Consider the matrix

$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 1 & 1 \end{pmatrix}.$$

1. Derive its singular value decomposition (SVD).
   (*Hint:* Compute $U\Sigma V^T$ to check your result.)

2. Write A as a sum of rank-1 matrices by using the singular values and vectors.

3. Is A invertible? Use the SVD to answer this question.

Solution:

$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 1 & 1 \end{pmatrix}$

SVD-Recipe: $\lambda \in \sigma(A^TA),\ \lambda \neq 0,\ \tilde{v}$ eigenvector

(i) $\sigma := \sqrt{\lambda}$

(ii) $v := \dfrac{\tilde{v}}{\|\tilde{v}\|}$

(iii) $u := \dfrac{1}{\sigma}Av$

1. Compute SVD and test:
   - $A^TA = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{pmatrix}$
   - Eigenvalues:
     $\det(A^TA - \lambda I) = (\frac{3}{2} - \lambda)^2 - \frac{1}{4} = 0 \quad \Leftrightarrow \quad \lambda \in \{1, 2\}$
   - Eigenvectors:

     a)

     $$(A^TA - \underbrace{\lambda_1}_{=2} I)\tilde{v} = 0 \quad \Leftrightarrow \quad \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \tilde{v} = 0$$
     $$\Leftrightarrow \quad \tilde{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

     b)

     $$(A^TA - \underbrace{\lambda_2}_{=1} I)\tilde{v} = 0 \quad \Leftrightarrow \quad \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \tilde{v} = 0$$
     $$\Leftrightarrow \quad \tilde{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

   - $\sigma_1 := \sqrt{\lambda_1} = \sqrt{2},\ \sigma_2 := \sqrt{\lambda_2} = 1,$
     $v_1 := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ v_2 := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix},$
     $u_1 := \frac{1}{\sigma_1} Av_1 = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$
     $u_2 := \frac{1}{\sigma_2} Av_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{2}{\sqrt{2}} \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

- Thus:

$$\Sigma = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{pmatrix}, \quad V = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

- Test:

$$U\Sigma V^T = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \underbrace{\begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}_{= \begin{pmatrix} \sqrt{2} & \sqrt{2} \\ 1 & -1 \end{pmatrix}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ \sqrt{2} & \sqrt{2} \end{pmatrix} = A \quad \checkmark$$

2.

$$\begin{aligned} A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T &= \frac{\sqrt{2}}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} (1, 1) + \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1, -1) \\ &= \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \quad \checkmark \end{aligned}$$

3. Yes, since $\sigma_1 \neq 0 \neq \sigma_2$.