Answer the following questions.

1. How is a *norm* $L : \mathbb{F}^n \to [0, \infty)$ defined?

2. Assume you are given the singular value decomposition $U\Sigma V^\top = A$ of some matrix $A \in \mathbb{R}^{m\times n}$, where the entries on the diagonal of Σ are given in a descending order. Denote the best rank-k approximation of A for $k \leq \min\{m, n\}$ and denote the criterion with respect to which this is the best approximation.

3. Give an example for a matrix $A \in \mathbb{R}^{2\times 2}$ where the LU-decomposition algorithm necessarily needs a permutation step.

4. Let $\lambda_1 \neq \lambda_2$ be two eigenvalues of a *symmetric* matrix $A \in \mathbb{R}^{n\times n}$, and let $v_1, v_2 \in \mathbb{R}^n$ be corresponding eigenvectors. Proof that $v_1^\top v_2 = 0$.

5. How is positive definiteness of a matrix $A \in \mathbb{R}^{n\times n}$ defined? What does this mean for the angle between a vector $x \in \mathbb{R}^n$ and the vector $z := Ax$?

Solution:

1. $(2P)$ $L : \mathbb{F}^n \to [0, +\infty)$ norm: $\Leftrightarrow$

$$\begin{aligned} &\text{i) } L(x) = 0 \quad \Rightarrow \quad x = 0 \\ &\text{ii) } L(\lambda x) = |\lambda| L(x) \\ &\text{iii) } L(x + y) \leq L(x) + L(y) \end{aligned}$$

2. $A = U\Sigma V^T = \sum_{j=1}^{\min(n,m)} \sigma_j u_j v_j^T$
   best rank-k approximation is given by truncated SVD

$$\begin{aligned} (1P) \quad A_k &:= \sum_{j=1}^{k} \sigma_j u_j v_j^T \text{ for which} \\ (1P) \quad A_k &:= \min_{B, \text{rank}(B)=k} \|B - A\|_F^2. \end{aligned}$$

3. $(2P)$ $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

4. $(2P)$ $A \in \mathbb{R}^{n\times n}$ symmetric $, \lambda_1 \neq \lambda_2 \in \sigma(A)$ with v_1, v_2

$$\begin{aligned} \Rightarrow \quad & v_1^T A v_2 = \lambda_2 v_1^T v_2 \\ \text{and} \quad & v_1^T A v_2 = v_2^T A^T v_1 = \lambda_1 v_1^T v_2 \\ \Rightarrow \quad & (\lambda_1 - \lambda_2) v_1^T v_2 = 0 \\ \overset{\lambda_1 \neq \lambda_2}{\Rightarrow} \quad & v_1^T v_2 = 0 \end{aligned}$$

5. $A \in \mathbb{R}^{n\times n}$ positive definite $:\Leftrightarrow \quad \forall x \in \mathbb{R}^n \setminus \{0\} : \; x^T A x > 0$

$$\begin{aligned} &\Rightarrow \quad 0 < x^T \underbrace{(Ax)}_{:=z} = \cos(\alpha) \underbrace{\|x\| \|Ax\|}_{\geq 0} \quad (1P) \\ &\Rightarrow \quad \cos(\alpha) > 0 \quad \Rightarrow \quad \alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \; |\alpha| < 90^\circ \quad (1P) \end{aligned}$$