1 $(n+1)$ vectors are always dependent

Consider arbitrary $n+1$ vectors $v_1, \dots, v_{n+1} \in \mathbb{F}^n$. Then there are coefficients $\alpha_i \in \mathbb{F}, i = 1, \dots, n+1$, which are not all zero and solve the equation $\sum_{i=1}^{n+1} \alpha_i v_i = 0$.

Solution:

If all vectors v_i are zero, the situation is trivial. Thus, we assume that at least one of the v_i is nonzero. Use all vectors v_i as columns of the matrix $V = [v_1, \dots, v_{n+1}] \in \mathbb{R}^{n\times(n+1)}$. Then build the REF (cf. theorem **??**) of V, i.e., construct P, L, U with $L \cdot U = P \cdot V$. Then U is of the form

$$U = \begin{bmatrix} u_{1,1} & \dots & u_{1,n} & u_{1,n+1} \\ & \ddots & \vdots & \vdots \\ & & u_{n,n} & u_{n,n+1} \end{bmatrix}$$

Thus, the equation $\sum_{i=1}^{n+1} \alpha_i v_i = 0$ is equivalent to $U[\alpha_1, \dots, \alpha_{n+1}]^\top = 0$. We can choose a permutation of columns Q such that we have for $\tilde{U} := UQ$

$$\tilde{U} = \begin{bmatrix} \tilde{u}_{1,1} & \dots & \tilde{u}_{1,\ell} & \dots & \tilde{u}_{1,n} & \tilde{u}_{1,n+1} \\ & \ddots & \vdots & \vdots & & \\ & & \tilde{u}_{\ell,\ell} & \dots & \tilde{u}_{\ell,n} & \tilde{u}_{\ell,n+1} \\ 0 & \dots & 0 & \dots & 0 & 0 \\ \vdots & & \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & \dots & 0 & 0 \end{bmatrix}, \quad \tilde{u}_{1,1}, \dots, \tilde{u}_{\ell,\ell} \neq 0$$

and we can choose the coefficients $\tilde{\alpha}_i$ in the following manner

$$\tilde{\alpha}_{n+1}, \dots, \tilde{\alpha}_{\ell+1} \neq 0 \text{ arbitrary}, \quad \tilde{\alpha}_i = -\frac{1}{\tilde{u}_{ii}} \sum_{k=i+1}^{n+1} \tilde{u}_{i,k} \tilde{\alpha}_k \text{ for } i = \ell, \dots, 1 \quad \text{and then } \alpha := Q\tilde{\alpha}$$