1 Questions Positive Definite

Definition. A matrix $A \in \mathbb{R}^{n\times n}$ is called positive definite (semi-definite) it $x^T Ax > 0 \ \ (\geq 0)$ for all $x \in \mathbb{R}^n \setminus \{0\}$.

Let $A \in \mathbb{R}^{n\times n}$ be a matrix and $\lambda \in \sigma(A)$. Please show:

1. If A is positive semi-definite, then $\lambda \geq 0$. (*Hint:* Proof by contradiction.)
2. If A is positive definite, then $\lambda > 0$. (*Hint:* Proof by contradiction.)
3. If A is symmetric and positive definite, then A is invertible. (*Hint:* Eigendecompostion.)
4. Show that $A := M^T M$ is positive semi-definite for all matrices $M \in \mathbb{R}^{m\times n}$.
   (*Hint:* $\|x\|_2^2 = x^T x \geq 0 \ \ \forall x \in \mathbb{R}^n$.)

Solution:

1. Assume $\lambda < 0$ exists with the associated eigenvector v. Then
$$v^T Av = \underbrace{\lambda}_{<0} \underbrace{v^T v}_{>0} < 0$$
$$\Rightarrow \ \ A \text{ is not semi-positive definite.}$$
2. Assume $\lambda \leq 0$ exists with the associated eigenvector v. Then
$$v^T Av = \underbrace{\lambda}_{\leq 0} \underbrace{\|v\|^2}_{>0} \leq 0$$
$$\Rightarrow \ \ A \text{ is not positive definite.}$$
3. $$\begin{aligned} A \text{ symmetric} &\Rightarrow \ \ A = V\Lambda V^T \\ A \text{ positive definite} &\Rightarrow \ \ \lambda > 0 \ \ \forall \sigma(A) \\ &\Rightarrow \ \ \Lambda \text{ is invertible} \end{aligned}$$
4. $$\begin{aligned} &A := M^T M, \text{ let } x \in \mathbb{R}^n \\ &\Rightarrow \ \ x^T Ax = x^T M^T Mx = \|Mx\|^2 \geq 0 \end{aligned}$$