The Four Fundamental Subspaces

1. Let $A \in \mathbb{R}^{m\times n}$. Show that
$$\forall y \in \mathrm{Im}(A), x \in \ker(A^T)\colon\ y^T x = 0$$
and
$$\forall y \in \mathrm{Im}(A^T), x \in \ker(A)\colon\ y^T x = 0.$$
2. Illuminate these findings on an example.

Remark: We say that $\ker(A^T)$ is the **orthogonal complement** of $\mathrm{Im}(A)$ and write

$$\ker(A^T)^\perp = \mathrm{Im}(A) \quad \text{or} \quad \ker(A^T) \perp \mathrm{Im}(A) \quad \text{or} \quad \ker(A^T) = \mathrm{Im}(A)^\perp.$$

Analogously $\ker(A)$ is the orthogonal complement of $\mathrm{Im}(A^T)$ and we write

$$\ker(A)^\perp = \mathrm{Im}(A^T) \quad \text{or} \quad \ker(A) \perp \mathrm{Im}(A^T) \quad \text{or} \quad \ker(A) = \mathrm{Im}(A^T)^\perp.$$

Solution:

1. Let $y = Av \in \mathrm{Im}(A)$ and $x \in \ker(A^T)$. Then
$$y^T x = (Av)^T x = v^T A^T x = v^T (A^T x) = 0.$$
Apply this to $C = A^T$ to show the other results.
2. Let us consider
$$A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}, \quad A^\top = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}.$$
Then we find

$$\begin{aligned}
\mathrm{Im}(A) &= \text{span} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \\
\ker(A) &= \{x \in \mathbb{R}^2 : Ax = 0\} \\
&= \{x \in \mathbb{R}^2 : x_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 6 \end{pmatrix} = 0\} \\
&= \{x \in \mathbb{R}^2 : 1x_1 + 2x_2 = 0\} \\
&= \{x \in \mathbb{R}^2 : x_1 = -2x_2\} \\
&= \text{span} \begin{pmatrix} -2 \\ 1 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\mathrm{Im}(A^\top) &= \text{span} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\
\ker(A^\top) &= \{x \in \mathbb{R}^2 : Ax = 0\} \\
&= \{x \in \mathbb{R}^2 : 1x_1 + 3x_2 = 0\} \\
&= \{x \in \mathbb{R}^2 : x_1 = -3x_2\} \\
&= \text{span} \begin{pmatrix} -3 \\ 1 \end{pmatrix}
\end{aligned}$$