**The SVD of Symmetric Matrices**

Let $A \in \mathbb{R}^{n\times n}$ be symmetric with eigenvalues $\lambda_1, \dots, \lambda_n \in \mathbb{R}$.

1. What are the singular values of $A$?
2. How does the SVD look like? What can you say about the SVD if in addition $A$ is positive definite?
3. *Optional:* Give a representation of the condition and rank of $A$ in terms of its eigenvalues.

**Solution:**

REMARK: $A \in \mathbb{R}^{n\times n}$, $(\lambda, v)$ eigenpair of $A$, then

$$A^k v \;=\; A^{k-1}\underbrace{(Av)}_{=\lambda v} \;= \dots = \lambda^k v \quad \forall k \in \mathbb{N}$$

$\Rightarrow (\lambda^k, v)$ is eigenpair of $A^k$.

1. $A$ symmetric $\;\Rightarrow\; A^T A = A^2 \;\Rightarrow\; \sigma(A^T A) = \{\lambda^2 : \lambda \in \sigma(A)\}$,
   $\Rightarrow$ singular values are $\sigma = \sqrt{\lambda^2} = |\lambda|$ for $\lambda \in \sigma(A)$ nonzero

   ("Singular values of a symmetric matrix are the absolute values of its nonzero eigenvalues!")

2. How does the SVD look like?
   Let $(\lambda_i, v_i)$ be eigenpairs of $A$, so that the $v_i$'s are orthonormal. We set

$$\sigma_i = |\lambda_i|, \quad (i = 1, \dots, r)$$
$$u_i = \begin{cases} \frac{1}{\sigma_i} A v_i = \frac{\lambda_i}{|\lambda_i|} v_i & : \text{ for } i = 1, \dots, r \;\; (\lambda_i \neq 0), \\ v_i & : \text{ for } i = r+1, \dots, n \;\; (\lambda_i = 0). \end{cases}$$

   (note: we have shown in the lecture that the $u_i$ as defined here are orthonormal and also note that $\ker(A) = \ker(A^\top)$).
   Then

$$A = V \Lambda V^\top \qquad \text{[eigendecomposition]}$$

$$= \underbrace{\underbrace{\begin{pmatrix} | & & | & | & & | \\ \frac{\lambda_1}{|\lambda_1|} v_1 & \dots & \frac{\lambda_r}{|\lambda_r|} v_r & v_{r+1} & \dots & v_n \\ | & & | & | & & | \end{pmatrix}}_{=:U} \underbrace{\begin{pmatrix} |\lambda_1| & & & & 0 \\ & \ddots & & & \\ & & |\lambda_r| & & \\ & & & \ddots & \\ 0 & & & & 0 \end{pmatrix}}_{=:\Sigma = \mathrm{diag}(\sigma_1, \dots, \sigma_r)}}_{=V\Lambda} \underbrace{\begin{pmatrix} - & v_1^\top & - \\ & \vdots & \\ - & v_n^\top & - \end{pmatrix}}_{V^\top} \qquad [SVD]$$

   $A$ symmetric and positive definite $\Rightarrow\; |\lambda_i| = \lambda_i > 0 \;\Rightarrow\; \frac{\lambda_i}{|\lambda_i|} = 1$. Thus:

   ("For spd matrices: SVD = Eigendecomposition!")

3. For the condition and the rank we obtain (by inserting $\sigma_i = |\lambda_i|$ )

$$\mathrm{cond}(A) = \frac{\max(\sigma_i)}{\min(\sigma_i)} = \frac{\max(|\lambda_i|)}{\min(|\lambda_i|)}$$

   and

$$\mathrm{rank}(A) = |\{i : \sigma_i > 0\}| = |\{i : |\lambda_i| > 0\}| = |\{i : \lambda_i \neq 0\}|.$$