1 Subset Property

Please show that the following assertions are equivalent.

1. $A \subset B$
2. $A \cap B = A$
3. $A \cup B = B$

Hint: It suffices to show $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$.

Solution:

- (i) $\Rightarrow$ (ii)
  $A \cap B \subset A$ (No need for assumptions.)

$$\begin{aligned} & x \in A \cap B \\ \Rightarrow & x \in A \wedge x \in B \\ \Rightarrow & x \in A \end{aligned}$$

  $A \subset A \cap B$

$$\begin{aligned} & A \subset B \\ \Rightarrow & x \in A \Rightarrow x \in B \\ \Rightarrow & x \in A \Rightarrow x \in A \wedge x \in B \\ \Rightarrow & A \subset A \cap B \end{aligned}$$

- (ii) $\Rightarrow$ (iii)
  $B \subset A \cup B$ (Again, no need for assumptions.)

$$\begin{aligned} & x \in B \\ \Rightarrow & x \in A \vee x \in B \\ \Rightarrow & x \in A \cup B \end{aligned}$$

  $A \cup B \subset B$

$$\begin{aligned} & x \in A \cup B \\ \Rightarrow & x \in A \vee x \in B \\ \Rightarrow & (x \in A \wedge x \in B) \vee x \in B \quad \text{(since by ii): } A \subset A \cap B) \\ \Rightarrow & x \in (A \cap B) \cup B \\ \Rightarrow & x \in B \end{aligned}$$

- (iii) $\Rightarrow$ (i)

$$\begin{aligned} & x \in A \\ \Rightarrow & x \in A \vee x \in B \\ \Rightarrow & x \in A \cup B \\ \Rightarrow & x \in B, \quad \text{(since by iii): } A \cup B \subset B) \end{aligned}$$