Curve Fitting

Assume you were given the following data:

z	-1	0	1
y	-1	3	1

1. **Polynomial Interpolation:**
   Find coefficients $c_i \in \mathbb{R}$ for the following parabola (polynomial of degree 2)
   $$f(z) = c_0 + c_1 z + c_2 z^2,$$
   such that $f(z_i) = y_i$ for all three measurements $i = 1,2,3$ from the table above. Therefore, set up and solve a linear system of the form $Ax = b$, where $x = (c_0, c_1, c_2)$. Draw the measurements and the parabola into one plot.

2. **Univariate Linear Regression:**
   Now assume a linear model of the form
   $$f(z) = c_0 + c_1 z.$$
   Consider $x = (c_0, c_1)$. Determine a matrix A and a vector b, such that
   $$\sum_{i=1}^{3} (f(z_i) - y_i)^2 = \|Ax - b\|_2^2.$$
   Solve the least squares problem
   $$\min_{x \in \mathbb{R}^2} \|Ax - b\|_2^2,$$
   to determine the coefficients $x = (c_0, c_1)$ by solving the *normal equation* (explanation follows later in the lecture)
   $$A^T A x = A^T b.$$
   Draw the measurements and the straight line into one plot.

Solution:

1. INTERPOLATION:
   - Aim: We want to find c_0, c_1, c_2, such that
     $f(z_1) = c_0 \cdot 1 + c_1 \cdot z_1 + c_2 \cdot z_1^2 = y_1,$
     $f(z_1) = c_0 \cdot 1 + c_1 \cdot z_2 + c_2 \cdot z_2^2 = y_2,$
     $f(z_1) = c_0 \cdot 1 + c_1 \cdot z_3 + c_2 \cdot z_3^2 = y_3.$
   - With $A = (a_{ij})_{ij} := (z_i^{j-1})_{ij}$, $x := (c_0, c_1, c_2)^T$, $b := (y_1, y_2, y_3)^T$ this is equivalent to:
     $$Ax = b.$$
   - Inserting the data points (z_i, y_i) gives:
     $$A = \begin{array}{c} \text{(I)} \\ \text{(II)} \\ \text{(III)} \end{array} \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}.$$

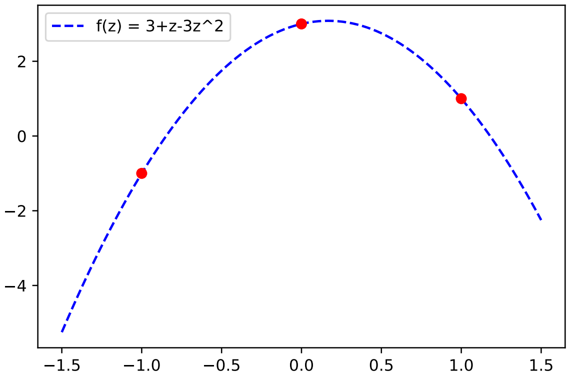
   (II) $\Rightarrow x_1 = 3$
   (I) $\Rightarrow \underbrace{x_1}_{=3} - x_2 + x_3 = -1 \quad \Leftrightarrow \quad x_3 = x_2 - 4$
   (III) $\Rightarrow \underbrace{x_1}_{=3} + x_2 + \underbrace{x_3}_{=x_2-4} = 1 \quad \Leftrightarrow \quad x_2 = 1$

- All in all:

$$x = \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -3 \end{pmatrix}.$$

- Thus, our fitting parabola is given by:

$$f(z) = 3 + z - 3z^2.$$



2. <u>LINEAR REGRESSION:</u>
   - Aim: Find c_0, c_1 such that

$$f(z_i) = c_0 + c_1 z_i \approx y_i \quad \forall i = 1, 2, 3.$$

   - As above we define

$$A := (a_{ij})_{ij} = (z_i^{j-1})_{ij} = \begin{pmatrix} 1 & z_1 \\ 1 & z_2 \\ 1 & z_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix},$$

$$x := \begin{pmatrix} c_0 \\ c_1 \end{pmatrix}, \quad b := \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

   - Then by construction of A, x and b we find:

$$\|Ax - b\|_2^2 = \left\| \underbrace{\begin{pmatrix} 1 & z_1 \\ 1 & z_2 \\ 1 & z_3 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix}}_{= \begin{pmatrix} c_0 + z_1 c_1 \\ c_0 + z_2 c_1 \\ c_0 + z_3 c_1 \end{pmatrix} = \begin{pmatrix} f(z_1) \\ f(z_2) \\ f(z_3) \end{pmatrix}} - \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \right\|_2^2 = \left\| \begin{pmatrix} f(z_1) - y_1 \\ f(z_2) - y_2 \\ f(z_3) - y_3 \end{pmatrix} \right\|_2^2 \overset{\text{Def. of } \|\cdot\|_2}{=} \sum_{i=1}^{3} (f(z_i) - y_i)^2.$$

   - We now solve the normal equation

$$A^T A x = A^T b$$

   to determine the solution of the least squares problem:

$$\min_{x=(c_0,c_1) \in \mathbb{R}^2} \|Ax - b\|_2^2 = \min_{c_0, c_1} \sum_{i=1}^{3} [(c_0 + c_1 z_i) - y_i]^2.$$

   - We find

$$A^T A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix},$$

$$A^T b = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

- Thus:

$$A^T Ax = A^T b \quad \Leftrightarrow \quad \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad \Leftrightarrow \quad \begin{matrix} c_0 = 1 \\ c_1 = 1 \end{matrix}.$$

- Therefore the best linear fit in the least squares sense is given by

$$g(z) = 1 + z.$$