Answer the following questions.

1. How is *injectivity* of a function $f\colon X \to Y$ defined?
2. How is a *scalar product* $P\colon \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}$ defined?
3. Assume you are given the singular value decomposition (SVD) $U\Sigma V^\top = A$ of some matrix $A \in \mathbb{R}^{m\times n}$. Determine the pseudoinverse A^+ of A with the help of the SVD. Proof that $A^+ = A^{-1}$, if A is invertible (*hint:* note that $m = n$ in this case).
4. Let $A \in \mathbb{R}^{n\times n}$ be a matrix. Are the notions of injectivity and surjectivity of A equivalent? Give a short justification.
5. What is the normal equation? Where is it applied?
6. Give an example of a vector space other than $\mathbb{R}^n$?
7. Let V be a vector space over the field $\mathbb{F}$. Give the definition of a basis.

Solution:

1. $(1P)$ $f: X \to Y$ injective $:\Leftrightarrow \quad f(x) = f(y) \quad \Rightarrow \quad x = y \quad \forall x, y \in X$
2. $P: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ scalar product, if

$$
\begin{aligned}
&(1P)\ \text{i)}\ \ \forall x,y \in \mathbb{R}^n:\ P(x,y) = P(y,x)\\
&(1P)\ \text{ii)}\ \ \forall x \in \mathbb{R}^n \setminus \{0\}:\ P(x,x) > 0\\
&(1P)\ \text{iii)}\ \ \forall x,y,z \in \mathbb{R}^n:\ P(x,y+z) = P(x,y) + P(x,z)\\
&(1P)\ \text{iv)}\ \ \forall x,y \in \mathbb{R}^n, \lambda \in \mathbb{R}:\ P(x,\lambda y) = \lambda P(x,y)
\end{aligned}
$$

3.
   - $(1P)$ pseudoinverse: $A^+ = V\Sigma^+U^T$, where $\Sigma^+ = \operatorname{diag}(\frac{1}{\sigma_i}:\ \sigma_i \neq 0)$
   - Let $A \in GL_n(\mathbb{R})$, then $(1P)$ $\sigma_{ii} \neq 0\ \forall i$ and $A^{-1} = (U\Sigma V^T)^{-1} = V\Sigma^{-1}U^T$ $(1P)$ $(V, U$ orthogonal$)$. Since $\Sigma^{-1} = \operatorname{diag}(\frac{1}{\sigma_i}) = \Sigma^+$ we find $A^{-1} = A^+$ $(1P)$
4.
5.
6. $(1P)$ $\mathbb{R}^{m\times n}$, $P_n(\mathbb{R}) := \{x \mapsto \sum_{i=0}^n \alpha_i x^i :\ (\alpha_0, \dots, \alpha_n) \in \mathbb{R}^{n+1}\}$
7. $\{v_1, \dots, v_n\} \subset V$ basis $:\Leftrightarrow$

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\begin{aligned}
&(1P)\ \text{i)}\ \ \{v_1,\dots,v_n\}\ \text{linearly independent}\ \ (\Leftrightarrow\ \textstyle\sum \lambda_j v_j = 0\ \Rightarrow\ \lambda_j = 0\ \forall j\\
&(1P)\ \text{ii)}\ \ \operatorname{span}(v_1,\dots,v_n) = V\ \ (\Leftrightarrow\ \{\textstyle\sum \lambda_j v_j:\ \lambda_j \in \mathbb{R}\} = V)
\end{aligned}
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