Gram–Schmidt Algorithm

Let $A \in \mathbb{R}^{n\times n}$ be defined as

$$A := \begin{pmatrix} 3 & 4 \\ 2 & 7 \end{pmatrix}.$$

1. Compute the QR-decomposition of A using the Gram-Schmidt algorithm.
2. Compute $QR = A$ to check your result.

Solution:

Let a_i denote the i-th column of A.

1. Compute QR-decomposition via Gram-Schmidt:

$$\begin{aligned}
\tilde{q}_1 &:= a_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \\
r_{11} &:= \|\tilde{q}_1\| = \sqrt{13} \\
q_1 &:= \frac{1}{\sqrt{13}} \begin{pmatrix} 3 \\ 2 \end{pmatrix} \\
r_{12} &:= a_2^T q_1 = (4,7) \begin{pmatrix} 3 \\ 2 \end{pmatrix} \frac{1}{\sqrt{13}} = \frac{1}{\sqrt{13}} 26 = 2\sqrt{13} \\
\tilde{q}_2 &:= a_2 - r_{12} q_1 = \begin{pmatrix} 4 \\ 7 \end{pmatrix} - 2\sqrt{13} \frac{1}{\sqrt{13}} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} \\
r_{22} &:= \|\tilde{q}_2\| = \sqrt{4+9} = \sqrt{13} \\
q_2 &:= \frac{1}{\sqrt{13}} \begin{pmatrix} -2 \\ 3 \end{pmatrix} \\
\Rightarrow \quad Q &= \frac{1}{\sqrt{13}} \begin{pmatrix} 3 & -2 \\ 2 & 3 \end{pmatrix} \quad , \quad R = \begin{pmatrix} \sqrt{13} & 2\sqrt{13} \\ 0 & \sqrt{13} \end{pmatrix} = \sqrt{13} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}
\end{aligned}$$

2. Test:

$$QR = \frac{1}{\sqrt{13}} \begin{pmatrix} 3 & -2 \\ 2 & 3 \end{pmatrix} \sqrt{13} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 2 & 7 \end{pmatrix} = A \;\; \checkmark$$