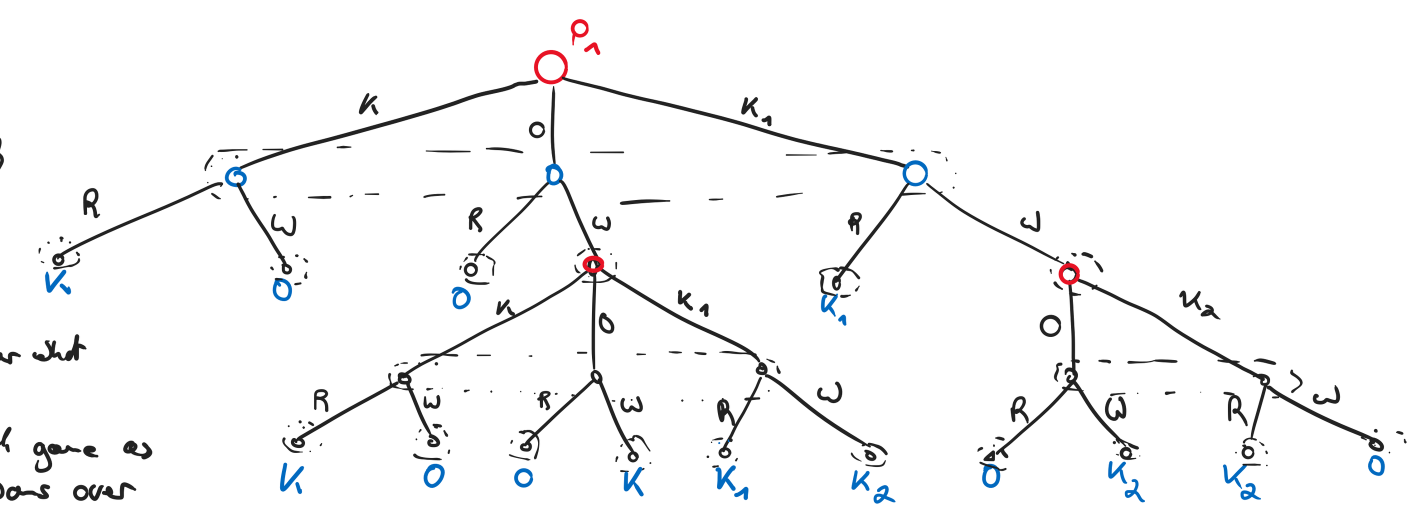
Robbery Game

"reduced extensive tree"



a) P1 & P2 actions:
$\mathcal{U} = \{K, O, K_1, K_2\}$
$\mathcal{U}_i \subset \mathcal{U}$
$\mathcal{V} = \{\text{Rob}, \text{Wait}\} := \{R, W\}$
$\mathcal{V}_i \subseteq \mathcal{V}$
outcome $:= \{K, O, K_1, K_2\}$
the game can be reduced, for example in the last stage it is clear what P1 & P2 will play.
Yes, it is a feedback game as
A) no information spans over multiple stages
B) P1 node is always root of subgame

b) we solve with backward induction. First the subtree $(u_1, V_1) = (O, W)$

	R	W
K	K	O
O	O	K
K_1	K_1	K_2

we can see that a NE is:
$$(\hat{u}_2, \hat{V}_2) = \left(\begin{bmatrix}1/2\\1/2\\0\end{bmatrix}, \begin{bmatrix}1/2\\1/2\end{bmatrix}\right) \text{ with value } \frac{K}{2}$$

thus, in the tree we can replace the subgame $(\underset{u_1}{O}, \underset{V_1}{W}) \to \frac{K}{2}$

Similarly, $(u_1, V_1) = (K_1, W)$ node can be replaced with

	R	W
O	O	K_2
K_2	K_2	O

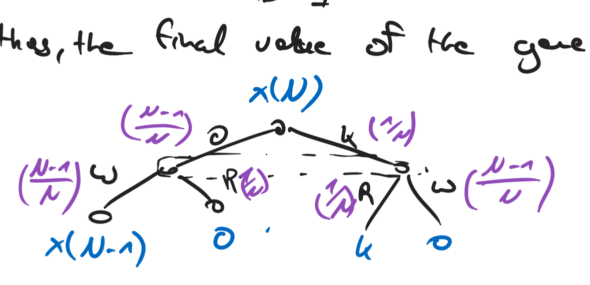
$\Rightarrow \text{value}(K_1, W) = K_2/2$

Thus, we can formulate the final subgame:

	R	W
K	K	O
O	O	$K/2$
K_1	K_1	$K_2/2$

a valid NE for this subgame is:
$$(\hat{u}_1, \hat{V}_1) = \left(\begin{bmatrix}1/3\\2/3\\0\end{bmatrix}, \begin{bmatrix}1/3\\2/3\end{bmatrix}\right)$$
thus, the final value of the game is $K/3$.

c) "all-in-strategy"
probabilities in purple



let $x(n)$, $n \in \{1, \dots, N\}$ be the expected outcome of the game where there are n days left. Then:

$$x(n) = \frac{1}{N}\left(\frac{1}{N}K + \frac{N-1}{N}O\right) + \frac{N-1}{N}\left(\frac{1}{N}\cdot O + \frac{N-1}{N}x(n-1)\right)$$

for $x(1)$ we have $x(1) = K$, $x(2) = K/2$, ... $\Rightarrow$ generally:

$$x(N) = \left(\frac{1}{N}\right)^2 K + \left(\frac{N-1}{N}\right)^2 x(N-1) = \left(\frac{1}{N}\right)^2 K + \left(\frac{N-1}{N}\right)^2\left(\left(\frac{1}{N}\right)^2 K + \left(\frac{N-1}{N}\right)^2 x(N-2)\right) = \dots$$

which can be solved inductively: $x(1) \to x(2) \to \dots \to x(N)$

Yes, it is a mixed strategy.

d) Yes, the behav. strategy would be

P1: $\mathcal{I}_n$: $\gamma^b(\mathcal{I}_n) = \begin{bmatrix}1/n\\1-1/n\end{bmatrix}$

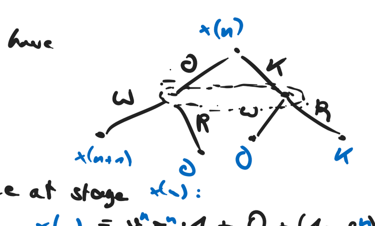
P2: Ω_n: $\omega^b(\Omega_n) = \begin{bmatrix}1/n\\1-1/n\end{bmatrix}$

for all n, so $n = N, N-1, \dots, 1$

e) actions: $\gamma^i \in \left\{\begin{bmatrix}\gamma_K^i\\1-\gamma_K^i\end{bmatrix}\right\}$, $\sigma^i \in \left\{\begin{bmatrix}\sigma_R^i\\1-\sigma_R^i\end{bmatrix}\right\}$ for P1, P2 respectively and γ_K^i, σ_R^i the probabilities of transport and robbing

at stage $n = N$ the NE strategies are obviously
$(\gamma^{*N}, \sigma^{*N}) = \left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\right)$ with value $x(N) = 1$
at $n \in 1 \dots N-1$:

we have



For game n:

	R	W
K	1	O
O	O	$x(n+1)$

Value at stage $x(n)$:
$$x(n) = \gamma_K^n \sigma_R^n \cdot 1 + O + (1-\gamma_K^n)(1-\sigma_R^n)\, x(n+1)$$
$$x(N) = 1$$

deriving w.r.t. the actions to min/max $x(n)$, we yield:

$\min_{\gamma_K^n} x(n)$: $\nabla_{\gamma_K^n} x(n) = \sigma_R^n - (1-\sigma_R^n)\, x(n+1) \overset{!}{=} 0$

$\max_{\sigma_R^n} x(n)$: $\nabla_{\sigma_R^n} x(n) = \gamma_K^n - (1-\gamma_K^n)\, x(n+1) \overset{!}{=} 0$

$$\Rightarrow \hat{\sigma}_R^n = \frac{x(n+1)}{1+x(n+1)} = \hat{\gamma}_K^n = \frac{1}{\frac{1}{x(n+1)} + 1} \quad (1)$$

we plug this back into $x(n)$:
$$x(n) = \left(\frac{x(n+1)}{1+x(n+1)}\right)^2 + \left(\frac{1}{1+x(n+1)}\right)^2 x(n+1)$$
$$= \frac{x(n+1)^2 + x(n+1)}{(1+x(n+1))^2}$$

using $x(N) = 1$ this sequence can be solved: $x(n) = \frac{1}{N-n+1}$

$\to x(N-1) = 1/2$
$x(N-2) = 1/3$
$\vdots$
$x(1) = 1/N$

thus we have found the value of the game $x(1) = 1/N$

We plug this value back into (1) and yield
$$\sigma_R^{*n} = \gamma_K^{*n} = \frac{1}{\frac{1}{x(n+1)} + 1} = \frac{1}{N-n+1}$$

From lecture 10 slide 32/35
A) NE strategies are played after stage n
B) ∀ strategies before stage n
is called subgame-perfect behav. saddle-point eq.
and is a behavioral saddle-point equilibrium.
since A) and B) are fulfilled, $\hat{T}$ is fulfilled.

Exercise 2

a) Reformulating in state space and induction

$$x_{k+1} = \underbrace{\begin{bmatrix}1&0\\0&1\end{bmatrix}}_{A} x_k + \underbrace{\begin{bmatrix}1\\1\end{bmatrix}}_{B_1} u_k + \underbrace{\begin{bmatrix}1\\1\end{bmatrix}}_{B_2} v_k \quad \text{where } x_k \begin{bmatrix}x_k^1\\x_k^2\end{bmatrix}$$

where $P_{2,1} = Q_1 = S_1 = \begin{bmatrix}1&0\\0&0\end{bmatrix}$, $S_2 = \begin{bmatrix}0&0\\0&1\end{bmatrix} = Q_2 = P_{2,1}$
$R_1 = R_2 = 1$ for S_1, S_2

I can use the best response strategies derived in the lecture slide 26/34

$$\hat{u}_0 = -\Big(1 + \underbrace{[1\ 1]\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}}_{1}\Big)^{-1} \underbrace{[1\ 1]\begin{bmatrix}1&0\\0&0\end{bmatrix}}_{[1\ 0]}\left(\begin{bmatrix}1&0\\0&1\end{bmatrix} x_0 + \begin{bmatrix}1\\1\end{bmatrix}\hat{v}_0\right)$$
$$= -\frac{1}{2}(x_0^1 + \hat{v}_0) \quad (*)$$

$$\hat{v}_0 = -\Big(1 + \underbrace{[1\ 1]\begin{bmatrix}0&0\\0&1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}}_{1}\Big)^{-1} [1\ 1]\begin{bmatrix}0&0\\0&1\end{bmatrix}\left(\begin{bmatrix}1&0\\0&1\end{bmatrix} x_0 + \begin{bmatrix}1\\1\end{bmatrix}\hat{u}_0\right)$$
$$= -\frac{1}{2}(x_0^2 + \hat{u}_0) \overset{(*)}{=} -\frac{1}{2}\left(x_0^2 - \frac{1}{2}x_0^1 - \frac{1}{2}\hat{v}_0\right) = \frac{1}{4}\hat{v}_0 - \frac{1}{2}x_0^2 + \frac{1}{4}x_0^1 \Rightarrow \hat{v}_0 = \frac{4}{3}\left(-\overset{\frac{2}{3}}{\frac{1}{2}}x_0^2 + \overset{\frac{1}{3}}{\frac{1}{4}}x_0^1\right)$$

Thus we have found the NE strategies:
$$(u_0^{NE}(x_0), v_0^{NE}(x_0)) = \cancel{\left(-\frac{1}{2}x_0^1, -\frac{1}{2}x_0^2\right)} = \left(-\frac{2}{3}x_0^1 + \frac{1}{3}x_0^2, -\frac{2}{3}x_0^2 + \frac{1}{3}x_0^1\right)$$

b) $J(x, u, v) = (x_0^1)^2 + (x_0^2)^2 + u_0^2 + v_0^2 + (x_1^1)^2 + (x_1^2)^2$
$= (x_0^1)^2 + (x_0^2)^2 + u_0^2 + v_0^2 + (x_0^1 + u_0 + v_0)^2 + (x_0^2 + u_0 + v_0)^2$

$$\nabla_{u_0} J = \cancel{2}u_0 + \cancel{2}(x_0^1 + u_0 + v_0) + \cancel{2}(x_0^2 + u_0 + v_0) \overset{!}{=} 0$$
$$\Rightarrow u_0^* = -\frac{1}{3}(x_0^1 + x_0^2 + 2v_0) \quad (1)$$

analogeously: $v_0^* = -\frac{1}{3}(x_0^1 + x_0^2 + 2u_0) \quad | (1)$
$$= -\frac{1}{3}\left(x_0^1 + x_0^2 - \frac{2}{3}(x_0^1 + x_0^2 + 2v_0^*)\right) = -\frac{1}{3}\left(\frac{1}{3}(x_0^1 + x_0^2) - \frac{4}{3}v_0^*\right)$$
$$v_0^*\Big(\underbrace{1 - \frac{4}{9}}_{5/9}\Big) = -\frac{1}{9}(x_0^1 + x_0^2) \quad \Big| \cdot \frac{9}{5}$$
$$v_0^* = -\frac{1}{5}(x_0^1 + x_0^2) = u_0^* \quad \text{by symmetry}$$

$$\left(\overset{u^{NE}}{-\frac{2}{3}x_0^1 + \frac{1}{3}x_0^2}, \overset{v^{NE}}{-\frac{2}{3}x_0^2 + \frac{1}{3}x_0^1}\right)$$

thus, the optimal cost to go is
$$J^*(x, u^*, v^*) = V^*(x_0) = (x_0^1)^2 + (x_0^2)^2 + 2\left(-\frac{1}{5}(x_0^1 + x_0^2)\right)^2 + \left(x_0^1 - \frac{2}{5}(x_0^1 + x_0^2)\right)^2 + \left(x_0^2 - \frac{2}{5}(x_0^1 + x_0^2)\right)^2$$

c) for the cost $J(x_0, u_0^{NE}, v_0^{NE}) = \cancel{(x_0^1)^2} + \cancel{(x_0^2)^2} + \left(\frac{1}{3}\right)^2\left[(-2x_0^1 + x_0^2)^2 + (-2x_0^2 + x_0^1)^2\right] + \left(\frac{2}{3}x_0^1 - \frac{1}{3}x_0^2\right)^2 + \left(\frac{2}{3}x_0^2 - \frac{1}{3}x_0^1\right)^2$

comparing with $J^* = \cancel{(x_0^1)^2} + \cancel{(x_0^2)^2} + 2\left(-\frac{1}{5}(x_0^1 + x_0^2)\right)^2 + \left(x_0^1 - \frac{2}{5}(x_0^1 + x_0^2)\right)^2 + \left(x_0^2 - \frac{2}{5}(x_0^1 + x_0^2)\right)^2$

$J^* - \!\!/ = \frac{2}{5^2}(x_0^1 + x_0^2)^2 + \left(\frac{3}{5}\right)^2 (x_0^1)^2 - 2\frac{3}{5}\frac{2}{5}x_0^1 x_0^2 + \left(\frac{2}{5}\right)^2 (x_0^2)^2 + \left(\frac{3}{5}\right)^2 (x_0^2)^2 - 2\frac{3}{5}\frac{2}{5}x_0^1 x_0^2 + \left(\frac{2}{5}\right)^2 (x_0^1)^2$

$J^{NE} - \!\!/ = \left(\frac{1}{3}\right)^2 (1+4+4+1)(x_0^1)^2 + \left(\frac{1}{3}\right)^2 (1+4+4+1)(x_0^2)^2 + \left(\frac{1}{3}\right)^2 (-4)\cdot 4\, x_0^1 x_0^2 + 2\cdot\frac{10}{9} x_0^1 x_0^2 - 2\frac{10}{9} x_0^1 x_0^2$
$= a_N (x_0^1 + x_0^2)^2 + b_N x_0^1 x_0^2$ where $a_N = \frac{10}{9}$, $b_N = -\frac{16}{9} - 2\frac{10}{9} = -\frac{36}{9}$

$J^* - \!\!/ = \left(\frac{2+9+4}{5^2}\right)(x_0^1)^2 + \left(\frac{2+9+4}{5^2}\right)(x_0^2)^2 + \left(\frac{4-12-12}{5^2}\right) x_0^1 x_0^2$

we can conclude: $J^{NE} > J^*$

d) see attached script ∇