GAME THEORY & CONTROL
ASSIGNMENT 3

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Please submit your solutions on Moodle.

You can either type them (in LaTeX, for example) or scan your handwritten solutions.

If you are scanning your handwritten solutions, please create a PDF that is readable and small in size. You can use any smartphone app that allows to scan documents to PDF (this feature is present in both the Dropbox app and the Google Drive app, plus many others). Do not send plain photos of the pages.

The deadline for this assignment is 20 December 2024, at 23:59.

Please write your solutions in a legible form, include all the steps of your reasoning, indicate what question you are answering, and don't include any solution attempt that you don't want us to grade.

Verify your answers by trying to get the same results in different ways, checking if they are compatible with the theoretical results that you know and with your intuition. You have plenty of time. Typos and errors by distraction are not acceptable.

This assignment contributes towards 10% of the final score for the course and must be done independently by each student. No group work is allowed.

Each "letter" question is worth 1 point, unless specified otherwise.
Make sure you answer everything to get the full point for each question. No fraction of points are awarded.
Take your time to double-check that you have not missed any part of each question.

Exercise 1 Robbery game

Consider the following game.

Player 1 goes to the bank every day, for N days. He has to transport K valuable objects to be put in a safety box. He can decide when to transport them, as long as all K of them are in the bank by the end of the N days. These are for example valid choices:

$$\begin{array}{ccccc} \text{day 1} & \text{day 2} & \text{day 3} & \dots & \text{day N} \end{array}$$
$$\begin{bmatrix} K & 0 & 0 & \dots & 0 \end{bmatrix}$$

or any strategy

$$\begin{array}{ccccc} \text{day 1} & \text{day 2} & \text{day 3} & \dots & \text{day N} \end{array}$$
$$\begin{bmatrix} K_1 & K_2 & K_3 & \dots & K_N \end{bmatrix}$$

as long as $\sum_{i=1}^{N} K_i = K$, and K_i are all non-negative integers.

Player 2 wants to rob Player 1. He can only rob him once. Therefore, everyday he can decide whether to rob him or to wait until next day, without knowing how many valuable items he is carrying. However, at the end of each day he knows how many items are left to be transported.

Perform the following tasks.

a) Represent the game in extensive form, for $N = 3$ and $K = 2$. The outcome of the game is the number of stolen items. Include all the necessary data: the information sets, the outcomes, the actions, etc. Is it a feedback game?

b) What is the value of the game for $N = 3$ and $K = 2$? What is the value of the subgame that you obtain when one day has passed, Player 1 didn't transport any item, and Player 2 didn't rob Player 1?

For the following tasks, consider generic N and K.

c) Consider generic N and K, and the following strategy:

   - Player 1, before day 1, picks with equal probability $1/N$ one of the N days, and plans to transport all the K items on that day.
   - Player 2, before day 1, picks with equal probability $1/N$ one of the N days, and plans the robbery on that day.

   Let us call this strategy the *all-in strategy*.

   - What is the expected outcome of this strategy?
   - Is this a mixed strategy?

d) Does an equivalent behavioral strategy (that is, a behavioral strategy that induces the same probability distribution on the leaves) exist? If yes, what is the equivalent behavioral strategy of Player 1 and of Player 2 at stage 1 (day 1)? What is the equivalent behavioral strategy when one day has passed, Player 1 didn't transport any item, and Player 2 didn't rob Player 1?

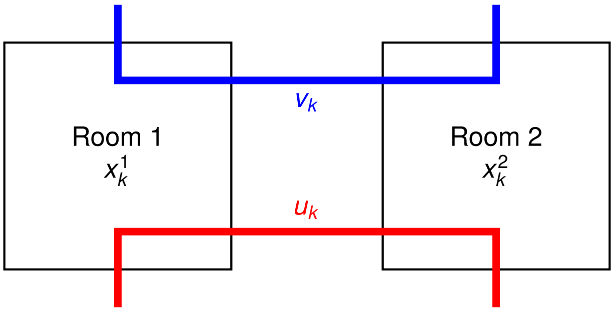
For the following task, consider a simplified version of this game in which Player 1 has only one item to transport ($K = 1$).

e) Prove that the value of the game is $1/N$, and that the feedback behavioral saddle point equilibrium at each stage (day) $n = 1, \dots, N$ is

   - for Player 1, to transport the item with probability $1/(N - n + 1)$
   - for Player 2, to rob Player 1 with probability $1/(N - n + 1)$

   where n is the number of stages (i.e., days) that have passed.

Exercise 2 Two-player LQR vs One-player LQR



Consider two agents aiming to regulate the temperature in their respective room, assumed to be perfectly insulated. Let x_k^1 and x_k^2 be the deviation from the desired temperature in room 1 and room 2, respectively. Since the heating/cooling system of room 1 goes through room 2 and vice versa, the discrete-time evolution of the temperature in room i is

$$x_{k+1}^i = x_k^i + u_k + v_k, \quad k = 0, \dots, K-1,$$

where u_k is the control input of agent 1 and v_k is the control input of agent 2. Assume that the heating/cooling system can provide both positive and negative energy, i.e., $u_k, v_k \in \mathbb{R}$.

Remark. *While we assumed no losses, the system dynamics can be interpreted as the linearization of a (possibly nonlinear) temperature evolution around an equilibrium point.*

Assume first that the system consists of a single stage, i.e., $K = 1$.

a) Agent 1 and 2 aim to minimize a combination of the quadratic deviation from the desired temperature and their control effort. In particular, their individual cost is

$$J^1(x,u) = (x_0^1)^2 + u_0^2 + (x_1^1)^2,$$
$$J^2(x,v) = (x_0^2)^2 + v_0^2 + (x_1^2)^2.$$

Find the Nash Equilibrium strategy $u_0^{\mathrm{NE}}(x_0), v_0^{\mathrm{NE}}(x_0)$, and the value of the game $V^i(x_0)$ for $i \in \{1,2\}$ through backward induction on the state.

b) Assume now that the two agents cooperate and aim to minimize their social cost, defined as

$$J(x,u,v) = J^1(x,u) + J^2(x,v).$$

Find the optimal inputs $u_0^*(x_0), v_0^*(x_0)$, and the optimal cost-to-go $V^*(x_0)$.

c) Compare the social cost resulting from the Nash Equilibrium inputs of a) and the optimal inputs of b). Give an expression for the price of anarchy

$$\mathrm{PoA} = \frac{J(x_0, u_0^{\mathrm{NE}}, v_0^{\mathrm{NE}})}{J(x_0, u_0^*, v_0^*)},$$

as a function of initial state x_0. Show that the price of anarchy is *strictly* larger than 1.

Hint. *You may assume $x_0 \neq 0$.*

Consider now the system with $K \in \{0, \dots, 100\}$ stages.

d) **(3 points)** Solve a), b), and c) numerically compute the price of anarchy for the initial condition $x_0 = (0.5, 0.5)$ for every $K \in \{0, \dots, 100\}$ and plot it.