1 Solving Linear Systems using the QR Decomposition

Let $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$ be given. Assume you already have the QR decomposition of A, i.e., an orthogonal matrix Q and an upper triangular matrix R, so that $A = QR$. Further assume that R has *nonzero* diagonal elements (i.e., $r_{ii} \neq 0$).

1. How can you use the QR decomposition $A = QR$ for solving a linear system of the form
$$Ax = b \ \ ?$$
2. Use your idea from 1. to solve the system $Ax = b$ where
$$A := \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = QR, \text{ with } \ Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}, \ R = \begin{pmatrix} \frac{2}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{3}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{2}{\sqrt{3}} \end{pmatrix} \text{ and } \ b := \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

   Remark: The square root terms will cancel out nicely while solving for x.

Solution:

1.
$$Ax = b \overset{A=QR}{\Leftrightarrow} QRx = b$$
$$\overset{\cdot Q^T}{\Leftrightarrow} Rx = Q^T b = \hat{b}$$

   Recipe:

   a) Compute $\hat{b} = Q^T b$

   b) Solve $Rx = \hat{b}$

2. a)
$$\hat{b} = Q^T b = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{2}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

   b)
$$\begin{array}{c} \text{(I)} \\ \text{(II)} \\ \text{(III)} \end{array} \begin{pmatrix} \frac{2}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{3}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{2}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$(\mathrm{III}) \quad \Rightarrow \quad x_3 = \frac{1}{2}$$

$$(\mathrm{II}) \quad \frac{3}{\sqrt{6}} x_2 + \frac{1}{\sqrt{6}} x_1 = \frac{2}{\sqrt{6}}$$

$$\Rightarrow \quad x_2 = \frac{\sqrt{6}}{3} \left(\frac{2}{\sqrt{6}} - \frac{1}{2\sqrt{6}} \right) = \frac{\sqrt{6}}{\sqrt{6}} \frac{1{,}5}{3} = \frac{1}{2}$$

$$(\mathrm{I}) \quad x_1 \frac{2}{\sqrt{2}} + x_2 \frac{1}{\sqrt{2}} + x_3 \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}}$$

$$\Rightarrow \quad x_1 = \frac{\sqrt{2}}{2} \left(\frac{2}{\sqrt{2}} - \frac{1}{\sqrt{2}} \right) = \frac{1}{2}$$

$$\Rightarrow \quad x = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$