1 Convergence Speed of Linear Iterations

Let $M \in \mathbb{R}^{n \times n}$ be *symmetric* with $\rho(M) < 1$, let $N \in \mathbb{R}^{n \times n}$ and $x_0, b \in \mathbb{R}^n$. Consider the fixed point iteration

$$x_{k+1} = Mx_k + Nb$$

and show the following convergence result

$$\|x_k - x^*\|_2 \leq \rho(M)^k \|x_0 - x^*\|_2,$$

where x^* is the associated fixed point. Thus, the smaller the spectral radius, the faster does the method converge.
Hint: For symmetric matrices $M \in \mathbb{R}^{n \times n}$ you can use $\|Mx\|_2 \leq \rho(M)\|x\|_2$ for all $x \in \mathbb{R}^n$.

Solution:

Since $\rho(M) < 1$, the iteration converges to the fixed point $x^* = Mx^* + Nb$. We use this representation in the formulas. We find

$$\|x^k - x^*\|_2 = \|Mx^{k-1} + Nb - (Mx^* + Nb)\|_2 = \|M(x^{k-1} - x^*)\|_2 \overset{\text{[Hint]}}{\leq} \rho(M) \underbrace{\|x^{k-1} - x^*\|_2}_{\leq \rho(M)\|x^{k-2} - x^*\|} .$$

By inserting the iteration instruction repeatedly we ultimately arrive at

$$\|x^k - x^*\|_2 \leq \rho(M)^k \|x^0 - x^*\|_2.$$