1 Sets and Functions

Let A, B be finite sets which contain the same number of elements, i.e., $|A| = |B|$ and let $f : A \to B$ be a function. Please show that the following statements are equivalent.

1. f is injective
2. f is surjective
3. f is bijective

Hint: (iii) follows immediately from (i) $\Leftrightarrow$ (ii).

Solution:

- (i) $\Rightarrow$ (ii)

$$\begin{aligned} i) &\Rightarrow \forall a_1, a_2 \in A : a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2) \\ &\Rightarrow |f(A)| = |A| \quad (\text{note: } A = \{a | a \in A\}, f(A) = \{f(a) | a \in A\}) \\ &\Rightarrow |f(A)| = |B| \quad (\text{since } |A| = |B|) \\ &\Rightarrow f(A) = B \quad (\text{since } f(A) \subset B \text{ and } B \text{ finite}) \\ &\Rightarrow ii) \end{aligned}$$

- (ii) $\Rightarrow$ (i)
  We know by surjectivity and $|A| = |B|$:

$$f(A) = B \quad \Rightarrow \quad |f(A)| = |B| \quad \Rightarrow \quad |f(A)| = |A| \quad (+)$$

  Now let $a_1 \neq a_2 \in A$ (to show $f(a_1) \neq f(a_2)$)
  Assumption: $f(a_1) = f(a_2)$, then

$$f(A) = \{f(a) : a \in A\} \subsetneq B \quad \Rightarrow \quad |f(A)| < |B| \quad \Rightarrow \quad |A| < |B| \quad \text{“contradiction to } (+) \text{ !"}$$

- (iii) $\Leftrightarrow$ (i) (or (ii))
  By definition we have

$$(iii) \Leftrightarrow (i) \wedge (ii) \Leftrightarrow (i)$$