**A matrix as a Linear Function**

Let $A \in \mathbb{F}^{m\times n}$ be a matrix. Then consider the mapping $f_A \colon \mathbb{F}^n \to \mathbb{F}^m, x \mapsto Ax$.

1. Show that

   $$f_A(\lambda x + y) = \lambda f_A(x) + f_A(y),$$

   for all $x, y \in \mathbb{F}^n$ and $\lambda \in \mathbb{F}$.

   *Hint:* A vector is a matrix with just one column, so you can make use of the computation rules given above.

   *Remark:* Functions satisfying this property are called **linear**.

2. Use this fact to show the following equivalence:

   $$\ker(A) := \{x \in \mathbb{F}^n \colon Ax = 0\} = \{0\} \quad \Leftrightarrow \quad f_A \text{ is an injective mapping,}$$
   $$\text{(i.e., } f_A(x) = f_A(y) \text{ implies } x = y\text{).}$$

   *Hint:* Split up the equality $\Leftrightarrow$ into $\Rightarrow$ and $\Leftarrow$ and prove each of them separately.

**Solution:**

1. Let $x, y \in \mathbb{F}^n$ and $\lambda \in \mathbb{F}$. Then

   $$f_A(\lambda x + y) = A(\lambda x + y) = A(\lambda x) + Ay = \lambda Ax + Ay = \lambda f_A(x) + f_A(y).$$

2. "$\Rightarrow$" Let $\ker(A) = \{0\}$
   (To show: $f_A$ is an injective mapping, i.e., $f_A(x) = f_A(y)$ implies $x = y$.)
   Let $x, y \in \mathbb{F}^n$ with $f_A(x) = f_A(y)$, which implies by definition $Ax = Ay$ and thus by linearity $A(x - y) = 0$.
   Thus, since $\ker(A) = \{0\}$, we conclude $x - y = 0$.

   "$\Leftarrow$" Let $f_A$ be an injective mapping, i.e., $f_A(x) = f_A(y)$ implies $x = y$.
   (To show:: $Ax = 0 \Leftrightarrow x = 0$ (here "$\Leftarrow$" is obvious).)
   Let $Ax = 0$, then we find

   $$f_A(0) = A0 = 0 = Ax = f_A(x).$$

   Thus, since $f_A$ is assumed to be injective, $x = 0$ (take "$y = 0$").