Solving Linear Systems using LU Decomposition
Consider the following linear systems.

1.
$$\begin{aligned} 2x_1 + x_2 + 3x_3 &= -3 \\ x_1 - x_2 - x_3 &= 4 \\ 3x_1 - 2x_2 + 2x_3 &= 5 \end{aligned}$$

2.
$$\begin{aligned} x_1 + 2x_2 + 2x_3 &= 1 \\ 2x_1 + x_3 &= 3 \\ 3x_1 + 2x_2 + 3x_3 &= 4 \end{aligned}$$

3.
$$\begin{aligned} x_1 + x_2 + 2x_3 &= 2 \\ 1x_1 - x_2 &= 0 \\ 2x_1 + 2x_3 &= 1 \end{aligned}$$

Cast them into the form $Ax = b$ and compute the LU-decomposition of A by rigorously applying Algorithm 1 (i.e., use the pivot element determined in Line 8):

- Find the matrices L, U and P such that $PA = LU$.
- Then determine the solution set $S := \{x \in \mathbb{R}^n : Ax = b\}$.

Hint: System 2. does not have a *unique* solution (but infinitely many). Determine the set of vectors $x \in \mathbb{R}^3$ for which the linear system 2. is valid.

Solution:

You can later use your Python Code to check your LU decomposition $PA = LU$ with all intermediate steps. Therefore we just copy the factors and solution here.

1. Unique solution:

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1/3 & -1/7 & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} 3 & -2 & 2 \\ 0 & 7/3 & 5/3 \\ 0 & 0 & -10/7 \end{pmatrix}$$

Solving steps yields: $x = (1, -2, -1)^T$, thus $S = \{(1, -2, -1)^T\}$

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INPUT: $A \in \mathbb{R}^{n \times n}$ (and $b \in \mathbb{R}^n$)
OUTPUT: LU decomposition $PA = LU$ (and if $Ax = b$ is uniquely solvable the solution $x \in \mathbb{R}^n$)

# FACTORIZATION
initialize piv = [1,2,...,n]
for $j = 1, ..., n-1$ do
    # Find the j-th pivot
    $k_j := \arg\max_{k \geq j} |a_{kj}|$
    if $a_{k_j j} \neq 0$ then
        # Swap rows
        A[$k_j$,:] $\leftrightarrow$ A[j,:]
        # by hand we also transform b on the fly
        b[$k_j$] $\leftrightarrow$ b[j]
        piv[$k_j$] $\leftrightarrow$ piv[j]
        # Elimination
        for $k = j+1, \ldots, n$ do
            $\ell_{kj} := a_{kj} / a_{jj}$
            $a_{kj} = \ell_{kj}$
            for $i = j+1, \ldots, n$ do
                $a_{ki} = a_{ki} - \ell_{kj} a_{ji}$
            end
            # by hand we also transform b on the fly
            $(b_k = b_k - \ell_{kj} b_j)$
        end
    end
end

# SOLVE
...
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Algorithm 1: In-place Gaussian Elimination with Row Pivoting: Factor and Solve

2. Infinitely many solutions: The algorithm outputs the following arrays

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 2/3 & -1 & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} 3 & 2 & 3 \\ 0 & 4/3 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

and we obtain

$$z := L^{-1}P^{\top}b = \begin{pmatrix} 4 \\ -1/3 \\ 0 \end{pmatrix}.$$

Therefore we see that the system $Ux = z$ has infinitely many solutions (last zero row in U and zero in same row in z). By solving $Ux = z$ we find

$$\begin{aligned} &\overset{\text{(II)}}{\Rightarrow} \quad 4/3x_2 + 1x_3 = -1/3 \quad \Rightarrow \quad x_2 = -\tfrac{1}{4}(1 + 3x_3) \\ &\overset{\text{(I)}}{\Rightarrow} \quad 3x_1 + 2x_2 + 3x_3 = 4 \quad \Rightarrow \quad x_1 = \tfrac{1}{3}(4 - 2x_2 - 3x_3) = \tfrac{3}{2} - \tfrac{1}{2}x_3 \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad S &:= \{x \in \mathbb{R}^3 : Ax = b\} \\ &= \{x \in \mathbb{R}^3 : x_1 = \tfrac{1}{2}(3 - x_3), \quad x_2 = -\tfrac{1}{4}(1 + 3x_3), \quad x_3 \in \mathbb{R}\} \\ &\overset{s:=x_3\in\mathbb{R}^3}{=} \left\{ \begin{pmatrix} \frac{3}{2} \\ -\frac{1}{4} \\ 0 \end{pmatrix} + s \begin{pmatrix} -\frac{1}{2} \\ -\frac{3}{4} \\ 1 \end{pmatrix} : s \in \mathbb{R} \right\}, \text{(i.e., we have infinitely many solutions!)} \end{aligned}$$

3. No solution: $S = \emptyset$

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/2 & -1 & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} 2 & 0 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

and we obtain

$$z := L^{-1}P^{\top}b = \begin{pmatrix} 1 \\ -1/2 \\ 1 \end{pmatrix}.$$

Thus, last row in U is a zero row but $1 \neq 0$ in z.