Rank/Image and Nullity/Kernel
Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

the column vector $\mathbf{1} := \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ (i.e., a (3×1) matrix) and the row vector $\tilde{\mathbf{1}} := (1\ 1\ 1)$ (i.e., a (1×3) matrix).

1. Show that $A = \mathbf{1}\tilde{\mathbf{1}}$.
2. Find two distinct nonzero vectors x and y, so that $Ax = 0$ and $Ay = 0$.
3. How does the image Im(A) look like? Characterize the set mathematically and also draw a picture. Find a basis of Im(A) and determine the rank of the matrix, i.e., rank(A).
4. How does the kernel ker(A) look like? Characterize the set mathematically and also draw a picture. Find a basis of ker(A) and determine its dimension.

Solution:

1. By applying the matrix-matrix product definition we multiply the matrix $\mathbf{1}$ with each column in $\tilde{\mathbf{1}}$ (here, a column is just the number 1). We obtain

$$\mathbf{1}\tilde{\mathbf{1}} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot (1\ 1\ 1) \left(1 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} 1 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} 1 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right) = A.$$

2. Since $a_1 = a_2 = a_3 = \mathbf{1}$, we have

$$0 = Ax = a_1x_1 + a_2x_2 + a_3x_3 = a_1(x_1 + x_2 + x_3) \Leftrightarrow x_1 + x_2 + x_3 = 0.$$

   Choose, e.g., $x = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and $y = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

3. By definition of the image we have

$$\begin{aligned} \text{Im}(A) &= \text{span}(a_1, a_2, a_3) \\ &= \{\lambda_1\mathbf{1} + \lambda_2\mathbf{1} + \lambda_3\mathbf{1} \colon \lambda_i \in \mathbb{R}\} \\ &= \{\lambda\mathbf{1} \colon \lambda \in \mathbb{R}\} \\ &= \text{span}(\mathbf{1}). \end{aligned}$$

   Since $\mathbf{1} \neq 0$, we have that $\{\mathbf{1}\}$ is a basis of length 1 for Im(A). In particular we find

$$\text{rank}(A) := \dim \text{Im}(A) = 1.$$

   (Note that two equal vectors $x = y$ are linearly dependent and that a single nonzero vector $x \neq 0$ is linearly independent.)

4. From 2. we already know

$$
\begin{aligned}
\ker(A) := \{x \in \mathbb{R}^3 \colon Ax = 0\} &= \{x \in \mathbb{R}^3 \colon x_1 + x_2 + x_3 = 0\} \\
&= \{x \in \mathbb{R}^3 \colon x_1 = -(x_2 + x_3)\} \\
&= \left\{ \begin{pmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix} : x_2, x_3 \in \mathbb{R} \right\} \\
&= \left\{ x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} : x_2, x_3 \in \mathbb{R} \right\} \\
&= \operatorname{span} \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}.
\end{aligned}
$$

Since $b_1 := \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and $b_2 := \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ are linearly independent (in fact, one can show $[b_1, b_2]x = 0$ implies $x = 0$), they form a basis of $\ker(A)$ and thus we have $\dim\left(\ker(A)\right) = 2$.