Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Assume you are given the least squares problem

$$\min_{x\in\mathbb{R}^n} \|Ax - b\|_2^2.$$

1. Which equation does a solution $\hat{x}$ of the above least squares problem solve?
2. Assume you are given the following data

z	-2	-1	0	1	2
y	3	-1	0	1	4

.

Solve the curve fitting problem

$$\min_{c_0,c_1\in\mathbb{R}} \sum_{i=1}^{5} (c_0 + c_1 z_i^2 - y_i)^2,$$

i.e., determine the minimizing parameters c_0 and c_1.

Solution:

1. $\hat{x}$ solves the normal equation $(1P)$: $A^T A\hat{x} \overset{(1P)}{=} A^T y$
2. In this case: $A = \begin{pmatrix} 1 & 4 \\ 1 & 1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 4 \end{pmatrix} \quad (2P)$,

$$A^T A = \begin{pmatrix} 5 & 10 \\ 10 & 34 \end{pmatrix} \quad (2P),$$

$$A^T y = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & 1 & 0 & 1 & 4 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 0 \\ 2 \\ 9 \end{pmatrix} = \begin{pmatrix} 7 \\ 28 \end{pmatrix} \quad (2P)$$

Normal equation:

$$\begin{pmatrix} 5 & 10 \\ 10 & 34 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 7 \\ 28 \end{pmatrix}$$

Gaussian elimination yields

$$\begin{pmatrix} 5 & 10 \\ 0 & 14 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 7 \\ 14 \end{pmatrix}$$

$$\Leftrightarrow \quad c_0 = -\tfrac{2}{3},\ c_1 = 1 \quad (1 + 1P)$$

$$\Rightarrow \quad \hat{x} = \begin{pmatrix} -\frac{2}{3} \\ 1 \end{pmatrix}$$