Rule of Sarrus

Derive the *Rule of Sarrus* for the determinant of a (3×3)-matrix by using the Laplace formula from the lecture with $n = 3$. Then compute the determinant of

$$A = \begin{pmatrix} 1 & 4 & 3 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{pmatrix}.$$

What does it tell us about the columns?

Solution:

Recall:

$$\begin{aligned} \det(A) &= \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}) \quad (i \in \{1, \dots, n\}, \text{ fixed}) \\ \det(a) &:= a \end{aligned}$$

Now consider $n = 3$ and let us fix $i = 1$. We indicate the submatrices A_{ij} by colors:

$$j = 1: \quad \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$j = 2: \quad \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$j = 3: \quad \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

By Laplace formula we obtain

$$\begin{aligned} \det(A) &= \underbrace{(-1)^{1+1}}_{=1} a_{11} \det(A_{11}) + \underbrace{(-1)^{1+2}}_{=-1} a_{12} \det(A_{12}) + \underbrace{(-1)^{1+3}}_{=1} a_{13} \det(A_{13}) \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{12}(a_{21}a_{32} - a_{31}a_{22}) \\ &= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22}. \end{aligned}$$

Note that we have exploited also the Laplace formula for 2×2 matrices. For the example matrix this yields $\det(A) = 2 - 2 = 0$, so that we can conclude that the columns are linearly dependent.