1 Estimate Number of Iterations

Consider the linear system $Ax = b$ with

$$A = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}.$$

Assume you want to solve this system with the Richardson, Jacobi and Gauß-Seidel method, respectively, without relaxation ($\theta = 1$). Estimate the number m of iterations that are needed for each method (if convergent) to reduce the error by a factor of $\varepsilon = 10^{-6}$, i.e.,

$$\|x_k - x\|_2 \leq \varepsilon \|x_0 - x\|_2.$$

Hint: Use the error estimate from previous exercises. You can later numerically verify your results.

Solution:

Since A is symmetric we can exploit the estimate

$$\|x_k - x\|_2 \leq \rho(M)^k \|x_0 - x\|_2,$$

where $\rho(M)$ denotes the spectral radius of the iteration matrix. Thus we want to find $m \in \mathbb{N}$ such that for all $k \geq m$, we have

$$\rho(M)^k \leq \varepsilon.$$

Applying logarithm we find that

$$m = \frac{\log(\varepsilon)}{\log(\rho(M))}.$$

Therefore let us compute the spectral radius for this case and each method. Consider the splitting $A = L + D + U$.

Richardson:
Here we have

$$M_R = I - A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}.$$

Thus

$$0 = \det(M_R - \lambda I) = (2 + \lambda)^2 - 1$$

gives $\lambda \in \{-1, -3\}$, so that $\rho(M_R) = 3$. Therefore Richardson without relaxation does not converge!

Jacobi:
Here we have

$$M_J = I - D^{-1}A = I - \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{3} \\ \frac{1}{3} & 0 \end{pmatrix}.$$

Thus

$$0 = \det(M_J - \lambda I) = \lambda^2 - \frac{1}{9}$$

gives $\lambda \in \{-\frac{1}{3}, \frac{1}{3}\}$, so that $\rho(M_J) = \frac{1}{3}$. Therefore Jacobi without relaxation does converge and

$$m = \frac{\log(\varepsilon)}{\log(\frac{1}{3})} \approx 13.$$

Gauß-Seidel:
Here we have

$$M_G = I - (D + L)^{-1}A$$

where

$$(D+L)^{-1} = \begin{pmatrix} \frac{1}{3} & 0 \\ \frac{1}{9} & \frac{1}{3} \end{pmatrix}$$

so that

$$M_G = I - \begin{pmatrix} \frac{1}{3} & 0 \\ \frac{1}{9} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{3} \\ 0 & \frac{1}{9} \end{pmatrix}.$$

Thus

$$0 = \det(M_G - \lambda I) = -\lambda(\tfrac{1}{9} - \lambda)$$

gives $\lambda \in \{0, \frac{1}{9}\}$, so that $\rho(M_G) = \frac{1}{9}$. Therefore Gauß-Seidel without relaxation does converge and

$$m = \frac{\log(\varepsilon)}{\log(\frac{1}{9})} \approx 7.$$