Orthogonal Projection

Prove the following statement: Let $V \subset \mathbb{R}^m$ be a linear subspace and $b \in \mathbb{R}^m$. Then

$$\widehat{z} = \arg\min_{z \in V} \|z - b\|_2^2 \quad \Leftrightarrow \quad \widehat{z} - b \in V^\perp := \{w \in \mathbb{R}^n : w^\top v = 0 \;\; \forall v \in V\}.$$

Hint: You can use: For all $x, y \in \mathbb{R}^m$: $\|x + y\|_2^2 = \|x\|_2^2 + \|y\|_2^2 \;\Leftrightarrow\; x^\top y = 0$.

Solution:

We use the hint with $x = \widehat{z} - b$ and $y := z - \widehat{z}$ for some $z \in V$ (note that $(z - \widehat{z}) \in V \; \forall z \in V$, since $\widehat{z} \in V$ and V subspace). More precisely, for a $\widehat{z} \in V$ we find

$$\begin{aligned}
\widehat{z} - b \in V^\perp \quad & \Leftrightarrow \forall z \in V : (\widehat{z} - b)^\top z = 0 \\
& \Leftrightarrow \forall z \in V : (\widehat{z} - b)^\top (z - \widehat{z}) = 0 \\
& \Leftrightarrow \forall z \in V : \|z - b\|_2^2 = \|\widehat{z} - b\|_2^2 + \|\widehat{z} - z\|_2^2 \\
& \Leftrightarrow \forall z \in V : \|\widehat{z} - b\|_2^2 \leq \|z - b\|_2^2 \\
& \Leftrightarrow \widehat{z} = \arg\min_{z \in V} \|z - b\|_2^2.
\end{aligned}$$