_1 and λ_2 to find Λ by solving $\chi_A(\lambda) = \det(A - \lambda I) = 0$.
2. Determine the corresponding eigenvectors v_1 and v_2 by solving $(A - \lambda_i I)v = 0$.
3. Normalize the eigenvectors to find Q by setting $\tilde{v}_i := \frac{v_i}{\|v_i\|_2}$ and $Q := [\tilde{v}_1, \tilde{v}_2]$. Test if Q^TQ equals I_2.
4. Test if $Q\Lambda Q^T$ equals A.

Solution:

First note that A is symmetric $(A = A^T)$. Thus, the theorem on eigendecomposition implies the existence of an orthogonal matrix Q and a diagonal matrix Λ, with $A = Q\Lambda Q^T$, which we will now determine.

1. Eigenvalues:
   $$0 = \det(A - \lambda I) = \det\left(\begin{pmatrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{pmatrix}\right) = (2-\lambda)^2 - 9$$
   $$\Leftrightarrow \quad (2-\lambda)^2 = 9 \quad \Leftrightarrow \quad 2 - \lambda = \pm 3 \quad \Leftrightarrow \quad \lambda = 2 \pm 3 \ \ (\lambda_1 := 5, \lambda_2 := -1)$$
   Thus we set
   $$\Lambda := \mathrm{diag}(\lambda_1, \lambda_2) = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}.$$

2. Eigenvectors:
   1) Determine an eigenvector corresponding to $\lambda_1 = 5$:
   $$\begin{aligned}(A - \lambda_1 I)v^1 = 0 &\Leftrightarrow \begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} v_1^1 \\ v_2^1 \end{pmatrix} = 0 \\ &\Leftrightarrow -3v_1^1 + 3v_2^1 = 0 \\ &\Leftrightarrow v_1^1 = v_2^1.\end{aligned}$$
   Choose, e.g., $v^1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

   2) Determine an eigenvector corresponding to $\lambda_2 = -1$:
   $$\begin{aligned}(A - \lambda_2 I)v^2 = 0 &\Leftrightarrow \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} v_1^2 \\ v_2^2 \end{pmatrix} = 0 \\ &\Leftrightarrow 3v_1^2 + 3v_2^2 = 0 \\ &\Leftrightarrow v_1^2 = -v_2^2.\end{aligned}$$
   Choose, e.g., $v^2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

3. Normalize eigenvectors to define Q:
   $$\tilde{v}_1 := \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tilde{v}_2 := \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Set

$$Q := [\tilde{v}_1, \tilde{v}_2] = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

We find that Q is orthogonal, more precisely,

$$Q^T Q = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = I.$$

4. Test:

$$\begin{aligned} Q \Lambda Q^T &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \underbrace{\begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}_{= \frac{1}{\sqrt{2}} \begin{pmatrix} 5 & 5 \\ -1 & 1 \end{pmatrix}} \\ &= \frac{1}{2} \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 5 & 5 \\ -1 & 1 \end{pmatrix}}_{= \begin{pmatrix} 4 & 6 \\ 6 & 4 \end{pmatrix}} \\ &= A \quad (\checkmark) \end{aligned}$$