Let

$$A := \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} \in \mathbb{R}^{2\times 2}.$$

1. Why do we know that an eigendecomposition $Q\Lambda Q^T$ of A exists?
2. What properties do Q and Λ have, if $Q\Lambda Q^T$ is an eigendecomposition of A?
3. Find such matrices Λ and Q.

Hint: Compute $Q\Lambda Q^T = A$ to check your result.

Solution:

First note that A is symmetric ($A = A^T$). So Theorem 3.3 implies the existence of an orthogonal matrix Q and a diagonal matrix Λ, with $A = Q\Lambda Q^T$, which we will now determine.

1. Eigenvalues:
   $0 = \det(A - \lambda I) = \det\left(\begin{pmatrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{pmatrix}\right) = (2-\lambda)^2 - 9$
   $\Leftrightarrow \quad (2-\lambda)^2 = 9 \quad \Leftrightarrow \quad 2-\lambda = \pm 3 \quad \Leftrightarrow \quad \lambda = 2 \pm 3 \;\; (\lambda_1 := 5, \lambda_2 := -1)$
   Set
   $$\Lambda := \mathrm{diag}(\lambda_1, \lambda_2) = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}.$$

2. Eigenvectors:

   1) Determine an eigenvector corresponding to $\lambda_1 = r$.
   $$\begin{aligned}(A - \lambda_1 I)v^1 = 0 &\Leftrightarrow \begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} v_1^1 \\ v_2^1 \end{pmatrix} = 0 \\ &\Leftrightarrow -3v_1^1 + 3v_2^1 = 0 \\ &\Leftrightarrow v_1^1 = v_2^1\end{aligned}$$
   Choose, e.g., $v^1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

   2) Determine an eigenvector corresponding to $\lambda_2 = r$.
   $$\begin{aligned}(A - \lambda_2 I)v^2 = 0 &\Leftrightarrow \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} v_1^2 \\ v_2^2 \end{pmatrix} = 0 \\ &\Leftrightarrow 3v_1^2 + 3v_2^2 = 0 \\ &\Leftrightarrow v_1^2 = -v_2^2\end{aligned}$$
   Choose, e.g., $v^2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

3. Normalize eigenvectors to define Q:
   $\tilde{v}_1 := \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tilde{v}_2 := \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$
   Set $Q := [\tilde{v}_1, \tilde{v}_2] = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.
   We find that A is orthogonal, more precisely,
   $$Q^T Q = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = I.$$

4. <u>Test:</u>

$$
\begin{aligned}
Q\Lambda Q^T &= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\underbrace{\begin{pmatrix}5 & 0\\ 0 & -1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}}_{=\frac{1}{\sqrt{2}}\begin{pmatrix}5 & 5\\ -1 & 1\end{pmatrix}}\\
&= \frac{1}{2}\underbrace{\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}5 & 5\\ -1 & 1\end{pmatrix}}_{=\begin{pmatrix}4 & 6\\ 6 & 4\end{pmatrix}}\\
&= A \quad (\checkmark)
\end{aligned}
$$