1 The SVD and the Rank of a Matrix

Let $A \in \mathbb{R}^{n\times n}$ with SVD $A = U\Sigma V^T$ and define $\text{rank}(A) :=$ "number of positive singular values". Show that A is invertible $\iff$ $\text{rank}(A) = n$.

Solution:

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\begin{aligned}
&\text{rank}(A) = n \quad \Leftrightarrow \quad \sigma_1, \dots, \sigma_n \neq 0 \\
&\Leftrightarrow \quad \Sigma = \begin{pmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{pmatrix} \text{ is invertible with } \Sigma^{-1} = \begin{pmatrix} \frac{1}{\sigma_1} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\sigma_n} \end{pmatrix} \\
&\Leftrightarrow A = U\Sigma V^T \text{ invertible with } A^{-1} = (U\Sigma V^T)^{-1} = V\Sigma^{-1}U^T.
\end{aligned}
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