1 Frobenius Norm, Trace and Singular Values

Recall that the Frobenius norm of a matrix $A \in \mathbb{R}^{m\times n}$ is defined by

$$\|A\|_F := \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2}$$

and the trace of a matrix $B \in \mathbb{R}^{n\times n}$ by

$$\operatorname{tr}(B) = \sum_{i=1}^{n} b_{ii}.$$

1. Show that for a matrix $A \in \mathbb{R}^{m\times n}$ we have
$$\|A\|_F^2 = \operatorname{tr}(A^T A).$$

2. Show that the trace is symmetric, i.e., for $A, B \in \mathbb{R}^{m\times n}$ we find
$$\operatorname{tr}(A^\top B) = \operatorname{tr}(B^\top A) = \operatorname{tr}(BA^\top).$$

   Remark: $\mathbb{R}^{m\times n} \times \mathbb{R}^{m\times n} \to \mathbb{R}, (A, B) \mapsto \operatorname{tr}(B^\top A)$ defines an inner product on $\mathbb{R}^{m\times n}$ and the Frobenius norm $\|A\|_F = \sqrt{\operatorname{tr}(A^T A)}$ denotes the corresponding norm. For example, Cauchy-Schwarz inequality holds.

3. Use these results and the singular value decomposition to show that the Frobenius norm of a matrix $A \in \mathbb{R}^{m\times n}$ can be expressed in terms of its singular values, i.e.,
$$\|A\|_F^2 = \sum_{i=1}^{\min(m,n)} \sigma_i^2 \quad \left(= \sum_{i=1}^{r} \sigma_i^2 = \|\Sigma\|_F^2 \right).$$

Solution:

1. First note that by definition of the matrix product we have
$$(A^\top A)_{jj} = \sum_{i=1}^{m} a'_{ji} a_{ij} = \sum_{i=1}^{m} a_{ij} a_{ij} = \sum_{i=1}^{m} a_{ij}^2.$$
   Therefore, by definition of the trace and the Frobenius norm, we have
$$\operatorname{tr}(A^\top A) = \sum_{j=1}^{n} (A^\top A)_{jj} = \sum_{j=1}^{n}\sum_{i=1}^{m} a_{ij}^2 = \|A\|_F^2.$$

2. By definition of the trace and the matrix product and exploiting the commutativity of the product in $\mathbb{R}$, we find
$$\begin{aligned}
\operatorname{tr}(A^\top B) &= \sum_{i=1}^{n} (A^\top B)_{ii} \\
&= \sum_{i=1}^{n}\sum_{k=1}^{m} a'_{ik} b_{ki} = \left(\sum_{i=1}^{n}\sum_{k=1}^{m} b'_{ik} a_{ki} = \sum_{i=1}^{n} (B^\top A)_{ii} = \operatorname{tr}(B^\top A) \right) \\
&= \sum_{i=1}^{n}\sum_{k=1}^{m} b_{ki} a'_{ik} \\
&= \sum_{k=1}^{m}\sum_{i=1}^{n} b_{ki} a'_{ik} \left(= \sum_{i=1}^{n} (BA^\top)_{kk} = \operatorname{tr}(BA^\top) \right)
\end{aligned}$$

3. Let $A = U\Sigma V^\top$, then

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\begin{aligned}
\|A\|_F^2 = \mathrm{tr}(A^\top A) &\overset{SVD}{=} \mathrm{tr}((U\Sigma V^\top)^\top U\Sigma V^\top) \overset{\text{reverse prod. of transpose}}{=} \mathrm{tr}(V\Sigma^\top U^\top U\Sigma V^\top) \\
&\overset{\text{U ortho.}}{=} \mathrm{tr}(V\Sigma^\top \Sigma V^\top) \overset{\text{symmetry of tr}}{=} \mathrm{tr}(\Sigma V^\top V\Sigma^\top) \\
&\overset{\text{V ortho.}}{=} \mathrm{tr}(\Sigma\Sigma^\top) \overset{\text{Def. tr}}{=} \sum_{i=1}^{n} (\Sigma\Sigma^\top)_{ii} \\
&\overset{\text{Def. mat. prod.}}{=} \sum_{i=1}^{m}\sum_{j=1}^{n} \Sigma_{ij}\Sigma_{ji}^\top = \sum_{i=1}^{m}\sum_{j=1}^{n} \Sigma_{ij}^2 \overset{\Sigma \in \mathbb{R}^{m\times n}\text{ diag.}}{=} \sum_{i=1}^{\min(m,n)} \Sigma_{ii}^2.
\end{aligned}
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