1 Eigenvalues Triangular Matrix

Use Theorem 12.2iii) to show that the eigenvalues of an upper triangular matrix $A \in \mathbb{R}^{n \times n}$ are given on its diagonal, i.e., that $\det(A - \lambda I) = 0$ for all $\lambda \in diag(A)$.

Solution:

Show: $\det(A - \lambda I) = 0 \ \ \forall \lambda \in \text{diag}(A) = \{a_{11}, \dots, a_{nn}\}$
Proof:
Since $(A - \lambda I)$ is itself upper triangular it remains to show, that

$$\det(U) = \prod_{i=1}^{n} u_{ii}$$

for any upper triangular matrix.(Because then: $\det(A - \lambda I) = \prod_{i=1}^{n}(a_i i - \lambda)$.)
Consider:

$$U = \begin{pmatrix} u_{11} & & & \\ 0 & u_{22} & & \\ \vdots & 0 & \ddots & \\ 0 & \cdots & 0 & u_{nn} \end{pmatrix}$$

Then by T.12.2: $\det(U) = \det([u_{11}])\det \underbrace{\begin{pmatrix} u_{22} & & \\ 0 & \ddots & \\ 0 & 0 & u_{nn} \end{pmatrix}}_{=:U_{11}}$

We can now apply the same idea to

$$U_{11} = \begin{pmatrix} u_{22} & & & \\ 0 & u_{33} & & \\ \vdots & 0 & \ddots & \\ 0 & \cdots & 0 & u_{nn} \end{pmatrix}$$

$$\begin{aligned} \det(U) &= \underbrace{\det([u_{11}])}_{=u_{11}} \underbrace{\det([u_{22}])}_{=u_{22}} \det \begin{pmatrix} u_{33} & & \\ 0 & \ddots & \\ 0 & 0 & u_{nn} \end{pmatrix} \\ &\vdots \\ &= \prod_{i=1}^{n} u_{ii} \end{aligned}$$