Eigenvalues and Positivity of a Matrix

Definition. A matrix $A \in \mathbb{R}^{n\times n}$ is called positive definite (semi-definite) it $x^T Ax > 0 \quad (\geq 0)$ for all $x \in \mathbb{R}^n \setminus \{0\}$.

Let $A \in \mathbb{R}^{n\times n}$ be any matrix. Please show:

1. If A is positive definite (semi-definite), then $\lambda > 0$ (≥ 0) for all eigenvalues $\lambda \in \sigma(A)$.
   (*Hint:* Rayleigh quotient.)
2. The reverse is not true: Find an example of a matrix, which has positive eigenvalues, but is *not* positive definite.
   (*Hint:* Consider, e.g., a triangular 2×2 matrix.)
3. The reverse is true for symmetric matrices: Let A be *symmetric* and $\lambda > 0$ (≥ 0) for all eigenvalues $\lambda \in \sigma(A)$, then A is positive definite (semi-definite).
   (*Hint:* Eigendecompostion.)
4. Let A be *symmetric*, then A is invertible if and only if $\lambda \neq 0$ for all eigenvalues $\lambda \in \sigma(A)$.
   (*Hint:* Eigendecompostion.)

Solution:

1. Let A be positive definite (semi–definite) and let (λ, v) be an eigenpair of A. Then by using the Rayleigh quotient we find
$$\lambda = \frac{v^\top Av}{v^\top v} = \frac{v^\top Av}{\|v\|_2^2} \quad \frac{>0 \quad (\geq 0)}{>0 \text{ since } v\neq 0} \quad > 0 \quad (\geq 0).$$
2. Take for example
$$A = \begin{pmatrix} 1 & -4 \\ 0 & 2 \end{pmatrix}$$
   which has positive eigenvalues $\sigma(A) = \{1,2\}$, but for $x = (1,1)^\top$ we have
$$x^\top Ax = -3 + 2 = -1 < 0.$$
3. For symmetric $A \in \mathbb{R}^{n\times n}$ we find an orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ and diagonal matrix $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ with $\lambda_i \in \sigma(A)$, so that
$$A = Q\Lambda Q^T \quad \text{(eigendecomposition)}.$$
   Since Q is orthogonal, any vector $x \in \mathbb{R}^n$ can be written as $x = Ix = QQ^\top x = Q\mu$, with coordinates $\mu := Q^\top x$ where $\mu \neq 0$ for $x \neq 0$. Therefore
$$x^\top Ax = \mu^\top Q^\top Q\Lambda Q^\top Q\mu = \mu^\top \Lambda\mu = \sum_{i=1}^{n} \mu_i^2 \lambda_i \geq 0.$$
   We have "$>$" if $x \neq 0$.
4. For symmetric $A \in \mathbb{R}^{n\times n}$ we find an orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ and diagonal matrix $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ with $\lambda_i \in \sigma(A)$, so that
$$A = Q\Lambda Q^T \quad \text{(eigendecomposition)}.$$
   Therefore:
$$A \text{ invertible} \overset{(*)}{\Leftrightarrow} \Lambda \text{ invertible} \overset{(**)}{\Leftrightarrow} \lambda \neq 0 \quad \forall \lambda \in \sigma(A).$$
   $(*)$ product of matrices is invertible if all factors are invertible and orthogonal matrices are invertible with inverse $Q^T = Q^{-1}$

   $(**)$ diagonal matrices are invertible, if and only if diagonal elements are nonzero

   Example: A symmetric and positive definite (spd) $\Rightarrow$ A invertible.