**The SVD of Rank-1-Matrices**

Let $u \in \mathbb{R}^m \setminus \{0\}$ and $v \in \mathbb{R}^n \setminus \{0\}$ be nonzero vectors and define $A := uv^T$. Find a reduced SVD of $A$ and shortly explain why $\text{rank}(A) = 1$.

**Solution:**

*We will in all detail derive the full and reduced SVD by following our recipe (your answer may be shorter!):*

We have

$$A^T A = \|u\|^2 vv^T.$$

Following the recipe for computing the SVD of $A$ we determine the eigenpairs of $A^T A$. We find

$$A^T A v = \|u\|^2 \|v\|^2 v,$$

which implies that $v$ is an eigenvector to the positive (note: $u, v \neq 0$) eigenvalue $\|u\|^2\|v\|^2$.

- Since $A^T A$ is symmetric we know all eigenvectors are mutually orthogonal.
- However, any vector $x$ orthogonal to $v$ is eigenvector to the eigenvalue $0$, since

  $$A^T A x = \|u\|^2 v \underbrace{v^T x}_{=0} = 0 \cdot x.$$

  Therefore the eigenvalues of $A^T A$ are given by

  $$\lambda_1 = \|u\|^2\|v\|^2,\ \lambda_2 = \cdots = \lambda_n = 0,$$

  with precisely $r = 1$ positive one ($\Rightarrow$ rank$(A) = 1$). Thus we find the singular values and right-singular vector by

  $$v_1 := \frac{v}{\|v\|},\ \sigma_1 = \|u\|\|v\| > 0.$$

  Extend $v_1$ to the orthogonal matrix $V = \begin{pmatrix} | & \\ v_1 & \dots \\ | & \end{pmatrix} \in \mathbb{R}^{n\times n}$ with orthonormal columns, where $v_2, \dots, v_n \in \ker(A)$. Also set

  $$\Sigma = \text{diag}(\sigma_1, 0, \dots, 0) \in \mathbb{R}^{m\times n}.$$

- Following the recipe, the corresponding left-singular vector is given by

  $$u_1 := \frac{Av_1}{\sigma_1} = \frac{1}{\|u\|\|v\|} uv^T \frac{v}{\|v\|} = \frac{u}{\|u\|} \underbrace{\frac{v^T v}{\|v\|^2}}_{=1} = \frac{u}{\|u\|}.$$

  Extend $u_1$ to the orthogonal matrix $U = \begin{pmatrix} | & \\ u_1 & \dots \\ | & \end{pmatrix} \in \mathbb{R}^{m\times m}$ with orthonormal columns, where $u_2, \dots, u_m \in \ker(A^T)$.
- All in all we then obtain the full and reduced (as sum of rank-1 matrices) SVD

  $$\Rightarrow \quad A = U\Sigma V^T = \|u\|\|v\| u_1 v_1^T.$$