1 Irrational $\sqrt{2}$

Please show that $\sqrt{2}$ is not a rational number.

Solution:

Proof We prove the assumption by contradiction. Let us assume $\sqrt{2} \in \mathbb{Q}$. Therefore there are $a, b \in \mathbb{N}$ such that a, b are relatively prime (i.e. their greatest common divisor is 1) and $\sqrt{2} = \frac{b}{a}$.
We see that $2a^2 = b^2$ which implies that b^2, and therefore b, are even. That means there is a $c \in \mathbb{N}$ such that $b = 2c$. This implies that $2a^2 = 4c^2$ which tells us that $a^2 = 2c^2$. Hence, a^2 and therefore a are even. That contradicts the assumption that a, b are relatively prime. Therefore $\sqrt{2}$ is not an element of $\mathbb{Q}$. $\square$