Let $A \in \mathbb{R}^{m \times n}$ be any matrix. Please show:

1. $A^T A$ is symmetric.
2. $A^T A$ is positive semi-definite.
3. $\ker(A) = \ker(A^T A)$.
   Hint: Show mutual subset relation.

Solution:

1. (2P) $(A^T A)^T \quad = \quad A^T (A^T)^T \quad = \quad A^T A$
2. (2P) $x^T (A^T A) x \quad = \quad (Ax)^T A x \quad = \quad \|Ax\|_2^2 \quad \geq \quad 0 \quad \forall x \in \mathbb{R}^n$
3. We show mutual subset relation:
   - (1P) "$\ker(A) \subseteq \ker(A^T A)$":
     Let $x \in \ker(A) \overset{\text{Def. } \ker(A)}{\Rightarrow} Ax = 0 \Rightarrow A^T A x = 0 \overset{\text{Def. } \ker(A^T A)}{\Rightarrow} x \in \ker(A^T A)$.
   - (1P) "$\ker(A^T A) \subseteq \ker(A)$":
     Let $x \in \ker(A^T A) \overset{\text{Def.}}{\Rightarrow} A^T A x = 0 \Rightarrow \underbrace{x^T A^T A x}_{= \|Ax\|_2^2} = 0 \overset{\text{norm} \|\cdot\|_2^2 \text{ is definite}}{\Rightarrow} Ax = 0 \overset{\text{Def.}}{\Rightarrow} x \in \ker(A)$.